JERK-FREE GENEVA WHEEL DRIVING

Dr. E. A. DIJKSMAN

Technological University, Eindhoven, The Netherlands

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Abstract—A jerk-free motion of an internal Geneva wheel is obtained by using a special driving mechanism. For this mechanism a four-bar mechanism is chosen from which the coupler point drives the wheel. By special choice of the four-bar linkage the coupler curve can be made symmetrical and containing two Ball's points. For each number of stations the dimensions of the mechanism are tabled for optimal transmission angles and next to it also in those cases where the coupler point lies in one line with the two moving turning-points.

Drawings of the intermittent mechanism are simplified in cases where the circling-point curve degenerates.

1. INTRODUCTION

NORMALLY a Geneva wheel is driven by a crank with a crankpin. At the beginning as well as at the end of the intermittent motion at least two grooves of the Geneva wheel are then tangential to the circle traced by the crankpin. At those positions the normal acceleration of the rotating crankpin will be transmitted to the wheel with an impact. Hence, the wheel has an infinitely great *jerk*† at such positions. This leads to an abrupt change in force on the wheel to the effect of undesirable vibrations in the mechanism.

With an internal Geneva wheel this drawback can be removed by using a *four-bar linkage* as a driving mechanism instead of a simple crank.

Such a four-bar linkage, indicated henceforth by (A_0ABB_0) , is a closed kinematic chain consisting of 4 links, one of them being the *fixed link* (A_0B_0) . The opposite link, the *coupler* (AB) , is the base of a *coupler-triangle*, the vertex of which, the *coupler-point* K , describes the *coupler curve* when the mechanism is moved (see Fig. 1).

If the driving pin is now situated in a coupler point it is possible to trace an inflection point at the moment of entering or leaving the Geneva wheel. The tangents to the coupler curve in the two symmetrically situated inflection points of the curve must then coincide with the corresponding grooves of the wheel.

To avoid undesirable turns in the coupler curve, it is still better to replace the inflection points by what is known as undulation points or Ball's points [4].

At a Ball's point the coupler curve has a four-point contact with its tangent, through which the coupler curve, at least for a great part, runs on one side of this tangent. As a consequence, the motion of the turning table can begin and end with a zero angular jerk.

† By jerk is understood the *time-derivative of the acceleration*.

considered. According to whether A_0B_0 , A_0A or AB is the smallest, we speak of a *drag-link*, a *crank-and-rocker* or a *double-rocker mechanism*. The coupler of the last mechanism must be driven directly from the fixed link; if no gear wheels are used this is only possible by bringing the driving axle of the rotor in the moving pivot A . This is an extra complication,

which however, can be avoided by using one of the cognate mechanisms corresponding to the proposition of Roberts instead of the double rocker mechanism. We then get a crank-and-rocker mechanism with exactly the same coupler curve as the one we started from.

Up to the time of writing, no usable solutions of the problem have been found for the drag-link mechanism. We can, therefore, as far as the solutions are concerned, confine ourselves exclusively to the crank-and-rocker mechanisms.

We know that symmetrical coupler curves are generated if $\overline{BA} = \overline{BB_0} = \overline{BK}$. The axis of symmetry thereby goes through the fixed turning-point B_0 and makes an angle of $((\pi/2) - \angle KAB)$ with the fixed link A_0B_0 . It is also perpendicular to A_0C_0 , a side of the fixed link triangle $A_0B_0C_0$, which is similar to the coupler-triangle ABK . Turning the coupler-triangle 180° about its coupler causes the axis of symmetry also to turn 180° about the fixed link. If the crankpoint arrives at the opposite point with respect to the fixed link, the coupler point coincides with its opposite position with respect to the axis of symmetry. Hence, the two positions of the crank-and-rocker mechanism with the crank-point A in line with A_0B_0 , have their corresponding coupler points at the intersections of the coupler curve with the axis of symmetry.

2. KINEMATIC FUNDAMENTAL THEORY [3]

The locus of those points of the moving coupler plane at which, at the moment of observation, the radius ρ of curvature in the described curves has a stationary value, is called the *circling-point curve*[†] and is indicated by k_u . Thus, in accordance with this definition, k_u is the locus of those coupler points for which $(d\rho/dt) = 0$. Its equation in the polar co-ordinates r and φ reads:

$$\frac{1}{r} = \frac{1}{l \sin \varphi} + \frac{1}{m \cos \varphi} \quad (1)$$

The abscissa axis of the corresponding Cartesian co-ordinates coincides with the pole tangent p , and the ordinate with the pole normal n (see Fig. 3). The positive sense of y is taken in the direction from P to the inflection pole W . The positive sense of x changes into the positive sense of y after rotating the abscissa axis counter-clockwise about P through $\pi/2$ radians.

If X_w is the intersection of a path normal PX with the inflection circle, the angle φ is determined by the equations: $\varphi = \angle X_w P x$ where $0 \leq \varphi < \pi$.

r is the distance from the origin P to the point of path X . If such a point is lying in the upper semi-plane, then r is considered positive, in the other case negative.

The length m is given by the equation:

$$m = -\frac{3\delta}{(d\delta/ds)}, \quad (2)$$

where δ is the diameter of the inflection circle and s the length of arc along the fixed polode. δ is always positive.

The length l is given by the equation

$$\frac{3}{l} = \frac{1}{R} + \frac{1}{\delta} = \frac{2}{R} - \frac{1}{R_0} \quad (3)$$

[†] Or cubic of stationary curvature.

FIG. 3.

where R and R_0 represent the ordinates of the centres of curvature of, respectively, the moving and the fixed polode in the pole. It is known that k_u has a double-point in the origin P with radii $\frac{1}{2}l$ and $\frac{1}{2}m$, the former coinciding with the pole normal, the latter with the pole tangent.

In accordance with the definition, the turning-points A and B of a four-bar linkage (A_0ABB_0) lie on the circling-point curve. Therefore, the circling-point curve can be considered to be the locus of possible points of choice A and B of the four-bar linkage.

The locus of those centres of curvature, of which the corresponding points of the path lie on the circling-point curve, is called the *instantaneous* centering-point curve and is indicated by k_a . The centering-point curve k_a is the circling-point curve of the inverse motion.

In accordance with the definition the centres of pivot on frame, A_0 and B_0 , of a four-bar linkage (A_0ABB_0) lie on the centering point curve, so curve k_a may also be seen as the locus of possible points of choice A_0 and B_0 of the four-bar linkage.

In the polar co-ordinates r_0 and φ , k_a is given by the equation

$$\frac{1}{r_0} = \frac{1}{l_0 \sin \varphi} + \frac{1}{m \cos \varphi}, \quad (4)$$

where l_0 is defined by the equation

$$\frac{3}{l_0} = \frac{2}{R_0} - \frac{1}{R} = \frac{1}{R_0} - \frac{1}{\delta} \quad (5)$$

or by

$$\frac{1}{l_0} = \frac{1}{l} - \frac{1}{\delta} \quad (6)$$

3. BALL'S POINT OR THE UNDULATION POINT U [4]

Ball's point is defined as that point of the path where the path tangent has a four-point contact with the path. This point lies on the inflection circle as well as on the circling-point curve. These curves generally intersect at $2 \times 3 = 6$ points, viz. the two isotropic points, the triple counting pole and a sixth point, Ball's point. If the pole is a triple intersection, a real Ball's point exists. In some special cases the inflection circle coincides with part of k_u and then any point of the inflection circle may be considered to be a Ball's point.

Generally Ball's point is identified by intersecting the inflection circle with the line through the pole in the asymptotic direction of the centering-point curve. This direction is fixed by the direction coefficient $-m/l_0$ (see Fig. 3).

4. DEGENERATIONS OF k_u AND k_a

Parts of this chapter which are printed in small type give rise to earlier derived driving mechanisms or to mechanisms unsuitable to our purpose.

4.1 The general cycloidal position of the four-bar linkage

If $d\delta/ds = 0$ (or $m^{-1} = 0$), it follows from (1) that either $\varphi = \frac{1}{2}\pi$, which is the equation of the *pole normal*, or that $r = l \sin \varphi$, which is the equation of a circle through P with its centre on the pole normal.

In this case the k_u curve is degenerated in the pole normal and a circle through $P(0, 0)$ and $L(0, l)$ with its centre on the pole normal n . In the same way we find the k_a curve to be

degenerated in the pole normal and a circle through $P(0, 0)$ and $L_0(0, l_0)$ with its centre on n .

The relation between the ordinates of the points $L(0, l)$, $L_0(0, l_0)$ and the inflection pole $W(0, \delta)$ is expressed by

$$\frac{1}{l} - \frac{1}{l_0} = \frac{1}{\delta}$$

(see Fig. 9). Hence, the three circles are interdependent. This interdependence can be designed as follows:

- choose X anywhere in the moving plane;
- connect X with P , L and W ;
- draw PY parallel to XW and find Y on XL ;
- draw YL parallel to PX and find L_0 on n .

Proof. From

$$\frac{\overline{PW}}{\overline{L_0P}} = \frac{\overline{PX}}{\overline{L_0Y}} = \frac{\overline{PL}}{\overline{L_0L}}$$

we find $\delta/(-l_0) = l/(l-l_0)$, which is another form of equation (6).

FIG. 4. Pericycloidal position: $R_0 < R < \frac{3}{2}R_0$, $-\delta < l < 0$, $-\frac{1}{2}\delta < l_0 < 0$.

FIG. 5. Pericycloidal position: $R = \frac{1}{2}R_0$, $l = -\delta$, $l_0 = -\frac{1}{2}\delta$, c.c.=cuspidal circle=inflection circle of the inverse motion.

FIG. 6. Cardioidal position (=kinematical inversion of the cardan position):
 $R = 2R_0$, $l = -\infty$, $l_0 = -\delta$.

FIG. 7. Evolvent position: $R = \infty$, $l = 3\delta$, $l_0 = -\frac{1}{2}\delta$.

FIG. 8. Elliptical or cardan position: $R = \frac{1}{2}R_0$, $l = \delta$, $l_0 = -\infty$.

FIG. 11. Hypocycloidal position: $\frac{4}{3}R_0 < R < R_0$, $0 < l < \frac{1}{2}\delta$, $0 < l_0 < \delta$.

The way in which the three interdependent circles increase and decrease is shown in a cycle of Figs. 4 to 11. Ball's point will here always be found in the inflection pole W , as the intersection of the inflection circle and the k_u curve.

4.1.1 Design of the driving mechanism. In the degenerated position of the four-bar linkage referred to above, there are three possibilities for assigning the places to the moving turning-points A and B on k_u (folding linkages not taken into account).

(a) B on n and A on the circle that is a branch of k_u . From the equation of Euler-Savary we have $\overline{PB}^2 = \overline{BB_0} \cdot \overline{BB_w}$, while $B_w = W = U = K$ and also $\overline{BA} = \overline{BB_0} = \overline{BK}$.

This leads to $\overline{PB}^2 = \overline{BW}^2$, so B coincides with the centre of the inflection circle, and $A = P$. From the theorem of Bobillier the centre of curvature corresponding to A lies on p . Generally speaking, however, the pole tangent p is not part of curve k_a . So in general no solution is possible. Only in the special case of a cardan position of the four-bar linkage, that is to say, if $R = \frac{1}{2}R_0$ (see Fig. 8), the circle forming part of k_a

FIG. 14.

(c) B on $c-k_u$ and A on n , as another branch of k_u (see Fig. 15). The construction then is as follows: Dependent on the desired angle 2τ , we start from a ratio l/l_0 , taken from Fig. 16, between the diameters of $c-k_u$ and $c-k_a$. Just as in case (b) we have $l > 0$ and $|l_0| \geq \delta$. (A negative value of l_0 means that $c-k_a$ and the inflection circle are on opposite sides of p .) With the aid of equation (6) we now draw the circles $c-k_u$, $c-k_a$ and the inflection circle. Moreover, the points P , B and B_0 lie on one straight line in such a way that B lies on $c-k_u$ and B_0 on $c-k_a$. On the assumption that $\overline{BB_0} = \overline{BK}$ we have

$$\left| \frac{\overline{PB}}{\overline{UB}} \right| = \left| \frac{\overline{PB}}{\overline{BK}} \right| = \left| \frac{\overline{PB}}{\overline{BB_0}} \right| = \left| \frac{l \sin \varphi}{(l_0 - l) \sin \varphi} \right| = \left| \frac{l}{(l - l_0)} \right| = \left| \frac{\delta}{l_0} \right| \leq 1.$$

Hence, besides $c - k_u$ a second locus can be indicated for point B , viz. the locus of those points for which the ratio of the distances to two fixed points U and P is given. This locus is the *circle of Apollonius* (see Fig. 15).

FIG. 16.

As $\overline{BA} = \overline{BB_0}$ the turning-point A is found at one of the two intersections of the pole normal with a circle with centre B and radius $\overline{BB_0}$. One of those intersections exactly coincides with the Ball's point, which can not be taken as turning-point A . The fixed point A_0 lies also on the pole normal and can be found by the equation $\overline{PA^2} = \overline{AA_0} \cdot \overline{AA_w}$. This follows from the Euler-Savary equation. The point A_w coincides with W , U and K . The corresponding coupler curve appears to have a path suitable for our purpose.

We are also able to determine the dimensions of the mechanism. We do this by using a number of equations, the derivation of which will, for the sake of brevity, be omitted. With

$$q = 1 - 2(l/l_0), \quad (7)$$

$$\tan^2 \tau = \frac{2q^3(q^2 + 1)(2q^2 + q + 1)^2}{(3q^2 + 1)^2} \quad (8)$$

Moreover,

$$\frac{a}{b} = \frac{2q^2}{3q^2 + 1} \left(\frac{q + 1}{2q^2 - q + 1} \right)^{\frac{1}{2}} \quad (9)$$

and

$$\frac{d}{b} = \frac{2}{3q^2 + 1} \left(\frac{2q^4 + 2q^3 + 4q^2 + q + 1}{q + 1} \right)^{\frac{1}{2}}, \quad (10)$$

while

$$\tan \gamma_1 = \frac{q - 1}{3q^2 + 1} (2q(q^2 + 1))^{\frac{1}{2}}. \quad (11)$$

Finally

$$\frac{\overline{B_0 E_0}}{b} = \frac{2}{(q + 1)(2q^2 + q + 1)} \left(\frac{2q(2q^2 - q + 1)(2q^4 + 2q^3 + 4q^2 + q + 1)}{q^2 + 1} \right)^{\frac{1}{2}} \quad (12)$$

and

$$\frac{\overline{E_0 U}}{b} = \frac{2q}{2q^2 + q + 1} \left(\frac{2q(2q^2 - q + 1)}{(q + 1)(q^2 + 1)} \right)^{\frac{1}{2}} \quad (13)$$

For a number of angles 2τ , the results are given in Table 1.

4.2 The cardan position of the four-bar linkage

When in addition to $d\delta/ds = 0$ (or $m^{-1} = 0$) also $R_0 = 2R$ (or $l_0^{-1} = 0$), the circling-point curve k_w is degenerated into the inflection circle and the pole normal. Then the centering-point curve k_a is degenerated in the pole tangent, in the pole normal and in the line at infinity (see Fig. 8). The cardan position may be presented as a special case of the general cycloidal position, as treated in section 4.1. Choosing the coupler point K also in the inflection pole W of the cardan position, we then have only one angle 2τ , for which the problem can be solved. In the cardan position however, the inflection circle forms part of k_w .

TABLE I

2τ	0°	15°	30°	45°	60°	72°	90°	105°	120°	135°	150°	165°	180°
a/b	0	0.0729	0.1629	0.2492	0.3262	0.3793	0.4427	0.4788	0.4978	0.4966	0.4698	0.4025	0
d/b	2	1.9254	1.8275	1.7241	1.6187	1.5334	1.4033	1.2915	1.1744	1.0482	0.9055	0.7263	0
γ_1	0°	-24.59°	-24.21°	-21.94°	-18.88°	-16.08°	-11.42°	-7.13°	-2.38°	$+3.03^\circ$	$+9.62^\circ$	$+19.03^\circ$	90°
$\mu_{\min}$	0°	4.63°	11.25°	18.77°	26.98°	33.99°	45.27°	47.95°	39.55°	32.02°	25.17°	18.63°	0°
$(\overline{B_0 E_0})/b$	0	0.8767	0.9473	0.9801	1.0036	1.0210	1.0488	1.0748	1.1039	1.1351	1.1646	1.1739	0
$(\overline{E_0 U_1})/b$	0	0.1537	0.2401	0.3017	0.3504	0.3837	0.4275	0.4600	0.4886	0.5114	0.5218	0.4966	0
l/l_0	$\frac{1}{2}$	0.4068	0.3549	0.3095	0.2654	0.2290	0.1694	0.1113	$+0.0398$	-0.0560	-0.2029	-0.5018	$-\infty$
l/b	1	0.8628	0.8215	0.8115	0.8130	0.8234	0.8546	0.8974	0.9611	1.0573	1.2142	1.5293	∞

So besides W , any point of the inflection circle is a Ball's point. Hence the whole inflection circle is at our disposal for choosing coupler point K , which means that, in the cardan position, any angle 2τ can be realised by means of a four-bar linkage.

4.2.1 Design of the driving mechanism. Restricting ourselves to finite lengths of the links, the turning-points A and B can not be chosen on the inflection circle. Point P is an exception to this rule, for in this point the radius of curvature of the path may take any value between 0 and ∞ . By not choosing A or B in P we get a *folding* linkage, in which all links are superimposed, and the minimum transmission angle $\mu_{\min}=0$. By choosing A in P we get the mechanism of Fig. 12, which has already been discussed in part (a) of section 4.1.1.

If B and P coincide, the construction is as follows (see Fig. 17):

- Draw the inflection circle i.e., the pole tangent p and pole normal PW .
- Determine angle γ_1 in Fig. 18 at a given angle 2τ . (For instance if $2\tau=90^\circ$ then $\gamma_1=16.47^\circ$.)
- Take $\angle WPK=2|\gamma_1|$. Choose one of the two possible positions of the free side of this angle. Let K be the point of intersection of this side with the inflection circle.
- Draw a circle with centre $P=B$ and radius $\overline{BK}=\overline{BB_0}=\overline{BA}$.
- Let A be the point of intersection of this circle with the pole normal (n) *outside* the inflection circle.
- Let B_0 be that point of intersection of the circle mentioned under (d) with p that lies on the same side of n as K if $\gamma_1>0$, and on the opposite side of n if $\gamma_1<0$.
- Determine the fixed turning-point A_0 with the aid of the equation $\overline{PA}^2=\overline{AA_w}\cdot\overline{AA_0}$, where $A_w=W$. (The points A_w and A_0 always lie on the same half of the path normal with respect to turning-point A .)

The coupler curve, traced by the coupler point of a four-bar linkage, constructed as described above, turns out to have a suitable shape in all cases. The choice of point A *inside* the inflection circle leads to a less suitable coupler curve and to a smaller minimum transmission angle $\mu_{\min}$; while the angles 2τ between 30.73° and 178.2° can not be realised.

In connection with the described design we can derive a number of equations with which the dimensions of the driving mechanism and the driven Geneva wheel can be determined. They are:

$$1 + \cos 2\gamma_1 = \tan(\tau + \gamma_1) \quad (14)$$

$$\frac{a}{b} = \frac{\cos 2\gamma_1}{1 + \cos 2\gamma_1} \quad (15)$$

$$\frac{d}{b} = \frac{1}{\sin(\tau + \gamma_1)} \quad (16)$$

$$\frac{\overline{B_0E_0}}{b} = \frac{1 - \sin 2\gamma_1}{\sin \tau} \quad (17)$$

FIG. 17.

FIG. 18.

$$\frac{\overline{E_0 U}}{b} = \left| \cos 2\gamma_1 - \frac{1 - \sin 2\gamma_1}{\tan \tau} \right| \quad (18)$$

It may be seen that always

$$\mu_{\min} \geq 2 \arcsin \frac{1}{4}(5^{\frac{1}{2}} - 1) \approx 36^\circ,$$

which is a reasonable and permissible angle.

It will be clear that only when $\gamma_1 = 0$ does the coupler point K coincide with the inflection pole W . This will be the case when $2\tau = 2 \arctan 2 = 126.87^\circ$. Only then is the solution according to the general cycloidal position identical with the one according to the cardan position. For significant values of 2τ the results according to the cardan position are given in Table 2.

4.3 Degeneration of the k_u curve, when $l^{-1} = 0$ (or $R = 2R_0$)

If $l^{-1} = 0$, it follows from equation (1) that either $\varphi = 0$, which is the equation of p , or $r = m \cos \varphi$, which is the equation of a circle through P with its centre on p . Hence, the k_u curve degenerates into p and a circle through $P(0, 0)$ and $M(m, 0)$ with its centre on p . The k_a curve keeps its general shape. The circle that is a branch of k_u , and the inflection circle intersect at the pole and at the only Ball's point.

FIG. 19.

TABLE 2

2τ	0°	15°	30°	45°	60°	72°	90°	120°	135°	150°	165°	175°	180°
γ_1	45°	$41^\circ 07'$	$36^\circ 57'$	$32^\circ 26'$	$27^\circ 33'$	$23^\circ 20'$	$16^\circ 28'$	$3^\circ 21' 22''$	$-4^\circ 11'$	$-12^\circ 43'$	$-23^\circ 04'$	$-32^\circ 46'$	-45°
a/b	0	0.11905	0.21723	0.29812	0.36402	0.40695	0.45632	0.49828	0.49734	0.47454	0.40932	0.29278	0
d/b	$2^\dagger$	1.33280	1.27000	1.22170	1.18510	1.16250	1.13830	1.11881	1.11920	1.13030	1.16140	1.22478	$2^\dagger$
$(AK)/b$	$2^\dagger$	1.50674	1.59840	1.68804	1.77334	1.83644	1.91796	1.99658	1.99468	1.95094	1.84010	1.68176	$2^\dagger$
$(B_0 E_0)/b$	0	0.07025	0.15165	0.24742	0.36000	0.46315	0.64606	1.01858	1.24000	1.48000	1.73580	1.91200	2
$(E_0 U_1)/b$	0	0.06549	0.13107	0.19615	0.26061	0.31113	0.38279	0.48232	0.51484	0.52004	0.46641	0.33076	0
μ_1	90°	$86^\circ 54'$	$83^\circ 56'$	$81^\circ 06'$	$78^\circ 28'$	$76^\circ 36'$	$74^\circ 16'$	$72^\circ 06'$	$72^\circ 09'$	$73^\circ 16'$	$76^\circ 30'$	$81^\circ 18'$	90°
$\mu_2 = \mu_{\min}$	90°	$74^\circ 44'$	$63^\circ 32'$	$55^\circ 00'$	$48^\circ 28'$	$44^\circ 24'$	$39^\circ 52'$	$36^\circ 09'$	$36^\circ 22'$	$38^\circ 17'$	$44^\circ 10'$	$55^\circ 33'$	90°
θ	45°	$48^\circ 37'$	$51^\circ 57'$	$54^\circ 56'$	$57^\circ 33'$	$59^\circ 20'$	$61^\circ 28'$	$63^\circ 21'$	$63^\circ 19'$	$62^\circ 17'$	$59^\circ 26'$	$54^\circ 44'$	45°
h/b	0	0.31788	0.55106	0.71964	0.83228	0.88446	0.90392	0.78142	0.64892	0.48266	0.24512	0.07556	0
$(\rho T_1)/b$	0	0.00003	0.00239	0.00867	0.02215	0.04122	0.09127	0.28398	0.48820	0.83400	1.81040	4.25120	0
$(\rho T_2)/b$	0	0.00930	0.07053	0.21388	0.43245	0.64040	0.96570	1.44623	1.64135	1.81850	2.04000	2.46830	0

4.3.1 *The four-bar linkage in the position, where $R=2R_0$.* In this degenerated position of the four-bar linkage there are three possibilities for placing A and B on k_u .

(a) *B on p and A on $c-k_u$.* From the theorem of Hartmann it follows that $B_0=P$. As, in addition, $\overline{BA}=\overline{BB_0}=\overline{BK}$, point B can only be taken in the centre of $c-k_u$ (see Fig. 19).

Point A can be chosen freely on $c-k_u$, while the centre of curvature A_0 corresponding to A is given by the equation $\overline{PA}^2=\overline{AA_w} \cdot \overline{AA_0}$. Since the coupler point K is a Ball's point, this point turns out to lie exactly on the axis of symmetry of the coupler curve. This means that in any case $2\tau=180^\circ$. Hence the mechanism can be applied to drive a two station Geneva wheel only [1].

(b) *A on p and B on c-k_u*. In this case $A_0 = P$ (see Fig. 20). Consequently, points B , B_0 and A_0 are in a straight line. In Fig. 21 the coupler corresponding to this position is indicated by A_1B_1 . The position in which the coupler point has arrived at the opposite point with respect to the axis of symmetry, corresponds to the position in which turning-point A has arrived at the opposite point A_2 with respect to the fixed link. It can now be seen from the figure that the two positions of the coupler can be obtained from one another by rotating it about the rotation centre $P_{12} = B_0$. Moreover, $\triangle B_0A_1B_1 \equiv \triangle B_0A_2B_2$. So $\star B_2A_2B_0 = \star B_1B_0A_2$. This indicates that in the second position the coupler runs parallel to the fixed link, in which case k_u and k_a are both degenerated: this is the case which we have already discussed in part (b) of section 4.1.1.

(c) *Both A and B on c-k_u* (see Fig. 22). As $\star QAP = \star BMP$ and as from the theorem of Bobillier it follows that $\star QPA = \star BPM$, $\therefore \triangle PQA \parallel \triangle PBM$. Hence, $\star PQA = \star PBM = 90^\circ$. This means that the collineation axis $PQ \perp AB$.

In Fig. 23 the coupler triangle corresponding to this position is indicated by $A_1B_1K_1$ (P_1Q_1 being assumed to be perpendicular to A_1B_1). By rotating the crank A_0A , the coupler triangle arrives at a second position $A_2B_2K_2$ if A_2 lies opposite to A_1 with respect to the fixed link. The corresponding coupler points K_1 and K_2 are symmetrical with respect to the axis of symmetry of the coupler curve, as can be seen from the fact that the two coupler triangles can reach each other's position by rotating them about the rotation centre $P_{12} = B_0$, and from the fact that the axis of symmetry goes through B_0 .

A third position of the coupler triangle, which is in an opposite position to the second with respect to the fixed link, is shown in the same figure. It will be clear that any geometrical property attached to four-bar linkage No. 3 can also be applied to four-bar linkage No. 2.

Now the first and the third four-bar linkages have the crank and the fixed link in common, while the remaining four sides form a rhombus. It will be shown by means of Fig. 24 that the collineation axes q_1 and q_3 of the linkages 1 and 3 are parallel, even in any other position of the common crank.

Proof. From $\triangle P_1A_1B_1 \parallel \triangle A_3P_3B_3$ it follows that $\overline{P_1B_1} \cdot \overline{P_3B_3} = \overline{A_1B_1} \cdot \overline{A_3B_3}$, so with $\triangle Q_1B_1B_0 \parallel \triangle B_0B_3Q_3$ we find that

$$\overline{Q_1B_1} \cdot \overline{Q_3B_3} = \overline{B_1B_0} \cdot \overline{B_3B_0} = \overline{A_1B_1} \cdot \overline{A_3B_3} = \overline{P_1B_1} \cdot \overline{P_3B_3}.$$

This may be written as

$$(\overline{P_3B_3}/\overline{Q_3B_3}) = (\overline{Q_1B_1}/\overline{P_1B_1}),$$

so with

$$\star P_3B_3Q_3 = \star Q_1B_1P_1$$

we have

$$\triangle P_3Q_3B_3 \parallel \triangle Q_1P_1B_1.$$

Hence $\star B_3P_3Q_3 = \star B_1Q_1P_1$, so q_1 is parallel to q_3 .

Since the collineation axis P_1Q_1 of linkage 1 is perpendicular to the coupler A_1B_1 , the collineation axis P_3Q_3 of linkage 3 will be perpendicular to the rocker B_3B_0 . The same property holds for linkage 2.

From the theorem of Bobillier we can then deduce that the crank A_2A_0 is perpendicular to the pole tangent p_2 . So circling point A_2 lies on n_2 . The equation of k_{u2} shows that if one circling point lies on n_2 , n_2 must be a branch of k_{u2} . Another branch of k_{u2} is a circle through P_2 with its centre on n_2 . So B_2 lies on $c-k_{u2}$ as long as position 2 is not a folded position. This leads to the position already investigated under part (c) of section 4.1.1, so that no new viewpoints arise from this consideration.

4.4 Degeneration of the k_a curve, when $l_o^{-1} = 0$ (or $R_o = 2R$)

In this case only the k_a curve degenerates into p and into a circle through $P(0, 0)$ and $M(m, 0)$ with its centre on p . Ball's point is found at the intersection of the inflection circle with a line through P in the asymptotic direction of k_a . This line coincides with the pole tangent p , so only the pole P can be a Ball's point. Since P is also a cusp, it is not suitable as a coupler point; so in this position there is no solution of the problem.

4.5 Degeneration of the k_u and k_a curve, when $P \rightarrow \infty$

Both the k_u and k_a curves are cubic. In general both curves pass through the two isotropic points and through one asymptotic point. If the double point of k_u and k_a , i.e. the pole P , becomes infinite then the line at infinity will intersect each cubic curve in not less than 5 points. This is only possible when the line at infinity is in itself a branch of k_u and k_a . Then the pole tangent is an asymptotic line of the remaining conic sections, since the two curves touch the pole tangent at P_∞ . Hence the two conic sections are hyperbolas.

The corresponding position of the four-bar linkage will be reached where the crank A_0A runs parallel to the rocker B_0B , as occurs if $P = P_\infty$ (see Fig. 25).

† Is similar to.

FIG. 22.

FIG. 23.

FIG. 25.

5. INVESTIGATION OF THE MAIN PROBLEM

In what precedes, all possibilities of degeneration of k_u have been investigated in connection with the production of V-shaped, symmetrical coupler curves by four-bar linkages.

When k_u is not of degenerate form, we get one more degree of freedom in design. This additional degree of freedom may then be used for different purposes. It allows us, for example, to choose the coupler point K on the produced part of the coupler for any given angle 2τ . (The choice corresponds to the condition $\gamma_1 = 0^\circ$.) The corresponding dimensions are calculated and given in Table 3. Besides, Figs. 29, 30, 31 and 32 show how the Geneva wheel moves with respect to time for $2\tau = 45^\circ, 60^\circ, 90^\circ$ and 120° .

TABLE 3. SOLUTIONS WITH B INSIDE THE INFLECTION CIRCLE AND $\gamma_1 = 0$

2τ	120°	90°	72°	60°	45°	30°
a/b	0.4988	0.4630	0.4187	0.3787	0.3152	0.2339
d/b	1.1514	1.3018	1.4015	1.4749	1.5773	1.6952
μ_1	68.80	56.14	48.94	44.10	37.73	30.58
μ_2	38.09	49.59	58.86	66.47	78.25	93.87
$\overline{B_0E_0}/b$	1.0688	0.8715	0.7597	0.6846	0.5861	0.4756
$\overline{E_0U_1}/b$	0.4870	0.4257	0.3856	0.3564	0.3154	0.2649
θ	62.78	58.75	55.77	53.62	50.82	47.90
h/b	0.7607	0.8746	0.9133	0.9219	0.9046	0.8379
$(\rho T_1)/b$	0.2877	0.1058	0.0546	0.0330	0.0154	0.0055
$(\rho T_2)/b$	1.4825	1.1346	0.8432	0.6260	0.3598	0.1468
φ_B	1.42	7.51	11.05	13.35	16.17	18.90
ε_m	7756.07	63.98	20.87	11.81	6.45	3.77

It is also possible to pursue an optimal $\mu_{\min}$ for each angle 2τ . The dimensions complying with this optimum are represented in Table 4. With the aid of the data taken from this table, drawings have been made of the driving mechanisms for $n=3, 4, 5, 6, 8$ and 12 stations (see Figs. 34 to 39). Moreover, graphs of the resultant motion of the Geneva wheel are shown in Figs. 40 to 43 for $n=8, 6, 4$ and 3 respectively.

The calculations behind the general case are based on the following construction (see Fig. 44):

- (a) Draw the pole tangent p , the pole normal n , and the inflection circle.

FIG. 29. Curves showing the motion of an 8-station Geneva mechanism driven by a four-bar linkage.

$$\begin{aligned}
 a/b &= 0.3152 & \theta &= 50.82^\circ \\
 d/b &= 1.5773 & h/b &= 0.9046 \\
 \mu_{\min} &= 37.73^\circ & \gamma_1 &= 0.00^\circ \\
 \frac{B_0 E_0}{E_0 U_1} b &= 0.5861 & 2\tau &= 45^\circ \\
 E_0 U_1 / b &= 0.3154
 \end{aligned}$$

For the meaning of the symbols see Fig. 33.

FIG. 30. Curves showing the motion of a 6-station internal Geneva mechanism driven by a four-bar linkage. $a/h = 0.3787$; $\theta = 53.62^\circ$; $d/b = 1.4749$; $h/b = 0.9219$; $\mu_{\min} = 44.10^\circ$; $\gamma_1 = 0.00^\circ$; $B_0 E_0/h = 0.6846$; $2\tau = 60$; $E_0 U_1/h = 0.3564$. For the meaning of the symbols see Fig. 33.

FIG. 32. For the meaning of the symbols see Fig. 33

FIG. 33.

TABLE 4. SOLUTIONS WITH B INSIDE THE INFLECTION CIRCLE AND $\mu_1 = \mu_2 = \mu_{\min}$

2τ	15°	30°	45°	60°	72°	90°	105°	120°	135°	150°	165°	175°
a/b	0.1251	0.2337	0.3210	0.3862	0.4236	0.4582	0.4692	0.4643	0.4419	0.3962	0.3098	0.1943
d/b	1.4086	1.3947	1.3772	1.3604	1.3492	1.3379	1.3340	1.3357	1.3433	1.3575	1.3798	1.4007
γ_1	+37.27°	+29.16°	+20.83	+12.42°	+5.76°	-3.97°	-11.77°	-19.22°	-26.30°	32.98°	-39.24°	-43.15°
$\mu_{\min}$	79.84	70.97	63.75	58.30°	55.13°	52.18°	51.24°	51.66°	53.57°	57.45°	64.68°	74.19°
$(B_0 \overline{E_0})/b$	0.0795	0.1915	0.3387	0.5173	0.6770	0.9316	1.1466	1.3551	1.5509	1.7287	1.8832	1.9677
$(E_0 \overline{U_1})/b$	0.0734	0.1586	0.2453	0.3228	0.3739	0.4293	0.4561	0.4657	0.4564	0.4225	0.3442	0.2250
θ	48.58°	51.66°	54.07°	55.82°	56.85°	57.98°	58.70°	59.33°	59.92°	60.48°	60.95°	60.98°
h/b	0.3507	0.6361	0.8284	0.9205	0.9281	0.8506	0.7271	0.5710	0.4007	0.2333	0.0878	0.0177
$(\rho T_1)/b$	0.0003	0.0029	0.0111	0.0287	0.0521	0.1078	0.1798	0.2871	0.4541	0.7450	1.4150	3.0960
$(\rho T_2)/b$	0.0104	0.0858	0.2690	0.5454	0.7981	1.1719	1.4534	1.7110	1.9743	2.3183	3.0140	4.7130
φ_B	0.50°	1.98°	4.08°	6.28°	7.85°	9.62	10.46°	10.69°	10.24°	9.03°	6.72°	3.91°
ε_m	10.61	8.00	8.88	12.66	20.36	73.50	2455.7	149.07	27.92	10.85	5.37	3.45

FIG. 34.

FIG. 35.

FIG. 36.

FIG. 37.

FIG. 38.

FIG. 39.

FIG. 40. Curves showing the motion of an 8-section internal Geneva mechanism driven by a four-bar linkage.

$$\begin{aligned} a/b &= 0.3210 & \theta &= +54.07 \\ d/b &= 1.3772 & h/b &= 0.8284 \\ \mu_{\min} &= 63.75 & \gamma_1 &= +20.83 \\ \frac{B_0 E_0}{b} &= 0.3387 & 2\tau &= 45^\circ \\ \frac{E_0 U_1}{b} &= 0.2453 & \mu_1 &= \mu_2 \end{aligned}$$

For the meaning of the symbols see Fig. 33

FIG. 41. Curves showing the motion of a 6-station internal Geneva mechanism driven by a four-bar linkage.

FIG. 42. Curves showing the motion of a 4-station internal Geneva mechanism driven by a four-bar linkage.

$$\begin{aligned} a/b &= 0.4582 & \theta &= 57.98 \\ d/b &= 1.3379 & h/b &= 0.8506 \\ \mu_{\min} &= 52.18 & \mu &= 3.97 \\ \overline{B_0 E_0}/b &= 0.9316 & \mu_1 &= \mu_2 \\ \overline{E_0 U_1}/b &= 0.4293 & 2\tau &= 90 \end{aligned}$$

For the meaning of the symbols see Fig. 33.

FIG. 43. For the meaning of the symbols see Fig. 33.

FIG. 44.

- (b) Choose polar co-ordinates r_B and φ_B of the turning-point B with respect to the pole and the pole tangent in such a way that

$$\frac{1}{2}\sin \varphi_B < r_B/\delta \leq \frac{1}{2}\sin \varphi_B \left[1 + \left(\frac{1 - \sin \varphi_B}{1 + 3 \sin \varphi_B} \right)^{\frac{1}{2}} \right],$$

if B is chosen *inside* the inflection circle, and

$$r_B/\delta \geq \frac{1}{2}\sin \varphi_B \left[1 + \left(\frac{1 + \sin \varphi_B}{1 - 3 \sin \varphi_B} \right)^{\frac{1}{2}} \right],$$

if B lies *outside* the inflection circle. (In any case we may confine ourselves to the first quadrant for the choice of B .)

- (c) Find B_0 from equation $\overline{PB}^2 = \overline{BB_0} \cdot \overline{BB_w}$.
- (d) Draw circle b with its centre at B and radius $\overline{BB_0} = b$.
- (e) Choose the coupler point in one of the intersections K_1 and K_2 of circle b with the inflection circle.
- (f) Find the Ball's point U in the chosen coupler point K .
- (g) Draw a line through K perpendicular to the path normal PK . The line segments which this line cuts from the two axes p and n form the sides of a rectangle with K as the fourth vertex.
- (h) Determine point B_* in a similar way, starting from point B .
- (i) Draw line K_*B_* which intersects p and n at M and L respectively. Then with $\overline{PM} = m$ and $\overline{PL} = l$, k_u is completely determined. (As is shown by Beyer [5], k_u is determined by p , n and two points of k_u , the points being U and B in the case under observation. This is done by determining the line ML as before, from which other points of k_u can be derived by reversing the process.)
- (j) Point A can then be located at one of the intersection points of k_u and circle b . As k_u is a circular curve, the isotropic points are to be counted as two intersection points. Since we forced Ball's point on b and k_u , this point of intersection may not be taken as a turning-point. So from the six intersection points of circle b with the cubic k_u , only three of them remain from which to choose turning-point A . Two of them may or may not be complex. If all three intersection points for the choice of A are real and b intersects the inflection circle in two different and real points K , six solutions exist in addition to the chosen polar coordinates r_B and φ_B of B .
- For a real point A on b , the added point A_* must lie on K_*B_* . Hence this leads to the construction of the real intersection(s) of the cubic b_* with K_*B_* , as can be easily effected by way of trial and error.
- (k) Find the point A_0 from point A using the equation $\overline{PA}^2 = \overline{AA_0} \cdot \overline{AA_w}$.
- (l) Draw the coupler curve, check the position of the axis of symmetry and judge the length of the straight guidance.

Remarks. The limitation in the choice of the position of turning-point B has three causes:

1. the equivalence of the left- and right-hand half plane;
2. the very existence of real intersections K_1 and K_2 of circle b with the inflection circle; and
3. the equivalence of solutions corresponding with the two different positions of the coupler point in the two Ball's points of the coupler curve.

The last point will require some further explanation: The two different inflection circles corresponding to the two positions of the coupler in which the coupler point coincides with a Ball's point, can be made of the same size by increasing or decreasing one of the two four-bar linkages. This makes it possible to indicate the positions of two points B producing equivalent solutions, in one moving plane (of drawing), in which the mutual relation between these points must be investigated.

Let the two positions of the coupler be represented by A_1B_1 and A_2B_2 . As is known, the points U_1 and U_2 are opposite points to one another with respect to the axis of symmetry of the coupler curve, the latter going through B_0 . The perpendicular bisectors of the distances $\overline{U_1U_2}$ and $\overline{B_1B_2}$ will then intersect at $P_{12} = B_0$. So A_1 and A_2 are opposite points to one another with respect to the fixed link A_0B_0 . To study the relationship between the two positions of the four-bar linkage, it is more convenient to replace the second by its image with respect to the fixed link. The positions under consideration will then be A_1B_1 and A_3B_3 , A_3 coinciding with A_1 . These positions are shown in Fig. 24. It is further proved in part (b) of section 4.3.1 that P_1Q_1 runs parallel to P_3Q_3 . From the theorem of Bobillier, it can then be seen that $\sphericalangle p_1P_1B_1 = \sphericalangle A_1P_1Q_1 = \sphericalangle A_3P_3Q_3 = \sphericalangle p_3P_3B_3$. Hence the two related points B , lying in one moving plane as mentioned above, are on the same path normal. It is also proved that $\overline{P_1B_1} \cdot \overline{P_3B_3} = \overline{B_1B_0} \cdot \overline{B_3B_0}$.

Now we have $\overline{P_1B_1^2} = \overline{B_1B_0} \cdot \overline{B_1B_{w1}}$ and $\overline{P_3B_3^2} = \overline{B_3B_0} \cdot \overline{B_3B_{w3}}$ so $\overline{B_1B_{w1}} \cdot \overline{B_3B_{w3}} = \overline{B_1B_0} \cdot \overline{B_3B_0}$,

or

$$(\delta_1 \sin \varphi_{B_1} - \overline{P_1B_1})(\delta_3 \sin \varphi_{B_3} - \overline{P_3B_3}) = \overline{B_1B_0} \cdot \overline{B_3B_0} = \overline{P_1B_1} \cdot \overline{P_3B_3},$$

consequently with $\varphi_{B_1} = \varphi_{B_3}$, for the two four-bar linkages the following relation holds:

$$\frac{\overline{P_1B_1}}{\delta_1} - \frac{1}{2} \sin \varphi_{B_1} = \frac{1}{2} \sin \varphi_{B_1} - \frac{(\overline{P_3B_3} \delta_1 / \delta_3)}{\delta_1}$$

In the moving plane of drawing the two couplers have the same inflection circle; so for the two related points B in this moving plane we have the equation

$$r_B - \frac{1}{2} \delta \sin \varphi_B = \frac{1}{2} \delta \sin \varphi_B - r_B^*.$$

Hence, the midpoint between two related points B of the moving plane must coincide with the intersection of a path normal with a circle which touches the pole tangent in P and passes through the centre of the inflection circle (see Fig. 45). (In general this midpoint does not coincide with P .) The equations referring to this general case have been omitted. They can be found in the author's thesis [2].

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