Four-bar Cognates of Special Forms of Watt’s Six-bar Mechanism

(Discussion on a paper of A. H. Soni about “coupler cognate mechanisms consisting of linkage parallelograms supported by four-bars”)

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Abstract
In a discussion of Soni’s paper[1] a geometric way has been proposed in deriving the cognate six-bar mechanisms as found by Soni. Moreover, it is proved that the curves produced by the six-bar mechanisms of the considered type and structure are ordinary four-bar coupler-curves of degree six and genus one.


Резюме — Родственность шарнирных четырехзвенных с частными формами шестизвенных шарнирных механизмов (Обсуждение статьи А. Г. Сони о родственных механизмах, образованных присоединением структурной группы к шатуну шарнирного четырехзвенного механизма параллелограмма): Е. Диксман.
В обсуждении статьи Сони объяснен чисто геометрический путь получения соиевыхских родственных шестизвенных механизмов, образованных присоединением структурной группы к шатуну шарнирного четырехзвенника. Кроме того, указывается, что шатунные кривые, описанные шатунными точками вышеупомянутых частных механизмов, также могут быть построены с помощью шарнирных четырехзвенников.

1. Introduction
In the Summer edition of this journal a very interesting paper appeared about coupler cognate mechanisms written by Professor A. H. Soni. In this paper three cognate six-bars of a particular type of Watt’s six-bar mechanism has been found. The type considered is the one, which has one linkage parallelogram as a kinematic sub-chain in the six-bar (see Fig. 1).

The manner in which he derived his cognates is partly geometric and partly algebraic through making use of the theory on complex numbers.

In order to emphasize these results, I intend to develop in this discussion another, purely geometric, way of obtaining the mentioned cognate mechanisms of Soni. Moreover I intend to prove too that the curves described by the coupler-points of this

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particular type of mechanism are identical with ordinary four-bar coupler curves of degree six and genus one.

![Diagram of a mechanism with labeled parts](image)

**Figure 1.**

2. The Six-Bar Cognate Mechanisms of Soni

A randomly chosen six-bar mechanism with the linkage-parallellogram $BCDK$, as shown in Fig. 1, will be termed the *source* mechanism in the course of this discussion. (With the exception of Section 3, it remains the same mechanism throughout the paper. In fact, however, the same purpose would be served by any linkage of this kind).

The source mechanism consists of a four-bar $A_0ABB_0$, the linkage-parallellogram $BCDK$, the arbitrarily chosen coupler-triangles $ABK$, $CDF$ and $KDE$ and, finally, of the rigid triangle $B_0BC$.

The curves described by the coupler-points $E$ or $F$ are under discussion. That is to say, we are looking for cognate six-bars of the same type as the source mechanism and of which the coupler-points $E$ or $F$ generate the same (identical) curve as generated by point $E$ (or $F$) of the source mechanism.

2.1 The generation of the curve produced by point $E$ of the source mechanism

Let us first consider the curve generated by coupler-point $E$. attached to link $KD$ of the mechanism. One of the (two) Soni cognate mechanisms may then be obtained by following the next design instructions: (See Fig. 2).

(a) Form the linkage-parallellograms $A_0ABA^\varphi$, $A^\varphi BBC_0B^\varphi$ and $A_0AKA^\varphi$. (Note that $\Delta A_0A^\varphi A^\varphi \equiv \Delta ABK$).

(b) Turn the four-bar $A_0A^\varphi B_0B^\varphi$ about $A_0$ over the angle $\alpha = \frac{\pi}{2} A_0A^\varphi A_0A^\varphi = \frac{\pi}{2} KAB$ and multiply the four-bar geometrically from $A_0$ by the factor $f_\alpha = A^\varphi A_0 A_0 = KAB/BA$ (One thus obtains the four-bar $A_0A^\varphi B_0B^\varphi$ which is similar to the four-bar $A_0A^\varphi B_0B^\varphi$).

(c) Frame the rigid triangle $A_0B_0B_0$ which is similar to the coupler-triangle $ABK$.

(d) Form the linkage-parallellogram $A^\varphi KEK^\varphi$. (Note that $\Delta K^\varphi A^\varphi B^\varphi$ is a rigid triangle.

This is due to the fact that $A^\varphi B^\varphi$ and $A^\varphi K^\varphi$ both have the identical angular velocity $\omega_{30}$ and also have the turning-joint $A^\varphi$ in common.)

(e) Next, choose a random turning-joint $D^\varphi$ in the moving plane attached to link $K^\varphi E$.

*The position of the mechanism is also chosen randomly.*
(f) Further, form the linkage-parallelogram $B^eK^eD^eC^e$. (Note that $\Delta B^eC^eB_0^e$ is a rigid triangle, since two sides of the triangle at all times have the same angular velocity $\omega_{10}$).

The source mechanism is now supplemented with a six-bar cognate mechanism of the same type and structure. The obtained cognate is indicated by hard-solid lines in Fig. 2. Like the source mechanism it has one degree of freedom of movement. Both mechanisms, the cognate and the source mechanism, have one kinematic sub-chain which is a linkage-parallelogram and both have the same coupler-point $E$ at all times coinciding. Therefore, both points $E$ generate the identical coupler-curve and the mechanisms may be called cognates.

(One may note that $D^e$, or otherwise $C^e$, may be chosen randomly. Therefore, there is a doubly infinite number of solutions of this kind.)

A second cognate of the source mechanism as found by Soni, may be obtained through the next sequence of design instructions: (See Fig. 3).

(a) Form the linkage-parallelogram $B_0^eBAB^ mass, B^mAA_0^eA^m, AKEK^m$ and $A_0AK^mA^m$.

(Note that $\Delta A_0^eA^m$ is a rigid one, since two sides of this triangle have the same angular velocity $\omega_{30}$ at any point of time).
(b) Turn the four-bar $A_oA^1B^2B_o$ about $A_o$ over the angle $\delta = < A^1A_0A^3$ and multiply the four-bar simultaneously from $A_o$ by the factor $f_o = A^1A_0A^3A_o$ (One thus obtains the four-bar $A_oA^1B^2B_o$ which is similar to the four-bar $A_oA^1B^2B_o$).

(c) Frame the rigid triangle $A_oB_oB_o^1$ which is similar to the triangle $A^1A^3A_o$, and form the rigid triangle $A^1B^2K^3$.

(d) Next, choose a turning-joint $D^1$ somewhere in the moving plane attached to link $K^3E$.

(e) Form the linkage-parallelogram $B^1K^3D^1C^1$ and

(f) Frame the rigid triangle $B_oB^1C^1$.

Again the source mechanism is supplemented by another six-bar cognate mechanism of the same structure. The obtained cognate is indicated by the hard-solid lines of Fig. 3. Point $E$ of the cognate six-bar generates a coupler-curve identical with the curve produced by point $E$ of the source mechanism.

Now, two cognates have been found based on an interchange in the sequence of the moving links of the initial four-bar $A_oA_0B_0B_o$. This initial four-bar, which is a kinematic sub-chain of the source mechanism, has the next sequence in the numbered links: $0-1-2-3$.

The first cognate of Soni is based on the sequence $0-2-3-1$ and the second cognate possesses the sequence $0-3-1-2$.

Clearly, three other possibilities exist, and they will probably correspond to as many cognates of Watt's type.

In order not to lengthen this discussion I do not intend to broaden the field of possible cognates. I will merely confine myself to a (geometric) discussion of the presented cognates of Soni.

One interesting point remains to be established, and that is the astonishing fact, that the curve generated by point $E$ of the source mechanism may also be generated by an ordinary four-bar mechanism.

To prove this—rather unexpected—fact, I intend to give the relational dimensions of such a special cognate in the next sequence of design instructions: (See Fig. 4).
(a) Turn the four-bar $A_oABB_0$ of the source mechanism about $A$ over the angle $\alpha = \gamma \cdot KAB$ and multiply the four-bar geometrically from $A$ by the factor $f_\alpha = KA/BA$. (One thus obtains the four-bar $HAKA'$ which is similar to $A_oABB_0$.)

(b) Form the rigid triangle $AA_oH$ which is similar to $\Delta AKB$ and note that link $HA'$ is a translating bar.

(c) Form the linkage-parallelograms $B_0BKB'$ and $A_oHA'A_0'$.

(d) Frame the rigid triangles $A_0B_0A'_0$ which is similar to the coupler-triangle $ABK$.

(e) Frame also the rigid triangle $KDB'$ which is similar to $\Delta ABCB'$.

(f) The coupler-point $E$ of the coupler-triangle $A'B'E$ attached to the coupler $A'B'$ of the four-bar $A_oA'B'B_0$ generates the same coupler-curve as produced by point $E$ of the source mechanism.

(g) Note that $\square A_oA'B'B_0 \sim \square A_oAB'B''B_0$ as long as point $B''$ is defined through the linkage-parallelogram $ABB_0B''$. (The center of similitude coincides with the fixed center $B_0$ and the multiplication factor $f_B = BK/BA$.)

(h) Note also that any point of the coupler-plane $3'$ of the source mechanism will be generated by the same four-bar $A_oA'B'B_0$. Therefore, there is a cognate generation of the entire coupler-plane of the source mechanism.

Clearly, two other four-bar cognates may be indicated through Roberts’ Law*. And they too, in turn, give rise to six-bar cognates of Watt’s type.

Anyway, we now have established the important fact, that four-bar coupler-curves may also be generated by an infinite row of six-bar cognates (of special type) of which the source mechanism is an example.

2.2 The generation of the curve produced by point $F$ of the source mechanism

Let us now consider the curve generated by the coupler-point $F$, attached to link $CD$ of the source mechanism. Soni proposes a cognate as shown in Fig. 4. Starting with the source mechanism again, the geometrical design of such a cognate may be obtained through the next sequence of instructions: (See Fig. 5).

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*Since $\Delta A_oA'B_0$ coincides with the fixed triangle in the configuration of Roberts, the other two four-bars have either $A_0B_o$ or $A_oA'_0$ as frame-link.
(a) Form the linkage-parallelograms $AB_{0}B''B'''$ and $B_{0}CFC''$.

(b) Turn the four-bar $A_{0}AB''B_{0}$ about $B_{0}$ over the arbitrarily chosen angle $\epsilon = \angle A'_{0}B_{0}A_{0}$ and multiply the four-bar geometrically from $B_{0}$ by the arbitrarily chosen factor $f_{\epsilon} = A'_{0}B_{0}/A_{0}B_{0}$. (One thus obtains the four-bar $A_{0}^'A^''B^{''}B_{0}$ which is similar to the four-bar $A_{0}AB''B_{0}$).

(c) Form the rigid triangles $A_{0}B_{0}A''_{0}$ and $B_{0}B''C''$.

(d) Choose the turning-joint $K^*_{0}$ arbitrarily in the moving plane attached to the link $A^*B^*$ (Thus $\Delta A^*B^*K^*_0$ is a rigid triangle.)

(e) Form the linkage-parallelogram $C''B''K''D''$ and

(f) Finally, form the rigid triangle $FC''D''$.

The obtained cognate, drawn with hard-solid lines in Fig. 4, has the same structure as the source mechanism. And the coupler-point $F$ describes the identical coupler-curve. (Note that the design of the alternative mechanism gives freedom of choice of the frame-center $A^{*}_{0}$ and also of the turning-joint $K_{0}$ (or $D_{0}$).)

Finally, we will show the surprising fact that the curve generated by point $F$ of the source mechanism may also be generated by three cognate four-bar linkages. One of them may be obtained as follows: (See Fig. 6)

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Figure 6.
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(a) Turn the four-bar $A_{0}ABB_{0}$ about $B_{0}$ over the angle $\beta_{0} = \angle CB_{0}B$ and multiply the four-bar geometrically from $B_{0}$ by the factor $f_{\beta_{0}} = CB_{0}/BB_{0}$. (One thus obtains the four-bar $A_{0}^*A'B'B_{0}$ which is similar to the initial four-bar $A_{0}AB_{0}$).

(b) Frame the rigid triangle $A_{0}B_{0}A''_{0}$ which is similar to $\Delta BB_{0}C$ and

(c) Form also the rigid triangle $A'CD$. (Note that $\square A''CDF$ forms a rigid quadrilateral).

(d) The curve produced by point $F$, may also be generated by the coupler-point $F$ of the coupler-triangle $A''CD$. (Note that $\square A''CDF$ forms a rigid quadrilateral).

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(b) Frame the rigid triangle $A_{0}B_{0}A''_{0}$ which is similar to $\Delta BB_{0}C$ and

(c) Form also the rigid triangle $A'CD$. (Note that $\square A''CDF$ forms a rigid quadrilateral).

(d) The curve produced by point $F$, may also be generated by the coupler-point $F$ of the coupler-triangle $A''CD$. (Note that $\square A''CDF$ forms a rigid quadrilateral).

Therefore, here too, the considered curve is an ordinary four-bar coupler-curve of degree six and genus one. The other two four-bar cognates* are to be found with the well-known Law of Roberts-Chebyshev.

*The first obtained four-bar cognate, solely, has the advantage of generating the entire coupler-plane $Z$ of the source mechanism.
3. The Source Mechanism as Part of the Generalized Pantograph

Finally, one may remark that, since the source mechanism contains a linkage-parallelogram, the mechanism must have something to do with the so-called generalized pantograph of Kempe-Burmester[3, 4]. (See Fig. 7).

This pantograph consists of a linkage-parallelogram with rigid triangles attached to each side of it.

In the design-position of the linkage, the four connected vertices of these triangles form a quadrilateral of which the sides pass through the turning-joints of the parallelogram. Moreover, the diagonals of the quadrilateral lie parallel to the sides of the parallelogram in this position.

Since 1888 it is known that, if one of the mentioned vertices should be made a center of pivot on the frame, the remaining three vertices will describe similar curves*. Now suppose, the source mechanism may be brought into the position† where the turning-joints $A$, $B$ and $B_0$ are in line. (Such a position will be called a design-position. If there is no such position, one draws the plan with the points $A_0$, $A$, $B$ and $B_0$ in line, similar to Cayley's plan for determining the link-lengths of the four-bar cognates of Roberts).

![Generalized pantograph](image)

Figure 7.

In the source mechanism, brought into the design-position, one so recognizes immediately part of the generalized pantograph. Therefore, the source mechanism in this position may be supplemented by the links $A_0'A'$ and $A_0''A''$ in accordance with the pantographic proposition. One may remark that for each position of the source mechanism, the quadrilateral $AA'A''B_0$ will always remain similar to itself‡.

Hence, a possible second design-position of the source mechanism does not furnish a pantographic solution differing from the one obtained with the first design-position.

The additional frame-centers $A_0'$ and $A''_0$ may be obtained through the similarity: $\Box A_0A_0'A''_0B_0 \sim \Box AA'A''B_0$.

As a result of the pantographic addition one recognizes the four-bar $A_0'A''CB_0$. Thus turning-joint $D$ (and any coupler-point attached to link $CD$) generates an ordinary coupler-curve. Since on the other hand the five-bar $A_0'A'DA''A''_0$ has two input-rods always rotating at equal angular velocity, any point attached to link $A'D$ or $KD$ will describe ordinary coupler-curves too.

* In the case under consideration, these curves are circles.
† If the crank $BB_0$ of the source mechanism may turn the full circle about the frame-center $B_0$, there generally are two such positions.
‡ This proposition was first recognized by Mr. A. B. Kempe in 1878.
Clearly, the application of the principle of the generalized pantograph of Kempe-Burmester leads also to the already found cognate four-bar linkages which generate the same curves. The reader may choose which way of obtaining he prefers.

References