Kempe’s (Focal) Linkage† Generalized, particularly in connection with Hart’s second straight-line mechanism

E. A. Dijksman‡

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Abstract

Very few kinematicians are aware of the existence of the focal linkage, which is an overconstrained one. Since there are many applications that could be derived from the linkage or from its derivatives, a thorough investigation has been made into the properties of the focal linkage. Here, a geometric approach as well as an algebraic one clears up some of the mystery that hangs around the mechanism. One of the main results that is achieved in the paper is the invention of a new eight-bar linkage containing a bar having rectilinear translation.

1. Introduction

One of the most fascinating linkages is the linkage that primarily has been assembled by Kemp in 1878[1] (see Fig. 1).

Immediately, the linkage draws one’s attention because the mutual motion of the links is hindered by the abnormally large number of links that constitute the linkage. This configuration is moveable only because of the particular choice of the link-lengths. Otherwise, the linkage would be a rigid structure and not a mechanism.

What must have fascinated even a great kinematician like Ludwig Burmester in his time, is the fact that the configuration does not seem to be as obvious as most other overconstrained linkages. This, perhaps, explains why he gave it such unusual attention; the paper he wrote on the subject contains some 30 pages!

In Burmester’s paper[2] the nature of Kempe’s overconstrained linkage has been thoroughly investigated. Nowadays, however, such a paper is difficult to understand for anyone who is not familiar with projective geometry. For this reason and maybe also because of the fact that wasn’t included in Burmester’s “Lehrbuch der Kinematik”, the linkage remained fairly unknown for at least half a century.

Figure 1. Kempe’s focal mechanism with constrained motion (1878).

†The focal properties have been derived by Burmester some 15 years after the first discovery of the linkage by Mr A. Kempe.
‡Visiting Lecturer, Department of Mechanical, Marine and Production Engineering, Liverpool Polytechnic, Byro Street, Liverpool L3 3AF, England. Lecturer, Eindhoven University of Technology, Eindhoven, The Netherlands.
However, in the present time, Wunderlich[3] who is a famous Austrian kinematician, again revived interest in the configuration. In fact, all three kinematicians, to wit Wunderlich, Burmester and Kempe, draw our attention to the fact that Hart’s[4] (second†)straight-line mechanism is a particular case of Kempe’s configuration (see Fig. 2). So, in a way, Kempe’s configuration is a generalized form of Hart’s straight-line mechanism, the latter being discovered by Hart a year before Kempe’s more general result.

One of the properties of Kempe’s linkage that has been revealed by Burmester, is that the quadruple joint $D$ that is connected to the four sides of the four-bar $(AoABBo)$, coincides with a focus of an inscribed conic section. In other words: the four sides of the four-bar $(AoABBo)$ are tangents to a conic section of which the quadruple joint $D$ is a focus (see Fig. 1).

Since a conic section is uniquely determined by five tangents, an infinite number of conic sections could be inscribed in the four-bar. Therefore, an infinite number of such foci exists, and so, there is an infinite number of quadruple joints $D$ that could be connected to the four sides of the four-bar.

According to Burmester, the locus joining all these foci is called the focal curve. In addition, the configuration itself, has been named (Burmester’s) focal mechanism. Though this seems only natural to do so, I propose to name it Kempe’s focal linkage, since it has been Kempe who discovered the linkage in the first place.

Apart from Hart’s straight-line mechanism that could be derived from the focal linkage, there are other technical applications. It could be used, for instance, to replace the four-bar coupler-motion by the coupler-motion of an alternative six-bar mechanism. For instance the coupler $AT$ of the six-bar $(AoFATDS)$ produces a motion identical to the one produced by the coupler $AB$ of the four-bar $(AoABBo)$ (see Fig. 1). So, the six-bar, which is part of the focal linkage, produces the same coupler-motion as the four-bar. It may, therefore, be called a coupler alternative mechanism. Although two more links are needed then to produce the (same) coupler-motion, it gives the designer more freedom to design the mechanism. In the given example, for instance, the designer could choose the double-joint $D$ anywhere on the focal curve, or, accordingly, the center $S$ anywhere on the line $AoBo$.

Similarly, he could use the focal linkage by replacing the four-bar by the six-bar $(BoCBTDS)$. Then, the coupler $TB$ is the common link. In a way he has replaced the center $Ao$ by a random point $S$ on the line $AoBo$.

So, summarizing, either the point $Ao$ or the point $Bo$ could be replaced by a random point $S$ on $AoBo$. (Later on we will show that Kempe’s linkage can be generalised with the consequence that the point $S$ is not necessarily restricted to the line $AoBo$, but can, in fact, be chosen anywhere in the frame. This can be seen immediately if we consider the mechanism shown in Fig. 8).

Another application of the focal linkage is one where we exploit the additional links in order to obtain better force-transmission throughout the linkage. The motion would then be more smooth.

![Figure 2. Hart’s second straight line mechanism (1877).](image)

†The first one, also named after Hart, represents his inversor.
Wunderlich [3] also used the linkage to drive a double rocker. With a double rocker, the cranks $A_0A$ and $B_0B$ merely oscillate. Kempe’s focal linkage now allows us to drive the four-bar with a single rotating crank ($SD$). In fact, we then drive the double rocker by the two crank-and-rockers $SDFA_0$ and $SDCB_0$ of which $SD$ is the common crank. (Naturally, the link $DT$ may then be omitted).

As already mentioned, other possibilities arise if we consider the generalized form of the focal linkage.

In this paper, attention will be given to the derivation and to the design of Kempe’s focal linkage as well as to Hart’s straight-line mechanism.

From a special case of the generalized form mentioned above, we will obtain a new eight-bar linkage containing a bar that produces an “exact” rectilinear translation. As is known [5], approximate rectilinear translations may be produced by six-bar linkages. These can be derived from four-bars in which a couplerpoint approximates a straight-line segment. The six-bar then is obtained by connecting a parallel moving rod to this point in a fashion described in [5].

Exact rectilinear translations, however, cannot be obtained this way, since there are no four-bars that generate a straight-line precisely. Linkages having a coupler-point that do generate an exact straight-line, at least contain six bars, as is demonstrated for example with Hart’s straight-line mechanism. Thus, starting from a six-bar that produces an exact straight-line, exact rectilinear translations of a coupler can only be generated by eight-bar linkages, since a dyad has to be adjoined to the six-bar to extract parallel motion.

Here, in this paper, we will obtain an eight-bar which generates exact rectilinear translation; this linkage is obtained from Hart’s straight-line mechanism in a manner similar to the way by which the generalized form is to be obtained from the focal linkage.

2. Kempe’s focal linkage (see Fig. 3)

We may derive the linkage from the four-bar $A_0FDS$ which is represented by the vector-identity

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = 0(t) \ldots (\square A_0FDS) \dagger$$

Here, all vectors represent the sides of the four-bar and have a constant length. Therefore, if we replace these vectors by complex numbers, the moduli of these numbers are constant. Clearly, the argument of the complex number that represents a side, is the angle between the side and some real axis in the fixed plane. (The fixed plane may be rigidly attached to the link $A_0S$ of the four-bar. However, this is not necessarily so).

We then reflect the four-bar into some fixed line, and multiply the image by a complex number $u$ of which the modulus, only, is a chosen constant. We so obtain the four-bar $DCBT$ which is then reflected similar to the four-bar $A_0FDS$. Thus, the identity

\[ \text{Figure 3. Complex numbers that represent Kempe's configuration.} \]  
\dagger This implies $A_0F = a_1$, $FD = a_2$, etc.
\[ u(\tilde{a}_1 + \tilde{a}_3 + \tilde{a}_2 + \tilde{a}_4) = 0 \ldots (\square DCBT) \] (2)

represents the newly obtained four-bar.

Here, a bar, placed on top of a vector, indicates the complex number that is \textit{conjugate} to the one that represents this vector. Clearly, all sides of \(\square DCBT\) have a constant length, since the modulus \(\sqrt{(u \bar{u})}\) of the factor \(u\) is a constant.

Next, if we define the points \(A\) and \(B_0\) through the relations

\[
A = (A_0F, TB) \\
B_0 = (A_0S, BC)
\]

we may observe the quadrilaterals \(AFDT\) and \(DCB_0S\) and find that their corresponding angles are equal. For instance, \(\angle FAT = \angle CDS\) since

\[
\angle(u_a, u \bar{u}_a) = - \angle(\bar{u}_a, u a_a) = - \angle(u \bar{u}_a, u \bar{u}_a)
\]

\[
= - \angle(u \bar{u}_a, a_3) = \angle(a_3, u \bar{u}_a).
\]

We are still free to choose the argument of \(u\). This freedom enables us to meet the equation

\[
\frac{B_0C}{B_0S} = \frac{DF}{DT}
\] (10)

which then causes the similarity between the quadrilaterals \(AFDT\) and \(DCB_0S\).

For reasons of symmetry, not explained in detail here, these quadrilaterals can be made four-bars by making the points \(A\) and \(B_0\) turning-joints of the configuration. In fact, we now have an overconstrained linkage in which the opposite four-bars are similar and stay that way in all positions. The four-bars \(DCB_0S\) and \(AFDT\) that are sub-chains in the linkage, may be represented by the identities:

\[
u a_1 + a_3 - \lambda a_4 - \lambda' u a_2 = 0 \ldots (\square DCB_0S)\] (3)

and

\[
a_2 + u a_4 - \mu u \bar{a}_3 - \mu' a_1 = 0 \ldots (\square AFDT)\] (4)

in which \(\lambda, \mu, \lambda'\) and \(\mu'\) are real numbers only.

Reflection of the latter gives its image represented by the equation

\[-\mu' \bar{a_1} - \mu \bar{u}_a + \bar{u}_a + \bar{a}_2 = 0.\]

Since the two four-bars that are represented by the respective equations (3) and (5) are \textit{directly} similar, we obtain the relations

\[
\frac{u}{-\mu'} = \frac{1}{-\mu} = \frac{-\lambda}{\bar{u}} = \frac{-\lambda' u}{1}
\] (6)

Therefore, the conditions

\[
\lambda \mu = \lambda' \mu' = \lambda' \mu u \bar{u} = 1
\] (7)

hold true if the four-bars \(DCB_0S\) and \(AFDT\) are reflected similarly, and conversely.

Because of symmetry, these relations also represent the conditions for similarity between the four-bars \(A_0FDS\) and \(DCBT\). Hence, the conditions (7) are sufficient and necessary for the opposite four-bars in the linkage, to be similar at every position. Geometrically, the conditions (7) represent the equations.
\[
\frac{AT}{TB} = \frac{A_0S}{SB_0} \quad (8)
\]
\[
\frac{A_0F}{FA} = \frac{B_0C}{CB} \quad (9)
\]
\[
\frac{B_0C}{B_0S} = \frac{DF}{DT} \quad (10)
\]

In the above, we have started by choosing the lengths of the vectors \(a_1, a_2, a_3, a_4, \) and \(u.\) However, in order to calculate the remaining dimensions, this would be a rather laborious way of proceeding. It would be easier to start with 5 other dimensions, such as, the four sides of the four-bar \(A_0ABB_0\) and \(\lambda\) which then determines the location of point \(S\) on the link \(A_0B_0.\) According to eqn (8) this simultaneously determines the location of the point \(T\) on the coupler \(AB.\) The calculation may then be continued, using the pentagonal loop or five-bar \(A_0ATDS\) that is contained in the configuration:

\[
(1 + \mu')a_1 + (a_3 + \mu u\bar{a}_3) + (a_4 - u\bar{a}_4) = 0. \quad (11)
\]

Hence,

\[
(1 + \mu')a_1 + (\frac{TBu}{u\bar{a}} + \mu TB) + a_4 - u\bar{a}_4 = 0
\]

Therefore,

\[
u = (1 + \mu')a_1 + \mu \frac{TB + a_4}{\bar{a}_4 - \frac{TB}{u\bar{a}}} = \frac{ST}{\bar{a}_4 - \frac{TB}{u\bar{a}}}
\]

Thus

\[
u\bar{u} = \frac{ST^2}{\left(\frac{TB}{u\bar{a}} - a_4\right) \left(\frac{TB}{u\bar{a}} - a_4\right)}.
\]

Using the expression for \(u\bar{u}\) from (7) yields:

\[
(a_4 + \mu')(a_3 + \mu' - TB) = \lambda \mu' ST^2 \quad (13)
\]

or

\[
a_4^2 \lambda^2 \mu'^2 - \lambda \mu'(ST^2 - 2a_4 \cdot TB \cdot \cos (AB, A_0B_0)) + TB^2 = 0 \quad (14)
\]

where \(a_4 = \sqrt{(a_0\bar{a}_4)}.\)

This equation represents a quadratic equation in the unknown \(\mu'.\) Hence, there are two roots \(\mu',\) each of them corresponding to a pair of points \((F, C).\) However, in case the discriminant of this equation is smaller than zero, no real points \((F, C)\) exist. Then, we have to shift the point \(S\) on \(A_0B_0\) until a real (focal) linkage is obtained.

Since \(\lambda \mu'\) is independent of position, so is the coefficient of \(\lambda \mu'\) in eqn (14). Therefore, we may calculate \(\mu'\) in the position for which \(A_0A\) is parallel to \(B_0B.\) Then, according to (8), \(ST\) becomes parallel to \(A_0A,\) and eqn (14) transforms into

\[
a_4^2 \lambda^2 \mu'^2 - \lambda \mu' \left\{ \frac{B_0B^2 + \lambda \cdot A_0A^2}{1 + \lambda} - \lambda \left( a_4^2 + AT^2 \right) \right\} + TB^2 = 0. \quad (15)
\]

Hence,

\[
(\mu' + 1)\left( \mu' + \frac{AB^2}{A_0B_0^2} \right) = \mu \left( \frac{B_0B^2 + \lambda \cdot A_0A^2}{\lambda (1 + \lambda)a_4^2} \right). \quad (16)
\]
Clearly, eqn (16) allows us to obtain two values for \( \mu' \) representing two pairs of points \((F, C)\) each available as a pair of joints incorporated in the focal linkage we wish to design.

It only remains to locate the quadruple joint \( D \). This can be accomplished, using the equations

\[
\begin{align*}
d^2 = \frac{\bar{u}_3}{\bar{u}_2} = \frac{TB^2}{\lambda \mu'} \quad (17) \\
a_2^2 = \frac{BC^2}{\lambda \mu'} \quad (18) \\
TD^2 = \lambda \mu'. a^2 \quad (19)
\end{align*}
\]

and

\[
CD^2 = \lambda \mu'. A_0 F^2. \quad (20)
\]

This determines all the dimensions of Kempe's focal linkage, provided we have made a choice of pairs \((F, C)\) that are coordinated to the pair \((S, T)\).

3. Hart’s (second) straight-line mechanism[4] (1877)

A particular case of the focal linkage is obtained when the center \( S \) lies at infinity. Then, the joint \( D \), that rotates about that center, will generate a straight-line segment with respect to the fixed link \( A_0 B_0 \). This line segment, that is produced by \( D \), is at a right angle to \( A_0 B_0 \), since the center \( S \) still joins the line \( A_0 B_0 \). The mechanism that is obtained this way is the straight-line mechanism named after Hart who discovered it in 1877.

According to eqn (8) of the last paragraph, the two points \( S \) and \( T \) simultaneously go to infinity. Therefore, the link \( DT \) does not exist in Hart’s mechanism, although the point \( D \) does produce a straight-line which is perpendicular to the coupler \( AB \) if we consider the motion with respect to the coupler instead of the fixed link.

Hart’s mechanism still gives us freedom to choose the dimensions of the basic four-bar which is \( A_0 A B B_0 \). The remaining dimensions then are calculated as follows:

For instance, with \( \lambda = \lambda a_4; a_4 = B_0 S; S A_0 = -1 \), eqn (16) transforms into the equation:

\[
(\mu' + 1)(\mu' + AB^2/A_0 B_0^2) = 0 \quad (21)
\]

where the root \( \mu' = -1 \) corresponds to a pair of points \((F, C)\) lying at infinity, which we do not want. Therefore, we may only use the other root, which is

\[
\mu' = -\frac{AB^2}{A_0 B_0^2}. \quad (22)
\]

Thus,

\[
CB_0 / CB = FA_0 / FA = A_0 B_0^2 / AB^2. \quad (23)
\]

Further, eqn (18) transforms into

\[
DF = \frac{CB}{AB} \cdot A_0 B_0. \quad (24)
\]

Finally, eqn (20) yields

\[
DC = A_0 F \cdot \frac{AB}{A_0 B_0}. \quad (25)
\]

Thus, starting from the four-bar chosen randomly, all other dimensions can be obtained using the foregoing equations.

Naturally, we want a resulting mechanism with reasonable transmission angles. Basically, the mechanism contains two such angles that are decisive. They are the angles
\[ \mu = \angle ABB_0 \quad \text{and} \quad \mu^* = \angle FDC. \]

If they stay larger than 30°, then, usually, the forces that are transmitted by the mechanism will cause a smooth motion. Otherwise, the mechanism may have a motion which gets out of hand at high velocities. With \( A_oA \) as the input-crank, however, we are always able to choose our dimensions such that the transmission angle \( \mu \) remains larger than 30°.

However, since the other angle

\[ \mu^* = \angle (a_2, u\bar{a}_1) = - \angle (u\bar{a}_2, u\bar{u}_1) = - \angle (a_1, u\bar{a}_2) = \angle (A_oA, BB_0) \]

we have to avoid positions where \( A_oA \) and \( B_oB \) are parallel.

This can be accomplished only if we take a double-crank as the basic four bar. Then, such positions do not occur. The result is shown in Fig. 4(a), where both the angles \( ABB_0 \) and \( FDC \) remain acceptable throughout the motion. Figure 4 also shows the curve-cognate† of Hart’s mechanism. This is obtained by observing the mechanism as a Stephenson-I six-bar.

As we know from the cognate theory described by Dijksman [9] in his paper “Six-bar cognates of a Stephenson-mechanism”, we are then dealing with a special case, since \( \Delta A_oAF \sim \Delta B_oBC \). However, the cognate then obtained in this case, again appears to be a mechanism of Hart’s configuration. So, in fact, nothing new arises out of it. As shown in Fig. 4(a), the cognate linkage will merely contain a double rocker if the basic four-bar is a double-crank in the source mechanism. How actually, the two curve-cognates are to be derived from one another, has been fully explained in the paper just mentioned.

Summarizing, we may conclude that all Hart’s linkages occur in pairs. Further, each pair represent two curve-cognates that produce the same line-segment which is normal to the common link which is the fixed link, \( A_oB_0 \) (see Fig. 4).

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*The name curve-cognate has been coined to distinguish between curve-cognates and coupler-cognates. Curve-cognates always have a common coupler-point, whereas coupler-cognates have a common coupler plane. Curve cognates are dissimilar mechanisms producing the identical curve. Coupler-cognates produce identical coupler motion.
4. Methods of design

(a) A construction of the focal linkage

As could be naturally expected, the geometric properties of the focal linkage are related to the reflected similarities between the opposite four-bars that are contained in the linkage. So, for example from

\[ \square A_o F D S \sim \square D C B T \quad \text{and} \quad \square S D C B_o \sim \square T A F D \]

we derive that

\[ \angle A_o D F = \angle D B C \quad \text{and} \quad \angle F D A = \angle C B_o D. \]

Consequently

\[ \angle A_o D A = \pi - \angle B_o D B. \quad (26) \]

Therefore, point \( D \) views opposite sides of the basic four-bar \( A_o A B B_o \) under the same angle or under angles that are each other's supplement. Clearly, this property defines the locus for the points \( D \) that are to be used as quadruple joints. In the literature\(^{[10, 11]} \) this locus is a well-known curve. As mentioned earlier, Burmester named it the focal curve. In the synthesis of planar mechanisms the curve plays an important role in connection with four position theory. Then, the curve is defined as the locus of points that view opposite sides of an opposite pole-quadrilateral under the same angle. It is then called the pole-curve, which is a circular curve of the third order. Therefore, all of its properties are thoroughly described in the literature (see for instance\(^{[12]} \)), and the given methods of constructing the curve can be used also to determine the focal curve in our case. Once the curve is known, we may choose point \( D \) on it and thereafter find the locations of the points, \( S, F, T \), and \( C \). This may be done, for instance, by using properties like \( \angle F D A = \angle B B_o D \), etc. However, this method implies drawing the focal curve. In order to avoid this, we would like to start the design by choosing the point \( S \) instead of \( D \). It means we have to use the properties of the focal curve in another way. Therefore, a design such as the one proposed, is based on the equation of the focal curve in a particular form.

To reveal this form, we observe the curve represented by the equation

\[ C_1 \cdot L_2 - C_2 \cdot L_1 = 0 \quad (27) \]

Here, \( C_1 \), for instance, represents a quadratic term that goes to zero if a point joins the circle \( C_1 \). (In other words, \( C_1 = 0 \) represents the equation of a circle).

Similarly, \( C_2 \) is a quadratic term denoting the circle \( C_2 \), whereas \( L_1 \) and \( L_2 \) are linear terms representing the lines \( L_1 \) and \( L_2 \).

In the case under consideration, \( C_1 \) is the circle joining the points \( A, A_o \) and \( Q = (A B, A_o B_o) \). Likewise, \( C_2 \) joins the points \( B, B_o \) and \( Q \) (see Fig. 5). Finally, \( L_1 \) and \( L_2 \) represent the lines \( A_o A \) and \( B_o B \), respectively.

Clearly, both curves, the one represented by eqn (27), as well as the focal curve, \( p \), are third order curves. They both join the points \( A_o, A, B, B_o, Q \) and \( P = (A_o A, B_o B) \), the circle-points \( I_1 \) and \( I_2 \) of the plane, and, finally, the focal point \( \Gamma \) of \( p \), which is another intersection\(^{†} \) of \( C_1 \) and \( C_2 \).

Equation (27) still gives us freedom to normalize the expressions for \( L_1 \) and \( L_2 \) such that a tenth point will be common to the two curves. However, if two third order curves intersect at more than \( 3 \times 3 = 9 \) points, the curves must be identical. Therefore, eqn (27) represents the focal curve if it is normalized in the right way.

Now, instead of eqn (27), we may also use the parametric representation

\[ \begin{align*}
C_x &= C_1 - \lambda C_2 = 0 \quad (28a) \\
L_x &= L_1 - \lambda L_2 = 0 \quad (28b)
\end{align*} \]

\(^{†}\text{From planar geometry we know that all four circles } C_1, C_2, C_3 \text{ (joining } A, B \text{ and } P \text{) and } C_4 \text{ (joining } A_o, B_o \text{ and } P \text{) that circumscribe the triangles of a quadrilateral have a common point, which is the focal point } \Gamma. \)
This is true, because if we eliminate the parameter \( \lambda \) from these equations, we again arrive at eqn (27). Here, eqn (28a) represents a pencil of circles \( C_\lambda \), whereas eqn (28b) represents a ray of lines \( L_\lambda \). Thus, if we intersect the circle \( C_\lambda \) with the corresponding line \( L_\lambda \), each time we obtain two points \( D \) and \( D' \) that are points of the focal curve. Clearly, \( C_\lambda \) joins the common points \( Q \) and \( \Gamma \), whereas \( L_\lambda \) has the pole \( P \) for its center.

Further, since \( \angle DSB_0 = \angle ATD \), the points \( Q, T, D \) and \( S \) all lie on the same circle \( C_\lambda \). Additionally the point \( D' \), defined by the crossed-parallelogram-chain \( TDSD' \), joins \( C_\lambda \). Therefore, \( D \) and \( D' \) both are points of \( p \) and \( C_\lambda \). Hence, they join the same line \( L_\lambda \) joining \( P \).

Since, in addition, \( DD' \) and \( ST \) are the diagonals of a crossed-parallelogram-chain, they are parallel. This property may be used to determine the direction of \( L_\lambda \) that is coordinated to the circle \( C_\lambda \). All this leads to a direct design of Kempe's focal linkage. The corresponding assignments are: (see Fig. 6).

(a) Choose the four-bar \( A_0ABB_0 \) randomly and do the same with the point \( S \) on \( A_0B_0 \).

(b) Determine point \( T \) on the coupler, such that \( AT/TB = A_0S/SB_0 \).

(c) Draw the circle \( C_\lambda \), joining the points \( S, T \) and \( Q = (AB, A_0B_0) \).

(d) Intersect \( C_\lambda \) and the line \( L_\lambda \) (joining \( P \) and being parallel to \( ST \)) at the quadruple points \( D \) and \( D' \). (In case there are no real intersections, a focal linkage cannot be assembled. Then, one has to re-locate the point \( S \) on \( A_0B_0 \) until real intersections occur.)

(e) Choose one of the available quadruple points \( D \) or \( D' \).

(f) Intersect \( C_\lambda \) and the circle \( C_1 \) joining \( A_0, A \) and \( Q \) at the focal point \( \Gamma \).

(g) Finally, intersect the circle \( C_\alpha \) joining \( P, D \) and \( \Gamma \) with \( A_0A \) at the point \( F \), and with \( B_0B \) at the point \( C \).

(h) Connect the point \( D \) with the points \( C, T, F \) and \( S \).

(i) Verify that \( C, T, F \) and \( S \) do join a circle. (This follows from the fact that \( \square SFTC \) resembles a cyclic quadrilateral.)
(b) A construction of Hart's straight-line mechanism

In the particular case where the points $S$ and $T$ vanish to infinity, we obtain Hart's straight-line mechanism. In this case, $C$, joining $S$ and $T$, degenerates into the line $\Gamma Q$ and into the line at infinity. The point $D$, which is the intersection of $\Gamma Q$ and the focal curve $p$, may then be determined using the focal construction of $p$ in which the focal axis $MM'$ and the focal point $\Gamma$ are the fundamental tools to carry it out.

For briefness' sake we will skip explaining the focal construction. Therefore, we will confine ourselves to the construction of Hart's linkage only.

The assignments are: (see Fig. 7).

(a) Connect the midpoints $M$ and $M'$ of the diagonals of the four-bar $A_0ABB_0$, chosen arbitrarily.

(b) Intersect the circles $C_3$ (joining $A$, $B$ and $P$) and $C_4$ (joining $A_0$, $B_0$ and $P$) at the point $\Gamma$.

(c) Intersect $MM'$ and $\Gamma Q$ at the point $M_\lambda$.

\[\text{Figure 6. Design of Kempe's configuration.}\]

\[\text{Figure 7. Design of Hart's straight-line mechanism.}\]

\[^{\dagger}\text{Here, any line that joins } \Gamma, \text{ intersects the focal curve at two other points having equal distances to the focal axis.}\]
(d) Determine $D$ on $\Gamma Q$ such that $M_4D = QM_4$.

(e) Finally, intersect the circle $C_4$ (joining $\Gamma$, $P$ and $D$) with $A_0A$ at the point $F$, and with $B_0B$ at the point $C$.

Remarks

(1) We may improve the accuracy of the construction using the fact that the midpoint $M''$ of the line-segment $PQ$ joins the focal axis $MM'$. 

(2) In addition, the location of $\Gamma$ may be made more accurate if we use the fact that the four circles $C_1$, $C_2$, $C_3$, and $C_4$ all have $\Gamma$ as their common point.

5. Kempe's focal linkage generalized

Kempe derived the focal linkage from a more general configuration that has escaped attention from most kinematicians. Only Bricard[13] hinted at it in his book *Leçons de Cinématique*. Kempe derived the general form by connecting two "conjugate" linkages both having the same "connecting diagram". The mathematics involved to carry this out, however, are rather complicated. For this reason we will derive the general form in a different way using a simpler procedure, such as the one that is based on spiral-similarities or stretch-rotations[7].

We do this, starting from the focal linkage and then stretch-rotate subsequently all opposite fourbars contained in the linkage with respect to the (successive) turning-joints of the basic four-bar $(A_0ABB_0)$. If we do this, the form obtained still contains opposite four-bars, but they no longer have a common (quadruple) joint. This joint, namely, has then been split up in four different joints that form a quadrilateral $(QRD'E)$ if we connect them (see Fig. 8).

The assignments to derive the general form from the focal linkage are as follows: (see Figs. 1 and 8)

(a) Stretch-rotate $\Box AFDT$ about $A$ using the complex multiplication-factor $AT'/AT$ chosen randomly. Then, we obtain the four-bar $AF'ET'$ that remains similar to $\Box AFDT$ if the triangles $AT'B$ and $AF'A_0$ are made rigid.

(b) Stretch-rotate $\Box BCDT$ about $B$ through multiplication with the complex factor $BT'/BT$. Hence, we obtain the four-bar $BC'QT'$ that remains similar to $\Box BCDT$ as long as the triangle $BC'B_0$ is a rigid one. In addition, the triangle $T'QE$ may also be made rigid since the angular rotations of the sides $T'E$ and $T'Q$ remain the same since they are not affected by the stretch rotations.

(c) Next, stretch-rotate $\Box B_0CDS$ about $B_0$ through multiplication with the factor $B_0C'/B_0C$ into the similar four-bar $B_0C'RS'$. Now, the triangles $B_0S'A_0$ and $C'QR$ may be made rigid.

\[\text{Figure 8. Kempe's configuration generalised.}\]

\[\text{Note that } \Gamma Q \text{ intersects } \Gamma \text{ at the three points } \Gamma, Q \text{ and } D. \text{ The way we have found the third point } D \text{ is based on the just-mentioned focal construction of } \Gamma.\]
(d) Finally, stretch-rotate AoFD’s about Ao into the similar four-bar AoF’D’S’ using the factor AoF’/AoF. Since S’” = S’ as will be proved hereafter, we are able to make ΔS’D’R a rigid triangle and so is ΔF’D’E.

In order to prove that indeed S’” = S’, it suffices to prove that AoS’/AoS = AoF’/AoF, since the stretch-rotation about Ao then indeed transforms S into S’.

According to the first three stretch-rotations we may note that:

\[ \frac{AoS'}{AoS} = \frac{AoS'}{AoS} \frac{BoS}{Bs} = \left(1 + \lambda \right) - \lambda \frac{Bc'}{Bc} \]

\[ = 1 + \lambda - \lambda \left\{ \frac{1 + \lambda'}{\lambda} \frac{1}{\lambda'} \right\} = 1 - \frac{\lambda}{\lambda'} \left(1 - \frac{BA + AT}{BT}\right) \]

\[ = 1 - \frac{\lambda}{\lambda'} \left(1 - \frac{AF'}{AF}\right) = 1 + \mu' - \mu \frac{AF'}{AF} = \frac{AoA + AF(AF/AF)}{AoF} \]

\[ = AoF'/AoF. \]

Hence, \( S'' = S' \).

All stretch-rotation factors that are used for the joints \( A_o, A, B \) and \( B_o \), therefore, are related through the equations:

\[ f_{ao} = 1 + \mu' - \mu f_a \]

\[ f_a = 1 + \mu - \mu f_a \]

\[ f_{bo} = 1 - \mu' + \mu f_a \]

(29)

Further, it may be proved that both the inner and the outer four-bar are similar and stay that way. To prove this, we observe the four-bar \( AoABBo \) as a closed vector-polygon and write

\[ (1 + \mu')u_a + (1 + \mu)u_a + (1 + \lambda u_a + (1 + \lambda)u_a = 0. \]

(30)

After reflection and multiplication with the factor \( u \), we obtain the four-bar represented by the identity:

\[ (1 + \mu')u_a + (1 + \mu)u_a + (1 + \lambda u_a + (1 + \lambda)u_a = 0. \]

Thus, according to eqn (7) we obtain immediately

\[ (1 + \mu')u_a + (1 + \lambda u_a + (1 + \mu')\mu a + (1 + \lambda)u_a = 0. \]

(31)

Hence,

\[ (1 + \mu')CD + (1 + \lambda)u' DS + (1 + \mu')\lambda FD + (1 + \lambda)DT = 0. \]

(32)

It is possible, therefore, to compose a four-bar of which the sides have the angular rotations of the four links joining the quadruple joint \( D \) of the focal linkage. The sides of the four-bar \( QRD'E \) have the same angular rotations. Thus, apart from a common factor, eqn (31) represents the four-bar \( QRD'E \). Consequently,

\[ \Box QRD'E \sim \Box AoABB_o. \]
Other properties that we may find are:

\[
\begin{align*}
\text{Area } \Delta RD'S' &= \text{Area } \Delta ABT' \\
\text{Area } \Delta QRC' &= \text{Area } \Delta A_0F'A \\
\text{Area } \Delta F'ED' &= \text{Area } \Delta C'BB_0 \\
\text{Area } \Delta QT'E &= \text{Area } \Delta S'A_0B_0
\end{align*}
\]  \hspace{1cm} (33)

Further, also

\[
\text{Area } \Delta A_0F'A + \text{Area } \Delta C'BB_0 = \text{Area } \Delta ABT' + \text{Area } \Delta S'A_0B_0. \quad (34)
\]

Finally, the proposition holds that each turning-joint of the general form, joins two rigid angles that are either equal or differ \( \pi \) radians.

6. Eight-bar linkage that contains a bar consistently rotating about a virtual center

If we look at the general form which is demonstrated in Fig. 8, we immediately see that the form is an overconstrained one. Because of this, we may loosen two turning-joints and still maintain a constrained motion. We must take care, however, to loosen joints only if they do not belong to the same triangle.

For instance, if we take away the turning-joints \( F' \) and \( B_o \), we obtain the mechanism as demonstrated in Fig. 9. In this linkage, the binary link \( BC' \) still has to rotate about the new virtual center \( B_o \). Actually, we have replaced the fixed center \( B_o \) with an eight-bar linkage that produces the rotation about that center.

The designer may use the linkage that way if there is lack of room in the neighbourhood of \( B_o \). Of course, he still has a considerable number of design-parameters at his disposal. There are 5 for the focal linkage and two additional free ones to obtain the general form from the focal linkage. So, in total, the designer has 7 free parameters at his disposal from which he may calculate the dimensions of the linkage, such as the one demonstrated in Fig. 9.

7. Eight-bar linkage containing a bar having rectilinear translation

We may obtain an entirely new mechanism if we generalize Hart's linkage using stretch rotations. This can be carried out in a manner similar to that used in obtaining the generalized form from the focal linkage. With Hart's linkage the points \( S \) and \( T \) are non-existent, since they are at infinity. Stretch-rotations that are carried out with respect to the turning-joints of the four-bar \( (A_0ABB_0) \) do not alter this. Therefore, the joints \( S' \) and \( T' \) are still points at infinity. Since link \( RD' \) rotates about \( S' \), it must now produce a rectilinear translation with respect to \( A_0B_0 \) which is the fixed link (see Fig. 10).

In this way, we have created a new linkage containing the bar \( RD' \) that oscillates rectilinearly. Actually, the mechanism that is obtained this way, contains two crank-and-slider mechanisms that are connected such that no slider is needed.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{eight-bar-linkage}
\caption{Eight-bar with a link that rotates about a fixed joint.}
\end{figure}
Figure 10. Eight-bar with rectilinear motion of the bar \( RD' \).

The mechanism as shown in Fig. 10, is obtained from Hart's linkage using for \( f_{ao} \) a stretch-rotation factor \( -i = -\sqrt{-1} \). In that case the link \( RD' \) translates rectilinear in a direction which is parallel to the direction of \( AoBo \).

Clearly, the designer is free to choose the complex value of the stretch-rotation factor \( f_{ao} \) in addition to the dimensions of the four-bar \( AoABBo \). The choice of \( f_{ao} \) enables him to govern the motion-direction of \( RD' \) as well as the displacement that is covered by the rectilinear translating link \( RD' \). The link \( RD' \), namely, moves in a direction enclosing the angle \( [(\pi/2) + \arg f_{ao}] \) with the frame-link \( AoBo \). Further, the displacement of \( RD' \) just equals \( |f_{ao}| \) times the displacement of point \( D \) of Hart's straight-line mechanism that has been used to obtain our eight-bar.

With regard to the dimensions of the newly obtained linkage, we note that still

\[
\lambda = \mu = -1 \quad \text{and} \quad 1/\lambda' = \mu' = -AB^2/AoBo^2.
\] (35)

Hence,

\[
f_n = f_\lambda = 1 + \lambda' - \lambda' f_{ao} \quad \text{and} \quad f_{bn} = f_{ao}.
\] (36)

From (36) and (9) it then follows that

\[
AoF'/AoA = BoC'/BoC
\]

Therefore,

\[
\Delta AAoF' = \Delta BBoC'.
\] (37)

Similarly,

\[
C'R/C'Q = F'D/F'E
\]

Hence,
The remaining dimensions of the linkage may further be determined as follows: Since

\[ B_0 R = f_{i_0} \cdot B_0 D \quad \text{and} \quad A_0 D' = f_{k_i} \cdot A_0 D \]

then, clearly, with \( f_{i_0} = f_{k_i} \), it follows that

\[ R D' = A_0 B_0 (f_{k_i} - 1) \]  \hspace{1cm} (39)

From this indeed, we may conclude that \( R D' \) can only move parallel to some fixed line in the frame. In the particular case that \( f_{k_i} = 1 \), the length of \( R D' \) reduces to zero. There is only a single point then that moves along a straight-line.

Naturally, since \( f_{k_i} = 1 \), we have not transformed Hart’s straight-line mechanism at all, and we are then still considering a six-bar instead of an eight-bar mechanism.

With

\[ A E = f_A \cdot A D \quad \text{and} \quad B Q = f_B \cdot B D \]

we, similarly, find that

\[ Q E = A B (f_A - 1) \]

Therefore,

\[ E Q = \lambda' A B (f_A - 1) \]  \hspace{1cm} (40)

From this, similar conclusions may be drawn as from eqn (39). We further find that

\[ Q R = C D (1 + \lambda' (f_{k_i} - 1)) \] \hspace{1cm} (41)

so, with (25) and (35) we derive for the modulus of \( Q R \) the expression

\[ Q R = - A_0 A \cdot \frac{A_0 B_0}{A B} |f_{k_i} - 1|. \]  \hspace{1cm} (41a)

Similarly, we find the relation

\[ D' E = D F (1 + \lambda' (f_{k_i} - 1)) \] \hspace{1cm} (42)

hence, with (25) and (35) we derive for the modulus of \( D' E \) the expression

\[ D' E = - B_0 B \cdot \frac{A_0 A}{A B} |f_{k_i} - 1|. \]  \hspace{1cm} (42a)

So, indeed

\[ Q R D' E \sim A_0 A B B_0 \] \hspace{1cm} (43)

which is a relationship that holds true also if the points \( S \) and \( T' \) are not lying at infinity.

In order to drive the linkage it would be advisable to let it have a crank that could rotate the full cycle. In other works: either a crank-and-rocker or a double-crank are the ones that have to be taken as our basic four-bar. Also, in order to obtain a smooth motion for all positions of the linkage, it is necessary to transmit the forces such that a maximum force component is available to produce a torque in an output member. To obtain this kind of situation in all positions of the linkage, the transmission angles [14] may never come below a permissible value. In the case
under consideration there are two such angles that are decisive for a smooth motion. These are the angles $\angle ABB_0$ and $\angle F'P_mC'$ if $A_0A'F'$ is the input-link (Here, $P_mC'= (EQ, D'R)$).

By choosing suitable dimensions for the basic four-bar, it is not difficult to keep the value of the transmission-angle $\angle ABB_0$ larger than a permissible value of, for example, $30^\circ$. The 2nd transmission angle, however, reaches the value of zero degrees twice because there are always two positions in which this occurs. These positions, therefore, are dead-center positions. If, namely, $F', C'$ and $P_mC'$ are aligned, the force transmitted between the links $RQC'$ and $D'EF'$ is directed along $C'F'$ and therefore, does not sustain the motion of the four-bar $QRD'E$ connected to the basic four-bar by the turning-joints $C'$ and $F'$.

Since $\Delta D'EF' \sim \Delta RQC'$, the configuration connected to these joints may be completed such that it resembles a Roberts' Configuration $C_R$. According to this configuration, there are two distinct possibilities that cause a zero angle for $\angle F'P_mC'$. They occur if either $ED'$ is parallel to $QR$ or otherwise if $\angle (EQ, D'R) \neq \angle EF'D'$.

According to eqn (43) the case $ED'\parallel QR$ occurs only if $B_bB\parallel A_0A$. For a crank-and-rocker mechanism this happens twice during a full rotation of the crank. For a double-crank, such a position does not occur. Further, according to eqn (43), the case $\angle (EQ, D'R) = \angle EF'R'$ occurs only if $\angle (A_0B_b, AB) = \angle A_0F'A$. Such a situation arises twice during the motion period of a double-crank and does not occur for a crank-and-rocker mechanism. Whatever our choice may be, therefore, there are always two positions for which the points $F'$, $C'$ and $P_mC'$ are aligned. In those positions, therefore, force-transmission to sustain the motion has to be enforced by other means.

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