An Experimental-Numerical Method for Nonlinear Structural Mechanics

A maximum likelihood estimator and the finite element method are used to construct a tool for estimating the state of continuous structures from measured structural data. These structural data can, for example, consist of displacements of material points and stresses or strains at material points. The statistical uncertainty of the measured data is supposed to be known. An approximation for the uncertainty of the estimated state can be given. The analyzing tool, elaborated for the geometrically nonlinear rigid plastic case, is useful for evaluation of feasibility and accuracy of experimental set-ups. An example of application of the tool is presented.

1 Introduction

In experimental mechanics the determination of the accuracy of the results gathered from measurements should be an integral part of the research. In cases in which the quantity of interest is directly measured, a prediction of the real value and an inaccuracy limit for this prediction follows straightforward from the measuring technique. An example of this is the experimental determination of the limit load of a structure by use of force transducers. In other cases the interesting figures are often derived from plain numerical manipulation of measured data values. Standard error propagation or probability analysis can be used to establish accuracy limits for the resulting data in an analytical way. A quite trivial example is the determination of the elastic moduli of some material from traditional material testing experiments like the tension test and the torsion test.

If the configuration under consideration is complex with a large amount of measured data, it may become necessary to combine numerical structural analysis with stochastic theories to be able to predict the values of desired data from measured data, as well as to assess the accuracy of the predicted values. Quite a lot research has already been performed in this field. A distinction can be made regarding developments making use of probability analysis and those making use of estimation theory.

The first category focusses on the prediction of statistical properties, resulting from variations in applied loads, material properties, and so on (e.g., Handa and Andersson, 1981; Drewniak and Pawicki, 1985; Liu et al., 1987). These methods are especially useful for reliability studies and risk analyses. The second category involves the prediction of the behavior of a structure or of uncertain parameters given a set of observations (e.g., Contreras, 1980; Mura et al., 1986; Oda and Shinada, 1987). These methods are often called (hybrid) experimental-numerical methods.

In this paper a development with respect to the second category is presented. It is a combination of the finite element method for structural analysis of isothermal quasistatic problems and the maximum likelihood method, known from estimation theory. It is suited for geometrically nonlinear problems, i.e., large displacements and rotations, with history-dependent material behavior, more specifically rigid plastic. The application is demonstrated in the field of contact and friction research. With the nonlinear method, the upsetting test is analyzed. The calculation gives directives for the quantities to be measured and their measuring accuracy needed for a reliable prediction of the contact stresses during upsetting.

In the following, a general description of the kind of problems is considered. Then the finite element method, for taking into account the physical behavior and the maximum likelihood method for dealing with the observation data, is discussed. After some remarks about their combined numerical implementation, the method is applied in the example of upsetting. Finally, some conclusions are given.

2 General Description of the Nonlinear Problem

An irreversible behaving body $B$ enclosed by boundary $\partial B$ is considered with a known initial state at some reference time $t_0$. Both the body and its boundary consist of an invariant set of material points. Due to mechanical loads the displacements of the material points are large, which causes geometrical non-linearity. Thermal effects, inertial effects, and distributed loads are supposed to be negligibly small.

The deformation of the body is considered during a time interval $t_0 \leq \tau \leq t$. At any time and at any internal point of
the body, local equilibrium requires the Cauchy stress tensor \( \mathbb{T} \) to satisfy (see, e.g., Hunter, 1976)
\[
\nabla \cdot \mathbb{T} = 0 \tag{1}
\]
with \( \nabla \) the gradient operator, \( \theta \) the zero vector, and \( \cdot \) the inner product. The actual value of \( \mathbb{T} \) at time \( t \) depends on the history \( (t_0 \leq \tau \leq t) \) of the deformation gradient \( \mathbb{F} \), expressible by means of a constitutive functional \( \mathcal{g} \)
\[
\mathbb{T} = \mathcal{g} \left[ \mathbb{F}(\tau) \right]. \tag{2}
\]
Mechanical properties figuring in \( \mathcal{g} \) should be known for the material under consideration. With three independent boundary conditions at each point of \( \partial B \), explicitly described during \( t_0 \leq \tau \leq t \), Eqs. (1) and (2) define a so-called direct problem (Kubo, 1988). The solution can be approximated by a straightforward analysis using a numerical scheme such as the finite element method.

In physical reality, however, sometimes boundary loads or displacements are not explicitly known or can only be measured with limited accuracy. To calculate for such cases the solution with a certain reliability, sufficient information about other quantities at the boundary or at interior points has to be determined and incorporated in the calculation, then called an inverse analysis. For this continuum problem a discretized approach, in space and in time, is adopted. Extension of the set of discretization points in space and time increases the accuracy of the approximation.

From the literature such types of analyses are known, especially in the field of contact problems between linear elastic bodies, e.g., Chambless et al. (1986), Oda and Shinada (1987), Turner et al. (1988). The (least squares or penalty) methods used are able to calculate the unknown interaction in the contact regions, but give neither indication on the weighting of the experimental data nor about the reliability of the results. Combined applications of numerical approximation techniques and estimation theories to contact problems with nonlinear materials involved are not known to the authors.

With the foregoing in mind, an incremental finite element formulation is chosen. The material behavior is assumed to be rigid plastic. As the estimation strategy for the incremental displacement field of the body with partly uncertain boundary conditions, the maximum likelihood method is used, which takes the stochastic behavior of the different boundary conditions into account. Information about the uncertainty of the displacement field and derivative quantities can be obtained.

3 Strategy for the Nonlinear Behavior

A well-known strategy used to solve approximately differential equations is the method of weighted residuals. The equilibrium Eq. (1) reads in the weak formulation
\[
\int_B \left( \nabla w \right)^T : \mathbb{T} dV = \int_{\partial B} w \cdot \mathbf{b} dA \tag{3}
\]
with \( : \) the double inner product, \( w \) the weighting vector field, and \( \mathbf{b} \) the external boundary load vector on the boundary \( \partial B \). This load is related to the material stress state by \( \mathbb{b} = \mathbb{T} \cdot n \) with \( n \) the unit outward normal vector on \( \partial B \). Substitution of the constitutive Eq. (2) yields an integral equation for the displacement field of the material points of the body. Elaboration of that equation depends on the nature of the constitutive equation. Here rigid plastic material behavior, obeying the Levy-Von Mises flow rule presupposing isotropic hardening, is chosen:
\[
\mathbb{T} = -p \| \nabla \epsilon \| - \frac{2}{3} \left( \frac{\mathbf{I}}{Y} \right) \mathbf{I} \mathbb{D} - \frac{1}{3} \text{tr} \left( \mathbb{T} \right) \epsilon \mathbb{I} = \left( \frac{2}{3} \mathbb{D} : \mathbb{D} \right)^{1/2} \tag{4}
\]
with \( \mathbb{I} \) the deformation rate tensor, \( p \) the pressure, \( \epsilon \) the equivalent plastic strain, and \( Y \) the yield stress. Equation (4) supposes \( \mathbb{I} \) unequal the second order null tensor. The Cauchy stress tensor \( \mathbb{T} \) satisfies Von Mises' flow criterion
\[
\frac{2}{3} \| \mathbb{D}^d \| : \mathbb{D}^d = Y^2, \quad \mathbb{T} = T + p \| \mathbb{T} - \frac{1}{3} \text{tr}(\mathbb{T}) \| \mathbb{I} \tag{5}
\]
where \( \mathbb{D}^d \) is the deviatoric stress tensor. For many metals under cold-forming conditions in noncyclic processes, the yield stress depends only on the history parameter \( \epsilon \). Elastic effects are not modeled by (4), as they may be neglected in case of large plastic strains. In the constitutive model the pressure \( p \) occurs as an extra unknown. This pressure should result from the requirement of incompressibility, \( \text{tr}(\mathbb{D}) = 0 \). This requirement is taken into account, weighted over \( B \) with a scalar weighting field \( q \)
\[
\int_B q \text{tr}(\mathbb{D}) dV = 0. \tag{6}
\]
The time domain \( t_0 \leq \tau \leq t \) is discretized in finite steps \( \Delta t \), called increments. For small incremental displacements \( u \), the tensor \( \mathbb{D} \) can be approximated by
\[
\mathbb{D} = \mathbb{A} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \tag{7}
\]
where the gradient operator \( \nabla \) is defined with respect to the configuration at the end of the increment. From (4) and (7) it follows for the end of the increment that
\[
\mathbb{T} = -p \| \nabla \Delta \epsilon \| - \frac{2}{3} \left( \frac{\mathbf{I}}{Y} \right) \mathbf{I} \mathbb{D} - \frac{1}{3} \text{tr} \left( \mathbb{T} \right) \epsilon \mathbb{I} \tag{8}
\]
where \( Y \) is the equivalent stress at the end of the increment, expressible in \( \Delta \epsilon \). The Eqs. (3), with (8) substituted in it, and (6), with (7) substituted in it, should hold at the end of each increment for arbitrary admissible weighting fields, \( w \) and \( q \), respectively.

The domain \( B \) is divided into material elements (Lagrangian formulation). In the elements the displacements, the pressure, and the weighting fields are interpolated between the nodal points using a Galerkin approach, i.e., equal interpolation functions for \( u \) and \( w \), as well as for \( p \) and \( q \). Requiring that the elaborated Eqs. (3) and (6) are satisfied for all admissible weighting fields results in a set of vector equations (related to equilibrium) and scalar equations (related to volume invariance) for each increment. With respect to some vector base, these equations can formally be written as
\[
\begin{bmatrix}
\mathbf{f}(\mathbf{u}, \mathbf{p}) \\
\mathbf{c}(\mathbf{u})
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix} \tag{9}
\]
where \( \mathbf{u} \) and \( \mathbf{p} \) are composed of the nodal values of the displacements and the pressures. As the set (9) is nonlinear in the nodal displacements and pressures, necessitating an iterative solution procedure, the iterative changes of \( \mathbf{f} \) and \( \mathbf{c} \) are of interest. From (9), analytical expressions for these iterative changes, \( \delta \mathbf{f} \) and \( \delta \mathbf{c} \), in the iterative changes, \( \delta \mathbf{u} \) and \( \delta \mathbf{p} \), can be obtained.

When displacement and pressure fields are known, the stresses at an arbitrary point of \( B \) can be calculated using (8) and the interpolations per element. To obtain stresses continuous over the element boundaries, a straightforward unweighted averaging of stresses in the nodal points at the element boundaries is used.

4 Strategy for the State Estimation

In the previous section the behavior of a body of rigid plastic material is approximated by a discretized incremental technique. Assuming the initial conditions and material behavior
to be known, the mechanical condition of the body after the first increment is described by $3n + m$ nodal unknowns composed of $3n$ displacements ($n$ displacement nodes) and $m$ pressures ($m$ pressure nodes), previously denoted by $u$ and $p$, respectively. The $3n$ displacements can arbitrarily be replaced by $3n$ nodal locations in the sequel.

For some $k$th increment the condition of the body at the end of the increment depends on $3n$ location degrees-of-freedom and $m$ pressure degrees-of-freedom more than its condition at the beginning of the increment. These degrees-of-freedom are gathered in the column $s(k)$, further called the incremental state extension of the $k$th increment. After $k$ increments, the total of $(3n + m)k$ unknowns is gathered in the accumulated state $s_N$

$$s_k^T = [s(1)^T s(2)^T \ldots s(k)^T]$$

(10)

which is the combination of the incremental state extensions $s(l)$ to $s(k)$.

To quantify the accumulated state $s_N$, belonging to a deformation process over a time period divided into $N$ increments, at least $(3n + m)N$ values of relevant quantities with respect to the condition of the body during the time period have to be known. A part $h_0$ of these quantities will be exactly known, the rest ($h_m$) of them will be measurable with only a limited accuracy. As relevant quantities, not only do the nodal degrees-of-freedom have to be considered, but also quantities like stresses, strains, and forces. The relevant quantities $h_c(k)$ and $h_m(k)$ of the $k$th increment can be uniquely expressed in the accumulated state $s_N$, so $h_c(k) = h_c(s_N)$ and $h_m(k) = h_m(s_N)$. The quantities $h_c(k)$ are called exact state-dependent quantities and $h_m(k)$ are the measurable state-dependent quantities. The actual data values for $h_c(k)$ compose the column $z_c(k)$ and are called exact observation data, and the actual data values for $h_m(k)$ compose the column $z_m(k)$ called measured observation data. Per increment, the incremental exact observation data $z_c(k)$ are equal to $h_c(k)$, whereas the incremental measured observation data $z_m(k)$ will deviate from $h_m(k)$ with an unknown stochastic column $v(k)$ containing incremental observation errors

$$z_c(k) = h_c(s_N), \quad z_m(k) = h_m(s_N) + v(k).$$

(11)

Note that modeling errors, e.g., due to the discretization, are assumed to be negligibly small. With the same notation convention (10), accumulated columns $h_{c,K}, h_{m,K}, z_{c,K}$ and $z_{m,K}$ are defined using the incremental columns $h_c(k)$, $h_m(k)$, $z_c(k)$, and $z_m(k)$. So it can be stated for the accumulated exact and measurable state-dependent quantities that $h_{c,K} = h_{c,K}(s_N), h_{m,K} = h_{m,K}(s_N)$.

(12)

The incremental observation errors of $k$ increments compose the accumulated observation error column $v_N = z_{m,K} - h_{m,K}(s_N)$.

The problem to be solved for the deforming body (over a time period divided into a total of $N$ increments) can now be stated as searching for that approximation $s_N$ for $s_N$ which obeys

$$z_m = h_m(s_N) + v_N.$$  

(13)

and satisfies some criterion expressed in $z_{m,N}, h_{m,N}, v_{N}$, and $s_N$. This formulation constitutes a static nonlinear estimation problem with constraints. A reliable estimate $s_N$ can be obtained with a maximum likelihood estimator (see, e.g., Schweppe (1973) and Norton (1986)). Generally for measuring errors, a Gaussian probability applies. Therefore, $v_N$ is supposed to have such a probability density $p(v_N)$ with zero mean and covariance matrix $R_N$. The likelihood function $p(z_N : s_N)$, defined as the density function of the observation data $z_N$ given state $s_N$, then results in

$$\exp \left( -\frac{1}{2} \left[ z_m - h_m(s_N) \right] R_N^{-1} \left[ z_m - h_m(s_N) \right] \right)$$

(14)

with $L$ the length of $v_N$, i.e., the total number of measured observations. Maximizing this function for the actual observation data $z_m$ under the constraints (13) yields the desired estimate $s_N$. Unicity of this estimate depends, among others, on the available exact and measurable state-dependent quantities and is not generally guaranteed. Physical insight and further research are needed to investigate this.

A disadvantage of the above method is of computational origin. To obtain a reasonable approximation of the nonlinear behavior of body $B$, the incremental state extension columns with each $3n + m$ nodal degrees-of-freedom and the total number of increments $N$ have to be relatively large. To deal with the accumulated state column $s_N$ of length $(3n + m)N$ in a maximization process of the function (14) would need large computation time and memory. For this reason, among others, in estimation theory for dynamic systems, filtering techniques are developed to determine estimates for successive time steps instead of simultaneously for the total set of time steps. The estimation of the incremental state extension $s(k)$ for the $k$th time step is then based on momentary observation data and some estimate $s_{k-1}$ for the accumulated state $s_{k-1}$. For the estimation of $s(k)$ in the considered case a simple “filter” is proposed in the sequel based on the previous incremental state extension estimate $s(k - 1)$ and the incremental observation data $z(k)$.

In the foregoing, $h_c(k)$ and $h_m(k)$ are introduced as functions of $s_N$, and thus of the combination of $s_{k-1}$ and $s(k)$. For the particular history-dependent material behavior these variables may be replaced by $s(k)$, $\Delta s(k)$ and $H(k)$, with $\Delta s(k)$, the change of the incremental state extension defined by

$$\Delta s(k) = s(k) - s(k - 1)$$

(15)

and $H(k)$, a column with the history parameters resulting after the $k$th increment, i.e., the equivalent plastic strains at a discrete number of material points. The choice of the material behavior enables that they can be expressed additively as

$$H(k) = H(k - 1) + \Delta H(s(k), \Delta s(k))$$

(16)

with $\Delta H$ a nonlinear function of $s(k)$ and $\Delta s(k)$. It is noticed that the change of the incremental state extensions contains the nodal incremental displacements and the changes of the pressure degrees-of-freedom of the $k$th increment. If an estimate $s(k - 1)$ for the incremental state extension $s(k - 1)$ is available, $s(k - 1)$ can be written as

$$s(k - 1) = \hat{s}(k - 1) + \delta s(k - 1)$$

(17)

with errors $\delta s(k - 1)$. Also, errors $\delta H(k - 1)$ will apply for the estimate $\hat{H}(k - 1)$. Assuming these errors are relatively small, the quantities $h_c(k)$ are written as

$$h_c(k) = h_c(s(k), s(k) - \hat{s}(k - 1), \hat{H}(k - 1) + \Delta H(s(k), s(k) - \hat{s}(k - 1))),$$

$$s(k) - \hat{s}(k - 1)) - \frac{\partial}{\partial \delta s} \left( H(z) \right) \delta s(k - 1)$$

+ \left( \frac{\partial}{\partial \delta H} \left( \delta H(k - 1) - \frac{\partial}{\partial \delta H} \left( H(z) \right) \delta s(k - 1) \right) \right)$$

(18)

where quadratic and higher-order terms in the errors are neglected. A similar relationship for $h_m(k)$ can be formulated. The only unknown in the zeroth-order terms of $h_c(k)$ and $h_m(k)$ is the incremental state extension $s(k)$. If the first-order terms in the errors are negligible, a maximum likelihood estimate $\hat{s}(k)$ for $s(k)$ can be obtained in a straightforward way. If the first-order terms are not negligible, the estimate $\hat{s}(k)$ solely based on the zeroth-order term will contain an extra bias

Journal of Applied Mechanics

DECEMBER 1993, Vol. 60 / 877
error, besides the bias error caused by the estimation method. As usual, the bias error $b(k)$ of the estimate $\hat{s}(k)$ is defined by $b(k) = E(\hat{s}(k)) - s(k)$ with $E(\hat{s}(k))$, the expected value of $\hat{s}(k)$, and $s(k)$, the true but unknown incremental state extension.

For the analysis of the upsetting experiment, the maximum likelihood estimator using only the zeroth-order terms is chosen, as the bias errors are expected to remain small. A part of the exact and measurable state-dependent quantities consists of positions of boundary nodes, which are linear in $s(k)$. This contributes to a bounding of the bias terms for two reasons. First, the maximum likelihood method delivers bias-free estimates for models with only linear state-dependent quantities. Secondly, such state-dependent quantities force the estimator to choose $\hat{s}(k)$ close to the true incremental state extension $s(k)$, despite possible bias errors in $\hat{H}(k - 1)$ caused by the maximum likelihood method. Assuming the estimate $\hat{s}(k)$ is unbiased and not too far from the true state, an approximation for its error covariance matrix $\Sigma$ can be given,

$$
\Sigma = E[\hat{s}(k) - s(k)](\hat{s}(k) - s(k))' = E[\hat{s}(k) - s(k)][\hat{s}(k) - s(k)]'.
$$

with $R = \Sigma(k)$, the covariance matrix of $v(k)$, and the Jacobian matrices $J_m^T = (\partial/\partial s_m)(\hat{h}(s))$ and $J_e^T = (\partial/\partial s_e)(\hat{h}(s))$. Note that the rows of $J_m^T$ span the orthogonal complement of the space spanned by the rows of $J_e$. Equations (19) offer a lower bound for the reliability of the resulting incremental state extension estimate.

### 5 Numerical Implementation

Because of the promising numerical results for incompressible flow, an axisymmetric triangular isoparametric $P_2 - P_1$ Crouzeix-Raviart element is applied for the numerical simulations. This element, with an extended quadrilateral interpolation for the shape and position field and a linear pressure field discontinuous over the element boundaries, provides accurate results for quite coarse meshes. It further shows the capability of numerical smoothing in the neighborhood of singular points. Such singularities can occur under sticking conditions at the outer edges of contact boundaries.

Various solution methods exist to tackle optimization problems (see, e.g., Gill et al., 1981). For the optimization considered here, the solution method has to minimize the following function

$$
F(s) = (z_m - h_m(s))^T R^{-1} (z_m - h_m(s))
$$

with respect to the unknowns $s$, with the nonlinear equality constraints

$$
z_e = h_e(s). 
$$

Again, unicity of the solution is not generally guaranteed. The penalty function approach is a very elegant and comprehensible method to deal with the constraints (21). This method transforms the constrained minimization of (20) into the unconstrained minimization of

$$
F(s) = (z_m - h_m(s))^T R^{-1} (z_m - h_m(s)) + (z_e - h_e(s))^T Q^{-1} (z_e - h_e(s)).
$$

The penalty matrix $Q^{-1}$, not necessarily diagonal, may be interpreted as a fictitious inverse covariance matrix of the exact observation data $z_e$. This interpretation is helpful in choosing reasonable values for $Q^{-1}$ in relation to $R^{-1}$, i.e., smaller covariances for $x_e$ than the covariances of comparable quantities of $x_m$ but not too small in order to avoid computational difficulties. Because of its robustness, the Newton method is used to perform the unconstrained minimization in an iterative manner. With the use of iterative equations as indicated in Section 3, the underlying equations are analytically known. Therefore, advantage can be taken of their structure, resulting in savings in computation time and memory. For the one-dimensional optimization along the search direction, a golden section search is employed, needing only the evaluation of function values.

In the upsetting experiment, given as an example in the next section, the boundary conditions are kinematical (prescribed positions of boundary nodes) and dynamical (prescribed stresses or forces). These quantities are partly considered as exact boundary conditions, depending on the problem studied. In the interior of the body there are constraints of zero nodal forces and equations following from volume invariance. The exact boundary conditions (except exact nodal point positions as these can be prescribed easily) and all constraints are taken in account as exact observation data.

### 6 Simulations of an Upsetting Experiment

It is the aim of the calculations to gain insight in the accuracy of the contact quantities which can be expected for different sets of boundary conditions, each representing a possible experimental set-up. With these results the upsetting experiment can be evaluated with respect to its usefulness for the measuring of contact behavior in forming processes.

The calculations are executed on an axisymmetric workpiece within the undeformed state, a diameter of 60 mm and a diameter-to-height ratio equal to 1. Because of axisymmetry and the assumption of symmetry with respect to the midplane of the workpiece, only one quarter of the cross-section has to be divided into elements. In Fig. 1 the different boundary parts are indicated. The workpiece is assumed to be made of pure aluminum. The yield stress $Y$ as function of the equivalent plastic strain is taken as

$$
Y = 32 + 120\varepsilon - 60\varepsilon^2 + 90\varepsilon^3 
$$

(23)

with $\varepsilon$ being the constant friction factor (0 ≤ $\varepsilon$ ≤ 1). The direction of the friction stress is equal to that of the relative displacement of the contact partner.

The nontrivial contact quantities of interest at the boundary part $\partial B_w$ are the radial displacements $\Delta r = r - r_0$ with respect to the reference configuration at time point $t_0$, the normal stress $\sigma_n$, and the (frictional) shear stress $\tau$. For the configuration considered, $\sigma_n$ acts in the negative axial direction and $\tau$ acts in the negative radial direction. For the boundary parts $\partial B_r$ and $\partial B_{\gamma}$, the boundary conditions, used for the calculations, are varied as indicated in Table 1. For all cases the boundary conditions along $\partial B_r$ and $\partial B_{\gamma}$ are reflecting the symmetry conditions ($\partial B_r$: radial displacement suppressed; $\partial B_{\gamma}$: axial displacement suppressed). Results from Case 1 will serve as input
Table 1  The boundary conditions of the five cases considered (crosses (x) indicate presence of a boundary condition)

<table>
<thead>
<tr>
<th>PART CONDITION</th>
<th>CASES</th>
</tr>
</thead>
<tbody>
<tr>
<td>S很少</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>SB ≡ MEASURED AXIAL DISPLACEMENT</td>
<td>X X X X X</td>
</tr>
<tr>
<td>VON MISES FRICTION MODEL</td>
<td>V</td>
</tr>
<tr>
<td>MEASURED NORMAL CONTACT STRESS</td>
<td>X X</td>
</tr>
<tr>
<td>MEASURED UPSETTING FORCE</td>
<td></td>
</tr>
<tr>
<td>S很少</td>
<td>6</td>
</tr>
<tr>
<td>MEASURED DISPLACEMENTS</td>
<td>X X X X X</td>
</tr>
</tbody>
</table>

Fig. 2 Deformed meshes after 3 mm (a) and 30 mm (b) upsetting

for the other calculations, which in fact should result in the same solution. The covariance estimates according to Eq. (19) are used to study for each case the reliability of this solution at the contact boundary. Case 3 is meant to study the influence of the measured upsetting force as extra boundary condition with respect to Case 2. Case 5 aims to investigate whether measured displacements of the outer unloaded surface contribute to an improvement of the contact quantities accuracy as resulting from Case 4. For all cases the standard deviation of the axial displacement at the contact boundary is supposed to be 0.01 mm, which is quite accurate. All measured quantities are supposed to be mutual independent. The experiences obtained from the different problem formulations are discussed sequentially.

Case 1. The boundary conditions as specified in Table 1, together with the symmetry boundary conditions, constitute a direct problem. For the constant friction factor $m$ a value of 0.1 is applied. The standard deviation of the friction stress calculated with this model is assumed to be 0.1 N/mm². In Fig. 2 the deformed meshes after 3 mm and 30 mm upsetting are shown. In the sequel the results will be presented always for these two upsetting stages.

As a measure for the reliability of the contact quantities a dimensionless standard deviation $s_d$ is introduced, which is comparable to the coefficient of variation from statistics. For each contact quantity $s_d$ is defined as a function of the position at the contact surface by

$$s_d = \frac{\delta}{\bar{p}_m}$$

with $\bar{p}_m$ the maximum of the absolute estimation of the quantity along the contact surface, and $\delta$ the locally estimated standard deviation. It can be interpreted as a relative error. The advantage of this definition is that the accuracy of the contact quantities, with mutually very different ranges, can be compared. The calculated values of $s_d$ for the radial displacement, normal stress, and friction stress after 3 mm and 30 mm upsetting are given in Fig. 3 as a function of the reference radial position $r_0$.

Case 2. In the second case, the knowledge on the contact model is not taken into account. For compensation, the measured axial and radial displacement of the free surface, with a standard deviation of 0.01 mm, are used as boundary conditions. It should be noted that for each increment the measured displacements are referenced to the original state, so it is assumed that no accumulation of measuring errors occurs. This second problem appears to be very ill-conditioned. Because of this, lack of convergence occurred in the calculation when this was started for each increment from a first guess for the solution which was too far from the actual solution. Therefore,
the incremental first guesses are taken to be equal to the incremental solutions of the direct problem of Case 1. The solution then calculated shows differences (increasing per increment) from the solution of the direct problem.

To illustrate this phenomenon, Fig. 4 shows the dimensionless absolute differences $\Delta_{21}$ between the solution of the current case and the solution of Case 1. The method to make the differences dimensionless is equal to the one used in Eq. (25). The differences are caused by the truncation of the displacements of the outer surface calculated for Case 1 before they are used as boundary conditions in the current case. In fact, these truncation errors act as very small measuring errors. The ill-conditioning is also illustrated by the resulting dimensionless standard deviations of the contact quantities, given in Fig. 5.

Case 3. With respect to Case 2, as extra input the measured upsetting force is used. It is supposed that this force (in the order of 0.1 MN) is very accurately known with a standard deviation of 100 N. To achieve convergence of the solution process the same procedure as in the previous case was necessary. The dimensionless standard deviations become smaller, see Fig. 6; however, not small enough to reach usefulness.

Fig. 4 Dimensionless absolute differences of Case 2 with respect to Case 1

Case 4. Instead of the knowledge of the contact model of Case 1 the measured normal stresses are applied as boundary conditions at the contact surface. The standard deviation of this stress is very conservatively supposed to be 10 N/mm². For this case as well as the next, the previously described special treatment of the incremental first guesses was no longer needed to obtain convergence. The resulting dimensionless standard deviations are given in Fig. 7. The large value of $s_d$ for the friction stress $\tau$ is caused by the small absolute values of $\tau$.

Case 5. In this case it is considered whether the estimates of Case 4 can be improved by measured contour data. As in the Cases 2 and 3, a standard deviation of 0.01 mm is supposed for the displacements of the contour points. In Fig. 8 improvements can be observed, especially for the high upsetting range and near the outer radius. In contrast to the reliability of the radial displacement and normal stress, the reliability of the friction stress is unsatisfactory. This has to be improved before calculated results for the friction stress can be used for quantifying contact models.

Three ways are mentioned to reach extra improvement. At first the standard deviation of the normal stress is taken to be quite large in the actual calculation. Measurement of the normal stress with a higher accuracy will reduce the standard deviation of the friction stress. Secondly, use can be made of
the smoothness of the contact stress pattern. Decrease in the number of contact stress unknowns will result in a higher accuracy for the contact quantities. This strategy was applied for calculations on a (linear elastic) stress measuring tool by Starmans et al. (1992). It turned out that the friction stress can be measured together with the normal stress, thus offering a third way for raising the reliability of the friction stress. The fitting quality of the (with a reduced number of contact stress unknowns) estimated contact stress pattern can be judged by using the reached minimum value of $F(s)$ according to Eq.
A quantitative measure for the fitting quality can be obtained by the $\chi^2$-test.

It is concluded that the use of measured displacements of the contour, the upsetting displacement, and the upsetting force only cannot lead to a successful estimate of the contact quantities. This is in accordance with experimental results as reported in the literature (Herbertz and Wiegels, 1981). Measurement of the stresses at the contact area is necessary. This can be done with a stress measuring tool as described by Starmans et al. (1992), or with some other method (see e.g., Tuncer and Dean (1987) for a review). Improvement of the results calculated from contact stress data may be obtained if additional quite accurate contour displacements are available.

7 Conclusions

In the foregoing a combined elaboration of the finite element method for quasi-static structural problems and the maximum likelihood method is described. It is applied to evaluate possible experimental set-ups in the field of contact and friction research in order to show its usefulness in experimental mechanics.

For the nonlinear problems an asymptotic unbiased estimation tool was formulated. This was, however, discarded on the basis of the current numerical limitations. A filter approach was presented instead, and elaborated for rigid plastic material behavior. Although this method needs still further evaluation it showed to be possible, in the example of the upsetting experiment, to take different kinds of measured data into account. Further, the covariance estimate proved useful to evaluate various experimental set-ups and to improve these set-ups. A further investigation of the behavior of this nonlinear tool is recommended. Via Monte Carlo simulations, the performance should be examined in further research. Also attention should be paid to the conditions for existence and unicity.

References


