GRADIENT ENHANCED DAMAGE FOR QUASI-BRITTLE MATERIALS

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SUMMARY

Conventional continuum damage descriptions of material degeneration suffer from loss of well-posedness beyond a certain level of accumulated damage. As a consequence, numerical solutions are obtained which are unacceptable from a physical point of view. The introduction of higher-order deformation gradients in the constitutive model is demonstrated to be an adequate remedy to this deficiency of standard damage models. A consistent numerical solution procedure of the governing partial differential equations is presented, which is shown to be capable of properly simulating localization phenomena.

KEYWORDS: continuum damage mechanics; localization; gradient dependence; finite element method

1. INTRODUCTION

Continuum damage models of quasi-brittle failure mechanisms generally exhibit strain softening. In conventional continuum theories, strain softening can cause local loss of ellipticity of the differential equations which describe the deformation process. As a consequence, the mathematical description becomes ill-posed and numerical solutions do not converge to a physically meaningful solution upon refinement of the spatial discretization.'

Some authors have proposed to abandon the principle of local action in the constitutive relations to remedy the problem of ill-posedness. The so-called non-local continuum models have been postulated to yield physically relevant and mesh-objective solutions.' On the other hand, some serious disadvantages of the procedure have been encountered, which are a direct consequence of the introduction of non-locality in the mathematical description. Theoretical and practical problems arise in the vicinity of construction edges, particularly if these are of a complex shape, and the implementation of a consistent numerical solution procedure requires drastic changes of existing computer codes, while inconsistent tangent operators deteriorate convergence characteristics to a dramatic extent.

As an alternative to non-local softening models, gradient-dependent descriptions have gained interest. Although strongly related to non-local theory, gradient-dependent models bear the significant advantage of being strictly local in a mathematical sense. The capabilities of gradient dependence have been investigated particularly for plasticity models. A detailed discussion of the incorporation of gradient terms in continuum damage models and the numerical implementation of such descriptions has not been presented hitherto.

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Here, a gradient formulation of a damage model for quasi-brittle fracture is derived from the non-local theory. With regard to numerical solutions, the spatial discretization of the governing equations is discussed and a consistent solution procedure of the resulting equations is presented. The effectiveness of the incorporation of gradient terms in the constitutive description with respect to mesh insensitivity is demonstrated by a one-dimensional example, for which both a numerical and an analytical solution are presented. Finally, a brief evaluation of the gradient damage model is given.

2. NON-LOCAL MODEL

This contribution is confined to quasi-brittle materials, which implies that damage evolution is considered to be the dominant dissipative mechanism. Viscous, thermal or other non-mechanical effects are not taken into account and strains and rotations are assumed to be small. Damage is considered to be isotropic; thus a scalar quantity, the damage variable $0 \leq D \leq 1$, suffices to describe the damage process. Undamaged material is characterized by $D = 0$, while $D = 1$ corresponds to complete loss of material coherence.

A strain-based formulation is adopted in the sense of Simo and Ju, which, starting from linear elastic material behaviour and utilizing the effective stress concept and the hypothesis of strain equivalence, leads to the stress–strain relation

$$\sigma = (1 - D)^4 \mathbf{H} : \varepsilon$$

with $\sigma$ the Cauchy stress tensor, $\varepsilon$ the linear strain tensor and $4\mathbf{H}$ the fourth-order Hookean stiffness tensor. The damage state is governed by the scalar history parameter $\kappa$, which represents the most severe deformation the material has experienced: $D = D(\kappa)$. In the non-local model, $\kappa$ is determined by an average scalar measure of the strain, the non-local damage equivalent strain $\bar{\varepsilon}_{eq} \geq 0$, through the Kuhn–Tucker relations

$$\dot{\kappa} \geq 0, \quad \bar{\varepsilon}_{eq} - \kappa \leq 0, \quad \dot{\kappa}(\bar{\varepsilon}_{eq} - \kappa) = 0$$

and the initial value $\kappa_0$. The value of the non-local equivalent strain $\bar{\varepsilon}_{eq}$ in a certain material point $x$ is a weighted average of the local equivalent strains $\varepsilon_{eq}(x) \geq 0$ over the surrounding volume $V$:

$$\bar{\varepsilon}_{eq}(x) = \frac{1}{V} \int_V g(\xi) \varepsilon_{eq}(x + \xi) \, dV,$$

with

$$\frac{1}{V} \int_V g(\xi) \, dV = 1$$

in which $g(\xi)$ is a weight function and $\xi$ denotes the relative position vector pointing to the infinitesimal volume $dV$.

3. GRADIENT DAMAGE FORMULATION

A gradient formulation can be derived directly from non-local theory. To this end, the local equivalent strain is expanded into a Taylor series according to

$$\varepsilon_{eq}(x + \xi) = \varepsilon_{eq}(x) + \nabla \varepsilon_{eq}(x) \cdot \xi + \frac{1}{2!} \nabla^2 \varepsilon_{eq}(x) \cdot (\xi)^2 + \frac{1}{3!} \nabla^3 \varepsilon_{eq}(x) \cdot (\xi)^3 + \frac{1}{4!} \nabla^4 \varepsilon_{eq}(x) \cdot (\xi)^4 + \cdots$$

as

$$\bar{\varepsilon}_{eq}(x) = \frac{1}{V} \int_V g(\xi) \varepsilon_{eq}(x + \xi) \, dV,$$
with $\nabla^{(n)}$ and $\cdot^{(n)}$ the $n$th order gradient operator and the $n$th order inner product, respectively; $\xi^{(n)}$ designates the $n$ factor dyadic product $\xi \ldots \xi$. With the assumption of isotropy, substitution of (4) into (3) yields

$$\bar{\varepsilon}_{eq} = \varepsilon_{eq} + c\nabla^2\varepsilon_{eq} + d\nabla^4\varepsilon_{eq} + \cdots$$

in which $\nabla^2$ denotes the Laplacian operator and the coefficients $c$, $d$, $\ldots$ are determined by the weight function $g(\xi)$ and the averaging volume $V$. Thus, neglecting higher-order terms in expression (5), definition (3) of the non-local equivalent strain can be replaced by

$$\bar{\varepsilon}_{eq} = \varepsilon_{eq} + c\nabla^2\varepsilon_{eq}$$

The gradient parameter $c$ is of the dimension length squared, so that an internal length scale is present in the gradient formulation.

Definition (6) of $\bar{\varepsilon}_{eq}$ is less suitable for numerical analyses. Because of the explicit dependence of $\bar{\varepsilon}_{eq}$ on the Laplacian of the local equivalent strain, finite element elaborations of (6) inevitably lead to $C^1$-continuity requirements of the displacement. This disadvantage can be avoided as follows. Differentiating expression (5) twice and reordering yields

$$\nabla^2\bar{\varepsilon}_{eq} = \nabla^2\varepsilon_{eq} - c\nabla^4\varepsilon_{eq} - d\nabla^6\varepsilon_{eq} + \cdots$$

substitution of which into (5) leads to

$$\bar{\varepsilon}_{eq} - c\nabla^2\varepsilon_{eq} = \varepsilon_{eq} + (d - c^2)\nabla^4\varepsilon_{eq} + \cdots$$

Comparing (5) and (8), it can be concluded that employing

$$\bar{\varepsilon}_{eq} - c\nabla^2\varepsilon_{eq} = \varepsilon_{eq}$$

as the definition of $\bar{\varepsilon}_{eq}$ will introduce an approximation of the same order of magnitude as the one induced by adopting (6). However, the treatment of $\bar{\varepsilon}_{eq}$ as an independent variable, which has to satisfy the partial differential equation (9), enables a straightforward, $C^0$-continuous finite element interpolation, as will be shown in the next section.

A difficulty of gradient models that has not been addressed so far, is the requirement of additional boundary conditions. In order to solve the averaging partial differential equation (9), boundary conditions concerning the equivalent strain $\bar{\varepsilon}_{eq}$ have to be specified. From a mathematical point of view, it is necessary to specify either $\bar{\varepsilon}_{eq}$ or the normal derivative $\nabla\bar{\varepsilon}_{eq} \cdot n$ ($n$ denotes the external normal unit vector) in every boundary point of the considered configuration. The physical interpretation of the additional boundary conditions remains an unresolved issue. The simple natural boundary condition

$$\nabla\bar{\varepsilon}_{eq} \cdot n = 0$$

has been adopted in this contribution, in correspondence with Lasry and Belytschko as well as Mühlhaus and Aifantis.

4. FINITE ELEMENT IMPLEMENTATION

The weak forms of the partial differential equations which govern deformation processes are derived according to the weighted residuals approach. The equilibrium equation

$$\nabla \cdot \sigma = 0$$
in which body forces have not been included for simplicity, is multiplied by the vectorial weight
function \( v(x) \) and subsequently integrated on the domain \( \Omega \). With the aid of the divergence
theorem, the resulting equation can be transformed to the weak form

\[
\int_{\Omega} (\nabla v)^T \sigma \, d\Omega = \int_{\Gamma} v \cdot p \, d\Gamma \quad \forall v \in \mathcal{V}^0
\]

(12)
of (11), with the superscript \( C \) denoting the conjugate operator, \( \Gamma \) the configuration boundary and
\( p = \sigma \cdot n \) the external stress vector acting on this boundary. The weak form of the averaging
equation (9) is obtained in a similar way. Multiplication with the weight function \( w(x) \) and integration
yields, again utilizing the divergence theorem and substituting the boundary condition (10),

\[
\int_{\Omega} (w \bar{\varepsilon}_{eq} + \nabla w \cdot \nabla \bar{\varepsilon}_{eq}) \, d\Omega = \int_{\Omega} w \bar{\varepsilon}_{eq} \, d\Omega \quad \forall w \in \mathcal{W}^0
\]

(13)

Departing from the weak forms, the finite element discretization is rather straightforward. Following the Galerkin approach, the displacement vector \( u \) and the weight function \( v \) are
discretized by

\[
u = \mathbf{g}^T \mathbf{N} \mathbf{u}, \quad v = \mathbf{g}^T \mathbf{N} \mathbf{v}
\]

(14)

with \( \mathbf{g} \) a column of unit vectors forming the co-ordinate system and the interpolation matrix
\( \mathbf{N} \) containing the shape functions; the columns \( \mathbf{u} \) and \( \mathbf{v} \) contain the nodal displacement and weight
vector components, respectively. Taking advantage of the symmetry of \( \sigma \) and storing its relevant
components in the column \( \mathbf{q} \), the kernel of the left-hand-side integral of (12) is elaborated
according to

\[(\nabla v)^T \sigma = \mathbf{v}^T \mathbf{B}^T \mathbf{q}
\]

(15)

with \( \mathbf{B} \) made up of the shape function derivatives. On the right-hand side of (12), \( v \cdot p \) is written as

\[v \cdot p = \mathbf{v}^T \mathbf{N}^T \mathbf{p}
\]

(16)
in which \( \mathbf{p} \) contains the components of the external load vector \( \mathbf{p} \). Substituting (15) and (16) into
(12) and taking into account that the resulting equation is to be satisfied for all \( \mathbf{u} \), the customary
discrete balance

\[f^u_{int} = f^u_{ext}
\]

(17)
of internal and external nodal forces is found, with

\[f^u_{int} = \int_{\Omega} \mathbf{B}^T \mathbf{q} \, d\Omega
\]

(18)

\[f^u_{ext} = \int_{\Gamma} \mathbf{N}^T \mathbf{p} \, d\Gamma
\]

(19)

A separate interpolation of the non-local equivalent strain and the corresponding weight
function is introduced:

\[\bar{\varepsilon}_{eq} = \mathbf{\bar{N}} \bar{\varepsilon}_{eq}, \quad w = \mathbf{\bar{N}} \mathbf{w}
\]

(20)
with \( \mathbf{N} \) consisting of interpolation functions; \( \bar{\varepsilon}_{\text{eq}} \) contains the nodal values of the average equivalent strain and \( \bar{w} \) those of the weight function \( w \). It is emphasized that the interpolation polynomials of \( u \) and \( \bar{\varepsilon}_{\text{eq}} \) need not be of the same order. Both discretizations only need to satisfy \( C^0 \)-continuity requirements. To avoid stress oscillations, the use of an interpolation for the displacements which is one order higher than that of the non-local equivalent strains seems advisable. Using the partial derivative columns

\[
\frac{\partial \bar{\varepsilon}_{\text{eq}}}{\partial \bar{x}} = \mathbf{B}_{\bar{\varepsilon}_{\text{eq}}}, \quad \frac{\partial \bar{w}}{\partial \bar{x}} = \mathbf{B}_{\bar{w}}
\]

the second term on the left-hand side of (13) is expanded as

\[
\mathbf{v}_w \cdot \mathbf{v} \bar{\varepsilon}_{\text{eq}} = \mathbf{w}^T \mathbf{B}^T \mathbf{B}_{\bar{\varepsilon}_{\text{eq}}}
\]

The other terms in (13) are elaborated simply by substitution of (20), eventually leading to

\[
\mathbf{K}^{\varepsilon} \bar{\varepsilon}_{\text{eq}} = \mathbf{f}^\varepsilon
\]

in which

\[
\mathbf{K}^{\varepsilon} = \int_{\Omega} \left( \mathbf{N}^T \mathbf{N} + \mathbf{B}^T \mathbf{B} \right) d\Omega
\]

\[
\mathbf{f}^\varepsilon = \int_{\Omega} \mathbf{N}^T \varepsilon_{\text{eq}} d\Omega
\]

With regard to the Newton–Raphson method, which is utilized to solve the discrete equations (17) and (23), the linearized change \( \delta \sigma_i \) of the stress column \( \sigma \) in iteration \( i \) is obtained starting from the matrix representation

\[
\delta \sigma_i = (1 - D) \mathbf{H} \delta \varepsilon_i
\]

Of the stress–strain relation (1):

\[
\delta \sigma_i = (1 - D_{i-1}) \mathbf{H} \delta \varepsilon_i - \delta D_i \mathbf{H} \delta \varepsilon_{i-1}
\]

For the first right-hand-side term, application of \( \varepsilon = b(u) = \mathbf{B} u \) simply yields

\[
\delta \varepsilon_i = \mathbf{B} \delta u_i
\]

The Kuhn–Tucker relations (2) imply that in case of increasing damage (\( \kappa > 0 \)) the history parameter satisfies \( \kappa = \bar{\varepsilon}_{\text{eq}} \), so \( \delta \kappa_i = \delta \varepsilon_{\text{eq},i} \). If no increase of damage occurs, \( \delta \kappa_i \) is given by \( \delta \kappa_i = 0 \). Whether or not damage is evolving, is determined by the actual value of the non-local equivalent strain compared to the converged value \( \kappa_0 \) of the history parameter in the previous increment. Thus, the change of damage \( \delta D_i \) can be linearized as

\[
\delta D_i = q_{i-1} \delta \varepsilon_{\text{eq},i} = q_{i-1} \mathbf{N} \delta \varepsilon_{\text{eq},i}
\]

in which

\[
q_{i-1} = \begin{cases} (\partial \mathbf{D} / \partial \varepsilon)_{i-1} & \text{if } \bar{\varepsilon}_{\text{eq},i-1} > \kappa_0 \\ 0 & \text{if } \bar{\varepsilon}_{\text{eq},i-1} \leq \kappa_0 \end{cases}
\]

With (28) and (29), expression (27) yields

\[
\delta \sigma_i = (1 - D_{i-1}) \mathbf{H} \mathbf{B} \delta u_i - \mathbf{H} \delta \varepsilon_{i-1} q_{i-1} \mathbf{N} \delta \varepsilon_{\text{eq},i}
\]
so that the iterative change of the internal nodal forces according to (18) may be written as

$$\delta f_{\text{int},i}^w = \int_{\Omega} B^T (1 - D_{i-1}) H B \, d\Omega \, \delta y_i - \int_{\Omega} B^T H_{\delta^i_{e_{i-1}}} q_{i-1} \tilde{N} \, d\Omega \, \delta \varepsilon_{\text{eq},i}$$  \hspace{1cm} (32)

Application of this expression in the discrete equilibrium equation (17) for iteration $i$ leads to

$$K_{i-1}^w \delta y_i + K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}} = f_{\text{ext},i}^w - f_{\text{int},i-1}^w$$  \hspace{1cm} (33)

with

$$K_{i-1}^w = \int_{\Omega} B^T (1 - D_{i-1}) H B \, d\Omega$$  \hspace{1cm} (34)

$$K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}} = - \int_{\Omega} B^T H_{\delta^i_{e_{i-1}}} q_{i-1} \tilde{N} \, d\Omega$$  \hspace{1cm} (35)

Applying the linearization

$$\delta \varepsilon_{\text{eq},i} = \xi_{\delta^i_{e_{i-1}}}^T \delta \varepsilon_{\delta^i_{e_{i-1}}} = \xi_{\delta^i_{e_{i-1}}}^T B \delta y_i$$  \hspace{1cm} (36)

in which

$$\xi_{\delta^i_{e_{i-1}}} = \left( \frac{\partial \varepsilon_{\delta^i_{e_{i-1}}}}{\partial \varepsilon} \right)_{i-1}$$  \hspace{1cm} (37)

equation (23) is elaborated as

$$K_{i-1}^w \delta y_i + K_{\varepsilon}^{w_{\delta \varepsilon_{\text{eq},i}}} = f_{\text{ext},i}^w - K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}}$$  \hspace{1cm} (38)

with $K_{\varepsilon}^w$ and $f_{\text{ext},i}^w$ according to (24) and (25), respectively, and

$$K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}} = - \int_{\Omega} \tilde{N}^T \xi_{\delta^i_{e_{i-1}}}^T B \, d\Omega$$  \hspace{1cm} (39)

The combination of equations (33) and (38) results in a square system of equations

$$\begin{bmatrix} K_{i-1}^w & K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}} \\ K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}} & K_{\varepsilon}^{w_{\delta \varepsilon_{\text{eq},i}}} \end{bmatrix} \begin{bmatrix} \delta y_i \\ \delta \varepsilon_{\text{eq},i} \end{bmatrix} = \begin{bmatrix} f_{\text{ext},i}^w \\ f_{\text{int},i-1}^w \end{bmatrix} - \begin{bmatrix} f_{\text{ext},i}^w \\ f_{\text{int},i-1}^w \end{bmatrix}$$  \hspace{1cm} (40)

which is rather similar to the system obtained by the De Borst and Muhlhaus\textsuperscript{10} for gradient plasticity. Expressions (35) and (39) for the partitions $K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}}$ and $K_{i-1}^{w_{\delta \varepsilon_{\text{eq},i}}}$ show that the tangent stiffness matrix is non-symmetric. However, the non-symmetry is caused by the damage formalism and not by the gradient enhancement. This becomes clear when the limiting case without gradient influence, i.e. $c = 0$, is considered. Then, the non-symmetry persists in (40). A symmetrized version could be used in computations, but this will probably lead to a loss of the quadratic rate of convergence, which is observed when the non-symmetric tangential stiffness matrix of equation (40) is used in a full Newton–Raphson procedure.

5. EXAMPLE

The merits of the gradient damage formulation are demonstrated using a simple, one-dimensional test problem. A bar of length $L$ is considered (Figure 1), which is subjected to a uniaxial, pure
tension loading by prescribed displacements at both ends. While all material characteristics are uniform for the entire bar, the cross-sectional area $A$ has been reduced by a factor $(1 - \alpha)$ between $x = -\frac{1}{2}l$ and $x = \frac{1}{2}l$ in order to trigger localization of deformation.

The equivalent strain is set equal to the axial strain $\varepsilon$ for the uniaxial, tensile stress situation. Thus, the averaging differential equation can be written as

$$\ddot{\varepsilon} - c \frac{d^2\dot{\varepsilon}}{dx^2} = \varepsilon$$  \hspace{1cm} (41)

or, employing the stress–strain relation

$$\sigma = (1 - D)Ec$$ \hspace{1cm} (42)

as

$$\ddot{\varepsilon} - c \frac{d^2\dot{\varepsilon}}{dx^2} = \frac{\sigma}{(1 - D)E}$$ \hspace{1cm} (43)

With respect to (43), the boundary condition

$$\frac{d\dot{\varepsilon}}{dx} = 0$$ \hspace{1cm} (44)

is imposed at the ends of the bar.

If the damage evolution law is chosen such that it represents the damage equivalent of perfect plasticity for a homogeneous strain distribution (Figure 2), i.e.

$$D(\kappa) = 1 - \frac{\kappa_i}{\kappa}$$ \hspace{1cm} (45)

an analytical solution can be derived for this problem. Because of the reduced cross-section, the actual stress in the weakened part of the bar is expected to exceed the homogeneous peak load $E\kappa_i$, while it remains below this value in the unweakened part. On the interval $(\frac{1}{2}l, \frac{3}{2}L)$—because of symmetry, only the right-hand part of the bar is considered—one can therefore set

$$\sigma = (1 - \beta)E\kappa_i, \text{ with } \beta > 0$$ \hspace{1cm} (46)
Figure 2. 'Perfect damage' model response for uniform strain

and in the weakened part, the length \( l \) of which is assumed to be smaller than the width \( w \) of the damage zone (Figure 1),

\[
\sigma = (1 + \gamma)E\kappa_i, \quad \text{with } \gamma > 0
\]  

(47)

In the undamaged part \((\frac{1}{2}w, \frac{1}{2}L)\), substitution of (46) and \( D = 0 \) in (43) yields the linear differential equation

\[
\bar{\varepsilon} - c \frac{d^2 \bar{\varepsilon}}{dx^2} = (1 - \beta)\kappa_i
\]  

(48)

which leads to the solution

\[
\bar{\varepsilon}(x) = (1 - \beta)\kappa_i + A_1 e^{(1/\sqrt{\beta/c})x} + A_2 e^{-(1/\sqrt{\beta/c})x} \quad \text{for } \frac{1}{2}w < x \leq \frac{1}{2}L
\]  

(49)

with \( A_1 \) and \( A_2 \) integration constants. In case of increasing damage, the history parameter \( \kappa \) equals the non-local strain \( \bar{\varepsilon} \) (cf. (2)). Using this equivalence, substitution of (45) in (43) yields for the damage zone \((0, \frac{1}{2}w)\) the differential equation

\[
\left( 1 - \frac{\sigma}{E\kappa_i} \right) \bar{\varepsilon} - c \frac{d^2 \bar{\varepsilon}}{dx^2} = 0
\]  

(50)

which, employing expressions (46) and (47), respectively, leads to solution parts of the forms

\[
\bar{\varepsilon}(x) = B_1 e^{\sqrt{\beta/c}ix} + B_2 e^{-\sqrt{\beta/c}ix}, \quad \text{for } \frac{1}{2}l < x \leq \frac{1}{2}w
\]  

(51)

and

\[
\bar{\varepsilon}(x) = C \cos(\sqrt{\beta/c}x) \quad \text{for } 0 \leq x \leq \frac{1}{2}l
\]  

(52)

Evidently, the sine-component is absent in (52) because of symmetry.

The integration constants \( A_1, A_2, B_1, B_2 \) and \( C \), the stress factors \( \beta \) and \( \gamma \) and the localization width \( w \) can be derived from boundary and continuity requirements. Through the differential equation (41), continuity of the displacement field imposes continuity of the non-local strain and its first derivative at \( x = \frac{1}{2}l \) and \( x = \frac{1}{2}w \). Furthermore, the discontinuity in the stress \( \sigma \) at \( x = \frac{1}{2}l \) should be in accordance with the surface reduction and the non-local strain \( \bar{\varepsilon} \) must equal the initial value \( \kappa_i \) of the history parameter for \( x = \frac{1}{2}w \). At the end of the bar, finally, the boundary condition (44) and a prescribed displacement of \( \frac{1}{2}\Delta L \) are applied. The latter quantity is related to
the non-local strain by
\[
\frac{1}{2} \Delta L = \int_0^{(1/2)L} \varepsilon \, dx = \int_0^{(1/2)L} \left( \varepsilon - c \frac{d^2 \varepsilon}{dx^2} \right) \, dx = \int_0^{(1/2)L} \bar{\varepsilon} \, dx - c \left. \frac{d \bar{\varepsilon}}{dx} \right|_{0}^{(1/2)L} = \int_0^{(1/2)L} \bar{\varepsilon} \, dx \tag{53}
\]
in which symmetry and the boundary condition (44) have been taken into account.

When \( A_1, A_2, B_1, B_2, C, \beta, \gamma \) and \( w \) have been computed for specific values of the geometry and material parameters, the strain and damage distributions can be determined in a straightforward fashion starting from the non-local strain solution (expressions (49), (51) and (52)). The strain is computed according to (41), which leads to
\[
\varepsilon(x) = (1 - \beta) \kappa_i \quad \text{for} \quad \frac{1}{2}w < x \leq \frac{1}{2}L \tag{54}
\]
is the undamaged part of the bar, and
\[
\varepsilon(x) = (1 - \beta) B_1 e^{\sqrt{\beta/c} x} + (1 - \beta) B_2 e^{-\sqrt{\beta/c} x}, \quad \text{for} \quad \frac{1}{2}l < x \leq \frac{1}{2}w \tag{55}
\]
\[
\varepsilon(x) = (1 + \gamma) C \cos\left(\sqrt{\gamma/c} x\right) \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}l \tag{56}
\]
in the process zone. In the undamaged part the damage variable satisfies
\[
D(x) = 0 \quad \text{for} \quad \frac{1}{2}w < x \leq \frac{1}{2}L \tag{57}
\]
while in the damage evolution zone substitution of \( \kappa = \bar{\varepsilon} \) according to (51) and (52) in the evolution law (45) yields
\[
D(x) = 1 - \frac{\kappa_i}{B_1 e^{\sqrt{\beta/c} x} + B_2 e^{-\sqrt{\beta/c} x}} \quad \text{for} \quad \frac{1}{2}l < x \leq \frac{1}{2}w \tag{58}
\]
and
\[
D(x) = 1 - \frac{\kappa_i}{C \cos\left(\sqrt{\gamma/c} x\right)} \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}l \tag{59}
\]

For finite element simulations, a bar of length \( L = 100 \text{ mm} \) and cross-sectional area \( A = 10 \text{ mm}^2 \) is considered. The length \( l \) of the weakened zone is set equal to 10 mm, the cross-section reduction factor \( \alpha = 0.1 \). The constitutive behaviour is described by Young's modulus \( E = 20000 \text{ N/mm}^2 \), the initial history parameter \( \kappa_i = 10^{-4} \) and the gradient parameter \( c = 1 \text{ mm}^2 \). Quadratic interpolation polynomials have been used for the displacement and linear polynomials for the non-local strain.

In Figure 3, the strain and damage distributions for \( \Delta L = 0.05 \text{ mm} \) have been plotted as computed with 20, 40 and 80 element meshes. Upon mesh refinement, both profiles rapidly converge towards the analytical solutions. The strain profile (Figure 3(a)) clearly shows the partially exponential and partially cosine distribution obtained in the analytical solution.

More extensive numerical analyses have been carried out employing the somewhat different damage evolution law
\[
D(\kappa) = \begin{cases} 
\kappa \frac{\kappa - \kappa_i}{\kappa_c - \kappa_i} & \text{if} \quad \kappa_i < \kappa \leq \kappa_c \\
1 & \text{if} \quad \kappa > \kappa_c 
\end{cases} \tag{60}
\]
which yields a linear softening stress–strain relation for a homogeneous strain field (Figure 4). Young's modulus is again chosen as \( E = 20000 \text{ N/mm}^2 \), while the initial and critical value of the history parameter are set as \( \kappa_i = 10^{-4} \) and \( \kappa_c = 0.0125 \), respectively.
For a reference value $c = 1 \text{ mm}^2$ of the gradient parameter, numerical solutions have been computed employing 80, 160, 320 and 640 element discretizations. The load–deflection curves obtained with these meshes have been plotted in Figure 5. The curves clearly show convergence to a meaningful, softening solution with a finite energy dissipation.
In Figure 6, the strain and damage evolution have been plotted for the 640 element mesh. The formation of an initially relatively large damaged area (Figure 6(b)) and the subsequent development of a narrow region of intense deformation (Figure 6(a)) seems to be an appropriate description of the process of initiation and growth of microcracks in a
relatively wide area and subsequent coalescence of some of these microcracks into one macro-
crack.

To examine the effect of gradient parameter variations on the response, the 640 element
simulation has been repeated for $c = 0.25, 0.50, 1.00, 2.00$ and $4.00$ mm$^2$. In Figure 7(a), the stress
in the bar has been plotted versus its elongation for $c = 0.25, 1.00$ and $4.00$ mm$^2$. Since the
spreading of deformation delays the onset of the damage process, a higher peak load is found for
increasing $c$. The less brittle behaviour for higher values of $c$, which is also observed in this plot,
can be related to the damage profiles at the moment of complete rupture (Figure 7(b)). For higher
values of $c$, and thus for a larger internal length, damage localizes in a larger region, and as
a consequence larger elongations are encountered. The dependence of the width of the damaged
zone on the internal length $\sqrt{c}$ introduced by the gradient dependence concept, is clearly
demonstrated by Figure 8(a). Another interesting relation is that between the internal length scale
and the energy which is dissipated in the fracture process. Figure 8(b) shows that the dependence
of the dissipated energy on the internal length is linear in the gradient model.

6. CONCLUSION

The merits of gradient dependence with respect to localization phenomena have been shown in
the context of continuum damage mechanics. The inclusion of a second-order gradient term in
the mathematical description of material behaviour implicitly introduces an internal length in the
constitutive model. On the occurrence of softening, the width of the zone in which deformation is
localized is determined by this internal length scale, as has been demonstrated by a simple,
one-dimensional example.

The gradient-dependent continuum damage model for quasi-brittle materials has been derived
from non-local theory. The averaging procedure which forms a part of the non-local model, has
been replaced by a partial differential equation, which has to be solved in addition to the
equilibrium equation. As a consequence, the 'non-local' scalar measure of strain which governs
damage evolution is considered as an additional independent variable. From the mathematical
viewpoint, boundary conditions with regard to damage evolution can be incorporated in
a natural fashion. For numerical solutions, the independent variables are interpolated separately.
As a result of the implicit incorporation of the gradient term, both discretizations need to
satisfy only $C^0$-continuity. A consistent solution strategy for the resulting equations has been
derived, which conforms to standard finite element procedures.

REFERENCES

5. R. de Borst, L. J. Sluys, H.-B. Mühlhaus and J. Pamin, 'Fundamental issues in finite element analysis of localization of