Element-free Galerkin methods in combination with finite element approaches

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Abstract

Element Free Galerkin (EFG) methods are methods for solving partial differential equations with the help of shape functions coming from Moving Least Squares Approximation. The EFG-method is more flexible than the Finite Element (FE) method, since it requires only nodal data and no element connectivity is needed. Because the EFG-method is computationally expensive, combinations of the EFG-method and the FE-method are considered. Implementations based on several weak forms of the problem description are studied. Numerical examples in elastostatics and fracture mechanics show that the implementations are able to compute both displacement and stress fields accurately.

1. Introduction

The Element Free Galerkin (EFG) method is a very promising method for the treatment of partial differential equations. Because of the absence of element connectivity, nodal points can be added easily to the part of the domain where the solution is (expected to be) steep. This makes the EFG-method more flexible than the Finite Element (FE) method. The method looks very promising for computations in fracture mechanics, since nodal points can be arranged around crack tips in order to obtain accurate stress intensity factors.

The method is based on Moving Least Squares Approximation (MLSA). For MLSA a set of nodal points, a set of basis functions and a set of weight functions should be chosen. MLSA results in shape functions which are used in Galerkin formulations of elasticity problems. Moreover, for numerical integration purposes, a cell structure for the problem domain should be chosen.

Moving Least Squares Approximation is computationally expensive, since for each point of the problem domain a linear system has to be solved. Moreover, a dense integration pattern is necessary to get accurate values for the resulting linear system. Therefore, from the viewpoint of computation time, it is more convenient to make use of EFG only on the part of the domain where one wants to achieve a better approximation of the solution, and to make use of the Finite Element (FE) method for the remaining part of the domain. This makes that in this paper attention is focused on combinations of the EFG-method and the FE-method.

The first form of the EFG method has been reported by Nayroles et al. [1]. Their method was called Diffuse Elements. In this method Moving Least Squares Approximation is used, but a term in the derivatives of the shape functions is neglected. Moreover, in the implementation of the method the MLSA-weight functions are not used for the determination for an integration point of the nodes which contribute to the stiffness matrix.

Belytschko et al. [2] and Lu et al. [3] reported the method in its current form. They did not omit a
term in the derivatives of the shape functions, they made use of the MLSA-weight functions for the
determination of the stiffness matrix for an integration point and they made use of a numerical
integration scheme by means of a background mesh of integration cells. Because of the circumvention
of the need for element data, they called this method element free. In their numerical examples they
focused on problems in fracture mechanics.

The current form was also studied by Hegen in [4]. In this study, it was reported that one has to be
careful with performing EFG for strongly nonconvex domains, and it was noticed that several
FE-approaches can be seen as an approach by EFG. Therefore, a suggestion was made for a simple way
of connecting FE-domains with EFG-domains.

This paper is the result of a study on several weak forms for the EFG-method, of a study on the
influence of the size of the supports of the (non-)singular weight functions which are used for EFG, of a
study on several combinations of EFG and FE and of a study on convergence rates for EFG and for
combinations of EFG and FE. The paper is organized as follows. In Section 2, Moving Least Squares
Approximation is described. In Section 3, weak forms are considered together with the discrete
equations for the EFG-method. Combinations of the EFG-method and the FE-method are considered
in Section 4. Numerical examples of several implementations of the method are presented in Section 5.
The paper will end with conclusions in Section 6.

2. Moving least squares approximation

In this section the main characteristics of Moving Least Squares Approximation (MLSA) are given. For
more details we refer to Lancaster et al. [5].

Consider a domain \( \Omega \) in space. To approximate a certain function \( u \) on \( \Omega \), a set of nodal points
\( \{ x_i \}_{i=1}^n \) is chosen together with a set of weight functions \( \{ w_i(x) \}_{i=1}^n \), such that \( x_i \in \Omega \), \( w_i \)
is non-negative and \( w_i(x) > 0 \) for each \( i \). Furthermore, a finite set \( \{ p_i(x) \}_{i=1}^m \) of so-called basis
functions is considered. This set can, for instance, be a finite subset of the set of monomials in the space
coordinates.

\[
\{ 1, x_1, x_2, \ldots, x_1^2, x_2^2, x_1 x_2, \ldots, x_1^2 x_2^2, \ldots \}.
\]

(2.1)

The Moving Least Squares approximant \( u^h \) of \( u \) for any \( x \in \Omega \) is given by

\[
u^h(x) = p(x)'a(x),
\]

(2.2)

where

\[
p(x)' = [p_1(x) \ p_2(x) \ \cdots \ p_m(x)]
\]

(2.3)

\[
a(x)' = [a_1(x) \ a_2(x) \ \cdots \ a_m(x)].
\]

(2.4)

The entries of \( a(x) \) are obtained by minimization of the sum

\[
J(a) = \sum_{i=1}^n w_i(x)(p(x)'a(x) - u_i)^2.
\]

(2.5)

Here, \( u_i \) is the value of \( u \) in \( x_i \). Stationarity of \( J(a) \) with respect to \( a \) results in a set of linear equations
for \( a(x) \)

\[
PW(x)P' a(x) = PW(x)u.
\]

(2.6)

with the entries of the matrices \( P \) and \( W \) given by

\[
P_{ji} = p_j(x_i), \ j = 1, \ldots, m, \ i \in \{ 1, \ldots, n \mid w_i(x) > 0 \}
\]

(2.7)

\[
W_{ji} = w_j(x_i) \delta_{ji}, \ i, j \in \{ 1, \ldots, n \mid w_i(x) > 0 \}.
\]

(2.8)
where \( \delta_{ij} \) is Kronecker's delta. The vector \( \mathbf{u} \) is the vector with entries \( u_i \). For simplicity, the following short-hand notation is introduced
\[
A(x) = PW(x)P'.
\] (2.9)

MLSA is well defined for the domain \( \Omega \) only when the system of equations (2.6) has a unique solution. It can be seen that this is the case if and only if the rank of the matrix \( P \) equals \( m \), the number of basis functions \( p_j \). Hence, a necessary condition for MLSA is that at least \( m \) weight functions are non-zero in each \( x \in \Omega \). In the sequel it will be assumed that MLSA is well defined for \( \Omega \).

The approximant \( u^h \) can also be expressed with the help of shape functions. One can write
\[
A(x) = \sum_{i=1}^{n} u_i \phi_i(x),
\] (2.10)
where the shape functions \( \phi_i \) are given by
\[
\phi_i(x) = \begin{cases} 
\sum_{j=1}^{m} p_j(x)(A^{-1}(x)PW(x))_{ji} & w_i(x) > 0 \\
0 & w_i(x) = 0.
\end{cases}
\] (2.11)

For examples of shape functions \( \phi_i \) we refer to Section 4. Eq. (2.11) shows that when \( w_i(x) = 0 \) also \( \phi_i(x) = 0 \). Hence, when \( w_i \) is only non-zero on a (small) subset of \( \Omega \), \( \phi_i \) is also only non-zero on this subset.

A disadvantage of MLSA is that for each point under consideration a linear system should be solved to obtain the value of the approximant \( u^h \) and the values of the shape functions \( \phi_i \). This is a burdensome task. Lu et al. [3] avoid matrix inversion by orthogonalisation of the basis functions with respect to the values of the weight functions in \( x \). They claim that this reduces the computational costs of solving the system. The computational costs of the orthogonalisation procedure, however, are of the same order as the costs of matrix inversion. From the viewpoint of accuracy, orthogonalisation of the basis functions is preferred over matrix inversion, since the orthogonalisation procedure is equivalent with solving the linear system (2.6) by means of the singular value decomposition of the matrix \( A \).

The smoothness of the approximant \( u^h \) and the smoothness of the shape functions \( \phi_i \), are related to the smoothness of both the basis functions and the weight functions. Let \( C^k(\Omega) \) be the space of \( k \) times continuously differentiable functions. Then, when \( w_i \in C^k(\Omega), \ i = 1, \ldots, n, \ k \in \mathbb{N}, \) and \( p_j \in C^l(\Omega), \ j = 1, \ldots, m, \ l \in \mathbb{N}, \) one has that \( u^h \in C^r(\Omega) \) and \( \phi_i \in C^r(\Omega), \ i = 1, \ldots, n, \) where \( r = \min(k, l) \). A proof of this can be found in [5] or [4].

When the function \( u \) is a linear combination of the basis functions \( p_j \), the approximant \( u^h \) equals this linear combination, as can be seen easily. Hence, the basis functions can be represented exactly by the shape functions. However, the shape functions are not a linear combination of the basis functions. For instance, when the basis functions are monomials in the space coordinates, the shape functions are not polynomial as can be seen in Section 4.

In general the shape functions \( \phi_i \) fail to have the selectivity property
\[
\phi_i(x_j) = \delta_{ij}.
\] (2.12)
Hence, \( u_i \) does not have the interpretation of nodal value of \( u^h \),
\[
u^h(x_i) \neq u_i.
\] (2.13)

There is, however, a possibility to obtain property (2.12). For this purpose, one should take the constant function on \( \Omega \) in the basis, and one should make use of weight functions in the form
\[
w_i(x) = s_i(x)|x - x_i|^{-\alpha_i} \quad x \neq x_i, \quad t = 1, \ldots, n,
\] (2.14)
where \( \alpha_i \in \mathbb{N} \) and \( \{s_i(x)\}_{i=1}^{n} \) are non-negative functions on \( \Omega \) such that \( s_i(x) > 0, \ i = 1, \ldots, n. \) Moreover, \( |x| \) is the Euclidean norm of \( x \). The weight functions are singular in the nodal points. For such weight functions MLSA is still well defined and the shape functions do have the selectivity property (2.12). For more details we refer to Lancaster et al. [5] or Hegen [4].
Other situations where the selectivity property (2.12) is fulfilled are situations where the shape functions coincide with shape functions coming from Finite Element interpolation. Such situations are described in Section 4.

In [4] it was remarked that one has to be careful with performing MLSA for a domain \( \Omega \) which is strongly non-convex. Here, one can think of a domain with a sharp concave corner. To achieve that MLSA is well defined for such a domain and to have that the shape functions are continuous on the domain, it is possible that shape functions become non-zero on parts of the domain (think of the opposite side of a corner) where it is more likely that they are zero. Hence, nodal points can influence the approximant \( u^b \) on parts of the domain where it is not really convenient to have this influence. In [2,3] solutions for this problem are presented for a domain with a crack. In this paper however, we focus on combinations of the EFG-method and the FE-method, and the domains which are considered in the numerical examples are (almost) convex.

3. Weak forms and discrete equations

3.1. Weak forms of the problem description

Consider the problem of the deformation of a linear elastic medium in three-dimensional space. Let \( \Omega \) be its undeformed configuration. The problem is described by the partial differential equations

\[
\sigma_{ij} + b_i = 0 \quad \text{on} \quad \Omega, \quad i = 1, 2, 3, \tag{3.1}
\]

where \( \sigma_{ij} \) are the entries of the Cauchy stress tensor \( \sigma \) and \( b_i \) are the entries of a volume force \( b \). All these entries are taken with respect to a fixed orthonormal basis. In (3.1) and in the sequel the Einstein convention of summation over repeated indices is employed and with \( \partial_i \) the partial derivative with respect to the \( x_i \)-coordinate is meant. Boundary conditions are given by

\[
\sigma_{ij} n_i = t_i^* \quad \text{on} \quad \Gamma_t, \quad i = 1, 2, 3, \tag{3.2}
\]

\[
u_i = u_i^* \quad \text{on} \quad \Gamma_u, \quad i = 1, 2, 3, \tag{3.3}
\]

where \( t_i^* \) and \( u_i^* \) are prescribed traction and displacement, respectively. \( \Gamma_t \) and \( \Gamma_u \) are disjunct, \( \Gamma_t \cup \Gamma_u = \Gamma \). The boundary of \( \Omega \) and \( n \) the unit outward normal on \( \Gamma \).

The Cauchy stress tensor \( \sigma \) corresponds with the displacement field \( u \) via a constitutive law of the form

\[
\sigma_{ij} = c_{ijkl} \epsilon_{kl} \quad i, j = 1, 2, 3, \tag{3.4}
\]

where the fourth-order tensor \( c \) is positive definite and left, right and fully symmetric. In (3.4), \( \epsilon \) is the linear Green Lagrange strain tensor. The entries of this tensor are given by

\[
\epsilon_{ij} = u_{(i,j)} \quad i, j = 1, 2, 3, \tag{3.5}
\]

where \( u_{(i,j)} \) is the symmetrical part of the gradient of the displacement field \( u \).

The Element Free Galerkin (EFG) method as well as the Finite Element (FE) method make use of weak forms of the problem description (3.1), (3.2) and (3.3). Some of these weak forms are presented in this section. For simplicity of the presentation we assume that \( \Gamma_t = \Gamma_t \) and \( \Gamma_u = \Gamma_u \) for \( i = 1, 2, 3 \).

The EFG-method makes use of the shape functions coming from Moving Least Squares Approximation (MLSA). Since these shape functions in general do not have the selectivity property (2.12), boundary conditions cannot be directly applied to nodal values. Resource however, can be made to a Lagrange multiplier form, see Belytschko et al. [2], of the problem description: Find \( \mathbf{u} \in (H^1(\Omega))^3 \) and \( \mathbf{\lambda} \in (H^{1/2}(\Gamma_u))^3 \) such that

\[
\int_{\Omega} \delta \mathbf{u} \cdot \mathbf{\sigma} \, d\Omega - \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{b} \, d\Omega - \int_{\Gamma_t} \delta \mathbf{u} \cdot \mathbf{t}^* \, d\Gamma - \int_{\Gamma_u} \delta \mathbf{\lambda} (\mathbf{u} - \mathbf{u}^*) \, d\Gamma - \int_{\Gamma_u} \delta \mathbf{u} \cdot \mathbf{\lambda} \, d\Gamma = 0 \tag{3.6}
\]
for all $\delta u \in (H^1(\Omega))^3$ and for all $\delta \lambda \in (H^0(\Gamma_u))^3$. With $H^i(\Omega)$ is meant the Sobolev space of order $i$. In (3.6) $\sigma$ is determined from $u$ with the help of (3.4) and (3.5). It can be shown that for a $C^1$-solution of (3.6) the equilibrium equations (3.1) and boundary conditions (3.2) and (3.3) are satisfied.

A slight modification of the weak form (3.6) is obtained when the Lagrange multiplier $\lambda$ is replaced by its physical meaning (see [3]). With the help of the integration theorem of Gauss it can be shown that the solution $\lambda$ of the weak form (3.6) can be identified with the stress vector $\sigma$ on $\Gamma_u$, i.e.

$$\lambda_i = \sigma_{ij} n_j = \sigma_i$$

(3.7)

Substitution of (3.7) into (3.6) results in the weak form: Find $u \in (H^1(\Omega))^3$ such that

$$\int_{\Omega} \delta u_{i(j)} \sigma_{ij} \, d\Omega - \int_{\Omega} \delta u_{i} b_{i} \, d\Omega - \int_{\Gamma_u} \delta u_{i} t_{i} \, d\Gamma - \int_{\Gamma_u} \delta u_{i} (u_{i} - u_{i}^*) \, d\Gamma - \int_{\Gamma_u} \delta u_{i} t_{i} \, d\Gamma = 0$$

(3.8)

for all $\delta u \in (H^1(\Omega))^3$.

When $\Gamma_u$ consists only of a set of discrete points, $\Gamma_u = \{y_1, \ldots, y_l\}$, a discrete set of Lagrange multipliers can be used to account for the essential boundary conditions: Find $u \in (H^1(\Omega))^3$ and $\lambda^k \in \mathbb{R}^3$, $k = 1, \ldots, l$, such that

$$\int_{\Omega} \delta u_{i(j)} \sigma_{ij} \, d\Omega - \int_{\Omega} \delta u_{i} b_{i} \, d\Omega - \int_{\Gamma_u} \delta u_{i} t_{i} \, d\Gamma - \sum_{k=1}^{l} \delta \lambda^k (u_i(y_k) - u_i^*(y_k)) \, d\Gamma - \sum_{k=1}^{l} \delta u_{i} (y_k) \lambda^k \, d\Gamma = 0$$

(3.9)

for all $\delta u \in (H^1(\Omega))^3$ and for all $\delta \lambda^k \in \mathbb{R}^3$, $k = 1, \ldots, l$.

To the weak forms (3.6) and (3.8) also terms coming from a discrete Lagrange multiplier description can be added when $\Gamma_u$ not only consists of a continuous part of the boundary, but also of a set of discrete points.

Essential boundary conditions can also be accounted for by means of a penalty formulation: Find $u \in (H^1(\Omega))^3$ such that

$$\int_{\Omega} \delta u_{i(j)} \sigma_{ij} \, d\Omega - \int_{\Omega} \delta u_{i} b_{i} \, d\Omega - \int_{\Gamma_u} \delta u_{i} t_{i} \, d\Gamma + 2\beta \int_{\Gamma_u} \delta u_{i} (u_{i} - u_{i}^*) \, d\Gamma = 0$$

(3.10)

for all $\delta u \in (H^1(\Omega))^3$. In (3.10), $\beta >> 1$ is a penalty parameter to enforce (3.3).

3.2. Discrete equations of the EFG-method

In order to introduce the discrete equations of the EFG-method, we focus on a problem for a linear elastic medium in two dimensions. It is assumed that nodal points $\{x_i\}_{i=1, \ldots, n}$, weight functions $\{w_i(x)\}_{i=1, \ldots, n}$ and basis functions $\{\phi_i(x)\}_{i=1, \ldots, n}$ are chosen for MLSA on $\Omega$.

Consider the weak form (3.6) of the problem description. As described in the previous section, MLSA results in shape functions $\phi_i(x)$, $i = 1, \ldots, n$. The following approximation structure for $u$ on $\Omega$ and for $\lambda$ on $\Gamma_u$ is adopted.

$$u_i = \sum_{a=1}^{n} \phi_a d_{ai} \quad i = 1, 2, 3$$

(3.11)

$$\lambda_i = \sum_{b=1}^{k} \psi_b I_{bi} \quad i = 1, 2, 3$$

(3.12)

where $\{\psi_b(x)\}_{b=1, \ldots, k}$ are approximation functions for $\lambda$ on $\Gamma_u$. For instance, for these functions one can take $\{\phi_a(x)\}_{a \in A}$, where $A$ is the set of indices $a$ of nodes $x_a$ which are on $\Gamma_u$.

Substitution of (3.11) and (3.12) into (3.6) yields the following linear system for $d$ and $l$, the vectors which are made up of $d_{ai}$ and $I_{bi}$, respectively,

$$\begin{bmatrix} K & G \end{bmatrix} \begin{bmatrix} d \n l \end{bmatrix} = \begin{bmatrix} f \n q \end{bmatrix}$$

(3.13)
The stiffness matrix $K$ and the matrix $G$ are built from $2 \times 2$ matrices $K_{ab}$ and $G_{ab}$, respectively, and the right-hand side vectors $f$ and $g$ are built from the $2 \times 1$ matrices $f_a$ and $q_b$. These matrices have the form

$$K_{ab} = \int_{\Omega} B_a^T DB_b \, d\Omega \quad a, b = 1, \ldots, n \quad (3.14)$$

$$G_{ab} = -\int_{\Gamma} \phi_a N_b \, d\Gamma \quad a = 1, \ldots, n, \quad b = 1, \ldots, k \quad (3.15)$$

$$f_a = \int_{\Omega} \phi_a b \, d\Omega + \int_{\Gamma} \phi_a t^a \, d\Gamma \quad a = 1, \ldots, n \quad (3.16)$$

$$q_b = \int_{\Gamma} \psi_b u^b \, d\Gamma \quad b = 1, \ldots, k \quad (3.17)$$

The matrices $B_a$ and $N_a$ are given by

$$B_a = \begin{bmatrix} \phi_{a,1} & 0 \\ 0 & \phi_{a,2} \end{bmatrix} \quad a = 1, \ldots, n \quad (3.18)$$

$$N_b = \begin{bmatrix} \psi_b & 0 \\ 0 & \psi_b \end{bmatrix} \quad b = 1, \ldots, k \quad (3.19)$$

and the $3 \times 3$ matrix $D$ represents the constitutive law (3.4) of the material.

Notice that $K_{ab}$ will be non-zero only when $\phi_a$ and $\phi_b$ are non-zero on a common part of $\Omega$. Hence, when each weight function (and therefore each shape function) is only non-zero on a small part of $\Omega$, the matrix $K$ will be sparse. Furthermore, it should be noticed that the entries of the solution $d$ of the linear system (3.13) do not represent the nodal displacements, since the shape functions $\phi_i$ in general fail to have the selectivity property.

The weak forms (3.8), (3.9) and (3.10) of the problem description together with the approximation structure (3.11) will lead to similar linear systems.

3.3. Numerical implementation

For the numerical implementation of the EFG-method the following weight functions with compact support are considered

- $C^0$-weight functions of Gaussian type with circular support (see Fig. 1)

![Fig. 1. $C^0$-weight functions of Gaussian type with circular support.](image)
\[ w_i(x) = \begin{cases} \exp(-\left(\frac{d_i}{c_i}\right)^{2k}) - \exp(-\left(\frac{r_i}{c_i}\right)^{2k}) & d_i \leq r_i \\
1 - \exp(-\left(\frac{r_i}{c_i}\right)^{2k}) & d_i \geq r_i, \end{cases} \] (3.20)

where \( d_i = |x - x_i|, k_i \in \mathbb{N} \) and \( r_i \) is the radius of the support of \( w_i \). The constant \( c_i \) controls the relative weights.

- \( C^0 \)-weight functions which are products of two 1D-weight functions of above type.
- Singular weight functions

\[ w_i(x) = s_i(x)|x - x_i|^{-\alpha_i}, \] (3.21)

where \( \alpha_i \in \mathbb{N} \) and where for \( s_i \), one of the previous weight functions is chosen.

In all numerical examples \( k_i \) and \( \alpha_i \) are chosen to be 1.

The \( C^0 \)-weight functions are differentiable except on the boundary of their supports. For the numerical implementation the derivatives on this boundary are set equal to zero. The Gaussian weight functions are then almost differentiable when \( r_i/c_i \geq 4 \). The radii of the supports of the weight functions are chosen such that MLSA is well defined for the domain \( \Omega \). In the implementation these radii are taken relatively large with respect to the grid size of the nodal distribution. This will be explained in the next section.

To evaluate the domain integrals and the boundary integrals in (3.14)-(3.17), \( \Omega \) is split up into volume cells and \( \Gamma_v \) and \( \Gamma_n \) are split up into boundary cells, see Fig. 2. In each cell numerical integration is performed. Since the shape functions are not piecewise polynomial a relatively dense pattern of integration points is necessary to obtain accurate values for the linear system (3.13). In the examples patterns are used such that the total number of integration points \( n_{\text{int}} \) in the domain is approximately given by

\[ n_{\text{int}} \approx \mu n, \]

where \( n \) is the number of nodes and \( 5 \leq \mu \leq 8 \).

The implementation of the EFG-method is organized according to:

1. Define nodes, basis functions and weight functions such that MLSA is well defined
2. Define cell structures for \( \Omega, \Gamma_v \) and \( \Gamma_n \), together with a number of integration points for each cell \( C \)
3. For each cell \( C \)
   (i) For each quadrature point \( y_q \) in \( C \)
      (a) Determine the nodes \( x_i \) such that \( w_i(y_q) > 0 \)
      (b) Compute \( \phi_i(y_q) \) and the derivatives \( \phi_i, i(y_q) \) for these nodes
      (c) Compute the contributions of \( y_q \) to (3.13)
      (d) Assemble these contributions in the cell contributions \( K_c, G_c, f_c \) and \( q_c \)
   (End of (i))
   (ii) Assemble the cell contributions in \( K, G, f \) and \( q \)
   (End of (3))
4. Solve system (3.13) for \( d \) and \( l \)
5. Define a set of points \( z_k \) in \( \Omega \)

![Fig. 2. Cell structure for \( \Omega \) and \( \Gamma \).](image)
(6) For each $z_k$
   (i) Determine the nodes points $x_i$ such that $w_i(z_k) > 0$
   (ii) Compute $\phi_i(z_k)$ and the derivatives $\phi_i'(z_k)$ for these nodes
   (iii) Compute the values of $u_i, u_{i,j}, u_{j,i}, \kappa_{ij}$ and $\sigma_{ij}$ in $z_k$
   (End of (6))

(7) Use the computed values in $z_k$ for determination of error estimates, stress intensity factors etc.
   Notice that this organization is similar to that of Finite Element programs. It differs only at 3(i)(a)
   and at 6; the nodes which contribute to the integration point stiffness differ within a cell and the set of
   points $z_k$ in $\Omega$ can be chosen independent of the nodes.

   In the implementation a subset of the shape functions $\{\phi_{j}\}_{j=1}^{n}$ is chosen for the approximation
   functions $\{\psi_{j}\}_{j=1}^{n}$ for the Lagrange multiplier.

   At step 2, $\Omega$ is split up into volume cells and the intersection of the boundaries of these cells with $I_u$
   and $I_v$ will form the cell structure for $\Gamma_v$ and $\Gamma_u$. For each cell, Gauss quadrature will be carried out with
   a fixed number of Gauss points for all the volume and boundary cells. For a domain $\Omega$ which has a
   curved boundary, the domain will be approximated by a set of volume cells which are constructed with the help of an isoparametric mapping. This approximation of the domain differs from the implementations of Belytschko et al. [2] and Lu et al. [3] where the domain is covered by the volume cells and an integration point is skipped when it is outside of the physical domain. An approximation of the domain
   by volume cells is however, preferred to obtain accurate values for the stiffness.

   In steps 3 and 7, orthogonalisation of the basis functions with respect to the weight functions, as
   described in [3], is used to determine the values of the shape functions and the values of the derivatives
   of the shape functions.

4. Combinations of EFG-method and FE-method

   For Moving Least Squares Approximation (MLSA) at each point of the domain the linear system
   (2.6) should be solved. Moreover, when MLSA-shape functions are used in an implementation of a
   weak formulation, a dense integration pattern is necessary to get accurate values for the resulting linear
   system. Hence, the Element Free Galerkin (EFG) method is computationally expensive. Therefore, it
   is more convenient to make use of EFG only on a part of the domain where one wants to achieve an
   accurate approximation of the solution, and to make use of the Finite Element (FE) method for the
   remaining part of the domain. In this section attention is focused on two ways of splitting up a domain
   in a part for EFG and in a part for FE. The first one is an approach where the FE-method is seen as a
   special approach by EFG and the second one is an approach where the FE-part and EFG-part are
   connected with the help of a Lagrange multiplier. Both approaches will now be described.

   Some FE-approaches can be seen as a subset of the EFG-method as can be seen from the following
   example. Consider four nodal points taken from a regular grid of nodal points, see Fig. 3(a). Let the
   monomials $1, x_1, x_2$ and $x_1 x_2$ form the set of basis functions, and let the weight functions $w_i$ for these
   four points be positive in the interior of the rectangle formed by these four points and zero on the sides
   of the rectangle which do not coincide with $x_i$. When no other weight functions of surrounding nodal
   points are positive on this rectangle, MLSA will result in bilinear interpolation. Hence, the shape

   ![Fig. 3. (a) Regular grid; (b) triangular grid.](image-url)
functions \( \phi_i \) are bilinear on the rectangle and they satisfy the selectivity property (2.12). These shape functions coincide with the bilinear FE-shape functions for the rectangle.

When the four nodal points do not form a rectangle, the shape functions \( \phi_i \) emerging from MLSA are still bilinear. However, the FE-interpolation for the grid via an isoparametric mapping will result in shape functions which are not bilinear.

For a regular grid of nodal points the shape functions coming from MLSA will look like FE-shape functions when the supports of the weight functions are relatively small with respect to the grid size \( h \) of the nodal grid. This can be seen in Fig. 4. For a regular grid of nodal points and a bilinear basis, weight functions are taken which are products of 1D Gaussian weight functions. Several sizes of the supports for these weight functions are considered. In Fig. 4, cross cuts of the several shape functions \( \phi \) and their derivative \( \phi_i \) for a particular nodal point are depicted.

Fig. 4 shows that for small radii the shape functions and the derivatives of the shape functions are strongly non-polynomial on some small parts of the domain. This makes these shape function not convenient for EFG from the viewpoint of numerical integration. This can be seen in the numerical examples presented in the next section. In the implementation it was also noticed that for small radii the matrix \( A \), see (2.9), can become poorly conditioned. Since for larger radii the shape functions and their derivatives are more rounded, one should work in the implementation with relatively large radii \( r_i \) (relatively large supports) for the weight functions. In the examples these radii were set to

\[
r_i = \alpha h,\]

with \( \alpha > 2 \). Fig. 4 also shows that in general the shape functions for a basis which consists of monomials are not piecewise polynomial.

Singular weight functions (3.21) for a regular grid result both for relatively small and relatively large supports in shape functions which look like FE-shape functions, as can be seen in Fig. 5. The shape functions and its derivatives are strongly nonpolynomial on small parts of the domain also for relatively large supports. Hence, these shape functions are not convenient for numerical integration. This can also be seen in the numerical examples in the sequel. Therefore, singular weight functions are not recommended for use in the EFG-method.

Another situation where MLSA-shape functions coincide with shape functions applied in a regular FE-method, is depicted in Fig. 3(b). Assume for this triangular grid that the monomials 1, \( x_1 \), \( x_2 \) form the set of basis functions. Let the weight functions \( w_i \) for the three points be positive in the interior of

Fig. 4. Cross cut of (a) shape functions \( \phi \); (b) their derivative \( \phi_i \) for regular grid.
the triangle formed by these points and be zero on the sides not coinciding with \( x_j \). When no other weight functions of surrounding nodal points are positive on this triangle, MLSA results in linear shape functions which coincide with the linear FE-shape functions for a three noded element.

Since there is no restriction to one type of weight function in the EFG-method on a given domain, the above considerations lead to a straightforward way to apply both EFG and FE on a single domain without the problem of coupling EFG-part with FE-part. This is illustrated in the next example.

Consider the domain with nodal points depicted in Fig. 6. The EFG-method for \( \Omega_1 \) will be equivalent with a regular FE-method, when
- For all the nodal points in \( \Omega_1 \) the weight functions are chosen such that their supports equal the union of the surrounding rectangles.
- For all the nodal points in \( \Omega_2 \) the weight functions are zero on \( \Omega_1 \).
- The set of basis functions is \( \{1, x_1, y_1, z_1, x_1^2, y_1^2, z_1^2\} \).
- The integration cells for \( \Omega_1 \) coincide with the rectangles and sufficient accurate Gauss quadrature is used for the cells.

\[ \Omega_1 \]

\[ \Omega_2 \]

Fig. 6. Domain handled by EFG-method and FE-method.
For $\Omega_2$ a general EFG-approach can be used. Of course all the weight functions should be chosen such that MLSA is well defined for $\Omega_2$. It can be seen easily that in this example there will be no problem of coupling the FE-domain $\Omega_1$ with the EFG-domain $\Omega_2$, because this approach can be seen as an EFG-approach for the entire domain.

A combination of EFG and FE as described above does not work very well as will be shown in the numerical examples. Due to the fact that the weight functions for the EFG-part have to be zero on the FE-part, some weight functions will have relatively small supports with respect to the mesh size of the nodal distribution. As already described this will lead to shape functions which are not convenient for numerical integration.

To circumvent the problem of the use of weight functions with relatively small supports, one can work with weight functions for the EFG-part which are non-zero on the FE-part. However, that will mean that the approach for the FE-part is not a FE-approach anymore, or when these non-zero weight functions are neglected, that one works with discontinuous shape functions (see remark about the smoothness of EFG-shape functions in Section 2).

Since a combination of EFG and FE where the FE-approach is seen as a special approach by EFG does not work well, we consider also a split up of the domain $\Omega$ in a part for FE and in a part for EFG, where these parts are now really coupled. Let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1$ and $\Omega_2$ are disjunct. Let $\Gamma_1$ and $\Gamma_2$ be the boundaries of $\Omega_1$ and $\Omega_2$, respectively.

For a combination of FE and EFG one should account for a coupling of the domains via the weak form. This is illustrated with an extension of the weak form (3.6): Find $u^1$, $u^2$, $\lambda^1$, $\lambda^2$ and $\gamma$ such that

$$
\int_{\Omega_1} \delta u_{i,j}^1 u^1_{j} \, d\Omega - \int_{\Omega_1} \delta u_{i,j}^1 b_j \, d\Omega - \int_{\Gamma_1} \delta u_{i,j}^1 t^1_j \, d\Gamma - \int_{\Gamma_1} \delta \lambda_{i}^1 (u^1_{i} - u^2_{i}) \, d\Gamma - \int_{\Gamma_1 \cap \Gamma_2} \delta u_{i,j}^1 \lambda^1_\gamma \, d\Gamma \\
- \int_{\Gamma_1 \cap \Gamma_2} \delta \gamma_{\lambda} u^1_{\gamma} \, d\Gamma = 0 \tag{4.1}
$$

$$
\int_{\Omega_2} \delta u_{i,j}^2 \sigma^2_j \, d\Omega - \int_{\Omega_2} \delta u_{i,j}^2 b_j \, d\Omega - \int_{\Gamma_2} \delta u_{i,j}^2 t^2_j \, d\Gamma \\
- \int_{\Gamma_1 \cap \Gamma_2} \delta \lambda_{i}^2 (u^1_{i} - u^2_{i}) \, d\Gamma - \int_{\Gamma_1 \cap \Gamma_2} \delta u_{i,j}^2 \lambda^2_\gamma \, d\Gamma + \int_{\Gamma_1 \cap \Gamma_2} \delta \gamma_{\lambda} u^2_{\gamma} \, d\Gamma = 0 \tag{4.2}
$$

for all $\delta u^1$, $\delta u^2$, $\delta \lambda^1$, $\delta \lambda^2$ and for all $\delta \lambda$. In this weak form the domains are connected via a Lagrange multiplier $\gamma$.

A split up of the domain where the domains are connected with the help of a Lagrange multiplier can also be formulated with the help of the weak form (3.9) or the weak form (3.8). The domains can also be coupled with the help of a penalty term which will lead to an extension of the weak form (3.10).

To come to discrete equations a FE-subdivision is assumed for $\Omega_1$, which leads to a set of nodal points $\{x^1_i\}_{i=1}^{n_1}$ together with a set of shape functions $\{\phi^1_i(x)\}_{i=1}^{n_1}$ which are piecewise polynomial over the elements. These shape functions have the selectivity property (2.12). For $\Omega_2$ nodal points $\{x^2_j\}_{j=1}^{n_2}$, weight functions functions $\{w_j(x)\}_{j=1}^{n_2}$ and basis functions $\{p_j(x)\}_{j=1}^{n_2}$ are assumed for MLSA on $\Omega_2$ which results in shape functions $\{\phi^2_a(x)\}_{a=1}^{n_2}$.

The following approximation structure for $u^1$ on $\Omega_1$ and $u^2$ on $\Omega_2$ is adopted.

$$
u^1_i = \sum_{a=1}^{n_1} \phi^1_a d^1_{a,i} \quad i = 1, 2, 3, \tag{4.3}
$$

$$
u^2_i = \sum_{a=1}^{n_2} \phi^2_a d^2_{a,i} \quad i = 1, 2, 3, \tag{4.4}
$$

Let the Lagrange multipliers $\lambda^1$, $\lambda^2$ and $\gamma$ be written according to

$$
\lambda^1_i = \sum_{h=1}^{k_1} \psi^1_h f^1_{h,i} \quad i = 1, 2, 3, \tag{4.5}
$$
\[
\lambda_i^2 = \sum_{b=1}^{k_i} \psi_i^b l_i^2, \quad i = 1, 2, 3, \tag{4.6}
\]
\[
\lambda_i = \sum_{b=1}^{k_i} \psi_i^b h_i, \quad i = 1, 2, 3, \tag{4.7}
\]

where \( \psi_i^b, \psi_i^c \) and \( \psi_i^d \) are approximation functions for \( \lambda_i^1, \lambda_i^2 \) and \( \gamma \).

Then substitution in (4.1), (4.2) yields a linear system of the form (compare with (3.13)),

\[
\begin{bmatrix}
K_1 & 0 & G_1 & 0 & H_1 \\
0 & K_2 & 0 & G_2 & H_2 \\
G_1^t & 0 & 0 & 0 & 0 \\
0 & G_2^t & 0 & 0 & 0 \\
H_1^t & H_2^t & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d^1 \\
d^2 \\
l^1 \\
l^2 \\
l_1
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
q_1 \\
q_2 \\
0
\end{bmatrix} \tag{4.8}
\]

In an implementation of this form of the problem, we take for \( \psi_i^b \) and \( \psi_i^d \) the restrictions of the FE-shape functions \( \phi_i^p \) to \( I_n \cap I_i \) and \( I_i \cap I_n \), respectively, and for \( \psi_i^c \) we take a subset of the EFG-shape functions.

Since the shape functions \( \{ \phi_i^p(x) \}_{i=1}^{n} \) have the selectivity property (2.12), the essential boundary conditions for \( \Omega \) can also be imposed directly, instead of making use of the Lagrange multiplier description in (4.1).

5. Numerical examples

In this section numerical results will be reported of implementations of the EFG-method and implementations of combinations of the EFG-method and the FE-method on 2D-problems. In each example the material behaviour will be isotropic linear elastic with \( E = 1 \) and \( \nu = 0.25 \). Moreover, in each example the volume force \( b \) is set equal to zero.

In the examples the values for the parameters of the EFG-method are based on the considerations in Section 4. The weight functions that are used, are given in Section 3. The sizes of the supports of these weight functions will have the value \( ah \) where \( a \geq 2 \) and \( h \) the grid size of the nodal distribution.

Moreover, as was pointed out, a relatively dense integration pattern is necessary in order to obtain accurate values for the resulting linear system.

5.1. Patch test

Consider the patch shown in Fig. 7. This patch has length 8 in the \( x_1 \)-direction and length 1 in the \( x_2 \)-direction. When a uniform axial stress is applied at the right end, \( u_1 = 0 \) and \( u_2 = 0 \) at \( x_1 = 0 \) and \( x_1 = 8 \), respectively, the solution for this problem in a situation of plane stress is given for \( 0 \leq x_1 \leq 8 \) and \( 0 \leq x_2 \leq 1 \) by

\[
\begin{align*}
u_i &= \frac{\sigma}{E} x_1, \tag{5.1}
\end{align*}
\]

![Fig. 7. Mesh of nodes and cell structure for patch test.](image)
\[ u_z = -\nu \frac{\sigma}{E} x_z. \]  

(5.2)

Since the solution is linear, a linear basis for MLSA is able to represent this solution. However, the shape functions coming from MLSA are not piecewise polynomial as in the FE-method (see Fig. 4) and therefore the numerical integration scheme does not yield sufficiently accurate values for the entries of the linear system (3.13). Hence, the accuracy of the calculated displacements will depend on the number of cells and the number of integration points.

This is illustrated for 27 nodal points in the domain, see Fig. 7, a linear basis; i.e., \{1, x_1, x_2\}, and \( C^1 \)-weight functions of Gaussian type with circular support. The weak form (3.6) of the problem description will be used and the number of integration cells and number of integration points will be varied. The results are depicted in Table 1.

In Table 1 \( \|u^h - u\| \) is the error of the displacements in the \( L^2 \)-norm,

\[ \|u\| = \left( \int_{\Omega} u_i u_i \, d\Omega \right)^{1/2}, \]  

(5.3)

and \( \|\epsilon^h - \epsilon\|_e \) is the error in the energy semi-norm,

\[ \|\epsilon\|_e = \left( \frac{1}{2} \int_{\Omega} \epsilon_{ij} \epsilon_{ij} \, d\Omega \right)^{1/2}. \]  

(5.4)

In the sequel also the Sobolev norms \( \|\| \|_k \) are used, where \( k \in \mathbb{N} \). For \( k = 2 \) this norm is, for instance, given by

\[ \|u\|_2 = \left( \int_{\Omega} u_i u_i + u_{ij} u_{ij} + u_{ij} u_{ij} \, d\Omega \right)^{1/2}. \]  

(5.5)

Note that the \( L^2 \)-norm is the Sobolev norm for \( k = 0 \). In the implementation all these norms are computed with the help of the cell structure for the domain integrals.

From Table 1 it is clear that with a dense integration pattern the results in this patch test are more accurate. Moreover, the results are far better for larger radii for the supports of the weight functions, since for larger radii the EFG-shape functions are more convenient for numerical integration (Section 4).

Consider now the nodal distribution of 51 nodes, see Fig. 8. We focus on the combinations of FE and EFG which were described in the previous section. Let the nodes on the left half \((x_i \leq 4)\) of the domain be nodes of a FE-mesh of linear triangles and let this FE-part be seen as a special approach by EFG as

<table>
<thead>
<tr>
<th>EFG-parameters</th>
<th>( |u^h - u|/|u| )</th>
<th>( |\epsilon^h - \epsilon|_e/|\epsilon|_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_i = 2, , c_i = r_i/4 ), 10 cells, (3,3) Gauss</td>
<td>( 3.39 \cdot 10^{-2} )</td>
<td>( 1.51 \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( r_i = 4, , c_i = r_i/4 ), 10 cells, (3,3) Gauss</td>
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<td>( 3.78 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>( r_i = 2, , c_i = r_i/4 ), 10 cells, (5,5) Gauss</td>
<td>( 7.95 \cdot 10^{-2} )</td>
<td>( 3.22 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>( r_i = 4, , c_i = r_i/4 ), 10 cells, (5,5) Gauss</td>
<td>( 2.82 \cdot 10^{-4} )</td>
<td>( 2.15 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>( r_i = 2, , c_i = r_i/4 ), 20 cells, (5,5) Gauss</td>
<td>( 1.13 \cdot 10^{-1} )</td>
<td>( 7.37 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>( r_i = 4, , c_i = r_i/4 ), 20 cells, (5,5) Gauss</td>
<td>( 4.47 \cdot 10^{-3} )</td>
<td>( 2.01 \cdot 10^{-2} )</td>
</tr>
</tbody>
</table>

Table 1

Results for problem with constant normal stress, Lagrange multiplier description

Fig. 8. Nodal distribution and cell structure for combination of FE and EFG.
described in the previous section. Hence, \( \{1, x_1, x_2\} \) is the set of basis functions and the weight functions for the nodes in the right half \( x_i = 4 \) are zero on the FE-part. Boundary collocation is used for the essential boundary conditions of the FE-part and the essential boundary conditions for the EFG-part are imposed by means of a Lagrange multiplier. Products of 1D Gaussian weight functions are used for the EFG-part and the integration structure for the EFG-part consists of 16 volume cells with \((5,5)\) Gauss quadrature for each cell. In Fig. 9(a) the relative error in \( \sigma_{11} \) can be found.

Since the basis is linear, the state of constant stress can be represented. However, we see that the stress emerging from this approach is inaccurate in the neighbourhood of \( x_i = 4 \). From the considerations of Section 4 this does not have to surprise us. To have that the FE-approach on the left half can be seen as an EFG-approach, the weight functions for nodes close to \( x_i = 4 \) have relatively small supports with respect to the grid size of the nodal distribution. This does result in shape functions which are strongly nonpolynomial on small parts of the domain and therefore inconvenient for numerical integration. Also in other examples it was noticed that an approach via a combination of FE and EFG, where FE is seen as a special approach by EFG, does not work well.

Consider the same problem where now the FE-part and EFG-part are connected with the help of a Lagrange multiplier as was explained in Section 4. The radii of the weight functions have been increased, since the weight functions may be nonzero on the FE-part. The relative error in the normal stress can be found in Fig. 9(b). It can be seen that the stress is far more accurate than the one for the other combination of FE and EFG. Also in other examples connection of FE- and EFG-domains by means of a Lagrange multiplier have proven to be far better than when the FE-approach is considered as a special EFG-approach. Therefore, in the sequel only such combinations of FE and EFG will be considered.

5.2. Higher order patch

Consider the patch shown in Fig. 7 with a linearly varying normal stress at the right end, i.e. \( \sigma_{11} = \sigma x_1 \), \( \sigma_{12} = 0 \) at \( x_1 = 8 \). An exact solution in a situation of plane stress is given for \( 0 \leq x_1 \leq 8 \) and \( 0 \leq x_2 \leq 1 \) by

\[
\begin{align*}
\sigma_1 &= \frac{6}{E} x_1 x_2 \\
\sigma_2 &= \frac{6}{E} x_1 + \frac{2}{E} x_2
\end{align*}
\] (5.6) (5.7)

Fig. 9. Relative error in \( \sigma_{11} \) for problem with constant normal stress. (a) FE+EFG with FE seen as EFG; (b) FE+EFG with connection by means of Lagrange multiplier.
Since the solution is quadratic, a quadratic basis for EFG will return this solution up to an error due to numerical integration.

Consider a nodal distribution of 51 points (17 points in the \( x_1 \)-direction and 3 points in the \( x_2 \)-direction), together with a bilinear basis, i.e. \{1, \( x_1, x_2, x_1x_2 \). Products of 1D Gaussian weight functions will be used. Results for the weak form (3.6) for different values of the radii and a constant number of integration points are depicted in Table 2. Like in the previous example \( u_1 \) is prescribed on \( x_1 = 0 \) and \( u_2 \) is prescribed on \( x_2 = 0 \).

In Table 2 the influence can be seen of the size of the supports of the weight functions on the accuracy of the EFG-method. For larger supports the results are more accurate due to the fact that the shape functions are more convenient for numerical integration. This example shows, what already has been remarked in Section 4, that the supports of the weight functions should not be taken too small.

In Section 2 singular weight functions (3.21) were introduced. These weight functions have the advantage that MLSA results in shape functions which have the selectivity property (2.12). Such singular weight functions have been considered for this problem. The results for different values for the radii are depicted in Table 3. When the results in Table 3 are compared with the ones in Table 2 we see that the use of singular weight functions leads to less accurate results, especially for relatively large supports. In Section 4 it was shown that also for relatively large supports of the singular weight functions the shape functions are strongly nonpolynomial on small parts of the domain. Therefore, these shape functions will not lead to an accurate performance of the method. Hence, singular weight functions are not recommended for use in the EFG-method.

In Section 3.1 several weak forms of the problem description were considered. All these forms have been implemented for this problem. The results are given in Table 4.

In Table 4 CL corresponds with the weak formulation by means of a Lagrange multiplier (3.6), PL with the formulation where the multiplier is replaced by its physical meaning (3.8), DL with the discrete Lagrange multiplier formulation (3.9) and PF with the penalty formulation (3.10). For the discrete

---

**Table 2**

<table>
<thead>
<tr>
<th>EFG-parameters</th>
<th>( |u^h - u|_H^2 / |u| )</th>
<th>( |u^h - u|_L^2 / |u| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^2_1 = r^2_2 = 1, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 1.14 \cdot 10^{-3} )</td>
<td>( 4.14 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.80 \cdot 10^{-4} )</td>
<td>( 2.03 \cdot 10^{-5} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 3, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.31 \cdot 10^{-5} )</td>
<td>( 3.12 \cdot 10^{-6} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 4, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.99 \cdot 10^{-6} )</td>
<td>( 3.71 \cdot 10^{-7} )</td>
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</table>

**Table 3**

<table>
<thead>
<tr>
<th>EFG-parameters</th>
<th>( |u^h - u|_H^2 / |u| )</th>
<th>( |u^h - u|_L^2 / |u| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^2_1 = r^2_2 = 1, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.26 \cdot 10^{-3} )</td>
<td>( 1.57 \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.30 \cdot 10^{-3} )</td>
<td>( 1.92 \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 3, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 3.29 \cdot 10^{-3} )</td>
<td>( 2.82 \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( r^2_1 = r^2_2 = 4, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 4.39 \cdot 10^{-3} )</td>
<td>( 3.72 \cdot 10^{-1} )</td>
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</tbody>
</table>

**Table 4**

<table>
<thead>
<tr>
<th>EFG-parameters</th>
<th>( |u^h - u|_H^2 / |u| )</th>
<th>( |u^h - u|_L^2 / |u| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL, ( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 1.28 \cdot 10^{-2} )</td>
<td>( 2.03 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>PL, ( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 2.36 \cdot 10^{-2} )</td>
<td>( 3.82 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>DL, ( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 1.78 \cdot 10^{-2} )</td>
<td>( 2.00 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>PF, ( r^2_1 = r^2_2 = 2, c^2_1 = r^2_1/4, 20 \text{ cells, } (5,5) \text{ Gauss} )</td>
<td>( 1.29 \cdot 10^{-2} )</td>
<td>( 2.03 \cdot 10^{-2} )</td>
</tr>
</tbody>
</table>
formulation DL the nodal points on $I_n$ are chosen as discrete points for boundary collocation and in the penalty formulation PF the penalty parameter $p$ has been given the value $1 \cdot 10^3$.

From Table 4 it is clear that all the formulations work quite well. The errors are of the same order. It should be noticed that a discrete formulation (DL) only guarantees that the essential boundary conditions are met in a set of discrete points on $I_n$. In between the nodes the essential boundary conditions are in general violated. Therefore, a weak formulation is preferred where the essential boundary conditions are met in the mean, like the other three formulations. In the next examples attention is focused on the weak formulations by means of a Lagrange multiplier (3.6) and (4.1), (4.2). It should be noticed that these formulations yield more unknowns than the PL- and the PF-formulation.

5.3. Shear force on a plate

Consider the beam plate, $0 \leq x_1 \leq l$, $0 \leq x_2 \leq d$, $l=8$, $d=1$, depicted in Fig. 7. At the right end a transverse shear force $P$ is applied. When the stresses at the right end are distributed according to $\sigma_{11}=0$ and $\sigma_{12}=-(P/2l)x_2(x_2-d)$, where $l=d^2/12$, a solution of the static problem for the beam-plate in a situation of plane strain is given by

$$u_1 = -\frac{P}{6EI} (1 + \nu) \left( x_2 - \frac{d}{2} \right) \left[ 3(1 - \nu)x_1(2l - x_1) + (2 - \nu)x_2(x_2 - d) \right]$$

and

$$u_2 = \frac{P}{6EI} (1 + \nu) \left[ (1 - \nu)(3l - x_1)x_1^2 + 3\nu(l - x_1) \left( x_2 - \frac{d}{2} \right)^2 + \frac{d^2}{4}(4 + \nu)x_1 \right],$$

where $0 < x_1 < l$ and $0 < x_2 < d$. The stresses in the beam-plate are given by

$$\sigma_{11} = -\frac{P}{l} (l - x_1) \left( x_2 - \frac{d}{2} \right)$$

$$\sigma_{12} = -\frac{P}{2l} x_2(x_2 - d)$$

$$\sigma_{22} = 0.$$

We consider the convergence of the EFG-method for this problem. Regular nodal distributions of 51, 100, 165 and 246 points are considered together with a linear and a quadratic basis. $C^0$-weight functions of Gaussian type with circular support are used with two different values for the radii $r_i$ of the supports: $r_i = 4h$ and $r_i = 6h$, where $h$ is the mesh size of the nodal distribution. For the two coarse nodal distributions 16 volume cells are used and for the two finer patterns 64 volume cells are used. In each cell (5.5) Gaussian quadrature is performed. The convergence rates $R$ for the error in the $L^2$-norm and for the error in the energy semi-norm, both relative to a Sobolev norm (see also [6]), are depicted in Figs. 10 and 11. For a definition of the Sobolev norms see equation (5.5).

In Figs. 10 and 11 also convergence rates are depicted for a FE-approach and for a combination of EFG and FE by means of a Lagrange multiplier. For the FE-approaches linear and quadratic triangles are used. For a combination of EFG and FE the left-hand side ($x_1 < 4$) is subdivided in linear and quadratic triangles and the right hand side is handled by EFG with $C^0$-weight functions of Gaussian type with circular support and radii $r_i$ equal to $6h$.

Both Figs. 10 and 11 show that the convergence rates of the EFG-method exceed those of the FE-method. Moreover, it is seen that the convergence rates of the combination of EFG and FE almost equal the ones of an approach by FE only. It is however expected that in a situation where a combination of EFG and FEM is used for a problem with a more non-polynomial solution, the convergence rates of such a combination will exceed the ones of the equivalent FE-approach. This can be seen in the next example.
5.4. Infinite plate with circular hole

Consider an infinite plate with a circular hole with radius \( a \); \( x_1^2 + x_2^2 \leq a^2 \). Let a uniform axial tension \( \sigma \) in the \( x_1 \)-direction be applied at infinity. The stress distribution in the plate for a situation of plane strain is then given by (see e.g. [7, p. 178])

\[
\sigma_{11} = \sigma \left( 1 - \frac{a^2}{r^2} \left( \frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \right)
\]  (5.13)
\[ \sigma_{r} = \sigma \left( -\frac{a^2}{r} \left( \frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \right) \]  

(5.14)  

\[ \sigma_{\theta} = \sigma \left( -\frac{a^2}{r} \left( \frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3a^4}{2r^4} \cos 4\theta \right) \]  

(5.15)  

where \( r \) and \( \theta \) are polar coordinates. The corresponding displacement field is

\[ u_1 = \frac{1 + \nu}{E} \sigma \left( (1 - \nu) r \cos \theta + 2(1 - \nu) \frac{a^4}{r} \cos \theta + \frac{1}{2} \frac{a^4}{r} \cos 3\theta - \frac{1}{2} \frac{a^4}{r} \cos 3\theta \right) \]  

(5.16)  

\[ u_2 = \frac{1 + \nu}{E} \sigma \left( -\nu r \sin \theta - (1 - 2\nu) \frac{a^4}{r} \sin \theta + \frac{1}{2} \frac{a^4}{r} \sin 3\theta - \frac{1}{2} \frac{a^4}{r} \sin 3\theta \right). \]  

(5.17)  

We see that both the displacements and the stresses show a steep behaviour in the neighbourhood of the hole.

A part, \( x_1 \leq 5 \) and \( x_2 \leq 5 \), of the upper right quadrant of the plate is considered, see Fig. 12. On \( x_1 = 5 \) and \( x_2 = 5 \) applied traction is prescribed according to (5.13)-(5.15), the circular boundary is traction free, \( u_1 = 0 \) on \( x_1 = 0 \) and \( u_2 = 0 \) on \( x_2 = 0 \). The radius \( a \) of the circular boundary is given the value 1.

First, a pure EFG-approach for this problem is considered. The nodal points will be spaced as depicted in Fig. 12(a). The nodes are regularly spaced in the \( \theta \)-direction and irregularly spaced in the \( r \)-direction. This to have more points in the neighbourhood of the circular boundary to achieve a better approximation of the steep displacements and stresses.

\( C^0 \)-weight functions of Gaussian type with circular support are used. For each nodal point \( x_i \) of the mesh a value \( s_i \) is computed such that the neighbouring nodal points of this mesh will be within the circle with radius \( s_i \) and centre \( x_i \). The neighbouring nodes are the (at most 8) neighbours in the \( r \) - and \( \theta \)-direction. The radius \( r_i \) of the weight function \( w_i \) is then set to a scalar multiple of \( s_i \),

\[ r_i = \alpha s_i. \]  

(5.18)

Fig. 12. Meshes of nodes and cell structures for plate with a circular hole: (a) EFG-approach; (b) FE + EFG with connection by means of Lagrange multiplier.
Hence, the radii differ from point to point and the smallest radii are obtained for the points close to the circular boundary.

We consider the convergence of the method and the approximation of the steep stresses for a linear and a quadratic basis. Nodal distributions of 56, 108 and 208 points are considered, the value of $\alpha$ is set equal to 2 and a cell structure (see Fig. 12(a)) of 36, 64 and 100 cells, respectively, is used with in each cell (5,5) Gauss quadrature. The convergence rates $R$ for the $L^2$-norm and the energy semi-norm are depicted in Fig. 13. In this figure the mesh parameter $h$ is taken to be the largest diameter of the quadrilaterals formed by the nodal points.

Fig. 13 shows that for both a linear and a quadratic approach, the convergence rates exceed the corresponding rates for an equivalent FE-approach, which are 2 and 1, and 3 and 2, respectively (see [6] or [8]). It is also shown that, in contrast with the previous example, the rates for a quadratic basis do not exceed the rates for a linear basis that much. This can be due to the fact that the radii of the weight functions are taken too small for a quadratic approach, since the same values were taken as for the linear case.

The approximation of the steep stress $\sigma_{y2}$ at $x_z = 0$ is depicted in Fig. 14 (the value of $\sigma$ was set equal to $1 \cdot 10^{-3}$). From this figure it is clear that this stress is well approximated for both a linear and a quadratic basis.

Combinations of the EFG-method and the FE-method are also considered. The part $x_1 \leq 3$, $x_2 \leq 3$ is handled by EFG and the other part of the domain is handled by FE. The domains are coupled by a Lagrange multiplier. The nodal points for the EFG-part are distributed similar to the distribution for the entire domain in the EFG-approach. In the FE-part four and nine noded quadrilaterals for a linear and a quadratic approach, respectively, are considered. This leads to nodal distributions and cell structures like the one for a linear approach depicted in Fig. 12(b).

In the EFG-part of the domain $C^0$-weight functions of Gaussian type with circular support are used. The radii of these weight functions are determined as in the EFG-approach. Nodal distributions of 114, 228 and 412 points are considered together with 36, 64 and 100 cells, respectively, in the EFG-part with in each cell (5,5) Gauss quadrature.

The convergence rates $R$ can be found in Fig. 15. The mesh parameter $h$ is again taken to be the largest diameter of the quadrilaterals formed by the nodal points.

We see that the convergence rates for a linear approach by FE+EFG exceeds the theoretical values for FE and that these rates do not differ that much from the rates for a pure EFG-approach.

![Fig. 13. Convergence rates $R$ for EFG for (a) the $L^2$-norm; (b) the energy semi-norm relative to a Sobolev norm, problem of plate with circular hole.](image)
Surprisingly a quadratic approach by FE+EFG is not as good as a linear approach. For a quadratic approach the convergence rates almost equal the theoretical rates for an equivalent approach by FE. Again we expect that this is due to the fact that for the quadratic approach in the EFG-part the radii of the weight functions are taken too small.

Also for a combination of FE and EFG for both a linear and a quadratic approach the stresses are well approximated as can be seen from Fig. 16 where $\sigma_{11}$ at $x_1=0$ is depicted.

5.5. Fracture example

Consider a finite plate with a crack under a Mode I loading situation; a loading perpendicular on the direction of the crack, see Fig. 17. The stresses in the neighbourhood of the crack tip are always of the form.
where $r$ and $\theta$ are polar coordinates around the crack tip. For every flat plate, regardless of plate geometry and plate dimensions, $f_i$ will be the same. However, $K_i$ will differ from plate to plate. $K_i$ is called the stress intensity factor for a Mode I loading situation. Such a stress intensity factor does depend on the plate geometry, the plate dimensions, the crack geometry and the applied tension. Stress intensity factors are often used in a criterion for crack growth.

Two Mode I loading situations will be considered; a rectangular plate with an edge crack and a rectangular plate with a central crack, both in a situation of plane strain, see Fig. 17. In an implementation only the upper half and upper right quadrant, respectively, of these plates will be considered.

For both cases the stress intensity factor has the form

$$
\sigma_{ij} = \frac{K_i}{\sqrt{\pi r}} f_j(\theta) \quad i, j = 1, 2,
$$

(5.19)
\[ K_i = F \left( \frac{a}{b} \frac{1}{b} \right) \sigma \sqrt{\pi a}. \] (5.20)

Values for the factor \( F \) for different values of the ratios \( a/b \) and \( l/b \) can be found for an edge crack in Rooke et al. [9] and for a central crack in Bowie et al. [10].

In an implementation the stress intensity factor for both situations will be computed from the first component of the \( J \)-vector, which components are given by

\[ J_k = \lim_{C \to 0} \int_{C} (\nabla \cdot \mathbf{u}_k - \sigma \cdot n_k) \, d\mathbf{s} \quad k = 1, 2. \] (5.21)

Here, \( C \) is a curve around the crack tip with begin and end points on the crack surfaces, \( W \) the energy density and \( n \) the outward pointing unit normal on \( C \). With \( C \to 0 \) shrinking to the crack tip is meant. The length of this vector has the interpretation of the amount of energy per new created crack surface which is available in the material for crack extension. Therefore, this length is often used in a criterion for crack growth. For the value of \( J_1 \) it is not necessary to take the limit in (5.21); \( J_1 \) is independent of the contour \( C \) around the crack tip. In a plane strain situation one has that

\[ J_1 = \frac{1 - \nu^2}{E} K_1. \] (5.22)

A linear approach by \( \text{FE} + \text{EFG} \), where the domains are connected by means of a Lagrange multiplier, will now be applied to the plates to approximate the stress intensity factors. Therefore, the width is set equal to \( b = 1 \), the length to \( l = 1 \) and the crack length is set equal to \( a = 0.4 \). A part of the plate around the crack tip, \( 0.2 \leq x_2 \leq 0.6 \) and \( 0 \leq x_1 \leq 0.2 \) will be handled by \( \text{EFG} \) and the remaining part of the plate is handled by \( \text{FE} \). Different numbers of nodal points in the \( \text{EFG} \)-domain will be considered. The number of finite elements will be fixed. Four noded quadrilaterals are used and regular and radial patterns are used in the \( \text{EFG} \)-part. This leads to nodal distributions and cell structures like the ones depicted in Fig. 18.

\( C^0 \)-weight functions of Gaussian type with circular support are used. For both a regular and a radial pattern in the \( \text{EFG} \)-part of the domain, the radii \( r \) of the weight functions are set equal to

Fig. 18. Meshes of nodes and cell structures for plate with a crack: (a) regular pattern for \( \text{EFG} \)-part; (b) radial pattern for \( \text{EFG} \)-part.
Fig. 19. Contour for computation of $J_1$.

Table 5

<table>
<thead>
<tr>
<th>EFG-parameters</th>
<th>Edge crack</th>
<th>Central crack</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 EFG-points regular, $\alpha = 2$</td>
<td>0.988</td>
<td>0.993</td>
</tr>
<tr>
<td>153 EFG-points regular, $\alpha = 2$</td>
<td>0.989</td>
<td>0.991</td>
</tr>
<tr>
<td>46 EFG-points radial, $\alpha = 2$</td>
<td>0.982</td>
<td>0.984</td>
</tr>
<tr>
<td>154 EFG-points radial, $\alpha = 2$</td>
<td>0.993</td>
<td>0.993</td>
</tr>
</tbody>
</table>

where in case of a regular pattern $h$ is the grid size of this pattern and in case of a radial pattern $h$ is the grid size in the radial direction. Eight volume cells are used in the EFG-part with in each cell $(7,7)$ Gauss quadrature. For the evaluation of $J_1$ the closed contour depicted in Fig. 19 is taken, where both $h_1$ and $h_2$ are set equal to 0.1. As described in [11] the contour integral (5.21) is converted into a domain integral over the interior of the contour. For the domain integration the interior has been split up in 8 integration cells. In each cell (5.5) Gauss quadrature has been performed.

Table 5 presents for the different nodal patterns in the EFG-part the values of $K_i^h/K_i$ for both the edge crack and the central crack. From this table we see that the stress intensity factors can be obtained within 2 per cent accuracy. This example shows that an EFG-approach can be used locally in a FE-mesh to approximate the steep displacements and stresses in the neighbourhood of a crack tip.

6. Conclusions

Combinations of the Element Free Galerkin (EFG) method and the Finite Element (FE) method have been considered. It was shown that a combination of EFG and FE where the FE-approach is seen as a special approach by EFG, does not work very well. This is due to the fact that one has to work for some nodal points with weight functions with relatively small supports with respect to the grid size of the nodal grid.

A combination of EFG and FE where the domains are connected by means of a Lagrange multiplier does work very well. As for a pure EFG-approach, it was shown that the rates of convergence for a combination of EFG and FE can exceed those of the FE-method. The problem of an infinite plate with a circular hole has shown that a combination of EFG and FE is very effective for a problem with steep displacements and stresses. Moreover, using EFG in the neighbourhood of a crack tip in a FE-mesh, resulted in stress intensity factors with a good accuracy. The results have shown that it is possible to use an EFG-approach locally in a FE-mesh.

The essential boundary conditions in the examples were imposed by means of a Lagrange multiplier.
However, essential boundary conditions for an approach by EFG and FE can also be imposed with the help of boundary collocation, by means of a penalty formulation or by means of a formulation where the Lagrange multiplier is replaced by its physical meaning.

A combination of EFG and FE has the following advantages over an approach by EFG for the entire domain:

- A combination of EFG and FE is from the viewpoint of computation time much cheaper, since for EFG for each point under consideration a linear system should be solved.
- To obtain accurate values for the stiffness matrix a dense integration pattern for EFG is necessary. For a combination of EFG and FE such a pattern is only necessary for a (small) part of the problem domain.
- The task of a correct definition of weight functions such that Moving Least Squares Approximation is well defined, falls for a combination solely on the EFG-part of the domain.

Moreover, for a combination of EFG and FE the flexibility of EFG over FE is kept on the part of the domain where one wants to achieve an accurate approximation of the solution.

The reported results and the above advantages over a pure EFG-approach make that combinations of the Element Free Galerkin method with the Finite Element method deserve consideration, for example in progressive crack growth.

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References