Long-Term Dynamics of Non-Linear MDOF Engineering Systems†

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Abstract—In this paper it is shown how the finite element technique has been integrated with numerical tools for the analysis of non-linear dynamical systems to obtain a tool for efficient dynamic analysis of multi-degree-of-freedom mechanical systems with local non-linearities. The influence of hard discontinuities like elastic stops can be taken into account. The developed methodology is applied to a number of mechanical engineering systems, i.e. a beam system with various discontinuous supports, a rotor model with a non-linear fluid-film bearing and a model of a portable CD player. It is shown that generally models with more than one degree of freedom are required for a sufficiently accurate representation of the system behaviour. © 1997 Elsevier Science Ltd

1. INTRODUCTION

In engineering practice, an increasing necessity can be observed for taking into account the influence of non-linearities in the description of the dynamic behaviour of complex mechanical systems. Quite often characteristic dynamic behaviour can be observed in complex mechanical systems which cannot be explained from analysis of linear models of such systems, often containing many degrees of freedom. Besides the fundamental need for a better understanding of the dynamic behaviour, also the wide availability of digital computers with rapidly increasing power is a very stimulating factor for non-linear analyses of complex dynamical systems.

Frequently, in mechanical engineering systems from a spatial or geometrical point of view, non-linearities have a local character. This means that from a spatial point of view, these local non-linearities constitute only a restricted part of the system under consideration. However, their presence in general has important consequences for the overall dynamic behaviour. Important examples of local non-linearities in mechanical engineering systems are: fluid-film bearings in rotating machinery, dry friction and backlash phenomena in certain connections of mechanical systems, non-linear spring and damper supports in piping or vehicle systems, etc.

A typical orbit of a dissipative linear or non-linear mechanical system driven by a periodic excitation consists of a transient, whereafter the system will settle in a post-transient state, the long-term or steady-state behaviour of the system. The long-term behaviour of non-linear systems is much more complex than the long-term behaviour of linearized systems. Long-term responses of linear systems will always have the same character as the excitation forces: if the excitation is sinusoidal, the long-term behaviour will also be

†Dedicated to Professor Franz Ziegler, Technical University of Vienna, on the occasion of his 60th birthday.
sinusoidal and will be unique. Also, the superposition principle holds for linear systems. This is not the case for non-linear systems, which makes analysis of the long-term behaviour of such systems much more complex and time consuming.

For the dynamic analysis of complex linear mechanical systems, the finite element method is a well-developed analysis tool. For mechanical systems with local non-linearities, the finite element technique has been integrated by the authors with numerical tools for the analysis of non-linear dynamical systems to obtain a tool for efficient dynamic analysis of non-linear multi-degree-of-freedom (MDOF) engineering systems. A description of the developed computational procedure for determining the long-term responses and evaluating their local stability will be given in Section 2.

In contrast with linear dynamic systems, non-linear dynamic systems can show different long-term responses (attractors) for one set of system parameters. Which attractor will be approached by the system depends on the initial state of the system and the global stability of the coexisting attractors. If the global stability of a specific attractor is low, small perturbations on the system will easily result in a jump to another attractor. From a practical point of view, it is important to determine which attractor will show up, because the maximum response of the different attractors can be very different and also response spectra can be very different. Large-amplitude responses may lead to large strains and stresses which can cause damage to the structure. In Section 3 computational procedures are outlined for determining the global stability of attractors of non-linear dynamic systems.

The above-indicated computational procedures have been applied by the authors to a number of MDOF mechanical engineering systems.

The first application refers to MDOF systems with strong stiffness discontinuities. Discontinuous systems are an important class of non-linear mechanical systems. In many engineering applications, discontinuities occur. An example of a discontinuous system is a satellite with solar panels [1]. While launching the satellite, the solar panels are in a folded position and dynamically loaded by a base excitation of the supports. In order to reduce the vibration levels of the panels, so-called one-sided snubbers are used leading to a discontinuous system. Such a discontinuous system is a typical example of a system with local non-linearities. Although the non-linearity is local, the overall dynamic response of the system changes drastically. A system with discontinuities cannot be linearized, so it is very difficult to approximate the system response through a linear analysis. Very often, such discontinuities create a response which contains more (mostly higher) frequencies than the excitation frequency, even in case the excitation frequency is low. Generally, this means that the higher eigenmodes of the linear parts of the system will be excited. Then, these higher eigenmodes will strongly influence the low frequency response and MDOF systems will be necessary to calculate the system behaviour accurately. In Sections 4 and 5, a beam system is analysed with various discontinuous supports, which is representative for the above class of non-linear engineering systems. Experimental results are reported to support the validity of the numerical results.

The second application of non-linear MDOF systems refers to rotordynamics. Journal bearings are an essential feature of all rotating machinery and provide the vital load-carrying capacity to support rotors against static and dynamic loads. The standard practice for dynamic imbalance response and stability analysis of rotor-bearing systems is to determine the linearized bearing stiffness and damping coefficients about a stationary equilibrium position. In this mathematical assumption, the synchronous motion about the stationary equilibrium position is described by ellipses in case of low values of unbalance. However, when the unbalance level becomes moderate to large, the rotor orbits are not necessarily elliptical, due to the non-linear characteristics of the bearings. Furthermore, in a linear system, operating above the threshold angular rotor speed, the amplitude of motion grows
exponentially with time and the orbits become unbounded. In an actual system this is not necessarily the case, because the non-linear effects can cause the motion to be bounded by limit cycles. Moreover, under some circumstances even a lower total response can be obtained by having a moderate level of rotor unbalance in the system.

The analysis of the non-linear effects in rotor-bearing systems is extremely difficult due to the fact that the number of DOFs is always large. The most commonly accepted method for non-linear dynamic analysis is to perform a time-transient integration method for the complete set of equations of motion. Therefore, even for relatively simple systems this becomes a formidable task since the Reynolds' equation for the bearings has to be evaluated at each time step [2]. In Section 6 of this paper the finite-length impedance bearing theory [3] is used for modelling the bearings of a simple rotor-bearing system. In this way rotor-bearing systems can be analysed efficiently because the current bearing forces in the equations of motion for the complete rotor system are calculated using analytical expressions. The finite-length bearing theory is accurate but only useful for plain cylindrical journal bearings. Because rotor-bearing systems are typical examples of systems consisting of large linear parts (shaft and disks) and local non-linearities (bearings), the models of these systems can be reduced considerably. Because of this, the long-term dynamics of a rotor-bearing system can be analysed efficiently by calculating periodic solutions by the methods outlined in this paper.

The third application of non-linear MDOF systems refers to consumer electronics. The handling of external shocks is one of the main problems in the design of consumer electronics. As an example, one can think of a portable compact disc (CD) player. Knowing the type of external disturbance, measures can be taken to guarantee a high-quality performance. This includes, for example, the design of appropriate suspensions of the internal mechanisms and optimization of the control systems and electronics. For a portable CD player, a special type of external disturbance is given by the motion which it experiences during jogging. The ability of the player to perform well under this condition is called 'joggability'. The measures necessary to assure joggability depend on the response of the player to this type of loading. In particular, the accelerations of the player are relevant to this evaluation. In Section 7, a simple model is presented which characterizes the non-linear behaviour of an idealized portable CD system. The jogging effect is represented by a harmonic excitation. The long-term responses and their global stability will be analysed by the computational method for MDOF systems outlined in Section 3.

2. COMPUTATION OF THE LONG-TERM RESPONSE

The mechanical systems to be analysed are divided into one or more linear components and local non-linearities. The linear components, which are supposed to be weakly or proportionally damped, are modelled by means of the finite element method, resulting in MDOF models for those components. These models are subsequently reduced using a component mode synthesis (CMS) method based on free-interface eigenmodes and residual flexibility modes, see Craig [4]. Only eigenmodes up to a certain cut-off frequency are kept in the reduced components. The residual flexibility modes guarantee unaffected (quasi-)static load behaviour of the reduced components.

The local non-linearities and the reduced linear components are coupled, leading to a reduced non-linear system which is an approximation of the unreduced system. The resulting reduction technique in general offers a large reduction of the CPU time needed for analysis, whereas simultaneously the decrease in accuracy of the system response up to the cut-off frequency is only small.
Elastic stops in the mechanical system are modelled using the non-linear Hertzian force-displacement law for the contact forces during the contact period.

The non-linear MDOF dynamic model, resulting from the above reduction and coupling process, is considered under periodic excitation. The long-term dynamics of this model can have a periodic, quasi-periodic or chaotic character.

The stable and unstable periodic solutions of the non-linear MDOF equations of motion for the above model are evaluated in a direct way by solving two-point boundary value problems obtained by supplementing the equations of motion with periodicity conditions for the solutions. Often the (multiple) shooting method [5] is used for solving the two-point boundary value problem. Because this method is very expensive from a computational point of view for systems with many degrees of freedom (DOFs), in literature usually only very simple systems with one or two DOFs are investigated. An alternative method to compute the periodic solutions of two-point boundary value problems has been developed by the authors [6] using the finite difference method with an equidistant time discretization mesh. This method is generally somewhat less accurate than a (multiple) shooting method but it is much more efficient, in particular for systems with many DOFs. On the other hand, the finite difference method with an equidistant discretization mesh has shown problems in handling systems with ‘hard’ discontinuities, i.e. discontinuities like dry friction and elastic stops, which cause a large change of the state of the system in a small time interval.

In the computational procedure, branches of periodic solutions are followed at changing system parameters by using a path-following technique. The local stability of these periodic solutions is investigated using Floquet theory. The long-term behaviour is also investigated by means of standard numerical time integration, in particular for determining chaotic motions.

The above-mentioned computational procedures for non-linear dynamic systems have been integrated into the commercial finite element package DIANA [7]. In this way the potential of the finite element method is used as a modelling basis for the class of non-linear systems under consideration. The DIANA package is used for all calculations in this paper.

3. COMPUTATION OF GLOBAL STABILITY

For determining the global stability of the attractors of a non-linear system, the boundaries of the so-called basins of attraction should be known. The basin of attraction of an attractor is the set of initial states of orbits that finally settle down on the attractor if time proceeds. Generally, the boundaries of the basins of attraction are the stable manifolds of unstable periodic saddle solutions. The stable (unstable) manifolds of an unstable periodic solution are the set of initial states for which theoretically the corresponding trajectories approach the unstable periodic solution if the time is increased (decreased). So by calculating the stable manifolds of these unstable periodic solutions, the global stability of stable attractors can be determined. Parker and Chua [8] developed a method for calculating stable and unstable manifolds of single-DOF systems. Generally, for multi-DOF systems, stable manifolds become planes in the Poincaré section. Therefore, it becomes very expensive to calculate these manifolds. However, often the unstable manifolds are still one-dimensional and can be calculated using the Parker and Chua method. Nevertheless, it is generally not possible to determine efficiently the global stability of MDOF systems using manifolds. The Parker and Chua method has been supplemented to the commercial finite element package DIANA mentioned in Section 2 by Van de Vorst [9].

The only technique to identify the basins of attraction of periodic solutions of arbitrary non-linear dynamic systems, which is available nowadays, is the cell mapping (CM) method,
established by Hsu [10, 11]. However, in its version developed by Hsu, the practical application of this method is restricted to systems with less than three DOFs for computational reasons. In the following, an extension of the existing cell mapping method is described, also enabling investigation of systems with more than two degrees of freedom.

The cell mapping method enables the attractors and their corresponding basins of attraction of a non-linear dynamic system to be found. Under the so-called simple cell mapping (SCM) method, a particular region $\Omega$ in the state space is discretized into a finite number, say $M$ cells. These cells are $N$-dimensional, where $N$ is the state space dimension. Each cell represents an indivisible state entity. The state of the system is then fully described by a cell index $z \in \{1, \ldots, M\}$ instead of a state vector $x = (x_1, \ldots, x_N)$.

The evolution of a system can now be described as a sequence of cell indices by inspecting its state at discrete equidistant times. Let $z(n)$ denote the cell containing the state of the system at $t = nT$, $n = 0, 1, \ldots$, with $T$ the time between two state inspections. The system evolution is then governed by

$$z(n + 1) = C(z(n)),$$

where the mapping $C: \mathbb{N} \rightarrow \mathbb{N}$ is called an SCM. For periodic systems, $T$ should be chosen equal to the system’s period to obtain a mapping $C$ which is independent of $n$.

We distinguish between cells which are periodic, i.e. cells $z^*$ with $C^m(z^*) = z^*$, for some $m \in \mathbb{N}$ which is called the period of $z^*$, and cells which are not. The latter are called transient cells, and they will be mapped onto a periodic cell in a finite number of steps or leave the region $\Omega$. Groups of periodic cells represent the system’s recurrent states (attractors, saddles and repellors). Although no aperiodic motion can occur because of the finite number of cells, chaotic motion is normally expected when dealing with periodic groups for a relatively long period.

When applying CM methods in their regular form to systems of many DOFs, problems of a computational kind can be expected. For a dynamic system of $l$ DOFs, the corresponding state space has the dimension $N = 2l$. The number $M$ of regular cells for an SCM application grows exponentially with $N$. Additionally, the necessary CPU time to determine a cell’s image grows linearly with $N$, since $N$ first-order ordinary differential equations need to be integrated for this purpose. This means that for $N > 3$, extremely high CPU times will occur.

Because of this computational restriction for the regular SCM method, a new technique is proposed, called MDOF cell mapping (MDCM). This technique will be illuminated subsequently.

Under regular SCM, the attractors and basins of attraction are determined in a region of interest $\Omega$ in the $N$-dimensional state space. However, a two-dimensional subspace $\Sigma$ always needs to be chosen for representation purposes. For large $N$, many choices for $\Sigma$ are possible. Hence, the user has to decide which subspace is most relevant, which means that much data will not be used in practice. Therefore, it is meaningful to make this choice beforehand and to determine the long-term behaviour only for the initial states lying in the subspace of interest. This point of view is the basis of MDCM.

The aim of the MDCM method is to determine the intersections of the basins of attraction of an $N$-dimensional dynamic system ($N \geq 3$) with a two-dimensional subspace $\Sigma$. For this purpose, the following steps are taken:

- The complete state space $\mathbb{R}^N$ is formally discretized into $N$-dimensional cells by introducing $N$ cell sizes $h_1, \ldots, h_N$. Each cell is indicated by a cell vector $z = (z_1, \ldots, z_N)$, where each index $z_i$, $i = 1, \ldots, N$, is an integer. By definition, a cell $z$ contains all states $x = (x_1, \ldots, x_N)$ with $(z_i - 1/2)h_i \leq x_i \leq (z_i + 1/2)h_i$, $i = 1, \ldots, N$.
- A two-dimensional subspace $\Sigma \subset \mathbb{R}^N$ is created by giving $N - 2$ cell indices a certain constant value.
• In $\Sigma$, a bounded region of interest $\Omega'$ is chosen by introducing an upper and a lower limit for the two remaining state variables.
• A set of cells $S$ is defined, covering $\Omega'$.
• For each cell $z \in S$, the group number $G(z)$ is determined by creating a cell processing sequence $z, C(z), C^2(z), \cdots$, see, for example, Hsu [11].

The intersections of the basins of attraction with $\Sigma$ are given by the cells in $S$ with equal group number. Since the image cell is determined only for cells in the processing sequence—instead of for all cells in $\Sigma$ under regular SCM—the CPU time and memory demand are reduced drastically in this way. Under MDCM, there is no real restriction on the system dimension.

An interesting aspect of MDCM is the possibility of re-using stored group numbers of processed cells. Having applied MDCM for a certain subspace $\Sigma^1$, use can be made of these group numbers when applying MDCM for another subspace $\Sigma^2$. When a cell sequence starting from a cell $z \in \Sigma^1$ leads to a cell $z'$ which has already been processed in the first application, the sequence can be terminated. All cells in the sequence then obtain the same group number as $z'$. In this way, the creation of each processing sequence is stopped at an early stage, yielding an extra gain of CPU time.

For more information on algorithmic aspects of MDCM the reader is referred to Van der Spek [12]. In Section 7, application of MDCM is performed on a two-DOF model of a portable CD player.

Finally it is remarked that a path-following technique for cell mapping has recently been developed by Van der Spek et al. [13], thus further increasing its potential.

4. BEAM SYSTEM SUPPORTED BY A ONE-SIDED SPRING

4.1. Numerical model and experimental set-up

Figure 1 shows a beam system which is analysed both numerically and experimentally. The beam is supported at both ends by leaf springs. In the middle of the beam a one-sided spring leaf spring support is present.

![Beam system with a one-sided spring](image)

$E = 2.0 \times 10^11 \text{ N/m}^2$

$h = 0.01 \text{ m} \\
h_b = 0.001 \text{ m} \\
\bar{h} = 0.006 \text{ m} \\
b = 0.09 \text{ m} \\
b_b = 0.075 \text{ m} \\
\bar{b} = 0.02 \text{ m} \\
\rho = 7746 \text{ kg/m}^3 \\
\rho_b = 7713 \text{ kg/m}^3 \\
\rho_b = 7746 \text{ kg/m}^3$

Fig. 1. Beam system with a one-sided spring.
In the numerical model the beam is assumed to be pinned at both ends, whereas the one-sided leaf spring is assumed to be linear and massless. In the middle of the beam a periodic excitation force is generated by a rotating mass. At this position the beam is also supported by a linear damper (not shown in Fig. 1) with a damping coefficient of $820\xi$ Ns/m (where $\xi = 0.0142$), causing dissipation of energy.

The pinned beam is modelled using the finite element method. Because of symmetry only half the beam is modelled. Subsequently, reduction is applied to a four-DOF model using the CMS method mentioned in Section 2. In the four-DOF model, three free-interface eigenmodes with eigenfrequencies of $f_1 = 13.1$, $f_2 = 117.5$ and $f_3 = 326.5$ Hz and one residual flexibility mode for the midspan displacement are included. Modal damping is added to the four modes with the damping coefficient $\xi_m$. The stiffness ratio of the one-sided spring and the beam is taken as 6.41.

To suppress the influences of the mass of the one-sided leaf spring in the experimental set-up, its dimensions are chosen such that its first eigenfrequency is much higher than the maximum excitation frequency. Furthermore, in the experiment the one-sided linear leaf spring is damped using a one-sided damper to achieve uniform collision between the beam and the one-sided leaf spring.

In the experimental set-up a sinusoidal excitation force is generated by a rotating mass in the middle of the beam, driven by an electric motor. Because of the high weight of the motor, the motor is not connected to the beam and a flexible coupling is used for coupling the mass unbalance to the rotor.

4.2. Results

In Fig. 2 some calculated and measured maximum absolute displacements of the periodic solutions in the middle of the beam are shown as a function of the excitation frequency $f_e$. In the numerical analysis the modal damping coefficient $\xi_m$ is taken as zero.

In the numerical results, apart from the harmonic resonance peak near 19 Hz (the first bilinear eigenfrequency), subharmonic resonances of $1/2$, $1/3$, $1/4$, $1/5$, $1/6$, $1/7$, $1/8$, $1/9$ and $1/10$ are also found. The highest $1/2$ and $1/3$ subharmonic resonances are related to the first harmonic resonance peak. The $1/3$ subharmonic near 28.5 Hz, the $1/5$ subharmonic near 47.5 Hz and the $1/7$ subharmonic near 70 Hz are related to the second superharmonic resonance peak near 9 Hz of the first 'eigenfrequency'. All other subharmonic resonances are also related to the superharmonic resonances of the first 'eigenfrequency'.

Besides the harmonic resonance peak, the harmonic response curve contains small to moderate superharmonic resonances near $1/2f_{b2}$, $1/3f_{b2}$ and $1/4f_{b2}$, where $f_{b2} = \text{second bilinear eigenfrequency} \approx 118$ Hz. To the right-hand side of the large $1/3$ subharmonic branch, the harmonic branch is unstable in a small frequency interval near 71 Hz. Numerical time integration shows that quasi-periodic behaviour and mode locking to chaos are found here. In a small frequency interval near 47.5 Hz, the $1/2$ subharmonic branch is divided into two parts. At this frequency interval a so-called intermittency route to chaotic behaviour is found. Both the intermittency route and the mode locking route are related to the (very small) fifth superharmonic resonance peak near 23 Hz of the second 'eigenfrequency' of the system.

The experimental results depicted in Fig. 2 correspond surprisingly well with the numerical results. Almost all subharmonics are observed in the experiments.

The experimental results in Fig. 2 show shadow peaks behind the harmonic, $1/2$

†Because the model is non-linear, one cannot speak of the eigenfrequencies of the model. Here, the term 'eigenfrequency' is used as the frequency at which the model resonates.
subharmonic and 1/3 subharmonic resonance peaks. These peaks are most probably caused by the coupling between the rotating mass and the electric motor.

Aperiodic signals could be measured at a small frequency interval near 47.5 Hz, where chaotic behaviour was found in the numerical analysis.

Finally it is remarked that a single-DOF model has also been constructed and analysed for the beam with a one-sided spring support. Although globally the 'amplitude'-frequency characteristic for the single-DOF model looks similar to Fig. 2, locally there are important differences. In particular, the superharmonic resonance peaks related to the higher 'eigenfrequencies' do not appear for the single-DOF model, as well as the related subharmonic and chaotic behaviour.

For the single-DOF model, an extensive discussion on the global stability using manifolds has been given by Van de Vorst et al. [14].

For a more detailed discussion of the results for the beam with a one-sided leaf spring support, the reader is referred to Fey et al. [6] and Van de Vorst et al. [15].

5. BEAM WITH AN ELASTIC STOP

5.1 Numerical model

The same beam system as in Section 4 is considered, however, the one-sided leaf spring is replaced by an elastic stop. If the displacement $y$ at the middle of the beam is positive, the beam hits a spherical elastic contact which is rigidly supported. Again energy dissipation is
established by a linear damper (not shown in Fig. 4) in the middle of the beam. The beam is modelled using the finite element method and a four-DOF model is constructed using the component mode synthesis method mentioned in Section 2.

The contact force $F_i$ between the beam and the spherical elastic contact is modelled using Hertz's law [16, 17], that is

$$F_i = \begin{cases} k_s y^{3/2} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

In eqn (2), $y$ is the transverse displacement of the beam at its middle, whereas the parameter $k_s$ is taken as $1.034 \times 10^{10}$ Nm$^{2/3}$ for a contact radius $r = 5$ mm and $E = 2.1 \times 10^{11}$ N/m$^2$ of the spherical contact.

5.2. Results

Figure 3 shows the calculated maximum absolute displacements of the periodic solutions in the middle of the beam as a function of the excitation frequency $f_e$. The modal damping coefficient $\xi_m$ is taken as 0.01. Furthermore, the amplitude of the excitation force is 1% of the value for the case of a beam with a one-sided spring. In contrast to the beam with a one-sided spring, the qualitative response of the beam with an elastic contact depends on the amplitude of the excitation force.

The response is dominated by the harmonic resonance peak near 23 Hz and 1/2 and 1/3 subharmonic resonance peaks near 45 and 68 Hz which are related to the harmonic resonance peak. In contrast to the solution for the one-sided spring support, the levels of...
subharmonic resonance peaks hardly decrease compared to the harmonic resonance peak. In both the 1/2 and 1/3 subharmonic peaks, the beam hits the elastic contact twice per period.

On the harmonic branch, superharmonic resonances exist which are related to the second and third ‘eigenfrequency’ of the system. These superharmonic resonance peaks result in additional resonance peaks on the 1/2 and 1/3 subharmonic solution branches. If the two extra peaks on the 1/2 and 1/3 subharmonic branches are projected back on the first harmonic resonance peak, these two peaks have to be caused by the fifth superharmonic resonance peak (27 Hz) of the second ‘eigenfrequency’ (see inset in Fig. 3). In the two extra peaks on the 1/2 and 1/3 subharmonic branches, the beam hits the contact once per period.

In the frequency areas 30–38 and 58–63 Hz, period doubling routes to chaos were found. Figure 5 shows the response of the system if the modal damping coefficient is decreased to \( \xi_m = 0.001 \). Compared to Fig. 3, now all sub- and superharmonic resonance peaks increase and in the frequency band where the harmonic resonance peak dominates, many superharmonic resonance peaks can be seen. These superharmonic resonance peaks are not related to one specific ‘eigenfrequency’ of the model. The superharmonic resonance peaks in the frequency domain 24–26 Hz are again dominated by the second and third ‘eigenfrequency’ of the model and they are responsible for the two extra resonance peaks on the 1/2 and 1/3 subharmonic branches. Again in these peaks the beam hits the contact once per period. The higher peaks in the frequency domain 22–23 Hz are dominated by the first and second ‘eigenfrequency’ of the model. Some peaks are unstable and also period doubling routes to chaos are found here. In the stable peaks the beam hits the contact once per period, and in the unstable peaks the beam hits the contact three times per period (Fig. 6). Also on other parts of the branches additional period doubling routes to chaos exist which did not appear for the higher damping (\( \xi_m = 0.01 \)) model.

For a more detailed discussion of the results for a beam with an elastic stop, and in particular for comparison with results for corresponding one-DOF and two-DOF models, the reader is referred to Van de Vorst et al. [18]. It appears from this discussion that the third ‘eigenfrequency’ in the four-DOF model still has a large influence on the system behaviour.

6. ROTORDYNAMIC SYSTEM

6.1. Numerical model

Figure 7 gives a schematic view of the rotor system which is analysed. The system consists of a rigid disk and a flexible shaft supported on one side by an oil journal bearing...
and pinned on the other side. As mentioned in Section 1, the oil journal bearing is modelled using the finite-length bearing theory. This theory is based on the impedance method [3]. In the long and short bearing theories, used for instance by Myers [19] and Khonsari and Chang [20], it is assumed that the half of the bearing for which cavitation exists depends

![Graph showing maximum displacements of periodic solutions of a beam with an elastic stop (εm = 0.001).](image)

Fig. 5. Maximum displacements of periodic solutions of a beam with an elastic stop (εm = 0.001).

![Graph showing a periodic solution of a beam with an elastic stop at f_s = 22.5 Hz (εm = 0.001).](image)

Fig. 6. Periodic solution of a beam with an elastic stop at f_s = 22.5 Hz (εm = 0.001).
only on the minimum oil film thickness. However, this assumption is only valid for low shaft velocities. The impedance method also uses the shaft velocity for calculating the cavitation region in the bearing which implies that this method is also valid for high shaft velocities. In Van de Vorst et al. [21], it has been shown that the influence of velocity on the cavitation region cannot be neglected in the non-linear analysis of such rotor-bearing systems.

The Sommerfeld number \( \sigma \) of the bearing is defined as

\[
\sigma = \frac{L_h R \mu \omega}{c^2 F}
\]

where \( L_h \) is the bearing length, \( R \) is the bearing radius, \( \mu \) is the lubricant viscosity, \( c \) is the bearing clearance, \( \omega \) is the angular rotor speed and \( F \) is the static external load force acting on the bearing. The system is investigated in the frequency range \( \omega = 0 - 1050 \text{ rad/s} \) and if the static force is taken as \( F = 8 \text{ N} \) (which is approximately the weight of the rotor), the Sommerfeld number of the bearing varies from 0 to 6.4.

Only the gyroscopic effect of the disk is taken into account; the gyroscopic effect of the shaft is neglected. The distance \( l \) between the oil journal bearing and the disk is taken as 0.038 m. This means that the disk is very close to the bearing, so the bearing is heavily loaded. The system is statically loaded by its own weight and dynamically by a mass unbalance of the disk, so the dynamic loads can be written as

\[
F = r_m \omega^2 \cos(\omega t); \quad F = r_m \omega^2 \sin(\omega t)
\]

where \( r_m = 0.0305 \text{ m} \) and \( m_r \) is the mass unbalance.

The shaft is assumed to be modally damped with modal damping coefficients \( \xi_m = 0.01 \) for all eigenmodes. The shaft is modelled using the finite element method and subsequently reduced using the component mode synthesis method mentioned in Section 2. In the reduction, two rigid body modes, eight residual flexibility modes (two for the bearing, two for the measurement locations \( y_m \) and \( z_m \) and four for the disk) and four free-interface
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6.2. Results: static loads, \( m_e = 0.0 \) g

Figure 8 shows the maximum absolute displacements of bearing position \( z_b \) as function of the angular rotor speed \( \omega \). For low angular rotor speeds, \( \omega < 590 \text{ rad/s} \), only stationary ('static') solutions are found (branch (a)). At \( \omega = 610 \text{ rad/s} \), these stationary solutions become unstable via a primary Hopf bifurcation. Further increasing the angular rotor speed leads to the occurrence of self-excited oscillations, normally referred to as \( 1/2\omega \) whirl or oil whirl in rotordynamics [19].

In Fig. 8 the maximum absolute displacements of the bearing position \( z_b \) occurring in the self-excited oscillations are also shown. The unstable \( 1/2\omega \) whirl (branch (b)) starts at the primary Hopf bifurcation and exists for somewhat lower angular rotor speeds than the threshold angular rotor speed for which the primary Hopf bifurcation exists. The branch becomes stable by a cyclic fold bifurcation and bends off to higher angular rotor speeds (branch (c)). Notice that stable and unstable periodic solutions exist for angular rotor speeds somewhat smaller than the threshold speed. Figure 10(a) shows the shaft positions in the bearing for the stable and unstable \( 1/2\omega \) whirl at \( \omega = 600 \text{ rad/s} \).

The solid line in Fig. 9 shows the frequencies \( \Omega \) of the periodic solutions. At the dotted line the frequency \( \Omega \) is half the angular rotor speed \( \omega \). As expected, the stable and unstable branches (b) and (c) have frequencies which are almost half the angular rotor speed. However, for high angular rotor speeds, \( \omega > 750 \text{ rad/s} \), the frequencies \( \Omega \) of the self-excited oscillations of branch (c) are much lower than half the angular rotor speed. The frequency goes to \( \Omega \approx 400 \text{ rad/s} \). It is known that the frequency \( \Omega \) of self-excited oscillations can move to the first eigenfrequency of the model [2]. The three branches of symbols in Fig. 9 represent the three lowest eigenfrequencies of the model linearized around the stationary solutions for varying angular rotor speed. The eigenfrequencies corresponding to the two lowest flexible eigenmodes of the shaft vary between 300 and 350 \text{ rad/s} and are much lower than the frequencies of the self-excited oscillations. However, the eigenfrequencies plotted
in Fig. 9 belong to the stationary solutions and the amplitude of the shaft motion in the bearing has a large influence on the eigenfrequencies of the model because the rotor is close to the bearing. In the self-excited oscillations the shaft curve is far from the stationary point [Fig. 10(a)]. Because of this the amplitude of the shaft motion in the bearing is large which will result in an increase of the ‘eigenfrequencies’ of the shaft. So in the periodic solutions the increased first flexible ‘eigenfrequency’ of the shaft can be expected to be very close to the frequencies of the self-excited oscillations.

Notice that the primary Hopf bifurcation occurs if the angular rotor speed is nearly two times the first eigenfrequency of the linearized model (this eigenfrequency will be equal to the ‘eigenfrequency’ of the non-linear model because the shaft motion is zero). Furthermore, the unstable branch of periodic solutions (b) starts with a free frequency $\Omega$ which is almost equal to the first eigenfrequency of the linearized model which could be expected (Fig. 9).

In the periodic solutions the shaft motion in the bearing is large. As mentioned above, the shaft motion strongly influences the ‘eigenfrequencies’ of the non-linear model and the growth of the shaft motion increases the ‘eigenfrequencies’. If the angular rotor speed is increased to two times the increased first ‘eigenfrequency’ of the shaft, the free frequencies

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Fig. 9. Frequencies of self-excited oscillations and eigenfrequencies of a linearized system for $m_0 = 0 \text{ g}$.

Fig. 10. Shaft positions in the bearing: (a) $m_0 = 0 \text{ g}$, (b) $m_0 = 1 \text{ g}$. 
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Ω of the periodic solutions are close to the increased first 'eigenfrequency'. Because of this the maximum response at the measurement point increases and because the free frequency is not near half the angular rotor speed, the shaft eccentricities in the bearing decrease (Fig. 8).

6.3. Results: static and dynamic loads, \( m_r = 1.0 \) g

In this section a mass unbalance of \( m_r = 1.0 \) g is added. Periodic solutions of the model are calculated for varying angular rotor speeds. Figure 11 shows the maximum absolute displacements of \( z_b \) occurring in the periodic solutions as a function of the angular rotor speed.

The dotted lines in Fig. 11 represent the dynamic response of the model linearized around the stationary solution as indicated in Fig. 8. Taking into account the fact that the stability of the linearized model equals the stability of the stationary solution in Fig. 8, it can be concluded that the linearized model is valid only for low angular rotor speeds. This is because for low angular rotor speeds the dynamic loads are small, which results in dynamic solutions close to the stationary solutions.

The dynamic response shows that at almost the same angular rotor speed for which the primary Hopf bifurcation in Fig. 8 occurred, the harmonic solution now becomes unstable via a secondary Hopf bifurcation. If the angular rotor speed is slightly increased, the harmonic solution becomes stable again via a second secondary Hopf bifurcation. In this small unstable frequency range quasi-periodic behaviour was found. A further increase of the angular rotor speed leads to a flip bifurcation. Here again quasi-periodic behaviour was found which locks on to 1/2 subharmonic behaviour if the angular rotor speed is further increased to 675 rad/s (see inset of Fig. 11). If the angular rotor speed is increased to \( \omega > 690 \) rad/s the 1/2 subharmonic behaviour changes again into quasi-periodic behaviour. In the frequency range \( \omega \approx 950-1020 \) rad/s the harmonic solution is stable again, and for \( \omega > 1020 \) rad/s, quasi-periodic behaviour was found.

Figure 10(b) shows the positions of the shaft in the bearing of the 1/2 subharmonic and harmonic solutions for \( \omega = 685 \) rad/s. Notice the large amplitude difference between the 1/2 subharmonic solution and the unstable harmonic solution.

For comparison of the results in Sections 6.2 and 6.3 with the results for different values
of the distance $L$ between the rotor and the oil journal bearing, the reader is referred to Van de Vorst et al. [22].

7. PORTABLE CD PLAYER

7.1. Numerical model

A CD player, schematically drawn in Fig. 12, mainly consists of two parts: the player $P$ (mass $m_2$) and a carrying strap containing a shoulder pad $B$ (mass $m_1$). The vertical displacement of $B$ is denoted by $q_1$. The two parts of the strap connecting $B$ with $P$ are considered massless and are modelled as one-sided linear springs and dampers, each with a stiffness $k$ and damping $d$. The displacement of $P$ is given by $q_2$, with $q_2 \ll q_1$ conforming to the situation in which the strap is stretched.

During jogging, the motion of the shoulder is assumed to be harmonic. The amplitude and frequency of this harmonic motion are given by $a$ and $f$, respectively. Hence, the motion of the shoulder is prescribed and given by $u(t) = a \sin(\omega t)$, with $\omega = 2\pi f$. The shoulder itself is modelled as a one-sided linear spring with stiffness $c$.

The equations of motion of the system are simple and piece-wise linear. Defining $\xi = q_1/a$, $\eta = q_2/a$, $\tau = \omega t$, the non-dimensional equations of motion are given by

$$\begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \end{bmatrix} = -\gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix} + F^b + F^s$$

![Fig. 12. Two-DOF model of a portable CD player under jogging condition.](image-url)
Here, \( \dot{\gamma} = \frac{d\gamma}{dt} \) and \( \gamma = g/a \omega^2 \), with \( g \) the acceleration due to gravity. \( F^b \) represents the strap forces, which are only non-zero when the strap is stretched \( (\xi > \eta) \). Hence,

\[
F^b = H(\xi - \eta) \left[ -\max \{0, \kappa_1(\xi - \eta) + \beta_1(\xi' - \eta')\} \right. \\
\left. \max \{0, \kappa_2(\xi - \eta) + \beta_2(\xi' - \eta')\} \right],
\]

where \( k_i = 2k/m_i \), \( \omega_i = 2\pi/m_i \omega (i = 1, 2) \), and \( H(x) \) is the Heavyside function. \( F^s \) represents the force which the shoulder exerts on \( \mathbb{B} \). Hence

\[
F^s = H(\sin \tau - \xi) \left[ \sigma(\sin \tau - \xi) \right]
\]

with \( \sigma = c/m_1 \omega^2 \).

### 7.2. Cell mapping approach

The MDCM method (see Section 3) is used to investigate the two-DOF model of the portable CD player, given by eqns (5)-(7). It is assumed that the strap stiffness is equal to the shoulder stiffness: \( k = c = 1000 \text{ N/m} \), while the strap damping is given by \( d = 4 \text{ Ns/m} \). The jogging amplitude \( a \) and frequency \( f \) are taken as \( a = 0.05 \text{ m} \) and \( f = 2 \text{ Hz} \), respectively. The masses of \( \mathbb{B} \) and \( \mathbb{P} \) are given by \( m_1 = 0.05 \text{ kg} \) and \( m_2 = 0.35 \text{ kg} \), respectively.

The aim of applying MDCM is to determine the possible types of response—in particular the acceleration—for the CD player. Looking at the background of the problem, it is easy to focus on the CD player and the influence of its initial state on its long-term behaviour. An appropriate choice for a two-dimensional subspace of relevant initial states is then given by, for example, \( \Sigma: \xi = \xi' = 0 \) (shoulder pad zero position and velocity).

The state of the system is given by \( \mathbf{x} = [x_1 \cdots x_4]^T \) with \( x_1 = \xi \), \( x_2 = \xi' \), \( x_3 = \eta \), \( x_4 = \eta' \). In this four-dimensional state space, a cell structure is defined by choosing four cell sizes: \( h_1 = h_3 = 0.01 \), \( h_2 = h_4 = 0.06 \). On \( \Sigma \), a region of interest \( \Omega' \) is defined by \( |x_3| \leq 0.5 \), \( |x_4| \leq 3 \). By means of MDCM, the long-term behaviour is determined for initial states \( \Omega' \). For the determination of image cells, a time integration interval of five forcing periods is used \( (\Delta \tau = 5/f) \).

In Fig. 13(a), the results of the MDCM application are shown. For the chosen region of interest \( \Omega' \), three different types of steady-state behaviour have been found. Cells denoted by \( (\bigcirc) \) lead to a harmonic solution, shown in Fig. 14(a). This attractor corresponds to a situation in which there is always contact between shoulder and pad. Cells denoted by \( (\bullet) \) lead to a coexisting harmonic solution [Fig. 14(c)], which appears to show two intervals of no-contact during each period. Finally, cells which are left blank in Fig. 13(a) lead to a large-amplitude quasi-periodic solution with small contact periods and large periods of free fall [Fig. 14(e)]. The trajectories in Fig. 14 are representations of the state of the player in original coordinates as a function of real time.

In Fig. 13(b), the basins of attraction are shown obtained by an additional MDCM application for a region \( \Omega'' \subset \Omega' \). Here, \( \Omega'' \) is defined by \( |x_3| \leq 0.05 \), \( |x_4| \leq 0.3 \). The cell sizes are given by \( h_1 = h_3 = 0.001 \), \( h_2 = h_4 = 0.006 \). From this ‘zoom window’ on \( \Omega'' \), it can be concluded that the basins of attraction have a fractal structure; changing the initial state only slightly may result in a totally different motion. Furthermore, an additional periodic group is found with a small basin of attraction \( (\ast) \). This group represents a 1/2 subharmonic solution [see Fig. 14(g)].

The accelerations of the CD player corresponding to the determined solutions are additionally shown in Fig. 14(b,d,f,h). The acceleration in the case of the full-contact solution is perfectly sinusoidal because the system remains linear. For the coexisting harmonic solution, the intervals of free motion are represented by intervals of constant
Fig. 13. Basins of attraction for $x_1$, $x_2$, 0; harmonic solution (.), coexisting harmonic solution (*), quasi-periodic solution (left blank), 1/2 subharmonic solution (++).
Fig. 14. Possible responses [m] and accelerations [m/s²] of the CD player as a function of time [s]: (a, b) full-contact harmonic solution; (c, d) coexisting harmonic solution; (e, f) quasi-periodic solution; (g, h) 1/2 subharmonic solution.
acceleration, g. It can be seen that the peak acceleration is less than 2g for this solution. In the case of quasi-periodic behaviour [Fig. 14(f)], however, accelerations of more than 13g are possible. Here, the motion of the player is characterized by large amplitudes (up to five times the shoulder amplitude). Finally, the peak acceleration for the 1/2 subharmonic is approximately 3g.

The results in Fig. 13(a) and (b) have been verified by determining the trajectory belonging to four different initial states, each corresponding to a different attractor. In Fig. 15, trajectories are shown obtained by integration starting at \( x = [0 \ 0 \ 0 \ x_4]^T \), with \( x_4 = 1.4, 1.2, -1.0, 0.18 \), respectively. The first three states indeed lead to the attractor as predicted by the MDCM method. For \( x_4 = 0.18 \), however, the wrong attractor is obtained. Although the transient behaviour of this trajectory is governed by the 1/2 subharmonic, it finally settles on the full-contact harmonic solution. This may be explained by the small basin of attraction of the 1/2 subharmonic solution and because of the fractal structure of the basin of attraction. Anyway, when the actual basins of attraction have a fractal structure, the representations obtained by means of CM should be interpreted with care. In such a situation, the uncertainties involved with CM have a larger impact than usual.

8. CONCLUSIONS

Efficient numerical tools for dynamic analysis of MDOF mechanical engineering systems with local non-linearities have been discussed. They enable the use of the potential of the finite element method as a modelling basis. Also, the influence of hard discontinuities like elastic stops can be taken into account in a convenient way. The effectiveness of the described methodology has been demonstrated for a number of representative mechanical engineering applications.

In the case of the beam system with a one-sided spring support, the numerical results appeared to match very well with the experimental results. In the case of the beam system with an elastic stop the need has been demonstrated for adding more DOFs (higher eigenmodes) for an accurate description of even the low-frequency response. Because in this paper the impact is modelled using a continuous contact law, no grazing bifurcations can be found. In the neighbourhood of grazing impact, i.e. the beam just touches the elastic contact, in many cases period doubling routes occur. However, this is not always the case and not every period doubling route is in the neighbourhood of a frequency for which grazing impact occurs. Finally it is remarked that very complex non-linear dynamic behaviour has been found for both (simple but realistic) beam systems.

For the rotor-bearing system the results show that calculating the response of the system linearized around the stationary solution is only accurate for small-bearing loads. Even in the case of only a static load, the stability of the systems cannot be determined using the linearized system because the results show that even for angular rotor speeds somewhat lower than the threshold angular rotor speed for which the primary Hopf bifurcation occurs, periodic solutions (stable or unstable) can coexist with the stable stationary solutions and the initial state of the system will determine which attractor will be approached if time proceeds. Because of the existence of an unstable branch the system will immediately show a large displacement jump if the angular rotor speed passes the threshold rotor speed.

Furthermore, the results show that the eigenfrequencies of the system have a large influence on the self-excited oscillations. In all cases the threshold angular rotor speed is almost two times the first eigenfrequency of the linearized system. For angular rotor speeds lower than the threshold speed an unstable 1/2\( \omega \) whirl exists, which starts at the primary Hopf bifurcation and becomes stable via a cyclic fold bifurcation and bends off to higher
Fig. 15. Verification of basins of attraction (initial state: $x_i = 0$, $i = 1,2,3$): (a) full-contact harmonic solution; (b) coexisting harmonic solution; (c) quasi-periodic solution; (d) full-contact harmonic solution with 1/2 subharmonic transient.
angular rotor speeds. Meanwhile, the shaft eccentricities in the bearing show a large increase.

The potential of the MDCM method has been demonstrated for application to a two-DOF model of a portable CD player. For an initial region of interest, three different responses were found to be possible: two harmonic solutions and one quasi-periodic solution. By focusing on a small part of the initial region, realized by an MDCM application with very small cells, also a 1/2 subharmonic solution with a small domain of attraction was detected. A second result of this zooming action was the confirmation of the fractal structure of these parts of the basins of attraction.

Unlike the (sub)harmonic solutions found, the quasi-periodic solution features large accelerations and a large-amplitude motion. Since this solution has a large basin of attraction, large accelerations are very likely to occur. For the assurance of joggability, it is therefore necessary to cope with these kind of accelerations, or to change one or more system parameters in such a way that only low-acceleration solutions are likely to occur.

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