CHAPTER 8

Analogy Theory for a Systems Approach to Physical and Technical Systems

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Abstract

Research in the use of analogies in science and technology during the last two centuries reveals that, besides the striking successes of applications of these analogies, there are also dramatic failures, not to speak of sophistic misuses, in the logic reasoning.

Searching for the reasons for success and failure forces us to lose ourselves in a mechanism of analogy-thinking that is governed by logic, this resulting in the creation of a sharp division between analogy-reasoning and analogy-application.

By realistically starting from the idea that an analogy supposes, besides similarities, also differences, it is justifiable to make use of them in systems science. In order to achieve this, in this chapter some conditions are first formulated. Next examples are given that show that the use of analogies is of good service for modeling. With the help of bond-graphs, an analogy can be critically applied, such that it can be seen visually, physically, and mathematically simultaneously and in a recognizable way. The application of algebraic topology in this field makes it possible to design a dual system from a certain original system. Finally, some rigorous conclusions are drawn and recommendations are made with respect to justified use of analogies in multidisciplinary systems approaches. With respect to engineering design, some expectations are entertained.

1. Introduction

1.1. Definition

Coming from the Greek word *ana-logon*, literally "in proportion," one can define concisely the word *analog* as a concurrence of some aspects of phenomena that are otherwise essentially different. It is not infrequent that an analogy points primarily to a very large difference and secondarily to some resemblance.

1.2. Analogy-Use and Analogy-Reasoning

One can make use of analogies in a correct and justified way in order to elucidate and comprehend a complicated problem, an abstract representation, or an obscure relation between certain facts or a system, which is difficult to approach. In system science it is required that the use of an analogy can only be correct and justified if it is rigorously proved.

Analogy-reasoning is reasoning from which a correspondence between two entities in a certain aspect is implied by a similar correspondence in another connected respect. We try to prove something with analogies. The analogy proof is an argument that starts from a partial similarity between two entities, in order to conclude their complete similarity. Generalizing in this way can be taken too far. In applications, analogies have failed because the analogy was not rigorously proved. The attribution of properties to an entity by means of analogy hopefully reveals new phenomena and thereby advances knowledge, but it is always fortuitous, and as a systematic approach to the advance of knowledge, it is, in a way, an extraordinarily clumsy technique. The price of analogy is, indeed, eternal vigilance.

1.3. Critical View of the Use of Analogies

Historical examples show that analogies have been turned to good advantage in the infancy of certain sciences, but failed in later research due to their imperfections. Analogies certainly form a reliable guide to the phenomena we can expect, but they never give a definite insight into what we will discover. One hypothesizes on the basis of an analogy and tests it critically [31, 55, 85].

1.4. Criterion for Analogy-Use in Systems Science

If one wishes to use analogies critically, one must restrict oneself to entities between which analogies have been determined or proved. From here, interpretations can be made and conclusions drawn that are only valid with regard to the connected aspects.

1.5. Requirements for the Application of Analogy

Before one decides to use an analogy, three requirements have to be fulfilled:

1. mathematical analogy, which implies that two models obey identical sets of equations;
2. physical analogy, which implies that generalized components obey identical physical laws; and
3. visual analogy, which implies a visual association of shape between elements or systems.

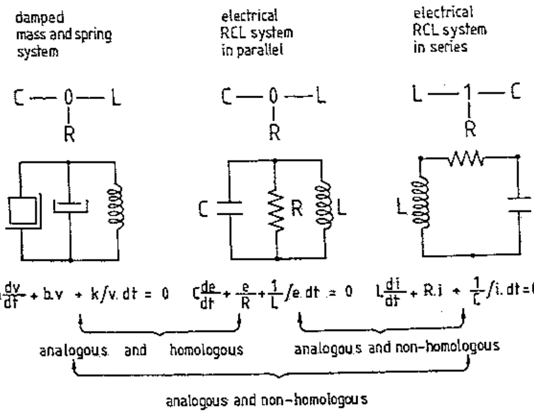


FIGURE 8.1. Distinction between analogy and homology.

1.6. Homology

Homology is defined as an abstraction of morphological resemblance between systems. By disregarding the component function, homology implies only a geometrical resemblance in system structure. The system and component functions are completely insignificant here. Analogous models that have identical mathematical functions need by no means be homologous. We can see this in Figure 8.1.

1.7. Duality

The peculiar developed relationship between analogy and homology can be explained by considering the concept of duality. Duality means ambiguity. On the one hand, one tries to formalize two different principles, conceptions, equations, or systems into one higher principle, broader conception, more universal equation, or more generalized system, respectively. On the other hand, if a system is modeled formally, one can often draw different possible conclusions about which seemingly contradictory (dual) interpretations can be made. In systems theory, duality results in a formalized system having two analogous, but nonhomologous models. In terms of the preceding example, the "series model" is a dual of the "parallel model."

2. Analogy Between Systems

In order to set up a system model quickly, a clear arrangement of thinking is required: *analogy-thinking*. It is therefore necessary to identify the analogy between systems and the analogy between treatment methodology. *Analogous systems* are systems obeying identical mathematical models, and *analogous treatment methods* are identical ways of treating systems, even if they are not

analogous. *Analogy-seeing* has to take place systematically by observing the above mentioned requirements. Before deciding on an analogy between systems, one has to check if each of the three partial analogies is fulfilled:

1. analogy between variables,
2. analogy between system components, and
3. analogy between system structures.

2.1. Analogy Between Variables

Firestone [22] introduced the distinction between two types of physical variable: *across-variables* and *through-variables*. Across-variables, like velocity, voltage, pressure, temperature, and concentration (chemical potential), are expressed as differences and are measured across two spatially distinct points. The corresponding through-variables, such as force, current, fluid flow, entropy flow, and mass flow, are measurable at a single point. The product of each corresponding variable pair has the dimension of power.

The work of Trent [84] provided a more rigorous basis for Firestone's "complete" analogies, as well as algorithmic rules for their construction, by the use of the linear graph concept and its associated matrix algebra. In Table 8.1, α- and τ-variables are displayed for some physically different systems.

The question whether this "across-through" concept is "correct," can be answered only after discussing another concept: the *effort-flow* concept. Based on this comparison, conclusions can be drawn with respect to the efficient use of one of these concepts.

2.1.1. Effort-Flow Concept (ef-analogy) [41]

In this concept a variable pair that consists of two variables is used, having an association with, respectively, a "flow" and an "effort."

TABLE 8.1. Through- and across-variables for physical systems. $\sigma = \int \tau \alpha t$; $\lambda = \int \alpha\, dt$.

System	Through-variable τ	Integrated through-variable σ	Across-variable α	Integrated across-variable λ
Mechanical-translational	Force F	Translational momentum p	Velocity difference v_{21}	Displacement difference x_{21}
Mechanical-rotational	Torque T	Angular momentum h	Angular velocity difference Ω_{21}	Angular displacement difference Θ_{21}
Electrical	Current i	Charge q	Voltage difference v_{21}	Flux linkage λ_{21}
Fluid	Fluid flow Q	Volume V	Pressure difference P_{21}	Pressure-momentum Γ_{21}
Thermal	Heat flow q	Heat energy $\mathscr{H}$	Temperature difference θ_{21}	Not used in general

TABLE 8.2. *ef*-analogy.

	Effort	Flow
Hydraulical	Pressure	Flow
Mechanical	Force	Velocity

TABLE 8.3. $\alpha\tau$-analogy.

	Across	Through
Hydraulical	Pressure	Flow
Mechanical	Velocity	Force

To be more illustrative, the energetic interpretation of these two variables in a hydraulical and mechanical energy domain is given in Table 8.2.

An argument in favor of this concept is the analogy in the example between the pressure p and the force F. Besides that, it is known that a hydraulic accumulator can be considered equivalent to a mechanical spring, by which it can be concluded that both are capacities. They store potential energy. For obvious reasons (from an electrical point of view), this concept is often called *mass-inductance analogy*.

Being correct, this *ef*-analogy is not efficient methodically, as we shall see later on.

2.1.2. *Across-Through Concept* ($\alpha\tau$-*analogy*) [22]

In this concept a variable pair is used that consists of two variables, having an association with, respectively, an "across-quantity" and a "through-quantity."

As with the *ef*-concept, Table 8.3 shows the interpretation of these abstract variables in a hydraulical and a mechanical energy domain. For obvious reasons (from an electrical point of view), this concept is often called *mass-capacitance analogy*.

The question to be asked is, What variable-pairs have most "natural correspondence" to one another? At first sight one should choose the *ef*-concept, because the pressure p and the force F represent "intensities" (how strong?), while flow Q and velocity v represent "extensities" (how much?). Regarding this choice, unfortunately there exists no unanimity in professional circles. Supporters of the *ef*-concept refer to fundamental-physical arguments, while supporters of the $\alpha\tau$-analogy give preference to analogy-aspects in the systematical and methodical approach, thereby using homology-considerations (this is analogy with special attention to form and structure resemblance).

Supporters of the $\alpha\tau$-concept consistently make use of analogy between "single-point" and "two-point" measurements, which emanates from the basic

idea that a system can be described only if it can be inspected and observed (by means of the two topologically describable variables!). Because of the topological nature of the $\alpha\tau$-concept, it belongs to cohomology theory, to be discussed further on.

The choice of the $\alpha\tau$-concept by the authors of this chapter comes from the pragmatic nature of systems science, which aims at uniformity in a systematical and methodical approach and consideration. In contrast with the $\alpha\tau$-concept, the *ef*-concept has to make use of two different methods (dual to each other!) in the modeling process [41]

2.2. Analogy Between System Components

In physically different energy domains, one comes across certain components with analogous functions. In Table 8.4 a functional classification is shown of analogous components with matching (generalized) function descriptions defined in terms of α- and τ-variables.

2.3. Analogy Between System Structures

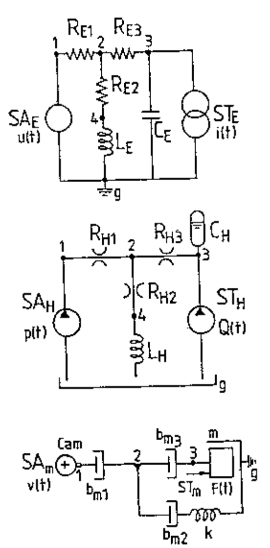
If two physically different systems display identical structures, they are called *structurally analogous*. This means that they possess identical structure equations.

2.4. Definition of Analogous Systems

If all three partial analogies are fulfilled, we speak of *analogous systems*. In Figure 8.2 analogous electrical, hydraulical, and mechanical systems are drawn, together with their three partial analogies.

TABLE 8.4. Analogous components.

Generalized resistor $\tau = \alpha/R$	with R = generalized ideal resistance $1/b$ = reciprocal translational damping $\mathbf{R}$ = electrical resistance $\mathbf{R}_f$ = fluid resistance $\mathbf{R}_t$ = thermal resistance
Generalized capacitor $\tau = C\ d\alpha/dt$	with C = generalized ideal capacitance m = mass $\mathbf{C}$ = electrical capacitance $\mathbf{C}_f$ = fluid capacitance $\mathbf{C}_t$ = thermal capacitance
Generalized inductor $\alpha = L \cdot d\tau/dt$	with L = generalized ideal inductance $1/k$ = reciprocal translational stiffness $\mathbf{L}$ = electrical inductance $\mathbf{L}_f$ = fluid inertance



1. *Variable analogy*

general	α	τ
electrical	U	I
hydraulical	P	Q
trans. mech.	V	F

2. *Component analogy*

$R_{E1} = R_{H1} = 1/b_{m1}$
$R_{E2} = R_{H2} = 1/b_{m2}$
$R_{E3} = R_{H3} = 1/b_{m3}$
$L_E = L_H = 1/K$
$C_E = C_H = m$
$SA_E = SA_H = SA_m$
$ST_E = ST_H = ST_m$

3. *Structure analogy*

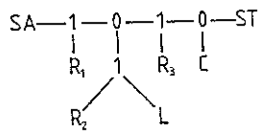


FIGURE 8.2. Complete analogy between electrical, hydraulical, and mechanical systems.

2.5. Analogy Between Methods of Derivation

In the systems science literature, one comes across two different methods for setting up so-called structure equations: linear graphs [13] and bond-graphs. It has to be mentioned that this chapter presupposes some knowledge of bond-graph theory. Outsiders who wish to become acquainted with $\alpha\tau$- and *ef*-bond-graphs are referred to

—$\alpha\tau$-bond-graphs: [10, 33, 34, 88];
—*ef*-bond-graphs: [41, 69, 78] and Chapter 9, "BondGraphs for Qualitative and Quantitative System Modeling" by Jean U. Thoma, in this volume.

Suppose that a bond-graph is set up with the power direction arrows pointing toward the elements, including sources, TFs, GYs, and multiports. Supposing then that the powers absorbed in and produced by the energy port are, respectively, positive and negative, a generalized method can be constructed for deducing the structure equations. Figure 8.3 illustrates this method.

Only the energy ports are numbered, not the bonds. Two groups of components are formed (Table 8.5). In the column vector on the left-hand side of the matrix structure equation, one stacks the through-variables of group $\leftharpoondown\!|$ on top of the across-variables of group $\rightharpoonup\!|$. In the column vector on the right-hand side, one stacks the across-variables of group $\leftharpoondown\!|$ on top of the across variables of $\rightharpoonup\!|$. This is done in index rank order.

Application of generalized laws of Kirchhoff ($\sum \tau_i = 0$ and $\sum \alpha_j = 0$) yields two submatrices B and C. Rearranging the above-mentioned variables in rank

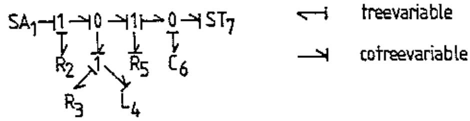


FIGURE 8.3. Causal bond-graph, an abstraction of the three analogous systems of Figure 8.2.

TABLE 8.5. Distinction between tree- and cotree-variables.

	Group	System components	Port numbers
Tree	$\leftarrow\!\!\dashv$	SA_1, R_2, R_3, C_6	1, 2, 3, 6
Cotree	$\rightarrow\!\!\dashv$	L_4, R_5, ST_7	4, 5, 7

order (first $\leftarrow\!\!\dashv$, then $\rightarrow\!\!\dashv$) gives the complete structure equation:

$$
\left.
\begin{aligned}
\underline{\tau}_t = \mathrm{B}\underline{\tau}_c \rightarrow \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_6\end{bmatrix} &= \begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}\tau_4\\ \tau_5\\ \tau_7\end{bmatrix}\\
\underline{\alpha}_c = -\mathrm{C}_t^{\mathrm{T}}\underline{\alpha}_t \rightarrow \begin{bmatrix}\alpha_4\\ \alpha_5\\ \alpha_7\end{bmatrix} &= \begin{bmatrix}-1 & -1 & -1 & 0\\ -1 & -1 & 0 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_6\end{bmatrix}
\end{aligned}
\right\}
\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_6\\ \alpha_4\\ \alpha_5\\ \alpha_7\end{bmatrix}
$$

$$
= \begin{bmatrix}
 & & & & 1 & 1 & 0\\
 & \underline{0} & & & 1 & 1 & 0\\
 & & & & 1 & 0 & 0\\
 & & & & 0 & 1 & -1\\
-1 & -1 & -1 & 0 & & & \\
-1 & -1 & 0 & -1 & & \underline{0} & \\
0 & 0 & 0 & 1 & & &
\end{bmatrix}
\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_6\\ \tau_4\\ \tau_5\\ \tau_7\end{bmatrix}
$$

with τ_t = tree-throughvariables, τ_c = cotree-throughvariables, α_c = cotree-acrossvariables, and α_t = tree-acrossvariables. A known network theory property is revealed: $\mathrm{B} = \mathrm{C}_t^{\mathrm{T}}$ (T = transposed).

The relationship with linear graphs is immediately clear. Indeed, one recognizes group $\leftarrow\!\!\dashv$ as edges of the "tree" of a linear graph, and group $\rightarrow\!\!\dashv$ as the edges of the "cotree." For bond-graph theory, one can obviously reap the fruits of network theory for linear graphs, that is, network synthesis, network optimization, interpretation possibilities by means of reciprocity, passivity and symmetry theorems, and network analysis of three- and multidimensional networks. Famous works such as Gabriel Kron's "Diakoptics" [46] can then be studied with less effort.

3. Duality Between Systems

In order to determine a duality between systems, it is also necessary to identify three partial dualities:

1. duality between variables,
2. duality between system components, and
3. duality between system structures.

3.1. Duality Between Variables

Because through- and across-variables can be formalized into the higher concepts of "physical quantities" the α-variables of a certain system can dually be understood as the τ-variables of another system, and vice versa. The energy nature remains unchanged.

3.2. Duality Between System Components

It is striking that, with regard to a generalized inductor, α equals a constant times the time derivative of τ, and that for a generalized capacitor τ equals a constant times the time derivative of α. By interchanging the αs and τs, one can obtain the dual system components, such as in Table 8.6.

3.3. Duality Between System Structures

In the dual case, interchanging αs and τs leads to dual structure equations, as given in Table 8.7.

3.4. Definition of Dual Systems

If two given systems contain all three partial dualities, then they are dual to one another.

TABLE 8.6. Component duals.

Original		Dual	
Impedance	Z	Admittance	Y
Admittance	Y	Impedance	Z
Inductor	L	Capacitor	C
Capacitor	C	Inductor	L
Resistor	R	Conductor	G
Conductor	G	Resistor	R
α-source	SA	τ-source	ST
τ-source	ST	α-source	SA
Transformer	TF	Dual transformer	TF
Gyrator	GY	Dual gyrator	GY

TABLE 8.7. Structure duals.

Original	Dual
0	1
1	0
B	$-C_t^T$
$-C_t^T$	B

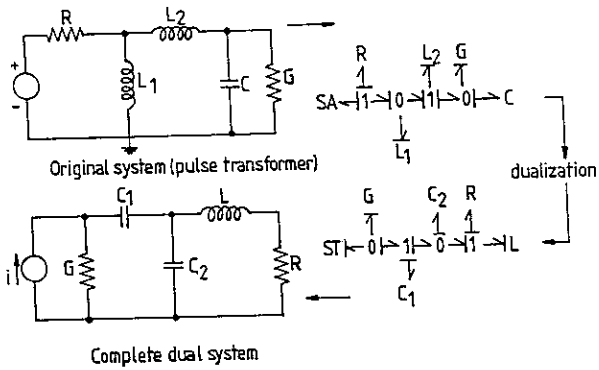


FIGURE 8.4. Dualization process.

4. Examples

4.1 *First illustrative example: Dualization process.* In this example the essence of a dualization process is briefly demonstrated with help of an electrical circuit, pictured in Figure 8.4.

Furthermore, it is possible to get the dual system component from the graphical characteristic of the original system component by constructing the corresponding dual characteristic.

4.2 *Second worked example: Thévenin versus Norton.* In Figure 8.5a and b, a Thévenin and a Norton network are shown. From electrical engineering it is known that these networks are each other's duals. This can be ascertained with help of Tables 8.6. and 8.7.

A more "abstract concept" for these networks appears to be the so-called "Kron-equivalent," portrayed in Figure 8.5c. It is exactly this equivalent that enables the incorporation of bond-graph theory and network theory, to be discussed later on, into cohomology theory.

4.3 *Third worked example: Constructing the dual from a pressure control valve.* In this example two devices are discussed that, after careful analysis,

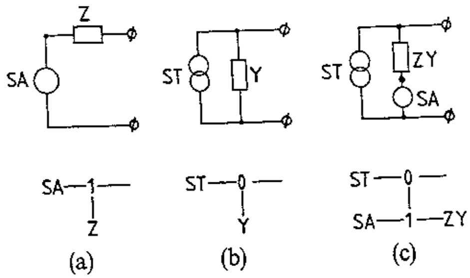


FIGURE 8.5. (a) Thévenin, (b) Norton, and (c) Kron equivalents.

appear to be each other's dual. The systems concerned are a pressure control valve and a flow control valve. By means of a systematic approach it is demonstrated that after

1. a carefully carried out analysis of the pressure control valve, resulting in a bond-graph,
2. a carefully carried out dualization process of this bond-graph, and
3. a carefully carried out synthesis of this bond-graph into a physical realization,

a flow control valve has been "designed." The analysis—respectively, synthesis—"route" stepwisely form each other's dual, while the procedure in both routes runs analogously!

Analogy theory has been successfully used here to design a dual system from a given system. Both processes are portrayed in Figures 8.6a and b. In the left part of this figure, the pressure control valve is shown. Here two "controllable" parameters can be localized, namely, the hydraulical resistance R_0 and the spring stiffness $1/L_0$. In the right part of the figure, the flow control valve is shown. Here the two "controllable" parameters are the hydraulical resistance R_d and the hydraulical inertance L_d.

Consistent handling of the duality-principle compels us to remark that "intensitivity" is the dual of "sensitivity." Now, in the case of the pressure control valve, R_0 and $1/L_0$, respectively, are the "insensitive" and the "sensitive" parameters, while in the case of the flow control valve, R_d and L_d, are the "sensitive" and the "insensitive" parameters, respectively.

In practice the adjustment parameters for the pressure control valve and the flow control valve, respectively, are $1/L_0$ and R_d. The subsequent steps in the illustrations are evident.

4.4 *Fourth worked example: Design of a controlled speed regulator.* In this example an analysis is made from a controlled frictional clutch, in order to construct a dual physical system from it. For this "original ↔ dual" transformation, in Figure 8.7, multiport representations are shown.

By reason of duality considerations, the dual system turns out to be a controlled speed regulator.

Now, how can it be constructed? From the abstract scheme of Figure 8.8,

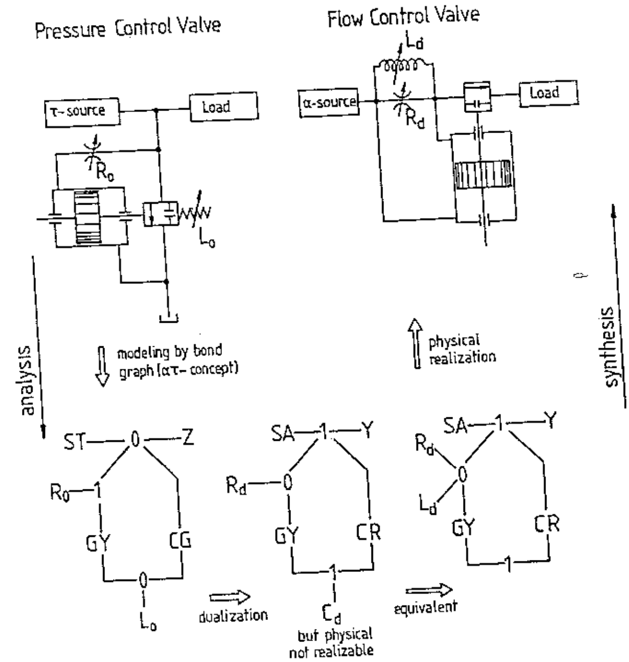


FIGURE 8.6. Designing a flow control valve from a pressure control valve.

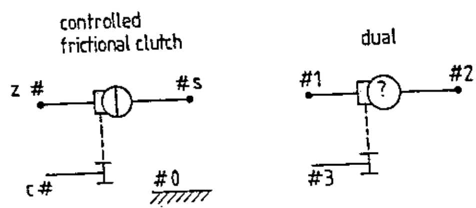


FIGURE 8.7. Controlled frictional clutch and its unknown dual.

a bond-graph is set up in Figure 8.9, which is transformed, after some pre-manipulation in Figure 8.10a, into the dual bond-graph in Figure 8.10b.

In Figure 8.9 CG is a controlled conductance and the section of the bond-graph associated with it should be read as; $\rightharpoondown\!|$ CG $\leftharpoondown\!|$, according to the equation for the torque due to coulomb friction:

$$T = T(N, \omega) = f.N.R. \qquad \text{where } f = \text{friction coefficient,}$$
$$N = \text{controlled normal force,}$$
$$R = \text{radius,}$$
$$\omega = \text{rotational speed,}$$
$$T = \text{torque.}$$

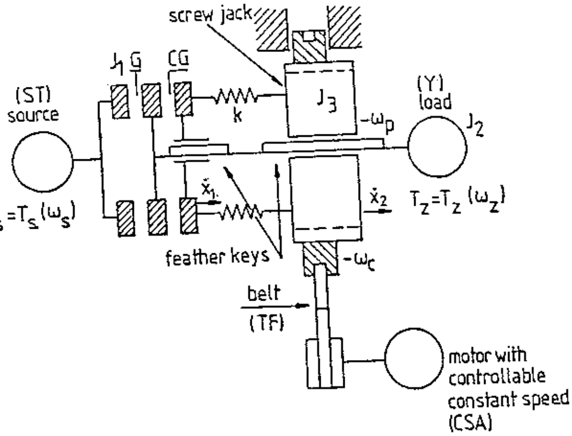


FIGURE 8.8. Controlled frictional clutch.

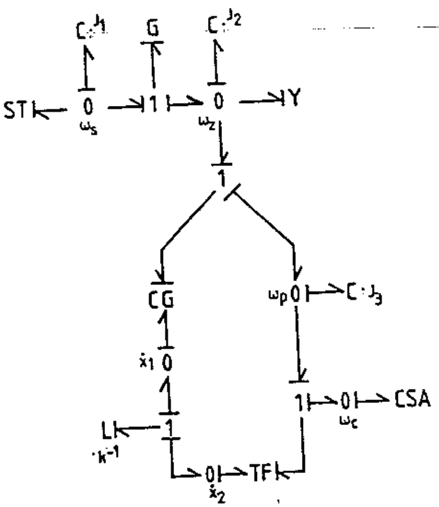


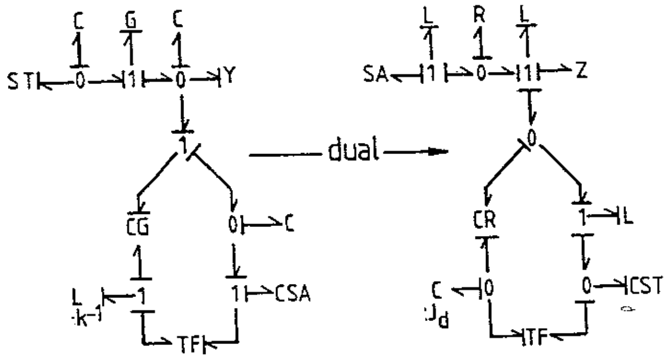
FIGURE 8.9. Bond-graph for Figure 8.8.

After reduction, the bond-graph looks like Figure 8.10a, where

$$C \leftarrow\!| 0 \qquad 0 \rightarrow\!| CST$$
$$\llcorner\!\rightarrow\!| TF \leftarrow \lrcorner$$

equals

$$C \leftarrow\!| 0 \qquad\qquad\qquad 0 \rightarrow\!| CST$$
$$\llcorner\!\rightarrow\!| GY_2 \leftarrow\!| 0 \mapsto GY_1 \leftarrow \lrcorner$$



(a) (b)

FIGURE 8.10. (a) Reduced bond-graph. (b) "Nonrealization" dual for (a).

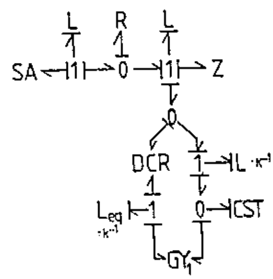
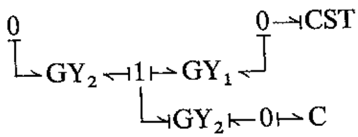
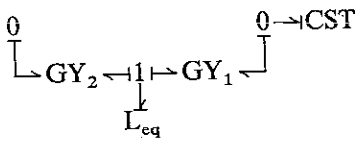


FIGURE 8.11. "Realizable" dual for Figure 8.10a.

equals



equals



In the dual bond-graph in Figure 10b, the controlled resistance CR is the dual of CG.

By finally combining $\vdash\!\rightharpoonup$CR$\,\vdash\!\!-0\vdash\!\rightharpoonup GY_2 \rightharpoondown\!\dashv$ to $\vdash\!\rightharpoonup$DCR$\rightharpoondown\!\dashv$(read as "to dual-controlled resistance") in Figure 10b, a transformation into a physical system is then constructed for the dual bond-graph in Figure 8.11. In Figure 8.12 the physical realization of Figure 8.11 is shown.

It is not the purpose of this example to claim that every dual system component can be realized physically. On the contrary, it appears that most

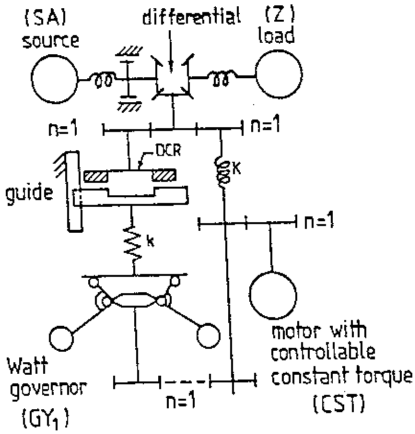


FIGURE 8.12. Centrifugal clutch: Realization of Figure 8.11

duals are physically not realizable. From a technical point of view, a dual from Figure 8.10b is more difficult to construct than a dual from Figure 8.11.

5. Choice Between Analogy and Dualogy

A *dualog* is defined as an analog of a dual or a dual of an analog, as shown in Figure 8.13.

The message of the contribution presented here, is to give preference to analogies above dualogies. First, the process of "analogy-seeing" with analogies is essentially easier than with dualogies, as is confirmed by the structural similarities between the systems in Figure 8.14. Second, for a "nonplanar system," as portrayed in Figure 8.15, a dualog cannot always be constructed, whereas an analog always can. The adjective *nonplanar* comes from graph theory and refers to a certain property of a graph. Here, a planar graph is a graph that can be embedded (and may be deformed) in a plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. The graph in Figure 8.16a is fundamentally nonplanar, as can be seen from Figure 8.16b. The linear graph of Figure 8.15 is, according to the definition, also nonplanar (see Figure 8.15c).

In 1932 Whitney [90] proved that from a nonplanar network one can never construct a dual. As long as a mechanical system can be represented by a planar graph, no practical difficulties arise from using analogs or dualogs because of the "planar graph theorem." When the graph is nonplanar, however, the mass$\leftrightarrow$inductance analogy is the one that fails. The mass$\leftrightarrow$capacitance analogy always applies because of its topological equivalence. "For pedagogical reasons, therefore, the mass$\leftrightarrow$inductance analogy owing its existence solely to the planar graph theorem, should be discarded in favor of the mass$\leftrightarrow$capacitance analogy which is fundamentally the correct one" [90]. According to Breedveld, this is too strong, since "the planar graph

FIGURE 8.13. Relations between duals, analogs, and dualogs.

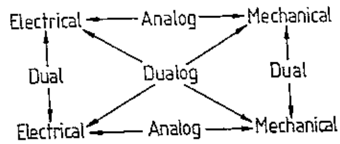
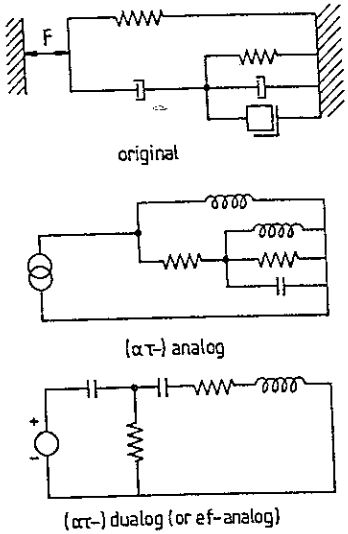


FIGURE 8.14. Problem of recognizing analog versus dualog.



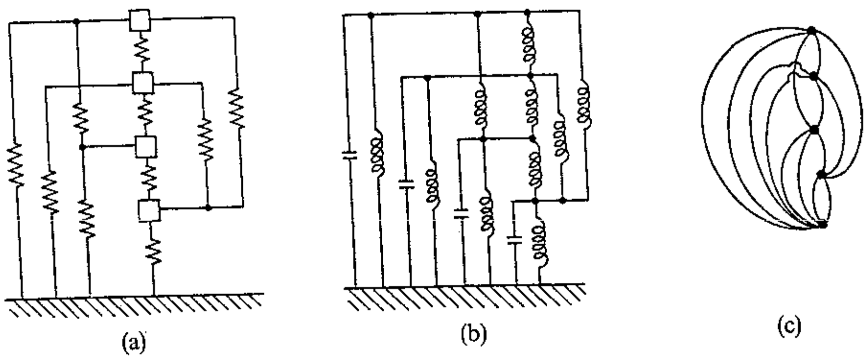
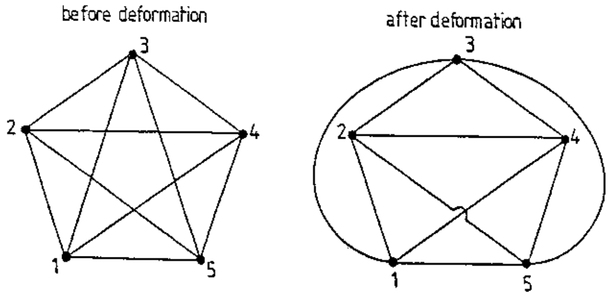


FIGURE 8.15. Mechanical system for which mass ↔ inductance analogy does not exist.

theorem states only planar graphs can be dualized. The fact however that some systems can be dualized and some not, does not provide a criterion to choose between the mass ↔ inductance or mass ↔ capacitance analogy" [13]. Indeed, no final verdict can be delivered, because the choice of analogs for mechanical systems depends on the requirements of the modeling techniques for the structure, which differ from application to application. Although it is



(a) (b)

FIGURE 8 16. Definition of a nonplanar graph.

pointless to discuss the "correct" analogy, since both are equally valid when they exist, the mass ↔ capacitance (or $\alpha\tau$- or mobility-) analogy has considerable advantages. This is specially so in the process of analogy-seeing:

1. The mass ↔ capacitance analogy is set up very easily;
2. for every possible mechanical combination (planar or nonplanar), an electric analog exists, whereas an electric dualog does not necessarily exist;
3. the topological concept allows us to construct analogs visually;
4. in the modeling process, it suffices to use only one procedure, whereas in mass ↔ inductance analogy, two procedures are necessary: one for mechanical and one for nonmechanical systems;
5. analogs of nonplanar systems are always physically interpretable (realizable), whereas duals or dualogs of nonplanar systems are not.

6. Analogy Between Physical Theories

As it appears to be possible to set up analogous systems in physically different domains, one's intuition suggests that a common analogy for physical theories must also exist, such that we may generalize electrical, hydraulical, mechanical, thermodynamical, etc., theories. This will become apparent, when we investigate how physical theories have been constructed over the last four centuries with the aid of geometry: There can hardly be a physical theory without geometry! [47, 66].

Just as Tonti [83] suggests, "in every physical theory there are basic physical quantities that are naturally referred to the most simple geometrical and chronometrical elements like points, lines, surfaces, volumes, time instants and time intervals and combinations of them," with the consequence that "in every physical theory there are basic physical laws that a physical quantity referred to a p-dimensional manifold ω like lines, surfaces, volumes, time intervals, etc. is equal to a physical quantity referred to its boundary $\partial\omega$". According to Tonti, this leads to "the possibility of doing the rational investigation of the

analogies between two physical theories according to the criterion that follows: to every physical quantity of one theory there corresponds that physical quantity of the other theory that is referred to the same geometrical entry."

Tonti proved that this was true. The instigator of these ideas was Gabriel Kron, who continually emphasized that there must be an underlying justification for the proliferation of electrical and mechanical analogs used in modeling a diverse range of physical phenomena. He introduced the use of homology theory (topology) in this context as the basis of his network and systems theory. By extending Kron's and more recent works into a unified theory beginning with a topological analysis of "system invariance," this leads to systematic procedures for deriving and solving matrix and tensor models, and efficient solution algorithms and a philosophy that should contribute toward the understanding and teaching methodology in general [11]. Branin [12] presents a treatment of the network concept not only in its usual context related to linear graph theory but also in relation to cohomology theory. This theory uses complexes and simplexes, homology and cohomology sequences, chains and cochains, and boundary and coboundary operators. The physical interest in the process of constructing sequential chains arises from the fact that the structure equations of every physcial theory state that "one chain is the coboundary of another."

In conclusion, the sensible way in which to achieve a deep understanding of cohomology theory appears to us to be to demonstrate, first with a simple example of network analysis and then by a more complex example from electromagnetism, that the whole of physics is based on topology (popularly said, "physics is geometry" and, moreover, that cohomology theory appears to be nothing other than a very consistent applied analogy theory). Here we attempt to give a new perspective on the work of Kron, Roth, Branin, and Nicholson [12, 46, 58, 70].

6.1. Simple Example: System Network Analysis

Take, for example, the generalized system model of Figure 8.2 with the associated causal bond-graph of Figure 8.3. By way of a particular procedure, as described in [7], [32], and [61], this bond-graph is transformed into the linear graph of Figure 8.17, with tree and cotrees as indicated. This linear

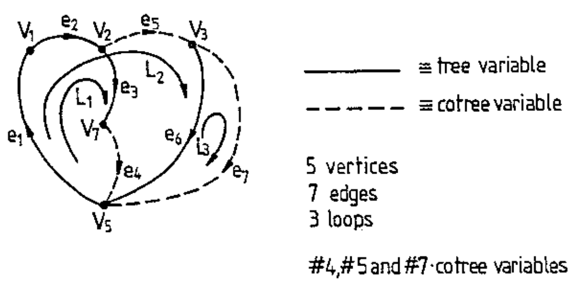


FIGURE 8.17. Linear graph for Figure 8.3.

graph is topologically representable only in terms of vertices ($\in$V), edges ($\in$e) and loops ($\in$L), such that the structure of it can be represented in the incidence matrices H(0, 1) and H(1, 2) below. These matrices, respectively, define the relationship between vertices and edges and between edges and loops, and thereby fully identify the system structure.

		Edges						
		e_1	e_2	e_3	e_6	e_4	e_5	e_7
Vertices	V_1	1	−1	0	0	0	0	0
	V_2	0	1	−1	0	0	−1	0
	V_3	0	0	0	−1	0	1	−1
	V_4	0	0	1	0	−1	0	0
	V_5	−1	0	0	1	1	0	1

		Loops		
		L_1	L_2	L_3
Edges	e_1	1	1	0
	e_2	1	1	0
	e_3	1	0	0
	e_6	0	1	−1
	e_4	1	0	0
	e_5	0	1	0
	e_7	0	0	1

From this follows

$$H(0,1) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = A^T;$$

$$H(1,2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C.$$

From this follows that

$$A^T\underline{\tau} = \underline{0}; \tag{1}$$

$$\underline{\alpha} = A\underline{\alpha}_0; \tag{2}$$

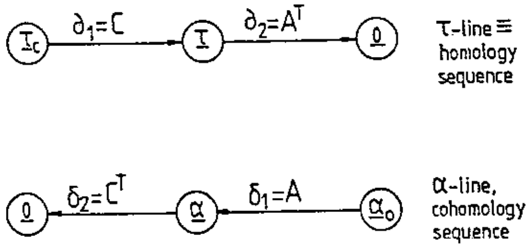


FIGURE 8.18. Topological string without impedance-link.

$$C^T\underline{\alpha} = \underline{0}; \tag{3}$$

$$\underline{\tau} = C\underline{\tau}_c. \tag{4}$$

with $\underline{\alpha}$ = across-variables with $\underline{\alpha}^T = [\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_4, \alpha_5, \alpha_7]$; $\underline{\tau}$ = through-variables with $\underline{\tau}^T = [\tau_1, \tau_2, \tau_3, \tau_6, \tau_4, \tau_5, \tau_7]$; $\underline{\alpha}$ = across-potentials with respect to a reference frame; and $\underline{\tau}_c$ = cotree through-variables with $\underline{\tau}_c^T = [\tau_4, \tau_5, \tau_7]$.

We shall now construct a signal flow diagram with the aid of eqs. (1) to (4). This diagram is called a (*topological*) *string*. Figure 8.18 shows two bilateral signal flows, one for τ to the right and the other for α to the left.

The multiplication by juxtaposition of the two operators on each line gives

$$A^TC = \underline{0} \quad \text{for } \tau\text{-line}; \tag{5}$$

$$C^TA = \underline{0} \quad \text{for } \alpha\text{-line}. \tag{6}$$

The proof of the correctness of (5) and (6) can only be given thermodynamically. Tellegen's theorem [63, 77] states the following:

$$P = \underline{\tau}^T\underline{\alpha} = 0 \tag{7}$$

and

$$P = \underline{\alpha}^T\underline{\tau} = 0. \tag{8}$$

Substitution of (2) and (4) in (7) gives

$$P = (C\underline{\tau}_c)^T A\underline{\alpha}_0 = \underline{\tau}_c^T C^T A\underline{\alpha}_0 = 0,$$

from which it follows that $C^TA = \underline{0}$. In a similar way (substitution of (2) and (4) in (8)), it follows that $A^TC = \underline{0}$.

Equations (5) and (6) are the interpretations of the first law of thermodynamics.

Finally, in order to obtain the constitutive relationship between the τ- and α-sequence, we introduce the *Kron-equivalent*. The Kron-equivalent is defined as a combination of one passive element Z and two (active) sources, SA and ST. This "Kron-element" can thus be used as a passive element and/or as a source. Figure 8.19 illustrates this abstract concept and the acausal bond-graph associated with this Kron-element. For this Kron-element, the following set of element equations holds:

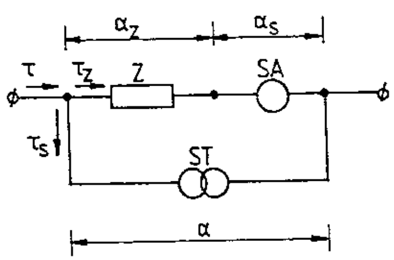
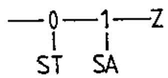


FIGURE 8.19. Visual definition of the Kron-element.



$$\left.\begin{aligned} \alpha_z &= Z\tau_z \\ \alpha_z &= \alpha - \alpha_s \\ \tau_z &= \tau - \tau_s \end{aligned}\right\} \text{ with }$$

Z = impedance,

α_z, τ_z = across- and through-variables of the passive elements (A, L, C),

α_s, τ_s = across- and through-variables of the source.

It appears that the Kron-element can be interpreted as a more highly abstracted version of Thévenin and Norton equivalents (see Example 4.2).

With the introduction of this type of generalized element, it is possible to consider every general bond-graph structure as being composed of n Kron-elements. In vector notation, the element equations now are

$$\left.\begin{aligned} \underline{\alpha}_z &= Z\underline{\tau}_z \\ \underline{\alpha}_z &= \underline{\alpha} - \underline{\alpha}_s \\ \underline{\tau}_z &= \underline{\tau} - \underline{\tau}_s \end{aligned}\right\} \quad \begin{aligned} &Z = \text{matrix impedance}, \\ &\text{underscore} = \text{vector symbol}. \end{aligned} \tag{9}$$

After applying a vector link Z between the τ-sequence and the α-sequence in the string of Figure 8.18, it holds that

$$(\underline{\alpha} - \underline{\alpha}_s) = Z(\underline{\tau} - \underline{\tau}_s). \tag{10}$$

This process is demonstrated in Figure 8.20. Premultiplication of (10) by C^T yields

$$C^T(\underline{\alpha} - \underline{\alpha}_s) = C^T Z(\underline{\tau} - \underline{\tau}_s). \tag{11}$$

After substitution of (3) and (4), we get

$$(\underline{0} - C^T\underline{\alpha}_s) = C^T Z C\underline{\tau}_c - C^T Z\underline{\tau}_s. \tag{12}$$

The Z-string in Figure 8.21 gives a signal flow interpretation of (12).

As far as the inversion of C^TZC $(=(C^TZC)^{-1})$ (which always exists [51]) is concerned, premultiplication of (12) with $(C^TZC)^{-1}$ and some additional manipulations leads to

$$-(C^TZC)^{-1}C^T\underline{\alpha}_s = (C^TZC)^{-1}C^TZ\underline{\tau} - (C^TZC)^{-1}C^TZ\underline{\tau}_s,$$

FIGURE 8.20. Z-link between the τ- and α- sequence.

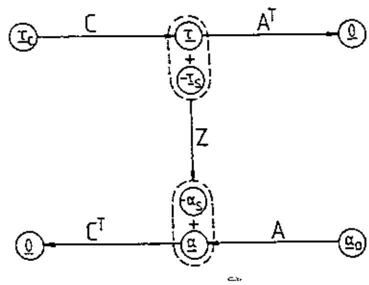
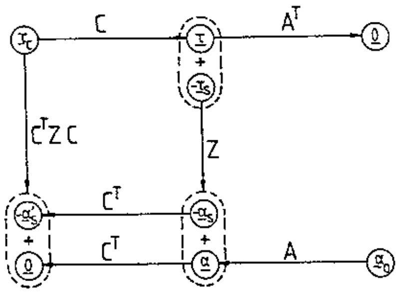


FIGURE 8.21. Z-string. $\underline{\alpha}'_s = C^T\underline{\alpha}_s$



which, after substitution of (4), gives

$$-(C^TZC)^{-1}C^T\underline{\alpha}_s = \underline{\tau}_c - (C^TZC)^{-1}C^TZ\underline{\tau}_s$$

or

$$-C(C^TZC)^{-1}C^T\underline{\alpha}_s = \underline{\tau} - C(C^TZC)^{-1}C^TZ\underline{\tau}_s. \tag{13}$$

By premultiplying the inverse of (10), $(\underline{\tau} - \underline{\tau}_s) = Y(\underline{\alpha} - \underline{\alpha}_s)$ (with Y = matrix admittance), by A^T and substituting (1) and (2), we arrive in a similar way at

$$(\underline{0} - A^T\underline{\tau}_s) = A^TYA\underline{\alpha}_0 - A^TY\underline{\alpha}_s \tag{14}$$

and thereafter at

$$-A(A^TYA)^{-1}A^T\underline{\tau}_s = \underline{\alpha} - A(A^TYA)^{-1}A^TY\underline{\alpha}_s. \tag{15}$$

In a similar way the corresponding Y-string for (15) is constructed and shown in Figure 8.22.

Figure 8.23 shows how Figure 8.21 and Figure 8.22 are superimposed upon one another.

Equations (13) and (15) are the "singular" answer to the "network" problem, formulated by Gabriel Kron and J. Paul Roth, which can be characterized in a generalized way as "given a network of known topology, known elements C, G, R, and/or L, and known across-and through-sources $\underline{\alpha}_s$ and $\underline{\tau}_s$, find the response across- and through-variables $\underline{\alpha}$ and $\underline{\tau}$ such that the constitutive laws and topological constraints are satisfied simultaneously" [70]. This can be translated as "given (1) a bond-graph, which determines the matrices A and

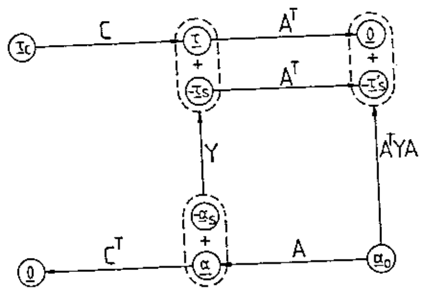


FIGURE 8.22. Y-string $\underline{\tau}'_s = A^T\underline{\tau}_s$.

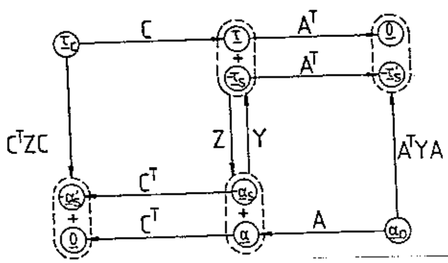


FIGURE 8.23. H-string: The combination of the Z-string and Y-string.

C, (2) the transformation matrix Z and/or its inverse Y, and (3) the arbitrary source vectors $\underline{\alpha}_s$ and $\underline{\tau}_s$, find the vectors $\underline{\alpha}$ and $\underline{\tau}$ such that (1) $\underline{\alpha} - \underline{\alpha}_s = Z(\underline{\tau} - \underline{\tau}_s)$ and/or $\underline{\tau} - \underline{\tau}_s = Y(\underline{\alpha} - \underline{\alpha}_s)$, (2) $A^T\underline{\tau} = 0$, and (3) $C^T\underline{\alpha} = 0$.

We call (C^TZC) and (A^TYA), respectively, the Roth-transformations of system impedance and system admittance, which are used by Kron as tensors [12, 14, 45, 70, 71, 74].

One recognizes (13) and (15), respectively, as the generalized "node" method and "loop" method, by which the network equations are set up (see, e.g., [9], [51], and [72]. In [51] these methods are combined in the so-called "mixed" method, of which Figure 23 is a visual representation.

There is also a third generalized method, which leads to state equations: the so-called MacFarlane method [50]. In general, one proceeds as follows: First,

$$C^T\underline{\alpha} = [C_t^T|I]\begin{bmatrix}\underline{\alpha}_t \\ \underline{\alpha}_c\end{bmatrix} = \underline{0} \qquad ([I] \equiv \text{unit matrix})$$

or

$$C_t^T\underline{\alpha}_t + \underline{\alpha}_c = \underline{0}. \tag{16}$$

Second,

$$A^T\underline{\tau} = [A_t^T|A_c^T]\begin{bmatrix}\underline{\tau}_t \\ \underline{\tau}_s\end{bmatrix} = \underline{0}$$

or

$$A_t^T\underline{\tau}_t + A_c^T\underline{\tau}_c = \underline{0}$$

or

$$\underline{\tau}_t + (A_t^T)^{-1}A_c^T\underline{\tau}_c = \underline{0}. \tag{17}$$

Premultiplication of (16) by $\underline{\tau}_c^T$ yields

$$\underline{\tau}_c^T C_t^T \underline{\alpha}_t + \underline{\tau}_c^T \underline{\alpha}_c = \underline{0}. \tag{18}$$

Premultiplication of (17) by $\underline{\alpha}_t^T$ gives

$$\underline{\alpha}_t^T \underline{\tau}_t + \underline{\alpha}_t^T (A_t^T)^{-1} A_c^T \underline{\tau}_c = \underline{0}. \tag{19}$$

Summation of (18) and (19) yields

$$-\underline{\alpha}_t^T[C_t + (A_t^T)^{-1}A_c^T]\underline{\tau}_c = \underline{\tau}_c^T\underline{\alpha}_c + \underline{\alpha}_t^T\underline{\tau}_t = \underline{\tau}_c^T\underline{\alpha}_c + \underline{\tau}_t^T\underline{\alpha}_t = \underline{\tau}^T\underline{\alpha}. \tag{20}$$

According to Tellegen's theorem, the right-hand side of eq. (20) equals zero It can be shown that

$$(A_t^T)^{-1}A_c^T = -C_t. \tag{21}$$

This is to say that, if

$$\underline{\alpha}_c = -C_t^T\underline{\alpha}_t,$$

then

$$\underline{\tau}_t = C_t\underline{\tau}_c, \tag{22}$$

which implies that

$$\begin{bmatrix} \underline{\tau}_t \\ \underline{\alpha}_c \end{bmatrix} = \begin{bmatrix} \underline{0} & C_t \\ -C_t^T & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{\alpha}_t \\ \underline{\tau}_c \end{bmatrix}. \tag{23}$$

Substitution of the element equations of the form

$$\underline{\alpha} = Z(\underline{\tau} - \tau_s) + \underline{\alpha}_s \quad \text{and/or} \quad \underline{\tau} = Y(\underline{\alpha} - \alpha_s) + \underline{\tau}_s$$

with $\underline{\tau}^T = [\underline{\tau}_t^T, \underline{\tau}_c^T]$ and $\underline{\alpha}^T = [\alpha_t^T, \alpha_c^T]$ in (23) gives, after elimination of certain algebraic equations, the state equations of the following form:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \quad \text{with } [\underline{x}]^T = [\underline{\alpha}_t^T, \underline{\tau}_c^T] \quad \text{and} \quad [\underline{u}]^T = [\underline{\tau}_{st}^T, \underline{\alpha}_{sc}^T], \tag{24}$$

where t = tree, c = cotree, st = tree-source, and sc = cotree-source.

6.2. Algebraic-Topological Interpretation of the Roth-Problem [12]

Given is the following system description:

1. a set of elements;
2. between these elements, certain relationships exist; and
3. certain properties are assigned to the elements and/or relations.

By interpreting Figure 8.2 and Fig. 8.17 topologically, this system description formally abstracts to a topological description that is characterized by three aspects:

1. a set V, consisting of vertices;
2. a set E, consisting of edges; and
3. a relationship between the edges and vertices.

In the Appendix, simplicial topology with its associated terminology, is presented "in a nutshell." The earlier treated example of Figure 8.2 is analyzed again, but now topologically.

Given the finite oriented two-dimensional simplicial complex K from Figure 8.2 and Figure 8.17, with

$$s_0 = 5 \quad \text{0-simplexes: } \sigma_{0;1}, \ldots, \sigma_{0;5} \quad (= V_1, V_2, V_3, V_4, V_5),$$

$$s_1 = 7 \quad \text{1-simplexes: } \sigma_{1;1}, \ldots, \sigma_{1;7} \quad (= e_1, e_2, e_3, e_4, e_5, e_6, e_7),$$

$$s_2 = 3 \quad \text{2-simplexes: } \sigma_{2;1}, \ldots, \sigma_{2;3} \quad (= L_1, L_2, L_3).$$

Application of the coboundary operator δ to the 0-simplexes yields

$$\delta_1(\sigma_{0;1}) = \sigma_{1;1} - \sigma_{1;2},$$

$$\delta_1(\sigma_{0;2}) = \sigma_{1;2} - \sigma_{1;3} - \sigma_{1;5},$$

$$\delta_1(\sigma_{0;3}) = -\sigma_{1;6} + \sigma_{1;5} - \sigma_{1;7},$$

$$\delta_1(\sigma_{0;4}) = \sigma_{1;3} - \sigma_{1;4},$$

$$\delta_1(\sigma_{0;5}) = -\sigma_{1;1} + \sigma_{1;6} + \sigma_{1;4} + \sigma_{1;7}.$$

In this way it is shown that the coboundary of the 0-simplex is a 1-cochain.

The coboundary of the coboundary is

$$\delta_2\delta_1(\sigma_{0;1}) = \delta(\sigma_{1;1} - \sigma_{1;2}) = \delta(\sigma_{1;1}) - \delta(\sigma_{1;2})$$
$$= (\sigma_{2;1} + \sigma_{2;2}) - (\sigma_{2;1} + \sigma_{2;2}) = 0$$

$$\delta_2\delta_1(\sigma_{0;2}) = \delta(\sigma_{1;2} - \sigma_{1;3} - \sigma_{1;5}) = \delta(\sigma_{1;2}) - \delta(\sigma_{1;3}) - \delta(\sigma_{1;5})$$
$$= (\sigma_{2;1} + \sigma_{2;2}) - (\sigma_{2;1}) - (\sigma_{2;2}) = 0$$

$$\delta_2\delta_1(\sigma_{0;3}) = \delta(-\sigma_{1;6} + \sigma_{1;5} - \sigma_{1;7}) = -\delta(\sigma_{1;6}) + \delta(\sigma_{1;5}) - \delta(\sigma_{1;7})$$
$$= -(\sigma_{2;2} - \sigma_{2;3}) + (\sigma_{2;2}) - (\sigma_{2;3}) = 0$$

$$\delta_2\delta_1(\sigma_{0;4}) = \delta(\sigma_{1;3} - \sigma_{1;4}) = \delta(\sigma_{1;3}) - \delta(\sigma_{1;4})$$
$$= (\sigma_{2;1}) - (\sigma_{2;1}) = 0$$

$$\delta_2\delta_1(\sigma_{0;5}) = \delta(-\sigma_{1;1} + \sigma_{1;6} + \sigma_{1;4} + \sigma_{1;7})$$
$$= -(\sigma_{2;1} + \sigma_{2;2}) + (\sigma_{2;2} - \sigma_{2;3}) + (\sigma_{2;1}) + (\sigma_{2;3}) = 0,$$

from which we may conclude that

1. the coboundary of the coboundary is zero;
2. a 1-cochain with coboundary zero is a 1-cocycle; and
3. this 1-cocycle represents the generalized across-law of Kirchhoff, $\sum \underline{\alpha} = 0$.

Application of the boundary operator ∂ to 2-simplexes yields the following:

$$\partial_1(\sigma_{2;1}) = (\sigma_{1;1} + \sigma_{1;2} + \sigma_{1;3} + \sigma_{1;4}),$$
$$\partial_1(\sigma_{2;2}) = (\sigma_{1;1} + \sigma_{1;2} + \sigma_{1;6} + \sigma_{1;5}),$$
$$\partial_1(\sigma_{2;3}) = (-\sigma_{1;6} + \sigma_{1;7}),$$

from which we find that the boundary of the 2-simplex is a 1-chain.

The boundary of the boundary is

$$\begin{aligned}\partial_2\partial_1(\sigma_{2;1}) &= \partial(\sigma_{1;1} + \sigma_{1;2} + \sigma_{1;3} + \sigma_{1;4})\\ &= (\sigma_{0;1} - \sigma_{0;5}) + (-\sigma_{0;1} + \sigma_{0;2}) + (-\sigma_{0;2} + \sigma_{0;4})\\ &\quad + (-\sigma_{0;4} + \sigma_{0;5})\\ &= 0\end{aligned}$$

$$\begin{aligned}\partial_2\partial_1(\sigma_{2;2}) &= \partial(\sigma_{1;1} + \sigma_{1;2} + \sigma_{1;6} + \sigma_{1;5})\\ &= (\sigma_{0;1} - \sigma_{0;5}) + (-\sigma_{0;1} + \sigma_{0;2}) + (-\sigma_{0;3} + \sigma_{0;5})\\ &\quad + (-\sigma_{0;2} + \sigma_{0;3})\\ &= 0\end{aligned}$$

$$\partial_2\partial_1(\sigma_{2;3}) = \partial(-\sigma_{1;6} + \sigma_{1;7}) = -(-\sigma_{0;3} + \sigma_{0;5}) + (-\sigma_{0;3} + \sigma_{0;5}) = 0,$$

from which the following conclusions can be drawn:

1. The boundary of the boundary is zero;
2. a 1-chain with boundary zero is a 1-cycle; and
3. this 1-cycle represents the generalized through-law of Kirchhoff, $\sum \underline{\tau} = 0$.

The Appendix shows how the incidence matrices can be derived. It appears that

$$\mathrm{H}(0,1) = \mathrm{A}^{\mathrm{T}} \quad \text{and} \quad \mathrm{H}(1,2) = \mathrm{C}$$

and also that

$$\partial_2\partial_1 = \mathrm{H}(0,1)\cdot\mathrm{H}(1,2) = \mathrm{A}^{\mathrm{T}}\mathrm{C} = \underline{0} \quad \text{and}$$
$$\delta_2\delta_1 = \mathrm{H}^{\mathrm{T}}(1,2)\cdot\mathrm{H}^{\mathrm{T}}(0,1) = \mathrm{C}^{\mathrm{T}}\mathrm{A} = \underline{0}$$

6.3. Abstract Extension to n-Simplexes [12]

So far we have only handled simplexes of up to and including order 2. The question arising here is, of course, whether we may proceed with simplexes of higher order; in other words, are there analogous forms of $\mathrm{A}^{\mathrm{T}}\mathrm{C}$ and $\mathrm{C}^{\mathrm{T}}\mathrm{A}$ for n-simplexes?

Completely analogous to Figure 23, and applying the theory in the Appendix, which in principle concerns n-simplexes, for example, a 3-complex

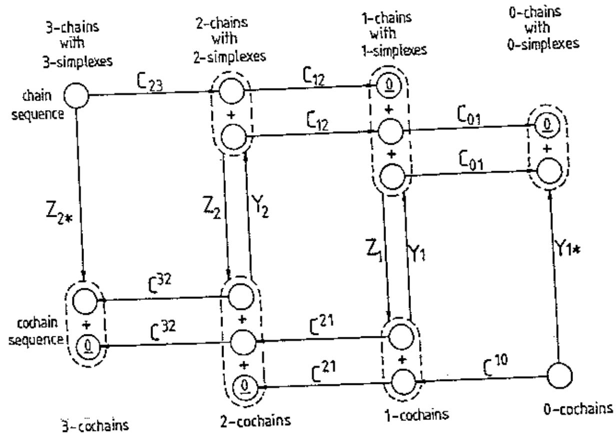


FIGURE 8.24. 3-complex string, where $Z_{2*} = C^{32}Z_2C_{23}$ and $Y_{1*} = C_{01}Y_1C^{10}$; $C^{32} = C^T_{23}$; $C^{21} = C^T_{12}$; $C^{10} = C^T_{01}$

(containing 3-, 2-, 1-, and 0-simplexes), as illustrated in Figure 8.24, can be constructed.

In conformity with simplicial topology, the next statement for Figure 8.24 has to hold: The (co)boundary of the (co)boundary is zero.

In Figure 8.26 this means that for each $i = 1 + j$ and $k = j + 1$ it holds that

$$C_{ij}C_{jk} = \underline{0} \quad \text{and} \quad C^{kj}C^{ji} = C^T_{jk}C^T_{ij} = \underline{0}. \tag{25}$$

6.4. Poincaré Duality [12]

The Appendix roughly indicates how the concept of duality can be incorporated in homology theory. With the help of Poincaré's duality proposition, Table 8.8 can be set up in analogy with Table 8.7.

In Figure 8.25 the chain sequence is replaced by a dual cochain. For the sake of clarity, the cochain and dual cochain sequences are, respectively, cohomology and homology sequences.

In Figure 8.26 a bare skeleton of the so-called cohomology "string" is shown.

6.5. Vector Calculus [12, 74]

Before physical theories are worked into the generalized string, first it has to be checked if the known rules of vector calculus can be applied to this string. Earlier it was made clear that the across- and through-variables can be considered as vectors and that the system structure can be incorporated in the incidence matrices. Now it is clear that it has to be checked whether vector calculus includes analogous forms of

$$A^TC = \underline{0} \quad \text{and} \quad C^TA = \underline{0}.$$

TABLE 8.8. Poincaré Duals, with $0 < p \leq n = 3$.

Primal 3-complex	Dual 3-complex
p-simplexes	$(3-p)$-simplexes
p-chains	$(3-p)$-cochains
p-cochains	$(3-p)$-chains
C_{01}	$\underline{\mathrm{C}}^{32} = \underline{\mathrm{C}}_{23}^{\mathrm{T}}$
C_{12}	$\underline{\mathrm{C}}^{21} = \underline{\mathrm{C}}_{12}^{\mathrm{T}}$
C_{23}	$\underline{\mathrm{C}}^{10} = \underline{\mathrm{C}}_{01}^{\mathrm{T}}$
$\mathrm{C}_{01}^{\mathrm{T}}$	$\underline{\mathrm{C}}_{23}$
$\mathrm{C}_{12}^{\mathrm{T}}$	$\underline{\mathrm{C}}_{12}$
$\mathrm{C}_{23}^{\mathrm{T}}$	$\underline{\mathrm{C}}_{01}$

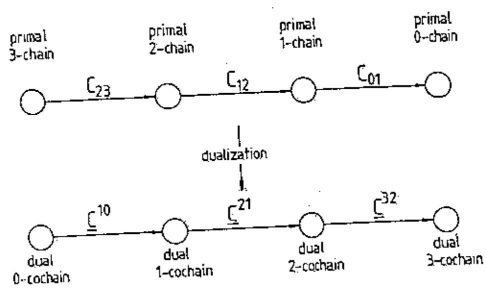


FIGURE 8.25. Dual of a chain sequence.

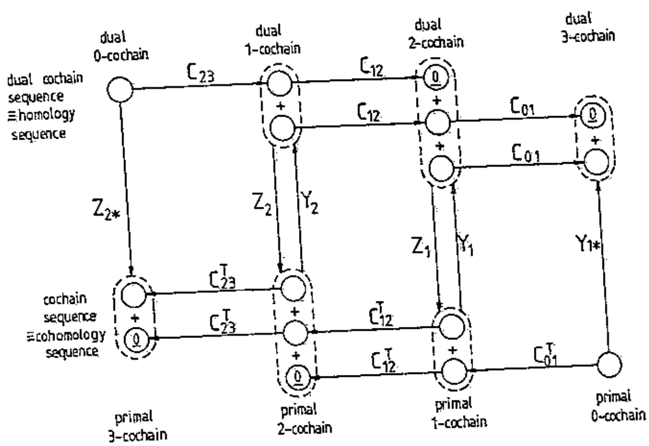


FIGURE 8.26. Cohomology string (generalized for a 3-complex).

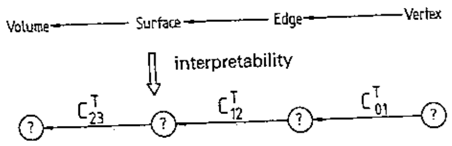


FIGURE 8.27. Interpretability of geometric transformations.

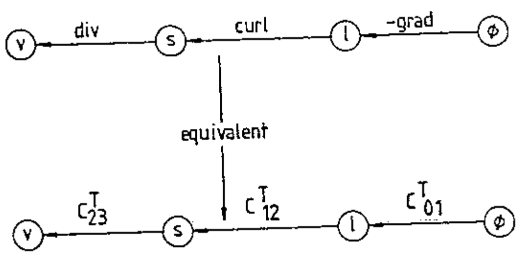


FIGURE 8.28. Integral-statements in a topological sequence.

In simplicial topology 0-, 1-, 2-, 3-, and n-simplexes identify with, respectively, vertex, edge, surface, volume, and hypervolume elements.

In Figure 8.27 the subsequent transformations of the geometric elements in rank order of dimension are briefly indicated.

The question now is how to interpret these transformations vector-algebraically. In rank order of dimension, the well-known integral-statements of vector calculus are

$$\phi(a) - \phi(b) = -\int_a^b \operatorname{grad} \underline{\phi} \cdot \mathrm{d}\underline{l} \qquad \text{(line integral)} \tag{26}$$

$$\oint \underline{A} \cdot \mathrm{d}\underline{l} = \int_S \operatorname{curl} \underline{A} \cdot \mathrm{d}\underline{s} \qquad \text{(Stokes's theorem)} \tag{27}$$

$$\int \underline{B} \cdot \mathrm{d}\underline{s} = \int_V \operatorname{div} \underline{B} \cdot \mathrm{d}\underline{v} \qquad \text{(Gauss's theorem)} \tag{28}$$

If we now consider gradient, curl, and divergence as linear operators, and if they are placed in a sequence in rank order of dimension, then Figure 8.28 shows clearly how these integral-statements are to be interpreted topologically.

The known vector identities

$$\operatorname{curl} \operatorname{grad}(\mathrm{f}) = 0 \tag{29}$$

and

$$\operatorname{div} \operatorname{curl} \underline{H} = 0 \tag{30}$$

confirm the correctness of the application of (25). These identities are obviously analogous interpretations of

$$A^T C = \underline{0} \quad \text{and} \quad C^T A = \underline{0}.$$

7. The Incorporation of Physical Theories in Cohomology Theory [3, 12]

The inclusion of Maxwell's equations of electromagnetism in cohomology theory is laid out here according to a logical scheme:

—first, the inclusion of the analyzed static cases of electromagnetism, for example, electrostatics and magnetostatics (which, as is known, are independent from each other), in their cohomology strings;
—next, the incorporation of the time derivatives of some particular quantities of the static cases; and
—finally, the coupling of these strings, as can be seen later in the schematized development phases in Figures 8.29 to 8.31.

7.1. Electrostatics

The quantities concerned are arranged in a certain rank order of dimension:

$$\phi \equiv \text{field potential [V]},$$

$$\underline{E} \equiv \text{electric field [V/m]},$$

$$\underline{h} \equiv \text{vector potential [C/m]},$$

$$\underline{D} + \underline{d} \equiv \text{dielectric displacement } [\text{C/m}^2] = [\text{As/m}^2],$$

$$\rho \equiv \text{charge density } [\text{C/m}^3] = [\text{As/m}^3].$$

Fundamental relations of electrostatics are

$$\operatorname{div} \underline{d} = 0, \tag{31}$$

$$\operatorname{div} \underline{D} = \rho(\underline{x}), \tag{32}$$

$$\operatorname{curl} \underline{E} = \underline{0}, \tag{33}$$

$$\operatorname{curl} \underline{h} = \underline{d}, \tag{34}$$

$$\underline{D} + \underline{d} = \varepsilon \underline{E}. \tag{35}$$

with ε = dielectric constant or permittivity [C/(V . m)].

The dimensions of the quantities concerned clearly determine their position in the string.

From this string-diagram (Figure 8.29), the relations below follow:

$$\rho(\underline{x}) = -\operatorname{div} \varepsilon \operatorname{grad} \phi(\underline{x}) \tag{36}$$

$$\phi(\underline{x}) = -(\operatorname{div} \varepsilon \operatorname{grad})^{-1} \rho(\underline{x}) \tag{37}$$

$$\phi(\underline{x}) = -(\operatorname{div} \varepsilon \operatorname{grad})^{-1} \operatorname{div} \underline{D} \tag{38}$$

$$\underline{E} = -\operatorname{grad} \phi(\underline{x}) = \operatorname{grad} (\operatorname{div} \varepsilon \operatorname{grad})^{-1} \rho(\underline{x}) \tag{39}$$

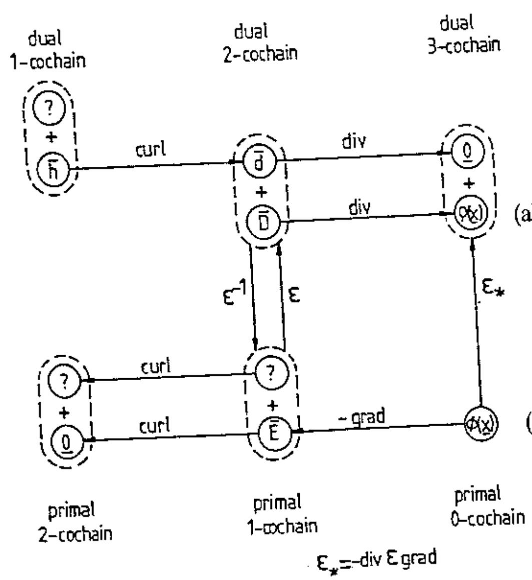


FIGURE 8.29. Electrostatic string. (a) τ sequence. (b) α sequence.

$$\operatorname{curl} \varepsilon^{-1} \operatorname{curl} \underline{h} = \operatorname{curl} \varepsilon^{-1} \underline{D} \tag{40}$$

$$\underline{h} = -(\operatorname{curl} \varepsilon^{-1} \operatorname{curl})^{-1} \operatorname{curl} \varepsilon^{-1} \underline{D} \tag{41}$$

$$\underline{d} = -\operatorname{curl}(\operatorname{curl} \varepsilon^{-1} \operatorname{curl})^{-1} \operatorname{curl} \varepsilon^{-1} \underline{D}, \quad \therefore \underline{d} = \underline{d}(\underline{D}) \tag{42}$$

7.2. Magnetostatics

The quantities concerned are arranged in the same way as in the previous case:

$$\underline{A} \equiv \text{vector potential } [\mathrm{W/m}] = [\mathrm{Vs/m}],$$

$$\underline{B} \equiv \text{inductive field } [\mathrm{W/m^2}] = [\mathrm{Vs/m^2}],$$

$$\underline{H} \equiv \text{magnetic field intensity } [\mathrm{A/m}],$$

$$\underline{J} \equiv \text{current density } [\mathrm{A/m^2}].$$

Fundamental relations of magnetostatics are

$$\operatorname{curl} \underline{H} = \underline{J} \tag{43}$$

$$\operatorname{curl} \underline{A} = \underline{B} \tag{44}$$

$$\operatorname{div} \underline{B} = 0 \tag{45}$$

$$\underline{B} = \mu \underline{H} \tag{46}$$

with $\mu =$ permeability $[\mathrm{W/(A \cdot m)}]$

In Figure 8.30 we can see how these relations are included. From this string-diagram, the known derivatives follow:

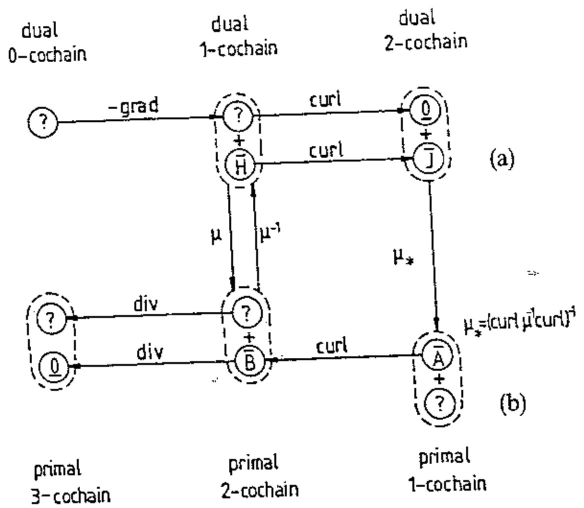


FIGURE 8.30. Magnetostatic string. (a) τ sequence. (b) α sequence.

$$\underline{J} = \text{curl}\,\mu^{-1}\,\text{curl}\,\underline{A} \tag{47}$$

$$\underline{A} = (\text{curl}\,\mu^{-1}\,\text{curl})^{-1}\underline{J} \tag{48}$$

$$\underline{B} = \text{curl}\,\underline{A} = \text{curl}(\text{curl}\,\mu^{-1}\,\text{curl})^{-1}\underline{J} \tag{49}$$

7.3. The Coupling of Electrostatic and Magnetostatic Strings

A comparison of the electrostatic and magnetostatic strings (Table 8.9) indicates that it is impossible to connect these strings "statically" to one another, since they are not dimensionally consistent. Looked at physically, this is the reason why electrostatics and magnetostatics are independent of each other.

In the dual 2-cochain $\underline{d}$ and $\underline{J}$ are dimensionally mutually incompatible: $[\underline{d}] \neq [\underline{J}]$. In the primal 1-cochain, this is also true of $\underline{E}$ and $\underline{B}$: $[\underline{E}] \neq [\underline{B}]$. Moreover, it is seen that the incompatibility arises from one dimension only, namely, that of time! This motivates us to introduce the time derivatives of ρ, $\underline{D}$, $\underline{d}$, $\underline{A}$, and $\underline{B}$.

In Figure 8.31 we can see how the electrodynamic and magnetodynamic strings can be linked to one another thanks to the time derivatives. The "coupled" strings yield the well-known equations of Maxwell, which comprise

1. Structure equations:

$$\text{curl}\,\underline{E} = -\partial\underline{B}/\partial t \qquad \text{(Faraday's law)}, \tag{50}$$

$$\text{curl}\,\underline{H} = \underline{J} + \partial\underline{D}/\partial t \qquad \text{(Ampere's law)}, \tag{51}$$

$$\text{div}\,\underline{B} = 0 \qquad \text{(Gauss' law)}, \tag{52}$$

$$\text{div}\,\underline{J} = -\partial\rho/\partial t \qquad \text{(Coulomb's law)}, \tag{53}$$

TABLE 8.9. Comparison between dimensions.

	Electrostatics	Magnetostatics
α sequence	[V] [V/m]	[W/m] = [Vs/m] [W/m^2] = [Vs/m^2]
τ sequence	[C/m] = [As/m] [C/m^2] = [As/m^2] [C/m^3] = [As/m^3]	[A/m] [A/m^2]

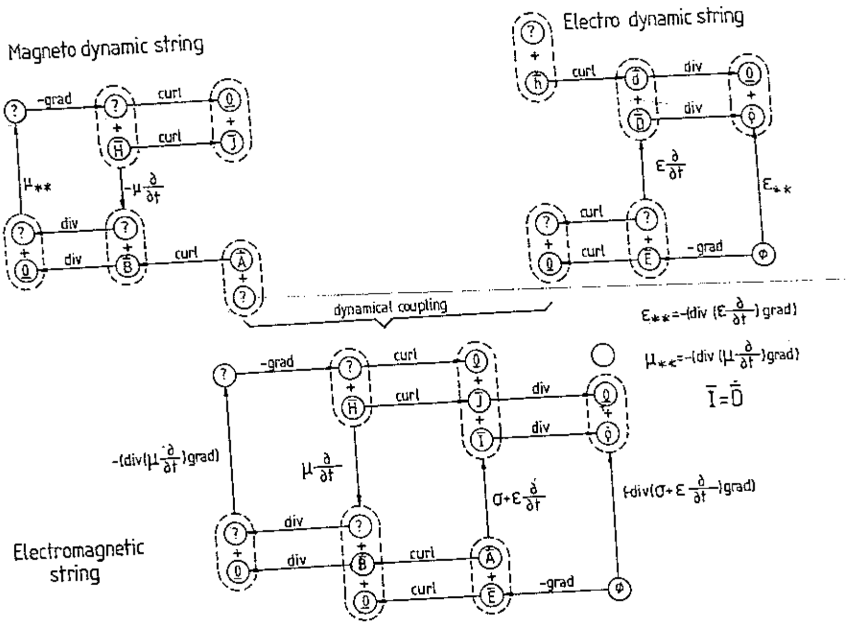


FIGURE 8.31. Electromagnetic string.

2. Constitutive laws:

$$\underline{D} = \varepsilon_0\underline{E} + \underline{P} = (1 + \chi_e)\varepsilon_0\underline{E} = \varepsilon\underline{E}, \tag{54}$$

$$\underline{B} = \mu_0\underline{H} + \mu_0\underline{M} = (1 + \chi_m)\mu_0\underline{H} = \mu\underline{H}, \tag{55}$$

$$\underline{J} = \sigma\underline{E}, \tag{56}$$

with σ = conductivity [A/(V $\cdot$ m)],

from which, for the sake of reminder, some important derivatives follow:

$$\mathrm{curl}\,\underline{B} = \mu\underline{J} + \varepsilon\mu\partial\underline{E}/\partial t \tag{57}$$

$$\dot{\rho} = \mathrm{div}(\sigma + \varepsilon\partial/\partial t)(\underline{E} + \dot{\underline{A}}) \tag{58}$$

$$\mathrm{div}\,\underline{E} = \rho/\varepsilon \quad \text{for } \sigma = 0 \text{ and } \dot{\underline{A}} = 0 \tag{59}$$

$$\operatorname{div} \underline{D} = \rho \tag{60}$$

$$\phi = -[\operatorname{div}(\sigma + \varepsilon\partial/\partial t)\operatorname{grad}]^{-1} \operatorname{div}[\underline{I} - (\sigma + \varepsilon\partial/dt)\underline{\dot{A}}] \tag{61}$$

$$\underline{H} = -[\operatorname{curl}[S_1 - [\operatorname{curl} T_2 \operatorname{curl}]^{-1}]\operatorname{curl}]^{-1}\operatorname{curl} S_1 \underline{I} \tag{62}$$

with $S_1 = (\sigma + \varepsilon\partial/\partial t)^{-1}$ and $T_2 = (\mu\partial/\partial t)^{-1}$. Moreover, we recognize some interesting "by-products":

—Application of the divergence operator to (50) gives, via

$$\operatorname{div} \operatorname{curl} \underline{E} = \operatorname{div}(-\partial B/\partial t) = -\partial(\operatorname{div} \underline{B})/\partial t = 0 \quad \text{(see (30))},$$

the expression $\operatorname{div} \underline{B} = 0$, which confirms (52).

—Application of the divergence operator to (51) leads, via

$$\begin{aligned}\operatorname{div} \operatorname{curl} \underline{H} &= \operatorname{div}(\underline{J} + \partial\underline{D}/\partial t) = \operatorname{div} \underline{J} + \operatorname{div}(\partial\underline{D}/\partial t)\\ &= \operatorname{div} \underline{J} + \partial(\operatorname{div} \underline{D})/\partial t = 0 \quad \text{and} \quad \operatorname{div} \underline{D} = \rho \qquad \text{(see (32))},\end{aligned}$$

to the result

$$\operatorname{div} \underline{J} + \partial\rho/\partial t = 0,$$

which confirms (53).

By carrying the latter argument further, it may be shown that eqs. (52) and (53) are consistent with (50) and (51).

From the local relations (50) to (55), we can derive a differential equation in vector notation for $\underline{E}$ and $\underline{B}$:

—By applying the curl operator to (51), combined with the substitution of (50), we obtain

$$\operatorname{curl} \operatorname{curl} \underline{H} = \operatorname{curl} \underline{J} + \operatorname{curl}(\partial\underline{D}/\partial t) = \operatorname{curl} \underline{J} + \partial(\operatorname{curl} \underline{D})/\partial t,$$

which with (55) and (54) and (50) yields

$$\operatorname{curl} \operatorname{curl} \underline{B} + \varepsilon\mu\partial^2\underline{B}/\partial t^2 = \mu \operatorname{curl} \underline{J}. \tag{63}$$

—In an analogous way, application of the curl operator to (50) gives

$$\operatorname{curl} \operatorname{curl} \underline{E} + \varepsilon\mu\partial^2\underline{E}/\partial t^2 = -\mu\partial\underline{J}/\partial t. \tag{64}$$

By supposing, for example, that $\underline{J}$ is a differentiable function of space and time at every point of a vacuum, we can in principle find the electric field $\underline{E}$ and the inductive field $\underline{B}$ by solving at least one of these differential equations, say, (63). By substituting the computed $\underline{B}$ field in (50) and (51), the associated $\underline{E}$ field can then be found. Equations (63) and (64) are nothing other than the electromagnetic wave equations, from which it may be shown that

$$(\varepsilon\mu)^{-1/2} = c,$$

with $c \equiv$ propagation velocity of the electromagnetic wave (=velocity of light).

The attentive reader will quickly remark that the magnetic space charge ρ_m is absent [48, 87]. This can be remedied by the use of the concept of symmetry. Figure 8.31 is actually symmetrical with respect to the central vertical axis. If we rotate this figure 180 degrees, then we have another string connection that describes a supplementary set of "magnetoelectric" field equations (sic!), wherein the magnetic space charge is fundamental rather than the electric space charge. Page and Adams have shown, not only that this possibility exists, but also that every linear combination of electric and magnetic space charges leads to a self-consistent theory of electromagnetism [62].

7.4. Implications for Extension of Cohomology Theory

To conclude, we have seen how Maxwell's laws may be incorporated, without difficulty, into cohomology theory, not losing sight of the fact that these laws describe "fields" and result in a field theory! It is logical to expect that we will produce "analog" field theories, in, among others, hydrodynamics, gravitational fields, acoustics, thermodynamics, magnetohydrodynamics, continuum mechanics, relativity theory, Schrödinger fields, meson fields, and thermoelastodynamics. They operate with "analog" Poisson, Laplace, diffusion, and Helmholtz equations (in fact, with partial differential equations of elliptic, parabolic, and hyperbolic type in tensor notation). This means that these field theories should be considered as different (read: specific) cohomology theories. It is to the honor of Tonti [83], who, on the basis of [79–82], has worked out a number of illustrative examples in different energy domains; he consistently develops physical theories from one cohomology theory.

8. Is Something Higher than Cohomology Theory Possible?

Before answering the question "Is there a higher (read: unifying) theory for the cohomology theories upon which the analogy between these last theories can be laid down 'visually, physically and mathematically'?", we must first mention recent developments in advanced physics. There are, up to the present, four sorts of forces of nature: gravitational force; electromagnetic force; strong nuclear force between the hadrons, built up with quarks; and weak nuclear force between hadrons as well as leptons (a hadron is a collective name for protons, neutrons, pions, and more exotic elementary particles, and a lepton is a collective name for, among others, electrons, muons, and neutrinos). These four forces of nature are, respectively, incorporated under the gravitational field, the electromagnetic field, and the Yang–Mills field, with and without Higgs field.

For the well-intentioned topologist, it is no longer surprising that the equations of the Yang–Mills field are analogies of those of Maxwell's theory of electromagnetic fields; in other words, they fit into the cohomology theory

and so have a topological string. So, for each of these four forces of nature we have a string. Just as we have seen how the electrodynamic and magnetodynamic strings may be linked to one another, we can also link these four strings together. So evolves a superstring theory as an answer to the last question posed. In theories such as that in [27], one even comes across "analog" forms of the Roth-transformations. Different sorts of symmetry are brought together under one "supersymmetry." One can follow up how far these analogous forms of $A^TC = \underline{0}$ and $C^TA = \underline{0}$ carry through in the superstring theory. One even comes across H-strings!

Such a superstring model, with various branchings and intertwinings, would be very appropriate for making a permanent link between physical theories in the form of their cohomology theories. It is exactly the complicated structure of the superstring model that provides so many difficulties at present that one must for now keep cohomology theories apart. The mathematical difficulties can certainly be eliminated provided very stringent conditions (i.e., supersymmetry (sic!)) are fulfilled.

Another interesting "analogy-seeing" is the analogous interpretation of the law of conservation. In the network of Figure 8.23 (as 2-complex), A^T and C^T symbolize the (dual) conservation laws of energy, namely, the first and second generalized laws of Kirchhoff, while in the 3-complex of electromagnetism of Figure 8.31, C_{01} and C_{23} represent the conservation laws of electric and magnetic charge, respectively. In the superstring model, there are about 12 laws of conservation: mass-energy, impulse, angular impulse, electron family number, muon family number, baryon family number, time inversion, space inversion, charge conjugation, combined space inversion and charge conjugation, strangeness, and isotopic spin. As the laws of conservation possess their validity everywhere and always, there are "inviolable" rules that determine the form of all interactions. A few of the conservation laws are in fact "invariance principles." Other conservation laws concern the family numbers, which may be viewed as analogies of the Euler–Poincaré formula for polyhedrons.

Totally independent from physics, simplicial topology ascertains that the conservation laws are in fact based on symmetries; in other words, conservation laws are symmetry laws. Why?

9. Symmetry and Invariance

In analogy theory, questions of symmetry and invariance have come to play an increasingly prominent part in the systematic classification of links, transformations, strings, and geometric configurations. It is interesting to note that implicitly, ideas of symmetry and invariance have governed the whole evolution of physics since Newton's days.

Qua definition, symmetry has two meanings: a geometric meaning and an algebraic meaning. In the first sense, we could reckon the following: translational symmetry as, for example, frieze decorations, rotational symmetry as

column and rosette decorations, and bilateral symmetry in the form of reflections, such as "right hand versus left hand," a symmetrically drawn heart, and handwriting versus mirror writing. Combinations of the above-mentioned symmetries also exist: In art we meet with examples such as the famous images of Escher, a jewel of a cutting-art of Bohemian crystal goblet, and different tapestry-symmetries; and in nature we find, for example, the antlers of a deer, symmetrically colored wings of a butterfly, and, not to be forgotten, a pretty girl with symmetry-beauties.

In the second sense, a whole is defined symmetric if it has interchangeable parts. The vagueness of this definition implies that many kinds of symmetry can exist. They differ in the number of interchangeable parts and in the operations that exchange the parts. Algebraically, we describe these different kinds of symmetry in terms of topological properties that remain invariant under a certain mapping of any set of elements. Symmetry properties of a system are characterized in terms of groups of transformations that leave the system unchanged. If a system proves invariant under a certain group of transformations, then this symmetry feature relates to the conservation of a certain "dynamical" quantity. That is why the existence of conservation laws is directly related to the symmetry of the laws of nature, that is, to their invariance or changelessness under various symmetry operations such as rotations, translations, and reflections of the spatial and temporal coordinates.

We may safely say that all a priori statements in physics have their origin in symmetry considerations [20, 21, 30, 59, 67, 68]. As Weyl [89] has phrased it, "if conditions which uniquely determine their effect possess certain symmetries, then the effect will exhibit the same symmetry." In such a way, Sophie Lie created the foundations for the science of symmetry of differential equations. Recently, on the occasion of the 150th anniversary of the birth of James Clerk Maxwell, Fushchick and Nikitin presented their work, titled "Symmetries of Maxwell's Equations" [24]. Obviously, the symmetry applications range from the direct link between one-dimensional symmetry groups and conservation laws (Noether's theorem) to the use of linear and nonlinear group representation theory in various contexts and in higher dimensions (such as, e.g., in superstring theory with supersymmetry).

The aim of the previous contemplation about symmetry is to apply symmetry consistently in analogy-thinking, analogy-seeing, and modeling of physical and technical systems. In this way, for such systems we can obtain complete symmetry between the set of through-variables and the set of across-variables, as shown in Table 8.10.

In Table 8.10 it is shown that the relationships between these two sets are symmetrical (or complementary or dually) to each other. This duality, complementarity, or symmetrical relationship between the through-variables and across-variables is emphasized qualitatively by the mere existence of the first law of thermodynamics.

In this context analogies, dualities, and dualogies may be considered as symmetries. Moreover, consequent use of symmetry in matrix theory, just as

TABLE 8.10. Symmetries between τ-set and α-set.

τ-set	α-set
$A^T\underline{\tau} = \underline{0}$	$C^T\underline{\alpha} = \underline{0}$
$A^TC = \underline{0}$	$C^TA = \underline{0}$
$\underline{\tau} = Y\underline{\alpha}$	$\underline{\alpha} = Z\underline{\tau}$

in network and bond-graph theory, leads to time-saving manipulations and computations.

Concerning analogy-thinking, the transition of ideas and models from one physical field to another must be based on invariances, forming analogous laws and relations [43].

We have seen that analogy-seeing, based on cohomology theory, can reduce the whole of physics to geometry. The intriguing question "What precisely is geometry?" leads us to the "Erlanger program" of Felix Klein [44]. The general idea of this program is resolved in Klein's definition, which may be rendered as follows: "A geometry is the study of those properties of a space which remain invariant when the elements of the space are subjected to the transformations of some group of transformations." This establishment has paramount scope, because, in the field of geometry, catastrophe theory [25, 91] (analyzing mathematical instabilities) and fractal theory [4–6, 52, 53, 65] (trying to understand chaos in nature) advance in the direction of physics. Thus, analogy theory can comprehend all these theories very nicely in the future.

10. Completeness and Consistency

Can this "higher" (physical) theory be consistent and complete? This question will definitely be answered negatively by virtue of Gödel's Incompleteness Theorem [26, 57]. Without proof and in a more popular way, this theorem states that

> All consistent axiomatic formulations of a theory include undecidable propositions.

When popularizing this theorem, the next analogies show striking resemblance:

— Epimenides's Paradox or the liar's paradox: "All Cretans are liars." More explicitly this statement can be written as "A liar says he lies; so he lies and he does not lie at the same time."
— Eubulides' Dilemma (approx. 400 B.C.) about the liar who says he lies, Electra who owns and does not own her brother Crestes, and so on.
— Maxwell's Demon [64].

— Ubbink's Cuckoo [18]
— Church's Undecidability Theorem [16].
— Turing's Halting Theorem [25].
— Tarski's Truth Theorem [76]

More precisely, the proof of Gödel's Incompleteness Theorem hinges on the formulation of a self-referential mathematical statement, precisely as Epimenides' Paradox is a self-referential statement of language. This means that it is impossible to accept the challenge of irrefutably proving that this theory is both consistent (free of contradictions) and complete (i.e., every true statement of the theory can be derived within the framework drawn up by this theory itself).

It is true, one can complete the (higher physical) theory concerned further on and make it more consistent (in terms of Gödel's theory: Increase the Gödel numbers); however, "completely complete" and 100% consistent, this theory shall never be. Even if the "superstring" model continues expanding, full completeness and consistency will never be reached.

Analogy theory itself can in this case subscribe to Gödel's theorem. If one theory is deduced from another theory in an analogical way (analogized), eventually the completeness and consistency will be increased, but their "full" completeness and consistency will not be reached. Analogy, namely, has, in this case, to indicate differences next to the aimed similarities. It is these very differences that are responsible for the incompleteness and inconsistency of the "analogized" theory. By incorporating physical theories in one cohomology theory, physical theories are in fact analogies of each other. In other words, with respect to each other they are incomplete and inconsistent. After all, these imperfections form no obstacle for the judgement of the acceptance, usefulness, and attractiveness of physical theories. The applicability of such a theory is, in fact, determined by pragmatic factors, such as agreement with observed facts, simplicity, and elegance, agreement with common sense (analogy-seeing and -thinking), fitness to support desirable human conduct, and, last but not least, agreement with purpose for which the theory has been developed.

11. Relationship Between Analogy Theory and Artificial Intelligence

Douglas R. Hofstadter summed up some essential abilities for intelligence:

— to respond to situations very flexibly;
— to take advantage of fortuitous circumstances;
— to make sense out of ambiguous or contradictory messages;
— to recognize the relative importance of different elements of a situations;
— to find similarities between situations despite differences which may separate them (sic!);

— to draw distinctions between situations despite similarities which may link them (sic!);
— to synthesize new concepts by taking old concepts and putting them together in new ways;
— to come up with ideas which are novel. [38]

These abilities can, on the basis of analogy with control theory, graph theory, information theory, and mathematical logic, more or less be incorporated in an artificial intelligence (AI) model. By virtue of Gödel's theorem, this AI-model will never be complete and consistent, but from psychology and psychiatry, we know that human intelligence is not consistent and complete either. This is the similarity between artificial and human intelligence. The difference lies in their respective Gödel numbers. If Gödel's number of AI is higher or lower than the one of human intelligence, then AI is superior, respectively, inferior, to human intellect. Hopefully the previous argument, developed with help of analogy theory, can throw light upon a controversy between Hofstadter and Lucas, as can be read in Hofstadter's magisterial book *Gödel, Escher, Bach: An Eternal Golden Braid* [38]. Lucas [49] claims that human intellect can principally not be imitated by a computer program. While his argumentation is completely based on Gödel's Incompleteness Theorem, this is not only disputed by Hofstadter but also by Arbib in his book *Brains, Machines and Mathematics* [2].

12. Epilogue

It is certain that the most general theory (read: in casu the analogy theory), which includes existing phenomena and can take up new phenomena, is best and most powerful; from such a theory, simple theories can always be deducted. This is, for example, how Newtonian mechanics can be considered as a more specific "case" of Albert Einstein's relativity theory.

The fertility of cohomology theory as analogy theory does not in the last instance manifest itself clearly in the unconstrained explanation, that it provides for already known phenomena. That is the reason why it is produced at all [17, 40, 74]. Of course, when such a theory gives a natural explanation of many phenomena, particularly if these are of widely different nature, then such a theory impresses; it lays down the unity, which leads to multiplicity. But such a theory shall gain yet more in convincingness when there are phenomena that can be explained, of which nothing is yet known at the time that the theory was set up. This is the nicest aspect of analogy theory, bearing in mind, however, that the analogy-use should critically be applied.

According to Maxwell, who, in his address to the British Association, stated that analogy was "not only a legitimate product of science, but capable of generating science in its turn ....." Analogy theory is apparently "science-forming" for physical theories and "model-forming" for technical systems.

It is now shown that the Analogy theory, due to its universality, can be

conceived as a "general" cohomology theory. In 1945 Eilenberg and Steenrod gave an axiomatic formulation of the homology [39 theory]. They set seven axioms on which a so-called theory should be based in order to be called real, and they, in general, derived the propositions that hold for such a theory. In that, we shall find again the stringent conditions for the justified use of analogies in systems science.

Analogy theory has indeed provided insight in the way analogies should be critically used. This, however, does not directly answer the pragmatic question, What do we want to do with analogies? The intentions of "analogy users" are very diverse, due to the wide applicability of analogy-thinking. The motives can roughly be classified as follows:

1. Analogies form a central concept, playing an important role in several different mathematical, physical, and other scientific specialized areas.
2. The transformation of new applications of the results and ideas from one area of scientific endeavor into another is stimulated by analogies.
3. Analogies have contributed largely to the influences that concepts, problems, and results of one field of inquiry have and have had on the development of another.

As a consequence of these three motivations, the following developments in the field of engineering design ("theory application" rather than "theory development") are currently observed:

— Analogical reasoning becomes more and more important from a design methodology point of view. The search for design methodologies and strategies to be used in a certain engineering field, equivalent to those already emerged in related engineering disciplines, has lead to multidisciplinary approaches. To this, a new, higher discipline owes its existence: systems engineering.
— The power of analogical reasoning for generating new ideas, concepts, and hypotheses has been appreciated in fields of very diverse nature, and it can therefore be used intelligently by design engineers especially in the conceptual stage of design.

It is therefore necessary to supply assistance in identifying and applying analogs most effectively. This assistance could be concretized by providing design engineers with

— data banks filled with analogous attributes and cases. These databases should contain sufficient details to support the analogical reasoning process.
— Strategies for applying analogs. This, as has been stated before, should be carefully and critically done.

In this way, the engineering design field, with its synthesizing nature, could benefit a lot from the modeling concepts with a more analytical nature. Morphological analysis schemes, which are helpful in engineering design,

could then be transformed into inductive "morphological synthesis tools," expressing the very essence behind the use of them.

Finally, because of the central theme of this book, *simulation*, a short implication of the previously described concepts will be given: As already stated before, every analogy can be used visually, physically, and mathematically. Due to the fact that physics can represent/explain mathematics as well as geometry, "mathematical" and "visual-topological" analogies remain as the elements of the basic description idiom for implementation in a computer program.

We have two types of computer programs at our disposal for implementing the analogy concept to be used in systems engineering:

1. network-oriented software, such as ECAP, NAP, (PC-)SPICE, μ-CAP; and
2. equation-oriented software, such as CSSL, CSMP, ACSL, PSI, TUTSIM, and MOSIS.

Two salient particularities can be observed:

1. Recently, "physical analogies," as in [56], are being used less and less.
2. Equation-oriented software with time-consuming integration methods has lower performance than network-oriented software.

From this it can be concluded that for systems simulation the future lies in network-oriented software, actually modeling the problem "topologically."

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Appendix

Simplicial Topology [8, 15, 23, 35–37, 39, 54, 86]

In simplicial topology, network figures (called complexes) are built up from a network of vertices, edges, triangles, tetrahedrons, and polyhedrons of higher dimensions; these components are determined completely by their vertices, and one can show that $n + 1$ vertices form an n-dimensional simplex (σ_n).

A 1-simplex ($\sigma_1 = \langle p_0 p_1 \rangle$), determined by the vertices p_0 and p_1, is provided with an orientation, from p_0 to p_1 or from p_1 to p_0, and is called, accordingly, $\sigma_1 = \langle p_0 p_1 \rangle$ or $\langle p_1 p_0 \rangle = -\langle p_0 p_1 \rangle$.

A 2-simplex ($\sigma_2 = \langle p_0 p_1 p_2 \rangle$, determined by vertices p_0, p_1, and p_2, is provided with an loop orientation and is called, accordingly, $\sigma_2 = \langle p_0 p_1 p_2 \rangle = \langle p_1 p_2 p_0 \rangle = \langle p_2 p_0 p_1 \rangle$ or $-\langle p_0 p_2 p_1 \rangle = -\langle p_1 p_0 p_2 \rangle = -\langle p_2 p_1 p_0 \rangle = \langle p_0 p_1 p_2 \rangle$.

Thus, n-simplexes can be described by $\sigma_n = \langle p_0 p_1 \cdots p_n \rangle$; a 0-simplex consists of an equal number of vertices $\sigma_0 = \langle p_0 \rangle$ or $\sigma_0 = -\langle p_0 \rangle$.

A formal sum of n-simplexes $c_n = a_1 \sigma_{n1} + a_2 \sigma_{n2} + \cdots + a_k \sigma_{nk}$, where a_j are natural numbers and the σ_{nj} are n-simplexes, is an n-chain. For example, if the 1-chain consists of 1-simplexes, one has a curve; the a_js indicate how often and in what direction (depending on the sign of a_j) the 1-simplexes are being passed along.

The boundary of the 1-simplex $\sigma_1 = \langle p_0 p_1 \rangle$ is called a 0-chain, $\partial \sigma_1 = \langle p_1 \rangle - \langle p_0 \rangle$.

The boundary of the 2-simplex $\sigma_2 = \langle p_0 p_1 p_2 \rangle$ is called a 1-chain, $\partial \sigma_2 = \langle p_1 p_2 \rangle - \langle p_0 p_2 \rangle + \langle p_0 p_1 \rangle$.

The boundary of the 3-simplex $\sigma_3 = \langle p_0 p_1 p_2 p_3 \rangle$ is called a 2-chain, $\partial \sigma_3 = \langle p_1 p_2 p_3 \rangle - \langle p_0 p_2 p_1 \rangle + \langle p_0 p_1 p_3 \rangle - \langle p_0 p_1 p_2 \rangle$.

In general, if $\sigma_n = \langle p_0 p_1 \cdots p_n \rangle$ then, by definition, the boundary of σ_n is $\partial \sigma_n = \Sigma(-1)^i \langle p_0 p_1 \cdots \hat{p}_i \cdots p_n \rangle$. The accent ^ above p_i means that the vertex p_i must be left out: $\langle p_0 p_1 \cdots \hat{p}_i \cdots p_n \rangle = \langle p_0 p_1 \cdots p_{i-1} p_{i+1} \cdots p_n \rangle$; ∂ is by definition a boundary operator. If $\sigma_n = +\langle p_0 p_1 \cdots p_n \rangle$, then $\sigma_{n-1,i} = (-1)^i \langle p_0 \cdots \hat{p}_i \cdots p_n \rangle$ such that $\partial \sigma_n = \Sigma \sigma_{n-i,i}$. For an n-chain, the boundary is defined as $\partial c_n = a_1 \partial \sigma_{n1} + a_2 \partial \sigma_{n2} + \cdots + a_k \partial \sigma_{nk}$, where $\partial \sigma_{nj}$ is the boundary of σ_{nj}.

The boundary of an n-chain, c_n, is an $(n-1)$-chain. An n-chain with boundary zero is an n-cycle. It can be shown that the boundary of a boundary is zero, so that each boundary is a cycle. One can further prove that the n-cycles that are not boundaries of an $(n+1)$-chain, form a group under the addition operator. The coboundary operator δ adds to every k-cochain a $(k+1)$-cochain, while every k-coboundary is a k-cocycle. It also holds that the coboundary of a coboundary is zero. Two (co-)cycles are called *homolog* if they differ by one boundary. If one considers homology n-(co-)cycles as equivalent, then one gets an n-(co-)homology group of an n-Betti-group of the complex.

The simplical homology just described is called homology theory.

Cohomology Theory

In (simplicial) cohomology theory, the building blocks are maps of simplexes into natural numbers rather than simplexes themselves. There is a close connection between the homology groups ($K_n(X)$) and the cohomology groups ($K^n(X)$) in a topological space. According to Poincaré's duality theorem for a particular class of spaces (the coherent oriented variety), the pth homology group and qth cohomology group are equal to one another if the dimension of the variety $n = p + q$

Incidence

The incidence between an oriented n-simplex $\sigma_n \in K(n > 0)$ and the $(n-1)$-simplexes $\sigma_{n-1;i} \in K$, with $i = 1, \ldots, \alpha_{n-1}$, is described by the incidence number:

$\mathrm{inc}(\sigma_{n-1;i}, \sigma_n) = 0$, if $\sigma_{n-1;i}$ and σ_n are not incident,

$= 1$, if $\sigma_{n-1;i}$ and σ_n are incident (in other words, $\sigma_{n-1;i}$ is the boundary simplex of σ_n) and coherently oriented,

$= -1$, if $\sigma_{n-1;i}$ and σ_n are incident and not coherently oriented.

These incidence numbers are the elements of an $\alpha_{n-1} \times \alpha_n$-matrix:

$$\mathrm{H}(n-1; n) = [\mathrm{inc}(\sigma_{n-1;i}, \sigma_{n;j})]$$

with row index number $i = 1, \ldots, \alpha_{n-1}$ and column index number $j = 1, \ldots, \alpha_n$.

First Incidence Matrix A^T

The matrix entry a_{ij} describes the incidence between the ith vertex $V_i(i = 1, \ldots, k)$ and the jth edge $e_j(j = 1, \ldots, t)$ as follows:

$a_{ij} = +1$, if V_i and e_j are incident and e_j is oriented toward V_i;

$a_{ij} = -1$, if V_i and e_j are incident and e_j is oppositely oriented toward V_i;

$a_{ij} = 0$, if V_i and e_j are not incident.

Second Incidence Matrix C

A loop orientation is assigned to every directed loop. The matrix entry c_{ij} describes the incidence between the ith loop $L_i(i = 1, \ldots, l)$ and the jth oriented edge $e_j(j = 1, \ldots, t)$ as follows:

$c_{ij} = 1$, if L_i and e_j are incident and if the loop orientation of L_i and the direction of e_j coincide;

$c_{ij} = -1$, if L_i and e_j are incident and if the loop orientation of L_i and the direction of e_j are opposed;

$c_{ij} = 0$, if L_i and e_j are not incident.