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PAIRS OF SLOWLY OSCILLATING FUNCTIONS OCCURRING IN ASYMPTOTIC PROBLEMS CONCERNING THE LAPLACE TRANSFORM

BY

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1. *Introduction.* A function $L(x)$, defined for $x > 0$, is called *slowly oscillating* if it is positive and measurable for all $x > 0$, and if, for every $p > 0$, we have $L(px)/L(x) \to 1$ $(x \to \infty)$.

This notion is due to KARAMATA [3] (with restriction to continuous functions; for the general case of measurable functions see [5]). The most important properties are (see sec. 4):

(i) If $0 < a < b < \infty$, then the relation $L(px)/L(x) \to 1$ $(x \to \infty)$ holds uniformly with respect to p for $a \leq p \leq b$.

(ii) There is a continuous function $\delta(x)$ such that $\delta(x) \to 0$ $(x \to \infty)$ and such that $L(x) \exp \{- \int_1^x \delta(t) t^{-1} dt\} \to 1$ $(x \to \infty)$.

Conversely, every positive measurable function L which has the property (ii), is automatically slowly oscillating.

In two recent papers by A. BÉKÉSSY [1] and E. E. KOHLBECKER [4] there occurred slowly oscillating functions in asymptotic problems, to the effect that the problem gave rise to the construction of a new slowly oscillating function, related to the first one by taking the inverse of a function.

This note is mainly dedicated to the remark that in both cases the relation between the original and the new slowly oscillating function is one-to-one (neglecting factors which have the limit 1). And there is a kind of duality: we shall express the relations between those functions in terms of a relation $L \to L^*$, which is involutory: $L^* \to L^{**} = L$.

2. *Pairs of conjugate slowly oscillating functions.*

Theorem 1. If $L(x)$ is slowly oscillating, then there exists a slowly oscillating function L^* such that

$$L^*(xL(x)) \cdot L(x) \to 1 \qquad (x \to \infty), \tag{2.1}$$

$$L(xL^*(x)) \cdot L^*(x) \to 1 \qquad (x \to \infty). \tag{2.2}$$

Moreover, L^* is asymptotically uniquely determined, i.e. if a slowly oscillating function M satisfies one of the relations $M(x(L(x))\cdot L(x) \to 1$, $L(xM(x))\cdot M(x) \to 1$ $(x \to \infty)$, then M and L^* are asymptotically equivalent: $M \sim L^*$, i.e. $M(x)/L^*(x) \to 1$ $(x \to \infty)$. The relation $L \to L^*$ is involutory, in the sense that L and $(L^*)^*$ are asymptotically equivalent. L^* will be called the conjugate of L.

Proof. According to (ii) (sec. 1) we can find a continuous function $\delta(t)$ $(0 \leq t < \infty)$, which satisfies $\delta(t) \to 0$ $(t \to \infty)$, $\delta(t) > -1$ $(0 \leq t < \infty)$, and such that $L_1(x) = \exp\{\int_1^x \delta(t)t^{-1}dt\}$ is asymptotically equivalent to L. It suffices to prove the theorem with L_1 instead of L. For if M is slowly oscillating, we have $M(xL(x)) \sim M(xL_1(x))$ by virtue of (i), whereas $L(xM(x)) \sim L_1(xM(x))$ is trivial.

In other words, we may and do assume that L itself has the form attributed to L_1.

Put $f(u) = \log\{e^u L(e^u)\}$. Then $f'(u) = 1 + \delta(e^u) > 0$ for all u, and $f'(u) \to 1$ $(u \to \infty)$. It follows that there is an inverse function $g(v)$, which has a continuous positive derivative, with $g'(v) \to 1$ $(v \to \infty)$. Putting $g'(v) = 1 + \delta^*(e^v)$, $L^*(x) = x^{-1}\exp(g(\log x))$, we easily verify that $L^*(x) = C\exp\{\int_1^x \delta^*(t)t^{-1}dt\}$, and $g(v) = \log\{e^v L^*(e^v)\}$. From $g(f(u)) = u$, $f(g(v)) = v$ we derive (2.1) and (2.2). And from the fact that $\delta^*(e^v) \to 0$ $(v \to \infty)$ we infer that L^* is slowly oscillating.

We now prove the uniqueness. If M satisfies $M(xL(x))\cdot L(x) \to 1$, we have $M(xL(x)) \sim L^*(xL(x))$. It follows that M and L^* are asymptotically equivalent, as $xL(x)$ increases continuously to infinity.

Secondly, if M is slowly oscillating, and $L(xM(x))\cdot M(x) \to 1$, then we have $xM(x) \to \infty$ $(x \to \infty)$. Furthermore,

$$L^*(x) \sim L^*\{xM(x)L(xM(x))\},$$

and the latter expression is $\sim \{L(xM(x))\}^{-1}$ according to (2.1). Finally, $\{L(xM(x))\}^{-1} \sim M(x)$ by virtue of our assumption about M. So $L^*(x) \sim M(x)$, q.e.d.

The fact that $(L^*)^*$ and L are asymptotically equivalent, is a consequence of the symmetry between (2.1) and (2.2) and of the uniqueness property.

If $(L(x), L^*(x))$ form a pair of conjugate slowly oscillating functions, in the sense of theorem 1, and if A and B are positive constants, then $(L(Ax), L^*(Bx))$ are again a pair of conjugates. For, the A and B do not influence the asymptotic behaviour at all. It is also easy to see that

$$(AL(x),\ A^{-1}L^*(x))$$

form a conjugate pair. For, (2.1) implies that

$$A^{-1}L^*(xAL(x))\cdot AL(x) \to 1 \quad (\text{since } L^*(xAL(x)) \sim L^*(xL(x))).$$

If α is a positive constant, and if $(L(x), L^*(x))$ form a conjugate pair, then

$$(\{L(x^\alpha)\}^{1/\alpha},\ \{L^*(x^\alpha)\}^{1/\alpha})$$

again form a conjugate pair. This follows from (2.1) if we replace x by x^α and if we raise the whole relation to the power $1/\alpha$.

The following remarks are essentially due to A. BÉKÉSSY ([1], Hilfssatz 2). Let $L(x)$ be slowly oscillating. Abbreviate $(L(x))^{-1} = k_1(x)$, and put $k_2(x) = k_1(x(k_1(x)),\ k_3(x) = k_1(xk_2(x)), \ldots, k_{n+1}(x) = k_1(xk_n(x)), \ldots$. It is easy to show that each k_n is slowly oscillating. Assuming that, for some n, we have $k_{n+1}(x) \sim k_n(x)$ $(x \to \infty)$, it is not difficult to show that $L^*(x) \sim k_n(x)$ $(x \to \infty)$. For, $k_{n+1} \sim k_n$ implies that

$$L(xk_n(x))k_n(x) = \{k_{n+1}(x)\}^{-1}k_n(x) \to 1,$$

and now $L^* \sim k_n$ follows from theorem 1 (see (2.2)).

If, especially, $L(x) = \exp\{\int_1^x t^{-1}\delta(t)dt\}$, with $\delta(t) = O((\log t)^{-1})$, we have $k_2 \sim k_1$, whence $L^*(x) \sim (L(x))^{-1}$.

3. *Applications.* E. E. KOHLBECKER proved an abelian theorem and its tauberian counterpart ([4], theorems 3 and 4) for integrals $f(s) = s\int_0^\infty P(u)e^{-su}du$. Let B and α be positive constants, and let $L(x)$ be a slowly oscillating function (which is assumed to be continuous). For every large u, there is at least one small solution s_u of the equation $u = B\alpha s^{-\alpha-1}L(s^{-1})$. The theorems state that $\log P(u) \sim (1+\alpha^{-1})us_u$ $(u \to \infty)$ implies $\log f(s) \sim Bs^{-\alpha}L(s^{-1})$ $(s \downarrow 0)$, and that the converse is true if $P(u)$ is assumed to be non-decreasing.

We can express these results in terms of a pair of conjugate functions. Define L_1 by $B\alpha L(x) = L_1(x^{\alpha+1})$, and put $(s_u)^{-\alpha-1} = v$. Then we have $u = vL_1(v)$, and it follows that $v \sim uL_1^*(u)$ $(u \to \infty)$. And the asymptotic equivalences for $\log P(u)$ and $\log f(s)$ become

$$\log P(u) \sim (1+\alpha^{-1})u^{\alpha/(\alpha+1)}\{L_1^*(u)\}^{-1/(\alpha+1)} \quad (u \to \infty), \tag{3.1}$$

$$\log f(s) \sim \alpha^{-1}s^{-\alpha}L_1(s^{-\alpha-1}) \quad (s \downarrow 0). \tag{3.2}$$

In order to compare these formulas with some generalizations, we prefer to use the parameter $\beta = \alpha/(\alpha + 1)$ instead of α, so $0 < \beta < 1$. Moreover, we introduce a new conjugate pair L_2, L_2^* by

$$L_1^*(u) = \{L_2(u^\beta)\}^{1/\beta},\ L_1(v) = \{L_2^*(v^\beta)\}^{1/\beta}$$

(cf. the end of sec. 2). Then (3.1) and (3.2) become

$$\log P(u) \sim \beta^{-1}u^\beta\{L_2(u^\beta)\}^{(\beta-1)/\beta} \qquad (u \to \infty) \tag{3.3}$$

$$\log f(s) \sim (1-\beta)\beta^{-1}s^{\beta/(\beta-1)}\{L_2^*(s^{\beta/(\beta-1)})\}^{1/\beta} \qquad (s \downarrow 0). \tag{3.4}$$

There are similar theorems in the cases $\beta < 0$ and $\beta > 1$. If $\beta < 0$ we can consider an integrand of the type $\exp(-su - g(u))$, where $u^{-\beta}g(u)$ is a slowly oscillating function of u^{-1}, and the theorem refers to $u \to 0$, $s \to \infty$. If $\beta > 1$, we can consider an integrand of the type $\exp\{su - g(u)\}$, where $u^{-\beta}g(u)$ is a slowly oscillating function of u, and the theorem refers to $u \to \infty$, $s \to \infty$. We can put these three possibilities, viz. $\beta < 0$, $0 < \beta < 1$, $\beta > 1$, into one single theorem:

Theorem 2. Let A, B, β be real constants, $A(1-\beta) > 0$, $B\beta(1-\beta) > 0$ (whence $\beta \neq 0$, $\beta \neq 1$, $A \neq 0$, $B \neq 0$). Assume that $P(u)$ is a real function, that $\int_0^R P(u)du$ exists in the Lebesgue sense for every positive R, and that $\int_0^\infty P(u)e^{-Asu}du$ converges for every $s > 0$. Put $f(s) = s\int_0^\infty P(u)e^{-Asu}du$. Let L be a slowly oscillating function, and let L^* be its conjugate. Then we have

(*a*) (Abelian part). If

$$\log P(u) \sim Bu^\beta\{L(u^\beta)\}^{(\beta-1)/\beta} \qquad (u^\beta \to \infty), \tag{3.5}$$

then

$$\log f(s) \sim B(1-\beta)\left(\frac{As}{B\beta}\right)^{\beta/(\beta-1)}\{L^*(s^{\beta/(\beta-1)})\}^{1/\beta} \quad (s^{\beta/(\beta-1)} \to \infty). \tag{3.6}$$

(*b*) (Tauberian part). If $P(u)$ is monotonic, then (3.6) implies (3.5).

The proof is, of course, very similar to the proof for the case $0 < \beta < 1$ in [4] (notice that (3.5) and (3.6) become (3.3) and (3.4), respectively, if we take $A = 1$, $B = \beta^{-1}$, $0 < \beta < 1$). We shall restrict ourselves to a rough outline of the proof. We first substitute $(As/\beta B)^{\beta/(\beta-1)} = w$, $u^\beta = v$, $P(u) = Q(v)$, $f(s) = g(w)$. Then we obtain

$$g(w) = (\beta B/\,|\beta|\,A)w^{(\beta-1)/\beta}\int_0^\infty \exp\{-\beta Bw^{(\beta-1)/\beta}v^{1/\beta}\}Q(v)v^{-1+1/\beta}dv,$$

and (3.5), (3.6), can be replaced by

$$\log Q(v) \sim Bv\{L(v)\}^{(\beta-1)/\beta} \qquad (v \to \infty) \tag{3.5'}$$

$$\log g(w) \sim B(1-\beta)w\{L^*(w)\}^{1/\beta} \qquad (w \to \infty). \tag{3.6'}$$

Notice that $L^*(w)$ and $L^*(s^{\beta/(\beta-1)})$ are asymptotically equal, since L^* is slowly oscillating.

The factors $(\beta B/|\beta|\,A)w^{(\beta-1)/\beta}$ and $v^{-1+1/\beta}$ can be neglected. We first consider the abelian part of the theorem. We want to find the approximate maximum of the integrand. To this end we consider $-\beta Bw^{(\beta-1)/\beta}v^{1/\beta} + Bv\{L(v)\}^{(\beta-1)/\beta}$. Taking the derivative with respect to v, we neglect the derivative of $L(v)$, and we get $-B(w^{(\beta-1)/\beta}v^{(1-\beta)/\beta} - \{L(v)\}^{(\beta-1)/\beta})$. This means that the maximum can be expected to be attained near the point where $w = vL(v)$, that is near $v_0 = wL^*(w)$. At $v = v_0$ the value of the logarithm of the integrand is about $B(1-\beta)w(L^*(w))^{1/\beta}$.

After this tentative discussion, a rigorous proof can be given as follows. If ε is given, there is a number $C(\varepsilon)$ such that

$$|\log (L(v)/L(v_0))| < \varepsilon\,|\log (v/v_0)| + \varepsilon,$$

provided that $v > C(\varepsilon)$, $v_0 > C(\varepsilon)$ (see (ii) in sec. 1). With this estimate it is not difficult to prove the necessary inequalities for the integrals from $C(\varepsilon)$ to v_0, and from v_0 to ∞, respectively.

For the tauberian part of the theorem we can argue as follows. Let $p > 1$, and assume that $B^{-1}\log Q(v) > pv(L(v))^{(\beta-1)/\beta}$ for a sequence of values $v_1, v_2, v_3, \ldots,$ with $v_n \to \infty$. Then it is not difficult to show that the contribution of the interval (v_n, pv_n) becomes much too large (notice that $B^{-1}\log Q(v)$ is increasing).

This argument shows that the upper limit of

$$B^{-1}\log Q(v)/(v\{L(v)\}^{(\beta-1)/\beta})$$

cannot exceed 1. In order to show that the lower limit is not less than 1, we have to examine an interval around $wL^*(w)$ and to use the fact that the upper limit is ≤ 1.

We leave it at these brief indications. After all, the case $0 < \beta < 1$ has already been treated in detail in [4].

A. Békéssy ([1], Satz 1) proved an abelian theorem on Laplace integrals, which can also be expressed in terms of conjugate slowly oscillating functions. He considers a continuous slowly oscillating function L, and puts $x(t) = \sup_{0\leq\tau\leq t}\tau^{\alpha}\{L(\tau^{-1})\}^{-1}$, where α is a positive constant. His theorem involves the inverse function $t(x)$ of $x(t)$.

If we define the slowly oscillating function L_1 by $L(t^{-1}) = \{L_1(t^{-\alpha})\}^{-1}$, then it is not difficult to show that

$$x^{1/\alpha}t(x) \sim \{L_1^*(x)\}^{-1/\alpha} \qquad (x\to\infty).$$

For, if we put $t^{-\alpha} = s$, then x becomes an increasing function of s. We have $x = \inf_{\sigma \geq s} \sigma L_1(\sigma)$, and what we have to prove is that $s \sim x L_1^*(x)$. Indeed, it is easy to show that if $\inf_{\sigma \geq s} \sigma L_1(\sigma) = s_1 L_1(s_1)$, then $s_1/s \to 1$ if $s \to \infty$.

Translated into our present terminology, Békéssy's theorem becomes the abelian part of

Theorem 3. Let a and α be positive constants, and let $f(t)$ be a positive measurable function for $0 < t \leq a$. For every $\delta (0 < \delta < a)$, we assume that $f(t)$ has a positive lower bound in the interval $\delta \leq t \leq a$. Put $F(x) = \int_0^a \exp(-xf(t))dt$ $(x > 0)$. Let L be a slowly oscillating function, and let L^* be its conjugate.

a. (Abelian part). If

$$f(t) \sim t^{\alpha}\{L(t^{-\alpha})\}^{-1} \qquad (t \downarrow 0), \tag{3.7}$$

then we have

$$F(x) \sim \Gamma(1 + \alpha^{-1})\{xL^*(x)\}^{-1/\alpha} \qquad (x \to \infty). \tag{3.8}$$

b. (Tauberian part). If $f(t)$ is non-decreasing in $0 < t \leq a$, then (3.8) implies (3.7).

It is not difficult to prove the tauberian part by the method of KARAMATA [3].

4. *Appendix.* A few things may be said about statements (i) and (ii) of sec. 1. KARAMATA [2] proved both (i) and (ii) for the case that L is continuous. For the case that L is only assumed to be measurable, (i) was proved in [5], but (ii) was not proved in this form. We shall show here how (ii) can be derived from (i).

Let $L(x)$ be slowly oscillating but not necessarily continuous. Then it follows from (i) that the function $f(t) = \log L(e^t)$ satisfies $f(t+q) - f(t) \to 0$ if $(t \to \infty)$, uniformly for $0 \leq q \leq 1$.

We can construct a continuously differentiable function f_1 such that $f_1(n) = f(n)$ $(n = 1, 2, \ldots)$, $f_1'(t) \to 0$ $(t \to \infty)$, $f_1(t) - f(t) \to 0$ $(t \to \infty)$. We can take

$$f_1(t) = f(n) + 6(f(n+1) - f(n)) \int_0^{t-n} u(1-u)du \quad (n \leq t \leq n+1),$$

whence

$$|f_1'(t)| < \tfrac{3}{2}\,|f(n+1) - f(n)| \qquad (n \leq t \leq n+1),$$

$$|f_1(t) - f(t)| \leq |f(t) - f(n)| + |f(n+1) - f(n)| \qquad (n \leq t \leq n+1).$$

As $f(t+q) - f(t) \to 0$ uniformly for $0 \leq q \leq 1$, we infer that $f_1(t) - f(t) \to 0$ $(t \to \infty)$. And $f_1(t) = f_1(1) + \int_1^t f_1'(u)du$ implies that $L_1(x) = \exp(f_1(\log x))$ has the form $C \exp\{\int_1^x \delta(s)s^{-1}ds\}$, with $\delta(s) \to 0$ if $s \to \infty$. It is easy to show that $L(x)/L_1(x) \to 1$ $(x \to \infty)$, and that $L_1(x)$ is slowly oscillating.

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