Some remarks on Tits geometries

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ABSTRACT

A result of Tits' in his paper "A local approach to buildings" is somewhat strengthened: it is shown that each geometry of type $D_8$ or $E_8$ is a building. A counterexample to the corresponding statement for $E_7$ is given. Moreover, a proof is given of the fact that any thick finite geometry of spherical type all of whose residues of type $C_3$ are buildings is itself a building.

In [10] Tits proves the following

1. **THEOREM.** Let $G$ be a geometry of type $M = A_n$, $B_2(=C_n)$, $D_n$, $E_8$, $E_7$, $E_6$, or $F_4$. Then $G$ is a building if and only if it has the following properties: if $M = C_n$, $D_n$ or $E_6$, properties (O) and (LL), if $M = E_7$, properties (O), (LL) and (LH), if $M = E_8$ or $F_4$, properties (O), (LL), (LH) and (HH).

For notation and terminology, the reader is referred to [10]. We only recall the meaning of the properties (O), (LL), (LH), (HH). For this we need the following labelling of the diagrams:

\[ A_n: \]
\[ \begin{array}{cccc}
  p & l_1 & l_2 & l_{n-2} & h \\
\end{array} \]

\[ C_n: \]
\[ \begin{array}{cccc}
  p & l_1 & l_2 & l_{n-2} & h \\
\end{array} \]
The elements of type $p$, $l_i$, and $h$ are called points, lines and hyperlines respectively.

(O) If two elements of type $l_i$ (for some $i$) have the same shadow in the set of all points, they coincide.

(LL) If two lines are both incident to two distinct points, they coincide.

(LH) If a line and a hyperline are both incident to two distinct points, they are incident.

(HH) If two distinct hyperlines are both incident to two distinct points, the latter are incident to a line.

Here we show that any geometry of type $D_n$ or $E_6$ is a building. For thick finite geometries of type $D_n$, this is stated in Timmesfeld [7], where the case $n = 4$ is attributed to Th. Meixner; the proof in [loc. cit.], however, is valid in general. Also, Tits [11] has observed that thick finite geometries of type $A_n$, $D_n$, $E_6$ or $E_7$ are buildings on the basis of a case-by-case argument. Our final proposition extends this observation to geometries of arbitrary spherical type whose residues of type $C_3$ are buildings by use of a unified proof, valid in all cases.

2. **COROLLARY.** (a) Every geometry of type $A_n$, $D_n$ or $E_6$ is a building.

(b) A geometry of type $E_7$ is a building if and only if it satisfies (LH).

(c) A geometry of type $E_8$ is a building if and only if it satisfies (LH) and (HH).
PROOF. By the above theorem, every geometry of type $A_n$ is a building. In order to apply Tits' theorem to the other cases, we prove $(O)$ and $(LL)$ for $D_n$, $E_6$ and $E_7$, and — assuming $(LH)$ — also for $E_8$.

For the sake of completeness we shall repeat the argument given by Timmesfeld [7] in case of $D_n$.

First, we introduce some terminology.

If $v_1, v_2, \ldots, v_t$ are elements of a geometry, we say that $v_1 - v_2 - \cdots - v_t$ is a chain whenever $v_i$ and $v_{i+1}$ are incident for each $i (1 \leq i < t)$.

For any object $X$ the set of points in $\text{Res}(X)$ is called the point shadow of $X$, and the set of lines in $\text{Res}(X)$ is called the line shadow of $X$.

We may assume $n>3$.

Step 1. If every geometry of type $D_n-1$ is a building, then any geometry of type $D_n$ satisfies $(LL)$.

Given lines $L$ and $L_1$ both incident with two distinct points $P$ and $P_1$, we can find a chain $L-M-H-L_1$, where $M,H$ have types $m,h$ respectively. (Such a chain exists, as may be seen, e.g., in $\text{Res}(P)$ which is a building of type $D_{n-1}$ by assumption.)

Now $\{M,H\}$ is a flag, and $\text{Res}(\{M,H\})$ is a projective space containing $P$ and $P_1$ and hence also a line $L_2$ incident with $P$ and $P_1$.

Moreover, $\text{Res}(M)$ is a projective space containing $P$, $P_1$ and the lines $L$, $L_2$ incident to both points, whence $L=L_2$. Similarly for $\text{Res}(H)$, we see that $L_1=L_2$, so that $L=L_1$.

Step 2. If a geometry of type $D_n$, $E_6$, $E_7$ or $E_8$ satisfies $(LL)$ and if $\text{Res}(P)$ satisfies $(O)$ for each point $P$, then the geometry itself satisfies $(O)$.

Suppose $M$ and $M_1$ are both objects of type $l_i$ for some $i$ with the same point shadow $S$. Let $L$ be a line incident to $M$.

Take distinct points $P$ and $P_1$ incident to $L$. Then $P$ and $P_1$ belong to $S$, so there is a line $L_1$ incident to $P$, $P_1$ and $M_1$.

In view of $(LL)$, we have $L=L_1$, so $L$ belongs to the line shadow of $M_1$. Thus the line shadow of $M_1$ contains the line shadow of $M$, and by symmetry, the line shadows of $M$ and $M_1$ coincide. Consequently, the point shadows of $M$ and $M_1$ in $\text{Res}(P)$ coincide. Since $(O)$ holds for $\text{Res}(P)$ by assumption, $M$ and $M_1$ coincide.

Step 3. Statement (a) holds.

For the case $D_n$, this is immediate from the two previous steps by induction on $n$.

Thus, consider a geometry $G$ of type $E_6$. In view of the theorem, Step 2 and statement (a), we need to show $(LL)$.

Let $L$ and $L_1$ be lines, both incident to the distinct points $P$ and $P_1$. Since
Res(P) is a building of type $D_5$, we can find a hyperline $H$ of $G$ in Res(P) incident to both $L$ and $L_1$. But Res(H) is a building of type $D_5$, too, so satisfies (LL). Since $P$ and $P_1$ (being incident to $L$) are elements of Res(H), this yields $L = L_1$.

Step 4. Every geometry of type $E_7$ satisfies (O) and (LL). In particular, statement (b) holds.

In view of the two previous steps, we need only verify property (LL) for a geometry of type $E_7$.

Given lines $L$ and $L_1$ both incident to two distinct points $P$ and $P_1$, we can find a hyperline $H$ in Res(P) which is incident to both $L$ and $L_1$. (For Res(P) is a building of type $E_6$.)

Since $P$ and $P_1$ are both elements of the building Res(H) of type $D_6$, it follows that $L = L_1$.

Step 5. Let $G$ be a geometry of type $E_8$. If $G$ satisfies (LL), then (O) holds. If $G$ satisfies (LH), then (LL) holds. In particular, statement (c) holds.

The first part follows from Step 2, Step 4 and the theorem. Suppose $G$ has property (LH). Let $P$ and $P_1$ be two distinct points, both incident to the lines $L$ and $L_1$. Take a hyperline $H$ incident to $L_1$. Then $P$ and $P_1$ are also incident to $H$, so applying (LH) to $L$ and $H$, we find that $L$ is incident to $H$. But Res(H) is a building due to (a), hence satisfies (LL). It follows that $L = L_1$.

This ends the proof of Step 5, and hence the proof of the corollary.

3. REMARKS. (i). Using a result of Buekenhout and Shult [1, 2] and an older (elementary) result of Tits [8] 7.3n, we can prove that a geometry of type $D_6$ is a building without recourse to the above theorem.

(ii). For a geometry of type $E_8$ the properties (LH) and (HH) can be replaced by the following property:

For each pair of distinct hyperlines, both incident to two distinct points, there is a line incident to both points and both hyperlines.

4. EXAMPLE. We exhibit a quotient of the unique building $G$ of type $E_7$ over the field $\mathbb{C}$ of complex numbers by a group of order 2, which is a geometry of type $E_7$, but not a building. This shows that Condition (LH) in the corollary is not superfluous. The example is an analogue of the ones given by Tits [10] for buildings of type $C_n$, and is given in terms of Ferrar's presentation [5] of $E_7$.

Let $J$ be the exceptional 27-dimensional Jordan algebra over the field $\mathbb{R}$ of the real numbers with positive definite trace form, and denote by $J$ its complexification, by $t$ complex conjugation with respect to the real form $J_1$, by $\langle \ldots \rangle$ the standard bilinear form on $J$ (positive definite on $J_1$), by $N(.)$ the standard cubic form, by $\langle \ldots \rangle$ its linearization, and by $\ast$ the cross product such that $\langle A \ast B, C \rangle = 6\langle A, B, C \rangle$ for $A, B, C$ in $J$. 396
The ternary algebra $M_1$ is the 56-dimensional real vector space $\mathbb{R} + \mathbb{R} + J_1 + J_1$, supplied with the alternating bilinear form \{ , \} given by

$$\{x_1, x_2\} = a_1 b_2 - a_2 b_1 + \langle A_1, B_2 \rangle - \langle A_2, B_1 \rangle$$

and the symmetric trilinear product $(x_1, x_2, x_3)\mapsto x_1 x_2 x_3$ obtained by linearizing the expression

$$xxx = 6(-a^2 b + a \langle A, B \rangle - 2N(b))$$

$$b^2 a - b \langle A, B \rangle + 2N(a),$$

$$\left(\begin{array}{l} \langle a, b \rangle - \langle A, B \rangle \end{array}\right) A - b B + B + (A^* A),$$

$$\left(\begin{array}{l} \langle a, b \rangle - \langle A, B \rangle \end{array}\right) B + a A^* A - A^*(B^* B),$$

where $x=(a, b, A, B)$ and $x_i=(a_i, b_i, A_i, B_i)$ in $M_1$ for each $i$ (thus $a_i, b_i$ in $\mathbb{R}$ and $A_i, B_i$ in $J_1$).

Then $Aut M_1$ is a real Lie group of type $E_7$. Denote by $M$ the complexification of $M_1$ and retain the notation for the linear extensions to $M$ of the bilinear form and the ternary multiplication on $M_1$.

Consider the subset $S$ of $M$ consisting of all members of $M$ for which $xxM$ is contained in $\mathbb{C} x$.

Instead of $\mathbb{C} x$, we shall also write $\langle x \rangle$. For $x_1, x_2$ in $M$ with $\langle x_1 \rangle, \langle x_2 \rangle$ distinct elements of $S$, we call $\langle x_1 \rangle, \langle x_2 \rangle$ adjacent — notation $\langle x_1 \rangle - \langle x_2 \rangle$ — whenever $Mx_1 x_2$ is contained in $\mathbb{C} x_1 + \mathbb{C} x_2$. For details on $M$ and $S$, the reader is referred to Faulkner [4] and Ferrar [5].

We are interested in the graph $(S, \sim)$. It has diameter 3, and two vertices $\langle x_1 \rangle$ and $\langle x_2 \rangle$ have distance at most 2 if and only if $\{x_1, x_2\} = 0$. Moreover, $(S, \sim)$ is isomorphic to the graph $(S', \tilde{\sim})$ obtained from the building $G$ by letting $S'$ be the set of points of $G$, and letting $P \sim Q$ for distinct $P, Q$ in $S'$ stand for the existence of a line of $G$ incident to both $P$ and $Q$, i.e., for collinearity of $P$ and $Q$.

It is known [3] that $G$ can be uniquely reconstructed from $(S', \tilde{\sim})$ up to isomorphism. Thus, any automorphism of $(S', \tilde{\sim})$ extends uniquely to an automorphism of $G$.

From now on we shall identify $(S, \sim)$ and $(S', \tilde{\sim})$.

Consider the semilinear transformation $f$ of $M$ given by

$$x^f = (-\bar{b}, \bar{a}, -B^*, A^*),$$

where $x=(a, b, A, B)$ in $M$.

The transformation $f$ preserves $S$ and induces an automorphism $f|S$ of $(S, \sim)$ of order two such that $P$ has distance 3 to $P^f$ in $(S, \sim)$ for any vertex $P$ of $S$. It readily follows that the unique automorphism of $G$ extending $f|S$ satisfies condition $(Q3)$ of [10] and that the group $F$ of order two which it generates acts freely on the set of all flags of $G$ of corank 2.

Hence, by [loc. cit.], the quotient $G/F$ is a geometry of type $E_7$, but not a building (e.g., since there are points in this quotient which are collinear to exactly two points of a line).

In order to deal with thick finite geometries we need the following lemma.
5. **Lemma.** Let $s$ be a natural number. Suppose $S$ is a finite regular connected graph with $v$ points, valency $k=0 \pmod{s}$ and diameter $d$ such that for all $i$ (1 ≤ $i$ ≤ $d$) and all $x, y$ in $S$ with mutual distance $d(x, y) = i$ we have

\[ \{z \in S | 1 = d(x, z) = i - d(y, z)\} \equiv 1 \pmod{s}, \]

and

\[ \{z \in S | 1 = d(x, z) = d(y, z) - i\} \equiv 0 \pmod{s}. \]

Then

(i) Each eigenvalue $w$ of the adjacency matrix $A$ of $S$ distinct from $k$ satisfies $w \equiv -1 \pmod{p}$ for every maximal ideal $p$ containing $s$ of the ring of algebraic integers generated by the eigenvalues of $A$. Moreover, $v \equiv 1 \pmod{s}$.

(ii) If $g$ is an automorphism of $S$ without fixed points such that each point in $S$ is not adjacent to its image under $g$, then $s = 1$.

**Proof.** (i). By induction with respect to $e$, one can show that for every $e \geq 0$ the $x, y$ entry of $(A + I)^e$ equals 0 if $d(x, y) > e$ and equals 1 (mod $s$) if $d(x, y) \leq e$.

Thus, there is a matrix $B$ with integral coefficients such that

\[ (A + I)^d = J + sB, \tag{*} \]

where $J$ is the 'all 1' matrix.

Since $S$ is regular of valency $k$, the matrices $A$, $J$, $I$ and $B$ commute and the 'all 1' vector $j$ is a common eigenvector.

By (4), we get $(k + 1)^d_j = (v + s)t_j$ for some integer $i$, and hence that $(k + 1)^d_v = v \pmod{s}$. It follows that $v \equiv 1 \pmod{s}$ as $k \equiv 0 \pmod{s}$ by assumption. This proves the last statement of (i).

Since $A$ and $J$ are symmetric and $S$ is connected, all other common eigenvectors are real and orthogonal to $j$ (with respect to the standard inner product).

Let $w$ be an eigenvalue of $A$ distinct from $k$ corresponding to the common eigenvector $b$, say. Then $Jb = 0$, so by (4), we get

\[ (w + 1)^d b = (sm)b \]

for some algebraic integer $m$.

Consequently, $w + 1 \equiv 0 \pmod{p}$ for each maximal ideal $p$ containing $s$, whence (i).

(ii). Let $M$ denote the permutation matrix of $g$ on the points of $S$ (considered as the basis of a real vector space). Suppose that $p$ is a maximal ideal containing $s$. Now, $A$ and $M$ commute as $g$ is an automorphism of $S$, so in view of (i) the matrix $(A + I)M$ has eigenvalues which are all 0 (mod $p$) except for the eigenvalue $(k + 1)$ of multiplicity 1 with eigenvector $j$. As $k \equiv 0 \pmod{p}$, we get trace $((A + I)M)^1 \equiv 1 \pmod{p}$. On the other hand, this trace equals 0 according to the assumption that no point is adjacent to or coincides with its image under $g$. This contradiction shows that $s$ does not belong to any maximal ideal, whence $s = 1$ and we are done.

6. **Remark.** Let $M$ be one of the diagrams in Theorem 1. Consider a thick finite building of type $M$. It can easily be checked that the (collinearity) graph
$S$ on the set of points in which two vertices $x, y$ are adjacent if and only if there is a line incident to both $x$ and $y$, satisfies the hypotheses of the above lemma, with $s$ a power of the characteristic of the field over which the building is defined. In particular, $s > 1$.

7. REMARK. We are indebted to F. Timmesfeld for pointing out to us the existence of a thick finite geometry of type $C_3$ associated with the alternating group on 7 letters, which is not covered by a building (in fact, the corresponding chamber system is 2-connected), see Kantor [6].

The involution exchanging opposite vertices of the octahedron (the thin building of type $C_3$) yields a quotient geometry of type $C_3$ which is not a building. In view of the existence of these geometries, the following result is to a certain extent best possible.

8. PROPOSITION. Any thick finite geometry of spherical type all of whose residues of type $C_3$ are covered by buildings is a building.

PROOF. For the types $A_n$, $D_n$ and $E_6$ this is immediate from the corollary. However, we shall not make use of this.

Let $G$ be a thick finite geometry of spherical type $M$. Without loss of generality, we may assume that $M$ is connected.

If $n \leq 2$, there is nothing to prove.

Suppose that $n \geq 3$.

If $M = H_3$ or $H_4$, the statement is easily verified by use of the Feit-Higman Theorem.

Since each residue of type $C_3$ is covered by a building, Theorem 1 of [10] yields that $G = D/A$ (up to isomorphism), where $D$ is a building of type $M$ and $A$ is a group of automorphisms of $D$ acting freely on the set of all flags of corank 2 and satisfying $(Q1)$ of [10]. Denote by $p : D \rightarrow G$ the canonical projection.

Let $X$ be an element of $D$. By induction on $n$, the residue of $X^p$ is a building, so by [10] 6.1.8, the restriction of $p$ to Res($X$) is an isomorphism onto Res($X^p$). Consequently, Res($X$) is a finite building, and so is $D$ by [9]. By $(Q1)$, the restriction of $p$ to Res($X$) is induced by the quotient map with respect to the stabilizer of $X$ in $A$. Thus, any automorphism in $A$ fixing $X$ fixes every element of Res($X$), and, by connectedness of $D$, is the identity.

Suppose $A$ contains a nontrivial automorphism $a$.

According to $(Q2')$ of [10] – a consequence of $(Q1)$ – there is no element $Y$ of $D$ incident to both $X$ and $X^p$. In particular, for each point $P$ of $D$ its image $P^a$ is not collinear with $P$. This, however, contradicts (ii) of Lemma 5 in view of Remark 6. It follows that $A$ is trivial, and that $G = D$ is a building.

9. REMARK. In the above proposition for geometries of type $C_n$ we can weaken the ‘thick’ part of the hypothesis by only requiring that lines are incident to at least three points.
10. ACKNOWLEDGEMENT

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Appendix

On the distance between opposite vertices in buildings of spherical type

by J. Tits

In [LA], 1.4, example (b), I indicated how, starting from a compact Lie group of type $B_n$ (and also from certain non compact groups), one can construct geometries of type $B_n$ which are not buildings (they are not simply connected). In the above paper (§ 4), a similar construction is described in the case of $E_7$. The main purpose of this appendix is to show, by a uniform proof, that the same method applies to a compact simple Lie group $G$ if and only if the connection index (order of the fundamental group = order of the centre of the universal covering) of $G$ is $\leq 2$; in other words, excluded are only the types $A_n (n \geq 2), D_n, E_6$. This shows that Corollary 2 (a) of the above paper is, in a certain sense, “best possible”.

Let $\Phi$ be a reduced irreducible root system in a real vector space $V$ (we suppose $V$ spanned by $\Phi$), let $A = \{a_1, \ldots, a_l\}$ be a basis of $\Phi$ and let $d = \sum_{i=1}^l d_i a_i$ denote the highest root. The following lemma is well known.
LEMMA. Let \( i \in \{1, \ldots, l\} \), and let \( a_{i_1}, \ldots, a_{i_m} \) denote the elements of \( A \) which are connected with \( a_i \) in the Dynkin diagram. Then, \( d_i \geq \frac{1}{2} \sum_{j=1}^{m} d_j \).

Let \( f: V \to R \) be the coroot associated with \( a_i \), that is, \( f(v) = 2(a_i, v)/(a_i, a_i) \). The number \( f(a_i) \) is equal to 2 if \( k = i \), is a strictly negative integer if \( k \in \{i_1, \ldots, i_m\} \), and is zero otherwise. Since \( d + a_i \) is not a root, we have

\[
0 \leq f(d) = 2d_i + \sum_{j=1}^{m} f(a_j) d_j \leq 2d_i - \sum_{j=1}^{m} d_j,
\]

hence the claim.

PROPOSITION. For \( i, j \in \{1, \ldots, l\} \), with \( i \neq j \), set \( \Phi(i, j) = \{ \sum c_k a_k \in \Phi \mid c_j \neq 0, c_j \neq 0 \} \) and let \( V(i, j) \) denote the subspace of \( V \) spanned by \( \Phi(i, j) \). Then, one has \( V(i, j) \neq V \) if and only if \( d_i = d_j = 1 \).

If \( d_i = d_j = 1 \), we have

\[
V(i, j) \subset R \cdot (a_i + a_j) + \sum_{k \neq i, j} R \cdot a_k \neq V.
\]

To prove the converse, we assume, without loss of generality, that \( i = 1 \), that \( d_j \geq 1 \) and that \( (a_1, a_2, \ldots, a_j) \) is the unique chain joining \( a_1 \) and \( a_j \) on the Dynkin diagram (i.e., that \( a_k \) and \( a_{k+1} \) are connected in that diagram for \( 1 \leq k \leq j-1 \)). Set \( b = a_1 + \ldots + a_j \) and let \( b_0 = b; b_1, \ldots, b_m = d \) be a sequence of roots such that \( b_{r} - b_{r-1} \in A \) for \( 1 \leq r \leq m \). Clearly, \( V(i, j) \) contains all \( b, (0 \leq r \leq m) \), hence also the set \( A' = \{ b, -b_{r-1} \mid 1 \leq r \leq m \} \). The smallest integer \( j' \in [1, j] \) such that \( d_{j'} \geq 2 \) whenever \( j' \leq k \leq j \) must be 1 or 2, otherwise the above lemma would imply \( 1 = d_{j-1} > \frac{1}{2}d_j \). Consequently, \( a_2, a_3, \ldots, a_j \) belong to \( A' \) and so do of course \( a_{j+1}, \ldots, a_i \). Therefore,

\[
V(i, j) \supset R \cdot b + \sum_{k=1}^{j} R \cdot a_k = V,
\]

q.e.d.

Let \( \Sigma \) denote the Coxeter complex associated with \( \Phi \), that is, “cut out” on the unit sphere of \( V \) (identified with its dual) by the kernels of the roots. Let \( \Delta \) be a building of type \( \Sigma \); here, “weak buildings” in the sense of [BN] are simply called “buildings,” so that we allow \( \Delta \) to be \( = \Sigma \). Remember that two vertices of \( \Delta \) are said to be in “generic position” if they belong to opposite chambers.

COROLLARY 1. Two vertices \( p, q \) of \( \Delta \) of nonopposite types \( i \) and \( j \) and in generic position are at distance at least two in the graph of vertices of \( \Delta \). They are at distance exactly two if and only if \( d_i = d_j = 1 \).

By [BN], 3.9, it suffices to consider the case where \( \Delta = \Sigma \). Let \( q' \) be the vertex of \( \Sigma \) opposite to \( q \) and let \( j' \) be its type. We assume, without loss of generality, that \( p \) and \( q' \) are the vertices of types \( i \) and \( j' \) of the “fundamental chamber”

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corresponding to the basis $A$. Clearly, the kernels of all elements of $\Phi(i,j)$ (cf. the Proposition) separate $p$ and $q$. Therefore, $p$ and $q$ are at distance at least 2 in the graph of $\sum$, and if they are at distance 2, any point connected with both of them must belong to the kernel of every element of $\Phi(i,j)$, which implies $d_i = d_j = 1$, by the Proposition. Conversely, if $d_i = d_j = 1$, the point $r \in \sum$ defined by

$$d(r) = a_k(r) = 0 \quad (k \neq i, j) \quad \text{and} \quad a_l(r) > 0$$

is connected with both $p$ and $q$ because it is separated from them by no kernel of root. Indeed, if a root $\sum c_k a_k$ is strictly positive on $r$, one has $c_i > c_j$ (because $a_i(r) + a_j(r) = d(r) = 0$), hence $c_i = 1$, $c_j = 0$ or $c_i = 0$, $c_j = -1$, and the root $\sum c_k a_k$ takes positive values in both $p$ and $q$, hence the corollary.

**COROLLARY 2.** The distance between a vertex of $\Delta$ of type $i$ and any opposite vertex is $\geq 3$; it is equal to 3 if and only if there exists $j \in \{1, \ldots, \widehat{i}, \ldots, I\}$ such that $d_i = d_j = 1$.

This is an immediate consequence of the previous corollary.

From Corollary 2 and [LA], 1.3 (cf. condition (Q3)), we deduce:

**COROLLARY 3.** Suppose that $d_i = 1$ for at most one value of $i$ (i.e. that $\sum$ is of type $C_n$, $F_4$, $E_7$ or $E_8$) and let $\Gamma$ be an automorphism group of $\Delta$ whose orbits consist of pairwise opposite vertices. Then $\Delta/\Gamma$ is a geometry of type $M$ (in the sense of [LA]), where $M$ is the Coxeter matrix of $\sum$.

**EXAMPLE.** Let $K$ be a field, $L$ a separable quadratic extension, $\Gamma$ the Galois group, $G$ a simple algebraic group, defined and anisotropic over $K$ and whose relative Weyl group over $L$ is of type $M = C_n$, $F_4$, $E_7$ or $E_8$ (e.g. $K = \mathbb{R}$, $L = \mathbb{C}$ and $G$ is a compact group of type $B_n$, $C_n$, $F_4$, $E_7$ or $E_8$), and $\Delta$ the building of $G$ over $L$. The proof of 4.7 in [GR] shows that if $P$, $P'$ are two parabolic subgroups of $G$ defined over $L$ and permuted by the nontrivial element of $\Gamma$, the unipotent radical $R_u(P \cap P')$ is defined over $K$. Since $G$ is $K$-anisotropic, it follows that $R_u(P \cap P')$ is trivial; in other words, $P \cap P'$ is reductive, which means that $P$ and $P'$ are opposite. Consequently, $\Delta/\Gamma$ is a geometry of type $M$, by Corollary 3.

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