Technische
Universiteit
Eindhoven

Instituut Wiskundige Dienstverlening Eindhoven

REPORT IWDE 89-01

TORSIONAL VIBRATIONS IN A CRANK SHAFT

J. Molenaar

January 1989

Den Dolech 2
Postbus 513
5600 MB Eindhoven

1. Introduction

In this report, we analyze the torsion in the crank shaft of a gas compressor with one crank. The geometry is as drawn in figures 1a and 1b. Two forces are exerted on the crank. First, the crank shaft is at one end driven by a motor. We assume that this happens with uniform angular velocity, irrespective of the load of the crank. Second, the piston meets with the gas force, which is passed on to the crank via the driving shaft. The geometry of the system implies, that the moment of inertia around the crank shaft of all moving parts together depends on the rotation angle ϕ of the crank shaft. The equation of motion for the torsion in this shaft has consequently periodically varying coefficients. A common approach to analyze this torsion is to average these coefficients over a period [1]. The resulting modelling equation can be dealt with in terms of eigenfrequencies, resonances and solutions, which are periodic in time. However, the solutions of the original equation of motion might be non-periodical and even show chaotic behaviour.

The purpose of this study is to compare the behaviour of the reduced model with that of the original one in order to check the reliability of the reduction in case of realistic compressor data.

Fig 1. Sketch of the geometry of crank, driving shaft and piston in side-view (a) and top-view (b).

2. Equation of Motion

The crank shaft is at one end driven with constant angular velocity ω and the rotation angle ϕ_0 at this end is given by

$$\phi_0(t) = \omega t. \quad (2.1)$$

The rotation angle $\phi(t)$ of the crank will deviate from ϕ_0 by the torsion angle $\Delta\phi$:

$$\Delta\phi(t) = \phi(t) - \phi_0(t). \quad (2.2)$$

Because all movements of crank, driving shaft and piston take place in a plane perpendicular to the crank shaft, the total impulse moment J of all these moments together is directed along the crank shaft. It can be written as

$$J(\phi, \dot{\phi}) = I(\phi) \dot{\phi} \quad (2.3)$$

with the total moment of inertia I given by

$$I(\phi) = I_c + I_p Z^2(\phi). \quad (2.4)$$

The constants I_c and I_p are defined in the appendix by (A.11b,c) and the dimensionless factor $Z(\phi)$ by (A.7).

The second law of Newton for rotating rigid bodies states that

$$\dot{J} = \sum_i M_i \quad (2.5)$$

with the summation over all momenta M_i under consideration. In the present model, three momenta are taken into account, namely the torsional, the gas and the frictional momentum. So, we may write

$$\dot{J}(\phi, \dot{\phi}) = M_{\text{torsion}} + M_{\text{gas}} + M_{\text{friction}}. \quad (2.6)$$

Following Hooke's law, we approximate M_{torsion} by

$$M_{\text{torsion}} = -k (\phi - \phi_0). \quad (2.7a)$$

The torsion constant $k(>0)$ depends on the material properties of the shaft.

In the appendix, expressions for M_{gas} and M_{friction} are derived. The former is given by

$$M_{\text{gas}} = F_g(\phi) R Z(\phi) \quad (2.7b)$$

with F_g the force exerted on the piston by the compressed and decompressed gas. F_g is a periodic function of ϕ .

An example of the dependence of F_g on ϕ for a realistic compressor is given in figure 2a.

Figure 2a. Plot of $F_g(\phi)$ for the data also used in figure 7a.

The expression for M_{friction} reads as

$$M_{\text{friction}} = -c(\phi) \dot{\phi} \quad (2.7c)$$

with $c(\phi)$ given in (A.15). It is a function of $|Z(\phi)|$.

Equation (2.6) with (2.7a, b and c) is a second order, inhomogeneous, non-linear, autonomous differential equation in ϕ . We are interested in the behaviour of the torsion angle $\Delta\phi(t)$, defined in (2.2). This torsion angle is in practice very small (< 0.01 radians). We may therefore linearize equation (2.6) around $\phi = \phi_0$ and use the approximations

$$I(\phi) = I(\phi_0) + I'(\phi_0) \Delta\phi$$

$$M_{\text{gas}}(\phi) = M_{\text{gas}}(\phi_0) + M_{\text{gas}}'(\phi_0) \Delta \phi \quad (2.8)$$

$$c(\phi) = c(\phi_0) + c'(\phi_0) \Delta \phi.$$

In addition to the linearization, we bring the equation of motion into dimensionless form by introducing the dimensionless variable

$$\tau = \omega t. \quad (2.8b)$$

If we neglect terms quadratic in $\Delta \phi$, we arrive at the linearized equation of motion (with $y \equiv \Delta \phi$ for notational ease):

$$\ddot{y} + a(\tau) \dot{y} + b(\tau) y = F(\tau). \quad (2.9)$$

Equation (2.9) may be read as the equation of motion of a particle with unit mass under influence of a harmonic force $-b y$, a frictional force $-a \dot{y}$ and a driving force F . Both the coefficients a and b and the driving force F are periodic in τ with period 2π . They are given by:

$$a(\tau) = \frac{2I' \omega^2 + c \omega}{I \omega^2} \quad (2.10a)$$

$$b(\tau) = \frac{I'' \omega^2 + c' \omega + k - M_{\text{gas}}'}{I \omega^2} \quad (2.10b)$$

$$F(\tau) = \frac{M_{\text{gas}} - I' \omega^2 - c \omega}{I \omega^2}. \quad (2.10c)$$

For convenience, we transform equation (2.9) into frictionless form by means of the substitution

$$y(\tau) = x(\tau) \exp\left(-\frac{1}{2} \int_0^\tau a(s) ds\right) \quad (2.11a)$$

If we differentiate both sides of (2.11a), we find for the torsion velocity

$$\dot{y}(\tau) = \left(\dot{x}(\tau) - \frac{1}{2} x(\tau) a(\tau)\right) \exp\left(-\frac{1}{2} \int_0^\tau a(s) ds\right). \quad (2.11b)$$

Transformation (2.11a) leads to the equation

$$\ddot{x} + d(\tau) x = \bar{F}(\tau) \quad (2.12)$$

with the definitions

$$d(\tau) = b(\tau) - \frac{1}{2} \dot{a}(\tau) - \frac{1}{4} a^2(\tau) \quad (2.13a)$$

$$\bar{F}(\tau) = F(\tau) \exp\left(+\frac{1}{2} \int_0^\tau a(s) ds\right). \quad (2.13b)$$

If we introduce the vectors

$$\mathbf{x} \equiv \begin{bmatrix} x \\ \dot{x} \end{bmatrix} ; \mathbf{F} = \begin{bmatrix} 0 \\ \bar{F} \end{bmatrix} , \quad (2.14)$$

equation (2.12) is cast into the standard form

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{F} \quad (2.15)$$

with the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -d(\tau) & 0 \end{bmatrix} . \quad (2.16)$$

It is easily checked by substitution, that the solution of (2.15) is given by

$$\mathbf{x}(\tau) = \phi(\tau, \sigma) \mathbf{x}(\sigma) + \int_{\sigma}^{\tau} \phi(\tau, s) \mathbf{F}(s) ds , \quad (2.17)$$

with $\mathbf{x}(\sigma)$ a known vector of initial values. The transition matrix ϕ is defined as the solution of the initial value problem

$$\begin{aligned} \dot{\phi}(\tau, \sigma) &= \mathbf{A}(\tau) \phi(\tau, \sigma) \\ \phi(\sigma, \sigma) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \end{aligned} \quad (2.18)$$

In (2.18) the derivative is taken with respect to the first argument. Note, that ϕ is completely determined by the homogeneous version of equation (2.15), thus with $\mathbf{F} \equiv 0$. If we take $\mathbf{F} \equiv 0$ in (2.17), we may directly conclude that

$$\phi(\tau, \tau') \phi(\tau', \sigma) = \phi(\tau, \sigma) \quad (2.19)$$

for arbitrary τ' . This implies, that ϕ is non-singular for all arguments, because it is non-singular for $\tau = \sigma$. If $\mathbf{A}$ would be constant, ϕ would be invariant with respect to a translation of both arguments. Thus we have

$$\phi(\tau, \sigma) = \phi(\tau - \sigma) \quad (2.20)$$

for constant $\mathbf{A}$. In the following section this property is used.

3. Approximation of $d(\tau)$

The coefficient $d(\tau)$ in (2.12) is a strongly varying function of $\tau \bmod 2\pi$. In fig. 2b a plot of d versus τ is given for a typical data set from practice.

*Figure 2b. Plot of the coefficient $d(\tau)$ for the data also used in figure 7a.
The dotted line represents the corresponding staircase function calculated from (3.1).*

In the literature, no method is known to determine the corresponding transition matrix analytically. Instead of resorting to numerical methods, we prefer to apply a technique, which is proven to be reliable in many cases [2]. This method is based on the observation that the contribution of the Fourier components of $d(\tau)$ to the calculation of ϕ strongly decreases with increasing frequency. This inspired Franks and Sandberg [3] to replace $d(\tau)$ by a more regularly shaped function with a Fourier spectrum, which closely coincides with that of $d(\tau)$ as far as the lower frequencies are concerned. We use a staircase function with N equal intervals. The height of the n -th step d_n can be calculated from the formula

$$d_n = \sum_{\substack{m \\ |m| < N/2}} \frac{(m \pi / N)}{\sin(m \pi / N)} e^{im\pi(2n-1)/N} \bar{d}_m. \quad (3.1)$$

The Fourier component $\bar{d}_m$ of $d(\tau)$ is, as usually, defined by

$$\bar{d}_m = \frac{1}{2\pi} \int_0^{2\pi} d(\tau) e^{-im\tau} d\tau. \quad (3.2)$$

To give an idea of this replacement procedure, we also present in fig. 2b the corresponding staircase function for $N = 16$.

The transition function of the staircase function is easily obtained. At the intervals $[t_{n-1}, t_n]$, with $t_n = \frac{n 2\pi}{N}$, $n = 1, \dots, N$, the matrix A reduces to

$$A_n = \begin{bmatrix} 0 & 1 \\ -d_n & 0 \end{bmatrix} \quad (3.3)$$

with transition matrix

$$\phi_n(\tau, \sigma) = \exp(A_n(\tau - \sigma)). \quad (3.4)$$

Evaluation of the exponent in (3.4) via its powerseries yields

$$\phi_n(\tau, \sigma) = \begin{bmatrix} \cos \sqrt{d_n}(\tau - \sigma) & \frac{1}{\sqrt{d_n}} \sin \sqrt{d_n}(\tau - \sigma) \\ -\sqrt{d_n} \sin \sqrt{d_n}(\tau - \sigma) & \cos \sqrt{d_n}(\tau - \sigma) \end{bmatrix} \quad \text{if } d_n \geq 0 \quad (3.5a)$$

$$= \begin{bmatrix} \cosh \sqrt{d_n}(\tau - \sigma) & \frac{1}{\sqrt{d_n}} \sinh \sqrt{d_n}(\tau - \sigma) \\ \sqrt{d_n} \sinh \sqrt{d_n}(\tau - \sigma) & \cosh \sqrt{d_n}(\tau - \sigma) \end{bmatrix} \quad \text{if } d_n \leq 0. \quad (3.5b)$$

If we use property (2.19), we may calculate ϕ for arbitrary arguments by evaluating products of ϕ_n matrices. This simplifies the evaluation of (2.17) considerably. It remains to evaluate the integral in (2.17) numerically, because the gas force F_g is not given in analytical form. A further simplification in the evaluation of (2.17) is obtained by writing it in an iterative form. If we want to evaluate the solution at grid points $\tau_i = i \Delta \tau$, $i = 0, 1, 2, \dots$, with $\Delta \tau$ a given increment, we may write

$$\mathbf{x}(\tau_{i+1}) = \phi(\tau_{i+1}, 0) \mathbf{x}(0) + \int_0^{\tau_{i+1}} \phi(\tau_{i+1}, s) \mathbf{F}(s) ds$$

$$= \phi(\tau_{i+1}, 0) [\mathbf{x}(0) + \mathbf{K}(0, \tau_{i+1})] \quad (3.6)$$

with $\mathbf{K}$ defined by

$$\mathbf{K}(\tau, \tau') = \int_{\tau}^{\tau'} \phi(0, s) \mathbf{F}(s) ds. \quad (3.7)$$

If we use that

$$\mathbf{K}(0, \tau_{i+1}) = \mathbf{K}(0, \tau_i) + \mathbf{K}(\tau_i, \tau_{i+1}) \quad (3.8a)$$

and

$$\phi(\tau_{i+1}, 0) = \phi(\tau_{i+1}, \tau_i) \phi(\tau_i, 0), \quad (3.8b)$$

we obtain the recursive formula

$$\mathbf{x}(\tau_{i+1}) = \phi(\tau_{i+1}, \tau_i) [\mathbf{x}(\tau_i) + \phi(\tau_i, 0) \mathbf{K}(\tau_i, \tau_{i+1})]. \quad (3.9)$$

Thus, if $\phi(\tau_i, 0)$ and $\mathbf{x}(\tau_i)$ are known at some gridpoint τ_i , the calculation of $\mathbf{x}(\tau_{i+1})$ and $\phi(\tau_{i+1}, 0)$ merely needs the analytical evaluation of $\phi(\tau_{i+1}, \tau_i)$, the numerical evaluation of $\mathbf{K}(\tau_i, \tau_{i+1})$, and some simple algebraic manipulations.

4. Discussion, Results and Conclusions.

To get some feeling for the possible complexity of the solutions of (2.9) with (2.10), it is instructive to interpret the different terms somewhat in detail. The frictional coefficient a contains a damping term proportional to c . This term stems from the physical friction in the familiar way. In (2.10a) also a term proportional to I' occurs. It changes sign and corresponds to a damping and an accelerating force alternatively. In the coefficient b in (2.10b) only the term proportional to k has definite sign, giving rise to a repulsive force in the familiar way. It is to be expected, that the solutions are stable if this term dominates the other ones. The generalized driving force F in (2.10c) contains, apart from the periodic gas force, a term proportional to I' , which changes sign, and one proportional to c , which tends to increase the torsional angle more and more in negative direction. A constant frictional coefficient c would lead to an increase of the average level of $\Delta\phi$ via F , and to a damping of the oscillations via a . However, the damping of the piston makes c to vary periodically and a driving force F , which varies periodically via c , may even excite the system.

It is impossible to analyze the complete behaviour of the system as a function of all parameters. We shall hereafter present a few illustrative examples. The parameters of our "standard" system are

$$\begin{aligned} I_{\text{crank}} &= 10 \text{ kg } m^2 \\ m_{d_1} &= 250 \text{ kg} \\ m_{d_2} &= 100 \text{ kg} \\ m_p &= 900 \text{ kg} \\ R &= 0.15 \text{ m} \\ R_c &= 0.15 \text{ m} \\ L &= 1.0 \text{ m} \\ \omega &= 40 \text{ rad / s} \\ k &= 5 \cdot 10^6 \text{ Nm.} \end{aligned} \tag{4.1}$$

In the subsequent calculation we shall modify this parameter set merely slightly and study the influence of averaging $I(\phi)$, friction, piston movement and gas force.

4A. Results for ϕ -independent $\bar{I}$

In the literature one often averages $I(\phi)$, using in (A.11a) the approximation:

$$\frac{1}{2\pi} \int_0^{2\pi} (\bar{Z})^2 d\psi = \frac{1}{2} \left(1 + \frac{R}{L}\right) \approx \frac{1}{2}. \quad (4.2)$$

This leads to a constant value $\bar{I}$. For the parameters in (4.1) $\bar{I}$ is equal to

$$\begin{aligned} \bar{I} &= I_{\text{crank}} + m_{d_1} R^2 + \frac{1}{2} (m_{d_2} + m_p) R^2 \\ &= 10 + (250 + \frac{1}{2} (100 + 900)) (0.15)^2 \\ &= 26.9 \text{ kg } m^2. \end{aligned} \quad (4.3)$$

In this approach (2.9) and (2.10) reduce to

$$\begin{aligned} \ddot{y} + a \dot{y} + by &= F \\ a &= c / (\bar{I} \omega) \\ b &= (k - M_{\text{gas}}') / (\bar{I} \omega^2) \\ F &= (M_{\text{gas}} - c \omega) / (\bar{I} \omega^2). \end{aligned} \quad (4.4)$$

In practice, we may assume

$$M_{\text{gas}}' \ll k.$$

This can, for example, be checked from fig. 2a with definition (2.7b) for M_{gas} . This implies, that b is nearly constant. In that case (4.4) describes an oscillator under a combination of a constant and a periodic force. The behaviour of the solutions of such a system are well-known [4]. The eigenfrequency ω_0 of the homogeneous ($F \equiv 0$) and frictionless ($a = 0$) equation is given by

$$\omega_0 = \sqrt{\frac{k}{\bar{I} \omega^2}} \quad \omega \quad \text{rad / s.} \quad (4.5)$$

The constant force $-c / \bar{I} \omega$ makes the system to oscillate around an average level $\bar{\Delta \phi}$ given by

$$\bar{\Delta \phi} = -c \omega / k \quad \text{rad.} \quad (4.6)$$

The force $M_{\text{gas}}' / \bar{I} \omega^2$ has period 2π . If it would be sinusoidal, it could lead to amplitudes $\Delta \phi_{\text{max}}$ given by

$$\Delta \phi_{\text{max}} = \frac{A_{\text{gas}}}{\bar{I}} ((\omega_0^2 - \omega^2)^2 + (c / \bar{I})^2)^{-\frac{1}{2}} \quad (4.7)$$

with A_{gas} the maximum amplitude of M_{gas} . From fig. 2a we see that M_{gas} is sinusoidal only in approximation. Still, we expect expression (4.7) to yield reasonably reliable values. From fig. 2a, (2.7b) and (A.7b) we find

$$A_{\text{gas}} \approx 6.10^4. \quad (4.8)$$

In fig. 5a we present the solution of (4.4) with $a \equiv F \equiv 0$ and initial values $y(0) = 0$, $\dot{y}(0) = 1$.

The obtained angular frequency is in exact agreement with the result of expression 4.5:

$$\omega_0 = 10.8 \omega \text{ rad / s.} \quad (4.9)$$

In this special case, the amplitude is completely determined by the initial values:

$$\Delta \phi_{\max} = \frac{\omega}{\omega_0} = 0.092 \text{ rad.} \quad (4.10)$$

In fig. 5b, we show the effect of friction. The parameters are as in fig. 5a except for $\dot{y}(0) = 0$ and $c_c = 10^4$, $c_p = 0$ (see A.15). The average level is in exact agreement with (4.6):

$$\overline{\Delta \phi} = -0.012 \text{ rad.} \quad (4.11)$$

In fig. 5c, the effect of the gas force is illustrated. The parameters are as in fig. 5a (no friction). The deviation of the sinusoidal form of M_{gas} is apparently present in the solution. From (4.7) with (4.8) we obtain for this case the value

$$\Delta \phi_{\max} = 0.012 \quad (4.12)$$

which is in fairly good agreement with the average amplitude in fig. 5c.

4B. Results for ϕ -dependent I

Here, we present solutions of (2.9) with (2.10) using expression (2.4) for $I(\phi)$. To isolate the effect of the piston, we first take $c \equiv M_{\text{gas}} \equiv 0$ (no friction, no gas force). The corresponding solution is given in fig. 6a. It clearly contains two signals with frequencies ω_0 and ω respectively. If we add friction the results are not very different. In fig. 6b crank friction is taken into account by choosing in (A.15) $c_c = 5000 \text{ Ns}$ and $c_p = 0$, while in fig. 6c piston friction is introduced by using in (A.15) $c_c = 0$ and $c_p = 5000 \text{ Ns}$.

Figures 7a,b and c are obtained with the same parameter data as fig. 6a,b and c respectively, except for introduction of the gas force corresponding to F_{gas} in fig. 2a. For this parameter set the gas force appears to have little effect on the amplitude.

4C. Comparison of the use of $\bar{I}$ and I for various values of k

If one uses the constant moment of inertia $\bar{I}$, the behaviour of the amplitude can be understood from (4.7). This expression shows resonance if

$$\omega_0 \approx \omega. \quad (4.13)$$

If we fix the values of ω and $\bar{I}$ in (4.5) at $\omega = 40 \text{ rad / s}$ and $\bar{I} = 26.9 \text{ kg m}^2$, we find that resonance occurs if

$$k \approx 4.3 \cdot 10^4 \text{ Nm.} \quad (4.14)$$

In the calculations above we used $k = 5 \cdot 10^6 \text{ Nm}$, which is far from resonance. To study the

behaviour of $\Delta \phi_{\max}$ as a function of k , we present in table 1 results for $\Delta \phi_{\max}$ for various values of k . The (a) column has been calculated using $\bar{I}$, whereas in the (b) column the ϕ -dependence of I is exactly taken into account. In both cases the gas force is applied, but no friction is present ($c \equiv 0$).

$k(Nm)$	$\Delta \phi_{\max}(rad)$ (a)	$\Delta \phi_{\max}(rad)$ (b)
$5.0 \cdot 10^6$	$1.2 \cdot 10^{-2}$	$4.0 \cdot 10^{-3}$ (8a)
$2.5 \cdot 10^6$	$2.4 \cdot 10^{-2}$	$4.0 \cdot 10^{-2}$ (8b)
$1.0 \cdot 10^6$	$6.0 \cdot 10^{-2}$	$1.0 \cdot 10^0$ (8c)
$1.0 \cdot 10^5$	$4.7 \cdot 10^{-1}$	∞ (8d)
$4.3 \cdot 10^4$	∞	∞

*Table 1. Values for the maximal amplitude $\Delta \phi_{\max}$ for various values of k .
The (a) column is for constant moment of inertia $\bar{I}$, the (b) column for ϕ -dependent $I(\phi)$.
The numbers (8a-d) indicate to the figures 8a-d, in which the solutions are plotted.*

From table 1 it is clear that averaging out of the ϕ -dependence of I leads to a considerable smaller region of resonance at the k -axis than found by exact calculations. This implies, that results obtained by the approximation are not reliable to indicate resonant behaviour of the crank-piston system. The choice of the parameters in constructing compressors should always be checked by a complete analysis as described in this report.

Figure Captions

In all figures the parameter values as given in (4.1) are used except for the values explicitly indicated.

Fig. 5: averaged moment of inertia $\bar{I} = 26.9 \text{ kg m}^2$;

- a) $m_{d_1} = 750$, $m_{d_2} = m_p = 0$; $y(0) = 0$, $\dot{y}(0) = 1.0$; no friction; no gas force
- b) as under a) with $\dot{y}(0) = 0$ and $c_c = 10000.0 \text{ Ns}$; no gas force.
- c) as under a) with $\dot{y}(0) = 0$ and gas force; no friction

Fig. 6: ϕ dependent moment of inertia $I(\phi)$; no gas force.

- a) no friction
- b) $c_c = 5000.0$, $c_p = 0.0 \text{ Ns}$
- c) $c_c = 0.0$, $c_p = 5000.0 \text{ Ns}$

Fig. 7: ϕ dependent moment of inertia $I(\phi)$; with gas force.

- a) no friction
- b) $c_c = 0.0$, $c_p = 5000.0 \text{ Ns}$
- c) $c_c = 5000.0$, $c_p = 0.0 \text{ Ns}$

Fig. 8: various k -values

- a) identical to fig. 7a
- b) as under a) with $k = 2.5 \cdot 10^6 \text{ Nm}$
- c) as under a) with $k = 1.0 \cdot 10^6 \text{ Nm}$
- d) as under a) with $k = 1.0 \cdot 10^5 \text{ Nm}$

Figure 5a. Averaged I, No Friction, No Gas Force

Figure 5b. Averaged I, Crank Friction, No Gas Force

Figure 5c. Averaged I, No Friction, With Gas Force

Figure 6a. Exact I, No Friction, No Gas Force

Figure 6b. Exact I, Crank Friction, No Gas Force

Figure 6c. Exact I, Piston Friction, No Gas Force

Figure 7a. Exact I, No Friction, With Gas Force

Figure 7b. Exact I, Piston Friction, With Gas Force

Figure 7c. Exact I, Crank Friction, With Gas Force

Figure 8a. Exact I, No Friction, With Gas Force

Figure 8b. Exact I, No Friction, With Gas Force

Figure 8c. Exact I, No Friction, With Gas Force

Figure 8d. Exact I, No Friction, With Gas Force

Appendix

In this appendix, we derive expressions (2.4), (2.7b) and (2.7c) used above.

Moment of Inertia

To simplify the derivations, we model the physical driving shaft with length L by a dumb bell of the same length, with all mass concentrated at the ends. See fig. 3.

Figuur 3. The driving shaft is modelled as a dumb bell with the same rotational properties.

To assure that the dumb bell has the same rotational properties as the driving shaft, the following conditions have to be satisfied:

$$\begin{aligned} m_{d_1} + m_{d_2} &= m_d \\ m_{d_1} L_1 &= m_{d_2} L_2 \end{aligned} \quad (A.1)$$

with m_d the total mass of the driving shaft. To obtain the moment of inertia corresponding to the parts, which move horizontally, we have to express the position and velocity of the piston in terms of the rotation angle ϕ and the angular velocity $\dot{\phi}$ of the crank. For the notation, see fig. 4.

Figuur 4. Indication of the meaning of the symbols R , L , x , ϕ , α , β used in the text.

The position x of the piston is given by

$$x = R \cos \phi + L \cos \alpha. \quad (A.2)$$

By differentiating both sides of (A.2), we find an expression for the velocity $\dot{x}$ of the piston

$$\dot{x} = -(R \dot{\phi} \sin \phi + L \dot{\alpha} \sin \alpha). \quad (A.3)$$

It remains to express $\dot{\alpha}$, $\cos \alpha$, and $\sin \alpha$ in terms of ϕ and $\dot{\phi}$. From the geometry in fig. 4 we immediately derive that

$$\sin \alpha = \frac{R}{L} \sin \phi \equiv \lambda \sin \phi. \quad (A.4a)$$

Thus

$$\cos \alpha = \sqrt{1 - \lambda^2 \sin^2 \phi}. \quad (A.4b)$$

If we differentiate both sides of (A.4a), we find

$$\dot{\alpha} = \frac{\lambda \cos \phi}{\cos \alpha} \dot{\phi}. \quad (A.5)$$

Substitution of (A.4) and (A.5) into (A.3) yields

$$\dot{x} = -R Z(\phi) \dot{\phi} \quad (\text{A.6})$$

with

$$Z(\phi) \equiv (\sin \phi + \frac{1}{2} \lambda \sin (2\phi) (1 - \lambda^2 \sin^2 \phi)^{-\frac{1}{2}}) \quad (\text{A.7a})$$

which, for $\lambda \ll 1$, reduces to

$$\bar{Z}(\phi) \equiv \sin \phi + \frac{1}{2} \lambda \sin 2\phi. \quad (\text{A.7b})$$

The kinetic energy of the horizontally moving masses m_{hor} is

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2} m_{\text{hor}} \dot{x}^2 \\ &= \frac{1}{2} m_{\text{hor}} R^2 Z^2(\phi) \dot{\phi}^2. \end{aligned} \quad (\text{A.8})$$

If we equate this to the general expression for kinetic energy

$$E_{\text{kin}} = \frac{1}{2} I_{\text{hor}} \dot{\phi}^2, \quad (\text{A.9})$$

we obtain for the moment of inertia of the horizontally moving parts

$$I_{\text{hor}} = m_{\text{hor}} R^2 Z^2(\phi). \quad (\text{A.10})$$

So, the total moment of inertia $I(\phi)$ of all moving parts can be written as

$$I(\phi) = I_c + I_p Z^2(\phi) \quad (\text{A.11a})$$

with I_c and I_p constants, given by

$$I_c = I_{\text{crank}} + m_{d_1} R^2 \quad (\text{A.11b})$$

$$I_p = (m_{d_2} + m_p) R^2. \quad (\text{A.11c})$$

I_{crank} is the moment of inertia of the crank itself and m_p the mass of the piston together with all parts rigidly connected to it.

Gas Momentum

Because the driving shaft is, of course, not rigidly connected to the piston or the crank, it can only pass on a force directed along itself. The component of the force $F_g(\phi)$, exerted on the piston by the gas, along the driving shaft is (see fig. 4)

$$F_g \cos \alpha. \quad (\text{A.12a})$$

The impact of this force on the impulse momentum of the crank is given by

$$M_{\text{gas}} = F_g R \cos \alpha \sin \beta. \quad (\text{A.12b})$$

To express this in terms of ϕ , we write

$$\begin{aligned}
 \sin \beta &= \sin(\pi - (\phi + \alpha)) \\
 &= \sin(\phi + \alpha) \\
 &= \sin \phi \cos \alpha + \cos \phi \sin \alpha.
 \end{aligned}
 \tag{A.13}$$

Expressions for $\sin \alpha$ and $\cos \alpha$ are given in (A.4). We arrive at the expression

$$M_{\text{gas}} = F_g R (1 - \lambda^2 \sin^2 \phi) Z(\phi) \tag{A.14a}$$

which for $\lambda \ll 1$ reduces to

$$M_{\text{gas}} = F_g R \bar{Z}(\phi). \tag{A.14b}$$

Frictional Momentum

We assume the friction to be composed of two parts, one of which being proportional to the velocity of the crank shaft and the other proportional to the velocity of the piston. We introduce therefore two frictional constants c_c and $c_p (\geq 0)$, which have to be determined experimentally, and write

$$\begin{aligned}
 M_{\text{friction}} &= -c_c R_c \dot{\phi} - c_p \dot{x} \\
 &= -(c_c R_c + c_p R |Z(\phi)|) \dot{\phi} \\
 &\equiv -c(\phi) \dot{\phi}.
 \end{aligned}
 \tag{A.15}$$

Note, that R is the radius of the crank and R_c the radius of the crank shaft. We used expression (A.6) for $\dot{x}$, but introduced the absolute value of $Z(\phi)$ rather than $Z(\phi)$ itself, to make sure that the friction is oppositely directed to the crank movement. This implies, that $c(\phi)$ is not differentiable if $\phi = 0$ or π . At these points, Z vanishes as long as $\lambda < 1$. Because this derivative is used in the linearized equation of motion, the function $Z(\phi)$ is numerically smoothed around these points.

References

1. Hafner K.E., Maas H., Torsionsschwingungen in der Verbrennungskraftmaschine, 1985, Springer Verlag Wien, ISBN 3-211-81793-X.
2. Richards J.A., Analysis of Periodically Time-Varying Systems, 1983, Springer Verlag Berlin, ISBN, 3-540-11689-3.
3. Franks, L.E., Sandberg I.W., Bell Syst. Techn. J. 89, 1960, pp. 1321-1350.
4. Kibble, T.W.B., Classical Mechanics, 1966, McGraw-Hill Publishing Company Limited.