Synchronization of Coupled Nonlinear Dynamical Systems with Time-delay: Analysis and Application

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Preface

This report treats the results of my internship on the Tokyo Metropolitan University in Japan. I have stayed for 15 weeks in the Control Engineering Laboratory of Prof. Dr. Ing. Toshiki Oguchi, part of the Mechanical Engineering Department. Hereby I want to thank Professor Oguchi for the opportunity to stay in his laboratory, as well as the supervision and useful tips. Moreover I want to thank him for all the effort to make this period a pleasant and interesting educational and cultural experience, for example by arranging a visa and making a reservation in the International House of the university.

I also want to thank the students of the Control Engineering Laboratory for helping me getting around in the university and helping me with the project.

Thijs Kniknie
Abstract

For a network of four unidirectionally coupled nonlinear systems with time-delay, full and partial synchronization conditions are derived. Therefore, the stability of the zero solution of the error dynamics of the network is investigated. A Lyapunov-Krasovskii functional is defined and the conditions, under which the time-derivative is negative are investigated. However, to do so, the nonlinear terms in the error dynamics have to be rewritten.

To deal with the nonlinear terms in the error dynamics, two methods are employed: using the boundedness of the trajectories and linearization of the nonlinear terms and, alternatively, using the sector condition. Formulating the problem as a generalized eigenvalue optimization problem gives the maximum allowable value of time-delay at a specific coupling gain, for which synchronization is still stable. A specific choice of formulating the error dynamics gives a condition for full or partial synchronization.

Two examples illustrate the two approaches to deal with the nonlinear terms. In both cases we can conclude that the approach is too conservative to give results that are comparable with simulations.

As an application of modeling this network, it is used to simulate synchronous neuron activity in the brain. A neuron model, suitable for analysis and simulations, is chosen. The Lur’’e type neuron model is most suitable in this case. The model is used in the network to simulate the Central Pattern Generator, which controls animal locomotion in the brain. Again, analysis and simulations differ a lot. However, simulations show that the network is indeed able to generate a variety of oscillation patterns, necessary for controlling animal gaits. Changing the time-delay in the coupling results in various phase shifts between the system trajectories, corresponding with animal gaits. Time-delay as a representation of speed is thus a acceptable modeling choice.

A slightly weaker condition for synchronization is used to obtain a stability diagram where the cases in which full and partial synchronization occurs is presented, obtained through simulations.
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Chapter 1

Introduction

1.1 Synchronization

In many situations in nature, the occurrence of synchronization can be observed. Examples are the synchronous flashing of fireflies, singing crickets and dancing people. The Dutch scientist Christiaan Huygens (1629-1695) was one of the first people to recognize synchronization in mechanical engineering. He discovered the synchronous motion of two pendulum clocks on a ship, while searching for a method to find longitude at sea (see Figure 1.1).

![Figure 1.1: The original drawing of Christiaan Huygens’ pendulum clocks.](image)

His discovery opened a great opportunity for applications. We can think of assembly and transportation systems, cooperating robots and remote control of mechanical systems.

Synchronization can be understood as the adaption of objects to eachother’s behavior, in such a way that their behavior is similar. This influence can be a result of some coupling between the objects, how weak it may be. Synchronization does not always imply that the behavior of objects is exactly copied; a rhythm of oscillations or a repeating sequence of events can also be considered as some kind of synchronization.
1.1.1 Synchronization of a network of neurons

The brain is an interesting system to study in this case. The occurrence of synchronous activity of parts of the brain can be observed and since the brain controls the body, synchronous behavior of the body is a logical consequence. From this point of view, the neurons in the brain can be considered as separate systems, that synchronize when necessary. This necessity appears for example in the case of locomotion of humans and animals. Locomotion requires a certain sequence of contraction of muscles in the body, and the control of these contractions happens in the brain.

1.1.2 Time-delay systems

Previous research on neuron models showed that in a network of neurons, neighboring neurons influence each other’s behavior. A neuron receives signals from its neighbors, and thus it is assumed that a network of neurons can be viewed as a network of coupled oscillators [4]. A nice photograph of a network of neurons is shown in Figure 1.2(a). A graphical interpretation of their interconnections is presented in Figure 1.2(b). The interest in time-delay systems arises when one considers that the transport of a signal from one neuron to another will take some time. Synchronization of time-delay systems is thus a useful topic to include, when investigating the behavior of a network of neurons. In this research, a method to find conditions for synchronization of coupled systems with time-delay will be employed.

(a) A network of neurons.  
(b) A graphical interpretation of a network of neurons.

Figure 1.2: Neurons.
1.2 Animal gaits

As stated above, parts of the brain control the movement of the body. It is widely accepted that animal locomotion is generated and controlled mostly by a so called central pattern generator (CPG), which is a network of neurons in the central nervous system, capable of producing rhythmic output [6]. The neurons generate a current, that is transported to the limbs in order to operate the muscles. For different kinds of gaits, different patterns of signals are necessary to let the limbs move in the desired sequence. For example, if we look at quadrupedal (four-legged) animals, we can distinguish different kinds of gaits at different speeds or even at the same speed but under different circumstances (see Figure 1.3 for some examples).

![Figure 1.3: Two gaits of a horse (taken from [4]).](image)

Of course one can think of control algorithms that operate in the brain, measuring for example speed and difficulty of the terrain, and selecting a gait with this information. But possibly, the brain exhibits some properties that allow it to automatically adapt to changes in the environment.

1.3 Problem formulation

This research aims to find conditions, under which a network of four coupled systems with time-delay synchronizes or partially synchronizes. Analytical stability conditions for the synchronization of the network will be derived, using a control theory approach. The analytical results will be compared with simulations, using examples.

The second part of this research will treat the possibility of simulating the control of animal gaits with a network of neuron models. The analytical results will be validated with simulations and a comparison will be made with the gaits of a quadrupedal animal.

1.4 Outline of the report

The outline of the report is as follows. In Chapter 2, a general approach to investigate the stability of (partial) synchronization is presented, together with an example, concerning the chaotic Lorenz system and a Lur’e type neuron model. Chapter 3 will present the analysis of the gaits of some animals, in particular the gaits of a horse. It will also treat the selection of
a suitable neuron model and the analysis of a network of neurons. Analysis of the stability of the synchronization of the network will be done and simulations will be used to validate the analysis. Furthermore, the ability to simulate animal gaits with this network will be investigated. In Chapter 4, conclusions and recommendations will be stated.
Chapter 2

Synchronization of coupled systems with time-delay

A great deal of research has been done on two coupled oscillators with time-delay [12], [17]. In this chapter a network of four coupled nonlinear systems with time-delay will be analyzed. The global structure of the network and the coupling will first be discussed. Secondly, a condition for global asymptotic stability of (partial) synchronization of the network will be derived, using a Lyapunov-Krasovskii functional. Furthermore, the theory will be applied to two examples, a network of Lorenz systems and a network of Lur’e type neuron models. The results will be validated numerically.

2.1 Design of the network

Consider a network of dynamical systems, as shown in Figure 2.1.

![Figure 2.1: A network of dynamical systems with time-delay](image)

The network consists of four identical systems with the input and output, denoted respectively as \( u_i(t) \) and \( y_i(t) \), for \( i = 1, 2, 3, 4 \). The circles indicate that the output signal experiences a time-delay. The systems are unidirectionally coupled in a ring. The choice for unidirectional
Next, consider the four continuous-time, nonlinear systems:

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i + f(x_i) + Bu_i \\
y_i(t) &= Cx_i \\
x_i(t) &= \phi_i(t) \quad t \in [-\tau, 0]
\end{align*}
\]  

(2.1)

where \( x_i \in \mathbb{R}^n, y_i \in \mathbb{R}^n, u_i \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, C \in \mathbb{R}^{n \times n} \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth function. Suppose we want to investigate full synchronization of this network. Then we have to define the condition for synchronization:

**Definition (2.1).** A pair of dynamical systems with state \( x_i(t) \) and \( x_j(t) \) respectively \((i \neq j)\), is synchronized if

\[
\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0.
\]

(2.2)

This implies that the error between the trajectories of the systems converges to zero no matter what initial conditions are taken. In other words, the solution \( e = 0 \) of the error dynamics is globally asymptotically stable. If the dynamics of (2.1) are used, the error dynamics of the network can be formulated as follows. Denote \( e(t) = x_i(t) - x_j(t) \). Then we obtain

\[
\begin{align*}
\dot{e}(t) &= Ae(t) + f(x_i) - f(x_j) + B(u_i - u_j) \\
e(t) &= \phi_i(t) - \phi_j(t) \quad t \in [-\tau, 0].
\end{align*}
\]

(2.3)

In the case of four unidirectionally coupled systems, Definition 2.1 becomes

\[
\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0, \quad i = 1, 2, 3, 4 \\
j = 4, 1, 2, 3.
\]

(2.4)

In order to formulate the error dynamics in terms of the error \( e \), the coupling terms have to be defined. With the aid of Figure 2.1 and considering the delayed input for each system, we obtain

\[
u_i(t) = -KC(x_i(t) - x_j(t - \tau)), \quad i = 1, 2, 3, 4 \\
j = 4, 1, 2, 3,
\]

(2.5)

with coupling gain \( K \in \mathbb{R}^{n \times 1} > 0 \), input matrix \( C \in \mathbb{R}^{1 \times n} \) and time-delay \( \tau > 0 \). One can easily see that in this case the coupling strength for all systems is equal. Combining (2.1), (2.4) and (2.5) and defining \( e_{ij} = x_i(t) - x_j(t) \) and \( e_{ij\tau} = x_i(t - \tau) - x_j(t - \tau) \), we obtain

\[
\begin{bmatrix}
\dot{e}_{14} \\
\dot{e}_{21} \\
\dot{e}_{32} \\
\dot{e}_{43}
\end{bmatrix}
= \begin{bmatrix}
a_0 e_{14} + a_1 e_{43\tau} + f(x_1) - f(x_1 - e_{14}) \\
a_0 e_{21} + a_1 e_{14\tau} + f(x_2) - f(x_2 - e_{21}) \\
a_0 e_{32} + a_1 e_{21\tau} + f(x_3) - f(x_3 - e_{32}) \\
a_0 e_{43} + a_1 e_{32\tau} + f(x_4) - f(x_4 - e_{43})
\end{bmatrix},
\]

(2.6)

with

\[a_0 = A - BKC, \quad a_1 = BKC\]
If we define the vectors
\[ e_t = \begin{bmatrix} e_{14} \\ e_{21} \\ e_{32} \\ e_{43} \end{bmatrix}, \quad e_\tau = \begin{bmatrix} e_{14\tau} \\ e_{21\tau} \\ e_{32\tau} \\ e_{43\tau} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \] (2.7)

(2.6) can be rewritten:
\[
\dot{e}_t = A_0 e_t + A_1 e_\tau + f(x) - f(x - e_t) \\
= A_0 e_t + A_1 e_\tau + \phi(x, e_t),
\] (2.8)

with
\[
A_0 = \begin{bmatrix} a_0 & O & O & O \\ O & a_0 & O & O \\ O & O & a_0 & O \\ O & O & O & a_0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} O & O & O & a_1 \\ a_1 & O & O & O \\ O & a_1 & O & O \\ O & O & a_1 & O \end{bmatrix}. \] (2.9)

It is easy to see that if \( e_t = 0 \) is an asymptotically stable equilibrium point of (2.8), synchronization of the network is asymptotically stable.

Linearizing the nonlinear term \( \phi(x, e_t) \) around \( e = 0 \), we obtain
\[
\dot{e}_t = A_0 e_t + A_1 e_\tau + D(x) e_t, \quad D(x) = \left( \frac{\partial \phi(x, e_t)}{\partial e_t} \right) |_{e_t=0}. \] (2.10)

It is well known that if \( e_t = 0 \) of (2.10) is asymptotically stable, then \( e_t = 0 \) of (2.8) is also asymptotically stable.

## 2.2 Stability of the synchronization

To investigate the asymptotic stability of \( e_t = 0 \) of (2.10), we present Theorem 1 of [16].

**Theorem (2.1).** The time delay system (2.10) is asymptotically stable for any delay \( \tau \) satisfying \( 0 < \tau < \bar{\tau} \), if there exist matrices \( P > 0, Q > 0, Z > 0, Y \) and \( W \) such that the following Linear Matrix Inequality (LMI) holds:
\[
\begin{bmatrix}
PA + A^T P + Y + Y^T + Q & PA_1 - Y + W^T & -\tau Y & \tau A^T Z \\
A^T P - Y^T + W & -W^T & -\tau W & \tau A^T Z \\
-\tau Y^T & -\tau W^T & -\tau Z & 0 \\
\tau Z A & \tau Z A_1 & 0 & -\tau Z
\end{bmatrix} < 0. \] (2.11)

With \( A_0 \) and \( A_1 \) as in (2.9) and \( A = A_0 + D(x) \).
Proof. Denote

\[ e = e(t + \theta) \quad -2\tau < \theta < 0 \]

and define a Lyapunov-Krasovskii functional as

\[ V(e) = V_1(e) + V_2(e) + V_3(e) \quad (2.12) \]

where

\[ V_1(e) = e(t)^T Pe(t) \]
\[ V_2(e) = \int_{-\tau}^{\tau} \int_{t+\beta}^{t} \dot{e}(\alpha)^T Z \dot{e}(\alpha) d\alpha d\beta \]
\[ V_3(e) = \int_{t-\tau}^{t} e(\alpha)^T Q e(\alpha) d\alpha. \]

By the Newton-Leibniz formula, we have \( e(t - \tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(\alpha) d\alpha \). Then the derivative of \( V(e) \) reads (see [16]):

\[ \dot{V}(e) = \dot{V}_1(e) + \dot{V}_2(e) + \dot{V}_3(e) \quad (2.13) \]

\[ \dot{V}_1(e) = \frac{1}{\tau} \int_{t-\tau}^{t} \left[ 2e(t)^T (PA + Y)e(t) + 2(e(t))^T (PA_1 - Y + WT)e(t - \tau) - 2e(t - \tau)^T We(t - \tau) - 2e(t)^T \tau Y \dot{e}(\alpha) d\alpha - 2e(t - \tau)^T \tau W \dot{e}(\alpha) d\alpha \right] d\alpha \]
\[ \dot{V}_2(e) = \frac{1}{\tau} \int_{t-\tau}^{t} \left[ e(t)^T \tau A^T Z A e(t) + 2(e(t))^T \tau A^T Z A_1 e(t - \tau) + e(t - \tau)^T \tau A_1^T Z A_1 e(t - \tau) - \dot{e}(\alpha)^T \tau Z \dot{e}(\alpha) \right] d\alpha \]
\[ \dot{V}_3(e) = \frac{1}{\tau} \int_{t-\tau}^{t} \left[ e(t)^T Q e(t) - e(t - \tau)^T Q e(t - \tau) \right] d\alpha, \]

with \( A = A_0 + D(x) \).

Combining these terms, we obtain

\[ \dot{V}(e) = \frac{1}{\tau} \int_{t-\tau}^{t} \zeta(t, \alpha)^T \Lambda(\tau) \zeta(t, \alpha) d\alpha, \quad (2.14) \]

with \( \zeta(t, \alpha) = [ e(t)^T \ e(t - \tau)^T \ \dot{e}(\alpha)^T ] \), and a symmetric matrix

\[ \Lambda(\tau) = \begin{bmatrix} PA + A^T P + Y + Y^T + \tau A^T ZA + Q & * & * \\
A_1^T P - Y^T + W + \tau A_1^T Z A_0 & -W - W^T + \tau A_1^T Z A_1 - Q & * \\
-\tau Y^T & -\tau W^T & -\tau Z \end{bmatrix}. \quad (2.15) \]

If \( \Lambda(\tau) < 0 \), the system is asymptotically stable. Using a Schur complement we obtain (2.11).
Note that in order to rewrite the error dynamics in state space notation, the nonlinear term \( \phi(x) \) is linearized around \( e = 0 \). However, the linearized term \( D(x) \) is still depending on the state \( x(t) \). This means that when solving LMI (2.11), information of the trajectories of the systems has to be at hand. To obtain this information, we present two approaches. The first one is using the linearized term \( D(x) \) and the boundedness of the trajectories \( x(t) \). The second one is not linearizing the term, but using the sector condition to rewrite the nonlinear term purely in terms of the error \( e \). This Chapter is concluded with examples of both methods.

2.3 Boundedness of the trajectories

In this section, we present an extension of the small-gain theorem for two bidirectionally coupled systems, as discussed in [17]. First, semi-passivity and semi-dissipativity are defined. For semi-dissipative systems, the small gain theorem will give proof of the boundedness of the trajectories of the systems. Finally, the bounds can be computed using Theorem 2.2.

2.3.1 Preliminaries

Consider the nonlinear system

\[
\dot{x}(t) = f(x, u), \quad y(t) = h(x) \quad (t \geq 0),
\]

(2.17)

with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}^m \).

According to semi-passivity as defined in [14] and [15], we introduce strictly semi-passivity and strictly semi-dissipativity as follows.

**Definition (2.2).** [strictly semi-passivity]

System (2.17) is said to be strictly semi-passive, if there exist a \( \mathcal{C}_1 \)-class function \( V : \mathbb{R}^n \to \mathbb{R} \), class-\( \mathcal{K}_\infty \) functions \( \underline{\alpha}(\cdot), \overline{\alpha}(\cdot) \) and \( \alpha(\cdot) \) satisfying

\[
\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)
\]

\[
\dot{V}(x) \leq -\alpha(\|x\|) - H(x) + y^T u
\]

(2.18)

(2.19)

for all \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \), where the function \( H(x) \) satisfies

\[
\|x\| \geq \eta \implies H(x) \geq 0
\]

(2.20)

for a positive real number \( \eta \).

**Definition (2.3).** [strictly semi-dissipativity]

System (2.17) is said to be strictly semi-dissipative with respect to the supply rate \( q(u, y) \), if there exist a \( \mathcal{C}_1 \)-class function \( V : \mathbb{R}^n \to \mathbb{R} \), class-\( \mathcal{K}_\infty \) functions \( \underline{\alpha}(\cdot), \overline{\alpha}(\cdot) \) and \( \alpha(\cdot) \) satisfying (2.18) and

\[
\dot{V}(x) \leq -\alpha(\|x\|) - H(x) + q(u, y)
\]

(2.21)

for all \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \), where the function \( H(x) \) satisfies (2.20).
Suppose Lemma 2.1 holds for the network of four unidirectionally coupled systems. Then,

**Lemma (2.1).** Suppose that system (2.17) is strictly semi-dissipative with respect to the supply rate \( \beta(\|u\|) \) for a class-\( \mathcal{K} \) function \( \beta(\cdot) \) and a real number \( \eta > 0 \) such that the response \( x(t) \) of (2.17) with the initial state \( x(0) = x_0 \) satisfies

\[
\|x\|_\infty \leq \max \{ \rho(\|x_0\|), \gamma(\|u\|_\infty), \eta \} \quad (2.22)
\]

\[
\limsup_{t \to \infty} \|x(t)\| \leq \max \{ \gamma(\limsup_{t \to \infty} \|u(t)\|), \eta \} \quad (2.23)
\]

for any input \( u \in \mathcal{L}_\infty^m \) and any \( x_0 \in \mathbb{R}^n \). Here functions \( \rho(\cdot) \) and \( \gamma(\cdot) \) are given by

\[
\rho(r) = \frac{\alpha^{-1} \circ \pi(r)}{r} \quad \gamma(r) = \frac{\alpha^{-1} \circ \pi \circ \alpha^{-1} \circ \kappa \beta(r)}{r} \quad (r \geq 0)
\]

where \( \kappa > 1 \), functions \( \alpha(\cdot), \pi(\cdot) \) and \( \alpha(\cdot) \) satisfy (2.18) and (2.21) and \( \eta \) satisfies (2.20).

**Proof.** The proof of this lemma can be found in [17].

### 2.3.2 Extension of the small-gain theorem

Suppose Lemma 2.1 holds for the network of four unidirectionally coupled systems. Then, similar to (2.22) and (2.23) we can state:

\[
\|x_i\|_\infty \leq \max \{ \rho_i(\|x_{0i}\|), \gamma_i(\|u_i\|_\infty), \eta_i \}, \quad i = 1, 2, 3, 4
\]

\[
\limsup_{t \to \infty} \|x_i(t)\| \leq \max \{ \gamma_i(\limsup_{t \to \infty} \|u_i(t)\|), \eta_i \}, \quad i = 1, 2, 3, 4
\]

with inputs \( u_i(t) = y_j(t - \tau) = C_j x_j(t - \tau), \quad i = 1, 2, 3, 4, \quad j = 4, 1, 2, 3 \).

Consider the following four systems

\[
\dot{x}_i(t) = f_i(x_i, u_i) \quad , \quad y_i(t) = C_i x_i \quad (t \geq 0),
\]

where state \( x_i \in \mathbb{R}^n \), input \( u_i \in \mathbb{R}^m \), output \( y_i \in \mathbb{R}^m \), \( C_i \in \mathbb{R}^{mn \times n} \) and \( f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) with initial conditions

\[
x_i(\theta) = \phi_i(\theta) \quad (-\tau \leq \theta \leq 0)
\]

\[
x_i(0) = \phi_i(0) = x_{0i},
\]

for \( i = 1, 2, 3, 4 \), respectively, where \( \phi_i : [-\tau, 0] \to \mathbb{R}^n \). Suppose that the systems (2.26) are strictly semi-dissipative. Then from Lemma 2.1, the systems have the properties (2.25). Now we consider the case in which these four systems are unidirectionally coupled with the following inputs:

\[
u_i(t) = y_j(t - \tau) = C_j x_j(t - \tau), \quad i = 1, 2, 3, 4, \quad j = 4, 1, 2, 3.
\]
Define class-$K$ functions as
\[ \pi_i(r) = \gamma_i(\sigma_{\text{max}}(C_j) \cdot r) \quad (r \geq 0), \]
where $\gamma_i(\cdot) = \overline{\sigma} \circ \alpha^{-1} \circ \kappa(\cdot)$ and $\sigma_{\text{max}}(\cdot)$ denotes the maximum singular value of a matrix. Then we obtain the following Lemma:

**Lemma (2.2).** For four systems (2.26), (2.27) coupled by (2.28), if the functions $\pi_1(\cdot)$, $\pi_2(\cdot)$, $\pi_3(\cdot)$ and $\pi_4(\cdot)$ in (2.29) satisfy
\[ \pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4(r) < r \quad \text{for all } r > 0, \]
then the trajectories $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ satisfy
\[
\begin{align*}
\limsup_{t \to \infty} \|x_1(t)\| &\leq \max\{\pi_1 \circ \pi_4 \circ \pi_3(\rho_2(\eta_2)), \pi_1 \circ \pi_4(\rho_3(\eta_3)), \pi_1(\rho_4(\eta_4)), \rho_1(\eta_1)\} \\
\limsup_{t \to \infty} \|x_2(t)\| &\leq \max\{\pi_2 \circ \pi_1 \circ \pi_4(\rho_3(\eta_3)), \pi_2 \circ \pi_1(\rho_4(\eta_4)), \pi_2(\rho_1(\eta_1)), \rho_2(\eta_2)\} \\
\limsup_{t \to \infty} \|x_3(t)\| &\leq \max\{\pi_2 \circ \pi_3 \circ \pi_2(\rho_1(\eta_1)), \pi_2 \circ \pi_3(\rho_2(\eta_2)), \pi_2(\rho_3(\eta_3))\} \\
\limsup_{t \to \infty} \|x_4(t)\| &\leq \max\{\pi_4 \circ \pi_3 \circ \pi_2(\rho_1(\eta_1)), \pi_4 \circ \pi_3(\rho_2(\eta_2)), \pi_4(\rho_3(\eta_3)), \rho_4(\eta_4)\},
\end{align*}
\]
where $\rho(\cdot) = \overline{\alpha}^{-1} \circ \alpha(\cdot)$ and $\eta_i$ satisfies (2.20).

**Proof.** Since the systems (2.26) are strictly semi-dissipative, from Lemma 2.1, there exist class-$K$ functions $\rho_i(\cdot)$ and $\gamma_i(\cdot)$ and positive real numbers $\eta_i$ such that the trajectories of (2.26) satisfy (2.25) for any inputs $u_1, u_2, u_3, u_4 \in L^m_{\infty}$.

First we show boundedness of the trajectories $x_i(t)$ for all $t \geq 0$. This is proven by contradiction. Suppose this is not true, then for any real number $R > 0$, there exists a time $T > 0$ such that the trajectories are defined on an interval $[0, T]$, and satisfy
\[ \text{either} \quad \|x_1(T)\| > R \quad \text{or} \quad \|x_2(T)\| > R \quad \text{or} \quad \|x_3(T)\| > R \quad \text{or} \quad \|x_4(T)\| > R. \]  
(2.32)

Here we choose $R$ satisfying
\[ R > M_i, \]  
(2.33)
with $i = 1, 2, 3, 4$ and $M_i$ is defined as:
\[
\begin{align*}
M_1 &= \max\{\|\phi_1\|, \rho_1(\|x_{01}\|), \pi_1(\|\phi_4\|, \rho_1(\|x_{04}\|), \pi_4(\|\phi_3\|, \rho_3(\|x_{03}\|), \pi_3(\|\phi_2\|, \rho_2(\|x_{02}\|), \rho_2(\eta_2)), \rho_3(\eta_3)), \rho_4(\eta_4)), \rho_1(\eta_1)\} \\
M_2 &= \max\{\|\phi_2\|, \rho_2(\|x_{02}\|), \pi_2(\|\phi_1\|, \rho_1(\|x_{01}\|), \pi_1(\|\phi_4\|, \rho_4(\|x_{04}\|), \pi_4(\|\phi_3\|, \rho_3(\|x_{03}\|), \rho_3(\eta_3)), \rho_4(\eta_4)), \rho_1(\eta_1), \rho_2(\eta_2)\} \\
M_3 &= \max\{\|\phi_3\|, \rho_3(\|x_{03}\|), \pi_3(\|\phi_2\|, \rho_2(\|x_{02}\|), \pi_2(\|\phi_1\|, \rho_1(\|x_{01}\|), \pi_1(\|\phi_4\|, \rho_4(\|x_{04}\|), \pi_4(\|\phi_3\|, \rho_3(\|x_{03}\|), \rho_3(\eta_3)), \rho_4(\eta_4)), \rho_1(\eta_1), \rho_2(\eta_2)) \} \\
M_4 &= \max\{\|\phi_4\|, \rho_4(\|x_{04}\|), \pi_4(\|\phi_3\|, \rho_3(\|x_{03}\|), \pi_3(\|\phi_2\|, \rho_2(\|x_{02}\|), \pi_2(\|\phi_1\|, \rho_1(\|x_{01}\|), \pi_1(\|\phi_4\|, \rho_4(\|x_{04}\|), \pi_4(\|\phi_3\|, \rho_3(\|x_{03}\|), \rho_3(\eta_3)), \rho_4(\eta_4)), \rho_1(\eta_1), \rho_2(\eta_2)), \rho_3(\eta_3)), \rho_4(\eta_4))\}.
\end{align*}
\]  
(2.34)

Let $T$ be such that (2.32) holds. Define the truncation of $x_i$ to the interval $[-\tau, T]$ by
\[
\begin{align*}
\hat{x}_i(t) &= x_i(t) \quad \text{if} \quad t \in [-\tau, T] \\
&= 0 \quad \text{if} \quad t > T,
\end{align*}
\]
and let $\hat{x}_1(t)$ denote the response of the system (2.26) for $i = 1$ with the initial condition $\phi_1$, for the input $\hat{u}_1(t)$. Since

$$\| \hat{u}_1 \|_{\infty} = \max_{-r \leq t \leq T - r} \| C_4 x_4(t) \| \leq \max_{-r \leq t \leq T} \| C_4 x_4(t) \| \leq \sigma_{\max}(C_4) \| \hat{x}_4 \|_{[-r, \infty)},$$

and the right hand side of this inequality is assumed to be bounded, we have

$$\| \hat{x}_1(t) \| \leq \max\{ \rho_1(\| x_{01} \|), \pi_1(\| \hat{x}_4 \|_{[-r, \infty)}), \rho_1(\eta_1) \} \quad \text{for all } t \geq 0.$$

Since, by causality,

$$\hat{x}_1(t) = x_1(t) \quad \text{for all } t \in [-\tau, T]$$

holds, we deduce that

$$\| \hat{x}_1 \|_{[-r, \infty)} \leq \max\{ \| \phi_1 \|_{c}, \rho_1(\| x_{01} \|), \pi_1(\| \hat{x}_4 \|_{[-r, \infty)}), \rho_1(\eta_1) \}. \quad (2.35)$$

For $i = 2, 3, 4$, similarly we obtain

$$\| \hat{x}_2 \|_{[-r, \infty)} \leq \max\{ \| \phi_2 \|_{c}, \rho_2(\| x_{02} \|), \pi_2(\| \hat{x}_4 \|_{[-r, \infty)}), \rho_2(\eta_2) \}, \quad (2.36)$$

$$\| \hat{x}_3 \|_{[-r, \infty)} \leq \max\{ \| \phi_3 \|_{c}, \rho_3(\| x_{03} \|), \pi_3(\| \hat{x}_2 \|_{[-r, \infty)}), \rho_3(\eta_3) \}, \quad (2.37)$$

$$\| \hat{x}_4 \|_{[-r, \infty)} \leq \max\{ \| \phi_4 \|_{c}, \rho_4(\| x_{04} \|), \pi_4(\| \hat{x}_3 \|_{[-r, \infty)}), \rho_4(\eta_4) \}. \quad (2.38)$$

Now, if $a = \max\{ b, c, \theta(a) \}$ and $\theta(a) < a$, then necessarily $\max\{ b, c, \theta(a) \} = \max\{ b, c \}$. Thus, replacing the estimates (2.36), (2.37) and (2.38) into (2.35) and using hypothesis (2.30) yields

$$\| \hat{x}_1 \|_{[-r, \infty)} \leq M_1.$$

Here we prove that $\pi_1, \pi_2, \pi_3, \pi_4$ are mutually interchangeable in (2.30). Since the function $\pi_1(\cdot) \in K$, set $r_i^* = \lim_{r \to -\infty} \pi_i(r)$.

1. In the case that $0 < r < r_1^*$, from (2.30), we obtain $\pi_2 \circ \pi_3 \circ \pi_3(r) < \pi_1^{-1}(r)$. Let $s \in (0, \infty)$ such that $s = \pi_1^{-1}(r)$, then $\pi_2 \circ \pi_3 \circ \pi_3(\pi_1(\pi_1^{-1}(r))) < \pi_1^{-1}(r)$ yields $\pi_2 \circ \pi_3 \circ \pi_3 \circ \pi_1(s) < s$ for all $s > 0$. One can repeat this, for example with $q = \pi_2^{-1}(s)$.

2. For $r \geq r_1^*$, set $p \in (0, r^*_1)$ such that $\pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4(r) < p$, then we obtain $\pi_2 \circ \pi_3 \circ \pi_3(r) < \pi_1^{-1}(p)$. Let $s = \pi_1^{-1}(p)$. Since $p < r$, $\pi_2(p) < \pi_2(r)$ holds. Therefore $\pi_2 \circ \pi_3 \circ \pi_3(p) < \pi_1^{-1}(p)$ holds and this yields $\pi_2 \circ \pi_3 \circ \pi_3(s) < \pi_1^{-1}(s)$ for all $s > 0$. Again, this procedure can be repeated to obtain different sequences.

Then, using (2.33), we have

$$\| \hat{x}_1 \|_{[-r, \infty)} \leq M_1 < R \quad \text{and}$$

$$\| \hat{x}_2 \|_{[-r, \infty)} \leq M_2 < R \quad \text{and}$$

$$\| \hat{x}_3 \|_{[-r, \infty)} \leq M_3 < R \quad \text{and}$$

$$\| \hat{x}_4 \|_{[-r, \infty)} \leq M_4 < R,$$
which contradicts (2.32). Now that we have shown that the trajectories are defined for all 
\( t \geq 0 \) and bounded, (2.25) yields

\[
\|x_i\|_\infty \leq \max\{\rho_i(\|x_{0i}\|), \pi_i(\|x_j\|_{[-\tau, \infty)}), \rho_i(\eta_i)\}.
\]  

(2.39)

From the initial conditions, we obtain

\[
\|x_i(t)\| = \|\phi_i(t)\| \leq \|\phi_i\| c \quad \text{for all} \ t \in [-\tau, 0].
\]  

(2.40)

Therefore, (2.39) and (2.40) yield

\[
\|x_i\|_{[-\tau, \infty)} \leq \max\{\|\phi_i\| c, \rho_i(\|x_{0i}\|), \pi_i(\|x_j\|_{[-\tau, \infty)}), \rho_i(\eta_i)\}
\]

Then combining these inequalities and using property (2.30), we obtain

\[
\|x_i\|_{[-\tau, \infty)} \leq M_i.
\]

Similarly, from (2.25), we have

\[
\limsup_{t \to \infty} \|x_i(t)\| \leq \max\{\pi_i(\limsup_{t \to \infty} \|x_j(t)\|), \rho_i(\eta_i)\}.
\]

Therefore combining them and using the property (2.30), we obtain (2.31).

2.3.3 Boundedness of the trajectories

Now we have proven the existence of bounds on \( x_i(t) \), we can state the following theorem:

**Theorem (2.2).** Define a class-\( \mathcal{K} \) function as \( \pi(r) \triangleq \gamma(\sigma_{\max}(C) \cdot r) \) for \( r \geq 0 \), where \( \gamma(\cdot) \) is defined as (2.24). If the function \( \pi(\cdot) \) satisfies

\[
\pi(r) < r \quad \text{for all} \ r > 0,
\]  

(2.41)

then the trajectories of the systems (2.6) converge to the bounded set

\[
\Omega = \{x \in \mathbb{R}^n \mid \|x\| \leq \rho(\eta)\}.
\]  

(2.42)

**Proof.** If (2.41) holds, then \( \pi \circ \pi(r) < r, \pi \circ \pi \circ \pi(r) < r \) and so on. This implies that (2.30) holds. Furthermore, since the systems are identical, \( \rho_i(\eta_i) = \rho_j(\eta_j) \) for all \( i, j \). Therefore, using Lemma 2.2 and (2.41) the trajectories of the systems (2.6) satisfy

\[
\limsup_{t \to \infty} \|x_1(t)\| \leq \max\{\pi_1 \circ \pi_4 \circ \pi_3(\rho_2(\eta_2)), \pi_1 \circ \pi_4(\rho_3(\eta_3)), \pi_1(\rho_4(\eta_4)), \rho_1(\eta_1)\}
\]

\[
\leq \rho_1(\eta_1).
\]  

(2.43)

This means all trajectories converge to \( \Omega \). \( \square \)

With the above derived conditions, we can calculate the bounds of the trajectories of each system in the network. In order to investigate the global asymptotic stability of the synchronization of the network, (2.11) must hold for all trajectories in the bounded set \( \Omega \). If this holds for trajectories, starting on the edge of this set, we can conclude that it holds for all trajectories, starting in \( \Omega \). To reduce computation time, \( \Omega \) will be defined by only a few strategically chosen points, as we will see in the first example in Section 2.8.1.
2.4 Sector condition

Another way to deal with the nonlinear term in (2.8) is to use the sector condition. The approach is based on the assumption that a nonlinear function is defined in a sector, which is bounded by linear functions. See for example Figure 2.2. One can easily see that the nonlinear function \( \phi(x) = \frac{1}{1+e^{-2x}} - \frac{1}{2} \) belongs to the sector, defined by the linear functions \( f_1(x) = 0 \) and \( f_2(x) = \frac{1}{2} x \).

\[ \begin{align*}
\phi(x) &\leq \alpha x \quad \forall \quad x \in \Omega.
\end{align*} \tag{2.44} \]

If \( \Omega = \mathbb{R} \), then \( \phi(t,x) \) satisfies the sector condition globally, in which case it is said that \( \phi(t,x) \) belongs to a sector \([\alpha, \beta]\). If \( \Omega = [a, b] \), it is said that \( \phi(t,x) \) belongs to a sector \((\alpha, \beta)\). Furthermore, we can easily derive the following condition:

\[ (\phi(t,x) - \beta x)(\phi(t,x) - \alpha x) \leq 0 \quad \forall \quad x \in \Omega. \tag{2.45} \]

Now suppose a simple Lyapunov candidate function is given, \( V(e_t) = e_t^T P e_t \). For the zero solution of the error dynamics to be asymptotically stable, we have to prove \( \dot{V}(e_t) < 0 \). Using (2.8), we obtain:

\[ \begin{align*}
\dot{V}(e) &= e^T P e + e^T P \dot{e} \\
&= 2(A_0 e_t + A_1 e_t + \phi(x, e_t))^T P e.
\end{align*} \tag{2.46} \tag{2.47} \]
Now, using (2.45), the following holds for all $\lambda > 0$:

$$
\dot{V}(e) = 2(A_0 e_t + A_1 e_{\tau} + \phi(x, e_t))^T P e - 2\lambda(\phi(x, e_t) - \beta e_t)(\phi(x, e_t) - \alpha e_t) \leq 0. \quad (2.48)
$$

This inequality can be rewritten as an LMI, with the nonlinear term $\phi(x, e_t)$ as a state variable. If this LMI holds, the system is asymptotically stable. Of course, the same procedure can be followed when choosing another Lyapunov candidate function.

### 2.5 Discussion

In section 2.3 and 2.4 we proposed two methods to simplify the nonlinear term in the error dynamics of the network. Both methods will make the stability criterion more conservative, since an approximation of the nonlinear term is made. In the first case, the bounds on the trajectories of the systems are approximated, and then used in a linearized term. In the second case, the nonlinear function is approximated by a set of linear functions. It is expected that these approximations will put more restrictions on the stability of the synchronization. However, the chosen Lyapunov-Krasovskii functional is the best available choice for this type of time-delay system. The functional should give less conservative results [16]. We can thus conclude that a big part of the benefit of using this functional will be diminished by the conservativeness, introduced by the approximations of the nonlinear term. Furthermore it should be noted that the choice of (2.12) as most suitable Lyapunov candidate function will not necessarily lead to the best results. Possibly this choice in combination with the LMI approach will give a conservative stability condition, so the analytical results will always differ from numerical or experimental results.

### 2.6 Generalized eigenvalue optimization

When analyzing the behavior of the network, it is interesting to investigate synchronization for varying gain and time-delay. The maximum amount of time-delay, which still ensures synchronization, can be computed rewriting the LMI as a so-called Generalized Eigenvalue Optimization Problem (GEVP). The algorithm gives a sufficient condition for synchronization by solving (2.11) for varying time-delay $\tau$. If the LMI holds, the network is stable for this value of time-delay. First we state the Theorem [12]:

**Theorem (2.3).** Let $\eta$ be the optimal solution of the following standard generalized eigenvalue problem:

$$
\min_{\mu > 0, P > 0, Q > 0, Z > 0} \mu \quad (2.49)
$$

subject to LMI (2.11) for all $x \in \Omega$. Then for any $\tau \in [0, 1/\mu]$, the error dynamics (2.10) is asymptotically stable.

To solve the GEVP for (2.11), we formulate the problem as follows:

Minimize $\mu$, subject to

$$
0 < B(x), \quad A(x) < \mu B(x). \quad (2.50)
$$
Below one can easily distinguish the positive definite matrices (2.51) and (2.52), which are grouped in $B(x)$. The terms in the LMI that contain $\bar{\tau}$ are put in $A(x)$, as can be seen in (2.53) and (2.54).

\[
0 < P, Q, Z \quad (2.51)
\]
\[
0 < \begin{bmatrix}
-PA_0 - A_0^T P - Y - Y^T - Q & * \\
-A_1^T P + Y^T - W & W + W^T + Q
\end{bmatrix} \quad (2.52)
\]
\[
\begin{bmatrix}
0 & * & * & * \\
0 & 0 & * & * \\
-Y^T & -W^T & -Z & * \\
ZA_0 & ZA_1 & 0 & -Z
\end{bmatrix} < \begin{bmatrix}
G_1 & * & * & * \\
G_2 & G_3 & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (2.53)
\]
\[
\begin{bmatrix}
G_1 & * \\
G_2 & G_3
\end{bmatrix} < \mu \begin{bmatrix}
-PA_0 - A_0^T P - Y - Y^T - Q & * \\
-A_1^T P + Y^T - W & W + W^T + Q
\end{bmatrix}. \quad (2.54)
\]

The GEVP can be solved using the LMI toolbox in MATLAB $[5]$ and yields the maximum value of $\mu = \frac{1}{\bar{\tau}}$, for which (2.11) is still satisfied.

### 2.7 Full and partial synchronization

In the introduction was already briefly noted that synchronization does not imply that the behavior of each system is exactly copied. For example, Huygens’ pendulum clocks were not swinging completely synchronous, but there was a fixed phase shift of half a period between the clocks. In this particular case this type of synchronization is called anti-phase synchronization. In general we can say the systems are partially synchronized.

In the case of four systems in a network, partial synchronization can also be observed. One can imagine that for a certain configuration of coupling parameters, a part of the network is synchronized and another part is not. To investigate the parameters for which this situation appears, the formulation of the error dynamics is crucial. For full synchronization, the error between all systems has to converge to zero. For partial synchronization, the error between specific systems has to converge to zero, while other errors do not. In the following examples, the formulation of the error dynamics as a way to distinguish between full and partial synchronization, will be discussed.
2.8 Examples

2.8.1 Four unidirectionally coupled Lorenz systems

In this section we will analyze the stability condition for four coupled Lorenz systems with time-delay. The network is constructed as in Figure 2.1. The Lorenz system is described by the following set of equations:

\[
\Sigma_i = \begin{cases}
\dot{x}_i = \sigma(y_i - x_i) + u_{ix}, \\
\dot{y}_i = r x_i - y_i - x_i z_i + u_{iy}, \\
\dot{z}_i = -b z_i + x_i y_i + u_{iz}, 
\end{cases} \quad i = 1, 2, 3, 4
\]

(2.55)

with output \( v = \bar{x}_i, \quad \bar{x}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^T \)

and \( \sigma = 10, \quad r = 28, \quad b = \frac{8}{3} \).

Unidirectional coupling is formulated as in (2.5):

\[
u_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \end{bmatrix} = -k C (\bar{x}_i - \bar{x}_j), \quad i = 1, 2, 3, 4, \quad j = 4, 1, 2, 3,
\]

(2.56)

with

\[
C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad k > 0.
\]

The Lorenz system with the parameters of (2.55) is a so-called chaotic system. The mathematical definition of a chaotic system is:

- It is highly sensitive to initial conditions. A small change in the initial conditions results in a large change in the time-response.

- The system is deterministic. This implies that the equations of motion of the system possess no random inputs. In other words, the irregular behavior of the system arises from non-linear dynamics and not from noisy driving forces.

- The trajectories of the system do converge to fixed points or periodic orbits.

One can imagine that for a network with this properties, synchronization is unlikely. Even if we consider a network of identical systems, changes in the initial conditions will completely change the behavior of the system. However, simulation results show that even chaotic systems can synchronize when coupled. In this example we will try to find a condition for global asymptotic stability of the synchronization of four Lorenz system with time-delay analytically.

Boundedness of the solutions

It is easy to see that the nonlinear terms \( x_i z_i \) and \( x_i y_i \) in (2.55) do not satisfy a sector condition. Therefore, the nonlinear terms have to be linearized and the bounds of the trajectories have to be used in order to derive a stability condition. Following the procedure in Section
we first investigate semi-passivity of the system. For $i = 1$, we define a storage function $V_1(\bar{x}_1) = \bar{x}_1^T \bar{x}_1$, where $\bar{x}_1 = [x_1 \ y_1 \ z_1 - \sigma - r]^T$. The derivative of $V$ along $\bar{x}_1$ is given by (see Appendix A):

$$
\dot{V}(\bar{x}_1) = -\alpha(\|\bar{x}_1\|) - H(\|\bar{x}_1\|) + \beta(\|\bar{x}_{4r}\|),
$$

with

$$
H(\bar{x}_1) = (\sigma - \epsilon)\bar{x}_1^2 + (1 - \epsilon)y_1^2 + (b - \epsilon)\left(\bar{z} - \frac{b - 2\epsilon}{2(b - \epsilon)}\right)^2 - \frac{b^2(\sigma + r)^2}{4(b - \epsilon)}
$$

so, according to Definition 2.2, the system is semi-passive. This implies that Lemma 2.1 holds. Since this choice of storage function $V_1(\bar{x}_1)$ implies $\bar{\alpha}(r) = \alpha(r)$, (2.24) is restated:

$$
\rho(r) = \pi^{-1} \circ \alpha(r) = r \quad \gamma(r) = \pi \circ \alpha \circ \alpha^{-1} \circ \kappa\beta(r) = \alpha^{-1} \circ \kappa\beta(r) \quad (r \geq 0).
$$

Using the Small-gain theorem and Theorem 2.2, we conclude

$$
\gamma_1(r_1) = \sqrt{\frac{\kappa b}{k + \epsilon}} r_1 < r_1.
$$

Therefore, for any finite number $k > 0$, setting $\kappa$ sufficiently close to 1, $\pi(r) < r$ holds for all $r$ and all trajectories $x_1(t)$ converge to

$$
\Omega = \{ \bar{x}_1 \in \mathbb{R}^n \mid \|\bar{x}_1\| \leq \eta_1 \}
$$

To find $\eta_1$, the maximum value of $\|x_1\|$ for which $\dot{V}(\bar{x}_1) < 0$ still holds, has to be found. Therefore, we analyse the terms in (2.57). First we show that $-\alpha(\|\bar{x}_1\|) + \beta(\|\bar{x}_{4r}\|) < 0$. Since $\gamma(r_1) < r_1$, we can rewrite $\beta(r_1)$ as follows:

$$
\alpha^{-1} \circ \kappa\beta(r_1) < r_1 \\
\kappa\beta(r_1) < \alpha(r_1) \\
\beta(r_1) < \frac{1}{\kappa} \alpha(r_1)
$$

As a consequence, following the proof of Lemma 2.1, presented in [17],

$$
- \alpha(\|\bar{x}_1\|) + \beta(\|\bar{x}_{4r}\|) \\
\leq - \alpha(\|\bar{x}_1\|) + \frac{1}{\kappa} \alpha(\|\bar{x}_1\|) \\
\leq - \frac{\kappa - 1}{\kappa} \alpha(\|\bar{x}_1\|) < 0, \quad \text{for all } \kappa > 1
$$
holds. This implies that $\dot{V}(\bar{x}_1) < 0$ if $H(\bar{x}_1) \geq 0$. To find the maximum value of $\|\bar{x}_1\|$ for which this holds, the following optimization problem has to be solved:

$$
\max_{0 < \epsilon < 1} \|\bar{x}_1\|
$$
subject to
$$H(\bar{x}_1) = 0$$

Solving the optimization problem for $\epsilon = 0.01$, we obtain

$$
\Omega = \{ \bar{x}_1 \in \mathbb{R}^n \mid \|\bar{x}_1\| \leq 39.4 \} \quad (2.61)
$$

A graphical representation of the bounds is given in Figure 2.3. The ball represents the computed bounds from (2.61). The stability analysis requires coordinates on this ball to fill in $D(x)$. To reduce computation time, the ball is approximated by a cube, containing the set $\Omega$. If analysis results in stability on the corners of the cube, the network is also stable for points inside $\Omega$.

![Figure 2.3: Bounds on the trajectories of the Lorenz system.](image)

**Synchronization**

In order to investigate synchronization of the network, the error dynamics are formulated. First, we distinguish between full and partial synchronization. Consider the network of Lorenz systems (2.55) with (2.56). Suppose we want to investigate full synchronization of the network. Recall (2.4) for full synchronization of four coupled systems. The error dynamics are now defined as follows. Defining the error state vector

$$
\xi = \begin{bmatrix} \xi_{12} \\ \xi_{23} \\ \xi_{34} \\ \xi_{41} \end{bmatrix} \quad \text{with} \quad \xi_{ij} = \begin{bmatrix} x_i - x_j \\ y_i - y_j \\ z_i - z_j \end{bmatrix}, \quad (2.62)
$$
we obtain
\[ \dot{\epsilon}(t) = A_0 \epsilon + A_1 \epsilon_r + \phi(\epsilon, x_i), \]  

(2.63)

where
\[
A_0 = \begin{bmatrix}
A_t & O & O & O \\
O & A_t & O & O \\
O & O & A_t & O \\
O & O & O & A_t 
\end{bmatrix},
\]

(2.64)

\[
A_1 = \begin{bmatrix}
O & O & O & A_r \\
A_r & O & O & O \\
O & A_r & O & O \\
O & O & A_r & O 
\end{bmatrix},
\]

(2.65)

\[
\phi(\epsilon, x_i) =
\begin{bmatrix}
0 \\
-e_{x12}^3 e_{x12} - e_{x12}^3 e_{x12} \\
0 \\
e_{x23}^3 e_{x23} - e_{x23}^3 e_{x23} \\
e_{x23}^3 e_{x23} + e_{y23}^3 e_{y23} + e_{x23}^3 e_{y23} \\
0 \\
e_{x34}^3 e_{x34} - e_{x34}^3 e_{x34} \\
e_{x34}^3 e_{x34} + e_{y34}^3 e_{y34} + e_{x34}^3 e_{y34} \\
e_{x41}^3 e_{x41} - e_{x41}^3 e_{x41} \\
e_{x41}^3 e_{x41} + e_{y41}^3 e_{y41} + e_{x41}^3 e_{y41} 
\end{bmatrix}.
\]

(2.66)

Linearizing \( \phi(\epsilon, x_i) \) around \( \epsilon = 0 \) gives:

\[ \frac{\delta \phi}{\delta \epsilon} |_{\epsilon=0} = D(x_i) \]

\[
D_i = \begin{bmatrix}
D_4 & O & O & O \\
O & D_1 & O & O \\
O & O & D_2 & O \\
O & O & O & D_3 
\end{bmatrix}
\]

with

(2.67)

Now, suppose we want to investigate partial synchronization. For example, system 1 and 3 synchronize and system 2 and 4 synchronize. Therefore, the following error dynamics must have \( \epsilon = 0 \) as an asymptotically stable equilibrium:

\[ \dot{\epsilon}(t) = A_0 \epsilon + A_1 \epsilon_r + \phi(\epsilon, x_3, x_4), \]  

(2.69)

where
2.8 Examples

\[ A_0 = \begin{bmatrix} A_t & O \\ O & A_t \end{bmatrix}, \quad A_t = \begin{bmatrix} -\sigma - k & \sigma & 0 \\ r & -1 - k & 0 \\ 0 & 0 & -b - k \end{bmatrix}, \quad (2.70) \]

\[ A_1 = \begin{bmatrix} O & -A_r \\ A_r & O \end{bmatrix}, \quad A_r = \begin{bmatrix} -k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{bmatrix} \quad \text{and} \quad (2.71) \]

\[ \phi(e, x_3, x_4) = \begin{bmatrix} 0 \\ e_{x13}e_{z13} - e_{x13}x_3 - e_{x13}z_3 \\ e_{x13}e_{y13} + e_{y13}x_3 + e_{x13}y_3 \\ 0 \\ -e_{x24}e_{x24} - e_{x24}x_4 - e_{x24}z_4 \\ e_{x24}e_{y24} + e_{y24}x_4 + e_{x24}y_4 \end{bmatrix}. \quad (2.72) \]

Linearizing \( \phi(e, x_3) \) around \( e = 0 \) gives:

\[ \frac{\delta \phi}{\delta e} |_{e=0} = D(x_i) = \begin{bmatrix} D_3 & O \\ O & D_4 \end{bmatrix} \quad \text{with} \quad D_i = \begin{bmatrix} 0 & 0 & 0 \\ -z_i & 0 & x_i \\ y_i & x_i & 0 \end{bmatrix}. \quad (2.73) \]

Note that if these error dynamics converge to zero, only a sufficient condition for partial synchronization is found. It can still happen that the network synchronizes fully. This is a drawback of this approach. However, since the condition for full synchronization can also be determined, we can still 'extract' a condition for partial synchronization. How this is done, will be explained in Chapter 3.

To investigate the stability of \( e = 0 \), recall (2.11):

\[
\begin{bmatrix}
PA + A^T P + Y + Y^T + Q & PA_1 - Y + W^T & -\bar{h}Y & \bar{h}A^T Z \\
A_1^T P - Y^T + W & -W - W^T - Q & -\bar{h}W & \bar{h}A_1^T Z \\
-\bar{h}Y^T & -\bar{h}W^T & -\bar{h}Z & 0 \\
\bar{h}ZA & \bar{h}ZA_1 & 0 & -\bar{h}Z
\end{bmatrix} < 0.
\]

Using \( A = A_0 + D(x_i) \) and \( A_1 \) as stated above, we solve the Generalized Eigenvalue Optimization Problem for partial and full synchronization and for varying coupling gain \( K \). Consequently, we obtain the stability diagrams in Figure 2.4.
Figure 2.4: Stability region of 4 unidirectionally coupled Lorenz systems with time-delay.
Discussion of the results

Clearly, Figure 2.4(a) shows that there is a minimum value of the coupling gain $K$, for which partial synchronization occurs. For $K < 10$, the maximum amount of time-delay $\tau_{\text{max}}$ considerably decreases. Around $K = 30$, $\tau_{\text{max}}$ increases a little till a maximum value of $\tau = 20[\text{ms}]$. The stability of the synchronization of the network is hardly affected by a further increase of the coupling gain. Calculating the stability region for full synchronization, the GEVP results show no value of time-delay $\tau$ for which the network synchronizes. However, simulation results contradict this outcome. As shown in Figure 2.4(b), we see that there is a region, for which the network synchronizes. The simulations show that at $K = 5$, the network synchronizes even for a time-delay $\tau = 70[\text{ms}]$. There is a minimum value $K$, for which the network synchronizes, but is it even lower than the one, calculated in the case of partial synchronization. Furthermore, partial synchronization is not observed in the simulations.

Apparently, The Lyapunov-Krasovskii functional in combination with the small-gain theorem results in a too conservative condition for stability. As discussed in section 2.5, Lyapunov stability in general is a strict condition for stability. Furthermore, linearization of the nonlinear terms in the Lorenz equations also introduces a difference between simulations and analysis of the network. Using another approach, possibly based on symmetries of the network, as discussed in [6] and [14], may result in a less conservative condition.

2.8.2 Four unidirectionally coupled Lure type neuron models

As a second example, consider a network of Lure type neuron models [8]

\[
\begin{aligned}
\dot{V}_i &= \begin{cases} 
\dot{v}_i(t) = c\phi(a v_i) - b v_i + u_0 - w_i + u_{ex} + u_{iv} \\
\dot{w}_i(t) = \rho[\phi(d(v_i + v_0)) - w_i] + u_{iw}
\end{cases} & i = 1, 2, 3, 4,
\end{aligned}
\]

(2.74)

with

\[
\begin{aligned}
V_i &= \begin{bmatrix}
v_i \\
w_i
\end{bmatrix}, \\
y_i(t) &= CV_i(t), \\
\phi(x) &= \frac{1}{1 + e^{2-4x}}.
\end{aligned}
\]

(2.75)

(2.76)

Assume full state coupling

\[
u_i = -kC(V_i - V_j\tau), \quad k > 0, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad i = 1, 2, 3, 4, \quad j = 4, 1, 2, 3.
\]

(2.77)

The parameters are chosen as follows:

\[
\begin{aligned}
a &= 1.8 & b &= 3 & c &= 2.2 & d &= 5 \\
u_0 &= -0.2 & v_0 &= -0.35 & \rho &= 0.3 & u_{ex} &= 0.04.
\end{aligned}
\]
Sector condition

First, we check whether the nonlinear terms satisfy the sector condition. There are two nonlinear functions:

\[ \phi_{i1}(v_i) = \phi(a v_i) \]
\[ \phi_{i2}(v) = \phi(d(v_i + v_0)). \]

In Figure 2.5 both nonlinear terms are plotted as function of \( v \).

![Figure 2.5: The nonlinear functions \( \phi_{i1} \) and \( \phi_{i2} \).](image)

On first sight, the functions do not satisfy a sector condition. A coordinate transformation will help in this case. The coordinate transformation will be conducted as follows:

\[ \hat{v}_i = v_i + \frac{5}{18} \]
\[ \hat{w}_i = w_i + \frac{5}{18} \]
\[ \hat{\phi}_i(\hat{v}_i) = \phi_i(\hat{v}_i) - \frac{1}{2}. \]

Substituting this transformation into (2.74) for \( i = 1 \), we obtain:

\[ \dot{\hat{v}}_1(t) = c[\phi(a(\hat{v}_1 - \frac{5}{18})) - \frac{1}{2}] - b(\hat{v}_1 - \frac{5}{18}) + u_0 - \hat{w}_1 - \frac{5}{18} + u_{ex} - K(\hat{v}_1 - \hat{v}_2) \]
\[ \dot{\hat{w}}_1(t) = \rho[\phi(d(\hat{v}_1 - \frac{5}{18} + v_0)) - \frac{1}{2} - \hat{w}_1 - \frac{5}{18}]. \]

Now, concerning the nonlinear functions, the following sector condition holds: there exists an \( \alpha > 0 \), such that \( 0 < \phi(y) \leq \alpha(y) \) for all \( y \in \{ y = Cx \mid x \in \Omega \} \). Then

\[ 0 \leq c[\phi(a(\hat{v}_1 - \frac{5}{18})) - \frac{1}{2}] = \hat{\phi}_1(\hat{v}_1) \leq c\hat{v}_1 \]
\[ 0 \leq \rho[\phi(d(\hat{v}_1 - \frac{5}{18} + v_0)) - \frac{1}{2}] = \hat{\phi}_2(\hat{v}_1) \leq \rho d\hat{v}_1 - \frac{7}{40}d. \]
One can see that $\phi_2$ still does not satisfy a sector condition. However, when formulating the error dynamics, all constant terms will disappear. Consider for example the error dynamics $\dot{v}_1 - \dot{v}_2$. Since $\dot{v}_1 - \dot{v}_2 = v_1 - v_2 = e_v$, we obtain:

\begin{align}
0 & \leq \hat{\phi}_1(\dot{v}_1) - \hat{\phi}_1(\dot{v}_2) \leq cae_v, \\
0 & \leq \hat{\phi}_2(\dot{v}_1) - \hat{\phi}_2(\dot{v}_2) \leq \rho de_v.
\end{align}

Furthermore, the following inequality holds:

\begin{align}
0 & \leq \hat{\phi}_1(\dot{v}_1) - \hat{\phi}_1(\dot{v}_2) - \hat{\phi}_2(\dot{v}_1) + \hat{\phi}_2(\dot{v}_2) \\
& \leq \rho de_v.
\end{align}

To check if conditions (2.78) and (2.79) hold, the behavior of the nonlinear terms in the error dynamics is simulated. In Figure 2.6 it can be seen that the sector conditions are satisfied.

**Figure 2.6:** The nonlinear error functions for $V_1 - V_2$.

### Error dynamics

The error dynamics for this network are formulated similarly as for the Lorenz network. For partial synchronization we obtain:

\begin{equation}
\dot{e}(t) = A_0 e + A_1 e_{\tau} + \Psi(\hat{v}_i), \quad i = 1, 2, 3, 4,
\end{equation}

where

\begin{align}
A_0 &= \begin{bmatrix} A_t & O \\ O & A_t \end{bmatrix}, \\
A_1 &= \begin{bmatrix} O & -A_\tau \\ A_\tau & O \end{bmatrix},
\end{align}

\begin{equation}
\Psi(\hat{v}_i) = \begin{bmatrix} \hat{\phi}_1(\dot{v}_1) - \hat{\phi}_1(\dot{v}_3) \\ \hat{\phi}_2(\dot{v}_1) - \hat{\phi}_2(\dot{v}_3) \\ \hat{\phi}_1(\dot{v}_2) - \hat{\phi}_1(\dot{v}_4) \\ \hat{\phi}_2(\dot{v}_2) - \hat{\phi}_2(\dot{v}_4) \end{bmatrix}.
\end{equation}

To investigate full synchronization, the errors between all systems are included.
Stability analysis

Again the following Lyapunov-Krasovskii functional will be used [16]:

\[ V(e) = V_1(e) + V_2(e) + V_3(e) \]

where

\[ V_1(e) = e(t)^T P e(t) \]

\[ V_2(e) = \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{e}(\alpha)^T Z \dot{e}(\alpha) d\alpha d\beta \]

\[ V_3(e) = \int_{t-\tau}^{t} e(\alpha)^T Q e(\alpha) d\alpha. \]

From the Newton-Leibniz formula, we have

\[ e(t-\tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(\alpha) d\alpha. \]

Furthermore the sector condition is formulated as in (B.3). Then, the time derivative of \( V(e_t) \) reads (see Appendix [B]):

\[ \dot{V}(e) = \frac{1}{\tau} \int_{t-\tau}^{t} \zeta(t, \alpha, \dot{v}_i)^T \Lambda(\tau) \zeta(t, \alpha, \dot{v}_i) d\alpha, \]

with

\[ \zeta(t, \alpha, \dot{v}_i) = \begin{bmatrix} e(t) & e(t-\tau) & \Psi(\dot{v}_i) & \dot{e}(\alpha) \end{bmatrix}, \]

and a symmetric matrix

\[ \Lambda(\tau) = \begin{bmatrix} PA_0 + A_0^T P + Y + Y^T + \tau A_0^T Z A_0 + Q & * & * & * \\ A_1^T P - Y^T + W + \tau A_1^T Z A_0 & -W - W^T + \tau A_1^T Z A_1 - Q & * & * \\ \tau Z A_0 + \lambda \eta C & \tau Z A_1 - R & -2\lambda & * \\ -\tau Y^T & -\tau W^T & -\tau R^T & -\tau Z \end{bmatrix}, \]

for any \( \lambda \geq 0. \)

Note that the nonlinear terms in \( \Psi(\dot{v}_i) \) are now state variables. The system is stable when \( \Lambda(\tau) < 0. \) Using the Schur complement [3], we obtain the following LMI concerning the stability of the network:

\[ \begin{bmatrix} PA_0 + A_0^T P + Y + Y^T + Q & * & * & * \\ A_1^T P - Y^T + W & -W - W^T - Q & * & * \\ P + R + \tau Z A_0 + \lambda \eta C & \tau Z A_1 - R & -2\lambda + \tau Z & * \\ -\tau Y^T & -\tau W^T & -\tau R^T & -\tau Z \end{bmatrix} < 0, \quad (2.85) \]

With system matrices \( A_0 \) and \( A_1 \) and \( \eta C \) (see Appendix [B]) as the implementation of the sector condition in the Lyapunov stability approach. Formulating this LMI as a Generalized Eigenvalue Problem and solving for \( \mu = \frac{1}{\tau} \) gives the stability diagram as shown in Figure 2.7

Again, we conclude that there is a big difference between the simulations and the calculations. Similar to the first example concerning the Lorenz systems, the stability diagram for partial synchronization gives a lower bound for the coupling gain. Full synchronization does not occur, according to the calculations. Again, full synchronization is observed for various values of coupling gain \( K \) and high values of time-delay \( \tau \) in the simulations. There is a peak
in the stability region at $K = 0.4$. Why this peak appears in the simulations is unknown. Furthermore, partial synchronization is only observed under very strict conditions, i.e. specific initial conditions and long simulation times. The combination of the sector condition with the Lyapunov stability approach also proves to be too conservative to obtain an analytic stability bound for full synchronization of the network.
Figure 2.7: Stability region of 4 unidirectionally coupled Lure systems with time-delay.
Chapter 3

A synchronization approach on animal gaits

In this chapter, the approach, explained in Chapter 2 will be used to investigate the synchronous behavior of a network of neurons. First, a view on animal locomotion will be given. Second, a representation of the control of this locomotion as a network of four neurons will be given. A suitable model of a neuron will be chosen and the synchronization of a network of neurons will be investigated. Finally, the results will be compared with simulations, which resemble the control of animal gaits in the brain.

3.1 Animal gaits

Previous studies on animal locomotion resulted in the observation of a series of gaits, depending on the animal’s body and the walking speed. A nice and clear example is the gait of a horse. With increasing walking speed, a different gait is required to maintain the desired speed. The diagram in Figure 3.1 shows the phase relations of the legs of a horse. The phase relations of the legs indicate the sequence of leg placement, which resembles a particular gait. Studies on these phase relations showed that they can be obtained naturally via Hopf bifurcation in small networks [4], [6]. This implies that the control of animal gaits can be represented by a simple network of oscillators, and changing parameters represents changing the phase relations and thus the gait. In fact, this leads to a widely accepted theory about animal locomotion, treated in the following section.

3.1.1 Central pattern generator

Animal locomotion is generated and controlled mostly by a so called Central Pattern Generator (CPG), which is a network of neurons in the central nervous system, capable of producing rhythmic output [6]. The neurons generate a current, that is transported to the limbs in order to operate the muscles. For different kinds of gaits, a different pattern of signals is necessary to let the limbs move in the desired sequence. It is assumed that the neurons have notion of neurons in their neighborhood when they pulse a signal. Moreover, the pulsing of neighboring neurons influences the state of the neuron itself. It may cause the neuron to pulse as well. In previous research a model of a network of oscillators, capable of generating different signal patterns, has been investigated [4], [6]. The neurons are coupled with a coupling term,
3.1.2 Time-delay

The time-delay between two neurons is assumed to be constant. This implies that whatever the circumstances, the transport of a current from one neuron to another will always cost a constant amount of time. However, when the speed of the animal increases, the reaction time of the network will be smaller. We can say that the brain has to think faster in order to keep up with the required gait at this speed. The time-scale of the actions of the network will change. This means that the influence of the time-delay on the reaction time changes. We can illustrate this with an example. Consider a linear continuous-time time-delay system

\[
\dot{x}(t) = Ax(t) + Bu(t - \tau).
\]

(3.1)

We can apply a time-scale transformation \( \theta = st, t = \frac{\theta}{s} \) as a representation of the demand to
think faster. Now (3.1) becomes

\[ \dot{x}(s) = Ax(s) + Bu(s - \tau). \]  

(3.2)

The influence of the time-delay \( \tau \) on the system will change under changing time-scale parameter \( s \). One can also use another approach and scale \( \tau \):

\[ \dot{x}(\theta) = Ax(\theta) + Bu(\theta - s\tau), \]  

(3.3)

which should give the same results as (3.2). A graphical representation of the time-scaling influence is shown in Figure 3.2.

![Figure 3.2: Effect of time-scaling on the influence of time-delay.](image)

The example illustrates that, while the time-delay in a network of neurons is constant, we can simulate a change in operation speed by varying it. We can see this operation speed as a measure of overall speed of the animal, since the animal has to react faster on changes in its environment.

3.2 A network of neuron models

A lot of scientific contributions treat the modeling of neurons. A model that nowadays is still the most complete one, has been proposed by Hodgkin and Huxley [7]. However, the model is quite complex. It is described by a set of four differential equations and has a large number of parameters. Numerical simulations thus are time-consuming and analysis of the model is difficult. As a result, a variety of reduced-order models is developed. These models may not be a perfect resemblance of real neurons, but most of them exhibit the same dynamic properties.

All models are based on the electrochemical properties of the neuron. In Figure 3.3, the parts of a neuron, responsible for its behavior, are named. The axon is responsible for generating potential difference \( V = V_i - V_{eq} \), by activation of sodium \( Na^+ \) and potassium \( K^+ \) ions, which results in a current flow through the membrane of the axon. Furthermore, there exists
a resting potential $V_{eq}$ and a leakage current $I_l$. Using their famous squid axon voltage clamp experiments, Hodgkin and Huxley proposed a model of a neuron, based on an electrical circuit. The parameters have been estimated using the experiments. Most of the other models are simplifications of this model, using the dynamical properties of the differential equations.

Since this research concerns the synchronization of oscillations of a network of neurons, the model should have some specific properties. First of all, simulations have to show whether the network can produce various oscillation patterns, resembling animal gaits. Since the analysis of the network requires the formulation of error dynamics, the model should be as simple as possible. This way, excessive computation times and complex error dynamics are avoided. We briefly discuss four models, with their advantages and disadvantages:

**FitzHugh-Nagumo model**

The FitzHugh-Nagumo model [12] is a second-order representation of the Hodgkin-Huxley model. This reduction is achieved by introducing an external random forcing input $I(t)$.

$$
\dot{v}(t) = -v(v-a)(v-1) - w + I(t) \\
\dot{w}(t) = \epsilon(v - bw) \\
y(t) = v(t)
$$

(3.4)
with constants $a, b,$ and $\epsilon$. This model shows nonperiodic oscillations for $I(t) \in [0.02, 0.04]$. However, the random input introduces some difficulties when formulating the error dynamics. One either has to set $I(t)$ equal in all systems, so it will vanish in the error dynamics, or keep a random term in the error dynamics. In the first case, the oscillatory behavior of the network will not be observed and in the second case, investigating Lyapunov stability of the network will require knowledge of the stability of stochastic systems. Investigating the stability of stochastic systems is out of the scope of this research.

**Morris-Lecar model**

The Morris-Lecar model [11], like the Hodgkin-Huxley model, is a very complete model. A lot of oscillation patterns can be generated, but unfortunately the model is very complex, due to a large number of nonlinear terms.

\[
\begin{align*}
\dot{v}(t) &= \frac{1}{C}[-g_K(n(v - V_k) - g_Ca m(v - V_Ca) - g_l(v - V_l) + I)] \\
\dot{n}(t) &= \lambda_n(v)(n_\infty(v) - n) \\
\dot{m}(t) &= \lambda_m(v)(m_\infty(v) - m) \\
y(t) &= v(t)
\end{align*}
\]

with

\[
\begin{align*}
n_\infty(v) &= \frac{1}{2}(1 + \tanh((v - V_3)/V_4)) \\
\lambda_n(v) &= \bar{\lambda}_n \cosh((v - V_3)/2V_4) \\
m_\infty(v) &= \frac{1}{2}(1 + \tanh((v - V_1)/V_2)) \\
\lambda_m(v) &= \bar{\lambda}_m \cosh((v - V_1)/2V_2)
\end{align*}
\]

and constants $g_K, V_k, g_Ca, V_Ca, g_l, \lambda_n, \lambda_m$, and $I$.

The error dynamics of a network of Morris-Lecar models will be very difficult to analyze.

**Lur’e type neuron model**

The Lur’e type neuron model [8] is a simplification of the Morris-Lecar model. The number of nonlinear terms is reduced to two, which can be chosen either piecewise linear or continuous. The model is also reduced to second-order.

\[
\begin{align*}
\dot{v}(t) &= c\phi(av) - bv + u_0 - w + u_{ex} \\
\dot{w}(t) &= \rho[\phi(d(v + V_0)) - w] \\
y(t) &= v(t)
\end{align*}
\]

with

\[
\phi(x) = \frac{1}{1 + e^{2-4x}}
\]

with constants $a, b, c, d, u_0, u_{ex}, \rho$ and $V_0$. The model has the same dynamics properties of the more complicated Morris-Lecar model [8]; it shows oscillatory behavior in various manners.
In Figure 3.4 the response of one system is plotted against time for the following parameters:

\[ a = 1.8 \quad b = 3 \quad c = 2.2 \quad d = 5 \]
\[ u_0 = -0.2 \quad v_0 = -0.35 \quad \rho = 0.3 \quad u_{ex} = 0.04, \]

(3.7)

Clearly the pulsing character of the system can be seen. The Lur’e model will be used in modeling a network of neurons.

![Figure 3.4: Response of the Lur’e neuron model.](image)

### 3.3 Synchronization of a network of Lur’e type neuron models

Now we have chosen a model, we construct a network as in the example of Chapter 2. Recall the dynamics of the Lur’e model in (3.6). For one system in the network we write:

\[
V_i = \begin{cases} 
\dot{v}_i(t) = c\phi(a v_i) - b v_i + u_0 - w_i + u_{ex} + u_{iv} \\
\dot{w}_i(t) = \rho[\phi(d(v_i + v_0)) - w_i] + u_{iw} \\
y_i(t) = CV_i(t) 
\end{cases} \quad i = 1, 2, 3, 4
\]

(3.8)

with

\[
\phi(x) = \frac{1}{1 + e^{2-4x}}.
\]

(3.9)

(3.10)
Since only the voltage \( v(t) \) is an output of the system, the network is coupled as follows:

\[
\begin{align*}
u_i &= -K(y_i - y_{j\tau}), \quad K > 0, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad i = 1, 2, 3, 4 \\
        &= - KC(V_i - V_{j\tau}) \\
        &= - K(v_i - v_{j\tau})
\end{align*}
\]  

(3.11)

Using the parameters from (3.7) and choosing initial conditions for both states randomly between \([0, 0.5]\), the network shows full and partial synchronization for varying coupling gain \( K \) and time-delay \( \tau \). To validate these simulations, we follow the same procedure as in Section 2.8.2 but now with only the voltage \( v(t) \) as output. The error dynamics are

\[
\dot{e}(t) = A_0 e + A_1 e_{\tau} + \Psi(\hat{v}_i), \quad i = 1, 2, 3, 4,
\]

where

\[
A_0 = \begin{bmatrix} A_t & O \\ O & A_t \end{bmatrix}, \quad A_t = \begin{bmatrix} -b - k & -1 \\ 0 & -\rho \end{bmatrix}, \\
A_1 = \begin{bmatrix} O & -A_{\tau} \\ A_{\tau} & O \end{bmatrix}, \quad A_{\tau} = \begin{bmatrix} -k & 0 \\ 0 & 0 \end{bmatrix}, \\
\Psi(\hat{v}_i) = \begin{bmatrix} \hat{\phi}_1(\hat{v}_1) - \hat{\phi}_1(\hat{v}_3) \\ \hat{\phi}_2(\hat{v}_1) - \hat{\phi}_2(\hat{v}_3) \\ \hat{\phi}_1(\hat{v}_2) - \hat{\phi}_1(\hat{v}_4) \\ \hat{\phi}_2(\hat{v}_2) - \hat{\phi}_2(\hat{v}_4) \end{bmatrix},
\]

for partial synchronization. For full synchronization the errors between all systems are included in the error dynamics. Using the Lyapunov-Krasovskii functional from (2.12) and condition (2.79), we obtain LMI (2.85). Formulating the Generalized Eigenvalue Problem and solving for varying coupling gain \( K \), we obtain Figure 3.5. Again we conclude what was seen in the example in Section 2.8.2: the calculations show a difference with the simulations. The Lyapunov stability approach in combination with the sector condition results in a too strict condition for full synchronization of the network.

Moreover, if we take a look at partial synchronization from a symmetry point of view [14], an interesting observation arises. Since all systems are coupled exactly the same with exactly the same coupling parameters, the network is highly symmetric. A permutation of systems should give the same results, since all systems are identical with identical couplings. Looking at the synchronization problem, this means that if we investigate synchronization of system 1 and 3, we also investigate synchronization of system 1 and 2 or any other combination. In the analysis this approach would mean that actually we investigate full synchronization. However, since the time-delay introduces a phase shift in between the systems, there still is some difference in the couplings between the systems. From simulations we can conclude that a certain value of time-delay introduces a phase lag of 180° in the trajectories of the coupled system and a phase lag of 360° for the system that is coupled after that. This implies that even though the network consists of identically coupled identical systems, the time-delay disturbs the input such, that we can not speak of exact symmetry in the network. It is however still interesting to look at synchronization from a symmetry point of view to obtain better results.
Figure 3.5: Stability region of 4 unidirectionally coupled Lure systems with time-delay, output $y_i = v_i$. 
3.4 Simulations with varying coupling gains

In order to get an overview on full and partial synchronization of the network, simulations are done with varying coupling gains and time-delay. The coupling gains in the network are now split into $K_0$ for $u_1$ and $u_3$, and $K_1$ for $u_2$ and $u_4$ (see Figure 3.6).

![Diagram of the network with varying coupling gains](image)

Figure 3.6: The network with varying coupling gains.

In these simulations, a weaker form of synchronization is stated, named practical synchronization:

**Definition 1** \((3.1)\). A pair of dynamical systems with state $x_i(t)$ and $x_j(t)$ respectively \((i \neq j)\), is practically synchronized if there exists a $\delta > 0$ such that

$$\lim_{t \to \infty} |x_i(t) - x_j(t)| \leq \delta, \quad (3.12)$$

for arbitrary initial conditions.

This definition is used, because in some cases, the error dynamics do not exactly converge to zero, but in practice we still can speak of synchronized systems. These results can not be compared with analytical results, because mathematically speaking, synchronization does not occurs in these cases. It is used only to look at the behavior of the network under varying coupling gain and time-delay.

In Figure 3.7 a 3-dimensional plot of the maximum allowable time-delay $\tau_{\text{max}}$ for $K_0$ and $K_1$ is shown. To investigate full synchronization, the errors between all systems are taken into account (Figure 3.7(a)). For partial synchronization, all combinations of pairs of systems are looked at. As discussed earlier, this condition for partial synchronization also contains the situation of full synchronization. Therefore we extract the results for full synchronization from the results for partial synchronization, resulting in Figure 3.7(b). Clearly, it can be seen that when $K_0$ equals $K_1$, full synchronization occurs. When they are different, partial synchronization is observed. In both cases, there is a minimum coupling gain for which any form of synchronization occurs. This graph shows that when the coupling gains differ from each other, partial synchronization is likely to happen. Full synchronization however, is restricted to equal coupling gains \((K_0 = K_1)\).
Figure 3.7: Synchronization, investigated through simulations, $\delta = 1 \cdot 10^{-5}$.
3.5 The network as central pattern generator

In this section we focus on numerical simulations of the network and the way it resembles the Central Pattern Generator (CPG). In animal locomotion, the limbs have to move in a particular manner at a particular speed. For example, when a horse is performing the 'walk' gait, each leg move with a quarter period phase shift with respect to its neighboring legs. In Figure 3.1, the phase relations of the legs of a horse are shown for all gaits. The CPG generates the pulsing patterns that the neurons have to send to the muscles, to let the legs perform the desired gait. These pulsing patterns are of course very complex, due to the large number of muscles that are involved in animal locomotion. However, the basic patterns should be the same for all separate muscles.

We want to check whether the constructed network of 4 systems, unidirectionally coupled with time-delay, can generate these patterns. In [4] and [6], the various patterns are generated by varying the coupling gains in the network separately. Furthermore, they conclude that for the network to generate all gaits patterns, it has to consist of 8 systems, coupled unidirectionally and bidirectionally in a particular manner. We try to obtain the same patterns with a network of 4 unidirectionally coupled systems with time-delay. The coupling gain is constant at $K_0 = K_1 = 1$. When varying only the time-delay $\tau$, a variety of oscillation patterns can be generated, due to the partial synchronization that occurs in the network. The simulation results are presented in Appendix C.

As seen in Appendix C, the network of 4 unidirectionally coupled systems with time-delay is able to generate most of the oscillation patterns that are required for controlling animal gaits by only varying the time-delay. The phase shifts between the legs of a horse, shown in Figure 3.1 can be distinguished in the simulations as well. For example, the 'walk' gait can be obtained by setting $\tau = 1.5$[ms] and the 'trot' or 'bound' gait can be observed at $\tau = 4$[ms], depending on which system represents which leg. As discussed in section 3.1.2, we thus can look at the time-delay as a measure of increasing speed, since the gaits are obtained in order of increasing speed. Of course this network is very simple and the brain is a much more complex network of neurons. However, we can conclude that the time-delay is an important factor in the dynamics of the network and has to be included in modeling the brain. It should be noted that when varying the coupling gains of the network separately, as done in [4] and [6], even more patterns could be generated.
Chapter 4

Conclusions and recommendations

4.1 Conclusions

A network of four unidirectionally coupled nonlinear systems with time-delay is thoroughly analyzed to find conditions, under which the network fully or partially synchronizes.

From the analysis of the network, we can conclude that using the Lyapunov-Krasovskii functional in combination with the linearization of the nonlinear terms and the boundedness of the trajectories or in combination with the sector condition is not a suitable method for investigating the stability of the synchronization. According to the analysis, full synchronization does not occur, while simulations show that it does for various coupling gains and time-delays. The reason for this difference is that the used method is too conservative to give a stability bound.

The network is used to simulate output of the Central Pattern Generator, a part in the brain that is responsible for the control of animal locomotion. The Lur’e type neuron model is a very suitable model to use in this case, because it has relatively simple equations of motion, which makes formulating error dynamics more easy. This simplification does not affect the dynamical properties of the neuron model much, so it still resembles real neuron behavior. Unfortunately, analysis of the network gives little information about the stability of synchronization of four of these models in a network.

Simulations of the network’s behavior under varying amounts of time-delay show that it can generate a variety of oscillation patterns. These patterns correspond well to the oscillation patterns, that are generated by a Central Pattern Generator. This observation is an indication that time-delay is an important parameter in modeling a network of neurons. Furthermore, the assumption that increasing time-delay is equivalent to changing the operation speed in the network, is justified.
4.2 Recommendations

More methods to investigate (partial) synchronization of a network of nonlinear systems with time-delay have to be investigated. In [14], synchronization, based on symmetries in the network, is investigated. This method can be of much use in this case, since the network is highly symmetric.

Using other models that the Lur’e type neuron model could be interesting, since the Lur’e type neuron model is a quite exotic model. More realistic neuron models like the Hodgkin-Huxley or Morris-Lecar model can give a more realistic representation of a network of neurons.

The network of unidirectionally coupled neuron models exhibits the properties to simulate animal gaits. However, one can expect that the influence of neurons on each-others behavior is bidirectional and nonlinear. Investigating the properties of a network of four bidirectionally coupled systems is an interesting research topic. It is also interesting to investigate the behavior of the network with all coupling gains varying and with varying time-delays for every coupling.
Bibliography


Appendix A

Semi-passivity of the Lorenz system

Recall the definition of semi-passivity:

**Definition** (strictly semi-passivity). System (2.17) is said to be strictly semi-passive, if there exist a $C^1$-class function $V: \mathbb{R}^n \to \mathbb{R}$, class-$K_\infty$ functions $\alpha(\cdot)$, $\overline{\alpha}(\cdot)$ and $\alpha(\cdot)$ satisfying

\[
\alpha(||x||) \leq V(x) \leq \overline{\alpha}(||x||)
\]

\[
\dot{V}(x) \leq -\alpha(||x||) - H(x) + y^T u
\]

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, where the function $H(x)$ satisfies

\[
||x|| \geq \eta \implies H(x) \geq 0
\]

for a positive real number $\eta$.

Consider the Lorenz system for $i = 1$:

\[
\Sigma_1 = \begin{cases} 
    \dot{x}_1 = \sigma(y_1 - x_1) - k(x_1 - x_{4r}) \\
    \dot{y}_1 = rx_1 - y_1 - x_1z_1 - k(y_1 - y_{4r}) \\
    \dot{z}_1 = -bz_1 + x_1y_1 - k(z_1 - z_{4r}),
\end{cases}
\]  

(A.1)

We define a storage function $V(x_1) = x_1^T x_1$ with $x_1 = [x_1 \ y_1 \ z_1 - \sigma - r]^T$. The derivative of $V(x_1)$ is

\[
\dot{V}(x_1) = x_1 \dot{x}_1 + y_1 \dot{y}_1 + (z_1 - \sigma - r) \dot{z}_1
\]  

(A.2)

\[
= -\sigma x_1^2 - y_1^2 - bz_1^2 + (\sigma + r)z_1
- kx_1(x_1 - x_{4r}) - ky_1(y_1 - y_{4r}) - k(z_1 - z_{4r})
+ \epsilon x_1^2 - \epsilon x_1^2 + \epsilon y_1^2 - \epsilon y_1^2 + \epsilon z_1^2 - \epsilon z_1^2
\]

With slack variable $0 < \epsilon < 1$. Here the derivative is split into two parts. The first part is rewritten as follows.

\[
-\sigma x_1^2 - y_1^2 - bz_1^2 + (\sigma + r)z_1 + \epsilon x_1^2 + \epsilon y_1^2 + \epsilon z_1^2
\]

\[
= - (\sigma - \epsilon)x_1^2 - (1 - \epsilon)y_1^2 - (b - \epsilon)z_1 - \frac{b(\sigma + r)}{(b - \epsilon)}z_1
\]

\[
= - (\sigma - \epsilon)x_1^2 - (1 - \epsilon)y_1^2 - (b - \epsilon)z_1 - \frac{b(\sigma + r)}{2(b - \epsilon)} + \frac{b^2(\sigma + r)^2}{4(b - \epsilon)}
\]

\[
= - H(x_1)
\]
For the second part we use the following expressions.

\[(x - y)^T K (x - y) = x^T K x + y^T K y - 2x^T K y \]
\[x^T K y = \frac{1}{2} x^T K x + \frac{1}{2} y^T K y - \frac{1}{2} (x - y)^T K (x - y) \]
\[x^T K (x - y) = \frac{1}{2} x^T K x - \frac{1}{2} y^T K y + \frac{1}{2} (x - y)^T K (x - y). \]

If \( K \leq 0 \), then \( \frac{1}{2} (x - y)^T K (x - y) \leq 0 \)
and \( x^T K (x - y) \leq \frac{1}{2} x^T K x - \frac{1}{2} y^T K y. \)

Using the above derived expressions, the second part of (A.2) becomes

\[- k x_1 (x_1 - x_{4\tau}) - k y_1 (y_1 - y_{4\tau}) - k z_1 (z_1 - z_{4\tau}) - \epsilon x_1^2 - \epsilon y_1^2 - \epsilon z_1^2 \]
\[= - k \| \mathbf{z}_1 \| (x_1 - x_{4\tau}) - \epsilon \| \mathbf{z}_1 \|^2 \]
\[= - \left( \frac{K}{2} + \epsilon \right) \| \mathbf{x}_1 \|^2 + \frac{K}{2} \| \mathbf{z}_{4\tau} \|^2 \]
\[= - \alpha (\| \mathbf{z}_1 \|) + \beta (\| \mathbf{z}_{4\tau} \|). \]

Combining these terms again, we conclude that the Lorenz system is semi-passive with respect to the supply rate \( \beta (\| \mathbf{z}_{4\tau} \|). \)
Appendix B

Derivation of LMI (2.85)

Set

\[ e_t = e(t + \theta), \quad -2\tau < \theta < 0 \]

and define a Lyapunov-Krasovskii functional as

\[ V(e_t) = V_1(e_t) + V_2(e_t) + V_3(e_t) \tag{B.1} \]

where

\[ V_1(e_t) = e(t)P e(t) \]

\[ V_2(e_t) = \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{e}(\alpha)^T Z \dot{e}(\alpha) d\alpha d\beta \]

\[ V_3(e_t) = \int_{t-\tau}^{t} e(\alpha)^T Q e(\alpha) d\alpha. \]

By the Newton-Leibniz formula, we have

\[ e(t-\tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(\alpha) d\alpha. \]

The error dynamics are formulated as \[ \dot{e}_t = A_0 e_t + A_1 e_{\tau} + \Psi(\hat{v}_i). \]

Furthermore we use the sector condition:

\[ \Psi(\hat{v}_i)^T (\Psi(\hat{v}_i) - \eta C e(t)) \leq 0, \tag{B.2} \]

with

\[ \eta C = \begin{bmatrix}
ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
pd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & ca & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & pd & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ca & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & pd & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ca & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & pd & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & ca & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & pd & 0 & 0 & 0 
\end{bmatrix} \quad \text{and} \quad \eta C = \begin{bmatrix}
ca & 0 & 0 & 0 \\
pd & 0 & 0 & 0 \\
0 & 0 & ca & 0 \\
0 & 0 & pd & 0 
\end{bmatrix}, \tag{B.3} \]

for full and partial synchronization respectively. The derivatives of the three terms in (B.1) then are:
\[
\dot{V}_1 = e(t)P \dot{e}(t) + \dot{e}(t)Pe(t) \\
= e(t)P(\dot{A}e(t) + A_1 e(t - \tau) + \Psi(\dot{v})) + (\dot{A}e(t) + A_1 e(t - \tau) + \Psi(\dot{v}))Pe(t) \\
= 2e(t)^T(\dot{PA})e(t) + 2e(t)^T(PA_1)e(t - \tau) + 2e(t)^TP\Psi(\dot{v}) - 2\lambda\Psi(\dot{v})^T(\Psi(\dot{v}) - \eta Ce(t)) \\
\]

\[
= 2e(t)^T(\dot{PA}_0)e(t) + 2e(t)^T(PA_1)e(t - \tau) + 2e(t)^TP\Psi(\dot{v}) - 2\lambda\Psi(\dot{v})^T(\Psi(\dot{v}) - \eta Ce(t)) \\
+ 2e(t)^TY(e(t) - 2e(t)^TYe(t - \tau) - 2e(t)^TY\int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha \\
+ 2e(t-\tau)^TW e(t) - 2e(t-\tau)^TW e(t - \tau) - 2e(t-\tau)^W \int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha \\
+ 2\Psi(\dot{v})^TRe(t) - 2\Psi(\dot{v})^TRe(t - \tau) - 2\Psi(\dot{v})^TR \int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha \\
\]

\[
= 2e(t)^T(\dot{PA}_0 + Y)e(t) + 2e(t)^T(PA_1 - Y + W^T)e(t - \tau) + 2e(t)^T(P + R^T + \lambda C^T\eta^T)\Psi(\dot{v}) - 2e(t-\tau)^T(W)e(t - \tau) - 2e(t-\tau)^T(R^T)\Psi(\dot{v}) \\
- \Psi(\dot{v})^T(2\lambda)\Psi(\dot{v}) \\
- 2e(t)^T(Y)\int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha - 2e(t-\tau)^T(W) \int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha - 2\Psi(\dot{v})^T(R) \int_{t-\tau}^{t} \dot{e}(\alpha)d\alpha \\
= \frac{1}{\tau} \int_{t-\tau}^{t} [2e(t)^T(\dot{PA}_0 + Y)e(t) + 2e(t)^T(PA_1 - Y + W^T)e(t - \tau) + 2e(t)^T(P + R^T + \lambda C^T\eta^T)\Psi(\dot{v}) - 2e(t-\tau)^T(W)e(t - \tau) - 2e(t-\tau)^T(R^T)\Psi(\dot{v}) \\
- \Psi(\dot{v})^T(2\lambda)\Psi(\dot{v}) \\
- 2e(t)^T(\tau Y)\dot{e}(\alpha)d\alpha - 2e(t-\tau)^T(\tau W)\dot{e}(\alpha)d\alpha - 2\Psi(\dot{v})^T(\tau R)\dot{e}(\alpha)d\alpha]d\alpha \\
\]

\[
\dot{V}_2 = \int_{t-\tau}^{\tau-\tau} [\dot{e}(t)^TZ\dot{e}(t) - \dot{e}(t + \beta)^TZ\dot{e}(t + \beta)] \\
= \int_{t-\tau}^{\tau-\tau} [\dot{e}(t)^TZ\dot{e}(t) - \dot{e}(\alpha)^TZ\dot{e}(\alpha)] \\
= \int_{t-\tau}^{\tau-\tau} [(A_0 e(t) + A_1 e(t - \tau) + \Psi(\dot{v}))^TZ(A_0 e(t) + A_1 e(t - \tau) + \Psi(\dot{v})) - \dot{e}(\alpha)^TZ\dot{e}(\alpha)] \\
= \frac{1}{\tau} \int_{t-\tau}^{t} [e(t)^T\tau A_0^TZA_0 e(t) + 2e(t)^T\tau A_0^TZA_1 e(t - \tau) + e(t-\tau)^T\tau A_0^TZA_0 e(t) + 2e(t-\tau)^T\tau A_0^TZA_1 e(t - \tau) - \dot{e}(\alpha)^TZ\dot{e}(\alpha)]d\alpha \\
\]

\[
\dot{V}_3 = e(t)^TQe(t) - e(t - \tau)^TQe(t - \tau) \\
= \frac{1}{\tau} \int_{t-\tau}^{t} [e(t)^TQe(t) - e(t - \tau)^TQe(t - \tau)]d\alpha \\
\]
Combining these three terms, we obtain
\[
\dot{V}(e_t) = \frac{1}{\tau} \int_{t-\tau}^{t} \zeta(t, \alpha, \hat{v})^T \Lambda(\tau) \zeta(t, \alpha, \hat{v}) d\alpha
\]
(B.4)
with
\[\zeta(t, \alpha, \hat{v}) = \begin{bmatrix} e(t)^T & e(t-\tau)^T & \Psi(\hat{v}) & \dot{\epsilon}(\alpha)^T \end{bmatrix},\]
and a symmetric matrix
\[
\Lambda(\tau) =
\begin{bmatrix}
PA_0 + A_0^T P + Y + Y^T + \tau A_0^T Z A_0 + Q & * & * & * \\
A_1^T P - Y^T + W + \tau A_1^T Z A_0 & -W - W^T + \tau A_1^T Z A_1 - Q & * & * \\
P + R + \tau Z A_0 + \lambda \eta C & \tau Z A_1 - R & -2\lambda & * \\
-\tau Y^T & -\tau W^T & -\tau R^T & -\tau Z
\end{bmatrix}.
\]
Applying a Schur complement as in [16], we obtain LMI (2.85).
Appendix C

Simulation results

Here, some simulation results are shown for the unidirectionally coupled network of Lur’e type neuron models with partial state coupling. For specific values of time-delay, the phase shifts, as shown in Figure 3.1, are compared with the ones shown here. The left graph shows the time response \( v(t) \) of the four systems. The graph to the right shown the time response of the errors of the two states between the systems.

In Figure C.1 the network is simulated for a time-delay \( \tau = 0.5 \) [ms]. Clearly, it can be seen that the network synchronizes fully for this value of time-delay. Only after 20 [ms] all systems pulse exactly similar. In the lower figure it can be seen that the errors between all systems converge to zero after 20 [ms]. If we assume that the amount time-delay represents the speed of the horse, low values of time-delay represent low speeds. Looking at the possible gaits at this speed in Figure 3.1, we can only conclude that for these values of time-delay, the horse stands still.

In Figure C.2 the time-delay in the coupling is increased to \( \tau = 1.5 \) [ms]. Now the systems do not synchronize fully, as can be seen very clearly if we look at the errors between the systems. However, we can see that the oscillations are similar, but only with a phase shift of a quarter of the period of the oscillation. This phase shift corresponds with the ‘walk’ gait from Figure 3.1, where we can also see the phase shifts of a quarter of the period.

If the time-delay is increased to \( \tau = 4 \) [ms], the pattern changes again, as we see in Figure C.3. Because the frequency of the oscillation increases, we zoom in on the last part of the simulation for a better view on the results. In this case, system 1 and 3 synchronize and system 2 and 4 synchronize as well. Between these two pairs, a phase shift of half a period appears. This oscillation pattern corresponds to three possible gaits in Figure 3.1, the ‘trot’, ‘pace’ and ‘bound’ gait.

Further increasing the time-delay, the ‘gallop’ gait is obtained for \( \tau = 8 \) [ms], as can be seen in Figure C.4.
Figure C.1: $K = 1$, $\tau = 0.5$ [ms], Stand.
Figure C.2: $K = 1, \tau = 1.5$ [ms], Walk.
Figure C.3: $K = 1$, $\tau = 4$ [ms], Trot/Pace/Bound.
Figure C.4: $K = 1$, $\tau = 8$ [ms], Gallop.