Literature Study on Periodic Solutions in Nonlinear Dynamic Systems

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Chapter 1

Introduction

Nonlinear dynamic systems represent a wide class of the systems. The dynamics of nonlinear systems are much richer than the dynamics of linear systems. Some specific phenomena can only take place in the presence of nonlinearity. Such phenomena are also called “essentially nonlinear phenomena” and examples of such phenomena are:

- **Finite escape time:** The state of an unstable linear system goes to infinity as time approaches infinity. However, the state of unstable nonlinear system can go to infinity even in finite time.

- **Multiple isolated equilibria:** A linear system can have only one isolated equilibrium point, hence, it can have only one steady-state operating point which attracts or repels the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

- **Limit cycles:** A linear autonomous system can exhibit steady state oscillations only if it has a pair of eigenvalues on the imaginary axis. This condition is almost never satisfied in real life systems, due to the presence of damping and perturbations. But even if that can be achieved, the amplitude of oscillations will be dependent on the initial state. In real life, steady-state oscillations in autonomous systems can only be produced by autonomous nonlinear systems. There are nonlinear systems which, after certain period of time when transient effects disappear, can exhibit steady-state oscillations of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as a stable limit cycle.

- **Subharmonic, harmonic or quasi-periodic oscillations:** A stable linear system subject to a periodic input (non-autonomous systems) produces an output of the same frequency. In nonlinear system, such solutions are called harmonic. Moreover, nonlinear systems subject to periodic excitation can also oscillate with frequencies, which are submultiples of the excitation frequency: subharmonic solution. It may even generate a quasi-periodic oscillation.
Chaos: A nonlinear system can also exhibit more complex steady-state behavior which is called chaos. From a practical point of view it can be defined as bounded steady-state behaviour which is not an equilibrium, a periodic oscillation or a quasi-periodic oscillation. The essential characteristic of such motion are the great sensitivity to the initial conditions and broad banded spectrum of the response.

Limit cycling, harmonic, subharmonic and quasi-periodic oscillation phenomena in dynamical engineering systems, often limits the performance of such systems. In this respect, one can think of stick-slip and whirl-type vibrations in oil-drilling rigs, vibrations in CD-player due to mass-unbalance of the CD, whirl vibrations in rotor-dynamic systems and many others. The same type of problem can be encountered in feedback controlled dynamic systems, for example due to saturation of the controller. Thus, the aims of this literature study are, firstly, to present some nonlinear physical phenomena that may cause periodic behaviour and, secondly, to present methods that may be used for the prediction and analysis of such periodic behaviour.

This literature study will be organized as follows. In chapter 2, some examples of real-life systems, that exhibit periodic behaviour, will be presented and the cause for such behaviour will be explained. Moreover, definitions of some basic notations, which are needed for understanding periodic behaviour, will be given. In chapter 3, the focus will lie on types of nonlinearities that can cause periodic behaviour and on conditions under which the periodic behaviour of nonlinear dynamic systems can occur. In chapter 4, tools for the prediction and estimation of periodic solutions will be presented. In chapter 5, conclusions will be presented.
Chapter 2.

Limit Cycling and Forced Oscillation Phenomena

Limit cycling and forced oscillation phenomena can occur in a various types of systems. In some systems, such behaviour is a desired feature, but in others it may be undesired. In order to explain the periodic behaviour, first some illustrative examples will be presented and, then, appropriate definitions of some basic notions will be given.

2.1. Example 1: Woodpecker Toy

The woodpecker toy is a toy that consists of a sleeve, a spring and the woodpecker (see figure 2.2.) [Pfeiffer and Glocker, 1996]. The hole of the sleeve is slightly larger than the diameter of the pole, thus allowing a kind of pitching motion, interrupted by the impacts with friction. The cause for the appearance of pitching motion is the combined presence of gravity force, dry friction force between the sleeve and the pole and the elastic spring.

Assume that the woodpecker is moving only in two-dimensional plane. In figure 2.1, a sequence of events of the woodpecker toy is shown, where the numbers 1-8 correspond to the steps depicted in the figure. In order to explain the sequence of events of the toy, let $t_k$ denote the time at the step $k$. Just before $t = t_1$ the sleeve is jamming and the woodpecker is rotating upward, thereby reducing the normal force between sleeve and pole in the contact point. At $t = t_1$, the sleeve starts sliding downward, due to the reduced normal contact force, and contact is lost at $t = t_2$. In the time interval $t_2 < t < t_3$, due to the presence of gravitational force, the toy is in free fall and is quickly gaining kinetic energy. The first upper sleeve impact occurs at $t = t_3$. But the contact immediately detaches. A beak
Figure 2.1. Sequence of events for a woodpecker toy (Step 1: Stick to slip transition at the lower sleeve, sleeve is sliding downward, body is rotating upward, Step 2: Detachment of the lower sleeve, sleeve is rotating upward, body is rotating upward (nearly no friction), Step 3: Impact with detachment of upper sleeve, sleeve is rotating downward after impact, body is rotating upward, Step 4: Beak impact with detachment, sleeve is rotating upwards after impact, body is rotating downward after impact, Step 5: Impact of upper sleeve, sleeve is sliding downward, body is rotating downward, Step 6: Detachment of upper sleeve, sleeve is rotating downward, body is rotating downward, Step 7: Impact of lower sleeve, sleeve is sliding downward, body is rotating downward, Step 8: Slip to stick transition of lower sleeve, jamming of sleeve, body is rotating downward).

impact may occur at $t=t_4$, which changes the direction of motion of the woodpecker. The beak impact is soon followed by the second upper sleeve impact at $t=t_5$. Detachment of the upper sleeve contact occurs at $t=t_6$. The toy is again in unconstrained motion during the time interval $t_6 < t < t_7$. Impact of the lower sleeve occurs at $t=t_7$, after which the sleeve is sliding down. The woodpecker is rotating downward, increasing the normal force, and jamming of the sleeve starts at $t=t_8$. The succession of jamming and sliding transfers the kinetic energy of the translational motion in $y$ direction, obtained during free falling, into rotational motion of the woodpecker. The woodpecker therefore swings backward when the lower sleeve contact jams, stores potential energy in the spring and swings forward again at $t=t_1 + T$, which completes the periodic motion.

Now, the following should be stressed. Due to the fact that the woodpecker is moving downward (in the $y$ direction according to the figure 2.2), it is not correct to call the motion of the woodpecker (explained in figure 2.1) a periodic motion. But, if we consider the gravity force to be constant with respect to downward movement and if we consider that the system is moving only in a two-dimensional plane, then it may be noticed that the behaviour of the woodpecker does not depend on $y$. Therefore all other coordinates but $y$ exhibit periodic behaviour. This is why the motion of the woodpecker, explained in figure 2.1, can be considered to be periodic motion.
2.2. Example 2: Ringing Bell

A ringing bell is an example of a system in which limit cycling appears. A scheme of the electric bell is shown in figure 2.3. The parts of a continuously ringing bell include the switch, an electromagnet, which is a device that acts as a magnet when a current runs through, and an armature, which is a movable metal part. A clapper is attached to the end of the armature. Moreover, a spring that rests against a screw is attached to the armature. Wiring connects the source of electric current, the switch (door button at the figure 2.3) and the electromagnet. Another wire connects the screw back and the source of the electric current. Together, the parts of the bell form an electric circuit.

![Figure 2.2. Schematic representation of the woodpecker.](image_url)

![Figure 2.3. Scheme of electric bell.](image_url)

When the armature returns to its original position, the spring comes in contact with the screw again and re-establishes the flow of electric current. The process repeats itself and the electric
bell keeps ringing as long as the door button is on; this causes limit cycling. In order to achieve such oscillations we need to have an appropriate electric source. This means that the electromagnetic coil should produce the electromagnetic force, when the current flows through the electric circuit, which should be high enough to be able to attract the clapper toward the bell. Moreover, when the clapper hits the bell the contact between the screw and the spring should be broken.

This is also an example periodic behaviour as a desired phenomenon.

2.3. Example 3: Hammering in Gear

Machines and mechanisms are built up from rigid or elastic bodies interconnected in such a way that certain functionality of these machines is realized. Couplings in machines are never ideal but may exhibit backlash or some properties which lead to stick-slip phenomena. The gear systems of fuel engines should be usually designed with large backlashes due to the operating temperature range of such engines [Pfeiffer and Glocker, 1996]. Therefore, the power transmission via such gear systems takes place discontinuously by an impulsive hammering process in all transmission elements. Undesired vibrations appear in the gear unit due to periodic excitations, mainly from the injection pumps, the existence of backlash and interaction between driving, the driven gear and the act of all possible external disturbances to the driving wheels. Additionally, in all other backlashes of the gear unit similar processes take place, where the state and the impacts in one gear unit with backlash considerably influence the state in all other gears. Such processes produce vibrations which may be periodic or quasi periodic (see [Pfeiffer and Glocker, 1996]).

Those oscillations are, of course an example of undesired forced oscillations.

Figure 2.4. Typical diesel engine driveline system and forces in a mesh of gears.

2.4. Example 4: Oscillators

In the previous three sections, three simple examples of systems that can exhibit periodic behaviour are presented. In this section we will consider oscillators in order to distinguish between oscillations that may appear in linear time-invariant systems and in nonlinear systems.
A system oscillates when it has a nontrivial periodic solution

\[ x(t + T) = x(t) \forall t \geq 0, \]

for some \( T > 0 \). The word "nontrivial" is used to exclude constant solutions corresponding to equilibrium points. Constant solutions satisfy the above equation, but it is not what we have in mind when we talk about oscillations or periodic solutions. The image of a periodic solution in the phase portrait is a closed trajectory (figure 2.5), which is usually called a periodic solution or a periodic orbit or a closed orbit.

![Phase portrait of Van der Pol oscillators](image)

*(a) A stable limit cycle; (b) an unstable limit cycle.*

A stable linear system subject to a periodic input produces an output of the same frequency as the input. It is also well known that a second-order linear time-invariant system with eigenvalues on the imaginary axis \(( \pm j \omega )\) can induce oscillations. The origin of that system is a center and the trajectories are periodic orbits. Such a system has a sustained oscillation with an amplitude that depends on the initial condition. It is usually referred to as the harmonic oscillator. If we think of the harmonic oscillator as a model of the linear LC circuit of figure 2.6, then the physical mechanism leading to these oscillations is a periodic exchange (without dissipation) of the energy stored in the capacitor electric field with the energy stored in the inductance. The periodic solutions in such systems are not robust. This means that infinitesimally small right-hand side (linear or nonlinear) perturbations will destroy the oscillation. That is, the linear oscillator is not structurally stable [Khalil, 1996]. In fact, it is impossible to build an LC circuit that realizes the harmonic oscillator, due to the existence of resistance in the electric wires that will eventually consume whatever energy was initially stored in the capacitor and inductor. Even if we succeed in building this linear oscillator, we would face the second problem: the amplitude of oscillation is dependent on the initial conditions.

Those two problems of linear oscillators can be eliminated in nonlinear oscillators. It is possible to build physical nonlinear oscillator such that the nonlinear oscillator is structurally stable and the amplitude of oscillation (in steady-state) is independent of initial conditions.

![Linear LC circuit](image)

*Figure 2.6. A linear LC circuit for the harmonic oscillator.*
The negative-resistance oscillator in figure 2.7 is an example of such a nonlinear oscillator. In [Khalil, 1996] it is shown that such circuit can be described by the differential equation:

\[ \ddot{x} - \epsilon (1 - x^2)\dot{x} + x = 0. \]  

Equation (2.1) is known as the van der Pol equation.

For \( \epsilon > 0 \) and for initial conditions \( |x(0)| > 1 \), the equation (2.1) behaves like a system with a stable equilibrium point at \( x = \dot{x} = 0 \). Thus, the system moves toward the origin. At the moment when \( |x(t)| < 1 \), then \( -\epsilon (1 - x^2) < 0 \) and the system (2.1) behaves like an unstable system and \( x \) starts moving toward infinity. However, at certain moment when \( |x(t)| > 1 \), the system will again behave like a stable system. Therefore, for this kind of system the following conclusion can be drawn [Khalil, 1996]: for \( \epsilon > 0 \) system, (2.1) possesses a periodic solution that attracts every other solution except the trivial one at the unique equilibrium point \( x = \dot{x} = 0 \).

In electronics these type of systems are called negative-resistance oscillators, and in mechanics negative-damping systems.

2.5. Definitions of Systems

In this chapter, we define different types of dynamical systems and present some useful facts, which are needed in the chapters that follow.

**Definition 2.1 Autonomous continuous-time dynamical systems** [Parker and Chua, 1989].

An nth-order autonomous continuous-time dynamical system is defined by the state equation

\[ \dot{x} = f(x, x(t_0)) = x_0 \]  

(2.2)

where \( x(t) \in \mathbb{R}^n \) is the state at time \( t \), and \( f: \mathbb{R}^n \to \mathbb{R}^n \) is called vector field.

The solution of (2.2) in [Parker and Chua, 1989] is called the flow and is often written as \( \varphi(t, x_0) \) in order to show the explicit dependence on the initial conditions. The set of points \( \{\varphi(t, x_0): -\infty < t < \infty\} \) is called the trajectory through \( x_0 \).

**Definition 2.2 Non-autonomous continuous-time dynamical system** [Parker and Chua, 1989].

An nth-order non-autonomous continuous-time dynamical system is defined by the state equation

\[ \dot{x} = f(x, t), x(t_0) = x_0. \]  

(2.3)

For non-autonomous systems, unlike autonomous systems, the vector field explicitly depends on time. The solution of (2.3) passing through \( x_0 \) at time \( t_0 \) is denoted by \( \varphi(t, t_0, x_0) \). If there exists some \( T > 0 \) such that \( f(x, t) = f(x, t + T) \) for all \( x \) and \( t \), then the system is said to
be time periodic with period $T$. The smallest of such $T$ is called the minimum period, but we will simply call it the period.

Furthermore, unless otherwise stated, due to the fact that we are interested in periodic behaviour of systems, all non-autonomous systems are assumed to be time-periodic and nonlinear (i.e. the vector field $f$, in (2.3), is time-periodic and nonlinear).

An $n$th-order time-periodic non-autonomous system with period $T$ can always be converted into an $(n+1)$th-order autonomous system by introducing an extra state $\theta$ and then, the appropriate autonomous system is given by

$$
\dot{x} = f\left(x, \frac{\theta T}{2\pi}, x(t_0) = x_0, \right) \tag{2.4a}
$$

$$
\dot{\theta} = \frac{2\pi}{T}, \theta(t_0) = \frac{2\pi t_0}{T}. \tag{2.4b}
$$

Since $f$ is time periodic with period $T$, the system (2.4) is periodic in $\theta$ with period $2\pi$. Hence, the points $\theta = 0$ and $\theta = 2\pi$ may be identified and the state space can be transformed from the Euclidean space $\mathbb{R}^{n+1}$ to the cylindrical space $\mathbb{R}^n \times S$, where $S = [0, 2\pi)$ denotes the circle.

The solution of (2.3) in the cylindrical state space is

$$
\begin{bmatrix}
x(t) \\
\theta(t)
\end{bmatrix} = \begin{bmatrix}
\phi(t, t_0, x_0) \\
\frac{2\pi t}{T} \mod 2\pi
\end{bmatrix} \tag{2.5}
$$

where the modulo function restricts $\theta(t)$ to $0 \leq \theta < 2\pi$. Using this transformation, the theory of autonomous systems can be applied to time-periodic non-autonomous systems.

Non-autonomous systems can be also considered as

$$
\dot{x} = f(x, u(t)), x(t_0) = x_0, \tag{2.6}
$$

where $u(t)$ is the input, which explicitly depends on time. Such non-autonomous systems are called forced systems. Due to the fact that only the periodic behaviour of systems is of our interest, unless otherwise stated, it will be assumed that the input to the non-autonomous systems is periodic. Then the non-autonomous system can be also modeled as

$$
\dot{x} = f(x, h(u(t))), x(t_0) = x_0, \tag{2.7a}
$$

$$
\dot{x}_u = f_u(x, u(t_0) = x_{u0} \tag{2.7b}
$$

where (2.7b) is an autonomous system with which is possible to obtain the periodic input $u(t) = h(x_u)$ for the system (2.7a).

### 2.6. Steady-state Behaviour

In the previous sections, we first explained some examples of real-life systems in which periodic behaviour can appear and second we formally defined types of systems, which will be considered. In this section, some definitions will be presented and types of periodic behaviour will be explained. In general, we are interested in steady state behaviour. The term steady state behaviour refers to the asymptotic behaviour of a system as $t \to \infty$. The difference between the trajectory and its steady state is called the transient.
Definition 2.3 \(\omega\) Limit set [Parker and Chua, 1989]. A point \(y \in \mathbb{R}^n\) is said to be an \(\omega\)-limit point of \(x\) if, for every neighborhood \(U\) of \(y\), the trajectory \(\varphi(t,x)\) of (2.2) repeatedly enters \(U\) as \(t \to \infty\). The set \(\omega(x)\) of all \(\omega\)-limit points of \(x\) is called the \(\omega\)-limit set.

Definition 2.4 Basin of attraction [Parker and Chua, 1989]. An \(\omega\)-limit set is attracting if there exist an open neighborhood \(U\) of \(\omega\) such that \(\omega(x) \subseteq \omega\) for all \(x \in U\). The union of such neighborhoods \(U\) represents the basin of attraction \(B_\omega\) of an attracting set \(\omega\).

These definitions are given for autonomous systems. These definitions can also be applied to non-autonomous systems after their conversion to autonomous systems (via transformation (2.4) or (2.7)).

There also exist reverse-time limit points and limit sets, which are defined as follows:

Definition 2.5 \(\alpha\) Limit set [Parker and Chua, 1989]. A point \(y \in \mathbb{R}^n\) is said to be \(\alpha\)-limit point of \(x\) if, for every neighborhood \(U\) of \(y\), the trajectory \(\varphi(t,x)\) of (2.2) repeatedly enters \(U\) as \(t \to -\infty\). The set \(\alpha(x)\) of all \(\alpha\)-limit points of \(x\) is called the \(\alpha\)-limit set of \(x\).

In general, nonlinear systems can have several attracting limit sets, each with a different basin of attraction. The initial condition determines which set is eventually reached.

In the sequel, we will present different types of steady-state behaviour.

2.6.1. Equilibrium Points

Definition 2.6 Equilibrium point [Parker and Chua, 1989]. An equilibrium point \(x_{eq}\) of a system is a constant solution of (2.3), i.e. \(x_{eq} = \varphi(t_0,x_{eq})\) for all \(t\). At an equilibrium point the vector field \(f\) vanishes and \(f(x,t) = 0\) implies that \(x\) is an equilibrium point.

In general, there is no such behaviour for non-autonomous systems.

2.6.2. Periodic Solutions

Definition 2.7 Periodic solution\(^1\) of autonomous systems [Parker and Chua, 1989]. \(\varphi(t,x_p)\) is a periodic solution of an autonomous system (2.2) if, for all \(t\),

\[
\varphi(t,x_p) = \varphi(t+T,x_p)
\]

(2.8)

for some minimum period \(T > 0\).

Furthermore, a periodic solution of an autonomous system will be denoted as \(\varphi_p(t,x_p)\), where \(x_p\) is a point on the periodic solution. The restriction \(T > 0\) is required to prevent the classification of an equilibrium point as a periodic solution. Note that \(x_p\) is not unique since any point lying on periodic solution satisfies (2.8).

A periodic solution is isolated if it possesses a neighborhood that contains no other periodic solution. For autonomous systems, an isolated periodic solution is called a limit cycle. Consequently, every limit cycle is a periodic solution. The converse, however, is not true: for instance, linear time-invariant systems may have periodic solutions, when it has a pair of eigenvalues on the imaginary axis \((\pm j\omega)\), but no limit cycles. In fact, one important difference between linear and nonlinear systems is that nonlinear systems may have limit

\(^1\) In [Khalil, 1996] a periodic solution is called a periodic orbit and in [Sastry, 1999] it is called a closed orbit.
cycles, but linear systems do not. This was explained in section 2.4, where examples of linear and nonlinear oscillators were presented.

A non-autonomous systems can be considered as a special case of an autonomous systems (see (2.4) and (2.7)), but it is useful to define periodic solutions for non-autonomous systems.

**Definition 2.8 Periodic solutions of non-autonomous systems** [Parker and Chua, 1989]. \(\varphi(t, t_0, x_p)\) is a periodic solution of a non-autonomous system (2.3) if, for all \(t\),

\[
\varphi(t, t_0, x_p) = \varphi(t + T, t_0, x_p)
\]

for some minimum period \(T > 0\).

Periodic solutions for non-autonomous systems, as was done for autonomous systems, will be denoted as \(\varphi_p(t, t_0, x_p)\), where \(x_p\) is a point on the periodic solution at the time instant \(t_0\).

A useful tool for visualization the steady-state behaviour of non-autonomous systems is the Poincaré section. In general, the Poincaré section can be defined as a hypersurface in the phase space that is transverse to the flow of a given system. For systems driven by a periodic excitation with period \(T_{ex}\), the Poincaré section is usually defined by the state \(x\) and \(\dot{x}\), at times \(t = iT_{ex}\) where \(i = 1, 2, \ldots\). This corresponds to a stroboscopic sampling of the state space with period time \(T_{ex}\).

It is well known that a stable linear time-invariant system, with a harmonic excitation, in the steady state, produces a state of the same frequency. However, the behavior of nonlinear systems, which are excited by a harmonic excitation are much more complicated than the behavior of linear time-invariant systems. For non-autonomous nonlinear systems the following types of periodic solution can appear:

- A periodic solution \(\varphi_p(t, t_0, x_p)\) is called a harmonic, fundamental or period 1-solution [Parker and Chua, 1989] if it has the same period time \(T\) as the excitation period \(T_{ex}\). An example of a harmonic solution is shown in figure 2.8 a, b, c. In the Poincaré section it is represented by one simple dot.

- A periodic solution \(\varphi_p(t, t_0, x_p)\) is called a period \(K\)-solution or \(K\)th-order subharmonic [Parker and Chua, 1989] if it has a period time \(T\) which is some integer multiple \(K\) of the excitation period \(T_{ex}\). An example of a period 3 solution is shown in figure 2.8 d, e, f and it can be seen that this solution appears in the Poincaré section as three dots.

- Besides the basic frequency, i.e. the lowest frequency occurring in the periodic solution, also frequencies occur which are a multiple of the basis frequency. In a (sub)harmonic orbit such a higher frequency may be dominant and may cause a resonance. This is called a superharmonic resonance.

It should be noted that a superharmonic solution i.e. a harmonic solution with a basis frequency larger than the excitation frequency, normally cannot exist.

---

3 More about Poincaré maps is presented in section 2.7

3 A harmonic solution in [Sastry, 1999] is called a harmonic orbit

4 A period \(K\)-solution in [Sastry, 1999] is called a \(\frac{1}{K}\) subharmonic orbit
2.6.3. Quasi-periodic Solutions

Definition 2.9 Quasi-periodic function [Parker and Chua, 1989]. A quasi-periodic function is one that can be expressed as a countable sum of periodic functions

\[ x(t) = \sum_{i} h_i(t) \]  

(2.10)

where \( h_i(t) \) has minimum period \( T_i \) and frequency \( f_i = \frac{1}{T_i} \). Furthermore there must exist a finite set of basis frequencies \( \{\hat{f}_1, \ldots, \hat{f}_n\} \) with the following two properties:

1. It is linearly independent, i.e. there does not exist a non-zero set of integers \( \{k_1, \ldots, k_n\} \) such that \( k_1\hat{f}_1 + \ldots + k_n\hat{f}_n = 0 \);
2. It forms a finite integral base for the frequencies $f_i$, i.e. for each $i$,
\[ f_i = \left| k_1 \hat{f}_1 + \ldots + k_n \hat{f}_n \right| \text{ for some integers } \{k_1, \ldots, k_n\}. \]

The basis frequencies are not defined uniquely. A quasi-periodic solution with $p$ base frequencies is called $p$-periodic. But the term $p$-periodic should not be confused with a period $K$-solution. The former is quasi-periodic and the latter is periodic. A quasi-periodic orbit (see figure 2.8) reveals itself in a Poincaré section as infinitely many points, filling up a closed curve.

2.6.4. Chaos

There is no widely accepted definition of chaos. From a practical point of view, it can be defined as "none of the above," i.e. as bounded steady-state behaviour that is neither an equilibrium point nor periodic nor quasi-periodic.

An example of a chaotic orbit is shown in figure 2.8j, k, l. It manifests itself in such a Poincaré section as a “cloud” of infinitely many points filling out a bounded region in an orderly manner, whereas no point does return precisely to any of the other points of the “cloud”.

2.7. Poincaré Maps

The Poincaré map replaces the flow of an $n$th-order continuous-time system by an $(n-1)$th-order discrete-time system. This technique is very useful for analyzing dynamical systems. The definition of Poincaré map ensures that its limit sets correspond to limit sets of the underlying flow. In the sequel, we will define the Poincaré map separately for autonomous and for nonautonomous systems.

2.7.1. Poincaré Map for Autonomous Systems

Consider an $n$-th order autonomous system with a limit cycle $\Gamma$ as shown in figure 2.9. Let $\mathbf{x}^*$ be a point on the limit cycle and let $\Sigma$ be an $(n-1)$-dimensional hyperplane transversal to $\Gamma$ at $\mathbf{x}^*$. For system (2.3), the notion “transverse to the flow” indicates

\[
\mathbf{n}^T(x) f(x) \neq 0,
\]

where $\mathbf{n}(x)$ is a column vector at $\mathbf{x}$ in phase space normal to the hyperplane $\Sigma$. The trajectory, starting from $\mathbf{x}^*$ will hit $\Sigma$ at $\mathbf{x}^*$ in $T$ seconds, where $T$ is the minimum period of the limit cycle. Trajectories starting on $\Sigma$ in a sufficiently small neighborhood of $\mathbf{x}^*$ will, in approximately $T$ seconds, intersect $\Sigma$ in the vicinity of $\mathbf{x}^*$. Hence, the solution $\varphi(t, \mathbf{x})$ of (2.2) and $\Sigma$ define a mapping $P_A$ of some neighborhood $U \subset \Sigma$ of $\mathbf{x}^*$ onto another neighborhood $V \subset \Sigma$ of $\mathbf{x}^*$. The mapping $P_A$ is called Poincaré map for autonomous systems.

The mapping $P_A$ is defined locally, in the neighborhood of $\mathbf{x}^*$. Therefore, it is not guaranteed that the each trajectory, starting from any point on $\Sigma$ will intersect $\Sigma$.

2.7.2. Poincaré Map for Non-autonomous Systems

In section 2.5 we showed that a time-periodic $n$th-order non-autonomous system with minimum period $T$ can be transformed into an $(n+1)$th-order autonomous system in the cylindrical state space $\mathbb{R}^n \times S$ via (2.4). Consider the $n$-dimensional hyperplane $\Sigma \in \mathbb{R}^n \times S$ defined by
\[
\Sigma := \{ (x, \theta) \in \mathbb{R}^n \times S : \theta = \theta_0 \}.
\] 

(2.12)

Every \( T \) seconds, the trajectory (2.5) intersects \( \Sigma \) (see figure 2.10). The resulting map \( P_N : \Sigma \to \Sigma \) is defined by

\[
P_N(x) := \varphi(t_0 + T, t_0, x).
\]

(2.13)

\textit{Figure 2.9. The Poincaré map for a third order autonomous system.}

\( P_N \) is called Poincaré map for non-autonomous systems, as was already discussed in section 2.6. such a Poincaré map is also called a Poincaré section and for systems driven by the periodic excitations the Poincaré section is usually defined by the state \( x \) and \( \dot{x} \), sampled with the period time of the periodic excitation.

\textit{Figure 2.10. The Poincaré map for a first order non-autonomous system.}
Chapter 3.

Causes for Periodic Behaviour of Systems

In the previous chapter, in order to obtain a better overview on the causes for periodic behaviour of nonlinear dynamic systems, several examples were presented. In this chapter, we will discuss some nonlinear effects that can cause such periodic behaviour.

3.1. Dry Friction

In general, mechanical systems exhibit friction. For some systems, the friction may be a desired characteristic as it is for brakes or musical instruments such as a violin. For other systems, however, friction is an undesired phenomenon (think of friction in servo systems). Extensive research on friction models and friction induced limit cycles has been done over the past decade [Armstrong-Hélouvry, 1994; Canudas de Wit, 1995; Gaffenert, 1997; Galvanetto, 1997; Leine, 2000; Nayfeh and Mook, 1979; Olsson, 1996a; Olsson, 1996b; Wallenborg, 1988].

In order to present the most commonly used dry friction models, a system as shown in figure 3.1 will be considered. It is supposed that the system is subject to a dry friction force $F_{dry}$ and an external force $F_{ex}$. The model of the system can be expressed by the following differential equation

$$m\ddot{v} = F_{ex} - F_{dry},$$

where $m$ is the mass and $v$ the velocity of the mass. The dry friction is defined as a force that resists relative tangential motion between two contacting surfaces of different bodies. The bodies “stick” to each other when the relative velocity between the contacting surfaces is zero. If the bodies slide over each other with a non-zero velocity, we speak of “slip”. In the stick phase, the friction force adjusts itself to enforce equilibrium with the external forces acting on the bodies. The bodies remain sticking as long as equilibrium is ensured. If the friction force in the stick phase exceeds a threshold, the so called the break-away friction force or maximum
static friction force, the bodies will begin to slip over each other. The maximum static friction force will be denoted by $F_s$. So the friction force $F_{dry}$ lies in the interval $-F_s \leq F_{dry} \leq F_s$ when the body stick. In the slip phase, the friction force is a function of the velocity $v$, i.e. $F_{dry} = F(v)$.

![Figure 3.1: Block on a floor.](image)

Many different models are proposed for the mathematical description of the dry friction. In [Olsson, 1996a], they are divided into the following three groups:

- **Static dry friction models**: Those models are the most commonly used and they regard friction as a function of velocity $v$.
- **Dynamics dry friction models**: Those models describe dry friction by differential equations and they are obtained through modification of the static friction models.
- **Special purpose models**: This category includes some models that give understanding of the physical mechanisms behind friction.

### 3.1.1. Static Dry Friction Models

The most commonly used dry friction model is the Coulomb friction model (figure 3.2a):

$$F_{dry} = \begin{cases} F_s \text{ sgn}(v) & \text{if } v \neq 0 \\ F_{ex} & \text{if } v = 0. \end{cases}$$

This friction model is rather straightforward.

In order to describe the stick phase, the following model was introduced (figure 3.2b):

$$F_{dry} = \begin{cases} F_d \text{ sgn}(v) & \text{if } v \neq 0 \\ F_{ex} & \text{if } v = 0, |F_{ex}| \leq F_s \\ F_s \text{ sgn}(F_{ex}) & \text{if } v = 0, |F_{ex}| > F_s. \end{cases}$$

where the friction force in the stick phase $F_s$ (static friction force) is higher than the Coulomb friction level $F_d$. Another improved dry friction model is contained in the model shown in figure 3.2.c. In this model, there exists an interval in which the dry friction force decreases from $F_s$ to $F_d$ as velocity $v$ increases. This effect is called the Stribeck effect and such a model can be expressed by
where $F(v)$ is an odd function modeling the Stribeck effect. It can be given either as a look-up table or as a parameterized curve that fits experimental data. A number of parameterizations of $F(v)$ have been proposed. The most common one is of the form

$$F(v) = F_d \left( \frac{v}{v_s} \right)^{\delta_s} \left( F_s - F_d \right) \left( v \right), \quad (3.5)$$

where $v_s$ is called the Stribeck velocity and $\delta_s$ is a constant which should be chosen such that (3.5) matches the experimental data ($\delta_s \in [0.5, 2]$, see [Olsson, 1996a]). Some other forms of $F(v)$, may be found in [Hensen, 2002; Leine, 2000; Olsson, 1996a].

The main problem of the friction models (3.2)-(3.5) is that they are all non-smooth and discontinuous and hence may cause a lot of computational problems when velocity $v$ is close to zero.

A smooth and continuous dry friction model is shown in Fig. 3.2d and it can be expressed as

$$F_{dry} = \frac{2}{\pi} \left( \frac{F_s - F_d}{1 + \delta^2 \Delta v^2} \right) \arctan(\Delta v), \quad (3.6)$$

where $\delta >> 1$ and $\delta > 0$. The slope of the function is very steep for small $|v|$. The friction force (3.6) has a maximum, which is approximately $F_s$ for large values of $\delta$. This maximum was already introduced as the static friction force. The upward ramp is followed by a downward ramp, which brings the friction force to an almost constant value $F_d$. This model has the advantage that the resulting equation of motion is a smooth ordinary differential equation. All standard integration routines can, therefore, be directly applied and the solution of the initial value problem always exists and is unique. But the disadvantages of this model are twofold. Firstly, the model cannot describe the stick phase properly. Namely, the friction force is zero at zero velocity. The mass will therefore always slip if any external force is present. The second disadvantage is the steep slope of the upward and downward ramps. The equation of motion with friction model (3.6) will be a very stiff differential equation, which is inconvenient from a numerical point of view.

Another dry friction model is proposed as

$$F_{dry} = \begin{cases} F(v) \text{sgn}(v) & \text{if } |v| > \eta \\ F_{ex} & \text{if } |v| \leq \eta, |F_{ex}| \leq F_s \\ F_s \text{sgn}(v) & \text{if } |v| \leq \eta, |F_{ex}| > F_s \\ \end{cases} \quad (3.7)$$

(figure 3.2e) which has a lot of advantages from numerical point of view (see [Leine, 2000]).

More information on dry friction models can be found in [Hensen, 2002; Leine, 2000; Olsson, 1996a].

### 3.1.2. Dynamic Dry Friction Models

A well-known dynamic friction model is the LuGre friction model [Canudas de Wit, 1995; Hensen, 2002; Olsson, 1996a]. It is a first-order dynamic model. The most commonly used form is
\[ \frac{dz}{dt} = v - \frac{v}{g(v)} z \]

\[ g(v) = \frac{1}{\sigma_0} \left( F_d + (F_s - F_d) \frac{v}{v_s} \right)^2 \]

\[ F_{dry} = \sigma_0 z + \sigma_1 \frac{dz}{dt}, \]

where \( \sigma_0, \sigma_1, F_d, F_s \) are positive constants (\( F_d, F_s \) are the same constants as in static dry friction models) and \( z \) is the additional state coordinate which can not be measured.

\[ \text{(a)} \]

\[ \text{(b)} \]

\[ \text{(c)} \]

\[ \text{(d)} \]

\[ \text{(e)} \]

*Figure 3.2: Static dry friction models.*
More about this model and other dynamic dry friction models can be found in [Hensen, 2002; Olsson, 1996a].

3.1.3. Special Purpose Models

Special purpose models include a category of friction models that describes contact forces using continuum mechanics. Another model is based on the hydrodynamics of lubricated contacts and there are also some special purpose models for road-tire friction and rock mechanics. Those types of models will not be considered here, but more about such models can be found in [Olsson, 1996a].

3.1.4. Dry Friction as Cause for Limit Cycling

In order to explain the mechanism for limit cycling in open-loop systems, we will consider the system shown in figure 3.3. That system may be described by the following differential equation

\[ m\ddot{x} + kx = F_{dry}, \]

where \( F_{dry} \) is a dry friction. When \( |kx| \leq F_s \), at the moment when \( \dot{x} = v_{dr} \), the system will enter the stick phase (see figure 3.2). Consequently \( F_{dry} = kx, \ m\ddot{x} = 0 \) and the mass \( m \) will move together with the driving belt (with the constant velocity \( v_{dr} \)). At the moment when \( |kx| > F_s \), then system enters the slip phase and it starts moving due to the force of a spring up to the moment when again the mass \( m \) enters the stick phase. This kind of motion in mechanical systems is called stick-slip motion.

There are a lot of examples of mechanical open-loop systems that exhibit limit cycling due to the presence of dry friction. In chapter 2, the woodpecker toy was described. It was noted that dry friction is also a necessary condition for the limit cycle occurring (the analysis of limit cycling in the woodpecker toy can be found in [Pfeiffer and Glocker, 1996]). Some more examples are shown in figures 3.3 and 3.4. Limit cycles caused by dry friction can be also found in a drilling process. In figure 3.4, the basic elements of a rotary drilling system are shown, but a more detailed description can be found in [Leine, 2000]. In this reference, it is stated that dry friction is not the only cause for limit cycle behaviour in the drilling process. In such systems at some point the transition from stick-slip (caused by dry friction) to whirl motion may appear. This transition is caused by some other nonlinear effects. More information about all possible effects that cause limit cycling in the drilling process can be found in [Leine, 2000].

In closed-loop systems, also stick-slip motion can be a reason for limit cycling. In order to explain the mechanism due to which limit cycling appears, we will consider the position-control closed-loop system shown in figure 3.5 with a PID controller. The angular position \( \theta \) should reach the desired angular position \( r \). Thus, the PID controller will transfer the control via the amplifier to the motor. At a certain moment, the error \( e \) will be so small that the torque of the motor, produced by the controller, will be lower than the torque due to friction. Then the position \( \theta \) will not change (the stick phase), but the error will still be non-zero. Due to the existence of the integral action in the PID controller, the value of the control action will increase or decrease, depending on the sign of the error. At a certain moment, the control to the motor will be high enough to produce a motor torque higher than the friction torque. Consequently, the motor will start moving (slip phase) and the absolute value of error will decrease. Now, the integral controller action might be such that the system overshoots \( e = 0 \). When the motor torque is again smaller than the torque produced by the dry friction, the system will enter the stick phase, and the complete event will repeat itself again and a limit cycling may be achieved [Hensen, 2002].
Figure 3.3. One degree of freedom system with dry friction.

Figure 3.4. Drilling rig.

Figure 3.5. a) Schematic picture of a position control system, b) Block diagram of the system.
There is also a wide variety of examples of mechanical systems that, in closed loop, exhibit limit cycling. Especially, dry friction is an important problem in position control systems. Similar examples of position-control closed-loop systems are analyzed in [Hensen, 2002; Leine, 2000; Olsson, 1996a; Olsson, 1996b; Wallenborg, 1988].

3.2. Dead Zone and Saturation

The dead zone (figure 3.6a) nonlinearity may be confused with the backlash nonlinearity. In the next section, the backlash nonlinearity will be presented and the difference between backlash and dead zone will be illuminated.

Servo systems with dry friction are often presented as linear systems with a voltage as the input and the (angular) position or (angular) velocity as the output, where the only nonlinearity is the friction force. This nonlinearity produces the following effect: if the (angular) velocity of the servo system is zero, it will be zero until the input voltage achieves a certain value $F_s$ (see figure 3.2), which depends on the dry friction. But when the (angular) velocity is not zero, then the system can be considered as a linear system if the input does not change direction. This is the reason, why systems with dry friction (figure 3.7a), are sometimes considered as linear systems, which exhibit a dead zone nonlinearity at the input (figure 3.7b) (see [Ortega, 2000]).

But the models from figure 3.7a and 3.7b are not equivalent. In order to show this, let us consider the system shown in figure 3.1 as a “linear system + dry friction force” (figure 3.7a), which can be expressed by the equation (3.1) with dry friction (3.2) Then the system (3.1) can be rewritten in the following form

$$m\ddot{v} = F,$$

where $F = F_{ex} - F_{dry}$, i.e.

$$F = \begin{cases} F_{ex} - F_s \text{sgn}(v) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}.$$  

(3.10)

The system in figure 3.7b can be presented by (3.9) where

$$F = \begin{cases} F_{ex} - F_s \text{sgn}(F_{ex}) & \text{if } |F_{ex}| \geq F_s \\ 0 & \text{if } |F_{ex}| < F_s \end{cases}.$$  

(3.11)

So, from (3.10 and 3.11) it is obvious that the systems in figure 3.7 are not equivalent.
However, due to the already mentioned reasons, this type of modeling dry friction effects was used for the analysis of closed-loop systems with dry friction, especially when the personal computers were not so powerful as nowadays and when approximation techniques for obtaining limit cycles were more popular than numerical techniques.

Due to the fact that in such systems dead zone is part of the driving amplifier (it is placed at the input of a servo system) and due to the fact that the amplifier in real life can not have an infinite amplification, in such systems dead zone (figure 3.6a) and saturation (figure 3.7b) should go together (figure 3.6c). Saturation is a nonlinearity which itself also may cause limit cycling.

The mechanism for limit cycling due to dead zone is similar to the mechanism of limit cycling due to existence of dry friction. The mechanism for limit cycling due to saturation is the following. Consider the closed-loop system shown in figure 3.7b which instead of dead-zone nonlinearity has the saturation nonlinearity as shown in figure 3.6b. In figure 3.6b x represents the input (external force) which acts to the system \( F_{\text{ex}} \) and y represents the input \( F \), which acts to the linear part of the system \( F \) is limited due to the existence of the saturation nonlinearity \( F_{\text{ex}} \) while is not limited). A limit cycle due to saturation may appear only if the slope (amplification) of the nonlinearity is steep enough that the closed loop system is unstable, when the saturation does not act. Then the output of the system increases as well as the control, thus at a certain moment the saturation will become active. That means that the amplification of the amplifier becomes lower (in fact it is zero) and, therefore, the system behaves as the stable system at that moment. Then the system enters again the zone where saturation does not active and, again, the system becomes unstable. This event can repeat itself and, consequently, the system may exhibit limit cycling.

An example of the qualitative analysis of a servo system with this type of nonlinearity (figure 3.6c) is presented in [Ortega, 2000]. In that paper, the closed-loop servo system (the analyzed system is similar to the system shown in figure 3.5) with a proportional controller is analyzed using the describing function method.

### 3.3. Backlash

Backlash is a nonlinearity which appears in mechanical systems coupled by gears and transmitions. A representation of this nonlinearity is shown in figure 3.8. This nonlinearity is different from the nonlinearity shown in figure 3.6a which represents the dead zone nonlinearity. A simple coupled system with backlash is shown in figure 3.9.

An example of an open loop system with backlash (diesel engine), which exhibits periodic behaviour, is presented in Section 2.3. In that system, the periodic behaviour is due to the presence of a periodic excitation. Suppose that the driving part in figure 3.9 is subject to a periodic driving torque. As a consequence, permanent impacts between the driven and driving
mass occurs due to the backlash. This may lead to the appearance of limit cycling. In Section 2.3, we have shown the diesel engine with a great number of this type of couplings with backlash. Such a system produces vibrations which may lead to periodic, quasi periodic or even chaotic behaviour [Pfeiffer and Glocker, 1996].

In order to explain the mechanism for limit cycling for closed-loop systems with backlash, we will consider a system which may be presented with the block diagram shown in figure 3.5. The system consists of a controller, amplifier and motor. The motor is coupled with another inertia with a backlash as shown in figure 3.9. For simplicity, we will assume that there is no dry friction in the motor. If the task is to control the position of the driven inertia, the following problem may appear. Due to existence of an error $e$, the controller will transmit the control action to the motor via an amplifier. That will cause the rotation of the driving inertia and the driven inertia. At the moment when error $e$ becomes equal to zero, due to the backlash and non-zero driven inertia, the driven inertia will continue to rotate. Then, the error $e$ will become non-zero. The controller will act again and the driving mass will contact with and transmit force to the driven inertia up to the moment when $e$ becomes equal to zero again. Then again, due to the presence of backlash and non-zero inertia, the driven inertia will keep rotating despite the fact that $e = 0$, and thus the whole event will repeat itself and limit cycling may appear.

An improved mathematical model of this nonlinearity can be found in [Pfeiffer and Glocker, 1996; Nordin, 2000]. In [Nordin, 2000], the design of the controller for a specific type of system with backlash is analyzed. Herein, some suggestions for designing the controller in such a way that the limit cycling may be avoided, are presented.

![Figure 3.8. Backlash nonlinearity.](image)

**Figure 3.9. Two masses coupled with the system with backlash.**

### 3.4. Relay feedback

Relay feedback is a nonlinearity that also may cause limit cycling. This type of nonlinearity is shown in figure 3.10 and may be represented using the sign-function, i.e. the control in relay feedback systems may be expressed as

$$u = U_0 \text{sgn}(y),$$

(3.12)

where $u$ is the input and $y$ is the output. Analogous to (3.2), the problem using this representation of relay feedback is that the control is not defined for $y = 0$. This problem is
solved by defining the set-valued sign function, with which the control of such systems is defined as (see [Anosov, 1959; Johansson, 1999; Leine, 2000; Tsupkin, 1984])

\[
u = u_0 \text{sgn}(y) \in \begin{cases} \{-u_0\} & \text{if } y < 0 \\ [-u_0, u_0] & \text{if } y = 0 \\ \{u_0\} & \text{if } y > 0. \end{cases} \tag{3.13}
\]

To explain the reasons for limit cycling behaviour of such systems, we consider a position-controlled closed-loop system shown in figure 3.5a, where the controller is a relay type controller (see figure 3.10). For simplicity, the motor does not exhibit any nonlinear effects (e.g. dry friction, backlash). When the error \(e\) is not zero, the relay type controller will give to the system an appropriate control action, with which the rotor shaft of the motor will move in such a way that at a certain moment \(e\) will be equal to zero. At the next time instant the control will change sign (because the error \(e\) will be equal to zero for a very short time). But, due to the inertia of the rotor of motor, the rotor will continue to rotate in the same direction up to the moment when \(\theta = 0\), and then it will start moving in the other direction. But at that moment \(e\) is already non-zero. Thus, the rotor will rotate in that direction until \(e\) becomes zero. Then, again, the control action will be changed (according to figure 3.10, the control action will change sign), but due to the rotor inertia the motor will keep rotating in the same direction and the whole event will repeat itself and a limit cycle may appear.

For this type of relay feedback, the analysis of periodic behavior is presented in [Johansson, 1999; Kowalczyk, 2001].

![Figure 3.10. Relay type nonlinearity.](image)

### 3.5. Hysteresis

A hysteresis nonlinearity can be expressed as in figure 3.11. Its main feature is that in a system, which has this type of nonlinearity, at least one of its states (\(y\) in a figure 3.11a) does not change in a same way when another its state (\(x\) in a figure 3.11a) increases and decreases. This kind of nonlinearity can be noticed in many dynamical systems.

If real life friction has this characteristic, it cannot be modeled with static dry friction models (see section 3.1.1). Even the LuGre friction model cannot model the closed hysteresis loop. However, there are some other friction models with which hysteresis behaviour of a friction force can be modeled (see [Olsson, 1996a]).

In practice switching behaviour, such as in the relay type nonlinearity, is usually presented with a hysteresis type of nonlinearity (figure 3.11b) but also with a nonlinearity which represents backlash nonlinearity (figure 3.8). This is due to the fact that in real-life systems switching between two states cannot occur instantaneously, but some delay does exist.
The mechanism of limit cycling when this type of nonlinearity is present, is easy to explain using an example of a system, which is used in the temperature control of a water. The switching device has a characteristic as shown in figure 3.11b. Suppose that the temperature of the water needs to be $T_{\text{set}}$. When the temperature of the water is below $T_{\text{max}}$ ($T < T_{\text{max}}$), the heating system is turned on ($c = 1$) and the temperature rises. When the temperature reaches the value $T_{\text{max}}$ (point $B$ in figure 3.11b) the heating system is turned off ($c = 0$ - point $C$ in figure 3.11b). Then, after certain time the temperature starts to decrease (assuming that the temperature of the space which surrounds the water is lower then the temperature of the water). When the temperature reaches $T_{\text{min}}$ (point $D$ in figure 3.11b), the heating system will be turned on (point $D$ in figure 3.11b) and the temperature will start to increase again and the whole event will repeat. Consequently, this may lead to a limit cycling. If hysteresis does not exist ($T_{\text{set}} = T_{\text{min}} = T_{\text{max}}$), switching will happen more often and that is neither needed nor good for the switching device.

More extensive analysis of closed-loop systems with a hysteresis nonlinearity can be found in [Gonçalves, 2001].

![Figure 3.11. Two examples of hysteresis type of nonlinearity.](a) (b)

### 3.6. Negative damping

An example of a system with negative damping is the system described by the van der Pol equation. In electronics, those systems are called negative-resistance systems. An example of such a system is explained in Section 2.4 (figure 2.7), where it is also explained what causes the limit cycling behaviour in those systems.

The friction Stribeck effect, as discussed in Section 3.1, is also an example of a system with negative damping. More examples are discussed in [Khalil, 1996] and [Sastry, 1999].

### 3.7. Different types of nonlinearities

It is very difficult to classify all possible nonlinearities that may induce periodic behaviour. In this section, we will mention some other examples of nonlinear systems which exhibit periodic behaviour, which cannot be classified using any of previously discussed groups.

A well-known example of a system which exhibits a periodic solution (which do not need to be a limit cycle) is the Predator-Prey system. In [Sastry, 1999], the following model is proposed:

$$
\dot{x} = ax - bxy,
\dot{y} = cxy - dy,
$$

(3.14)
where $x$ is the population of the prey, $y$ is population of the predators and $a$, $b$, $c$, $d$ are positive constants. Some other models of Predator-Prey systems can also be found in [Sastry, 1999].

Whirl motion is a nonlinear phenomena which appears in rotor-dynamic system. More about that phenomenon may be found in [De Kraker, 2000]. The analysis of periodic behaviour in such systems is performed in [De Vrande, 2001]. A similar phenomena (whirl effect) which appears in oil-drilling process are analyzed in [Jansen, 1993; Leine, 2000; Pavone, 1994; Van den Steen, 1997; Van der Heijden, 1994].
Chapter 4.

Tools for the Prediction and Estimation of Periodic Solutions

The prediction and estimation of periodic solutions is important for gaining understanding of the circumstances under which these periodic orbits can occur. The dynamics of physical systems is usually described by a set of differential equations. Often, the term dynamic system is used for a system which is expressed by a set of differential equations. As already stated in the Chapter 2, nonlinear systems may show a different types of steady-state behaviour such as equilibrium points, periodic solutions (having a fixed amplitude and period time in steady state), quasi-periodic solutions (where the motion is dominated by two or more incommensurate frequencies).

Various techniques exist for the determination (approximation) of periodic solutions. Moreover, it is also important to determine the stability of such solutions. Therefore, in this chapter, we will present various algorithms for predicting periodic solutions and some algorithms for the determination of the stability of such solutions are discussed.

4.1. Floquet Multipliers

Suppose that the non-autonomous dynamic system is described by (2.3) and that it has a periodic solution \( \phi_p(t, t_0, x_p) \), \( x_p \) is a point on the periodic solution at time \( t_0 \), such that

\[
\dot{\phi}_p(t, t_0, x_p) = f(t, \phi_p(t, t_0, x_p)).
\]

(4.1)

For the periodic solution \( \phi_p(t, t_0, x_p) \) the periodicity condition holds i.e.

\[
\phi_p(t + T, t_0, x_p) = \phi_p(t, t_0, x_p).
\]

(4.2)
In order to derive stability conditions of the periodic solution, we compute the Poincaré map (see section 2.7). In general, this is not a straightforward task since it involves integrating the differential equation and computing the first return time explicitly. However, the eigenvalues of the linearization of the Poincaré map may be computed by linearization the flow $\varphi(t, t_0, x)$ around the limit cycle $\varphi_p(t, t_0, x_p)$. This is referred to as the Floquet technique. The linearization of (2.3) around $\varphi_p(t, t_0, x_p)$ can be written as

$$\varphi_p(t, t_0, x_p) + \Delta \varphi(t, t_0, x_p) = \varphi_p(t, t_0, x_p) + \Delta \varphi(t, t_0, x_p)$$

(4.3)

where $h.o.t$ denotes higher-order terms. Due to the fact that we are only interested in the perturbation $\Delta \varphi(t, t_0, x_p)$ at a specific time $t$ ($t = \text{const}$), then in (4.3) $\Delta t = 0$ and (4.3) becomes

$$\varphi_p(t, t_0, x_p) + \Delta \varphi(t, t_0, x_p) = \varphi_p(t, t_0, x_p) + \frac{\partial f}{\partial x}(t, \varphi_p(t, t_0, x_p)) \Delta \varphi(t, t_0, x_p) + h.o.t.$$  

(4.4)

Inserting (4.1) into (4.4), yields the following differential equation for the perturbation $\Delta \varphi(t, t_0, x_p)$

$$\Delta \varphi(t, t_0, x_p) = \frac{\partial f}{\partial x}(t, \varphi_p(t, t_0, x_p)) \Delta \varphi(t, t_0, x_p) + h.o.t.$$  

(4.5)

Since $\varphi_p(t, t_0, x_p)$ is time-dependent and periodic, $\frac{\partial f}{\partial x}(t, \varphi_p(t, t_0, x_p))$ is a time-dependent and periodic Jacobian matrix, which is denoted as $A(t, t_0, x_p)$, and we have that

$$A(t, t_0, x_p) = A(t + T, t_0, x_p).$$  

(4.6)

Consequently, (4.5) can be written in the following form:

$$\Delta \varphi(t, t_0, x_p) = A(t, t_0, x_p) \Delta \varphi(t, t_0, x_p) + h.o.t.$$  

(4.7)

Denote the fundamental matrix solution of this system by $\Phi(t, t_0, x_p)$, with $\Phi(t_0, t_0, x_p) = I$. The fundamental solution matrix is dependent on time $t$ but also on the initial condition $x_p$ (which is on the periodic solution). The fundamental matrix relates $\Delta \varphi(t, t_0, x_p)$ to $\Delta \varphi(t_0, t_0, x_p)$ by

$$\Delta \varphi(t, t_0, x_p) = \Phi(t, t_0, x_p) \Delta \varphi(t_0, t_0, x_p) + h.o.t.$$  

(4.8)
The higher-order terms vanish when (infinitely) small perturbations are considered, thus (4.8) becomes

\[ \Delta \varphi(t, t_0, x_p) = \Phi(t, t_0, x_p) \Delta \varphi(t_0, t_0, x_p). \]  

(4.9)

Now, the transition property of the fundamental solution matrix will be presented. Namely, using (4.9), we can write

\[ \Delta \varphi(t_1, t_0, x_p) = \Phi(t_1, t_0, x_p) \Delta \varphi(t_0, t_0, x_p) \]  

(4.10a)

\[ \Delta \varphi(t_2, t_0, x_p) = \Phi(t_2, t_1, x_p) \Delta \varphi(t_1, t_0, x_p), \]  

(4.10b)

where \( x_p \) is a point on the periodic solution at the time instant \( t_0 \) and \( x_p1 \) is a point on the periodic solution at the time instant \( t_1 \). Consequently, the following relation holds

\[ \Delta \varphi(t_2, t_0, x_p) = \Phi(t_2, t_1, x_p1) \Phi(t_1, t_0, x_p) \Delta \varphi(t_0, t_0, x_p) = \Phi(t_2, t_0, x_p) \Delta \varphi(t_0, t_0, x_p) \]

and thus

\[ \Phi(t_2, t_0, x_p) = \Phi(t_2, t_1, x_p1) \Phi(t_1, t_0, x_p), \]  

(4.11)

which represents the transition property of the fundamental solution matrix.

According to (4.4) - (4.8), we can conclude that in order to study the stability of the periodic motion of the nonlinear system (2.3), we can study the stability of the origin of the linear time-variant system (4.7) without the higher-order terms. But first the following lemmas will be introduced:

**Lemma 4.1:** Let \( A \) and \( B \in \mathbb{R}^{n \times n} \) be square matrices. Then, the matrices \( C_1 = AB \) and \( C_2 = BA \) have identical eigenvalues.

**Proof:** Let \( \lambda_{ii}, u_{ii}, i = 1, \ldots, n \), be the eigenvalues and eigenvectors of \( C_1 \), such that

\[ \lambda_{ii} u_{ii} = ABu_{ii}. \]

Multiplication with \( B \) gives

\[ \lambda_{ii} Bu_{ii} = BABu_{ii}. \]

Hence, \( \lambda_{ii} \) is also the eigenvalue of \( C_2 = BA \) with eigenvector \( u_{2i} = Bu_{ii} \).

**Lemma 4.2:** Let \( \varphi_p(t, t_0, x_p) \) be a periodic solution with period \( T \) and with the periodic solution matrices \( \Phi_1 = \Phi(t_1 + T, t_1, x_p1) \) and \( \Phi_2 = \Phi(t_2 + T, t_2, x_p2) \). Then \( \Phi_1 \) and \( \Phi_2 \) have the same eigenvalues.

**Proof:** Splitting the fundamental solution matrices using (4.11) the following is obtained:

\[ \Phi_1 = \Phi(t_1 + T, t_2, x_p2) \Phi(t_2, t_1, x_p1) \]
and
\[ \Phi_2 = \Phi(t_2 + T, t_1 + T, x_{p3}) \Phi(t_1 + T, x_{p2}), \]
where
\[ x_{p1} = \Phi_p(t_1, t_0, x_p), \]
\[ x_{p2} = \Phi_p(t_2, t_0, x_p), \]
\[ x_{p3} = \Phi_p(t_1 + T, t_0, x_p). \]
Due to periodicity property of \( \Phi_p(t, t_0, x_p) \) we have that \( \Phi(t_2 + T, t_1 + T, x_{p3}) = \Phi(t_2, t_1, x_{p1}) \)
and therefore
\[ \Phi_1 = \Phi(t_1, t_2, x_{p2}) \Phi(t_2, t_1, x_{p1}) \]
and
\[ \Phi_2 = \Phi(t_2, t_1, x_{p1}) \Phi(t_1, t_2, x_{p2}). \]

Then, from lemma 4.1 it follows that \( \Phi_1 \) and \( \Phi_2 \) have an identical set of eigenvalues. \( \square \)

**Lemma 4.3:** The fundamental solution matrix of (4.7) (when h.o.t. vanishes), with (4.6) may be written as
\[ \Phi(t, t_0, x_p) = K(t, t_0, x_p) e^{B(t-t_0)}, \] (4.12)
where \( K(t, t_0, x_p) = K(t + T, t_0, x_p) \in \mathbb{R}^{n \times n}, \ K(t_0, t_0, x_p) = I \) and \( B = \frac{1}{T} \log(\Phi(t_0 + T, t_0, x_p)). \)

**Proof:** Using the definition of \( B \), define
\[ K(t, t_0, x_p) = \Phi(t, t_0, x_p) e^{-B(t-t_0)}. \]

Now (4.11) follows from the definitions. To prove the periodicity of \( K(t, t_0, x_p) \) note that
\[ K(t + T, t_0, x_p) = \Phi(t + T, t_0, x_p) e^{-B(t + T - t_0)} \]
\[ = \Phi(t + T, t_0 + T, x_p) \Phi(t_0 + T, t_0, x_p) e^{-B(t + T - t_0)} \]
\[ = \Phi(t + T, t_0 + T, x_p) e^{BT} e^{-B(t + T - t_0)} \]
\[ = \Phi(t + T, t_0 + T, x_p) e^{-B(t - t_0)} \]
\[ = \Phi(t, t_0, x_p) e^{-B(t - t_0)} \]
\[ = K(t, t_0, x_p) \] \( \square \)
From this lemma, it follows that the long-term behaviour of the fundamental solution matrix \( \Phi(t, t_0, x_p) \) is determined by the constant matrix \( B \) and hence by the monodromy matrix \( \Phi_T \), which is defined by

\[
\Phi_T := \Phi(t_0 + T, t_0, x_p) = e^{BT}.
\] (4.13)

According to (4.10) we have that

\[
\Delta \Phi(t_0 + T, t_0, x_p) = \Phi(t_0 + T, t_0, x_p) \Delta \Phi(t_0, t_0, x_p) = \Phi_T \Delta \Phi(t_0, t_0, x_p).
\] (4.14)

Thus, the monodromy matrix \( \Phi_T \) maps an initial perturbation condition \( \Delta \Phi(t_0, t_0, x_p) \) to a perturbation \( \Delta \Phi(t_0 + T, t_0, x_p) \) one period later. Namely, according to (4.13) the following can be obtained

\[
\Delta \Phi(t_0 + kT, t_0, x_p) = \Phi_T^k \Delta \Phi(t_0, t_0, x_p), \quad k = 0, 1, \ldots.
\] (4.15)

This equation indicates that the evolution of \( \Delta \Phi(t, t_0, x_p) \) (for \( t = kT \)) depends on the eigenvalues of monodromy matrix \( \Phi_T \). The eigenvalues of monodromy matrix are called the Floquet multipliers.

From (4.12) and (4.13) it is seen that

\[
\Phi(t + kT, t_0, x_p) = \Phi(t + (k-1)T, t_0, x_p) \Phi_T = \cdots = \Phi(t, t_0, x_p) (\Phi_T)^k.
\] (4.16)

The consequence of this equation is that the fundamental solution \( \Phi(t, t_0, x_p) \) is known at any time if it is known for \( t_0 \leq t \leq t_0 + T \).

Now we will say something about the stability of (4.7) and therefore the stability of the periodic solution \( \phi(t, t_0, x_p) \). According to (4.16) the Floquet multipliers determine the exponential growth or decay of perturbation around the periodic solution in the eigendirections of the monodromy matrix \( \Phi_T \) and hence the stability of the periodic solution \( \phi(t, t_0, x_p) \). If a Floquet multiplier has a magnitude larger (smaller) than one, then a perturbation of the initial condition in the corresponding eigendirection will grow (decay) after one period (expression (4.15)).

Further, we will discuss a property which holds only for autonomous systems. Let us introduce an infinitely small perturbation on the initial condition in the direction of the vector field, \( \Delta \phi(t_0, t_0, x_p) = \alpha f(x_p) \), \( \alpha << 1 \). Then, the resulting perturbation after one period will be identical to the initial perturbation \( \Delta \phi(t_0 + T, t_0, x_p) = \Delta \phi(t_0, t_0, x_p) \), due to the periodicity of the periodic solution\(^5\). Consequently, we have that

\[
\Delta \phi(t_0 + T, t_0, x_p) = \Phi(t_0 + T, t_0, x_p) \Delta \phi(t_0, t_0, x_p) \Rightarrow f(x_p) = \Phi_T f(x_p).
\]

\(^5\) This is not true for nonautonomous systems
This means that for system with a periodic solution, one eigenvector equals \( f(x_p) \) with the corresponding eigenvalue (Floquet multiplier) \( \lambda_1 = 1 \). This does not hold for the non-autonomous systems.

As mentioned in section 2.7, the nonlinear non-autonomous time-periodic system (2.3) can be converted into a nonlinear autonomous system via (2.4) or (2.7). In doing so, we obtained an autonomous system of a higher order than the initial one, but which is also time-periodic. According to the previous conclusion, such system will also have at least one eigenvalue equal to 1. That Floquet multiplier correspond to (2.4b) (or (2.7b)) which is a part of the system (2.4) (or (2.7)).

In figure 4.1, two different sets of Floquet multipliers of two different periodic solutions are depicted in the complex plane. The Floquet multipliers in the figure 4.1b indicate an unstable periodic solution because one Floquet multiplier lies outside the unit circle.

![Floquet multipliers in the complex plane.](image)

In the sequel, we present methods which may be used for the prediction of limit cycles. First, we introduce some approximating methods (the method of harmonic balance, the describing function method and the perturbation methods) that appeared at the middle of the previous century. These methods were useful in those times due to the fact that at that time the possibilities for extensive numerical computations were virtually absent. Afterwards, we will introduce several numerical methods which are used for finding periodic solutions.

### 4.2. Harmonic Balancing Method

In order to introduce the harmonic balancing method, the following nonlinear system will be considered

\[
\ddot{x} = f(x),
\]

where \( x \in \mathbb{R} \) and \( f \in \mathbb{R} \) is nonlinear vector field. Suppose that (4.17) has a periodic solution around the origin and that the vector field \( f \) can be expanded in a power series and approximated by
\[ f(x) = \sum_{i=1}^{N} \alpha_i x^i, \alpha_i = \frac{1}{i!} f^{(i)}(0), \]  

with \( f^{(i)} \) denotes the \( n \)th derivative with respect to the argument. Due to the fact that we want to find a periodic solution of the system \((4.17)\), the solution \( x \) will be assumed to be of the following form:

\[ x(t) = \sum_{m=0}^{M} A_m \cos(m\omega t + m\phi_0). \]  

(4.19)

Substituting \((4.19)\) into \((4.18)\) and \((4.17)\) and equating the coefficients of each of the lowest \( M + 1 \) harmonics to zero, we obtain a system of \( M + 1 \) algebraic equations relating \( \omega \) and the \( A_m \). Usually these equations are solved for \( A_0, A_2, A_3, \ldots, A_M \) and \( \omega \) in terms of \( A_1 \). The accuracy of the resulting periodic solution depends on the value of \( A_1 \) and the number of harmonics in the solution \((4.19)\).

In order to illustrate the method, suppose that the periodic solution of \((4.17)\) is of the form \((4.19)\) for \( M = 1 \) and that \( f(x) \) can be approximated by \((4.18)\) for \( N = 3 \). Inserting

\[ x(t) = A_0 + A_1 \cos(\omega t + \phi_0) \]

into \((4.18)\) \((N = 3)\) the following is obtained:

\[
\begin{align*}
\alpha_1 A_0 + \alpha_2 A_0^2 + \frac{1}{2} \alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2} \alpha_3 A_0 A_1^2 \\
+ \left[ -\left(\omega^2 - \alpha_1\right) A_1 + 2\alpha_2 A_0 A_1 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4} \alpha_3 A_1^3 \right] \cos(\omega t + \phi_0) \\
+ \frac{1}{2} \alpha_2 A_1^2 + \frac{3}{2} \alpha_3 A_0 A_1^2 \right] \cos(2\omega t + 2\phi_0) + \frac{1}{4} \alpha_3 A_1^3 \cos(3\omega t + 3\phi_0) &= 0
\end{align*}
\]

With the previous procedure, due to the fact that \((in \( (4.19) \)) M = 1\), we need to solve \( A_0 \) and \( \omega \) in terms of \( A_1 \). Equating the constant term and the coefficient of \( \cos(\omega t + \phi_0) \) to zero, we have

\[
\alpha_1 A_0 + \alpha_2 A_0^2 + \frac{1}{2} \alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2} \alpha_3 A_0 A_1^2 = 0
\]

\[-\left(\omega^2 - \alpha_1\right) A_1 + 2\alpha_2 A_0 A_1 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4} \alpha_3 A_1^3 = 0
\]

For finding the periodic solution, we need to solve those equations. The solutions will be \( A_0 = A_0(A_1) \) and \( \omega = \omega(A_1) \). For small \( A_1 \) we have that

\[
A_0 = -\frac{\alpha_2}{2\alpha_1} A_1^2 + \mathcal{O}(A_1^4)
\]

\[
\omega = \sqrt{\alpha_1} \left( 1 + \frac{3\alpha_3 \alpha_1 - 4\alpha_2^2}{8\alpha_1^2} A_1^2 \right)
\]

33
Now in order to find the solution, we need to know the value for $A_1$. For obtaining that value we need to have \textit{a priori} knowledge about the periodic solution (in this case about $A_1$). Even if we know $A_1$, the question is if the obtained solution is a good approximation of the exact periodic solution. For larger $N$ and $M$ the obtained solution will be more accurate. But the question is how large $N$ and $M$ should be chosen in order to obtain a satisfactory approximation. The larger $N$ and $M$ in (4.18) and (4.19) are, the more complicated the system of nonlinear algebraic equations becomes. Therefore, in [Nayfeh and Mook, 1979] this method is not preferred for use.

4.3. Describing Function Method

In this section, we will describe the idea of the describing function method (the complete method is described in [Khalil, 1996]). The describing function method is basically cosideres the scalar nonlinear systems, which can be represented by a block scheme as depicted in figure 4.2. Although it is an approximation technique, it can be used for analysing self-sustained oscillations in dynamical systems with nonlinearities [Armstrong-Hérouvry, 1994; Basso, 1997; Ortega, 2000; Ollson 1996a; Ollson, 1996b, Wallenborg, 1988].

![Figure 4.2. Class of nonlinear systems for which the describing function method can be applied.](image)

Consider nonlinear systems represented by a feedback connection of a linear time-invariant dynamic system described by the transfer function $W(s)$ and a static nonlinear element

\[ y(t) = F(x(t), x(t)), \]

as shown in figure 4.2, and assume that $r(t)$ is constant. Suppose that this system exhibits a limit cycle with unknown amplitude $A$ and angular frequency $\omega$. In order to determine $A$ and $\omega$, it is supposed that the linear part $W(s)$ behaves as a low-pass filter. Due to the assumption that the system exhibits limit cycling, the output $y(t)$ of the nonlinear element may be represented as a signal that consists of harmonics with frequencies $\omega, 2\omega, 3\omega, \ldots$. Because the linear system is assumed to behave as a low-pass filter, $x(t)$ can be approximated by

\[ x(t) = x_0 + x^* = A \sin(\omega t), x_0 = \text{const}. \]

Then, the output of the nonlinear element will be

\[ y(t) = F(A \sin(\omega t), A\omega \cos(\omega t)) = a_0 + \sum_{i=1}^{\infty} (a_i \sin(i\omega t) + b_i \cos(i\omega t)), \]

34
where $a_0$, $a_i$, $b_i$, $i=1, 2, \ldots$ are the Fourier coefficients. Due to the low-pass filter characteristics of the linear part $W(s)$ of the system, the higher-order harmonics will be much more suppressed than the basic harmonics. Consequently, the output (4.22) of the nonlinear element (4.20) may be approximated as

$$y(t) = F(A \sin(\omega t), A \omega \cos(\omega t)) = a_0 + a_1 \sin(\omega t) + b_1 \cos(\omega t),$$

(4.23)

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} F(A \sin(\phi), A \omega \cos(\phi)) \, d\phi,$$

(4.24)

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\phi), A \omega \cos(\phi)) \sin(\phi) \, d\phi,$$

(4.25)

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\phi), A \omega \cos(\phi)) \cos(\phi) \, d\phi,$$

(4.26)

are Fourier coefficients.

From figure 4.2, we see that the following holds:

$$X(s) + W(s)(Y(s) - R(s)) = 0,$$

(4.27)

where $X(s)$, $Y(s)$ and $R(s)$ represents the Laplace transform of the signals $x(t)$, $y(t)$ and $r(t)$, respectively. From (4.21), we have that

$$\mathcal{L}(x(t)) = X(s) = \frac{x_0}{s} + \chi(s) \text{ and}$$

$$\mathcal{L}(\sin(\omega t)) = \frac{\chi'(s)}{A},$$

(4.28)

(4.29)

and from the fact that $r = \text{const}$ we have that

$$R(s) = \frac{r}{s}.$$

(4.30)

From (4.23), the following expression may be obtained for $Y(s)$:

---

6 This assumption is, in general, not true. Even if the linear part of the system has the low-pass filter characteristic, the cut-off frequency is not always such that it can suppress all higher-order harmonics. Thus assumption (4.21) represents one more limitation of the describing function method.
\[
\mathcal{L}(y(t)) = Y(s) = \frac{a_0}{s} + a_1 \mathcal{L}(\sin(\omega t)) + b_1 s \mathcal{L}(\sin(\omega t)) \leftrightarrow \\
\Rightarrow Y(s) = \frac{a_0}{s} + \frac{a_1}{A} X^*(s) + \frac{b_1}{A} s X^*(s)
\]

Inserting (4.28) and (4.31) into (4.27) yields

\[
\frac{x_0}{s} + X^*(s) + W(s) \left( \frac{a_0}{s} + \frac{a_1}{A} X^*(s) + \frac{b_1}{A} s X^*(s) - \frac{r}{s} \right) = 0.
\]

In order to find the periodic solution, take \( s = j\omega \). Then, (4.32) will be zero if

\[
x_0 + W(0)x_0 - W(0)r = 0,
\]

\[
1 + \left( \frac{a_1}{A} + \frac{b_1 j\omega}{\omega A} \right) W(j\omega) = 0.
\]

From (4.24), (4.25) and (4.26) it can be concluded that \( a_0 = a_0(x_0, \omega, A) \), \( a_1 = a_1(A, \omega, x_0) \) and \( b_1 = b_1(A, \omega, x_0) \). Equation (4.33) represents one nonlinear algebraic equation and (4.34) two nonlinear algebraic equations with three unknowns \((A, \omega \) and \( x_0 \)), which need to be solved in order to find the approximating periodic solution (4.21).

If \( r(t) \neq \text{const} \), a procedure for finding the limit cycle exists for the case that \( r(t) \) is a slowly varying signal. However, due to the fact that this is an approximate technique and that nowadays many powerful numerical tools exist for approximating periodic solutions, this procedure is hardly ever used. Therefore, it will not be presented here.

Equation (4.33) is known as the (first-order) harmonic balance equation, or simple harmonic balance equation. The function

\[
H(A, \omega) = \frac{a_1}{A} + \frac{b_1 j\omega}{\omega A}
\]

is called the describing function of the nonlinearity (4.20).

It should be noticed that when the system in steady-state has zero mean value, then

\[
r = a_0
\]

and from (4.33) we have that

\[
x_0 = 0.
\]

This means that the value of \( r \) will not influence the result obtained by (4.34). Therefore, because of simplicity, for systems of the form shown in figure 4.2, which have a mean value of \( e(t) \) equal to zero in steady state, the describing function method is performed for \( r(t) = 0 \) [Armstrong-Hélovry, 1994; Basso, 1997; Ortega, 2000; Ollson 1996a; Ollson, 1996b, Wallenborg, 1988]. Then in order to find the periodic solution only (4.34) need to be solved.
The describing function method (for $a_0 = x_0 = 0$) states that if (4.34) has a solution $(A, \omega) = (A_s, \omega_s)$ then there "probably" exists a periodic solution of the system with frequency and amplitude7 (those parameters are the parameters of the signal (4.21)) close to $\omega_s$ and $A_s$. Conversely, if (4.34) has no solutions, then the system "probably" does not have a periodic solution. More analysis is needed to replace the word "probably" with "certainly" and to quantify the phrase "close to $A_s$ and $\omega_s$" when there exists a periodic solution. More about this subject and about the describing function method can be found in [Khalil, 1996].

4.4. Perturbation Methods

Perturbation methods are also approximation methods. According to these techniques, the solution can be represented by the first few terms of an asymptotic expansion. The expansion may be carried out in terms of a parameter. Such expansions are called parameter perturbations. Alternatively, the expansions may be carried out in terms of a coordinate. These are called coordinate perturbations [Nayfeh, 1973].

Suppose that the nonlinear differential equation $f(x, \dot{x}, \ddot{x}, \ldots, x^{(n)}, \epsilon) = 0$ needs to be solved, where $x$ is a scalar or vector independent variable and $\epsilon$ is a parameter. In general, this problem cannot be solved exactly. However, if there exists an $\epsilon = \epsilon_0$ ($\epsilon$ can be scaled such that $\epsilon_0 = 0$) for which the problem described above can be solved exactly, then the problem may be solvable, using perturbation techniques, by expanding the function $x(t, \epsilon)$ in a power series of powers of $\epsilon$, as:

$$x(t, \epsilon) = \sum_{i=0}^{\infty} \epsilon^i x_i(t).$$

(4.38)

Herein, $x_i(t)$ is independent of $\epsilon$ and $x_0(t)$ is the solution of the problem for $\epsilon = 0$. One can then substitute this expansion into $f(x, \dot{x}, \ddot{x}, \ldots, x^{(n)}, \epsilon) = 0$, expand it for small $\epsilon$ and collect coefficients of each power of $\epsilon$. For simplicity and improved understanding, this technique is demonstrated by means of the van der Pol oscillator

$$\frac{d^2 x}{dt^2} + x = \epsilon \left(1 - x^2 \right) \frac{dx}{dt},$$

(4.39)

for small $\epsilon$. If $\epsilon = 0$, (4.39) reduces to

$$\frac{d^2 x}{dt^2} + x = 0.$$

(4.40)

The solution of (4.40) is

$$x = A \cos(t + \varphi),$$

(4.41)

7 Here, the term "amplitude of the periodic solution" represents "the amplitude of the first harmonic of the signal $x(t)$" (see figure 4.2).
where $A$ and $\varphi$ are unknown constants. To determine an improved approximation to the solution of (4.39), a perturbation expansion in the form of (4.38) will be pursued. Substituting (4.38) into (4.39), the following equation is obtained

$$\frac{d^2 x_0}{dt^2} + x_0 + \varepsilon \left( \frac{d^2 x_1}{dt^2} + x_1 \right) + \varepsilon^2 \left( \frac{d^2 x_2}{dt^2} + x_2 \right) + \ldots =$$

$$= \varepsilon \left( 1 - x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \right) \left( \frac{dx_0}{dt} + \varepsilon \frac{dx_1}{dt} + \varepsilon^2 \frac{dx_2}{dt} + \ldots \right).$$

Expanding for small $\varepsilon$, we obtain

$$\frac{d^2 x_0}{dt^2} + x_0 + \varepsilon \left( \frac{d^2 x_1}{dt^2} + x_1 \right) + \varepsilon^2 \left( \frac{d^2 x_2}{dt^2} + x_2 \right) + \ldots =$$

$$= \varepsilon \left( 1 - x_0 \right) \frac{dx_0}{dt} + \varepsilon^2 \left( \left( 1 - x_0 \right) \frac{dx_1}{dt} - 2x_0 x_1 \frac{dx_0}{dt} \right) + \ldots.$$  \hspace{1cm} (4.42)

Since $x_i(t)$ is independent of $\varepsilon$, (4.42) will hold if

$$\frac{d^2 x_0}{dt^2} + x_0 = 0,$$  \hspace{1cm} (4.43)

$$\frac{d^2 x_1}{dt^2} + x_1 = \left( 1 - x_0 \right) \frac{dx_0}{dt},$$  \hspace{1cm} (4.44)

$$\frac{d^2 x_2}{dt^2} + x_2 = \left( 1 - x_0 \right) \frac{dx_1}{dt} - 2x_0 x_1 \frac{dx_0}{dt}.$$  \hspace{1cm} (4.45)

The solution of (4.43) is given by (4.41). By inserting this solution into (4.44), we may find the particular solution of (4.44), and subsequently the particular solution of (4.45) (see [Nayfeh, 1973]). Inserting those solutions into (4.38), we obtain an approximation of the solution of the van der Pol equation (4.39), which depends on $A$ and $\varphi$. In order to find the values for $A$ and $\varphi$ we need to have a good guess of one point on the periodic solution.

Perturbations techniques are often used to find periodic solutions. The method, however, is only accurate for weakly nonlinear systems. In our example, due to the fact that $\varepsilon$ is small and $\varepsilon$ multiplies the nonlinear part of (4.39). The system (4.39) represents a weakly nonlinear system.

### 4.5. Shooting method

The shooting method is a widely used numerical method for finding periodic solutions of nonlinear systems [Ascher, 1995; Leine and Van de Wouw, 2001; Nayfeh and Balachandran, 1995; Parker and Chua, 1989; Van Campen, 2000]. In order to find a numerical solution, we consider the autonomous nonlinear system of the form

$$\dot{x} = f(x)$$  \hspace{1cm} (4.46)
and we need to solve the following equation

$$H(x_0, T) = x(t_0 + T, x_0) - x_0 = 0, x(t_0 + T, x_0) = \varphi_p(t_0 + T, x_0),$$  \hspace{1cm} (4.47)

where $\varphi_p(t, x_0)$ is the periodic solution of (4.46) for the initial value $x_0$, which is on the periodic solution. Assuming that the periodic solution exist, the single shooting method can be solved using the Newton–Raphson procedure. Therefore, in order to explain the method, first the Newton–Raphson algorithm will be explained.

**Intermezzo: Newton-Raphson procedure**

Consider a nonlinear function $f(x)$ with a zero at $x = x^*$. Starting from an initial guess $x^{(0)}$, the zero $x^*$ can be found using the following iterative expression

$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})} \text{ (see figure 4.3).}$$  \hspace{1cm} (4.48)

In order to obtain convergence for an arbitrary function $f(x)$, the initial guess should be “close enough”, i.e., in a neighbourhood of $x = x^*$ in which the function $f(x)$ is either locally convex or locally concave.

The Newton-Raphson algorithm can also be applied to find a zero of a multi-dimensional function $f(x)$. Then, each iteration step is expressed by the following set of linear equations

$$\frac{\partial f}{\partial x^{(i)}} \Delta x = -f(x^{(i)}),$$  \hspace{1cm} (4.49)

with which the update $x^{(i+1)} = x^{(i)} + \Delta x$ is obtained.

Using the Newton-Raphson algorithm (4.49), equation (4.47) can easily be solved. Thus,

$$\frac{\partial H}{\partial x_0} (x_0, T) \Delta x_0 + \frac{\partial H}{\partial T} (x_0, T) \Delta T = -H(x_0, T),$$  \hspace{1cm} (4.50)

which is equivalent to
For autonomous systems, (4.51) gives (see [Ascher, 1995; Leine and Van de Wouw, 2001; Nayfeh and Balachandran, 1995; Parker and Chua, 1989; Van Campen, 20001])

\[
(\Phi_T(x_0) - I)\Delta x_0 + f(x(t_0 + T, x_0))\Delta T = x_0 - x(t_0 + T, x_0). \tag{4.52}
\]

Equation (4.52) represents a system of \( n \) equations with \( n + 1 \) unknowns (\( n \) components of \( x_0 \) and the period \( T \)), which means that it cannot be uniquely solved. The reason for this is that the phase of a periodic solution belonging to an autonomous system is not fixed. Any point on the periodic solution gives a solution of (4.47). In order to remove this arbitrariness, some suggestions are proposed in [Nayfeh and Balachandran, 1995]. One of the suggestions is the use of an additional equation of the form

\[
(f(x_0))^T \Delta x_0 = 0. \tag{4.53}
\]

This means that the update \( \Delta x_0 \) is restricted to be orthogonal to the vector field \( f(x_0) \). Therefore, condition (4.53) is called an orthogonality condition. Then, the shooting method solves at each iteration \( n + 1 \) linear equations

\[
\begin{bmatrix}
\Phi_T(x_0^{(i)}) - I & f(x(t_0 + T^{(i)}, x_0^{(i)}))\\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x_0^{(i)} \\
\Delta T^{(i)}
\end{bmatrix}
= \begin{bmatrix}
x_0^{(i)} - x(t_0 + T^{(i)}, x_0^{(i)}) \\
0
\end{bmatrix}, \tag{4.54}
\]

and then updates

\[
\begin{bmatrix}
x_0^{(i+1)} \\
T^{(i+1)}
\end{bmatrix}
= \begin{bmatrix}
x_0^{(i)} \\
T^{(i)}
\end{bmatrix}
+ \begin{bmatrix}
\Delta x_0^{(i)} \\
\Delta T^{(i)}
\end{bmatrix}, \tag{4.55}
\]

with initial guesses \( x_0^{(0)} \) and \( T^{(0)} \).

In (4.54) \( x(t_0 + T^{(i)}, x_0^{(i)}) \) and the fundamental matrix \( \Phi_T(x_0^{(i)}) \) should be found. The value for \( x(t_0 + T^{(i)}, x_0^{(i)}) \) can be found using numerical integration of (4.46) for the initial condition \( x_0^{(i)} \). In order to find the fundamental matrix we start with (4.8). According to that equation, it can be seen that

\[
\frac{\partial \Phi_P}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0). \tag{4.56}
\]

Differentiating (4.46) with respect to the initial value \( x_0 \) the following is obtained

\[
\frac{\partial \Phi_P}{\partial x_0}(t, t_0, x_0) = \frac{\partial \Phi}{\partial x}(\Phi_P(t, t_0, x_0)) \frac{\partial \Phi_P}{\partial x_0}(t, t_0, x_0)
\]

\[
\frac{\partial \Phi_P}{\partial x_0}(t, t_0, x_0). \tag{4.57}
\]

From (4.56) it can be seen that
Therefore, equation (4.57) can be rewritten as

\[
\frac{\partial \Phi_p}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0).
\]

Therefore, equation (4.57) can be rewritten as

\[
\Phi(t, t_0, x_0) = \frac{\partial f}{\partial x} \left( \Phi_p(t, t_0, x_0) \right) \Phi(t, t_0, x_0).
\]  

(4.58)

where \( \Phi(t_0, t_0, x_0) = I \) (see lemma 4.3). According to (4.58), the fundamental solution matrix \( \Phi(t, t_0, x_0) \) can be obtained by numerical integration for the initial value \( \Phi(t_0, t_0, x_0) = I \). Then, according to (4.13) \( \Phi_T \left( x_0^{(i)} \right) \) can be found by solving (4.58), for \( x_0 = x_0^{(i)} \), \( t = t_0 + T \) and \( \Phi(t_0, t_0, x_0) = I \) using numerical integration techniques.

A graphical interpretation of the shooting algorithm is depicted in figure 4.4.

Periodic solutions of the non-autonomous system (2.3) can be found by transforming the non-autonomous time-periodic system into an autonomous system of which the periodic solution can be found using the shooting method for autonomous systems. Moreover, in non-autonomous systems, which exhibit limit cycling, the explicit time-dependent term have period \( T_{ex} \) which is known or can be determined. Then, the period \( T \) of the whole system is a rational multiple of \( T_{ex} \), thus, \( T \) is not an unknown quantity, \( \Delta T = 0 \) and according to (4.54) and (4.55) we only need to solve at each iteration \( n \) linear equations

\[
\left( \Phi_T \left( x_0^{(i)} \right) - I \right) \Delta x_0^{(i)} = x_0^{(i)} - x_0^{(i)}(t_0 + T, x_0^{(i)}).
\]  

(4.59)

and then to update by

\[
x_0^{(i+1)} = x_0^{(i)} + \Delta x_0^{(i)}.
\]  

(4.60)

Figure 4.4. Graphical interpretation of the shooting method.
4.6. Finite Difference Method

The finite difference method represents a periodic solution solver, which approximates the periodic solution by a large number of linear segments [Ascher, 1995; Nayfeh and Balachandran, 1995; Parker and Chua, 1989; Van Campen, 2000] as shown in figure 4.5. The method uses a sequence of $N$ points $X=[x_1 \ x_2 \ \cdots \ \ x_N]$ equally spaced in the time with step length $h=\frac{T}{N}$. The segment starting from $x_k$ connects to the succeeding segment when

$$x_{k+1} = x_k + hf(x_k).$$

Herein, the forward Euler scheme is used. The finite difference method with a forward Euler scheme is based on finding the solution of the following set of equations

$$H(X,T)=
\begin{bmatrix}
-x_1 + x_N + hf(x_N) \\
-x_2 + x_1 + hf(x_1) \\
-x_3 + x_2 + hf(x_2) \\
\vdots \\
-x_N + x_{N-1} + hf(x_{N-1})
\end{bmatrix}
= 0, \quad X =
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_N
\end{bmatrix}.$$  \quad (4.61)

![Graphical interpretation of the finite difference method.](image)

In (4.61), the equation

$$-x_1 + x_N + hf(x_N) = 0, \quad (4.62)$$

means that the final point of the last segment $(x_N + hf(x_N))$ should match with the starting point of the first segment $(x_1)$.

For autonomous systems, the solution of this equation, may found using the Newton-Raphson algorithm by solving in each iteration step the set of equations

$$\frac{\partial H}{\partial X}(X,T)\Delta X + \frac{\partial H}{\partial T}(X,T)\Delta T = -H(X,T). \quad (4.63)$$
together with the equation

\[ g^T \Delta X = 0, \]  

(4.64)

where

\[ \Delta X = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_N \end{bmatrix}. \]  

(4.65)

The additional equation (4.64) is used for the same purpose as in the previous section. In order to satisfy the orthogonality condition \( g \) should be chosen similarly as in (4.53), i.e.

\[ g = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}. \]  

(4.66)

From (4.61) the partial derivatives can be evaluated as

\[
\frac{\partial H}{\partial X}(X,T) = \begin{bmatrix}
-I & 0 & \ldots & 0 & I + \frac{T}{N} \frac{\partial f}{\partial x}(x_N) \\
I + \frac{T}{N} \frac{\partial f}{\partial x}(x_1) & -I & \ldots & 0 & 0 \\
0 & I + \frac{T}{N} \frac{\partial f}{\partial x}(x_2) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & I + \frac{T}{N} \frac{\partial f}{\partial x}(x_{N-1}) & -I
\end{bmatrix},
\]  

(4.67)

\[
\frac{\partial H}{\partial T}(X,T) = \frac{1}{N} \begin{bmatrix}
f(x_N) \\
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{N-1})
\end{bmatrix}.
\]  

(4.68)

The set of \( Nn \) linear equations (4.63) together with orthogonality condition (4.64) gives the following set of \( Nn + 1 \) set of linear equations with \( Nn + 1 \) unknowns (\( \Delta X \) and \( \Delta T \)):

\[
\begin{bmatrix}
\frac{\partial H}{\partial X}(x^{(i)},T^{(i)}) & \frac{\partial H}{\partial T}(x^{(i)},T^{(i)}) \\
g^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta X^{(i)} \\
\Delta T^{(i)}
\end{bmatrix} =
\begin{bmatrix}
-H(x^{(i)},T^{(i)}) \\
0
\end{bmatrix}.
\]  

(4.69)

Then, updates can be computed using
The finite difference method can be improved by taking better difference schemes. This method does not use any integration algorithm. As a consequence, the finite difference method is computationally more efficient than the shooting method. However, the set of equations, which has to be solved, may be very large. Another disadvantage, compared to the shooting method, is that the method can only be applied to smooth systems. Moreover, it is not obvious how to choose the initial guess \( x'(0) \). The advantage of the method is that the convergence is usually relatively good and the basin of the convergence is large. While the shooting method uses one point, which makes the method very vulnerable for bad initial guesses, the finite difference method uses a number of points along the periodic solution, which damps out the influence of one "bad" point.

For a non-autonomous system, as we explained in the previous section, \( T \) is a known quantity. Consequently, \( \Delta T = 0 \) and we need to solve \( Nn \) linear equations

\[
\frac{\partial H}{\partial x}(x^{(i)},T^{(i)})\Delta x^{(i)} = -H(x^{(i)},T^{(i)}),
\]

and to implement the following update

\[
x^{(i+1)} = x^{(i)} + \Delta x^{(i)}.
\]

### 4.7. Multiple Shooting Method

The advantage of the finite difference method compared to the shooting method initiated the idea to combine those two methods which resulted in the multiple shooting method. In the multiple shooting method one uses (as in the finite difference method) a number of points \( x_1, x_2, \ldots, x_N \) as an initial guess for the periodic solution. However, the segments are the solution \( \phi(t,x_i), \quad i = 1,2,\ldots,N \) of the nonlinear equation \( \dot{x}(t) = f(x(t)) \), if we consider autonomous systems, with the initial value \( x_0 = x(t_0) \). Then, in order to find the periodic solution, the following set of equations needs to be solved

\[
\begin{bmatrix}
- x_1 + \phi(h, x_N) \\
- x_2 + \phi(h, x_1) \\
- x_3 + \phi(h, x_2) \\
\vdots \\
- x_N + \phi(h, x_{N-1})
\end{bmatrix} = 0, \quad h = \frac{T}{N}.
\]

The values \( \phi(h, x_i) \) are the coordinates of the final value of the \( i \)-th segment (the starting point of each segment is \( x_i \) - see figure 4.6). \( \phi(h, x_i) \) can be obtained by numerical integration of the nonlinear equation \( \dot{x}(t) = f(x(t)) \) for which we need to find the limit cycle with initial conditions \( x_1, x_2, \ldots, x_N \). Applying the Newton-Raphson algorithm gives the set of equations (4.63). Again an additional equation has to be added (see (4.64) and (4.66)). The partial derivatives \( \mathbf{H}(\mathbf{X}, T) \) (see 4.73) can be evaluated as
where

\[
\frac{\partial H}{\partial x}(X,T) = \begin{bmatrix}
-1 & 0 & \cdots & 0 & \Phi_h(x_N) \\
\Phi_h(x_1) & -1 & \cdots & 0 & 0 \\
0 & \Phi_h(x_2) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \Phi_h(x_{N-1}) & -1
\end{bmatrix},
\]

(4.74)

\[
\frac{\partial H}{\partial T}(X,T) = \frac{1}{N} \begin{bmatrix}
f(\varphi(h,x_N)) \\
f(\varphi(h,x_1)) \\
f(\varphi(h,x_2)) \\
\vdots \\
f(\varphi(h,x_{N-1}))
\end{bmatrix},
\]

(4.75)

Finally, using (4.70) new values need to be updated.

According to the (4.56) and to the fact that \( \varphi(t,x_0) \) is the solution of the nonlinear equation \( \dot{x}(t) = f(x(t)) \) for initial value \( x_0 \), \( \Phi_h(x_i) \) is the fundamental matrix of the system \( \dot{x}(t) = f(x(t)) \), which can be found numerically (see section 4.5).

For non-autonomous systems, \( \Delta T = 0 \) because \( T \) is known.

Here, it should be also noted, that if we use Euler scheme with one integration step for the numerical integration of \( \dot{x}(t) = f(x(t)) \), the multiple shooting method is identical to the finite difference method. On the other hand, for \( N = 1 \), this method becomes the shooting method.

Figure 4.6. Graphical interpretation of the multiple shooting method.
4.8. Sequential Continuation Method

The most important drawback of all methods for computing the periodic solutions, which were discussed in the sections 4.5.-4.7., is that these methods will only converge if the initial guess is close “enough” to the periodic solution. This means that those methods do not find periodic solutions, but refine an already good guess of the periodic solution. Thus, it may be concluded that the use of such methods may be very limited. However, once a periodic solution is found for a certain parameter set, one can use this information in order to find the periodic solution when a parameter of the system is varied. The variation of this parameter yields a branch of solutions in the solution-parameter space. Let $\mu^*$ denote the parameter of interest. If the periodic solution at $\mu = \mu^*$ is known, the periodic solution at $\mu = \mu^* + \Delta \mu$ may be found using the solution at $\mu = \mu^*$ as initial guess. The periodic solver is likely to converge for small $\Delta \mu$. This type of continuation is called sequential continuation [Nayfeh and Balachandran, 1995] and it is depicted in figure 4.7.

![Figure 4.7. Graphical interpretation of sequential continuation method (\(\mu\) parameter of interest, A some characterization of the magnitude of the limit cycle).](image)

The sequential continuation method proceeds step-wise, due to the fact that each previous periodic solution is the initial guess for finding the next periodic solution. With this method it is possible to follow branches of both stable and unstable periodic solutions. However, a problem may occur at bifurcation points. If the branch undergoes a fold bifurcation, at which the branch turns around (figure 4.7), the previous periodic solution is not a good guess for the next periodic solution. In such a case, it may happen that even for a very small $\Delta \mu$ the periodic solver does not converge. Even if the periodic solver converges, a part of the branch is not followed. This means that the method do not follow the branch.

4.9. Arclength Continuation Method

There are some improvements of the continuation method that can overcome the problem of following the branch even when encountering bifurcation points [Nayfeh and Balachandran, 1995]. The arclength continuation method is one of such improved methods and it is based on a prediction step tangent to the branch and subsequent corrector steps to converge to the branch (see figure 4.9). Such a method can indeed round a turning point, which was not the case with the sequential continuation method. This means that the algorithm not only has to
solve for the new update $\Delta x_0$ and $\Delta T$ during the corrector step but also for $\Delta \mu$. In the sequel, the idea of the use predictor and corrector steps will be illuminated.

### 4.9.1. Predictor Step

The idea of the predictor step is not to vary only the parameter of interest $\mu$, but also the initial value $\Delta x_0$ and $\Delta T$ in such a way that the predictor step is locally tangent to the branch (see figure 4.8). The branch is defined by $H(x_0, T, \mu) = 0$, the tangent vector by

$$p = \begin{bmatrix} p_{x_0} \\ p_T \\ p_\mu \end{bmatrix}.$$  

If we apply the Newton-Raphson algorithm to $H(x_0, T, \mu) = 0$, the following can be obtained

$$\frac{\partial H}{\partial x_0}(x_0, T, \mu)p_{x_0} + \frac{\partial H}{\partial T}(x_0, T, \mu)p_T + \frac{\partial H}{\partial \mu}(x_0, T, \mu)p_\mu = -H(x_0, T, \mu) = 0. \tag{4.77}$$

![Figure 4.8. Graphical interpretation of sequential continuation method with predictor.](image)

Expression (4.77) represents a system of $n$ linear equations with $n+2$ unknowns ($n$ unknowns from $p_{x_0}$ and two more unknowns $p_T$ and $p_\mu$). Therefore, two more equations are needed in order to obtain the coordinates of vector $p$. One can be the orthogonality condition $f^T(x_0, \mu)p_{x_0} = 0$, and one coordinate may be fixed, due to the fact that we are only interested in the direction of vector $p$, thus, we can set $p_\mu = p_{\text{fix}}$ where $p_{\text{fix}}$ is some arbitrarily chosen value for $p_\mu$, and solve the following system of linear equations

$$\begin{bmatrix} \frac{\partial H}{\partial x_0}(x_0, T, \mu) & \frac{\partial H}{\partial T}(x_0, T, \mu) \\ f^T(x_0, \mu) & 0 \end{bmatrix} \begin{bmatrix} p_{x_0} \\ p_T \end{bmatrix} = \begin{bmatrix} -\frac{\partial H}{\partial \mu}(x_0, T, \mu)p_{\text{fix}} \\ 0 \end{bmatrix}, \tag{4.78}$$
for $p_{x_0}$ and $p_T$. Then the vector $p$ will be

$$p = \begin{bmatrix} p_{x_0} \\ p_T \\ p_{\text{fix}} \end{bmatrix}.$$  

The predictor $\left(x_p^{(i)}, T_p^{(i)}, \mu_p^{(i)}\right)$ in continuation step $i$ can be found by taking a step in the direction of $p$ (see figure 4.9) as

$$\begin{bmatrix} x_p^{(i)} \\ T_p^{(i)} \\ \mu_p^{(i)} \end{bmatrix} = \begin{bmatrix} x_0^{(i)} \\ T_0^{(i)} \\ \mu_0^{(i)} \end{bmatrix} + \alpha^{(i)} \begin{bmatrix} p_{x_0} \\ p_T \\ p_{\text{fix}} \end{bmatrix}, \quad (4.79)$$

where $\alpha^{(i)}$ is the step length and the $x_0^{(i)}, T_0^{(i)}$ and $\mu_0^{(i)}$ represent the solution of the equation $H(x_0,T,\mu)=0$ in the step $i$. To assure that the branch is followed in the same direction one has do demand acute angle of succeeding predictor steps:

$$(\alpha^{(i)} p^{(i)})^T (\alpha^{(i-1)} p^{(i-1)}) > 0. \quad (4.80)$$

From this condition the sign of the step length $\alpha^{(i)}$ can be determined by

$$\text{sgn}(\alpha^{(i)}) = \text{sgn}(\alpha^{(i-1)} p^{(i)} p^{(i-1)}). \quad (4.80)$$

### 4.9.2. Corrector Step

The corrector step is very similar to the single shooting method scheme, except that the value of $\mu$ is not fixed. The corrector step tries to find a zero of $H(x_0,T,\mu)$ by a Newton-Raphson procedure:

$$\frac{\partial H}{\partial x_0}(x_0,T,\mu) \Delta x_0 + \frac{\partial H}{\partial T}(x_0,T,\mu) \Delta T + \frac{\partial H}{\partial \mu}(x_0,T,\mu) \Delta \mu = -H(x_0,T,\mu), \quad (4.81)$$

which, after evaluating the partial derivatives, gives the following equation (see section 4.5)

$$\left(\Phi_T(x_0) - I\right) \Delta x_0 + f(x(t_0 + T,x_0)) \Delta T + \frac{\partial H}{\partial \mu} \Delta \mu = x_0 - x(t_0 + T,x_0). \quad (4.82)$$

To solve this system of equations uniquely, two additional equations are needed: one being the orthogonality condition (4.53) and the other being a condition on the direction of the corrector step. One possible condition on the direction to the corrector step is to take the corrector step perpendicular to the predictor step (see figure 4.9), which is more or less the shortest route to the branch,

$$p_{x_0}^T \Delta x_0 + p_T \Delta T + p_{\mu} \Delta \mu = 0. \quad (4.83)$$
Then, the Newton-Raphson procedure together with the orthogonality condition and the condition on the direction of the corrector gives the set of $n + 2$ linear equations

$$
\begin{bmatrix}
\Phi_T(x_0) - I & f(x(t_0 + T, x_0)) & \frac{\partial H}{\partial \mu} \\
(f(x_0))^T & 0 & 0 \\
p_{x_0}^T & p_T & p_\mu
\end{bmatrix}
\begin{bmatrix}
\Delta x_0 \\
\Delta T \\
\Delta \mu
\end{bmatrix} =
\begin{bmatrix}
x_0 - x(t_0 + T, x_0) \\
0 \\
0
\end{bmatrix},
$$

(4.84)

which has to be solved in each corrector step $j$. When this set of equations is solved, we can update to corrector point $j + 1$

$$
\begin{bmatrix}
x_0^{(j+1)} \\
T^{(j+1)} \\
\mu^{(j+1)}
\end{bmatrix} =
\begin{bmatrix}
x_0^{(j)} \\
T^{(j)} \\
\mu^{(j)}
\end{bmatrix} +
\begin{bmatrix}
\Delta x_0^{(j)} \\
\Delta T^{(j)} \\
\Delta \mu^{(j)}
\end{bmatrix},
$$

(4.85)

with the predictor as starting point ($j = 0$). The derivative \( \frac{\partial H}{\partial \mu} \) has to be calculated numerically in the following way: for a small change $\Delta \mu$ of parameter $\mu$, we compute $\Delta H$ (using (4.47) or (4.61) or (4.73) depending on the used periodic solver) and approximate $\frac{\partial H}{\partial \mu}$ with $\frac{\Delta H}{\Delta \mu}$.

---

4.10. Event Mapping Method

The event mapping method is based on the methods that were used in seismology to understand the mechanism that causes earthquakes, but was used and adapted in [Galvanetto, 1997] in order to analyze the periodic behaviour of mechanical systems with dry friction. Even the term "event" which was used by Galvanetto has a seismic origin. With this method a
one-dimensional iterated mapping is obtained. In order to explain the method, we consider van der Pol oscillator described by (2.1).

Figure 4.10. A stable limit cycle for the Van der Pol oscillator.

In section 2.4, we have explained that for $\varepsilon > 0$ the system exhibits a stable limit cycle and an unstable equilibrium point and for $\varepsilon < 0$ an unstable limit cycle and a stable equilibrium point. Now, we will derive a one-dimensional mapping for $\varepsilon > 0$ in a following way. We will make a series $d_i = x_i^{(i)}$, $i = 1, 2, \ldots$ where $x_i^{(i)}$ is the $x_i$ coordinate of the point in a coordinate plane, which crosses through the line $x_2 = 0$ for $x_1 > 0$ (see figure 4.10), at the moment $t_i$, $i = 1, 2, \ldots$. The moment $t_i$, when we pick up a member of the series $d_i$, is called an event. Then, if we plot the function

$$d_{i+1} = P(d_i),$$

(4.86)

for an initial condition outside of the limit cycle, the points $O_i^+$, $i = 1, 2, \ldots$, shown in figure 4.11a, can be obtained. If the initial condition is inside the limit cycle, the points $I_i^-$, $i = 1, 2, \ldots$ shown in the same figure, are obtained.

The stability of equilibrium points $d_{eq}$ of the system is determined by the stability of nonlinear difference equation (4.86) i.e. is determined by the eigenvalues of $\frac{\partial P}{\partial d}|_{d=d_{eq}}$. In particular, if the eigenvalues have modulus less than 1, the equilibrium point is stable and if the eigenvalues have modulus greater than 1, than the equilibrium point is unstable [Sastry, 1999]. In figure 4.11a, it may be seen that

$$\lim_{d \to \pm d_{\infty}} \left| \frac{\partial P}{\partial d} \right| < 1,$$

(4.87)

and

$$\lim_{d \to 0} \left| \frac{\partial P}{\partial d} \right| > 1.$$

(4.88)
Consequently, $d_{eq} = \pm d_{\infty}$ is stable and $d_{eq} = 0$ unstable equilibrium points of the system described by (4.86). Due to the fact that points $I_i^+$ are going of the point $(x_1, x_2) = (0, 0)$ toward the point $(x_1, x_2) = (d_{\infty}, 0)$ and also $O_i^+$ are oriented toward the $(x_1, x_2) = (d_{\infty}, 0)$, this means that $(x_1, x_2) = (0, 0)$ is unstable equilibrium point and the point $(x_1, x_2) = (d_{\infty}, 0)$ represents a point of the stable limit cycle.

In figure 4.11a, points $I_i^-$ and $O_i^-$ are obtained if the points $x_i^{(i)}$, which crosses through the line $x_2 = 0$ for $x_1 < 0$ (figure 4.10), are considered.

If we perform the same procedure for the case when $\varepsilon < 0$, figure 4.11b can be obtained. Then, we have that

$$\lim_{d \to d_{\infty}} \left| \frac{\partial \mathcal{P}}{\partial d} \right| > 1$$

and

$$\lim_{d \to 0} \left| \frac{\partial \mathcal{P}}{\partial d} \right| < 1,$$

which means that $(x_1, x_2) = (d_{\infty}, 0)$ represents the point of the unstable limit cycle and $(x_1, x_2) = (0, 0)$ is the stable equilibrium point.

The figures 4.11a and 4.11b are called event maps.

![Event map for van der Pol oscillator](image)

**Figure 4.11. Event map for van der Pol oscillator shown in figures 2.5 and 4.10: a) $\varepsilon > 0$, b) $\varepsilon < 0$.**

The event map technique can be very easily defined for a second order system. But for higher order systems, it is not always possible to define a one-dimensional iterated mapping. An example of the event mapping technique for a fourth order system is presented in [Galvanetto 1997].
From this example it may be concluded that the event map is infact the Poincaré map. But due to the fact that in this method the emphasis lies on choosing the event (in order to obtain one-dimensional iterated mapping) and not always explicitly on choosing the hyperplane, it is called event map.

The advantage of this technique is that if we can define a one-dimensional iterative mapping, it is possible to investigate the stability properties of the equilibrium set and the existence of the limit cycles with their stability properties. The main disadvantage is that it is not always possible to define such a one-dimensional iterated mapping for the systems with an order higher than two. Consequently, the use of this technique is limited to a small class of nonlinear systems.
Chapter 5

Conclusions

The aim of this literature study is to explain limit cycling and forced oscillation phenomena and to present techniques for predicting such phenomena. Therefore, in sections 2.1 and 2.2 the example of the woodpecker toy and electric bell, the limit cycling phenomenon is introduced. In section 2.3, an example of the diesel engine system is presented, where quasi-periodic and even chaotic behaviour may appear. In section 2.4, on the example of oscillators, the difference between a close orbit and a limit cycle is explained. With those four straightforward examples, the aim was to identify reasons why limit cycling may appear in real-life systems. In sections 2.5 and 2.6, some definitions of phenomena, which are connected with the limit cycling and forced oscillation phenomena, are given.

Chapter 3 gives an overview of physical nonlinear phenomena that may cause limit cycling. For every nonlinearity mentioned, a physical system, in which such a non-linearity occurs, is described and reasons why a limit-cycle or a forced oscillation phenomena appears, are explained.

In chapter 4, firstly, the harmonic balancing method, the describing function method and the perturbation method are presented. Then, single shooting, finite difference, multiple shooting, sequential continuation, arclength continuation and the event mapping method are illuminated. Of course, this is not a complete list of methods, which can be used for predicting periodic behaviour, but the methods discussed here are often applied for predicting periodic behaviour.

In this way, this literature may be a good starting point for further investigation of periodic behaviour in mechanical systems.
References


