A SECOND ORDER FORMULATION FOR THE ANALYSIS OF SLENDER, ELASTIC BEAMS

L.Ph.J. Frenken

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Eindhoven University of Technology
Department of Mechanical Engineering
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Lambert Frenken
ABSTRACT

A second order formulation for the analysis of elastic beams is presented. Shear deformation and distortion of the cross section is not considered in the present theory. Based on energy considerations the analysis is able to predict the bifurcation-type buckling condition for slender beams taking into account the prebuckling deformation. As an example a closed form solution for the lateral buckling of a simply supported beam subjected to uniform bending is presented.
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LIST OF SYMBOLS

A = cross-sectional area

$b_y, b_z$ = monosymmetry parameters

$\psi$ = warping parameter

$C_1, C_2$ = amplitudes of displacement variations

E = Young's modulus of elasticity

G = shear modulus of elasticity

$I_o$ = polar moment of inertia about the shear center

$I_y, I_z$ = moments of inertia about the Y and Z axis, respectively

J = St. Venant's torsion section constant

J = Jacobian

L = beam length

M = moment

$M^*_a$ = bimoment

$M_{cr}$ = critical moment

$M_x, M_y, M_z$ = moments about the X, Y and Z axis, respectively

$M^*_a$ = bimoment per unit length
LIST OF SYMBOLS (cont'd)

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<tr>
<td>( m_x, m_y, m_z )</td>
<td>moments per unit length about the X, Y and Z axis, respectively</td>
</tr>
<tr>
<td>( n )</td>
<td>vector normal to the surface of the cross section</td>
</tr>
<tr>
<td>( P )</td>
<td>total potential energy</td>
</tr>
<tr>
<td>( P_0, P_y, P_z )</td>
<td>critical loads for buckling about the shear center, Y and Z axis, respectively</td>
</tr>
<tr>
<td>( Q_x, Q_y, Q_z )</td>
<td>forces in the x, y and z-direction, respectively</td>
</tr>
<tr>
<td>( q_x, q_y, q_z )</td>
<td>loads per unit length in the x, y and z direction, respectively</td>
</tr>
<tr>
<td>( R )</td>
<td>rotation matrix</td>
</tr>
<tr>
<td>( S_{ij} )</td>
<td>Piola-Kirchhoff stress tensor of the second kind</td>
</tr>
<tr>
<td>( s )</td>
<td>surface coordinate of the cross section</td>
</tr>
<tr>
<td>( T_o )</td>
<td>polar fourth moment of cross-sectional area about the shear center</td>
</tr>
<tr>
<td>( U )</td>
<td>strain energy</td>
</tr>
<tr>
<td>( u )</td>
<td>the x component of the displacement of the origin when no warping occurs</td>
</tr>
<tr>
<td>( v, w )</td>
<td>the y and z components, respectively, of the displacement of the shear center</td>
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<table>
<thead>
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<tr>
<td>( \bar{u}, \bar{v}, \bar{w} )</td>
<td>displacements in the x, y and z direction respectively of any point in the cross section</td>
</tr>
<tr>
<td>( \bar{u}_i )</td>
<td>generalized displacement</td>
</tr>
<tr>
<td>( X, Y, Z )</td>
<td>coordinate axes</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>coordinates of any point of the beam, prior to deformation</td>
</tr>
<tr>
<td>( y_0, z_0 )</td>
<td>coordinates of the shear center prior to deformation</td>
</tr>
<tr>
<td>( \alpha, \beta, \gamma )</td>
<td>angles of rotation about the X, Y and Z axis respectively</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>warping constant</td>
</tr>
<tr>
<td>( \dot{\gamma}<em>{xy}, \dot{\gamma}</em>{yz}, \dot{\gamma}_{xz} )</td>
<td>strain components</td>
</tr>
<tr>
<td>( \varepsilon_{x}, \varepsilon_{y}, \varepsilon_{z} )</td>
<td>strain components</td>
</tr>
<tr>
<td>( \delta, \delta^2 )</td>
<td>first and second variations</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>axial strain</td>
</tr>
<tr>
<td>( \varepsilon_{ij} )</td>
<td>Green-Lagrange strain tensor</td>
</tr>
<tr>
<td>( \bar{\varepsilon}_{ij} )</td>
<td>linearized deformation tensor</td>
</tr>
<tr>
<td>( \mu )</td>
<td>geometric constant of beam</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson's ratio</td>
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LIST OF SYMBOLS (cont'd)

\[ \begin{align*}
\sigma_x, \sigma_y, \sigma_z & = \text{stress components} \\
\tau_{xy}, \tau_{yz}, \tau_{xz} & = \text{Cauchy stress tensor} \\
\psi & = \text{warping function} \\
\omega_{ij} & = \text{linearized rotation tensor}
\end{align*} \]

**Superscripts**

* = reference to the local coordinate system \( x^*, y^*, z^* \).

**Subscripts**

The summation index is adopted to repeated lower case subscripts. Subscripts preceded by a comma denote differentiation with respect to these subscripts.
1. INTRODUCTION

Mechanical nonlinearities, such as plasticity effects, more than geometric effects are usually considered to be important in the load-carrying capacity of beams. Nevertheless, in the case of slender beams, buckling occurs in the elastic range and the nonlinear stability problem has not been widely investigated.

In the classical buckling analysis of beams, it is assumed that the prebuckling displacements are small enough to be neglected in the derivation of the governing differential equations [1-4]. However, for example in aluminium extrusions with open cross sections where the ratios of the major axis flexural stiffness to the minor axis flexural stiffness and the torsional stiffness may be less than three, the actual flexural-torsional buckling load may exceed the classical predictions by up to 25%, due to the prebuckling displacements [5-11]. Also for thin walled beams, interaction phenomena between local and global modes of buckling may affect the load-carrying capacity of beams considerably [12-17].

The purpose of this paper is to present a set of nonlinear equilibrium equations for slender beams with undeformed cross section under various loading and support conditions. A straightforward way for deriving the equations governing the bifurcation-type buckling condition taking into account the prebuckling deformation is also presented. The analysis is used to obtain a closed form solution for the lateral buckling of a simply supported beam subjected to uniform bending.
2. ELEMENTS OF BEAM THEORY

We consider an initially straight, prismatic, homogeneous beam of length L and cross-sectional area A, subjected to end, surface and volume loading. The beam is referred to rectangular cartesian coordinates x, y, z, where the X-axis is the longitudinal axis, as shown in Fig. 1.

The object of beam theory is to reduce a three-dimensional problem to an approximate one dimensional one. Slender beam theory may be derived in terms of the following simplifying approximations.

1. During bending and/or stretching, cross sections normal to the undeformed longitudinal axis are assumed to remain plane, normal and undeformed, so that transverse normal and shearing strains may be neglected in deriving the beam kinematic relations.

2. During twisting the cross sections of the beam rotate about the shear-center axis, while the normal displacement of any point in the cross section is equal to the product of the angle of twist per unit length, and the so called warping function $\psi(y,z)$, which is a function of the cross section geometry only.

3. Transverse normal stresses are assumed to be small compared with the other normal stress component, so that they may be neglected in the stress-strain relations.

These assumptions are known as the Bernoulli-Euler-Vlasov assumptions.

Let us now consider the beam in a slightly deformed configuration, as shown in Fig. 2. Let $u$ denote the x-component of the displacement of the origin when no warping occurs, and let $v$ and $w$ denote the y and z components, respectively, of the displacement of the shear center. In the following we will restrict ourselves to the intermediate class of deformations, which is defined by the limitation that the strains be small compared with unity, and rotations moderately small (see appendix A). As a consequence rotations may be described by a vector, letting $\alpha$, $\beta$, $\gamma$ denote the x, y and z components of the rotation of the shear center respectively. Then, as a consequence of the first two approximations, the displacement
components at any point in the beam $u, v, w$, may be expressed by the relations

$$u = u - yv_x^* - zw_x^* + \psi(y,z)a_x^*$$

$$v = v - (z-z_o^*)a^* - \frac{1}{2}(y-y_o)a^2$$

$$w = w + (y-y_o)a^* - \frac{1}{2}(z-z_o)a^2$$

Here $y_o$ and $z_o$ denote the location of the shear center prior to deformation. Subscripts preceded by a comma denote differentiation with respect to these subscripts. The superscript $^*$ is used to denote quantities with respect to the local coordinate directions $x^*, y^*, z^*$ of the beam in deformed configuration. These quantities are related to the global coordinate directions $x, y, z$ through the rotation matrix $[R]$, i.e.

$$\begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix} = \begin{bmatrix} \cos(x^*,x) & \cos(x^*,y) & \cos(x^*,z) \\ \cos(y^*,x) & \cos(y^*,y) & \cos(y^*,z) \\ \cos(z^*,x) & \cos(z^*,y) & \cos(z^*,z) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

in which $\cos(\ , \ )$ indicates the directional cosine of the two axes. For small angles of rotation the matrix $[R]$ may be presented in the linear form (see also appendix B)

$$[R] = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}$$

So

$$\begin{bmatrix} \alpha_x^* \\ \beta_x^* \\ \gamma_x^* \end{bmatrix} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \beta_x \\ \gamma_x \end{bmatrix}$$

and
Furthermore, neglecting the shear deformation,

\[
\begin{align*}
\beta &= -w, \\
\gamma &= v, \\
\end{align*}
\tag{2.6}
\]

Substituting Eqs (2.6) into Eqs (2.4) and (2.5) and neglecting terms involving products of \( u, x, \) since \( u, x = \theta, xx \) is small compared with unity (see appendix A), yields

\[
\begin{align*}
\alpha, x &= \alpha, x - v, x w, xx + w, x v, xx \\
v, x &= v, x + w, x \\
w, x &= -v, x + w, x \\
\end{align*}
\tag{2.7}
\]

The second order equations (2.1) may be considered as an attempt to construct a general theory of deformation of slender beams, based on Ref. [18] Eqs (VI.49) and (VI.83). The linearized form of the deformation field (2.1) is used in a majority of the earlier studies (Bleich, 1952; Vlasov, 1961; Timoshenko and Gere, 1961; Galambos 1968). However, in contrast with the linearized form, the present nonlinear deformation field (2.1) makes it possible to investigate the stability of slender beams, taking into account the prebuckling deformation.

Neglecting the shear deformation (Wagner-hypothese), then for the intermediate class of deformations the \( \tilde{\varepsilon}, x, \tilde{\gamma}, xy, \) and \( \tilde{\gamma}, xz \) components of the strain-displacement relations for a three-dimensional medium are (see appendix A):

\[
\tilde{\varepsilon}, x = \tilde{u}, x + \frac{1}{2}(\tilde{v}, x^2 + \tilde{w}, x^2)
\]
\[ \gamma_{xy} = u_x' + v_y' + \dot{v}_x y' + \dot{w}_x y', \quad (2.8) \]

\[ \gamma_{xz} = u_z' + \dot{w}_x z' + \dot{v}_x z', \quad (2.8) \]

where \( \gamma_{xy}, \gamma_{xz} \) are extensional and shearing strain components at any point through the cross section. Introduction of Eqs (2.1) and (2.5) gives

\[ \begin{align*}
\dot{\varepsilon}_x &= u_x' - y(v_{x,x} + \dot{w}_{x,x}) - z(w_{x,x} - \dot{v}_{x,x}) \\
&\quad + \psi(\alpha, x) - v, w_{x,x} + w, v, x
\end{align*} \]

\[ + \left[ \frac{1}{2} v_x^2 + w_x^2 + 2z_v x_{x,x} - 2y_w x_{x,x} \right] \alpha_x^2 \]

\[ + \left[ (y - y_o)^2 + (z - z_o)^2 \right] \alpha_x^2 \]

\[ (2.9) \]

\[ \dot{\psi}_{xy} = [\psi_y - (z - z_o)](\alpha_x - v, x, w, v, v, x, x) \]

\[ \dot{\psi}_{xz} = [\psi_z + (y - y_o)](\alpha_x - v, x, w, v, v, x, x) \]

where higher order terms have been neglected. Eqs (2.9) are the kinematic relations for the beam.

The generalized Hooke's law for the strain components \( \dot{\varepsilon}_x, \gamma_{xy}, \gamma_{xz} \) in a three dimensional isotropic medium has the form

\[ \begin{align*}
\dot{\varepsilon}_x &= \frac{1}{E} [\vec{\sigma}_x - \nu(\vec{\sigma}_y + \vec{\sigma}_z)] \\
\gamma_{xy} &= \frac{1}{G} \vec{r}_{xy} \\
\gamma_{xz} &= \frac{1}{G} \vec{r}_{xz}
\end{align*} \]

\[ (2.10) \]

where

\[ G = \frac{E}{2(1+\nu)} \]

\[ (2.11) \]

\( E \) is the modulus of elasticity, \( G \) is the shear modulus and \( \nu \) is Poisson's ratio. The symbols \( \vec{\sigma}_x, \vec{r}_{xy} \) etc. denote stress components at any point.
through the cross section. As a consequence of the third approximation $\tilde{\sigma}_y$ and $\tilde{\sigma}_z$ are negligibly small. Omission from Eqs (2.8) and rearrangement gives the relations

$$\tilde{\sigma}_x = E \tilde{\epsilon}_x$$

$$\tilde{\tau}_{xy} = G \tilde{\gamma}_{xy}$$

$$\tilde{\tau}_{xz} = G \tilde{\gamma}_{xz}$$

(2.12)
3. TOTAL POTENTIAL ENERGY

The total potential energy $P$ of a beam subjected to end, surface and volume loading is the sum of the strain energy $V$ and the potential energy of the applied conservative load $Q$ [20]:

$$ P = U + Q $$  \hspace{1cm} (3.1)

The strain energy for a three-dimensional isotropic medium referred to arbitrary orthogonal coordinates may be written

$$ U = \frac{1}{2} \iiint_V (\bar{\sigma}_x \varepsilon_x + \bar{\sigma}_y \varepsilon_y + \bar{\sigma}_z \varepsilon_z + \bar{\tau}_{xy} \gamma_{xy} + \bar{\tau}_{xz} \gamma_{xz} + \bar{\tau}_{yz} \gamma_{yz}) \, dx \, dy \, dz $$  \hspace{1cm} (3.2)

Omission of $\bar{\sigma}_y$, $\bar{\sigma}_z$ and $\bar{\tau}_{yz}$ in accordance with the basic approximations of slender beam theory, introduction of Eqs (2.12) and rearrangement gives

$$ U = \frac{1}{2} \iiint_V [E \varepsilon_x^2 + G \gamma_{xy}^2 + \gamma_{xz}^2] \, dx \, dy \, dz $$  \hspace{1cm} (3.3)

Taking the principal axes for the cross section coordinates $(y,z)$, and the shear center $O$ as the pole of the normalized warping, then, by definition:

$$ \int_A y \, dy \, dz = \int_A z \, dy \, dz = \int_A \psi \, dy \, dz = 0 $$

$$ \int_A y \psi \, dy \, dz = \int_A z \psi \, dy \, dz = 0 $$  \hspace{1cm} (3.4)

Substituting Eqs (2.10) into Eq. (3.3), integrating with respect to $y$ and $z$, and making use of Eqs (3.4) yields

$$ U = \frac{1}{2} \int_0^L \{ E A [u_x^2 + (v_x^2 + w_x^2 + 2z_o v_x \alpha_x - 2y_o w_x \alpha_x + \frac{1}{A} \alpha_x^2) u_x] + \frac{1}{4} (v_x^2 + w_x^2 + 2z_o v_x \alpha_x - 2y_o w_x \alpha_x)^2 \} \, dx $$
\[ + \frac{1}{2}(v_x^2 + w_x^2 + 2z_0 v_x' \alpha_x - 2y_0 w_x' \alpha_x) \frac{I_o}{A} \alpha_x^2 \]

\[ + \frac{T_o}{4A} \alpha_x^4 \]

\[ + EI_y(w_{xx} - v_{xx} \alpha - b_y \alpha_x^2)(w_{xx} - v_{xx} \alpha) \]

\[ + EI_z(v_x + w_{xx} \alpha - b_z \alpha_x^2)(v_{xx} + w_{xx} \alpha) \]

\[ + E \Gamma (a_{xx} - v_x w_{xxx} + v_{xxx} + b_\psi \alpha_x^2)(a_{xx} - v_x w_{xxx} + w_x v_{xxx}) \]

\[ + GJ (a_{xx} - v_x w_{xx} + w_x v_{xx})^2 \] dx \quad \text{(3.5)}

where

\[ b_y = \frac{1}{I_y} \iint (y^2 + z^2) zdydz - 2z_0 \]

\[ b_z = \frac{1}{I_z} \iint (y^2 + z^2) ydydz - 2y_0 \]

\[ b_\psi = \frac{1}{I} \iint (y^2 + z^2) \psi dydz \]

\[ I_o = \iint [(y - y_0)^2 + (z - z_0)^2] dydz \]

\[ I_y = \iint z^2 dydz \]

\[ I_z = \iint y^2 dydz \]

\[ T_o = \iint [(y - y_0)^2 + (z - z_0)^2] dydz \]

\[ J = \iint [(\psi_x - (z - z_0))^2 + (\psi_z + (y - y_0))^2] dydz \]
The warping function $\psi$ is determined by the well-known Neumann-problem (see appendix C)

$$\psi_{yy} + \psi_{zz} = v^2 \psi = 0 \quad (3.7a)$$

within the cross section, and

$$\left[ \psi_y (z - z_o) \right] \cos(n,y) + \left[ \psi_z (y - y_o) \right] \cos(n,z) = 0$$

$$\left[ \psi_y (z - z_o) \right] z_s - \left[ \psi_z (y - y_o) \right] y_s = 0 \quad (3.7b)$$

on the surface of the cross section. The normal vector $n$ is positive outward, and the surface coordinate $s$ is positive as indicated in Fig. 3.

The position of the shear center $O$ according to the energetic definition (see appendix C), is given by the equations:

$$y_o = \frac{1}{A} \int_A \int \left( \Phi_z + y \right) dydz$$

$$z_o = \frac{1}{A} \int_A \int \left( -\Phi_y + z \right) dydz \quad (3.8)$$

The notation and sign conventions for positive forces and moments acting on a beam element are illustrated in Fig. 4, where $Q_x$, $Q_y$, and $Q_z$ are the longitudinal and transverse components of the force acting on the cross section and $M_x$, $M_y$, and $M_z$ are the components of the moment acting on the cross section. The bimoment $M^*$ is not a real moment, its dimension is $[AL]^2$. $q_x$, $q_y$, $q_z$, $m_x$, $m_y$, $m_z$ and $m_a$ are the forces and moments per unit length, acting on the beam. The potential energy of the applied loads for a conservative system is the negative of the work done by the loads as the structure is deformed. Consequently, the potential energy for the surface loads as shown in Fig. 4 may be written

$$Q = - \int_0^L \left[ q_x u + q_y v + q_z w + m_x x' - m_y x' + m_z z' \right] dx$$
assuming that longitudinal and transverse loads are applied at the centroid and shear center of the cross section, respectively. For other load application points, the relations (2.1) have to be introduced.
4. THE MINIMUM POTENTIAL ENERGY CRITERION.

For equilibrium the total potential energy \( P \) must be stationary; i.e. its first variation \( \delta P \) must equal zero. Substituting Eqs (3.5) and (3.7) into Eq. (3.1) and applying the minimum potential energy criterion yields

\[
\delta P = \int_{0}^{L} \left[ EA[u_x, x] + \frac{1}{2}(v_x^2, x + w_x^2, x + 2z_0 v_x, x, x - 2y_0 w_x, x, x \right.
\]

\[
+ \left. \frac{I}{2A_x} \alpha_x^2 \right] \delta u_x, x \, dx
\]

\[
+ \int_{0}^{L} \left[ \left( \frac{1}{2} (v_x^2, x + w_x^2, x + 2z_0 v_x, x, x - 2y_0 w_x, x, x \right. \right.
\]

\[
+ \left. \frac{I}{2A_x} \alpha_x^2 \right] \delta v_x, x \, dx
\]

\[
- E_I (w_x, xx - v_x, xx \alpha - \frac{1}{2} b_x^2) \delta v_x, xx
\]

\[
+ E_I (v_x, xx + w_x, xx \alpha - \frac{1}{2} b_x^2) \delta v_x, xx
\]

\[
- E_I (\alpha_x xx - v_x, xx w_x, xx + w_x, xx v_x, xx + \frac{1}{2} b_x^2) (w_x, xxx \delta v_x, x - w_x, x \delta v_x, xxx) \]

\[
- GJ(\alpha_x x - v_x, x w_x, xx + w_x, x v_x, xx) (w_x, xx \delta v_x, x - w_x, x \delta v_x, xx) \, dx
\]

\[
+ \int_{0}^{L} \left[ \left( \frac{1}{2} (v_x^2, x + w_x^2, x + 2z_0 v_x, x, x - 2y_0 w_x, x, x \right. \right.
\]

\[
+ \left. \frac{I}{2A_x} \alpha_x^2 \right] \delta w_x, x \, dx
\]

\[
+ E_I (w_x, xx - v_x, xx \alpha - \frac{1}{2} b_x^2) \delta w_x, xx
\]

\[
+ E_I (v_x, xx + w_x, xx \alpha - \frac{1}{2} b_x^2) \delta w_x, xx
\]

\[
- E_I (\alpha_x xx - v_x, xx w_x, xx + w_x, xx v_x, xx + \frac{1}{2} b_x^2) (v_x, x \delta w_x, xxx - v_x, xxx \delta w_x, x) \]

\[
- GJ(\alpha_x x - v_x, x w_x, xx + w_x, x v_x, xx) (v_x, x \delta w_x, xx - v_x, xx \delta w_x, x) \, dx
\]
By integration by parts and by application of the lemma of DuBois-Reymond one obtains the equilibrium equations for a beam element

\[ (EA[u, x] + \frac{1}{2}(v, x)^2 + w, x)^2 + 2z_o v, x^2 \alpha, x - 2y_o w, x^2 \alpha, x \]

\[ + \frac{I_o}{A} \alpha, x^2 \] \[ + \frac{I_o}{I} \alpha, x^2 \] \[ = 0 \] \[ (4.2a) \]
\[- \left( \mathbf{E}_A \left[ u_x + \frac{1}{2} (v_x^2 + w_x^2 + 2z_0 v_x \alpha_x + 2y_0 w_x \alpha_x \right), x \right.
\left. + \frac{I_0}{A} \alpha_x \right] \right) (v_x + z_0 \alpha_x \right) \right) \right) \right), x \right.
\left. - \left[ \mathbf{E}_y \left( w_{xx} - v_{xx} \alpha - \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xx \right.
\left. + \left[ \mathbf{E}_z \left( v_{xx} + w_{xx} \alpha - \frac{1}{2} b_z \alpha_x^2 \right) \right] \right), xx \right.
\left. + \left[ \mathbf{E}_x \left( \alpha_{xx} - v_{xx} \alpha_{xxx} + w_{xx} \alpha_{xxx} + \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xxx \right.
\left. - \left[ \mathbf{E}_x \left( \alpha_{xx} - v_{xx} \alpha_{xxx} + w_{xx} \alpha_{xxx} + \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xxx \right.
\left. + \left[ \mathbf{G}_x \left( \alpha_x - v_x \alpha_{xx} + w_x v_x \alpha_{xx} \right) \right] \right), x \right.
\left. + \left[ \mathbf{G}_x \left( \alpha_x - v_x \alpha_{xx} + w_x v_x \alpha_{xx} \right) \right] \right), xx \right.
\left. - q_y - m_z \alpha_x - (m_y \alpha_{xx}, x) \right) \right) \right), \right. x - \left( m_y \alpha_x \right) \right), xx = 0 \right) \quad (4.2b) \right.
\left. \left\{ \mathbf{E}_A \left[ u_x + \frac{1}{2} (v_x^2 + w_x^2 + 2z_0 v_x \alpha_x + 2y_0 w_x \alpha_x \right), x \right.
\left. + \frac{I_0}{A} \alpha_x \right] \right) \right) \right) \right), x \right.
\left. - \left[ \mathbf{E}_y \left( w_{xx} - v_{xx} \alpha - \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xx \right.
\left. + \left[ \mathbf{E}_z \left( v_{xx} + w_{xx} \alpha - \frac{1}{2} b_z \alpha_x^2 \right) \right] \right), xx \right.
\left. + \left[ \mathbf{E}_x \left( \alpha_{xx} - v_{xx} \alpha_{xxx} + w_{xx} \alpha_{xxx} + \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xxx \right.
\left. - \left[ \mathbf{E}_x \left( \alpha_{xx} - v_{xx} \alpha_{xxx} + w_{xx} \alpha_{xxx} + \frac{1}{2} b_y \alpha_x^2 \right) \right] \right), xxx \right.
\left. + \left[ \mathbf{G}_x \left( \alpha_x - v_x \alpha_{xx} + w_x v_x \alpha_{xx} \right) \right] \right), x \right.
\left. + \left[ \mathbf{G}_x \left( \alpha_x - v_x \alpha_{xx} + w_x v_x \alpha_{xx} \right) \right] \right), xx \right.
\left. - q_z + m_{y_x} + (m_y v_x, xx) \right) \right) \right), x - \left( m_y v_x \right) \right), xx = 0 \right) \quad (4.2c) \right.\]
With the dynamic boundary conditions:

\[
Q_x = EA[u_x + \frac{1}{2}(v_x^2 + w_x^2 + 2z_o v_x + \alpha_x - 2y_o w_x + \alpha_x) + \frac{I_o}{A} \alpha_x^2] \quad (4.3a)
\]

\[
Q_y = EA[u_y + \frac{1}{2}(v_y^2 + w_y^2 + 2z_o v_y + \alpha_y - 2y_o w_y + \alpha_y) + \frac{I_o}{A} \alpha_y^2] (v, x + o \alpha_x, x) \quad (4.3b)
\]
\[-23-\]

\[+ [EI_y(w_{xx} + v_{xx} \alpha - \frac{1}{2} b_y \alpha^2)]_x, x \]

\[- [EI_z(v_{xx} - w_{xx} \alpha - \frac{1}{2} b_z \alpha^2)]_x, x \]

\[+ [E_G'(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx} + \frac{1}{2} b \psi \alpha^2)]_x, x \]

\[- E_G'(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx} + \frac{1}{2} b \psi \alpha^2)]_x, x \]

\[- [GJ(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx})]_x \]

\[- GJ(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx})_x \]

\[+ m_z + \frac{m}{a_{xx}^*}(m \psi v_x)_x, x \]

\[Q_z = E_A[u_{xx} + \frac{1}{2}(v_{xx}^2 + w_{xx}^2 + 2z_{xx} v_{xx} \alpha_{xx} - 2y_{xx} w_{xx} \alpha_{xx})
\]

\[+ \frac{I_{0_A \alpha_{xx}^2}}{A}(w_{xx} - y_{xx} w_{xx}) \]

\[- [EI_y(w_{xx} - v_{xx} \alpha - \frac{1}{2} b_y \alpha^2)]_x, x \]

\[- [EI_z(v_{xx} + w_{xx} \alpha - \frac{1}{2} b_z \alpha^2)]_x, x \]

\[- [E_G'(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx} + \frac{1}{2} b \psi \alpha^2)]_x, x \]

\[+ E_G'(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx} + \frac{1}{2} b \psi \alpha^2)]_x, x \]

\[+ [GJ(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx})]_x \]

\[+ GJ(\alpha_{xx} - v_{xx} w_{xxx} + w_{xx} v_{xxx})_x \]

\[+ m_y - \frac{m}{a_{xx}^*}(m \psi v_x)_x - \frac{m}{a_{xx}^*}v_{xx} \]

\[M_x = E_A[u_{xx} + \frac{1}{2}(v_{xx}^2 + w_{xx}^2 + 2z_{xx} v_{xx} \alpha_{xx} - 2y_{xx} w_{xx} \alpha_{xx})
\]

\[+ \frac{I_{0_A \alpha_{xx}^2}}{A}(z_{xx} v_x - y_{xx} w_x) \]
\begin{align}
+ & \mathbf{EA}[u, x] + \frac{1}{2}(v, x^2 + w, x^2 + 2z, v, x, x - 2y, w, x, x, x
+ & \frac{T_o}{I_o} \frac{2I}{2I_o} a, x, x
- & \mathbf{EI}_y (w, x, x - v, x, x - \frac{1}{2} b, x, x b, y, x, x
- & \mathbf{EI}_z (v, x, x + w, x, x - \frac{1}{2} b, x, x b, z, x, x
- \left[ \mathbf{EI}_y (v, x, w, x, x, x + w, x, v, x, x, x + \frac{1}{2} b, x, x) b, y, x, x
+ & \mathbf{EI}_z (v, x, x, x, x, x + w, x, v, x, x, x + \frac{1}{2} b, x, x) b, z, x, x
+ & \mathbf{GJ}(\alpha, x - v, x, w, x, x, x + w, x, v, x, x, x
- & m \frac{a, x}{m}
\end{align}

\begin{align}
M_y &= \mathbf{EI}_y (v, x, x - v, x, x - \frac{1}{2} b, x, x)
+ & \mathbf{EI}_z (v, x, x + w, x, x - \frac{1}{2} b, x, x)
+ \left[ \mathbf{EI}_y (v, x, w, x, x, x + w, x, v, x, x, x + \frac{1}{2} b, x, x) v, x, x
- & \mathbf{GJ}(\alpha, x - v, x, w, x, x, x + w, x, v, x, x, x
+ & m \frac{a, x}{m} v, x, x - m \frac{a, x}{m} v, x, x
\end{align}

\begin{align}
M_z &= -\mathbf{EI}_y (w, x, x + v, x, x - \frac{1}{2} b, x, x)
+ & \mathbf{EI}_z (v, x, x - w, x, x - \frac{1}{2} b, x, x)
- \left[ \mathbf{EI}_y (v, x, w, x, x, x + w, x, v, x, x, x + \frac{1}{2} b, x, x) w, x, x
+ & \mathbf{GJ}(\alpha, x - v, x, w, x, x, x + w, x, v, x, x, x w, x, x
- & m \frac{a, x}{m} w, x + m \frac{a, x}{m} w, x, x
\end{align}
$$M_a^* = E^*(\alpha_{xx} - v_{xw}^{xxx} + w_{xv}^{xxx} + \frac{1}{2}b^2 \psi \alpha_x^2)$$  \hfill (4.3g)
5. THE TREFFTZ CRITERION.

According to the Trefftz criterion for loss of stability the critical load for a continuous structural system is defined as the smallest load for which the second variation of the total potential energy of the system is no longer positive definite. At this load the equilibrium changes from stable to unstable [20-22]. The linear differential equations for determination of the bifurcation-point load are obtained by integrating the expression for \( P \) by parts and by application of the lemma of Dubois-Reymond.

The linear stability equations may also be obtained by application of the equivalent adjacent-equilibrium criterion. Let

\[
\begin{align*}
    u &= u_0 + u_1 \\
    v &= v_0 + v_1 \\
    w &= w_0 + w_1 \\
    \alpha &= \alpha_0 + \alpha_1
\end{align*}
\]  

(5.1)

where \( u_0, v_0, w_0, \alpha_0 \) denotes the configuration whose stability is under investigation, and where the variations \( u_1, v_1, w_1, \alpha_1 \) are admissible virtual increments. Introduction into Eqs. (4.1) is seen to give terms that are linear and nonlinear in the \( u_0, v_0, w_0, \alpha_0 \) and \( u_1, v_1, w_1, \alpha_1 \) displacement components. In the new equations, the terms in \( u_0, v_0, w_0, \alpha_0 \) alone add to zero because \( u_0, v_0, w_0, \alpha_0 \) is an equilibrium configuration and terms that are nonlinear in \( u_1, v_1, w_1, \alpha_1 \) may be omitted because of the smallness of these incremental displacements. Thus the resulting equations are homogeneous and linear in \( u_1, v_1, w_1, \alpha_1 \) with variable coefficients in \( u_0, v_0, w_0, \alpha_0 \).
6. APPLICATION OF THE THEORY.

The governing differential equations for buckling of slender beams are complicated and closed form solutions can only be obtained for single members with simple axial loading arrangement and boundary conditions. For some other cases it is possible to make simplifying assumptions, so that approximate closed form solutions can be obtained. However, for more complicated structures under general in-plane loading, a numerical technique must be used.

6.1 Simply supported beam in uniform bending.

An approximate closed form solution can be obtained from the flexural-torsional buckling of a simply supported double-symmetric beam subjected to two equal and opposite end moments (see Fig. 5) [11]. For the specified loading conditions the prebuckling displacements are

\[ u_o'x = -\frac{1}{2}w_o'^2 \]
\[ v_o = 0 \]
\[ w_o'xx = \frac{M}{EI} \]
\[ \alpha_o = 0 \]

Critical conditions occur when \( \delta^2 p = 0 \). The boundary conditions in terms of variations in \( u, v, w \) and \( \alpha \) are:

\[ u_1'x + w_0'xw_1'x = 0 \]
\[ v_1 = v_1'xx = 0 \]
\[ w_1 = w_1'xx = 0 \]
\[ \alpha_1 = \alpha_1'xx = 0 \]

Since the in-plane displacements components \( u_1 \) an \( w_1 \) only occur in positive definite form in the second variation and can therefore only increase the calculated value of the critical moment, these displacement will be zero in the buckling mode. A solution for the buckling mode can be obtained by
assuming displacement functions for \( v_1 \) and \( \alpha_1 \) which satisfy the prescribed boundary conditions. Therefore assuming

\[
v_1 = C_1 \sin \frac{\Pi}{L} x
\]

\[
\alpha_1 = C_2 \sin \frac{\Pi}{L} x
\]

(6.3)

where \( C_1 \) and \( C_2 \) define the absolute magnitudes of the displacement variations, substituting Eqs (6.2) and (6.3) into Eqs (5.1) and following the procedure described in section 5, the resulting equation can be expressed in matrix form as

\[
\begin{bmatrix}
a_1 & a_2 + a_3 \\
a_2 + a_3 & a_4
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(6.4)

where

\[
a_1 = EI_z \left( \frac{\Pi}{L} \right)^4
\]

\[
a_2 = M \left( \frac{\Pi}{L} \right)^2 \left( 1 - \frac{I_z}{I_y} \right)
\]

(6.5)

\[
a_3 = -\frac{1}{2} \frac{M}{EI_y} \left( \frac{\Pi}{L} \right)^2 [GJ + E\Pi (\frac{\Pi}{L})^2]
\]

\[
a_4 = -\frac{M}{EI_y} (1 - \frac{I_z}{I_y}) + GJ (\frac{\Pi}{L})^2 + E\Pi (\frac{\Pi}{L})^4
\]

For a nontrivial solution to this homogeneous equation system the determinant of the coefficients of \( C_1 \), \( C_2 \) must equal zero. Hence, neglecting \( a_3^2 \) since it is small compared with other terms, the critical moment is given by

\[
M_{cr} = \left( \frac{p_o \frac{I}{P} A}{\mu} \right)
\]

(6.6)
where

\[ \mu = \left(1 - \frac{P_z}{P_y}\right) \left[1 - \frac{P_o}{A \left(\frac{L}{L}\right)}\right] \]

\[ P_o = [GJ + E' \left(\frac{L}{L}\right)] A \left(\frac{L}{L}\right) \]  

\[ P_y = EI_{y} \left(\frac{L}{L}\right)^2 \]

\[ P_z = EI_{z} \left(\frac{L}{L}\right)^2 \]

When \( I_y \gg I_z \), the solution reduces to the well known classical solution

\[ M_{cr} = \frac{I_0}{P_o, P_z, A} \]  

(6.8)

When \( I_y \) is not very much greater than \( I_z \), the dominant term in the denominator is \( (1 - \frac{I_z}{I_y}) \) and the percentage increase in the critical moment is almost independent of the span.

For a monosymmetric beam the critical moment is found to be [7]:

\[ M_{cr} = \frac{P_o b_{yz}^2}{2\mu} \pm \left[\frac{P_o b_{yz}^2}{2\mu} + \frac{P P_z A}{\mu}\right] \]  

(6.9)
7. **Conclusions.**

The presented second order formulation for the analysis of slender beams makes it possible to predict the bifurcation-type buckling condition under various loading and support conditions, taking into account the prebuckling deformation. A closed form solution, obtained for the lateral buckling of a simply supported beam subjected to uniform bending, agrees with earlier analyses: if the ratios of the major axis flexural stiffness to the minor axis flexural stiffness is less than three, the in-plane deformations may effect the buckling load considerable.
REFERENCES


Fig. 1: Prismatic beam

Fig. 2: Beam in deformed configuration
Fig. 3: Orientation of the normal vector \( n \) and the surface coordinate \( s \).

Fig. 4: Loading arrangement.
Fig. 5: Beam subjected to uniform bending
APPENDIX A: The intermediate class of deformations.

An elastic body is deformed so that a generic point $P$ in the reference state
displaces to $P'$.

Let $x_i$ be the Cartesian coordinates of the reference state and let $x'_i$ be the
Cartesian coordinates of the deformed state, then the displacement field is
given by

$$
\bar{u}_i = x'_i - x_i \tag{A.1}
$$

In what follows, derivatives are with respect to the original state, and are
indicated by $(\big),_i = \partial(\big)/\partial x'_i$, while a repeated index will imply a
summation.

The Green-Lagrange strain-tensor is given by [18]

$$
\varepsilon_{ij} = \frac{1}{2} (\ddot{u}_{ij} + \ddot{u}_{ji} + \ddot{u}_{ki} \ddot{u}_{kj}) \tag{A.2}
$$

while the corresponding Piola-Kirchhoff stress tensor of the second kind is

$$
\bar{S}_{ik} = J \left( \frac{\partial x_k}{\partial x'_j} \right) \frac{\partial x'_j}{\partial x'_i} \bar{\sigma}_{jl} \tag{A.3}
$$

when $J$ is the Jacobian for the transformation between current and original
coordinates, and $\bar{\sigma}_{lj}$ is the physical (Cauchy) stress tensor. Defining the
linearized deformation tensor as

$$
\bar{\theta}_{ij} = \frac{1}{2} (\ddot{u}_{ij} - \ddot{u}_{ji})
$$

and the linearized rotation tensor as

$$
\bar{\omega}_{ij} = \frac{1}{2} (\ddot{u}_{j'i} - \ddot{u}_{i'j}) \tag{A.4}
$$

the Green-Lagrange strain tensor may be written

$$
\dot{\varepsilon}_{ij} = \dot{\theta}_{ij} + \frac{1}{2} (\dot{\theta}_{ki} - \dot{\omega}_{ki})(\dot{\theta}_{kj} - \dot{\omega}_{kj}) \tag{A.5}
$$
For most stability problems, $\theta$ as well as $\bar{w}$ are small compared with unity, while $\bar{\theta}$ is of the same order as, or higher than $\bar{w}^2$. For this intermediate class of deformations, the Green-Lagrange strain tensor may be written approximately [19]

$$\varepsilon_{ij} = \theta_{ij} + \frac{1}{2} \bar{w}_{ki} \bar{w}_{kj}$$  \hspace{1cm} (A.6)

Written out the $\varepsilon_{xx}$, $\varepsilon_{xy}$ and $\varepsilon_{xz}$ components, for a three-dimensional medium are

$$\varepsilon_{xx} = \theta_{xx} + \frac{1}{2} (\bar{w}_{xx} + \bar{w}_{xy} + \bar{w}_{xz})$$

$$\varepsilon_{xy} = \theta_{xy} + \frac{1}{2} (\bar{w}_{xx} \bar{w}_{xy} + \bar{w}_{xy} \bar{w}_{yy} + \bar{w}_{xz} \bar{w}_{yz})$$  \hspace{1cm} (A.7)

$$\varepsilon_{xz} = \theta_{xz} + \frac{1}{2} (\bar{w}_{xx} \bar{w}_{xz} + \bar{w}_{xy} \bar{w}_{yz} + \bar{w}_{xz} \bar{w}_{zz})$$

Substituting Eqs (A.3) and (A.4) into Eq. (A.7) and neglecting the shear deformation ($\bar{\theta}_{xy} = \bar{\theta}_{xz} = 0$) yields

$$\varepsilon_x = \bar{u}_x + \frac{1}{2} (\bar{v}_x + \bar{w}_x)$$

$$\gamma_{xy} = \bar{u}_y + \bar{v}_x + \bar{v}_x \bar{v}_y + \bar{w}_x \bar{w}_y$$  \hspace{1cm} (A.8)

$$\gamma_{xz} = \bar{u}_z + \bar{w}_x + \bar{v}_x \bar{v}_z + \bar{w}_x \bar{w}_z$$

where

$$\varepsilon_x = \varepsilon_{xx}$$

$$\gamma_{xy} = 2 \varepsilon_{xy}$$  \hspace{1cm} (A.9)

$$\gamma_{xz} = 2 \varepsilon_{xz}$$
APPENDIX B: The rotation matrix.

If the rotation of the cartesian coordinate system \(x, y, z\) to the cartesian coordinate system \(x^*, y^*, z^*\) is relatively large, it can not be described by a vector and it is treated by means of modified Euler angles. If it is described by a finite rotation \(\gamma\) about the Z axis, followed by a rotation \(\beta\) about the Y axis followed by a rotation \(\alpha\) about the X axis (see fig. B-1), then the rotation matrix \([R]\) is given by [20]

\[
[R] = \begin{bmatrix}
\cos(x^*,x) & \cos(x^*,y) & \cos(x^*,z) \\
\cos(y^*,x) & \cos(y^*,y) & \cos(y^*,z) \\
\cos(z^*,x) & \cos(z^*,y) & \cos(z^*,z)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos\beta \cos\gamma & \cos\beta \sin\gamma \\
-cos\alpha \sin\gamma + \sin\alpha \sin\beta \cos\gamma & \cos\alpha \cos\gamma + \sin\alpha \sin\beta \cos\gamma \\
\sin\alpha \sin\gamma + \cos\alpha \sin\beta \cos\gamma & -\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma
\end{bmatrix}
\]

The rotation matrix \([R]\) can be approximately written in antisymmetric form as

\[
[R] = \begin{bmatrix}
\cos\beta \cos\gamma & \sin\gamma & -\sin\beta \\
-sin\gamma & \cos\alpha \cos\gamma & \sin\alpha \\
\sin\beta & -\sin\alpha & \cos\alpha \cos\beta
\end{bmatrix}
\]

(B.2)

Noting that the \(Y^*\) and \(Z^*\) axes are perpendicular to the shear center deflection curve and taking the effect of axial elongation into consideration, we have

\[
tg \beta = \frac{-W'_{X}}{1+\varepsilon} \quad tg \gamma = \frac{V'_{X}}{1+\varepsilon}
\]

(B.3)

where \(\varepsilon\) is the axial strain.
Making use of the relations

\[
\sin \alpha = \frac{\tan \alpha}{1 + \tan^2 \alpha} \quad \cos \alpha = \frac{1}{1 + \tan^2 \alpha} \quad (B.4)
\]

the rotation matrix is expressed in the form

\[
[R] = \begin{bmatrix}
\frac{(1+\varepsilon)^2}{[(1+\varepsilon)^2 + v_x^2]} & \frac{1}{[(1+\varepsilon)^2 + v_x^2]} & v_{x, x} \\
\frac{1}{[(1+\varepsilon)^2 + v_x^2]} & \frac{1}{[(1+\varepsilon)^2 + v_x^2]} & \cos \alpha \\
\frac{1}{[(1+\varepsilon)^2 + w_x^2]} & -\sin \alpha & \frac{1+\varepsilon}{[(1+\varepsilon)^2 + w_x^2]}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{[(1+\varepsilon)^2 + w_x^2]} \\
\sin \alpha \\
\frac{1+\varepsilon}{[(1+\varepsilon)^2 + w_x^2]}
\end{bmatrix}
\]

(B.5)

If we assume that the deformations are small, then, Eq. (B.5) reduces to the simple form

\[
[R] = \begin{bmatrix}
1 & v_{x, x} & w_{x, x} \\
-v_{x, x} & 1 & \alpha \\
-w_{x, x} & \alpha & 1
\end{bmatrix} = \begin{bmatrix}
1 & \gamma & -\beta \\
-\gamma & 1 & \alpha \\
\beta & -\alpha & 1
\end{bmatrix}
\]

(B.6)

and the rotation may be treated as a vector.
Fig. B-1: Orientation of the $X, Y, Z$ and $X', Y', Z'$ coordinate systems.
APPENDIX C: The torsion problem.

The total potential energy $P$ for a prismatic beam in uniform torsion is (see section 3)

$$P = \frac{1}{2} GL \alpha \int \int_A \left\{ [\psi_y' - (z-z_0)]^2 + [\psi_z' + (y-y_0)]^2 \right\} dydz$$

$$- M_t \int_0^L \alpha_x' dx$$

(C.1)

Application of the minimum potential energy criterion by variation of $\psi$ yields

$$GL \alpha \int \int_A \left\{ [\psi_y' - (z-z_0)] \delta \psi_y' + [\psi_z' + (y-y_0)] \delta \psi_z' \right\} dydz = 0$$

(C.2)

By application of Green's theorem and the lemma of Dubois-Reymond one obtains:

$$\psi_y'' + \psi_z'' = v^2 \psi = 0$$

(C.3a)

within the cross section, and

$$[\psi_y' - (z-z_0)] \cos(n,y) + [\psi_z' + (y-y_0)] \cos(n,z) =$$

$$[\psi_y' - (z-z_0)] z_s - [\psi_z' + (y-y_0)] y_s = 0$$

(C.3b)

on the surface of the cross section. The normal vector $n$ is positive outward, and the surface coordinate $s$ is positive as indicated in Fig. 3. Multiplying Eq. (C.3b) by $z$, integration over the surface and application of Stoke's theorem yields

$$y_0 = \frac{1}{A} \int \int_A (\psi_z' + y) dydz$$

(C.4a)

This equation can also be obtained by varying Eq. (C.1) with respect to $y_0$ and applying the lemma of Dubois-Reymond.
Analogous one finds

\[ z_0 = \frac{1}{A} \iint_A (-\psi_y + z) \, dydz \]  \hspace{1cm} (C.4b)