Actuation Principles of Permanent Magnet Synchronous Planar Motors
A Literature Survey

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CBT 534-05-2717
DCT 2005.149

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Eindhoven, December 2005
Abstract

At Philips Applied Technologies, research is being done on the control and design of high-performance electromechanical systems. The actuators in a wafer or reticle stage used in the lithography industry are a good example of such a system. Due to increasing requirements related to the performance and accuracy of the wafer scanners used for producing IC’s, other types of electromechanical actuators have been proposed. To actuate the wafer stages in a plane, the conventional actuator is composed of linear motors placed in a H-bridge configuration. The EUV project demands the actuator to operate in a complete vacuum and therefore a so-called synchronous permanent magnet planar motor is proposed to replace the existing actuator configuration in the wafer stage. This actuator levitates and propels itself above a permanent magnet array. Because the actuator is magnetically levitated, there is no need for air or other types of bearings and the actuator can operate in vacuum.

Although the servo performance of the planar motor is satisfactory in the view of the present EUV applications, the true potential of the planar motor is not used yet. This full potential will be needed in future EUV applications. The servo errors are presently dominated by components that can be attributed to various disturbances related to the particular actuation and commutation principle. Accompanying electronics can introduce disturbances in the actuator. To counteract these phenomena, a physical model of the most relevant sources for the disturbances in the servo error must be derived. The objective of this literature study is to gain insight in all physical phenomena and mechanisms playing a role in electromechanical actuators. Then, with the use of this theory the synchronous planar motor can be analysed and sources for the disturbances can be identified and resolved.

This report is divided into two parts. In the first part, the electromagnetic field is analysed and the different components in electromechanical actuators are investigated. These include the electric coils and permanent magnets. The interaction between the two magnetic fields leads to the generation of forces that can be used for actuation. Modelling techniques to model the electromagnetic field and the generated forces are also discussed in this part. The second part is merely devoted to the actuation principles of the synchronous planar motor itself. The motor principle and the associated commutation principles are explained.

The planar motor is based on the three-phase/four-pole actuation principle. With the appropriate distribution of the magnetic field and coil currents, the actuator forces to drive and lift the actuator can be decoupled. In this way, both components can be controlled individually. Possible sources for actuator-related disturbances are commutation errors, amplifier inaccuracies, unknown machine parameters, and the generation of additional forces and torques purely due to the actuator design and end effects of coils and permanent magnets.
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The constants, variables with their SI units, general subscripts and abbreviations used in this report are listed in the following tables.

### Constants

<table>
<thead>
<tr>
<th>Constant</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g)</td>
<td>Gravitational constant</td>
<td>9.80665 (m/s^2)</td>
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<td>(\varepsilon_0)</td>
<td>Permittivity of free space</td>
<td>(8.84518 \cdot 10^{-12} , F/m)</td>
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<tr>
<td>(\mu_0)</td>
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<td>(\pi)</td>
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### Variables

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<th>SI Unit</th>
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<td>(A)</td>
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<tr>
<td>(B, B)</td>
<td>Magnetic flux density</td>
<td>(T = Wb/m^2)</td>
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<tr>
<td>(B_r)</td>
<td>Magnetic remanence</td>
<td>(T)</td>
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<tr>
<td>(C)</td>
<td>General Curve</td>
<td>–</td>
</tr>
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<td>(D, D)</td>
<td>Electrical flux density</td>
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<td>(e)</td>
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<td>(E, E)</td>
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<td>Induced electric field</td>
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</tr>
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<td>(E_{st})</td>
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<tr>
<td>(F_C)</td>
<td>Force</td>
<td>(N)</td>
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<td>(H_c)</td>
<td>Magnetic coercivity</td>
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<td>(A)</td>
</tr>
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<td>Unit</td>
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<td>--------------------------------------------------</td>
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<tr>
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<td>Magnetisation</td>
<td>$A/m$</td>
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<td>Polarisation</td>
<td>$C/m^2$</td>
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<tr>
<td>$P$</td>
<td>Electric Power</td>
<td>$Watt$</td>
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<td>Electric charge</td>
<td>$C$</td>
</tr>
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<td>Position vector</td>
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<td>$\Omega$</td>
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<td>$S, S, s, s$</td>
<td>Surface normal vector, Surface</td>
<td>$m^2$</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>$s$</td>
</tr>
<tr>
<td>$U, u$</td>
<td>Potential difference</td>
<td>$V$</td>
</tr>
<tr>
<td>$U_h$</td>
<td>Potential difference in Hall sensor</td>
<td>$V$</td>
</tr>
<tr>
<td>$v, v$</td>
<td>Velocity</td>
<td>$m/s$</td>
</tr>
<tr>
<td>$V$</td>
<td>Volume</td>
<td>$m^3$</td>
</tr>
<tr>
<td>$w$</td>
<td>Coil width</td>
<td>$m$</td>
</tr>
<tr>
<td>$W_{pm}$</td>
<td>Permanent magnet field energy</td>
<td>$Watt$</td>
</tr>
<tr>
<td>$x$</td>
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</tr>
<tr>
<td>$y$</td>
<td>Cartesian y-coordinate</td>
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<tr>
<td>$z$</td>
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<tr>
<td>$\theta$</td>
<td>Angle</td>
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<td>$\vartheta$</td>
<td>Current phase angle of coil</td>
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<tr>
<td>$\mu$</td>
<td>Magnetic permeability</td>
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<td>$\mu_r$</td>
<td>Relative magnetic permeability</td>
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<td>$\mu_{rc}$</td>
<td>Recoil permeability</td>
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<tr>
<td>$\mu_B$</td>
<td>Magnetic dipole moment</td>
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<td>$\rho$</td>
<td>Quantity density</td>
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<td>$\rho_{ms}$</td>
<td>Magnetic surface charge density</td>
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</tr>
<tr>
<td>$\tau$</td>
<td>Magnetic pitch</td>
<td>$m$</td>
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<tr>
<td>$\phi$</td>
<td>General scalar field</td>
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<tr>
<td>$\phi$</td>
<td>Force angle in linear synchronous actuator</td>
<td>$rad$</td>
</tr>
<tr>
<td>$\phi_m$</td>
<td>Magnetic scalar potential</td>
<td>$A$</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Flux linkage</td>
<td>$Wb$</td>
</tr>
<tr>
<td>$\Phi_D$</td>
<td>Electric Flux</td>
<td>$C$</td>
</tr>
<tr>
<td>$\Phi_B$</td>
<td>Magnetic Flux</td>
<td>$Wb$</td>
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</table>
Electric susceptibility
Magnetic susceptibility
Differential operator

General Subscripts

B Magnetic field
E Electric field
ext External
mech Mechanical
n Normal component
t Tangential component
x Cartesian x-coordinate
y Cartesian y-coordinate
z Cartesian z-coordinate

Abbreviations

DOF Degree of Freedom
EUV Extreme Ultra Violet
FEM Finite Element Method
IC Integrated Circuit
ILC Iterative Learning Control
PM Permanent Magnet
PMA Permanent Magnet Array
PMSM Permanent Magnet Synchronous Motor
SPMPM Synchronous Permanent Magnet Planar Motor
Chapter 1

Introduction

1.1 Background

At Philips Applied Technologies\footnote{Formerly Philips Centre for Industrial Technology (CFT).}, research is being done on high-performance, six degrees of freedom (6-DOF) motion systems. A typical machine in need of these high-precision motion systems is a wafer scanner, which is used in the photolithography industry for the mass-production of integrated circuits (IC's). A wafer scanner is responsible for the lithographic printing process of the IC patterns on a silicon disc, the so-called wafer. This wafer contains a light sensitive layer. By the use of short-wavelength light, IC patterns can be etched in the wafer. This is done with the use of a so-called reticle, containing the original IC pattern. This reticle is positioned before a high-precision and advanced lens system, which focusses the light on the wafer. Due to the IC pattern on the reticle, it will only pass light to the corresponding areas of the wafer that need to be removed. In Figure\ref{fig:wafer_scanner} the scanning process in a wafer scanner is schematically depicted. In this picture the reticle stage, the wafer stage and the lens assembly can be clearly distinguished. The reticle as well as the wafer stage can be actuated, to guide the light through the reticle and on the wafer. After exposure, the pattern can be recovered from the wafer with the use of a chemical process. Because an IC contains multiple layers with different patterns, the etching process needs to be performed multiple times on the same wafer. Because of limitations of the lens system, only parts of the silicon disc can be exposed at the same time. These conditions require that the reticle as well as the wafer need to be positioned with respect to each other and the lens system. The reticle as well as the wafer are fixed to so-called stages, the reticle stage and the wafer stage. The stages are able to position the respective stages in all degrees of freedom. Due to the dimensions of the chips, the position accuracies need to be in the order of \textit{nm}. During exposure, movement of both stages needs to be suppressed as much as possible. Of course high throughput is required, which forces the stages to position, accelerate, decelerate and move as fast as possible. If the accuracy requirements are not met an erroneous IC will result.

To achieve the required accuracy and performance a (wafer-)stage traditionally is made out of two devices: A 6-DOF short-stroke stage and a 3-DOF long-stroke stage. The short-stroke device is placed on top of the long-stroke device. The short-stroke device usually consists of a stack of at least two linear motors. In this configuration, a high moving mass and a limited mechanical stiffness limit the bandwidth, and therefore the performance of the actuator. Due
to increasing performance requirements with respect to accuracy, directly coupled to the IC dimensions, light sources with a smaller wavelength have been proposed to etch the wafer. The proposed type of light source is in the range of the Extreme Ultra-Violet, or EUV. Because of interference of this small wavelength light with the ambient air, which deteriorates the accuracy and required exposure, the stage and actuator are required to operate in a complete vacuum. The demand for a complete vacuum requires a different type of long-stroke actuator. The actuators used to position the stage make use of air bearings. It is obvious that air bearings can not be used in a vacuum. Also other mechanical bearings that need lubricants are not suitable because the lubricant can pollute the vacuum. On the other hand, magnetic bearings do not need any lubricants and also work in vacuum. Therefore so-called magnetic planar motors are proposed to replace the classical long stroke actuators. A magnetic planar motor or surface actuator generates forces that can levitate and propel a stage over long strokes in the horizontal plane and short strokes in the vertical plane. Also small rotations around all three axis can be controlled. In this way six degrees of freedom can be actuated frictionless and in vacuum.

Until now, many different planar motor types have been designed. All types are based on the interaction between magnetic and electric fields. This interaction results in the required force to lift the rotor or mover and to propel it with respect to the stator. Depending on the principles of a certain planar motor it usually can be classified into three types: variable reluctance planar motor, induction planar motor, and permanent magnet planar actuator. The variable reluctance planar motor or Sawyer planar stepper-motor, patented by B.A. Sawyer in 1968 [30], was the first planar motor used in the industry. The major drawback of this type of motor is that it moves in a step-wise manner, it is relatively slow and it needs additional
air-bearings to levitate the platform [5]. Later, other types of planar motors were designed. In an induction planar motor the stator consist of a homogeneous conductive plate surrounding a bulk ferromagnetic plate. The mover contains the electromagnetic coils and induces a magnetic field in the stator, as described in [16, 27]. The absence of permanent magnets and slots removes cogging effects. But overall, the performance of the induction motors is relatively low. Especially induction planar motors that need a mechanical bearing have a low performance, due to the required larger air-gap [5]. The motors of the permanent magnet planar motor type use a permanent magnet array in combination with properly oriented coils. By inducing a magnetic field with the electric coils close to the permanent magnets forces are generated. Both parts, permanent magnets and coils, can be implemented in the stator as well as the mover. If the mover consists of the permanent magnet array and the stator contains the coils, the motor is referred to as ‘inverted planar motor’, with the advantage that there are no moving cables. A special case of permanent magnet planar actuators is the SPMPM, or Synchronous Permanent Magnet Planar Motor. Besides the fact that the SPMPM does not need an additional bearing system, it has the advantages of low cost, a simple structure, and a high force density.

1.2 Motivation

Within Philips Applied Technologies, the SPMPM is proposed to replace the existing wafer stage actuator in future wafer scanners. Due to the tight requirements on position errors, relatively high speeds, and large loads, the control of such an actuator is challenging. Moreover, all six degrees of freedom have to be controlled simultaneously. Various controllers for planar motors have been developed and tested, see [34, Chapter 5] and [32, 36]. Despite the implemented control structure in the test set-up, the servo errors of the planar motor are presently dominated by low-frequent components. The behaviour of the servo error can be seen as a result of the chosen actuator layout and commutation principle of the motor. The disturbance causing the low-frequent servo error can therefore be regarded as an ‘actuator related’ disturbance. The disturbances depend on the position of the actuator and the disturbance frequency is mainly dependent on the velocity of the planar motor. Other disturbance sources are due to imperfect knowledge of the system parameters, like amplifier gains and offsets. Future specifications on the servo error and settling time justify the elimination of these actuator related disturbances. To some extent this can be done with the use of suitable feedback controllers [35]. This solution, however, becomes less effective for higher velocities, tighter error specifications, and tighter settling time requirements. Both these issues will arise in future actuators and therefore a more thorough study of the actuator-related disturbances is required. Then, with the use of an appropriate feedforward control structure, the disturbances can be compensated more effectively.

In order to classify and predict actuator related disturbances, first a study of the mechanisms arising inside electromechanical actuators, like the SPMPM, must be performed. First of all, the fundamental properties involved in electromechanical actuators have to be defined. Based on the description of these properties, the basic electromagnetic phenomena arising in these actuators can be analysed and quantified. Once the mechanisms are known, the individual components of the actuator can be modelled and analysed. By combining the individual components a total model for the electromechanical actuator can be obtained. Complete
motor analysis, including mechanics and dynamics, is performed for various types of planar actuators, see [9, 10, 12]. A more general approach for analysing a planar motor can be found in [31]. Once the mechanisms behind the planar motor are characterised, a model for the actuator-related disturbances can be developed.

To be able to fully understand the fundamental mechanisms arising in electromagnetism it is chosen to start searching for literature that describes electromagnetism in its most fundamental form, the theory of vector fields. Electromagnetic phenomena like induction, Eddy current damping, and Lorentz force generation are based on the interaction between magnetic and electric fields. Both fields are of the vector field type, and a very thorough study of vector fields is therefore required. With the use of the theory for vector fields, a description of the electromagnetic phenomena can be given. The SPMPM is based on the Lorentz force generation principle which states that forces are exerted on moving electric charges placed in an external magnetic field. These forces are used to lift the platform with respect to the magnet plate and to control all six degrees of freedom. When current is fed to electromagnetic coils electric charges will move through these coils. When these coils are placed close to a permanent magnet, which produces the external magnetic field, forces acting on the coils as well as the permanent magnets will be generated. Therefore the generation of magnetic fields arising from electromagnetism as well as permanent magnets must be investigated. Furthermore, all individual components in the actuation chain of the actuator together will determine how the total generated force, and therefore need to be studied as well. These components include the coils and the permanent magnets itself, but also components like the commutation algorithm and the amplifiers.

1.3 Problem Statement

This literature study is based on the following problem statement:

Based on the available literature, make a critical overview of the fundamental physical properties or phenomena and their modelling methodologies arising in electromagnetic actuators in order to characterise actuator related parasitic effects in synchronous permanent magnet linear or planar actuators.

Based on the discussion in the previous section, the following questions can be distinguished with respect to this literature survey:

- What are the properties of the electric, magnetic, and electromagnetic field, how can they be characterised, and how can they be described mathematically?

- What sources for the (electro)magnetic field exist and how can they be qualified?

- What techniques exist to model the fundamental (electro)magnetic phenomena laying the foundation for electromechanical actuators?

- What is the working principle of the synchronous permanent magnet planar motor and what are the main components in the actuation chain of the SPMPM at Philips Applied Technologies?
1.4 Outline of the Report

The outline of this report follows the approach stated at the end of Section 1.2. In the first part, Chapters 2 through 5, general fundamental physical properties of the electromagnetic field will be discussed. The second part, Chapter 6, is entirely focused on actuators of the linear and planar permanent magnet synchronous type.

In Chapter 2, the mathematical description of general scalar and vector fields is treated. Section 2.3 deals with the properties of general vector fields as well as some basic vector field operations. This section also deals with the classification of vector fields based on their divergence and curl. The general boundary conditions along an interface, that vector fields must satisfy, are discussed in the Section 2.5. The fundamental theory explained in Chapter 2 will be used in the subsequent chapters.

In Chapter 3, the most general equations describing the electromagnetic field, the equations of Maxwell, will be stated. Together with the continuity equations and two constitutive equations, the electromagnetic field can be described uniquely. Using the theory of Chapter 2, the properties and class of the electric and magnetic field will be derived. The chapter ends with the general boundary conditions for the electromagnetic field.

Subsequently, in Chapter 4, fundamental electromagnetic phenomena that arise in electromechanical actuators will be discussed. Phenomena based on Faraday’s law, are treated in this chapter. Also the magnetic field produced by current-carrying conductors and the generation of force following the Lorentz principle is given some attention. At the end of Chapter 4, the general procedure to model electromagnetic coils is discussed. An approach to obtain the generated magnetic field is given. Also the electric equation governing the behaviour of the coil will be discussed.

Then, in Chapter 5, the phenomenon of permanent magnetism is treated. Some magnetic materials are able to produce a magnetic field in the absence of any currents. The chapter starts with a short discussion on the origin of permanent magnetism and different types of magnetic materials. Then, the general properties of permanent magnets will be discussed using the hysteresis curve. The stability and behaviour of permanent magnets during operation will be treated in Sections 5.6 and 5.7. The final section deals with the modeling of permanent magnets.

In Chapter 6, the results from the preceding chapter are used to describe the linear and planar actuator. Due to the widespread use of the concepts and theories treated in Chapters 2 through 5, only the planar motor that actually has been build within Philips Applied Technologies will be discussed; the three-phase/four-pole synchronous linear or planar actuator. First, the working principle of the linear actuator is discussed, using a two-dimensional model. Then, different types of magnet arrays and their influence on the actuator behaviour and performance will be discussed. Methods to analyse the generation of forces and magnetic flux are discussed and the commutation principle of the motor is explained. In Section 6.8, the theory is expanded to the three-dimensional planar case.

Finally, in Chapter 7, the main conclusions are drawn and recommendations are given.
Chapter 2

Static Vector Fields

2.1 Introduction

To be able to characterise electric and magnetic vector fields, first the definition and properties of vector fields in general must be discussed. In this chapter the basics of vector analysis and vector calculus will be presented. It is assumed that the reader is familiar with basic vector operations, like vector and dot products, spatial derivatives and integrals, and different types of three-dimensional coordinate systems. These will not be treated in this chapter. The chapter starts with the definition of the scalar and vector fields. Then, in Section 2.3 the basics of vector calculus are discussed and the most important theorems and identities will be treated. Furthermore a classification of the different types of vector fields is given. In the subsequent section the sources of vector fields will be discussed. Finally, the boundary or interface conditions for general vector fields will be derived using the theory presented in the earlier sections.

2.2 Definition of Fields

A field in its most general definition is described as ‘a distribution of any quantity in space’. The field can be time-dependent and it can be defined over the whole space or a specific part of space. Based on the characteristics of the quantity that is studied, two different types of fields are distinguished in this report. These are the scalar field and the vector field. A field can be described mathematically as the function of a set of variables, including time, in a given space. This chapter deals with vector fields described in the Cartesian coordinate system only, but it can be shown that the results of this chapter are independent of the chosen coordinate system [19,21]. If a quantity in every point in a (sub)space can be characterised by a scalar, the accompanying field of this quantity is called a scalar field. A general expression for a static scalar field is given by:

\[ \phi = \phi(r) = \phi(x, y, z) \]  

(2.1)

where \( r \) is a vector in three-dimensional space expressed in the Cartesian coordinates \( x, y, z \). If a quantity in every point in a (sub)space is characterised by a value and direction, the accompanying field is called a vector field. A typical expression for a static vector field is
given by:

\[ F(\mathbf{r}) = f_1(\mathbf{r})\mathbf{e}_x + f_2(\mathbf{r})\mathbf{e}_y + f_3(\mathbf{r})\mathbf{e}_z \] (2.2)

or

\[ F(\mathbf{r}) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \]

where again \( \mathbf{r} \) is a vector expressed in Cartesian coordinates and \( \mathbf{e}_x, \mathbf{e}_y, \) and \( \mathbf{e}_z \) are the three unit vectors in the Cartesian coordinate system. A vector field can be visualised using so-called field lines. A field line is defined as a curve which, at every point through which it passes, has the same direction as the field \( F \). The curve expressed in coordinates \( x, y, z \), corresponding to a Cartesian coordinate system, can be found by solving the set of equations:

\[
\begin{align*}
\frac{dx}{d\alpha} &= g(\alpha)f_1(x(\alpha), y(\alpha), z(\alpha)) \\
\frac{dy}{d\alpha} &= g(\alpha)f_2(x(\alpha), y(\alpha), z(\alpha)) \\
\frac{dz}{d\alpha} &= g(\alpha)f_3(x(\alpha), y(\alpha), z(\alpha))
\end{align*}
\] (2.3)

where \( g(\alpha) \) is an unknown and arbitrary function of the parameter \( \alpha \). For some vector fields this set of equations can be solved analytically, giving one single expression for the field lines in a vector field. In Figure 2.1 field lines of an arbitrary vector field are depicted. The direction of the vector field itself is illustrated with arrows. Every field line is represented by a curve to which the vector field is tangent in every point. Although a field line represents the direction of the vector in every point, it does not give any information about the magnitude. Therefore, more quantitative representations of vector fields are needed. These will be derived in the next sections.

### 2.3 Vector Field Calculus

This section deals with theorems and identities well known in the field of vector calculus. They will be used to derive the main tools to describe a scalar or vector field. With the use of the tools discussed in this section, so-called scalar and vector potentials can be used to uniquely describe scalar and vector fields. These potential functions can be very useful
2.3 Vector Field Calculus

Figure 2.2: Example of a vector field with nonzero divergence (a) and with nonzero curl (b).

for the derivation of an analytical description of a (physical) vector field. In this section the theorems and identities needed in the rest of this report will be briefly discussed. They are stated without any proof (for proofs or treatment of vector operations the reader is referred to [1, Chapter 15-16]).

2.3.1 The Vector Differential Operator

In order to compute differentials of vector or scalar fields, the so-called ‘vector differential operator’ is used. This operator is a symbolic vector and its expression is dependent on the used coordinate system of the field on which it is applied. For example, the vector differential operator, \( \nabla \), for a Cartesian coordinate system is given by:

\[
\nabla = \frac{\partial}{\partial x} e_x + \frac{\partial}{\partial y} e_y + \frac{\partial}{\partial z} e_z
\]

where \( e_x \), \( e_y \), and \( e_z \) are the three unit vectors in the Cartesian coordinate system. The vector differential operator, also-called del or nabla operator, can be applied to scalar as well as vector fields. By applying the vector differential operator to a scalar or a vector field, the gradient, divergence or curl of the corresponding field can be obtained.

Definition 1 (Gradient, divergence and curl)

Let \( \phi \) be a scalar field and \( \mathbf{F} \) be a vector field. Then the gradient, divergence and curl of a scalar or vector field are defined by:

\[
\text{grad } \phi = \nabla \phi
\]

\[
\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}
\]

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F}
\]

Note that the gradient of a scalar field is a vector field, the divergence of a vector field yields a scalar and that the curl of a vector field leads to a vector again. The gradient of the scalar field gives the maximum rate of change, or slope, in any point for which the scalar field is defined. The value of the divergence of a vector field in a certain point is a measure of the rate
Static Vector Fields

at which the field ‘diverges’ or ‘spreads away’ from this point. In Figure 2.2(a) an example of a two-dimensional vector field with a nonzero divergence is depicted. The arrows in the figure give the direction as well as the magnitude of the vector field quantity \( \mathbf{F} \). It is obvious that the vector field diverges from the point \((0,0)\) in all directions. The field magnitude increases with the distance to the point \((0,0)\) and hence the field has a nonzero divergence. Using the mathematical expression for the field and Equation 2.8, it can be concluded that the divergence equals 2 in every point. The curl of a vector field in a certain point can be best explained by the extent to which the vector field ‘swirls’ around this point. In Figure 2.2(b) a typical vector field that possesses a curl is depicted. Again the direction and magnitude of the vector field are indicated by arrows. As can be seen from the figure, the vector field swirls around point \((0,0)\) in counterclockwise direction. The field magnitude increases with the distance from the point \((0,0)\) and therefore the field is said to have a nonzero curl. With the use of Equation 2.9 it can be computed that the curl of this vector field is constant and can be expressed as \(2e_z\), where \(e_z\) is the unit vector directed out of the paper.

2.3.2 Identities and Theorems

There are numerous identities involving \(\text{grad } \phi\), \(\text{div } \mathbf{F}\) and \(\text{curl } \mathbf{F}\). The important identities in the analysis of scalar and vector fields are collected in the following theorems.

**Theorem 1 (Identities)**

Let \(\phi\) be a scalar field and \(\mathbf{F}\) a vector field, all assumed to be sufficiently smooth such that all the partial derivatives in the identities are existent and continuous. Then the following identities hold:

\[
\begin{align*}
(a) & \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0 \\
(b) & \quad \nabla \times (\nabla \phi) = 0
\end{align*}
\]

The first identity states that the divergence of the curl of an arbitrary vector field \(\mathbf{F}\) yields zero in all cases. The second identity says that the curl of the gradient of an arbitrary scalar field \(\phi\) is always zero. The following two theorems are widely used in vector calculus, to rewrite or simplify expressions involving vector fields or integrals of vector fields.

**Theorem 2 (Divergence Theorem)**

Let \(V\) be a regular, three-dimensional domain whose boundary \(S\) is an oriented, closed surface with unit normal \(\mathbf{n}\) pointing out of \(V\). If \(\mathbf{F}\) is a smooth vector field defined on \(V\) then

\[
\int_V \nabla \cdot \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot d\mathbf{S}
\]

(2.12)

where the vector \(d\mathbf{S}\) has a magnitude of an infinitesimal surface element \(dS\) and a direction normal to the surface element.

In words, this theorem states that the surface integral out of a closed surface \(S\) of a vector field \(\mathbf{F}\) is equal to the volume integral of the divergence of \(\mathbf{F}\) over the volume \(V\) enclosed by \(S\). Its most important use is the conversion of volume integrals of the divergence of a vector field into integrals over a closed surface.
2.3 Vector Field Calculus

**Theorem 3 (Stokes’ Theorem)**

Let $S$ be a piecewise smooth, oriented surface in three-dimensional space, having unit normal $\mathbf{n}$, and having a boundary $C$ consisting of one or more piecewise smooth, closed curves with orientation inherited from $S$. If $\mathbf{F}$ is a smooth vector field defined on an open set containing $S$, then

$$\oint_C \mathbf{F} \cdot d\ell = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \quad (2.13)$$

where the vector $d\mathbf{S}$ has a magnitude of an infinitesimal surface element $dS$ and a direction normal to the surface element.

This theorem states that the line integral around a closed curve $C$ of a vector field is equal to the surface integral of the curl of the vector field $\mathbf{F}$ through any surface of which $C$ is the rim.

**Theorem 4 (Helmholtz Theorem)**

A vector field is uniquely defined (within an additive constant) by specifying its divergence and its curl.

To understand this theorem, consider an arbitrary vector field $\mathbf{F}$. A vector field can always be decomposed into two terms: the gradient of a scalar function and the curl of a vector function.

$$\mathbf{F} = \nabla \phi + \nabla \times \mathbf{G} \quad (2.14)$$

The divergence of the vector field $\mathbf{F}$ from Equation (2.14) equals:

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \phi) + \nabla \cdot (\nabla \times \mathbf{G}) \quad (2.15)$$

The second term on the right hand side is zero, from the identity given in Equation 2.10. The first term is in general a nonzero scalar function, which is denoted by $\rho$. Thus the divergence of the vector field is $\nabla \cdot \mathbf{F} = \rho$. The curl of the vector field in Equation (2.14) equals:

$$\nabla \times \mathbf{F} = \nabla \times (\nabla \phi) + \nabla \times (\nabla \times \mathbf{G}) \quad (2.16)$$

Now the first term on the right hand side is zero, due to the identity given in Equation 2.11. The second term can generally be seen as a nonzero vector, which will be denoted by $\mathbf{J}$. Then the curl of the vector field $\mathbf{F}$ is defined as $\nabla \times \mathbf{F} = \mathbf{J}$. Because $\rho$ and $\mathbf{J}$ are unique descriptions of the divergence and curl of the vector field $\mathbf{F}$, together they describe the entire vector field $\mathbf{F}$ uniquely, within an additive constant. Note that an arbitrary constant $c$ can be added to the field $\mathbf{F}$ without changing the divergence or the curl.

### 2.3.3 Field Types

In the previous section it is stated that a vector field is uniquely defined by its divergence and curl. The divergence and the curl of a vector field can be both zero or nonzero. A vector field is classified according to these criteria and therefore four different classes exist. If the divergence of a vector field is zero, the vector field is said to be solenoidal, and if the divergence is nonzero the field can be classified as nonsolenoidal. Furthermore, if a vector field has zero
curl, it is called irrotational. On the other hand, vector fields that have a nonzero curl are said to be rotational. From the Helmholtz Theorem it can be concluded that if the curl of a vector field is zero (the field is irrotational), the field can be described solely by the gradient of a scalar field. Likewise, the vector field can be described by solely the curl of another vector field, if the divergence of the field is zero (the field is solenoidal). This observation defines the scalar potential and the vector potential of a vector field.

Definition 2 (Scalar potential)
Let $F$ be a smooth vector field in the domain $D$ and $\phi$ be a scalar field defined on $D$. If the vector field is irrotational, and thus $\nabla \times F = 0$, the field can be described solely by the gradient of a scalar field. Then this gradient $\nabla \phi$ is defined as the scalar potential of $F$, if the following relation holds:

$$F = \nabla \phi$$  \hspace{1cm} (2.17)

Definition 3 (Vector potential)
Let $F$ be a smooth vector field in the domain $D$ and $\phi$ be a scalar field defined on $D$. If the vector field is solenoidal, and thus $\nabla \cdot F = 0$, the field can be described solely by the curl of a vector field. Then the vector field $G$ is defined as the vector potential of $F$, if the following relation holds:

$$F = \nabla \times G$$  \hspace{1cm} (2.18)

If the divergence and the curl of a vector field are both nonzero, it does not have an explicit scalar or vector potential. If both potential functions are zero, from Equation 2.14 it can be concluded that the vector field can only be constant in space. The scalar and vector potentials can be considered as the origin or sources of 'sudden changes' in the direction or magnitude of the vector field. So, if a vector field has a scalar potential, the source for the existence of the field has a scalar origin. A vector field that possesses a vector potential is created by sources that have a vectorial origin. If a vector field does not have a scalar or vector potential, it can have sources of scalar as well as vectorial origin or it can have no sources at all. All four types of vector fields are gathered in the following definition.

Definition 4
In total four different types of vector fields can be defined, based on their divergence and curl:

- A nonsolenoidal, rotational vector field, $\nabla \cdot F \neq 0$, $\nabla \times F \neq 0$. This is the most general vector field possible and it has no scalar or vector potential. The field has both a scalar and a vector source.

- A nonsolenoidal, irrotational vector field, $\nabla \cdot F \neq 0$, $\nabla \times F = 0$. This vector field has a scalar potential and only a scalar source.

- A solenoidal, rotational vector field, $\nabla \cdot F = 0$, $\nabla \times F \neq 0$. This vector field has a vector potential and only a vector source.

- A solenoidal, irrotational vector field, $\nabla \cdot F = 0$, $\nabla \times F = 0$. This vector field possibly has no scalar or vector potential. The field has neither scalar nor vector sources.
2.3.4 Conservative and Nonconservative Fields

Another classification of vector fields can be based on the conservativeness of vector fields. The conservativeness of a vector field is related to the circulation of the vector field. The left hand side of Equation (2.13) the line integral of the tangential component of $\mathbf{F}$ around $C$ is also referred to as the circulation of $\mathbf{F}$ around $C$.

**Definition 5 (Circulation of a vector field)**

Let $C$ be a closed curve and $\mathbf{F}$ a smooth vector field defined on an open set containing $C$. Then the circulation of the vector field $\mathbf{F}$ around $C$ is defined by:

$$\oint_C \mathbf{F} \cdot d\ell$$

(2.19)

The circulation of a vector field around any closed path can be zero or nonzero, depending on the type of vector field. Both types are important in the analysis of fields and are defined as follows [19]:

**Definition 6 (Conservative and nonconservative fields)**

A vector field whose circulation around any arbitrary closed path is zero is called a conservative field. In a force field, the line integral represents work. A conservative field in this case means that the total work done by the field or against the field on any closed path is zero.

A vector field whose circulation around any arbitrary closed path is nonzero is called a non-conservative field. In terms of forces, this means that moving in a closed path requires net work to be done either by the field or against the field.

With the use of the Theorem of Stokes and the definition of the circulation another condition can be formulated which determines if a vector field is conservative.

$$\oint_C \mathbf{F} \cdot d\ell = \int_S \nabla \times \mathbf{F} \cdot dS = 0 \quad \rightarrow \quad \nabla \times \mathbf{F} = 0$$

(2.20)

However the circulation of a vector field can be computed to check the conservativeness another, often much easier, option is to compute the curl of the vector field. From this result it can also be concluded that a field that is irrotational is also conservative and vice versa, because they are both defined by the curl of the vector field. From Definition 6 it can be concluded that a conservative vector field can be used to represent a force field with an associated energy. If the field is conservative, it will have zero curl. Therefore, with the use of the Helmholtz Theorem it can also be concluded that the vector field can be described by a scalar potential only. This potential $\Phi$ of the vector field equals the potential energy of the force. The line integral along a open curve in a conservative vector field equals the potential energy difference between the end points of the curve.
2.4 Sources of Vector Fields

In Section 2.3.3 a classification in vector fields is made based on the existence of the divergence and curl of a vector field. These properties represent sources for the vector field that can be of scalar or vectorial origin. This section gives a more elaborate discussion on vector field sources.

The divergence of a vector field is associated with scalar sources and sinks of the field. A source is a region in space from which field lines emerge and flow outwards, whereas a sink is a region towards which field lines converge. These sources or sinks can be concentrated in a single point or can be distributed along a line, surface, or volume within space. In the same manner, the curl of a vector field can be associated with a source of vectorial origin. Recall that the curl of a vector field is defined by another vector field, meaning it has a magnitude and direction in every point in space. The magnitude of this curl determines the amount of rotation of the vector field around this vector. The direction of the vector quantity determines whether the vector field swirls in clockwise or counterclockwise direction. Again, the curl of a vector field can be limited to a point, but it can also be distributed along a line, surface, or volume. Below a quantitative expression for the scalar and vectorial sources of vector fields is derived using the Stokes’ and Divergence Theorems.

A vector field is described in terms of its divergence and curl, also called the differential form of a vector field. With the use of the Divergence Theorem and Stokes’ Theorem, also an integral form of a vector field can be derived. In Section 2.3.2 it is shown that the divergence of a vector field can be expressed as \( \nabla \cdot \mathbf{F} = \rho \), whereby \( \rho \) may be considered as a source for the vector field of scalar origin. Substituting this relation in the Divergence Theorem yields:

\[
\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, dV = \int_V \rho \, dV \tag{2.21}
\]

In this representation, the parameter \( \rho \) can be regarded as the volume density of a certain quantity in the vector field. The right hand side of Equation 2.21 therefore equals the total amount of quantity \( Q \) inside the volume \( V \). However, the total amount can also be located in a point or distributed along a line or surface. In these cases, the right hand side of the Equation 2.21 must be transferred into the appropriate integral form. The curl of a vector field can be described by the relationship \( \nabla \times \mathbf{F} = \mathbf{J} \). Substituting this relation in Stokes’ Theorem yields:

\[
\oint_C \mathbf{F} \cdot d\ell = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{2.22}
\]

Now the parameter \( \mathbf{J} \) can be regarded as a surface density of the flow of the vector field \( \mathbf{J} \) through the surface \( S \). Therefore, the right hand side of the relation equals the total flow of the vector field through the surface \( S \). Again, the appropriate integral form has to be chosen in the case the total flow is located in a point or distributed along a line. Equation 2.21 and 2.22 are known as the integral form of a vector field and are analogous to the differential form discussed earlier. In the same manner as the divergence and curl, both equations describe the vector field uniquely.
2.5 Boundary Conditions

In the previous sections, vector fields are classified into four types. This section deals with the regions of space in which such fields can exist. Normally, all fields in physics are only defined within a certain region and therefore they are terminating on outer boundary surfaces. Conditions within a field itself can also change discontinuously, because of the presence of other types of materials that change the behaviour of the vector field. This section deals with the discontinuities of vector fields across surfaces. To derive analytical relationships of boundary conditions of vector fields, a distinction is made between the behaviour of a vector field component tangential and normal to the surface of discontinuity. As will be shown in this section, the separate components can be related directly to the definition of the divergence and curl of a vector field.

**Tangential Component**

The derivation of the boundary condition for the tangential component of a vector field, with respect to the surface discontinuity, a situation as depicted in Figure 2.3(a) is assumed. This figure shows two regions in space, 1 and 2, that have an interface $S$. It is assumed that the interface has a uniformly distributed scalar quantity density $\rho_s$ and a vector flow density $J_s$, which is directed into the paper. The flow of quantity is directed into the paper, as depicted in the figure. The interface has normal components $n$ and $-n$ at either sides. The vector field $F$ exists at both sides of the surface with directions $F_1$ and $F_2$. The curl of the vector field is defined as $\nabla \times F = J$, and from Section 2.3.3 we know that also the following relation holds:

$$\oint_C F \cdot d\ell = \int_S J \cdot dS \quad (2.23)$$

This relation must, of course, also hold across the interface surface. To evaluate Relation 2.23, consider Figure 2.3(b) that shows a closed loop $abeda$ over an infinitesimal interface element $dS$. The loop has two sides parallel and infinitely close to the boundary surface.

$$\oint_{abeda} F \cdot d\ell = \int_S J \cdot dS \quad (2.24)$$

The right hand side of Equation 2.23 equals the total flow of quantity through the loop $abeda$. The scalar line flow density is defined by $J_s$. Note that, in general, $J_s$ is a flow distributed.
over a surface. However, here the flow is distributed over a line and therefore the right hand side of Equation (2.24) is integrated over the line \( ab \). Allowing the distances \( bc \) and \( da \) to tend to zero, the total contribution due to this part of the contour is zero. Only the integration along \( ab \) and \( cd \) contributes to the left-hand side of (2.24). Finally, the product \( \mathbf{F} \cdot d\ell \) means that only the tangential components are used and Equation (2.24) becomes:

\[
\int_{ab} F_1 t d\ell - \int_{cd} F_2 t d\ell = \int_{ab} J_s d\ell \tag{2.25}
\]

Integrating over the two segments \( ab \) and \( cd \) and setting \( ab = cd \), we get:

\[
F_1 t - F_2 t = J_s \tag{2.26}
\]

This is the first condition at the interface. The discontinuity of the tangential component of a vector field is equal to the local surface flow density at the interface. The flow density responsible for the discontinuity is always directed perpendicular to the local tangential component. It can also be seen that this flow density is responsible for a change in magnitude and direction of the vector field. Therefore local flow density can be seen as a vectorial source of the vector field, see Section 2.4.

**Normal Component**

For the derivation of the boundary condition for the normal component of the vector field, the divergence of the vector field is used. The divergence of the vector field is given by \( \nabla \cdot \mathbf{F} = \rho \). In this case the following relation also holds for the vector field, see Section 2.3.3:

\[
\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \rho \, dV \tag{2.27}
\]

Now consider Figure 2.4 where again a part of the interface between two regions in space is depicted. Again the interface has a uniformly distributed scalar quantity density \( \rho_s \) and a vector flow density \( \mathbf{J}_s \), directed into the paper. The vector field on both sides of the interface is defined by \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). To apply Relation (2.27) to the interface an infinitesimal cylindrical
volume at the interface is defined, as depicted in the figure. The right hand side in Equation [2.27] is equal to the total quantity enclosed in the volume \( V \). The quantity is located entirely on the interface between the two regions on the surface \( S \) within the cylinder:

\[
\int_V \rho \, dV = \rho_s S
\]  

(2.28)

Let the height of the cylinder tend to zero, so the flux through the lateral surface of the cylinder equals zero. Then the total flux of the vector field \( \mathbf{F} \) out of the volume \( V \) is defined by the fluxes through the surface \( S_1 \) and \( S_2 \) only. Equation [2.27] then becomes:

\[
\int_{S_1} \mathbf{F} \cdot d\mathbf{s}_1 + \int_{S_2} \mathbf{F} \cdot d\mathbf{s}_2 = \rho_s S
\]  

(2.29)

The vector product \( \mathbf{F} \cdot d\mathbf{S} \) means that only the normal components of the vector field \( \mathbf{F} \) are taken into account:

\[
\int_{S_1} F_{1n}d\mathbf{s}_1 - \int_{S_2} F_{2n}d\mathbf{s}_2 = \rho_s S
\]  

(2.30)

The second term on the left hand side becomes negative because the normal of the surface \( S_2 \) has opposite direction. By setting \( S_1 = S_2 = S \) the boundary condition for the normal component of a vector field along an interface discontinuity becomes:

\[
F_{1n} - F_{2n} = \rho_s
\]  

(2.31)

Hence, the change, or discontinuity, in the normal component of a vector field at the interface between two regions is equal to the local surface quantity density at that interface. Again the direction and magnitude of the field change, and hence, the charge density can be regarded as a scalar origin for vector fields.

**Refraction**

The change in direction of a vector field at a boundary or an interface is called refraction. In Figure [2.5] the refraction of the vector field \( \mathbf{F} \) is clearly visible. With the use of this figure the following expression for the refraction can be obtained:

\[
\tan \theta_1 = \frac{F_{1t}}{F_{1n}} \quad \text{and} \quad \tan \theta_2 = \frac{F_{2t}}{F_{2n}}
\]

(2.32)

\[
\tan \frac{\theta_1}{\theta_2} = \frac{F_{1t}F_{2n}}{F_{2t}F_{1n}}
\]

In the next chapter it will be shown that the refraction is directly related to the material properties at the interface.
Figure 2.5: Refraction at an interface.
Chapter 3

Electromagnetic Field Equations

3.1 Introduction

In the previous chapter vector algebra and vector calculus were used to define general vector fields. Different classes of vector fields were discussed and general interface conditions were derived. In the present chapter the theory from the previous chapter will be used to characterise the electromagnetic field. The chapter starts with the equations of Maxwell which describe the electromagnetic field in a unique way. Because this set of equations is valid for the whole range of electromagnetic phenomena, it can be seen as the most general description of the electromagnetic field. However, it will become obvious that the equations in this form are rather complicated and that in some cases simplifications can be used find an analytical solution of the electromagnetic field. In this context, a few special field conditions will be mentioned which will considerably simplify the equations. The second part of this chapter consists of the derivation of boundary and interface conditions for the electromagnetic field. The theory and analytical representation of the electromagnetic field treated in this chapter will be used in the remainder of this report to find solutions for specific electromagnetic problems.

3.2 Maxwell’s Equations

The field equations of Maxwell presented in this section are stated without any proof to keep the discussion short. The intention is to state the most important equations in the electromagnetic field theory, in order to use them instead of proving them. Numerous text books deal with the derivation and background of the Maxwell equations. For an elaborate discussion on (static) electromagnetic fields the reader is referred to [19, Chapter 3-10]. First, the equations of Maxwell will be discussed, followed by constitutive and continuity equations.

The electric and magnetic fields are all governed by one set of equations, that define the curl and divergence of the field quantities and are known as the Maxwell’s equations. The Maxwell equations were first presented by James Clerk Maxwell in 1864. The importance of these equations follows from the fact that they uniquely define the link between electric and magnetic fields, constituting the electromagnetic field. The quantities that describe the electric vector field are the electric field strength $\mathbf{E}$ and the electric flux density $\mathbf{D}$. Likewise, the magnetic field is described by the magnetic flux density $\mathbf{B}$ and the magnetic field strength $\mathbf{H}$. 
The exact definitions of these quantities, from a physical point of view, are given in Appendix A. With these definitions, the equations of Maxwell can be written in differential as well as integral form:

\[
\oint_C E \cdot d\ell = -\frac{\partial}{\partial t} \int_S B \cdot dS \quad \nabla \times E = -\frac{\partial B}{\partial t} \tag{3.1a}
\]

\[
\oint_C H \cdot d\ell = \int_S J \cdot dS + \frac{\partial}{\partial t} \int_S D \cdot dS \quad \nabla \times H = J + \frac{\partial D}{\partial t} \tag{3.1b}
\]

\[
\int_S D \cdot dS = \int_V \rho \, dV \quad \nabla \cdot D = \rho \tag{3.1c}
\]

\[
\oint_S B \cdot dS = 0 \quad \nabla \cdot B = 0 \tag{3.1d}
\]

where \(\rho\) corresponds to the scalar free charge density and \(J\) is the vector free current density at any point in the region. Note that the equations contain four vector variables, \(E, D, B,\) and \(H\) which all have three components in space. The first two equations are vector equations, which is equivalent to six scalar equations, whereas the latter two are scalar equations. Thus in total there are twelve unknowns and eight scalar equations. Moreover, with the use of the continuity equation

\[
\oint_S J \cdot dS = -\frac{\partial}{\partial t} \int_V \rho \, dV \quad \nabla \cdot J = -\frac{\partial \rho}{\partial t} \tag{3.2}
\]

it can be shown that Maxwell’s equations are dependent [19]. Only the first two equations are independent, reducing the number of independent scalar equations to six. The continuity equation used here is also known as the law of conservation of electric charge. It is a result of the property that electric charge cannot be created or destroyed. The total flow of charge through a closed surface \(S\) equals the change of total charge concealed within the volume \(V\) spanned by \(S\). This is exactly what is described in Equation 3.2. The dependency of the equations of Maxwell requires two extra independent vector equations to solve the system. The field vectors \(E\) and \(D\) and also \(B\) and \(H\) are related by the properties of the material in which they are present. These constitutive properties are given by:

\[
D = \varepsilon_0 E + P \tag{3.3}
\]

\[
B = \mu_0 (H + M) \tag{3.4}
\]

where \(P\) and \(M\) are, respectively, the polarisation and magnetisation vector inside materials, as defined in Appendix A. The polarisation and magnetisation vectors inside a material tend to strengthen the electric or magnetic field, due to the alignment of electric and magnetic dipoles inside the material. The parameters \(\varepsilon_0\) and \(\mu_0\) are the permittivity and the permeability of vacuum. The polarisation and magnetisation vectors are dependent on the electric and magnetic field strength, respectively. Therefore the constitutive relations can also be written as:

\[
D = \varepsilon (E) \, E \tag{3.5}
\]

\[
B = \mu (H) \, H \tag{3.6}
\]
where $\varepsilon$ is the permittivity, and $\mu$ is the permeability of the material. Assuming that the material properties are known, Equations 3.1a and 3.1b together with 3.5 and 3.6 compose a system with twelve unknowns and twelve scalar equations. Hence, the system can be solved. For linear, homogenous and isotropic materials, $\varepsilon$ and $\mu$ are constants. Together with these constitutive equations, the differential form of the Maxwell equations both define the divergence and curl of the electric and magnetic vector field. From the Helmholtz Theorem treated in Section 2.3.2 it can be concluded that the electromagnetic field is defined uniquely. With the use of the constitutive equations, vector fields $\mathbf{E}$ and $\mathbf{H}$, describing the electric field, can be classified in the nonsolenoidal, rotational fields. This means that it has both scalar sources, in the form of charges or charge distributions, and vector sources. The vector source of an electric field is given by the time-dependency of the magnetic flux density. The magnetic field vectors $\mathbf{B}$ and $\mathbf{H}$ are of the nonsolenoidal, irrotational class and only have sources of vectorial origin. These are given by currents or current densities and the time-dependency of the electric flux density.

Now all equations governing the electromagnetic field have been stated, a short discussion on the equations of Maxwell will be given. To explain the equations of Maxwell, first the definitions of the electric and magnetic flux are given:

$$\Phi_D = \int_S \mathbf{D} \cdot d\mathbf{S} \quad (3.7)$$
$$\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{S} \quad (3.8)$$

**Faraday’s law**

Equation 3.1a is also known as Faraday’s law for electromagnetic induction. Together with the definition of electric flux in Equation 3.7, the law of Faraday can be paraphrased as follows. The law states that the line integral of the electric field strength along a closed conductor equals the time derivative of the magnetic flux through the surface spanned by the loop. The total magnetic flux through the surface can change due to a changing magnetic flux intensity $\mathbf{B}$ through the surface or due to change in the surface dimensions. Thus from this law it can be concluded that a time-varying magnetic field creates a time-varying electric field. Faraday’s law is often used to calculate the induced potential or electro motive force (EMF), as will be discussed in Chapter 4.

**Ampère’s Law**

The second equation of Maxwell, Equation 3.1b, is better known as the generalised law of Ampère. It says that the line integral of the magnetic field strength $\mathbf{H}$ along a closed contour equals the current through any surface defined by the contour. The current through the surface $S$ is not equal to solely the surface integral of the vector surface current density $\mathbf{J}$. Also the change in time of the total electric flux through the surface must be accounted for, as a direct consequence of enforcing the continuity equation. The law of Ampère also states that a time-varying electric field generates a time-varying magnetic field. The extension of the law of Ampère with the electric flux term is done by Maxwell and can be used to explain the behaviour and existence of electromagnetic waves. Without the extension of Maxwell this is not possible.
Gauss’ Law

The law of Gauss, Equation 3.1c, is basically a relationship between the sources inside a closed surface and the field they produce through this entire surface. The law of Gauss for an electric field states that the total charge inside a volume \( V \) spanned by a closed volume \( S \) is equal to the surface integral of the electric flux density \( D \) over \( S \).

Nonexistence of Magnetic Monopoles

An analogous relation as the law of Gauss for electric fields must also exist for magnetic fields. But due to the fact that magnetic monopoles do not exist, this relation is a little different. In the electric field positive and negative charges can be isolated, while this is not the case for magnetic fields. A magnetic north and south pole will always coexist within any volume and therefore the total magnetic ‘charge’ inside a volume is always equal to zero. This is described by the fourth Maxwell equation. The magnetic flux through an arbitrary closed surface, the surface integral of the magnetic flux density over \( S \), is always zero. Sometimes Equation 3.1d is also referred as Gauss’ law for magnetic fields.

This completes the discussion on the equations of Maxwell and the physical background of these equations. In the next section some, special cases of the equation of Maxwell will be treated.

3.3 Special Field Conditions

As discussed in the previous section, a system of twelve equations must be solved to find a solution to the general electromagnetic field. Moreover, due to the coupling between the magnetic and electric field and due to the time dependency of these equations, a system of at least six time-dependent partial differential equations must be solved. This is a very complex task. Therefore, analytical solutions for the general electromagnetic field are rather complex, sometimes even impossible to obtain. In some cases, however, the equations of Maxwell can be considerably simplified by assuming special field conditions. This section deals with these special cases of the electric, magnetic, and electromagnetic field. In this section, only the differential form is used to describe the fields, but by using the Divergence and Stokes’ Theorem the integral form can always be obtained.

The Electrostatic Field

Only the electric field is considered, therefore the magnetic field strength and flux density are assumed to be zero. Consider the definition of the electric field, Equation 3.1a and 3.1c and assume static conditions. All time dependencies cancel from the equations. The resulting expressions are known as the equations for the electrostatic field:

\[
\nabla \cdot D = \rho \quad (3.9)
\]
\[
\nabla \times E = 0 \quad (3.10)
\]

Again with the use of the constitutive relations, the field is uniquely defined. If the material properties in which the electric field is present are linear and Relation 3.6 holds, it can be concluded that the static electric field is solenoidal and irrotational. The only sources for the electric field are of scalar nature and are given by charges or charge densities.
The Static Magnetic Field

The magnetic field is uniquely described by Equations \(3.1b\) and \(3.1d\). To obtain the static magnetic field, again all time dependencies must be omitted from the equations. Then the Maxwell equations for the magnetic field reduce to:

\[
\nabla \cdot \mathbf{B} = 0 \quad (3.11)
\]
\[
\nabla \times \mathbf{H} = \mathbf{J} \quad (3.12)
\]

The constitutive equation relating magnetic flux density \(\mathbf{B}\) and the magnetic field strength \(\mathbf{H}\) can be used to classify the magnetic field. By considering the divergence and curl of the magnetic field, it can be concluded that it must be classified into the solenoidal, rotational field. This is the same class as the time-depending magnetic field. This means that its only sources are of vectorial origin, and in the case of the magnetic field, these sources are vector currents or vector current densities.

Magnetic Field of Permanent Magnets

Once again consider the equations describing the magnetic field. Apart from the assumption that the field is static, now also the absence of any current or current density is assumed. This can be the case if only permanent magnets are considered which are obviously current-free. But still a magnetic field exist, which can be described by the following relations:

\[
\nabla \cdot \mathbf{B} = 0 \quad (3.13)
\]
\[
\nabla \times \mathbf{H} = 0 \quad (3.14)
\]

This type of field is of the solenoidal, irrotational class and therefore has zero divergence and is curl-free. It has no sources of scalar or vectorial origin.

3.4 Boundary Conditions

The space in which the electromagnetic field usually exists is bounded by external boundaries. On these boundaries the field equations have to satisfy certain boundary conditions. Moreover, the electromagnetic field has to satisfy so-called interface conditions across interfaces between two different materials. These boundary or interface conditions will be discussed in this section using the theory presented in Section 2.5.

3.4.1 Interface Conditions between two Materials

Now the field equations are given by the Maxwell equations, it is possible to derive the general interface conditions between two materials in the electromagnetic field. If the electromagnetic field is defined in a certain domain containing materials with different properties, boundary conditions on the interfaces between these materials must be specified in order to solve the entire electromagnetic field. Hence, an interface is defined as an infinitely thin boundary between two materials, with no properties of its own. On each side of the interface a material with certain properties is present. The derivation of the interface conditions for the electromagnetic field can be found in Appendix B. During the derivation of the interface conditions it is assumed that the material properties are constant and given by Equations...
Electromagnetic Field Equations

**Table 3.1:** Interface conditions for the general electromagnetic field.

<table>
<thead>
<tr>
<th>Component</th>
<th>Electric field</th>
<th>Magnetic field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential</td>
<td>$E_{1t} = E_{2t}$</td>
<td>$H_{1t} - H_{2t} = J$</td>
</tr>
<tr>
<td></td>
<td>$D_{1t} = D_{2t}$</td>
<td>$B_{1t} - B_{2t} = J$</td>
</tr>
<tr>
<td>Normal</td>
<td>$D_{1n} - D_{2n} = \rho_e$</td>
<td>$B_{1n} = B_{2n}$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_e$</td>
<td>$\mu_1 H_{1n} = \mu_2 H_{2n}$</td>
</tr>
</tbody>
</table>

However, the dependency of these properties on the electric or magnetic field strength is omitted in the notation. The interface conditions resulting from Appendix B are summarised in Table 3.1. From the table the following conclusions can be drawn:

- The tangential component of the electric field strength $E$ is continuous across the interface surface, regardless of the charge densities on the interface between two materials.
- The tangential component of the electric flux density $D$ is discontinuous across the interface. The discontinuity is equal to the ratio of the permittivities of the materials.
- The normal component of the electric field strength $E$ and flux density $D$ are discontinuous across the interface. The discontinuity of the flux density equals the electric charge density $\rho_e$. In the absence of charge densities, the normal component of the electric flux density is continuous across the interface.
- The tangential component of the magnetic field strength $H$ and flux density $B$ are discontinuous across the interface. The discontinuity of the field strength is equal to the surface current density $J$. If surface currents are absent on the interface, the normal component of the magnetic field strength is continuous across the interface.
- The normal component of the magnetic flux density $B$ is continuous across the interface, regardless of the surface current densities present on the interface between two materials.
- The normal component of the magnetic field strength $H$ is discontinuous across the interface. The discontinuity is equal to the ratio of permeabilities of the materials.

Furthermore, it can be concluded that the interface conditions for the electric field do not depend on the magnetic field and vice versa. The interface conditions stated in this section are valid for the general electromagnetic field, so also for the special cases treated in Section 3.3.

### 3.4.2 Refraction

With the interface conditions from Table 3.1 an expression for the refraction of the electric and magnetic field can be given. Recall from Section 2.5 and Figure 2.5 that the refraction is given by the ratio of the angle of incidence $\theta_1$ and the angle of refraction $\theta_2$:

$$\tan \theta_1 \tan \theta_2 = \frac{F_{1t} F_{2n}}{F_{2t} F_{1n}}$$  (3.15)
3.5 Equivalent Charge Density

Assuming there are no charge densities, the refraction of the electric field across an interface can be expressed as:

\[ \frac{\tan \theta_1}{\tan \theta_2} = \frac{E_1 E_{2n}}{E_2 E_{1n}} = \frac{E_2}{E_1} \frac{\varepsilon_1}{\varepsilon_2} \]  \hspace{1cm} (3.16)

So, the refraction of the electric field is determined by the ratio of permittivities. Likewise, the refraction of the magnetic field across an interface in the absence of surface current densities is given by:

\[ \frac{\tan \theta_1}{\tan \theta_2} = \frac{H_1 H_{2n}}{H_2 H_{1n}} = \frac{H_2}{H_1} \frac{\mu_1}{\mu_2} \]  \hspace{1cm} (3.17)

For this case, the refraction of the magnetic field is determined by the ratio of the permeabilities of the materials. In physics, the permeabilities of materials can differ by a great amount, allowing for large refraction angles of the magnetic field. For both fields it holds that if the permittivity or permeability increases across the interface, the direction of the field will rotate in the direction of the interface. If the permittivity or permeability decreases across the interface, the field direction bends away from the interface.

3.5 Equivalent Charge Density

Sometimes it is convenient to consider the effect of a boundary as being due to charges or currents that lie along the boundary line or surface. These charges or currents, that do not actually exist but have the same effect on the field distribution as the boundary itself, are called equivalent charge or current densities. Although both representations are equivalent, current densities are more difficult to handle and therefore will not be treated in this section.

Assume a magnetic field across a boundary or interface \( I \) as given in Figure 3.5. At any point on the boundary, let \( H_n \) be the normal component of an externally applied magnetic field intensity and let \( H'_n \) be the normal component of the magnetic field produced by the equivalent surface charge density \( \rho_{ms} \) present on the surface. The latter term acts in the same direction as the externally applied field. To get the total magnetic field on both sides of the boundary, the two individual fields must be summed. Since the normal component of the magnetic flux density vector is continuous across a boundary, see the boundary conditions in Table 5.1 the following relation must hold:

\[ \mu_1 (H_n + H'_n) = \mu_2 (H_n - H'_n) \quad \iff \quad H'_n = \left( \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \right) H_n \]  \hspace{1cm} (3.18)
The equivalent surface charge density can be derived easily. The magnetic field $H'_n$ is directed in either direction normal to the boundary and so the surface charge density can be expressed as:

$$\rho_{ms} = 2\mu_0 H'_n$$  \hspace{1cm} (3.19)

By eliminating $H'_n$ from this expression with the use of Relation [3.18] and by defining the total magnetic field in region one as $H_{n1} = H_n + H'_n$ the magnetic surface charge density can be written as:

$$\rho_{ms} = \frac{\mu_0}{\mu_2} (\mu_2 - \mu_1) H_{n1}$$  \hspace{1cm} (3.20)

So now the effect of the boundary is given by the equivalent magnetic surface charge density in terms of the total magnetic field in one region and the permeabilities of the boundary materials. In the absence of electric surface charge densities, the equivalent electric surface charge density $\rho_{es}$ can be defined in the same manner and is given by:

$$\rho_{es} = \frac{\varepsilon_0}{\varepsilon_2} (\varepsilon_2 - \varepsilon_1) E_{n1}$$  \hspace{1cm} (3.21)
Chapter 4

Electromagnetism

4.1 Introduction

The electromagnetic field is composed of the electric and magnetic field and can be fully described by the equations of Maxwell, as discussed in the previous chapter. The electric field is generated by stationary electric charges or time-varying magnetic fields. The electric field gives rise to the electric force which causes static electricity and drives the flow of electric current in electrical conductors. Forces resulting from the magnetic field are more fundamental forces that arise due to the movement of electrical charge. Magnetism is observed whenever electrically charged particles are in motion. This can arise either from movement of electrons in an electric current, resulting in ‘electromagnetism’, or from the quantum-mechanical orbital motion and spin of electrons, resulting in what are known as ‘permanent magnets’ which are discussed in Chapter 5.

The present chapter deals with electromagnetism, and especially the phenomena that can be observed in the electromagnetic field. The electromagnetic field is produced by the flow or movement of electric charges through or along the surface of electrical conducting materials. First these conductors and their characteristics are discussed. In Section 4.3 various electromagnetic phenomena are presented. The resulting magnetic field of a current-carrying conductor, Lorentz force, electromagnetic induction, and Eddy currents will be discussed successively. The law of Ampere or Biot-Savard’s law can be used to describe the magnetic field produced by moving charges or currents. The Lorentz force equation is derived which relates the force exerted on a current-carrying conductor in a magnetic field. Faraday’s law is used to explain the phenomena of electromagnetic induction and Eddy currents. Finally, in Section 4.4 techniques to solve the electromagnetic field and to determine the field produced by electromagnets are treated.

4.2 Conductors

Electric conduction exists in materials that allow the exchange of electrons. This current originates from the movement of electrons in the outer electron shells of each atom. The electrons can move randomly from atom to atom and can be considered as an electrical current. If an electric field is applied to a conducting material, a force \( F = QE \) will work on
the electrons. In this case, the movement of electrons inside the material will not be random anymore, but will follow some pattern according to the electric field and field lines. The electrons start to accelerate due to the exerted force. In their motion they will encounter atoms, collide, and slow down. The result is that, in spite of the acceleration due to the electric field, the electrons travel at a fixed velocity, called the drift velocity \( v \). Therefore the macroscopic net current or flow of charges is observed to be constant. The drift velocity is proportional to the applied electric field. The number of available charges inside a material is dependent on the material itself. On the other hand, the current inside a material is proportional to the velocity of charge. The electric field and the current density inside a conductor can therefore be related by:

\[
J = \sigma (E) E \tag{4.1}
\]

where \( J \) is the conduction current density and \( \sigma \) is referred to as conductivity of the material. The conduction current density can be defined in a point by taking the limit of electrical current per unit of surface:

\[
J = n \lim_{\Delta S \to 0} \frac{\Delta I}{\Delta S} \tag{4.2}
\]

where \( n \) is the normal vector of the infinitesimal surface element \( \Delta S \) and \( \Delta I \) is the current trough \( \Delta S \). With this definition, the total current through any surface can be calculated by the integration over the surface:

\[
I = \int_S J \cdot dS \tag{4.3}
\]

This relation indicates that the total current through a surface \( I \) equals the integral of the current density \( J \) over an area \( S \). Because current density is independent of the cross section through which it is defined, it is a more fundamental quantity than current and in the context of electromagnetism it is often more useful. Only when the current is confined to a known cross section, as in the case of current through a wire, it becomes convenient to use the current itself.

### 4.3 Electromagnetic Phenomena

This section deals with four phenomena that can arise in the electromagnetic field. First the law of Biot-Savard is used to describe the magnetic field produced by a current-carrying conductor. Then the forces exerted on current-carrying conductors inside a magnetic field are treated using the Lorentz force equation. The last two sections deal with electromagnetic induction and Eddy currents, which will be explained using Faraday’s law.

#### 4.3.1 Magnetic Field of Moving Charges

In Appendix A the law of Biot-Savard is used to define the magnetic flux density in an arbitrary point in space, as a result of moving charges or electric currents. Of course, this definition can also be used to calculate the magnetic field of a current-carrying conductor.
element. The total magnetic field can be determined by integrating over the total conductor volume. The total magnetic flux density can be expressed as:

\[ B(r) = \frac{\mu_0}{4\pi} \int_V \frac{J(r') \times (r' - r)}{|r' - r|^3} dV \] (4.4)

where \( r \) is the vector directed to the point in which the magnetic flux density must be calculated, \( r' \) is the vector pointing to the source element \( dV \) which has a current density \( J \). If the conductor can be considered infinitely thin, or in other words the conductor is considered to be of the filamentary type, the Biot-Savard law in terms of a current \( I \) through the conductor can be used:

\[ B(r) = \frac{\mu_0}{4\pi} \int_l \frac{I d\ell \times (r' - r)}{|r' - r|^3} \] (4.5)

where \( r \) is the vector directed to the point in which the magnetic flux density must be calculated, \( r' \) is the vector pointing to the source element \( d\ell \) which carries a current \( I \), and \( l \) is the total conductor length. With the use of Equations 4.4 or 4.5, the total magnetic flux density can be computed in an arbitrary point in space. The analytical solution for a straight filamentary conductor element using the law of Biot-Savard is given in [3]. The total flux density distribution of a filamentary wire then can also be computed by summing the contributions of each wire element.

### 4.3.2 Lorentz Force

The most general representation of forces acting on a charged point or particle in an arbitrary point in space is given by the following relation:

\[ F = Q(E + v \times B) \] (4.6)

where \( Q \) is the point charge, \( v \) the speed of the particle, \( E \) and \( B \) are the local electric field strength and the local magnetic flux density, respectively, and \( F \) is the total force on the particle. The total force on a particle can be divided into two components: the electric Coulomb force \( F_C \) and the magnetic Lorentz force \( F_L \).

\[ F_C = QE \] (4.7)
\[ F_L = Qv \times B \] (4.8)

The Coulomb force is dependent on the electric field and is obviously conform the definition of the electric field strength, see Appendix A. The Lorentz force is dependent on the speed of the particle and the local magnetic flux density. As can be seen from Equation 4.6, this force component is perpendicular to the local speed of the particle and the magnetic flux density at that point. Equation 4.8 is known as the Lorentz-Coulomb force equation or sometimes only as the Lorentz force equation. However, the Lorentz force itself is the force generated by the magnetic field only. From Equation 4.8, it can be concluded that the Lorentz force acting on a charged particle is perpendicular to the local magnetic flux density and the direction of motion of the particle.
4.3.3 Electromagnetic Induction

The principle of electromagnetic induction can be explained best by means of an example.

Example 1

Consider an infinitely long densely wound solenoid, through which a current $I$ is flowing, as in Figure 4.1. This solenoid results in a magnetic field, characterised by $B$. In a certain fixed point in space, $P$, with position vector $r_P$, a particle with charge $+Q$ is located. Suppose that the magnetic field induced by the solenoid is constant, due to a constant current $I$, and that the solenoid is moving with respect to the particle which remains in point $P$. From the view of an observer that moves together with the solenoid, a magnetic force is acting on the charge that equals $F = v \times B$. From the view of an observer fixed to the coordinate system of the particle, a changing force is acting on the charge $Q$. However a changing magnetic field is present in point $P$. The definition of the electric field $\mathbf{F} = Q \mathbf{E}$ makes the force to be regarded as a cause from a (changing) electric field. Now consider the situation that the solenoid is fixed, as well as the particle, but now a changing magnetic field is created by changing the current through the solenoid in time, $I(t)$. However the mechanism for changing the magnetic field in point $P$ differs. The conclusions from an observer fixed to the particle remain the same. Because the particle is not moving, an electric field is observed.

From this simple example the conclusion can be drawn that with a time-varying magnetic field, no matter the cause for the variation, an electric field can always be associated. The time-varying electric field is referred to as ‘induced electric field’. The phenomenon itself is called electromagnetic induction. If a charged particle is situated in a static and induced electric field simultaneously, the total force on the particle equals:

$$\mathbf{F} = Q (\mathbf{E}_{\text{st}} + \mathbf{E}_{\text{ind}})$$

(4.9)

where $\mathbf{E}_{\text{st}}$ is the static magnetic field and $\mathbf{E}_{\text{ind}}$ is the induced magnetic field. If a particle is moving with a velocity $\mathbf{v}$ with respect to the source of a magnetic field, the induced electric field and the total force on the particle are given by:

$$\mathbf{E}_{\text{ind}} = \mathbf{v} \times \mathbf{B}$$

(4.10)

$$\mathbf{F} = Q (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(4.11)

which is equal to the Lorentz force Equation. Because currents are sources of magnetic fields, time-varying currents will cause time-varying magnetic fields. Hence, time-varying currents are sources for induced electric fields as well. Assume a current density distribution $\mathbf{J}$, which
4.3 Electromagnetic Phenomena

Figure 4.2: The principle of mutual induction.

can be a function of time as well as position, in an arbitrary volume $V$. The induced electric field then equals [19]:

$$E_{ind} = -\frac{\partial}{\partial t} \left( \frac{\mu_0}{4\pi} \int_V J dV \right) \quad (4.12)$$

where $r$ is the distance between the volume element $dV$ and the point where the induced electric field is being determined. So, if the distribution of time-varying currents is known the induced electric field can be determined in every point. Most of the times no analytical solution can be found, however, due to complex geometries.

**Faraday’s Law**

Faraday’s law is an equation that relates the total electromotive force (EMF) induced in a closed conductor loop and the induced electric field and is given by:

$$\text{EMF} = \oint_C E_{ind} \cdot d\ell = -\frac{d}{dt} \int_S B \cdot dS = -\frac{d\Phi_B}{dt} \quad (4.13)$$

where $E_{ind}$ is the induced electric field, $C$ is the closed conductive contour, $B$ is the magnetic flux density distribution, and $S$ is the surface spanned by the contour. The definition of the magnetic flux, Equation 3.8, is used to rewrite the right hand side of the equation. This law states that the total EMF induced in a closed conductor loop equals the time derivative of the magnetic flux through the surface spanned by the loop. The proof of this law is omitted in this report, but can be found in [28]. The induced electric field, which is the cause of the EMF, is not dependent on the mechanism by which the magnetic field changes. The change can be due to mechanical motion in a magnetic field or time-varying current sources.

**Self and Mutual Inductance**

Consider two coils $C_1$ and $C_2$ shown in Figure 4.2. When a time varying current $I_1(t)$ flows through the first coil, it creates a time-varying magnetic field, characterised by $B$ as well as a time-varying induced electric field, $E_{1,ind}$. The latter field produces an EMF $e_{12}(t)$ in the second coil, given by:

$$e_{12}(t) = \oint C_2 E_{1,ind} \cdot d\ell_2 \quad (4.14)$$
For linear media it is known that the magnetic flux density vector is proportional to the current that causes the magnetic field. So, the flux $\Phi_{12}$ through $C_2$ caused by the current $I_1(t)$ in $C_1$ is also proportional to $I_1(t)$:

$$\Phi_{12}(t) = M_{12}I_1(t) \quad (4.15)$$

where $M_{12}$ is referred to as mutual inductance between the two coils. This constant parameter only depends on the geometry of the coils and the properties of the medium that surrounds the system. Using this definition and Faraday’s law, the induced electromotive force in coil 2 can also be written as:

$$e_{12}(t) = -\frac{d\Phi_{12}(t)}{dt} = -M_{12}\frac{dI_1(t)}{dt} \quad (4.16)$$

If the situation is reversed, so let coil $C_2$ carry a time-varying current and induce a time-varying induced electric field $E_{2,\text{ind}}$, the mutual inductance parameter $M_{21}$ can be defined. It can be proven that $M_{12} = M_{21}$ always holds. So, the mutual inductance parameter describes the relationship between the current in one coil or current loop and the total magnetic flux through a second coil or current loop.

As mentioned at the beginning of this section, a coil which carries a current that varies in time induces a magnetic and electric field. The induced electric field exists everywhere and therefore it also induces an EMF in the coil itself. The process is known as self induction. In analogy with mutual induction, self induction of a coil can be described as:

$$\Phi_{\text{self}} = LI(t) \quad (4.17)$$

$$e(t) = \frac{d\Phi_{\text{self}}}{dt} = -L\frac{dI(t)}{dt} \quad (4.18)$$

where $\Phi_{\text{self}}$ is the total magnetic flux through the coil, $L$ is the self inductance of the coil which is only dependent on the geometry of the system and properties of the medium, and $I(t)$ the time-varying current through the coil. The relation between mutual inductance $M$ between two coils with self inductances $L_1$ and $L_2$ is often written as:

$$M = k\sqrt{L_1L_2} \quad -1 \leq k \leq 1 \quad (4.19)$$

where $k$ is referred to as coupling coefficient. This relationship also learns that the largest possible value of the mutual inductance is the geometric mean of the self inductances.

### 4.3.4 Eddy Currents

In the previous section the induced EMF, or induced voltage can be generated in a loop or any conducting wire, regardless of its shape. If the loop is closed, the induced EMF will result in an induced current. However, Faraday’s law, as written in Equation 4.13, does not require the existence of a physical loop. A changing magnetic field will always produce an induced electric field and if this induced electric field passes through a conductor an induced EMF will be the result. To explain the physics behind Eddy currents, consider Figure 4.3. A changing magnetic field $B(t)$ passes through a conductor of arbitrary shape. Because electrons can move freely inside the conductor, an induced EMF will result from the induced electric field. Due to this induced electric field, electrons start to swirl around the changing magnetic field.
4.4 Modelling Electric Coils

Section 4.3.1 dealt with the magnetic field produced by moving charges. The magnetic field produced by a coil, which is in fact a conductor with multiple windings, can be computed using Equation 4.1. This computation, however, always involves a complicated integral over a cross product between two vectors. In the next section, the magnetic vector potential is introduced which can, in some cases, simplify the solution of the magnetic flux density. This section furthermore treats the voltage equation and flux linkage of an electric coil or electromagnet.

4.4.1 The Magnetic Vector Potential

Because the magnetic flux density has zero divergence in all cases, it can be defined uniquely by a vector potential. The existence of such a potential function is derived in Section 2.3.3 and the definition of the vector potential is given by Definition 3. The magnetic vector potential $A$ is defined as:

$$ B = \nabla \times A $$

(4.20)

The definition of $A$ is based entirely on the mathematical properties of the magnetic flux density vector $B$ and not on its physical characteristics. The magnetic vector potential does not have a simple physical meaning in the sense that it is not a measurable physical quantity like $B$. Once the vector potential is known, the magnetic flux density can always be obtained very easily through Equation 4.20. In Appendix C, an analytical expression for the magnetic vector potential in a region where only a current density $J$ is present is derived as:

$$ -\nabla^2 A = \mu J - \mu \varepsilon \frac{\partial^2 A}{\partial t^2} $$

(4.21)
where all material properties are assumed linear. Now, instead of solving for a vector product, a second order differential equation has to be solved to find an expression for the magnetic flux density.

4.4.2 The Voltage Equation

The electrical equation describing an electric coil is given by:

\[ u = i(t)R + \frac{\partial \Phi_B(t)}{\partial t} \]  

(4.22)

where \( u \) is the potential difference, \( R \) is the total electric resistance of the coil, \( i(t) \) is the current through the coil as a function of time, and \( \Phi_B(t) \) is the total flux linked to the coil. The first term on the right hand side is known as Ohm’s law and the second term is Faraday’s law treated in Section 4.3.3. Because the EMF can be treated as a potential difference or electric voltage, the induction mechanism is included directly into the voltage equation. The flux \( \Phi_B \) is referred to as the total flux linking the coil, meaning the total flux that passes through the loops of the coil. The time derivative of this linked flux or flux linkage determines the induced voltage in the coil. How the flux linkage can be computed is treated in the next section.

4.4.3 Flux Linkage

The total flux linkage of a coil can of course be computed using the definition of flux as given in Section 3.2. However, using Stokes’ theorem, this expression for the flux linkage can be converted into a closed contour integral. The result becomes:

\[ \Phi_B = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\ell \]  

(4.23)

where \( C \) is the contour bounding the surface \( S \) spanned by the coil loops. This relation shows that the total flux through a surface \( S \) equals the line integral of the magnetic vector potential along the contour of the surface. The computation of the magnetic flux density is not necessary to find the flux linkage of the coil. The relation also gives a physical meaning of the magnetic vector potential as a measure of flux.
Chapter 5

Permanent Magnetism

5.1 Introduction

In physics, permanent magnetism is a phenomenon by which materials exert an attractive or repulsive force on other materials. In Chapter 4 it is explained that magnetism arises whenever charged particles are in motion. Permanent magnetism arises due to the orbital motion and spin of electrons in materials. Some well known materials that exhibit easily detectable magnetic properties are iron and certain steels. However, all materials are influenced to some extent by the presence of a magnetic field, although in most cases the influence is too small to detect without special equipment. This chapter starts with a brief discussion of the classes of magnetic materials. Each class of material can be distinguished by its behaviour in the presence of an externally applied magnetic field. In Section 5.3 the theory used to describe the strength of permanent magnets is discussed. Section 5.5 deals with the properties of magnetic materials using the hysteresis curve, which relates the magnetic flux density inside the material with an externally applied field. Also the behaviour of the permanent magnet in dynamic operation is treated. A general approach to derive a model to describe the magnetic field produced by a permanent magnet is discussed in Section 5.8. Finally an analytical description for cuboidal magnets and magnet arrays is given, and the method to include other materials, like iron, in the model, are discussed.

5.2 Classes of Magnetic Materials

To start the discussion on the classification of magnetic materials, first the concepts of the magnetisation vector and magnetic dipole moment must be known. Both concepts are treated in Appendix A. All materials are made out of atoms. Because these atoms consist of a core with negatively charged electrons orbiting around this core, an atom can be regarded as a microscopic magnetic dipole. Most solids are non-magnetic, however, due to the formation of electron pairs with opposite spin within one atom. The magnetic moments due to the spin of the electrons cancel each other and the total magnetic moment of the electron pair equals zero. The magnetic moment of atoms with completely filled electron-shells normally is cancelled totally by electron pairs. Therefore, only atoms with partially filled electron shells can have a net magnetic moment. The way local atomic moments are coupled or linked to each other, provides a way of characterising magnetic materials. The atomic moments may be randomly oriented with respect to each other or form certain interaction patterns. These
interaction patterns result in changes in permeability of materials. Dependent on the sign and magnitude of the permeability, magnetic materials can be classified into diamagnetic, paramagnetic or ferromagnetic materials. All materials can be classified into one of these groups, except vacuum which is considered to be nonmagnetic. In this section, the three classes are discussed. For a more thorough treatment of magnetic materials and permanent magnets in particular, the reader is referred to [23].

**Diamagnetism**

Diamagnetic materials possess a very weak form of magnetism. The magnetic field produced by the material is only present in combination with an externally applied magnetic field. Diamagnetism is the result of a change in orbital motions of the electrons due to an external magnetic field. The macroscopic induced magnetic moment is opposing the external field. In diamagnetic materials under normal conditions, the magnetic moments due to orbiting electrons and electron spins cancel each other. When placed in a magnetic field, the field due to orbiting electrons is slightly smaller than that of spins, causing a magnetic field that opposes the externally applied field. Diamagnetism is found in all materials, but because it is so weak it can only be found in materials that do not exhibit other forms of magnetism. Since their presence slightly reduces the magnetic field, diamagnetic materials evidently have a permeability slightly smaller than one ($\mu_r < 1$). An exception to the weak behaviour of diamagnetism is when a material becomes superconducting. Superconductors are perfect diamagnets, meaning that the relative permeability equals $\mu_r = -1$. The magnetic flux density in superconductors is always zero.

**Paramagnetism**

Paramagnetism is found in materials that possess a relative permeability slightly greater than one. Paramagnetism results from the alignment of magnetic dipoles inside a material with each other and an externally applied field. In contrast to diamagnetic materials, paramagnetic materials have orbital and spin moments that do not cancel each other. Because these magnetic moments are oriented randomly inside the material, the observed external field is zero. When a field is applied, the magnetic moments of the atoms align themselves with the field. In an external field, paramagnetic materials attract and repel like normal permanent magnets. Therefore, the atomic magnetic dipole must be unequal to zero and hence the atoms must have partially filled electron shells. The alignment of atomic magnetic dipoles with the external magnetic field tends to strengthen it. A result of this property is that the permeability of the material is slightly greater one: $\mu_r > 1$.

**Ferromagnetism**

In ferromagnetic materials, the magnetic dipoles do not only align with an externally applied magnetic field, but also tend to align spontaneously. This purely quantum-mechanical effect is called ‘exchange interaction’ [6]. The alignment of dipoles only happens to atoms close to each other. Over long distances the alignment effect vanishes and dipoles will try to anti-align. Unlike other materials, the individual electron spins are oriented and aligned in so-called domains, instead of being randomly oriented. The transition between two domains is called a domain wall which is a surface over which the orientation of the magnetic moment changes abruptly. The magnetisations in different domains have different directions, allowing the
vector sum over the total sample to vanish. This is the reason that normal non-magnetised ferromagnetic materials do not have a net magnetic moment. An ordinary piece of iron generally has little or no net magnetic moment. However, when an external magnetic field is applied, the magnetic domains that are aligned with this field will move their domain walls, causing them to grow at the expense of neighbouring domains. This results in a net magnetic moment, aligned with the externally applied field. Soft ferromagnetic materials will lose the orientation whenever the external field is removed. Hard ferromagnetic materials, however, will remain re-oriented when the field is removed and a net magnetic moment remains present. When a magnetic moment or magnetisation inside a material exists without the presence of an externally applied magnetic field, this piece of material is referred to as permanent magnet.

The magnetisation as a function of the external field is described by the hysteresis curve, as discussed in Section 5.5. Beyond a certain temperature, called the ‘Curie temperature’, the thermal vibrations in the material completely prevent parallel alignment of molecule magnetic moments within a domain and ferromagnetic materials become paramagnetic.

### 5.3 Magnetisation and Demagnetisation

Permanent magnets are characterised by the fact that they possess a net magnetic moment unequal to zero, without the presence of an externally applied magnetic field. The direction of this magnetic moment is the result of anisotropy inside the material. Magnetocrystalline anisotropy exists if the crystal lattice structure of the material has preferred directions for magnetic moments. But also the shape of the sample itself can generate preferred magnetic moment directions, which is called shape anisotropy. Furthermore stress or atomic pair ordering can also result in certain types of anisotropy [26]. If a material possesses anisotropy which results in a net magnetic moment \( \mu_B \), then its magnetisation \( M \) is given by, see Appendix A:

\[
M = \lim_{dV \to 0} \sum \frac{\mu_B}{dV}
\]

(5.1)

where \( dV \) is a volume element inside the material. When a magnetised material has surfaces through which flux lines emerge or enter with a normal component, so-called ‘free’ or surface poles exist at these surfaces. This means that the surfaces get magnetically ‘charged’, due to the magnetisation of the material. Dependent on the type of pole, a surface can be treated as a north or as a south pole. A magnetic field always emanates from a north pole and terminates at a south pole. This field can travel inside as well as outside the material and the field that passes through the material itself opposes its magnetisation, as depicted in Figure 5.1. The magnetisation vector in the sample results in magnetic poles at both ends of the magnetic sample. These poles induce a so-called demagnetising magnetic field \( H_{dm} \) in the sample. It is important to distinguish the external magnetic field strength \( H_{ext} \) that exists in the absence of any magnetised material and the total or internal magnetic field strength \( H \). The latter is given by:

\[
H = H_{ext} + H_{dm}
\]

(5.2)

The relation between the magnetic flux density, the total magnetic field strength, and the magnetisation inside a material, given by Equation 5.3 then still holds:

\[
B = \mu_0 (H + M)
\]

(5.3)
5.4 Equivalent Magnetic Charge and Current Density

From the law of Ampere we know that a magnetic field is generated by moving currents. In permanent magnet however, it is obvious that electric currents are absent. It can be shown that the magnetisation inside a material can be described by so-called magnetisation currents using the following relations:

\begin{align*}
\nabla \times M &= J_m \\
M \times n &= J_{ms}
\end{align*}

where \( J_m \) is the equivalent magnetic volume current density, \( J_{ms} \) is the equivalent magnetic surface current density, and \( n \) the surface normal containing \( J_{ms} \). For proofs see [6, Section 1.3] and [19, Section 9.2]. The magnetisation inside a material may also be expressed in terms of so-called magnetic charge densities which are defined by:

\begin{align*}
\nabla \cdot M &= \rho_m \\
M \cdot n &= \rho_{ms}
\end{align*}

where \( \rho_m \) is the magnetic volume charge density, \( \rho_{ms} \) is the magnetic surface charge density, and \( n \) the surface normal containing \( \rho_{ms} \). From these relations it can be concluded that a magnetised material can always be modelled by a surface and a volume current density. The volume density only exists if the magnetisation within the material is nonuniform. Actually, surface charge densities are the source for the formation of the magnetic poles of a magnet. By using Relation 5.6 an expression for the magnetisation inside a magnet and the magnetic field produced by a magnet can be derived, as will be shown in Section 5.8.

5.5 Hysteresis

The behaviour of ferromagnetic materials, when placed in an externally applied magnetic field, is examined using the magnetic hysteresis loop of the material. In such a loop the magnetic flux density of a ferromagnetic sample as a function of an externally applied field is given. A typical example of such a hysteresis curve is given in Figure 5.2. Figure 5.2(a) gives the magnetic flux density in a hard ferromagnetic material as a function of the externally applied field strength. Figure 5.2(b) gives the magnetisation inside the material at corresponding instances. Normally, the flux density, magnetisation, and field strength are local...
Figure 5.2: The typical hysteresis loop for magnetic materials.
variables and can vary throughout the material. However, in the figure the magnetisation and flux density are regarded as an average rather than a local property in a domain. Consider a magnetic material in demagnetised state, so $B = H = M = 0$. Now, an external magnetic field given by $H_{\text{ext}}$ and slowly increasing in strength is applied to the ferromagnetic sample. The application of this (initially weak) field produces motion of domain walls to expand the volume of the domains that are aligned with $H$. This process results in a net magnetic moment, and hence the material gets magnetised. The magnetisation will produce a magnetic flux density $B$ in the material. The initial magnetisation curve is drawn as line $Oa$ in both figures. The slope of the curve decreases as the applied field strength increases, because there are fewer domains left to align. When most domain wall motion is completed, some domains remain that have nonzero components of magnetisation perpendicular to the field direction. The magnetisation of these domains must be rotated into the field direction to minimise the total potential energy. The domains will keep rotating until they are all aligned with the local magnetic field. At this point, $a$ in Figure 5.2, the material gets saturated. When a material is saturated, an increasing magnetic field $H$ does not result in an increased magnetisation vector $M$ inside the material, as can also be seen from Figure 5.2(b). In a saturated material, the total magnetic flux density $B$ only increases due to the increasing magnetic force: $dB = \mu_0 dH$.

When the magnetic domains inside the material rotate, a kind of ‘friction’ exists between them. If the magnetic field is slowly decreased from the saturation point, the domains cannot rotate to their original positions, because they cannot overcome this friction. After the magnetic force has reached zero again, a certain amount of magnetisation remains present in the material. The flux density remaining in the sample, when the applied field is zero, is called the remanent flux density $B_r$ or the magnetic remanence. To reduce the magnetic flux density to zero, a magnetic field in the opposite direction has to be applied. This process is called demagnetisation. The reverse magnetic field needed to restore the flux density inside the sample to zero is called the coercive field intensity $H_c$ or the magnetic coercivity. The intrinsic coercivity $H_{ic}$ of the magnetic material is equal to the magnetic field required to reduce the magnetisation in the sample to zero. The remanence inside a magnet is a measure of its strength while the coercivity of a magnet is a measure for the magnet’s ability to maintain its strength in the presence of demagnetising fields. Further increasing the demagnetising field causes a negative flux density. When the domains have relaxed to a random pattern at $H_c$, they now start to align with the magnetic field in opposite direction. Eventually, all domains will be aligned with this field and saturation occurs again. Cyclic application of an external field causes the magnetic material to respond as described by the loop $aB_rH_bB'_rH'_c$. This is called the hysteresis loop and is caused by the ‘friction’ between the magnetic domains inside the material. The total area within the hysteresis loop is a measure for the amount of energy that is used to complete one loop.

The hysteresis loop gives the relation between $B$ and $H$ and so the slope of the curve at any point gives the permeability of the ferromagnetic material at that field level. The slope depends on the location of the curve, and therefore the behaviour of the material depends on the history of the magnetisation inside the material. The permeability of ferromagnetic materials in general is a nonlinear function of the magnetic field intensity, as is also stated in Section 3.2.
5.6 Stability of Permanent Magnets

To examine the suitability of a type of permanent magnet for a certain application, its stability must be observed. The stability of the permanent magnet is defined as the dependency of the magnetic field it produces on external parameters like time, temperature, and demagnetising fields. For each application, different levels of stability may be allowed for the permanent magnet to fulfill its purpose. Any permanent magnet will be demagnetised if it is exposed to certain conditions such as high temperatures or demagnetising fields close to the coercivity of the magnetic material. A few different processes that can alter the strength of a magnet can be distinguished.

Any magnet is likely to lose strength slowly in time. Some magnetic materials lose strength faster than others. These effects can be measured in the first few seconds up to hours and even days after magnetising the material. This process is usually described as magnetic viscosity and referred to as natural ageing. Artificial means of speeding up the natural ageing process, such as deliberately demagnetising the permanent magnet by a small amount or exposing them to a temperature cycle, can be used to stabilise the ageing process. When the temperature of a permanent magnet changes, there is usually a reversible decrease of magnetisation with increasing temperature. The magnet returns to its original state and strength after the temperature has reached the initial value. If a magnet is cooled, the magnetisation will initially increase reversibly. In addition to reversible temperature changes in magnetisation, there can be irreversible losses. These losses may be caused by the accelerated natural ageing process. The domain walls get activated thermally and induce irreversible changes in the material. Another cause of irreversible losses in magnetic materials is the change in shape of the hysteresis curve, most likely a reduction coercivity of the magnet.

As already stated in this section, each application requires other forms and levels of stability. In [23, Chapter 5], the stability or the dependency of the magnet strength on time and temperature is experimentally examined, for different types of magnetic materials.
5.7 Permanent Magnet Behaviour during Operation

The second quadrant of the hysteresis curve describes the region where permanent magnets usually operate. In this region, the magnet has an internal magnetic flux density in the presence of a demagnetising field. In this configuration, it can deliver work. The energy stored in a magnet is related to the area inside the second quadrant of the loop. The maximum value of the product of $B$ and $H$ is called the maximum energy product, $BH_{\text{max}}$, and is a measure of the maximum amount of useful work that can be performed by the magnet. This constant is used as a figure of merit for permanent magnet materials. The higher the remanence and coercivity of a magnet, the higher the rating and power density of the magnet. So, the performance and behaviour during normal operation of a permanent magnet is given by its demagnetisation curve. A general permanent magnet has a demagnetisation curve as shown in Figure 5.3(a). The permanent magnet has a remanence $B_r$ and coercivity $H_c$. Whenever the demagnetising field starts to increase, the working point of the permanent magnet travels along the demagnetisation curve. Assume that the working point of the magnet is given by the point $P$. This point is reached by reducing the external field strength from zero to $H_p$. The magnetic flux density in this point equals $B_p$. A permanent magnet is subjected to dynamical operation if its working point changes as a result of a change in the external field. The working point of a permanent magnet in dynamic operation usually is not found on the outer hysteresis loop, but on an inner loop such as the one shown in Figure 5.3(a). The inner loop may lie entirely inside the second quadrant or extend into the first. The inner loops are very thin and for most purposes they can be approximated by a straight line. The slope of this line is known as the relative recoil permeability $\mu_{rc}$. Analytical models of this general demagnetisation curve are presented in [23] and [17, Appendix A].

Due to continuous research and the need for stronger permanent magnetic materials, new types of permanent magnets have been developed in the recent years. These new type of permanent magnets are so-called ‘rare-earth’ permanent magnets and are characterised by a much higher energy product than most other magnetic materials. They possess a high remanence, which results in a relatively strong magnetic field and a very high coercivity. These new permanent magnets give rise to new types of design and new applications for permanent magnets. Most rare-earth permanent magnets have a demagnetisation curve as shown in Figure 5.3(b) and show a linear and therefore much more ideal behaviour of the magnetic flux density as a function of the externally applied magnetic field. Again, the remanence is given by $B_r$ and the coercivity by $H_r$. The slope of the demagnetisation curve again is given by the recoil permeability $\mu_{rc}$ as given in Figure 5.3(b). Unlike normal permanent magnetic materials, the working point of rare-earth permanent magnets in static and dynamic operation is always located on this demagnetisation curve. For modern rare-earth magnets, the recoil permeability, or just relative permeability $\mu_r$, is very close or equal to unity. In Appendix A, it is shown that for magnetic materials the following relations hold:

$$\mu_r = 1 + \chi_m$$

(5.8)

where $\chi_m$ is the magnetic susceptibility. The susceptibility defines the dependency of the magnetisation inside a material and an externally applied magnetic field. Because the relative permeability for a rare-earth magnet is close or equal to unity, its susceptibility must be zero. Therefore, it can be concluded that the magnetisation inside the rare-earth magnetic material...
is constant and independent of the externally applied magnetic field. This is a very useful property as will be shown in the next sections.

5.8 Modelling of Permanent Magnets

Permanent magnets are used in electromechanical actuators, because they produce a magnetic field. The model of a permanent magnet therefore comes down to a spatial description of the magnetic field produced by the permanent magnet. In the previous section, it is shown that the magnetic field or flux density produced by the magnet is dependent on the existence and strength of an external magnetic field and the magnetisation of the magnetic material. In the modelling procedures treated in this section, it is assumed that the magnetisation of the magnetic material is known. How this magnetisation is realised is not included in the model. Saturation effects are also excluded from the model. Based on this magnetisation, an analytical description of the field distribution is derived. The analysis of permanent magnets starts with the definition of the scalar magnetic potential.

5.8.1 The Scalar Magnetic Potential

With the use of the equations of Maxwell, a model for the spatial distribution of the magnetic field can be derived. When only permanent magnets are considered, current densities are absent. Furthermore it is assumed that the magnetic field produced by the magnet is time independent. In the absence of electric currents and time-varying fields, Maxwell’s equations for the magnetic field reduce to:

$$\nabla \times \mathbf{H} = 0$$  \hspace{1cm} (5.9)
$$\nabla \cdot \mathbf{B} = 0$$  \hspace{1cm} (5.10)

From Chapter 2 we know that a curl-free vector field can be described by a scalar potential. The curl of the magnetic field strength $\mathbf{H}$ is zero in the case of a permanent magnet, so the field produced by the magnet can be described by:

$$\mathbf{H} = -\nabla \phi_m$$  \hspace{1cm} (5.11)

where $\phi_m$ is the magnetic scalar potential. Substituting the relation between the magnetic flux density, magnetisation, and the total field strength $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ in Equation 5.11 and using Equation 5.10 gives:

$$\nabla^2 \phi_m = \nabla \cdot \mathbf{M}$$  \hspace{1cm} (5.12)

This equation is also known as Poisson’s equation for the magnetic field. In all regions external to a magnet where the magnetisation does not exist, the right hand side vanishes. Also in the case of uniformly magnetised volume, the right hand side of the equation equals zero. For these cases, the Laplace’s equation for the magnetic scalar potential is obtained:

$$\nabla^2 \phi_m = 0$$  \hspace{1cm} (5.13)

In literature, a few methods are used to find a solution of the magnetic scalar potential or magnetic field strength $\mathbf{H}$. In [3, Chapter 3.4] analytical solutions to the Laplace and Poisson equations are derived, using separation of variables. In [7, 11], complex Fourier series or
space harmonics are used to describe the distribution of the magnetisation of the permanent magnet. This solution method can also be used to describe the magnetisation current densities instead of the magnetisation itself to obtain a solution. Together with the appropriate boundary conditions, the magnetic potential and magnetic field can be solved using Laplace’s or Poisson’s equation.

Another method to find an analytical solution for the magnetic vector field is the usage of the magnetic charge density \( \rho_m \) defined in Section 5.4. According to the definition of the charge density by Equation 5.6, the Poisson equation for the magnetic scalar potential can also be written as:

\[
\nabla^2 \phi_m = \rho_m
\]  

(5.14)

The solution of Equation 5.12 in terms a known charge density in the permanent magnet is given by [26, Section 2.6]:

\[
\phi_m(r) = -\frac{1}{4\pi} \int \frac{\rho_m(r')}{|r - r'|} \, dX
\]  

(5.15)

where \( r' \) is the position vector along a volume, surface, or curve for which the charge density is specified, and \( X \) is the total volume, surface, or curve. Of course, an expression must be known for the magnetic charge density in order to find a solution. Once the scalar magnetic potential is known, the magnetic field strength \( H \) can be determined by computing the gradient of the magnetic scalar potential, as given in Equation 5.11:

\[
H(r) = -\frac{1}{4\pi} \nabla \int \frac{\rho_m(r')}{|r - r'|} \, dX
\]  

(5.16)

Because the gradient is defined in global space, the nabla operator may also be written inside the integral resulting in the following relation for the magnetic field:

\[
H(r) = -\frac{1}{4\pi} \int \rho_m \frac{(r - r')}{|r - r'|^3} \, dX
\]  

(5.17)

This solution method will be used the in next section to obtain an analytical solution of the magnetic field produced by cuboidal magnets.

5.8.2 The Magnetic Field of Cuboidal Magnets

For permanent magnets that have a uniform magnetisation volume charge density, \( \nabla \cdot \mathbf{M} \neq 0 \) only occurs on the faces of the magnet. These faces are said to possess a surface charge density \( \rho_{ms} = \mathbf{M} \cdot \mathbf{n} \), see Section 5.3. If the magnet is cuboidal and its magnetisation is directed along one of the principle axes, only the two faces perpendicular to this magnetisation have a uniform surface pole density, see Figure 5.4(a). Consider the positively charged surface \( S \), which has dimensions \( 2a \) and \( 2b \) and a charge density \( \rho_{ms} \), as depicted in Figure 5.4(b). The origin of the Cartesian coordinate system coincides with the centre of the surface.
An expression for the magnetic field strength outside the magnet can be computed using Equation 5.15 and can be expressed as:

\[ H(r) = \frac{1}{4\pi} \int_{S'} \rho_{ms} \frac{(r - r')}{|r - r'|^3} dS' \]

\[ = \frac{\rho_{ms}}{4\pi} \int_{-b}^{b} \int_{-a}^{a} \frac{(r - r')}{|r - r'|^3} dxdy \]  

(5.18)

where \( r' \) is a position vector defined on the surface \( S' \). The derivation of the analytical solution is very complex and be denoted as follows [2]:

\[ H(x, y, z) = \frac{\rho_{ms}}{4\pi} \sum_{i,j=0}^{1} (-1)^{i+j} \epsilon(S_i, T_j, R) \]  

(5.19)

The three components of \( H \) in Cartesian space are referred to as \( H_x, H_y, \) and \( H_z \). The expressions for all three components depend on the function. \( \epsilon(S_i, T_j, R) \). They are given by:

\[ \epsilon_x = \ln (R_{i,j} - T_j) \]  

(5.20a)

\[ \epsilon_y = \ln (R_{i,j} - S_i) \]  

(5.20b)

\[ \epsilon_z = \arctan \left( \frac{S_i T_j}{R_{i,j}^2} \right) \]  

(5.20c)

where:

\[ S_i = x - (-1)^i a \]

\[ T_j = y - (-1)^j b \]

\[ R_{i,j} = \sqrt{S_i^2 + T_j^2 + z^2} \]

The magnetic field strength in an arbitrary point produced by the cuboidal magnet can be calculated by summing the contributions of the positively and negatively charged surfaces.
5.8.3 Presence of Iron

The presence of permanent magnets in combination with soft magnetic materials like iron can be modelled by the method of images, presented in [3, Section 2.2.1]. The procedure, also-called magnet mirroring, states that influence of boundaries on the applied magnetic field by a permanent magnet can be calculated by distributing images of the applied magnetic field. The images are mirrored in the boundary line. The total field is given by the sum of the applied field and the image field. With the use of the theory for the equivalent magnetic surface charge density treated in Section 3.5 the effect of the boundary can be related to the material permeabilities on both sides of the boundary. Consider a permanent magnet characterised by a magnetic surface charge density $\rho_{ms}$ placed in a region with permeability $\mu_1$ that is placed close to a magnetic material with permeability $\mu_2$. This situation is depicted in Figure 5.5. Two magnets with horizontal and vertical magnetisation are placed next to a boundary surface $S$. The surfaces of the magnets that contain a surface charge density are represented by $S_1$ to $S_4$. Now, for every surface of the magnet that contains a magnetic surface charge with density $\rho'_{ms}$, an image surface exists with magnetic surface charge density $\rho_{ms}$. The image charge density can be computed using:

$$\rho'_{ms} = \frac{-\mu_2 - \mu_1}{\mu_2 + \mu_1} \rho_{ms} \quad (5.21)$$

Each surface of the original magnet is mirrored in the boundary line to make up for the image magnet. In Figure 5.5 the image surfaces are given by $S'_1$, $S'_2$, $S'_3$, and $S'_4$. The field produced by the magnet in combination with the other boundary is given by summing the fields produced by the original magnets and their images.

A boundary material like iron has a very high permeability with respect to the permeability of permanent magnets. In that case, the system of permanent magnets and iron can be replaced by the magnets and their exact images, as is illustrated in Figure 5.6. On the left side of the figure, four permanent magnets on top of an iron plate are shown. The magnetisation inside each magnet is indicated by an arrow and assumed uniformly throughout the volume of the magnet. On the right side, the representation of the same magnets and their images is shown. Both representations are equivalent and result in the same magnetic field distribution. If the magnets are cuboidal, the total magnetic field can be found by using the method discussed in the previous section. From the equivalence of the back-iron and the image magnets it can also be concluded that the back-iron increases the magnetic flux density above the magnets. In
fact, the image of the magnets with a magnetisation perpendicular to the boundary surface doubles the height of the magnet and increases the resulting flux density distribution and field strength.

Figure 5.6: Magnet mirroring for back-iron.
Chapter 6

Synchronous Linear and Planar Actuators

6.1 Introduction

In the preceding chapters, the theory and background needed to describe and analyse electromagnetic actuators is presented. In general, actuators based on electromagnetic principles convert electric energy into mechanical energy or vice versa, through an intermediate medium, i.e. the magnetic field. In literature, numerous types of electromechanical actuators for a wide range of applications are discussed. One way of categorising electromechanical actuators is according to their operating principle, which is based on the electromagnetic mechanism used to generate forces. In general three types of electromagnetic principles are applied in electromechanical actuators: Lorentz force, induction force, and reluctance force [24]. These force mechanisms can, for example, lead to stepper motors, induction motors, or Lorentz actuators. A good overview of operating principles can be found in [4, Chapter 20] and [24, Chapter 3]. Another way of characterising electromechanical actuators is based on their mechanical output. This leads to the distinction between rotary actuators which deliver a torque and linear actuators which deliver a force. Direct linear actuators in general are unfolded versions of their rotary counterparts. Various types of rotary actuators are discussed in detail in [25] while an overview linear actuators is given in [5].

When movements in a horizontal plane are required a stack of two linear motors or planar motor can be used. A planar motor uses the linear motor technology in two directions. Linear actuators can translate in one direction, while planar actuators are capable of moving over long ranges in a plane. Dependent on the design of the planar motor, the other degrees of freedom, one translation and three rotations, can be controlled over a small range. Electromechanical actuators usually consist of one moving and one static part. In rotary electromechanical actuators these two parts are indicated as the rotor and the stator. For linear and planar motors, the moving part of the actuator is referred to as mover. Because the mover translates with respect to the stator, a bearing is needed to minimise the friction. The mover can be equipped with normal mechanical bearing with lubricants or air bearings. Another option is given by the generation of extra electromechanical forces that counteract the gravitational forces. This type of actuator generates a force with two components. The first component counteracts gravity and can be used to control small vertical movements. The
second component translates the mover in horizontal direction. Because the actuator is lifted without making any contact to the magnet array, frictionless movements can be obtained.

Planar motors can have different types and operating principles. The first planar motor was the Sawyer stepper motor, which is capable of stepping in two directions [30]. Other types of planar motors include the induction planar motor, described in [16, 27], and permanent magnet planar actuators. Two types of permanent magnet planar motors exist: the moving-magnet and moving-coil configuration. A moving-magnet permanent magnet planar actuator consists of a mover containing the permanent magnets while the static part is made of current-carrying conductors or coils. Moving-magnet planar actuators are presented in [15]. The moving-coil planar actuator has a mover that contains the coils and the stator contains the permanent magnet array. A disadvantage of a moving-coil planar actuator with respect to a moving-magnet planar actuator is that the cables feeding the coils have to be attached to the moving part, which influences the dynamics of the actuator. A big advantage of permanent magnet planar motors is the fact that they do not necessarily need additional bearings, because they can be levitated magnetically. Different types of moving coil permanent magnet planar actuators are discussed in [12]. A complete overview and comparison between different types of planar actuators can be found in [20, 24].

In this chapter only the planar actuator build at Philips Applied Technologies will be discussed. This synchronous permanent magnet planar motor generates forces based on the Lorentz principle. Forces are exerted on current-carrying coils in an external magnetic field which can be used to actuate and lift the mover. The fact that the actuator type includes permanent magnets means that the external magnetic field is generated by an array of permanent magnets, instead of electromagnets. Moreover, the actuator is of the synchronous type, meaning that speed of the motor is directly proportional to the frequency of the AC current that feeds it. This is the case when the currents in the coils of the actuator follow the periodicity of the flux distribution of the magnet array.

The working principle of the linear or planar actuator will be explained using a two-dimensional simplification. First, one single coil above an array of magnets is discussed in Section 6.2. Then the two-dimensional synchronous linear actuator will be discussed. The two-dimensional representation, together with a few assumptions regarding the magnetic field of the permanent magnets, allows the derivation of an analytical relation between the current in the coils of the actuator and the generated forces. This is done in Section 6.3. The field produced by the permanent magnet array plays an important role in the final performance of the actuator. In Section 6.4 different types of magnet arrays will be discussed and analysed. In Section 6.5 a method to model the field produced by permanent magnet arrays is presented along with three general techniques to calculate the forces and torques generated in the actuator. The commutation principle and the amplifiers in synchronous actuators will be discussed in Section 6.6. Then, in Section 6.7 a first discussion on possible sources for actuator related disturbances is presented. Finally, in Section 6.8 the theory for linear motors is extended to planar motors.
Figure 6.1: Lorentz force in linear actuators.
6.2 Linear Actuators with Permanent Magnets

To explain the working principle of a permanent magnet linear or planar actuator, consider Figure 6.1(a). A current-carrying conductor is placed in a magnetic field produced by a Permanent Magnet Array (PMA). The current flowing through the conductor is directed out of the paper. The conductor and the PMA are considered to be infinitely long and therefore end effects can be neglected. For now, assume that the flux lines of this magnetic field follow the pattern given in the figure. How such a magnetic flux distribution can be generated will be discussed in Section 6.4. The gravitational force is directed in negative vertical direction. The magnetic flux density is symmetric and periodic with periodicity $2\tau$ also known as magnetic period, where $\tau$ is referred to as the magnet pitch. From the Lorentz force equation \[ F = Bq Iv, \] we know that a force is acting on a current-carrying conductor when placed in a magnetic field. The direction of the force is always perpendicular to the local magnetic field and the current direction. The magnitude is linear with the current through the conductor and the local magnetic flux density. In Figure 6.1(a) the direction of the force acting on the conductor, for various positions of the conductor with respect to the magnet array, is drawn. Because the magnetic field direction rotates as a function of the horizontal position, the force acting on the conductor also rotates when it moves in horizontal direction with respect to the magnet array. If the current through the conductor is reversed the force on the conductor will rotate $180^\circ$.

Now consider Figure 6.1(b) where an electric coil is placed in the magnetic field. The coil consists of two conductors with an opposite current direction. The parts of the conductors that close the coil are omitted and their effect is neglected. The width of the coil is given by $w$. The current in the left part of the coil is directed out of the paper and in the right part directed into the paper. The forces acting on both parts are drawn in the figure. If the position of the coil with respect to the magnetic field is as given in Figure 6.1(b), the horizontal components of the forces cancel each other and hence no net force is generated in horizontal direction. Only a vertical force remains that lifts the coil above the magnet array. Because the coil consists of conductors at a fixed distance and the magnetic field distribution is symmetric and periodic, the direction of the forces in both parts of the coil can be related by a phase difference given by $\Delta\phi = \pi + \pi w / \tau$. The first term is the result of the change in current direction between the two coil parts and the second term results from the magnetic field distribution. For a given width of the coil, the magnitude of the net force acting on the coil remains constant and rotates as a function of the horizontal direction. In Figure 6.1(c) and Figure 6.1(d) for example, the position of the coil with respect to the magnet array is changed. The coil in Figure 6.1(c) is positioned in such a way that no net vertical force is acting on the coils. In this configuration, a net moment is working on the coil in counterclockwise direction. The coil in Figure 6.1(d) generates a net force that is directed in negative vertical direction. In this configuration no net torque is acting on the coil. It can be concluded that the magnitude of this torque is position dependent. The dimensions of the coil relative to the magnetic period or magnet pitch determines the magnitude of the net force. For example, if the dimension of the coil is chosen as $w = \tau$, the phase difference between the two forces becomes $2\pi$ and both forces will be directed into the same direction, resulting in a maximum net force.
It is obvious that the use of a single coil in a linear actuator will result in a continuously rotating force, which is undesirable. Especially the force in the vertical direction should be remain fixed in order the make the actuator hover at a constant distance above the magnet array. The width of the coil determines the magnitude of the net force. For a linear actuator to operate properly, multiple coils per actuator are needed. The total number of coils and the total width of all coils together with respect to the periodicity of the magnetic flux distribution determine the final behaviour of the motor. By choosing the number and width of the coils different linear actuators can be designed. In Section 6.3 a linear actuator with three coils spanning a total width of two times the magnetic period will be discussed. The layout of this type of actuator results in a relatively simple relation between current and force and therefore is widely used in various applications.

6.3 Synchronous Linear Actuators

The discussion presented in this chapter follows from the theory and design of a synchronous planar motor as in [12]. It starts with the derivation of the force formulas of the linear synchronous motor. The so-called three-phase/four-pole synchronous linear actuator is schematically depicted in Figure 6.2. The actuator consists of three coils 1, 2, and 3 placed next to each other above a permanent magnet array. The magnetic field produced by the magnet array is also drawn. Again, gravity is directed into negative vertical direction. The total width of all the coils together equals four times the magnet pitch \( \tau \). The coils can move in the \( x \) and \( z \)-directions. The total force generated by all three coils acting on the centre of mass is also drawn in the figure. This force consists of a normal force \( F_z \) that lifts the actuator above the magnet array and counteracts the gravity force and a tangential force \( F_x \) that generates the \( x \) translation. The total torque generated by the actuator around its centre of mass is given by \( T_y \). The angle of the resultant force \( \mathbf{F} \) with the vertical direction is symbolised by \( \theta \). Although the magnet array and the coils of the actuator are finite in \( y \)-direction, end effects are neglected in the following derivation and the pure two-dimensional case is assumed. In Section 6.8 the results of this section will be used in the discussion of planar actuators.
From Chapter 4 we know that the electrical equation for a coil can be described as:

$$ u(t) = i(t)R + \frac{\partial \Phi(t)}{\partial t} $$  \hspace{1cm} (6.1)

where $u$ is the voltage over the coil, $R$ is the total electric resistance of the coil, $i(t)$ is the current through the coil as a function of time, and $\Phi(t)$ is the total magnetic flux linked to the coil. This flux can be divided into flux generated by the coil itself $\Phi_c$ and flux generated by the permanent magnet array $\Phi_m$. The flux generated by the permanent magnet array linked to one of the coils is dependent on the actual position of the coil above the array. It is assumed that the flux $\Phi_m$ is constant in time, and so the electrical equation can also be written as:

$$ u = i(t)R + \frac{\partial \Phi_c(t)}{\partial t} + \frac{\partial \Phi_m(x, z)}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi_m(x, z)}{\partial z} \frac{dz}{dt} $$ \hspace{1cm} (6.2)

Now, the first term is distinguished as Ohm’s law, the second term contains the influences of self and mutual inductances between the coils, and the third term contains the influence of the permanent magnet array. If the electrical equation is multiplied by the current, the power equation follows:

$$ P = i^2(t)R + i(t)\frac{\partial \Phi_c(t)}{\partial t} + i(t)\frac{\partial \Phi_m(x, z)}{\partial x} \frac{dx}{dt} + i(t)\frac{\partial \Phi_m(x, z)}{\partial z} \frac{dz}{dt} $$ \hspace{1cm} (6.3)

In this equation, the first term is a result of the Ohmic losses, the second term equals the change of stored magnetic energy in the coil, and the last two terms represent the mechanical power delivered in $x$ and $z$-direction respectively. Mechanical power is usually expressed in terms of forces and velocities. The mechanical power in the $x$ and $z$-directions can also be written as:

$$ P_{\text{mech},x} = i(t)\frac{\partial \Phi_m(x, z)}{\partial x} \frac{dx}{dt} = F_x \frac{dx}{dt} $$ \hspace{1cm} (6.4)

$$ P_{\text{mech},z} = i(t)\frac{\partial \Phi_m(x, z)}{\partial z} \frac{dz}{dt} = F_z \frac{dz}{dt} $$ \hspace{1cm} (6.5)

With this result, the force constants $K_x$ and $K_z$ of the coil are defined as:

$$ K_x = \frac{F_x}{i(t)} = \frac{\partial \Phi_m}{\partial t} $$  \hspace{1cm} (6.6)

$$ K_z = \frac{F_z}{i(t)} = \frac{\partial \Phi_m}{\partial t} $$  \hspace{1cm} (6.7)

The force constants are determined by the position dependency of the linked magnetic flux produced by the permanent magnet array. For now, it is assumed that the position dependency of the linked flux, $\Phi_m$, in $x$-direction is harmonic with a period of $2\tau$. Pure harmonic characteristics in the $x$-direction are an ideal case however. How periodic magnet arrays can be produced and to what extent they approximate the ideal case, will be treated in Section 6.4. The magnetic flux produced by the PMA is harmonic with the same period, as depicted in Figure 6.2. The magnetic flux being purely harmonic has the advantage that it will result in a very simple and straightforward commutation mechanism, as will become clear later on.
6.3 Synchronous Linear Actuators

in this section. An exponential dependency of the linked flux in $z$-direction is most likely and therefore a realistic function for the flux linkage is [12]:

$$\Phi_m(x, z) = \hat{\Phi}_m e^{-\alpha z/\tau} \sin \left( \frac{\pi x}{\tau} \right)$$  \hspace{1cm} (6.8)

where $\hat{\Phi}_m$ is the top value of the flux linkage, $\alpha$ is a geometry-determined constant, and $\tau$ is the magnet pitch. With the expression for the linked flux produced by the permanent magnets, the force constants for a single coil can be written as:

$$K_x = \frac{\pi \hat{\Phi}_m}{\tau} e^{-\alpha z/\tau} \cos \left( \frac{\pi x}{\tau} \right) = \hat{K}_x(z) \cos \left( \frac{\pi x}{\tau} \right)$$ \hspace{1cm} (6.9)

$$K_z = -\frac{\alpha \hat{\Phi}_m}{\tau} e^{-\alpha z/\tau} \sin \left( \frac{\pi x}{\tau} \right) = \hat{K}_z(z) \sin \left( \frac{\pi x}{\tau} \right)$$ \hspace{1cm} (6.10)

where $\hat{K}_x$ and $\hat{K}_z$ are, apart from the constants $\alpha$, $\tau$, and $\phi_m$ a function of $z$ only. The linear actuator consists of three coils and each coil has its own force constant dependent on the actual position of the coil with respect to the magnet array. For each of the three coils, ($j = 1, 2, 3$), the force constants are given by:

$$K_{x,j} = \hat{K}_x(z) \cos \left( \frac{\pi x}{\tau} + \vartheta_j \right)$$ \hspace{1cm} (6.11)

$$K_{z,j} = \hat{K}_z(z) \sin \left( \frac{\pi x}{\tau} + \vartheta_j \right)$$ \hspace{1cm} (6.12)

where $\vartheta_j$ is the phase angle of each coil with respect to the magnet array. Take the horizontal position of the linear actuator to be equal to the horizontal position of the first coil. Then, because of the chosen three-phase/four-pole layout of the linear actuator, the phase angle for each coil is given by:

$$\vartheta_j = \frac{4\pi}{3} (j - 1)$$ \hspace{1cm} (6.13)

because every successive coil is shifted over a distance $4\tau/3$ in the positive horizontal direction.

The force constants of the coils change while the actuator is moving over the magnet array. Therefore, in order to generate a constant force, the currents through the coils should also change accordingly. To create a constant force, the amplifier of the linear actuator should generate the currents $i_j$ in the following manner:

$$i_j = \hat{I} \sin \left( \frac{\pi x}{\tau} + \vartheta_j + \phi \right)$$ \hspace{1cm} (6.14)

where $\hat{I}$ is the amplitude of the currents through each coil, $\vartheta_j$ is the phase angle of each coil given by Equation 6.13 and $\phi$ determines the ratio between the force components, scaled by their force constants. This can be illustrated by the computation of the total force that is generated by the linear actuator. This can be done by summing the forces generated by the individual coils:

$$F = \begin{bmatrix} F_x \\ F_z \end{bmatrix} = \begin{bmatrix} K_{x,1} & K_{x,2} & K_{x,3} \\ K_{z,1} & K_{z,2} & K_{z,3} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \frac{3}{2} \hat{I} \begin{bmatrix} \hat{K}_x \sin(\phi) \\ \hat{K}_z \cos(\phi) \end{bmatrix}$$ \hspace{1cm} (6.15)
From this expression it can be seen that $\phi$ and $\theta$ (with $\theta$ defined in Figure 6.2), are not the same angles. The force direction angle $\theta$ gives the direction of the total force generated by the linear actuator and the angle $\phi$ gives the ratio between the scaled force components in horizontal and vertical direction. However, both angles are related by:

$$\theta = \arctan \left( \frac{F_x}{F_z} \right) = \arctan \left( \frac{\hat{K}_x \sin(\phi)}{\hat{K}_z \cos(\phi)} \right) = \arctan \left( \frac{\hat{K}_x \tan(\phi)}{\hat{K}_z} \right)$$

(6.16)

If the force constants for the horizontal and vertical direction are equal both angles will also be the same. From Relation 6.15 it becomes obvious that the amplitude of the current determines the magnitude of the total force and the angle $\theta$ controls the direction of the total force. The current amplitude $\hat{I}$ and the phase angle $\phi$ needed to generate the required force components in horizontal and vertical direction, $F_x$ and $F_z$, can be computed using the following formulations:

$$\hat{I}(t) = \frac{2}{3} \sqrt{\left( \frac{F_x(t)}{\hat{K}_x} \right)^2 + \left( \frac{F_z(t)}{\hat{K}_z} \right)^2}$$

(6.17)

$$\phi(t) = \arctan \left( \frac{F_x(t)}{\hat{K}_x} \frac{\hat{K}_z}{F_z(t)} \right)$$

(6.18)

The conclusion from this derivation is that, with the assumption regarding the harmonic distribution of the magnetic field, the force generated by the actuator can be decoupled into a horizontal and vertical force. Both components can be regulated individually, which is a desirable situation in the field of control. Because the vertical force can be kept constant, the actuator can hover at a constant distance above the magnet array in all positions. However, as already mentioned, the harmonic distribution of the magnetic flux above the permanent magnet array is an ideal case. In practice it is observed that real harmonic flux distributions are very hard to obtain and other arrays that approximate the harmonic distribution are used in linear actuators.

### 6.4 Magnet Arrays

In the analysis presented in the Section 6.3, the magnet flux density produced by the magnet array exhibits a harmonic dependency on the horizontal position. This magnet array plays an important role in the final behaviour and performance of the drive. Only for the harmonic distribution, the relation between current through the coils of the actuator and force produced by the actuator can be expressed in a very simple form, see Equation 6.15. This section deals with the procedure to make magnetic fields which are harmonic or at least periodic. It will be shown that a pure harmonically distributed magnetic field is very hard to obtain and therefore other permanent magnet structures have been developed to approximate the harmonic field distribution. A permanent magnet array usually consists of individual permanent magnets stacked next to or on top of each other. The individual magnets can have a different magnetisation magnitude and direction. With the use of these magnets and by creating a stacking order, the field distribution of the stack or array can be customised. By positioning the magnets in a certain repeating order, the magnetic flux density above the magnet array can be made periodic.
6.4 Magnet Arrays

6.4.1 Conventional Magnet Array

Periodicity of the magnetic field can be constructed by ordering independent permanent magnets in a periodic way. Consider a magnet array in air with an iron back plate, as depicted in Figure 6.3. This two-dimensional magnet array is constructed of individual rectangular magnets. The magnetisation of two adjacent magnets is equal in magnitude, but opposite in direction. In Figure 6.3 the direction of magnetisation inside the permanent magnets is represented by arrows. All magnets have the same dimensions. The resulting magnetic flux lines above and below the array are schematically represented in the figure. The magnetic flux below the magnet array is contained within the iron, because of its high magnetic permeability with respect to air. This iron plate actually shields the region below the iron from any magnetic field. As a consequence of the shielding, the magnetic field strength above the magnet array increases. Due to the repetition of magnetised material, the magnetic flux density above the magnet array is periodic with the horizontal direction. However, it can be computed that the magnetic field above the array is not purely sinusoidal, see [33], which complicates the operation of a linear drive. Instead of an ideal distribution with periodicity equal to the magnet spacing, disturbances in the harmonic magnetic flux density distribution are present. This makes the operation of the drive more complex. Another disadvantage of this type of magnet array is that parts of the magnetic flux produced by the magnets are unused, because flux is flowing through the iron. In this way, not all the potential energy of the permanent magnets can be transferred towards the coils of the linear motor. Consequently, other magnet arrays have been developed that produce more efficient and sinusoidal flux density distributions, as will be discussed in the next sections.

6.4.2 Ideal Halbach Array

A permanent magnet structure which generates a purely harmonic magnetic flux density distribution is first presented in [18] by K. Halbach. The key concept of this so-called Halbach array is that the magnetisation vector in the magnetic material rotates continuously as a function of position. The position dependency of the magnetisation vector is aligned with the intended force direction. Only then the magnetic flux density distribution is inherently sinusoidal. In the ideal case, the Halbach array will consist of one piece of magnetic material with a continuously changing magnetisation. Such a magnetic material is schematically represented in Figure 6.4(a). If the vector rotates continuously, the field on one side of the array will vanish, whereas the other side will have the strongest field possible. Halbach arrays are therefore said to be self-shielding, which means that the need for a back-iron is not essential.
Dependent on the direction of rotation, the magnetic field is created on one of the sides of the array. In Figure 6.4(a), when moving in positive $x$-direction the magnetisation vector is rotating in counterclockwise direction, resulting in a magnetic flux density distribution above the magnetic material. In contrast with a conventional magnet array, all magnetic flux is concentrated on one side of the array, which improves the utilisation of magnetic material and increases the magnetic field strength and power density. Although the Halbach array has the ideal harmonic flux distribution, the process to create such an array is very complex. It includes a combination of moulding and sintering and a sophisticated magnetising procedure.

### 6.4.3 Segmented Halbach Array

Because of the complex process to fabricate an Halbach array, in practice magnetic arrays are constructed that approximate the ideal case. So-called segmented Halbach arrays are used which consist of an assembly of permanent magnets with uniform magnetisation. The magnetisation of each consecutive magnet is rotated with respect to the previous one. In this case, a close approximation of the ideal Halbach array can be achieved. An example of a segmented Halbach array is given in Figure 6.4(b). Instead of magnetisation in vertical direction that changes $180^\circ$ in every next magnet as in the conventional magnet array, now the magnetisation rotates with steps of $90^\circ$. This pattern is obviously a closer approximation to the ideal Halbach array. The more segments per magnetic pole are used, the closer the approximation to the Halbach array. If the dimensions of each magnet are equal and square, the magnetic field strength of the array such as the one shown in Figure 6.4(b) is stronger by a factor of $\sqrt{2}$ than the conventional array and lies within 90% of the strength of an array that has the ideal structure [33]. Instead of concentrating the magnetic flux on one side of the array, the segmented array has a strong and a weak side, again dependent on the direction of the rotation of magnetisation. The ideal Halbach array results in a purely sinusoidally dis-
tributed magnetic field which contains no higher harmonics. Because the segmented Halbach array still contains discrete steps in the magnetisation direction as a function of position, the magnetic flux density distribution will contain higher harmonics. But these harmonics decay fast with the distance from the array, [37]. Although the use of a higher number of segments improves the field distribution, it remains an approximation and therefore reduces the ease of operation of the drive with respect to a drive with an ideal magnet array. The segmented Halbach array, however, still remains favourable over the conventional array.

Recently a lot of work is done on the three-dimensional magnetic field description as a result of two-dimensional magnetic arrays. For more information on Halbach arrays in different types of synchronous machines, the reader is referred to [33]. In this reference, also a quantitative comparison with respect to the magnetic field strength of ideal and segmented Halbach arrays is given. In [37], a review of machines using a permanent magnet Halbach array is given. Furthermore, in [8, 11] different types of permanent magnet segmented Halbach arrays that are harmonic in two directions are analysed and compared. The latter type of array can be used in the planar motor, which will be discussed in Section 6.8.

6.5 Actuator Analysis

In Section 6.3 the relation between current and generated force for a two-dimensional linear permanent magnet actuator is derived. The relation is shown to be very straightforward and simple. Unfortunately, the relation only holds for the assumption that the magnetic field and flux density is distributed harmonically and end effects due to finite length of coils and magnets are neglected.

When a three-dimensional actuator needs to be designed or thoroughly analysed one can start by solving the Maxwell equations analytically. However, first of all this requires an analytical description of the geometry, dimensions and material properties of all elements. Due to the often very complex geometry and layout in electromechanical actuators, an analytical solution of the Maxwell equations does not always exist. In this case, the problem sometimes can be simplified by using the electric or magnetic potential functions, as discussed in Chapters 4 and 5. As an alternative, Finite Element Methods (FEM) can be used to find a solution for the magnetic field, flux, and force distribution. But for three-dimensional structures, like linear and planar motors, FEM analysis is a time consuming process, so for a fast evaluation or analysis it is not suitable. However, other methods which are based on the analytical equations can be used in stead of FEM packages.

6.5.1 Force and Torque Calculation

For the evaluation of the electromagnetic forces in electromechanical actuators, several methods exist in literature. The different methods used to evaluate the generated forces will be discussed in this section.

Virtual Work

One alternative to FEM packages or the use of the Maxwell equations is derived through the evaluation of the magnetic energy at a macroscopic level. The virtual work method uses
the principle of virtual displacement which will be applied to magnetic energy and magnetic co-energy to derive the force and torque equations. The definition of magnetic co-energy, (which does not exist physically) is based on the fact that it exactly complements the real physical magnetic field energy. So, it is referred to as complementary magnetic field energy, in short ‘co-energy’. The force or torque acting on parts of the actuator is based on the change of energy and co-energy in an electromechanical system. The so-called macroscopic magnetic force equation represents the total summed magnetic force in a predefined direction and can be derived using the general expressions for the energy and co-energy. For example, the total force acting on a part moveable in the \(x\)-direction in a system that contains a permanent magnet, a coil, and a yoke part and assuming no hysteresis in the permanent magnet is given by [24]:

\[
F_x = \frac{\partial \Phi_{pm}}{\partial x} + \frac{1}{2} I^2 \frac{\partial L}{\partial x} - \frac{\partial W_{pm}}{\partial x} \tag{6.19}
\]

where \(F_x\) is the force in \(x\)-direction acting on the movable part, \(I\) is the current trough the coils, \(\Phi_{pm}\) is the flux of the permanent magnet linked to the coil, \(L\) is the self induction of the coil, and \(W_{pm}\) is the magnetic field energy due to the permanent magnet. For an elaborate derivation of the macroscopic force formula, the reader is referred to [24, Appendix A.3]. The first term on the right hand side describes the interaction between the coil and the permanent magnet. The second term is due to the coil solely, while the third term is only a result of the permanent magnet. The equation can be applied directly to calculate the forces and moments. For this, the individual components of Equation 6.19 can be evaluated analytically or numerically. For the calculation of the forces and torques in all six principle directions one needs to substitute the variable \(x\), with the appropriate degree of freedom. For the calculation of torques, the degree of freedom is obviously one of the three rotation angles. The equations that follow can easily be extended to systems with multiple coils, magnets, and moveable parts. Based on the macroscopic force, sophisticated analytical models or formulas can be derived for electromechanical actuators.

Maxwell Stress

A second alternative is the use of the co-called Maxwell stress tensor. The electromagnetic force can be derived from the field density in the air gap by means of a surface integral over a closed surface. The closed surface then has to completely contain the part of the actuator on which the force needs to be calculated. This method makes the calculation of field-energies and flux linkages unnecessary. The total electromagnetic force can be calculated by specifying a fictitious force density \(f\), called Maxwell tension, in this surface. This force density is given by [24]:

\[
f = \left( B \cdot n \right) \frac{B}{\mu_0} - \frac{1}{2} n \left( \frac{B^2}{\mu_0} \right) \tag{6.20}
\]

where \(B\) is the total magnetic flux density distribution and \(n\) is the unity vector perpendicular to a local surface element. For a system in Cartesian coordinates, the Maxwell stress tensor can be derived as:

\[
f = \frac{1}{\mu_0} \begin{bmatrix}
\frac{1}{2} (B_x^2 - B_y^2 - B_z^2) & B_x B_y \frac{1}{2} (B_y^2 - B_x^2 - B_z^2) & B_z B_x \\
B_x B_y \frac{1}{2} (B_y^2 - B_x^2 - B_z^2) & \frac{1}{2} (B_y^2 - B_x^2 - B_z^2) & B_y B_z \\
B_z B_x & B_y B_z & \frac{1}{2} (B_z^2 - B_x^2 - B_y^2)
\end{bmatrix} \tag{6.21}
\]
When the magnetic flux density is known over the closed surface, the total force can be computed by integrating over that surface:

\[ F = \oint_S f dS \]  \hspace{1cm} (6.22)

The torques or moments acting on the part of the actuator can be computed by multiplying the forces with a position vector that describes the location of the local force with respect to the position where the torque needs to be calculated.

**Lorentz Force**

The Lorentz force equation, as described in Section 4.3.2, is actually the basis of all magnetic forces. The Lorentz force is a result of the relativistic effects that occur between moving charges and therefore its origin is defined on atomic level. A general expression for the Lorentz force is given by Equation 4.8,

\[ F_L = Qv \times B \]  \hspace{1cm} (4.8)

The Lorentz force equation is the simplest method used to compute the forces acting on the individual parts of an electromechanical actuator. However, the practical application of the Lorentz force equation is limited to cases where the surrounding media have a relative permeability equal or close to one. Media like air and modern rare-earth permanent magnets fulfill this property. Once the magnetic field distribution and the current through a coil are known, the total force experienced by the coil can be computed using the equation:

\[ F = \int_V J \times B \, dV \]  \hspace{1cm} (6.23)

where \( F \) is the force on the coil, \( J \) is the current density in the coil material, \( B \) the local magnetic flux density in the coil, and \( V \) the volume of the coil. The total torque, \( T \), working on the coil can be expressed by:

\[ T = \int_V r \times J \times B \, dV \]  \hspace{1cm} (6.24)

where \( r \) is the position vector.

The net generated forces and torques in an actuator are highly dependent on the dimensions and shape of the coils and permanent magnets. The Lorentz force method discussed above is used in [12] to analyse the synchronous linear actuator discussed in Section 6.3. It is shown that apart from a force in horizontal and vertical direction, also a torque around the \( y \)-axis is produced. This result is not obtained when the forces are computed based on the electrical and mechanical equations alone. All the methods discussed in this section can be used to compute the torques and forces generated by the actuator. In this way an analysis of unwanted or additional forces and torques can be made. In some cases, these additional torques can be decreased by changing the design or lay-out of the actuator.
6.5.2 Evaluation of the Magnetic Field

The methods to calculate forces and moments generated in the actuator discussed in Section 6.5.1 have one thing in common: all the methods require a description of the distribution and strength of the magnetic flux density and field strength. Again, in literature a few different methods are presented. FEM packages can be used to calculate the magnetic field distribution, but it still has the disadvantage of being relatively slow. Also solutions of the magnetic vector potential can be used to determine a solution for the magnetic field.

The Field of the Magnet Array

If only the magnetic field distribution due to the permanent magnet array is considered, the magnetic scalar potential can be used. In Section 5.8 a solution for the magnetic flux density distribution of permanent magnets is derived. Therefore, the modelling of the permanent magnet array starts with the method mentioned in Section 5.8. First, the magnetic scalar potential must be determined. However, if all magnets in the array are of the cuboidal shape, the analytical solution for one single magnet, given by Equation 5.19 in Section 5.8.2 can be used. The magnetic flux distribution of an array of magnets can be calculated by calculating the magnetic flux contributions of all individual magnets in the array. By shifting and rotating the individual magnets to their place in the magnet array and summing the individual magnetic fields the total magnetic field generated by the array can be obtained, as described in [13]. Because the complexity of the analytical solution increases very fast with an increasing number of magnets in the array, numerical techniques can be used to evaluate the magnetic flux distribution. The method is only valid if the magnetisation of the permanent magnets is not influenced by each other. The magnetisation of each magnet determines the magnetic field strength. If the magnetisation of a permanent magnet changes due to the presence of another permanent magnet, the total magnetic field is different from the sum of the individual fields. When the actuator is operated, the coils will also generate magnetic fields that can change the magnetisation inside the magnets. These effects can introduce disturbances in the magnet array model. In Section 5.7 it is proven that a permanent magnetic material possesses a constant magnetisation if its relative permeability equals one. This is true for new rare-earth magnetic materials. In the following example, the magnetic field of a Halbach permanent magnet array is evaluated.

Example 2

This example deals with a two-dimensional segmented Halbach array consisting of four magnets per magnetic period, similar to the one depicted in Figure 6.7(a). The dimension of the magnets with magnetisation in z-direction is 30 × 30 mm and the dimension of the magnets with horizontal magnetisation, in x or y-direction, is 30 × 10 mm. The total size of the array equals 170 × 170 mm. All magnets have a thickness of 10 mm. The array is shielded at the bottom side by an iron back plate to maximise the flux density on the upper side. All magnets have the same material constants and have a magnetic remanence \( B_r = 1.0 \) T and hence a magnetisation \( M = 1.0/\mu_0 = 7.97 \times 10^6 \) A/m. Furthermore, it is assumed that the magnets have a relative permeability that can be approximated by one. This means that the total field of the array can be calculated by summing the magnetic fields of all individual magnets in every point. Because all magnets are of the cuboidal type, their magnetic flux distribution is given by Equation 5.19. The iron back plate can be modelled by using the method of images discussed in Section 5.8.3. To evaluate the magnetic field produced by the array and because
6.6 Commutation

Commutation inside electromechanical actuators can be defined as the control of current through the coils of an actuator to produce a desired force or torque. In permanent magnet motors, a constant or optimal force or torque can be produced when the current is channelled to the proper windings or coils. When the motor is moving, the position of the windings relative to the magnets change and the current through each coil has to change in order to keep producing the optimal force or torque. Therefore, a commutation law is always dependent on the actual position of the actuator with respect to the permanent magnets. For a synchronous planar or linear actuator above a harmonic magnet array to generate a constant force, the current through each coil must satisfy the condition given by Equation \((6.14)\)

\[
i_j = \hat{I} \sin \left( \frac{\pi x}{\tau} + \vartheta_j + \phi \right)
\]
where  is the current amplitude, is the phase angle in each of the three coils, and is an extra phase angle to control the direction of the generated force. If the current through each of the coils satisfies this relationship, the generated force by the actuator is constant and can be controlled very easily. Equation 6.14 is also referred to as the commutation law for the linear synchronous actuator. For the commutation of the actuator to work properly, the position of the actuator coils with respect to the permanent magnets must be known. Then, this position can be substituted in the commutation law. If the required force profiles are determined, the value for the current amplitude and phase angle can be computed from Equations 6.17 and 6.18. In the case of a linear or planar actuator, the position measurement must be absolute. If the position measurement would be incremental, it would not be able to operate during startup, because it does not have information about its actual position with respect to the magnet array. An absolute position measurement can be made by using the magnetic field of the permanent magnets itself to obtain the position information. Actually, as can also be observed from the commutation law, it is only necessary to know the position of the coils within twice the magnetic pitch . Therefore, sensors that can measure the magnetic field strength in a certain point are widely used as a basis for electronic commutation. If the magnetic field in a point is known, it can be translated into a position. In [14], different types of sensors have been examined, one of them is the Hall sensor which is used for most electronically commutated motors. This sensor is discussed in the next section.

6.6.1 The Hall Sensor

To explain the principle on which a Hall sensor is based, consider Figure 6.6. Assume a current  is flowing through a thin layer of semi-conductor material and that a magnetic field is present that passes through that thin layer. Two forces are present on the charges or electrons inside the material. A Coulomb force as a result of the electric field generates the actual displacement of charge and induces a current through the material. Due to this movement and the presence of the magnetic field, a Lorentz force is acting on the charges as well. This force is directed perpendicular to the current direction and the magnetic field, inherently forcing the electrons towards one side of the Hall sensor material. This sideways movement of electrons induces a potential difference over the edges of the Hall sensor which can be detected. The potential difference over the sides of the Hall sensor can be written as:

\[ U_h = kIB\sin(\alpha) \]
where \( k \) is a constant dependent on geometric dimensions, material properties, temperature, etcetera, \( I \) is the current through the Hall sensor, \( B \) is the local magnetic flux density, and \( \alpha \) is the angle between the magnetic flux and the semiconductor material. As can be concluded from the Equation 6.25, the potential difference \( U_h \) is linearly related to the current \( I \) and the normal component of the magnetic flux. Therefore, assuming a constant current \( I \) through the sensor, the signal \( U_h \) is a measure for the local magnetic flux density distribution.

Three Hall sensors can be used to determine all three components of the magnetic flux density distribution, \( B_x, B_y, \) and \( B_z \), in a Cartesian frame. However, this magnetic field is produced by the permanent magnets as well as the current-carrying coils. The Hall signal is only suitable for position measurement based on the magnetic field produced by the permanent magnets. Hence, the Hall sensor signal must be compensated for the presence of the coils and the resulting magnetic field. This can be done by measuring the influence of the current-carrying coils on the Hall sensor signal for various coil currents. If the magnetic field is unique in every point within one magnetic pitch, then the position can be reconstructed very easily using a look-up table. Another option is given by [12] where arrays of multiple Hall sensors are used to generate two sine waves, shifted \( 90^\circ \) in phase, which are used to extract the absolute position of the actuator.

### 6.6.2 The Current Amplifier

To generate a desired motion profile, the actuator must be fed with the currents computed in the commutation algorithm. The electrical equations for the three-phase linear actuator moving in \( x \)-direction depicted in Figure 6.2 are given by:

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
= 
\begin{bmatrix}
  R_1 & 0 & 0 \\
  0 & R_2 & 0 \\
  0 & 0 & R_3
\end{bmatrix}
\begin{bmatrix}
  i_1 \\
  i_2 \\
  i_3
\end{bmatrix}
+ 
\begin{bmatrix}
  \frac{d}{dt} 
\end{bmatrix}
\begin{bmatrix}
  L_1 & M_{12} & M_{13} \\
  M_{21} & L_2 & M_{23} \\
  M_{31} & M_{32} & L_3
\end{bmatrix}
\begin{bmatrix}
  i_1 \\
  i_2 \\
  i_3
\end{bmatrix}
+ \hat{\Phi}_m e^{-\alpha x/\tau}
\begin{bmatrix}
  \sin \left( \frac{\pi x}{\tau} \right) \\
  \sin \left( \frac{\pi x}{\tau} + \frac{4\pi}{3} \right) \\
  \sin \left( \frac{\pi x}{\tau} + \frac{8\pi}{3} \right)
\end{bmatrix}
\]

(6.26)

where the indices 1, 2, and 3 refer to the three coils of the actuator, \( u \) is the voltage over the coil, \( i \) is the current through the coil, \( R \) is the electrical resistance of the coil, \( L \) and \( M \) are the self and mutual inductances of the coils, \( \Phi_m \) is the top value of the flux induced by the permanent magnets linked to the coil, \( \alpha \) determines the vertical position dependency of the linked flux, \( \tau \) is the magnet pitch of the permanent magnet array, and \( x \) is the horizontal position of the actuator. The equations are based upon the same assumptions on the flux density distribution as in Section 6.3. This means that the flux linkage of each coil is assumed to be harmonic in horizontal direction and exponential in vertical direction. The geometry-based phase difference of \( 4\pi/3 \) in flux linkage between the three coils also enters the electrical equation.
The currents computed in the commutation algorithm are fed to the coil by the current amplifier. The actual force generated by the actuator is highly dependent on the amplifier specification concerning drift, offset, and gain per phase. Improper parameters can lead to position dependent loop gains or generation of additional forces or torques by the actuator. The dynamic response of the amplifier can also change the behaviour of the linear actuator.

6.7 Parasitic Effects

In the previous section, the commutation algorithm and the generation of the coil currents by the amplifier are discussed. Both components are located in the actuation chain of the actuator and therefore influence the generated forces. Together with the coil dimensions, flux density distribution generated by the permanent magnets and the orientation of the coils with respect to the permanent magnets, they determine the total generated force. Errors in one of these components can introduce deviations from the desired force profile. Commutation errors, for example, are introduced if the measured horizontal position entered in the commutation law differs from the true position of the coil with respect to the permanent magnet array. Also the temperature dependency of the Hall sensor needs to be examined to obtain a complete overview of Hall sensor commutation errors. Other disturbances in force generation can be introduced by wrong amplifier gains or offsets, as can be concluded from Equation 6.15. If the current trough a coil differs from the ideal value, disturbances are introduced. Also inaccuracies in the coil dimensions or partially short-circuited coils change the force constant and introduce force disturbances. Other sources for disturbances are, as already discussed in Section 6.4, deviations in the assumed harmonic flux density distribution and demagnetisation of the permanent magnets. All disturbance sources mentioned above give rise to so-called actuator related disturbances and are very reproducible over a certain time-span. Most of the time these disturbances, or force ripples, are a periodic function of the position. How the disturbances or inaccuracies in each component of the action chain influence the total generated force differs for each actuator and therefore needs to be investigated individually.

Other approaches to compensate for force ripples are based on experimental data, and different methods have been published. In [29] a ripple compensation for a synchronous linear motor based on a fourier series approximation is derived. First the force ripple is measured by measuring the thrust force of the linear motor at different horizontal positions. Then a approximation of this force ripple is generated by approximating the thrust force by a fourier series approximation and using data fitting optimisations. Once the force ripple is known, it is compensated by using input-output linearisation. The actual compensation consists of a current dependent component and a current independent component. Both components however are dependent on the actual position of the actuator. Another approach is given in [22] where an iterative learning control (ILC) structure is chosen to compensate for the force ripple in a permanent magnet linear motor (PMLM). In combination with a normal PID controller which stabilises the system, the ILC feedforward controller enhances the tracking performance of the actuator by using the experience gained from the repeated execution of the same operations.
6.8 Synchronous Planar Actuators

So far, only two-dimensional synchronous linear actuators have been discussed. The principle of the linear motor moving in one direction can be extended into the so-called planar motor which can move in $x$ as well as $y$-directions. The top view of a such a synchronous permanent magnet planar motor is represented in Figure 6.7(a) and is described in [9–11]. The layout of the permanent magnet array can be observed immediately. The magnets are not just periodically assembled in one direction, but a checker-board-like configuration is applied. The magnetisation in each magnet is indicated again by an arrow. It can also be noticed that the array is of the segmented Halbach type with two magnets per magnetic pitch, which is defined by $\tau$. Because the total width of the coils in the actuator equals four times the magnetic pole, a synchronous three-phase/four-pole linear actuator is created. Apart from the force in $z$-direction, that lifts the coils from the magnet array, the coil assembly can also generate a force in the $x$-direction. Furthermore, the two ends of the coil will also generate forces that are perpendicular to the current through the coil. This results in additional forces in $x$, $y$, and $z$-directions. How these additional or disturbance forces affect the performance of the drive has to be investigated for each single actuator. The influence is highly dependent on the specific dimensions and layout of the coils and permanent magnets in the actuator.

To produce a planar actuator that can move in $x$ as well as $y$-directions, the given set of coils has to be combined with a similar set of coils rotated over $90^\circ$ with respect to the magnet array. Then, the two sets of coils together can generate forces in $x$ and $y$-directions, due to the periodicity of the magnetic flux density distribution in both $x$ and $y$-directions. In Section 6.4, it is stated that segmented Halbach arrays do not possess a purely harmonic distribution, however. The disturbances in the magnetic flux density distribution influence the behaviour of the drive. The influence of the spatial higher harmonics present in the flux linked to the coils can be minimised by rotating the coils $45^\circ$ with respect to the magnet array [12]. Due to the spatial integration of the flux density over the coils, the disturbances
are averaged out to a maximal extent. This situation is depicted in Figure 6.7(b). In this case, the magnetic pitch $\tau$ is defined in a different way and therefore also the total width of the three coils is changed. This is done to maintain a synchronous three-coil actuator.

The experimental setup of the SPMPM build at Philips Applied Technologies is depicted in Figure 6.8. It contains four synchronous linear actuators of the type given in Figure 6.7(b). Together they can produce forces in the horizontal plane as well as the vertical direction. The base plate containing the permanent magnet array and the planar motor carrier can be clearly distinguished. Also the cable arm, which contains the current carrying wires to the coils and the cooling infrastructure can be observed. The theories presented in this chapter, as well as the fundamental properties of the electromagnetic field discussed in Chapters 2 to 5 now can be used to investigate the sources of the actuator related disturbances present in this planar actuator.
Chapter 7

Conclusions and Recommendations

This literature survey concerns the actuation principles in permanent magnet synchronous planar motors. To this end, four questions have been formulated in Section 1.3:

- What are the properties of the electric, magnetic, and electromagnetic field, how can they be characterised, and how can they be described mathematically?
- What sources for the (electro)magnetic field exist and how can they be qualified?
- What techniques exist to model the fundamental (electro)magnetic phenomena laying the foundation for electromechanical actuators?
- What is the working principle of the synchronous permanent magnet planar motor and what are the main components in the actuation chain of the SPMPM at Philips Applied Technologies?

7.1 Conclusions

The conclusions drawn from this report answering the questions above are as follows:

- The electric field is characterised by the electric field strength $E$ and the electric flux density $D$, while the magnetic field is defined by the magnetic field strength $H$ and the magnetic flux density $B$. The electric field can be classified as a nonsolenoidal, rotational field, which means it can have scalar as well as vectorial sources. The magnetic field is nonsolenoidal and irrotational and only has sources of vectorial origin. The electric and magnetic field, also electromagnetic field, can be described uniquely with the use of the equations of Maxwell. With the use of the Divergence and Stokes’ theorems general boundary conditions for the electromagnetic field can be obtained.

- Magnetism is observed whenever electrically charged particles are in motion. This can arise either from movement of electrons in an electric current, resulting in ‘electromagnetism’, or from the quantum-mechanical orbital motion and spin of electrons, resulting in what are known as ‘permanent magnets’. The electromagnetic field is generated by the following sources: electric or magnetic charge or charge densities (scalar) or electric current densities (vectorial).
Conclusions and Recommendations

- To model the mechanisms or phenomena in electromechanical actuators, a distinction must be made between electromagnetic and permanent magnetic phenomena.
  
  - The main electromagnetic phenomena are: the generation of a magnetic field by electrical currents, Lorentz force generation, electromagnetic induction, and the generation of Eddy currents. To find the magnetic field produced by electric coils of arbitrary shape, the law of Biot-Savard can be used. The magnetic vector potential can also be used for very complex geometries. To model the coil electrically the principles of Ohmic losses, induction, and flux linkage must be modelled.
  
  - In a permanent magnet the orientation of the magnetic moments inside the material aligns with an externally applied magnetic field and stays aligned when the field is removed. Permanent magnets can be modelled using the theory of equivalent surface charge density and the magnetic scalar potential. The method of magnet mirroring can be used to find a solution for the magnetic field of permanent magnets in combination with other materials.

- The force generation in the SPMPM itself is based on a two-dimensional three-phase/four-pole synchronous permanent magnet linear motor. A segmented Halbach array is used to generate a harmonic flux density distribution. By generating appropriate harmonic currents through each coil, the two main force components generated by the actuator can be decoupled and can be controlled independently. The main components in the actuation chain of the SPMPM at Philips Applied Technologies are a set of properly oriented coils in combination with permanent magnets, the commutation algorithm and its sensors, and the amplifier.

7.2 Recommendations

With the theories and methodologies treated in this literature study a start can be made to investigate and model the individual components of the SPMPM. In this way their effect on the generated forces and possible introduction of disturbances can be analysed. The following recommendations are given with respect to the modelling and compensation of actuator related disturbances, which is the topic of the forthcoming MSc project.

- Examine the effect of disturbances originating inside the actuator, and the total propagation of the these actuator related disturbances through the actuation chain. Build a physical model for the disturbances, based on the electromagnetic phenomena discussed in this literature survey. The following considerations create a good starting point:
  
  - Due to the actuator design and the end effects of the coils and permanent magnets additional forces and torques are generated in the actuator. By using one of the techniques discussed in Section 6.5.1 these additional forces and torques can be determined.
  
  - Offset and gain errors in the Hall sensors affect the generation of position information for the commutation algorithm. This will also influence the force generation inside the actuator.
  
  - Amplifier errors, like drift, gain, and offset errors result in non-ideal force distributions.
7.2 Recommendations

- Because the actuator needs to be fed with three-phase currents, cables are attached to the actuator platform. Besides Eddy damping, these cables will introduce additional stiffness and damping in the actuator.

- Due to the use of a segmented Halbach array, the magnetic flux density distribution of the permanent magnet array is not purely sinusoidal. Additional harmonics will affect the force generation in the actuator and can be a source for the observed periodic actuator disturbances.

- Investigate the possible position, time, and temperature dependency of the actuator related disturbances and add them to the disturbance model.

- Synthesise an appropriate feedforward compensation scheme based on the model for the actuator related disturbances. This feedforward control structure can be used in machine-in-the-loop optimisation schemes or iterative learning control to optimise or identify the parameters disturbance model.
Appendix A

Definition of Field Quantities

The present appendix given the definitions of the electric and magnetic field quantities as presented in [19]. Section \[A.1\] deals with the electric field quantities and Section \[A.2\] deals with the definition the magnetic field.

A.1 Electric Field

Coulomb’s Law

The relation between the force that acts on two stationary point charges in vacuum is found experimentally by Coulomb and has a similar form as the well known gravitational force acting on two masses. The force in both laws is directed along the line through the centre of the two charges or masses. The law of Coulomb is defined as:

\[ F_E = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{r^2} e_r \]  

(A.1)

where \( Q_1 \) and \( Q_2 \) are the two charges considered, \( \varepsilon_0 \) is the permittivity of vacuum, \( r \) is the distance between the two charges, and \( e_r \) is the unity vector in the direction of the force. Unlike electrical charges attract each other and alike charges repel each other.

Electric Field Strength

The electric field in an arbitrary point, as a consequence of a charge or a group of charges, is defined as the force that acts on a stationary charge divided by the magnitude of that charge.

\[ E = \frac{F_E}{Q} \]  

(A.2)

Here, \( E \) is the electric field in an arbitrary point and \( Q \) is the source of the electric field. Note that the electric field has the same direction as the force that acts on an arbitrary charge in that point. Since the principle of superposition is valid for electric fields, the total electric field of a group of charges equals the vector summation of the individual electric fields.

Electric Flux Density

Dielectric materials are characterised by the fact that the molecules inside a dielectric material tend to align with an externally applied electric field. When a molecule or atom of a dielectric
material is placed in an electric field, positive and negative charges will move with respect to each other. This causes the molecule to become an electric dipole with an electric dipole moment $\mu_E$. The polarisation vector is defined as the total dipole moment per unit volume and can be found by summing all dipole moments of the individual molecules:

$$P = \lim_{dV \to 0} \frac{\sum \mu_E}{dV}$$  \hspace{1cm} (A.3)

When polarised, a dielectric material is a source for an electric field itself. Consequently, the polarisation of the dielectric body is dependent on the primary external electric field, but also on its own polarisation. Therefore the electric flux density $D$ is defined as:

$$D = \varepsilon_0 E + P$$  \hspace{1cm} (A.4)

When the dielectric material is linear and isotropic, the polarisation can be written as:

$$P = \chi_e(E) \varepsilon_0 E$$  \hspace{1cm} (A.5)

where $\chi_e(E)$ is referred to as electric susceptibility that can be dependent on the electric field strength. The relation shows that the polarisation is proportional to the electric field in every point of the dielectric. For all dielectrics, $\chi_e > 0$. Only for vacuum, $\chi_e = 0$. Furthermore, the electric flux density then can be written as:

$$D = \varepsilon_0 E + \chi_e(E) \varepsilon_0 E = \varepsilon_0 (1 + \chi_e(E)) E = \varepsilon_0 \varepsilon_r(E) E = \varepsilon(E) E$$  \hspace{1cm} (A.6)

where $\varepsilon_r(E)$ is the relative permittivity of the dielectric and $\varepsilon(E)$ is the permittivity of the dielectric. For linear and isotropic materials these properties will be constants.

### A.2 Magnetic Field

**Magnetic Flux Density**

In a magnetic field, forces are exerted on currents or moving charges. To derive the relation between moving charges and forces exerted on them, Figure A.1 is used. Two current loops are depicted carrying the currents $I_1$ and $I_2$. The loops are divided into small current elements of length $d\ell_1$ and $d\ell_2$. The distance between the two elements is $r$ and the unit vector pointing...
A.2 Magnetic Field

**Figure A.2:** A magnetic dipole is defined from a current-carrying loop.

from one element to the other is \(e_r\). The magnetic force that the two current elements apply on each other follows from the experiments of Biot and Savard and is given by:

\[
dF_{B,12} = I_2 d\ell_2 \times \left( \frac{\mu_0 I_1 d\ell_1 \times e_r}{r^2} \right)
\]

(A.7)

where \(\mu_0\) is the permeability of the vacuum. From this equation it follows that two vector cross products are needed to compute the force \(F_{B,12}\). The magnetic field will be defined by the magnetic flux density \(B\). The magnetic flux density element \(dB\), caused by the element \(d\ell\) carrying a current \(I\), in an arbitrary point in space is defined as:

\[
dB = \frac{\mu_0 I d\ell \times e_r}{4\pi |r|^2}
\]

(A.8)

Now the unit vector \(e_r\) is pointing from the current element to the point \(P\) in space for which the magnetic flux density is calculated, as represented in Figure A.2. The superposition principle also holds for magnetic fields. So the total magnetic flux density in an arbitrary point is given by the summation of all individual contributions of current elements:

\[
B(r) = \frac{\mu_0}{4\pi} \int \frac{I d\ell \times (r' - r)}{|r' - r|^3}
\]

(A.9)

where \(r\) is the vector directed to the point in which the magnetic flux density must be calculated, \(r'\) is the vector pointing to the source element \(d\ell\), and use is made of:

\[
e_r = \frac{r' - r}{|r' - r|}
\]

(A.10)

Expression (A.9) is also known as Biot-Savard’s law.

**Magnetic Field Strength**

The presence of magnetic dipoles in atoms or on a macroscopic view is less obvious than for the electric dipoles. Negatively charged electrons moving around a positively charged core inside atoms can macroscopically be seen as a complicated system of current loops. All these current loops can be regarded as microscopic magnetic dipoles with magnetic dipole moment \(\mu_B\). The magnetic dipoles will also try to align with an externally applied magnetic field. The magnetisation vector \(M\) describes the total vector magnetic dipole moment per unit volume in a magnetic material at a given point:

\[
M = \lim_{dV \to 0} \frac{\sum \mu_B}{dV}
\]

(A.11)
When a magnetic material is magnetised, it will contribute to the total magnetic field strength \( H \) which is defined as:

\[
H = \frac{B}{\mu_0} - M
\]  
(A.12)

If the magnetic material is linear and isotropic, the following relation between the magnetic field strength and the magnetisation can be defined:

\[
M = \chi_m(H)H
\]  
(A.13)

where \( \chi_m(H) \) is referred to as the magnetic susceptibility and is generally a function of the magnetic field strength \( H \). With this formulation and the definition of \( H \) the magnetic flux density can be expressed in terms of the magnetic field intensity and material constants:

\[
B = \mu_0 (H + M) = \mu_0 (H + \chi_m(H)H) = \mu_0 (1 + \chi_m(H))H = \mu_0 \mu_r(H)H = \mu(H)H
\]  
(A.14)

Here, \( \mu_r(H) \) is the relative permeability and \( \mu(H) \) is the permeability of the material. Materials can be diamagnetic, for which \( \chi_m < 0 \) and \( \mu_r < 1 \), and paramagnetic, for which \( \chi_m > 0 \) and \( \mu_r > 1 \). A special case of materials are the ferromagnetic materials for which \( \chi_m \gg 0 \). For linear and isotropic materials the magnetic susceptibility is constant and therefore also the relative permeability and permeability are material constants.
Appendix B

Interface Conditions

To derive an expression for the interface conditions which the electric and magnetic field must obey, the theory treated in Section 2.5 is used. The condition for the electric and magnetic field will be derived separately.

B.1 Conditions in the Electric Field

Assume a situation as is depicted in Figure B.1(a). The electric field strength is discontinuous across a boundary. The regions on both sides of the boundary line are characterised by the permittivities $\varepsilon_1$ and $\varepsilon_2$. According to the first Maxwell equation, the following relation holds for the contour $abcda$:

$$\oint_{abcd} E \cdot d\ell = -\frac{\partial}{\partial t} \int_S B \cdot dS$$  \hspace{1cm} (B.1)

where $S$ is the surface enclosed by the contour. The magnetic flux density $B$ in the right hand side of this equation is defined on a surface. In the limit the distances $bc$ and $da$ go to zero and therefore also the enclosed volume tends to zero. Hence, in the limit the right hand side of Equation B.1 equals zero also. Only the tangential components of the electric field

---

**Figure B.1:** Tangential interface conditions in the electric field.
along the sides $ab$ and $cd$ contribute to the left hand side and the boundary condition for the tangential component of the electric field becomes:

$$E_{1t} - E_{2t} = 0 \iff E_{1t} = E_{2t} \quad (B.2)$$

The derivation of the normal component of the interface condition in the electric field starts with Figure [B.1(b)]. An infinitesimal cylinder across the boundary is considered. The third equation of Maxwell states that the following holds for the cylinder:

$$\oint_S D \cdot dS = \int_V \rho \, dV \quad (B.3)$$

with $S$ enclosing the volume $V$. The total charge contained in the volume of the cylinder is solely located on the boundary surface $dS$. If the height of the cylinder tends to zero, the total flux through the surface of the cylinder is given by the normal components of the electric field on the top and bottom surfaces $dS_1$ and $dS_2$. In analogy with the derivation for a general vector field, the interface condition for the normal component becomes:

$$D_{1n} - D_{2n} = \rho \quad (B.4)$$

Equations [B.2] and [B.4] together define the interface conditions for the magnetic field.

### B.2 Conditions in the Magnetic Field

To derive the interface conditions for the magnetic field, the same procedure as in the previous section is followed. According to the second Maxwell equation, the following holds for the line integral of the magnetic field strength along a closed path $abcd$:

$$\oint_{abcd} H \cdot d\ell = \int_S J \cdot dS + \frac{\partial}{\partial t} \int_S D \cdot dS \quad (B.5)$$

The situation with the relevant parameters is given in Figure [B.2(a)]. As the electric flux density, the magnetic flux density is defined on a surface. Because the surface enclosed by
the loop $abcda$ tends to zero, the second term on the right hand side of Equation B.5 equals zero in the limit. The current density $J$, however, is defined as a line density in this case and therefore the first term on the right hand side is nonzero. By defining the scalar current density $J$ the boundary condition for the tangential component on the magnetic field strength becomes:

$$H_1 - H_2 = J$$

(B.6)

The derivation of the interface condition for the normal component of the magnetic field starts with the fourth equation of Maxwell which states that:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

(B.7)

To analyse this situation, Figure B.2(b) is considered. The fourth equation of Maxwell states that the total flux through a closed surface $S$ equals zero. Contributions to the flux only come from the surface $dS_1$ and $dS_2$. Together with Equation B.7 the boundary condition becomes:

$$B_{1n} - B_{2n} = 0 \iff B_{1n} = B_{2n}$$

(B.8)

Equations B.6 and B.8 together give the interface conditions inside the magnetic field. The meaning of the interface conditions is discussed in Section 3.4.1.
Appendix C

The Magnetic Vector Potential

In this appendix, the magnetic vector potential is used to obtain an equivalent of the Maxwell’s equations. The derivation starts with the definition of the magnetic vector potential $A$:

$$ B = \nabla \times A $$

(C.1)

When the definition of the magnetic vector potential is substituted in the first and third equation of Maxwell, i.e., Equation 3.1a and 3.1c the following relations are obtained:

$$ \nabla \times E = -\frac{\partial (\nabla \times A)}{\partial t} $$

(C.2)

$$ \nabla \frac{1}{\mu} \nabla \times A = J + \frac{\partial D}{\partial t} $$

(C.3)

The nabla operator and the time derivative are interchangeable, so the result for the first equation can also be written as:

$$ \nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0 $$

(C.4)

The term in the parentheses is curl free and it may be written as the gradient of the electric scalar potential $\nabla V$ which is defined as:

$$ E + \frac{\partial A}{\partial t} = -\nabla V $$

(C.5)

By rearranging this equation, the electric field strength $E$ can be written as:

$$ E = -\frac{\partial A}{\partial t} - \nabla V $$

(C.6)

If this equation for the electric field is substituted into Equation C.3, we get:

$$ \nabla \times \frac{1}{\mu} \nabla \times A = J + \frac{\partial}{\partial t} \varepsilon \left( -\frac{\partial A}{\partial t} - \nabla V \right) $$

(C.7)

Now we assume that the material in which the relation is defined is linear, such that the permeability $\mu$ is independent of $B$ and that the permittivity $\varepsilon$ is independent of $E$. This gives:

$$ \nabla \times (\nabla \times A) = \mu J - \mu \varepsilon \frac{\partial^2 A}{\partial t^2} - \mu \varepsilon \frac{\partial}{\partial t} (\nabla V) $$

(C.8)
The left hand side of this equation can be expanded using the identity \( \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \). The nabla operator and the time derivative are interchangeable and so Equation C.8 can be rewritten as:

\[
\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu J - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \mu \varepsilon \frac{\partial V}{\partial t} \right) \quad (C.9)
\]

Now, a relation between the magnetic vector potential and the electric scalar potential is obtained. To solve the equation in terms of the magnetic vector potential, a relation for the electric scalar potential is necessary. Possibilities to remove the term containing the electric scalar potential are given by assuming it zero, independent of time, or constant in space. This is, however, not a general property of the electric field and therefore the assumptions cannot be made in general. In some special cases, however, the assumptions can be used to obtain a expression in \( \mathbf{A} \) solely. Another option is given by specifying the divergence of the magnetic vector potential. According to the Theorem [4] the Helmholtz theorem, a vector field is defined uniquely by its curl and its divergence. The curl of the magnetic vector potential is given by its definition in Equation C.1. If the divergence of the magnetic vector potential is chosen as:

\[
\nabla \cdot \mathbf{A} = -\mu \varepsilon \frac{\partial V}{\partial t} \quad (C.10)
\]

the last term in Equation C.9 vanishes and we get:

\[
-\nabla^2 \mathbf{A} = \mu J - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad (C.11)
\]

This is the relation for the magnetic vector potential given in Section 4.4.1. Relation C.10 is known as the Lorentz condition or Lorentz gauge. It can be shown that the Lorentz gauge is consistent with the field equations and the principle of conservation of charge, making it a correct condition. In the case static fields are considered, the Coulomb gauge can be used to determine a solution for the magnetic vector potential:

\[
\nabla \cdot \mathbf{A} = 0 \quad (C.12)
\]

In the static case and with the use of the Coulomb gauge, Equation C.9 reduces to:

\[
-\nabla^2 \mathbf{A} = \mu J \quad (C.13)
\]
Bibliography


