TECHNICAL NOTE

Newsboy Characterizations for the Optimal Reorder Levels of Multi-Echelon Inventory Systems with Fixed Batch Sizes

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Abstract

This note considers an $N$-stage serial production/inventory system where materials flow from one stage to another in fixed batches. Linear holding and penalty costs (for backorders) are assumed. By Chen [2000], echelon stock $(R, nQ)$ policies are optimal for such systems. Based on the results of Chen [1998], we show that the optimal reorder levels satisfy newsboy inequalities (equalities) when the demand has a discrete (continuous) distribution. The newsboy inequalities/equalities show a direct relation between the probability of no-stockout at the most downstream point and the cost parameters. Thus, they contribute to the understanding of optimal control. Also, they are easy to explain to managers and non-mathematical oriented students.

Keywords: Inventory/Production: multi-echelon, serial systems, $(R, nQ)$ policy, newsboy characterizations.

Area of review: Operations and Supply Chain Management

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1. Introduction

This technical note considers an $N$-stage serial inventory/production system facing stochastic demand of the customers at the most downstream stage (stage 1). The stages are numbered such that stage 1 orders from stage 2, 2 from 3, ..., and stage $N$ from an external supplier with ample stock. The order size at each stage is required to be a nonnegative integer multiple of a base quantity specific for that stage. Further, there is an integer-ratio constraint implying that the base quantity at some stage should be a positive integer multiple of the base quantity of the immediate successor stage. The leadtimes between the stages are constant. Any unfulfilled demand is backlogged and a penalty cost is incurred. We assume centralized control and periodic review of the inventories. The objective is to minimize the average expected holding and penalty costs of the system in an infinite horizon.

For the system under study, Chen [2000] has shown that an optimal ordering policy for each stage is to follow an echelon stock $(R, nQ)$ policy: whenever the echelon inventory position at stage $i$ is at or below the reorder level $R_i$, a minimum integer multiple of its base quantity ($Q_i$) that brings the echelon inventory position above $R_i$ is ordered from stage $i + 1$. By Chen [1998], the optimal reorder points are calculated by solving $N$ single-stage $(R, nQ)$ models sequentially. In other words, $N$ single dimensional average cost functions are minimized successively. This note focuses on the optimal reorder points. We heavily draw on the results of Chen [1998]. Introducing a new representation based on the concept of shortfall, we are able to derive alternative expressions for cost functions, which lead to newsboy characterizations (newsboy inequalities/equalities) for the optimal reorder levels. Newsboy inequalities/equalities are expressions that show a direct relation between the probability of no-stockout at stage 1 (as a result of a given reorder level at some stage $i$) and the cost parameters.
Our contribution in this study is as follows. We show that the optimal reorder levels in an N-stage serial inventory/production system with fixed batch sizes satisfy newsboy inequalities (equalities) when the demand distribution is discrete (continuous). Newsboy characterizations are appealing because: (i) they provide new insights and contribute to the understanding of optimal control, (ii) they are relatively easy to explain to non-mathematical oriented students (e.g., MBA students) and managers when compared to recursive expressions that one finds in the literature.

Under continuous demand, newsboy equalities have been derived for multi-echelon serial systems without batching, serial systems with fixed replenishment intervals, and divergent systems under balance assumption by Van Houtum and Zijm [1991], Van Houtum, Scheller-Wolf and Yi [2003], and Diks and De Kok [1998], respectively. Doğru, De Kok, and Van Houtum [2004] extended the results of Diks and De Kok [1998] to the discrete demand case where newsboy inequalities instead of newsboy equalities are obtained. This study generalizes the newsboy characterizations of Van Houtum and Zijm [1991] in two directions; our model incorporates fixed batch sizes and can handle discrete demand distributions.

In a recent paper, Shang and Song [2005] consider the same model that we study in this note. They construct upper and lower bound functions for the cost functions that have to be minimized sequentially in order to calculate optimal reorder points. The upper and lower bound functions are single-stage newsboy functions with modified holding and penalty costs. The minimization of these functions leads to lower and upper bounds for the optimal reorder levels, which follow from newsboy characterizations. In contrast, we derive newsboy characterizations for the optimal reorder levels themselves. By our newsboy characterizations, one obtains an alternative proof for the property that the variables $r_i^l$ as defined by Shang and Song [2005] are lower bounds for the optimal reorder levels; see Remark 3.

The rest of the paper is organized as follows. We introduce the model in §2. The results from Chen and Zheng [1994a], and Chen [1998] that we use in our analysis are presented in §3.1.
Newsboy characterizations are derived in §3.2. We dedicate §3.3 for the discussion of our results when the demand process is continuous, and conclude in §4.

2. Model

We consider a serial $N$-stage inventory/production system under periodic review. The most downstream stage, stage 1, orders from stage 2, 2 from 3, ..., $N - 1$ from $N$, and stage $N$ from an exogenous supplier (called stage $N+1$) with ample stock. Any order of stage $i$ is an integer multiple of a base order quantity $Q_i$. Further, we assume that $Q_{i+1} = n_i Q_i$ for $i = 1, ..., N - 1$ where $n_i$ is a positive integer (integer-ratio assumption). Because of the integer-ratio assumption, we presuppose that the initial on-hand stock at stage $i$ is an integer multiple of $Q_{i-1}$ for $i = 2, ..., N$. Stage 1 faces the stochastic demand of the customers. Demands in different periods are i.i.d., discrete, nonnegative random variables. Any unfulfilled customer demand at stage 1 is backlogged and a penalty cost is incurred. There are deterministic leadtimes between the stages, and between the supplier and stage $N$. Holding cost at every stage, and penalty cost at stage 1 are linear. We assume centralized control and the objective is to minimize the average expected holding and penalty costs of the system in the long-run.

The following sequence of events takes place during a period: (i) inventory levels at all stages are observed and the current period’s ordering decisions are made (at the beginning of the period), (ii) orders arrive following their respective leadtimes (at the beginning of the period), (iii) demand occurs during the period, (iv) holding and penalty costs are assessed on the period ending inventory and backorder levels (at the end of the period).

We follow the same notation and assumptions as Chen [1998], except that we consider a periodic review setting with i.i.d. demand. For details, we refer to §2 and §3 of Chen [1998]. As indicated by Chen, his analysis and the results also hold for periodic review models with i.i.d. demand. However, we need to modify the definitions of some variables and introduce new notation. In
addition to the notation introduced above, we define:

\( \mathbb{Z} = \) set of integers; \( \mathbb{Z}^+ \) is the set of positive integers, and \( \mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\} \).

\( \mathbb{R} = \) set of real numbers.

\( t = \) index for periods, \( t \in \mathbb{Z}^+ \).

\( L_i = \) leadtime from stage \( i + 1 \) to stage \( i \), \( L_1 \in \mathbb{Z}_0^+ \) and \( L_i \in \mathbb{Z}^+ \) for \( i = 2, ..., N \).

\( l_i = \) total leadtime from the outside supplier to stage \( i \), \( l_i = \sum_{j=1}^{N} L_j \) for \( i = 1, ..., N \).

\( h_i = \) echelon holding cost per unit per period at stage \( i \), \( h_i > 0 \) for \( i = 1, ..., N \).

\( H_i = \) installation holding cost per unit per period at stage \( i \), \( H_i = \sum_{j=i}^{N} h_j \) for \( i = 1, ..., N \).

\( p = \) penalty cost per backlogged unit per period, \( p > 0 \).

\( D(t) = \) discrete demand in period \( t \), which is distributed over \( \mathbb{Z}_0^+ \) with \( \Pr\{D(t) = 1\} > 0 \).

\( \mu = \) expected one-period demand, \( \mathbb{E}[D(t)] = \mu \) \( \forall t, \mu > 0 \).

\( D_i(t) = \) demand during the periods \( t + l_{i+1}, ..., t + l_i \).

\( D^-_i(t) = \) demand during the periods \( t + l_{i+1}, ..., t + l_i - 1 \).

\( F = \) cumulative distribution function of one-period demand defined over \( \mathbb{Z}_0^+ \).

\( B(t) = \) backorder level at stage 1 at the end of period \( t \).

\( IL_i(t) = \) echelon inventory of stage \( i \) at the end of period \( t \), i.e., on-hand inventory at stage \( i \) plus inventories in transit to or on-hand at stages 1, ..., \( i - 1 \) minus backorders at stage 1, \( i = 1, ..., N \).

\( IL^-_i(t) = \) echelon inventory of stage \( i \) at the beginning of period \( t \) just after the receipt of the incoming order, but before the demand, \( IL_i(t) = IL^-_i(t) - D(t), i = 1, ..., N \).

\( IP_i(t) = \) echelon inventory position at stage \( i \) at the beginning of period \( t \) just after ordering, but before the demand, i.e., \( IL^-_i(t) + \) inventories in transit to stage \( i \), \( i = 1, ..., N \).

When the period index \( t \) in variables \( D_i(t) \), \( D^-_i(t) \), \( B(t) \), \( IL_i(t) \), \( IL^-_i(t) \) and \( IP_i(t) \) is suppressed, the notation represents the corresponding steady state variables. While \( D^-_i \) denotes \( L_i \) period demand in steady state, \( D_i \) stands for \( L_i + 1 \) period demand in steady state.
Remark 1: We assume $\Pr\{D(t) = 1\} > 0$ in order to obtain an irreducible Markov chain that describes the behavior of the system. All results derived in the rest of this paper are valid under this condition. However, the results can be shown to hold under weaker conditions on the demand, see the appendix of Chen [2000].

3. Analysis

This section is composed of three parts. We review some of the main results from Chen [1998] in §3.1; as a matter of fact, some results stem from Chen and Zheng [1994a]. These are used in §3.2 to derive the newsboy characterizations. The case of continuous demand and its consequences on the newsboy characterizations are discussed in §3.3.

3.1 Preliminaries

We express the expected holding costs via echelon holding cost parameters. At the end of period $t$, the expected holding and penalty cost of the system is

$$
\sum_{i=1}^{N} h_i IL_i(t) + (p + H_1)B(t).
$$

Consider the following chain of actions that starts with the ordering decision of stage $N$ in period $t$. After ordering, the echelon inventory position of stage $N$ is $IP_N(t)$. This order is received by stage $N$ at the beginning of period $t + l_N$ and the echelon inventory of stage $N$ at that epoch is $IL_N(t + l_N)$. Note that $IL_N(t + l_N)$

(i) bounds the ordering decision of stage $N - 1$ in period $t + l_N$ from above, i.e., $IP_{N-1}(t + l_N) \leq IL_N(t + l_N)$, and

(ii) determines the echelon holding cost of stage $N$ at the end of period $t + l_N$, i.e., $h_N IL_N(t + l_N)$. 


Similarly, the order placed by stage \( N-1 \) in period \( t+l_N \) arrives in period \( t+l_{N-1} \) and it bounds the ordering decision of stage \( N-2 \) from above, i.e., \( IP_{N-2}(t+l_{N-1}) \leq IL_{N-1}^-(t+l_{N-1}) \), and effects the echelon \( N-1 \) cost \( h_{N-1}IL_{N-1}(t+l_{N-1}) \). Apply this reasoning for the rest of the stages.

The sum of the costs as a consequence of the chain of actions that starts with the ordering decision of stage \( N \) in period \( t \) is

\[
\sum_{i=1}^{N} h_i IL_i(t+l_i) + (p + H_1)B(t+l_1).
\]  

(2)

Note that, under the average cost criterion in an infinite horizon, (1) and (2) are equivalent. Further, when the system reaches the steady state, the time index can be suppressed for the calculation of the expected value of (2).

Chen [2000] showed that echelon stock \((R, nQ)\) policies are optimal for each stage of the system under study. Hence we are interested in finding optimal echelon reorder levels, \( R^* = (R^*_1, ..., R^*_N) \), that minimize

\[
C(R) \defeq E \left[ \sum_{i=1}^{N} h_i IL_i + (p + H_1)B \right],
\]  

(3)

where \( C(R) \) is the expected average cost of the system in a steady state when echelon reorder points \( R = (R_1, ..., R_N) \) are used (cf. Chen [1998], p. S225).

In order to set the stage for new results, we now review some important findings that originate from Chen and Zheng [1994a] and Chen [1998]. First, note that

\[
IL_i = IP_i - D_i, \quad i = 1, ..., N
\]  

(4)

From Lemma 1 of Chen and Zheng [1994a],

\[
IP_i = O_i(IL_{i+1}^-), \quad i = 1, ..., N
\]

where

\[
O_i[x] \defeq \begin{cases} 
  x & \text{if } x \leq R_i + Q_i \\
  x - mQ_i & \text{if } x > R_i + Q_i,
\end{cases}
\]
and \( m = \max\{b | x - bQ_i > R_i, \ b \in \mathbb{R}^+ \} \).

We define the following random variables:

\[
Pr\{U_i = u\} = \frac{1}{Q_i}, \quad u = 1, ..., Q_i, \ i = 1, ..., N
\]

\[
Pr\{Z_i = z\} = \frac{1}{n_i}, \quad z = 0, ..., n_i - 1, \ i = 1, ..., N - 1.
\]

These random variables are independent of each other and independent of the demand process. If two random variables \( X \) and \( Y \) have the same distribution, then we denote it by \( X \overset{d}{=} Y \). From Lemma 1 in Chen [1998], it holds that

\[
U_{i+1} \overset{d}{=} Z_i Q_i + U_i \quad \text{for } i = 1, ..., N - 1. \tag{5}
\]

Chen [1998] introduced the concept of effective reorder point at stage \( i \), denoted by \( V_i \). The effective reorder points are defined recursively as:

\[
V_N = R_N, \quad V_i = \min\{R_i, V_{i+1} + Z_i Q_i - D_{i+1}^-\} \quad \text{for } i = N - 1, ..., 1.
\]

(Notice that when a recursion starts from some stage \( i \) and continues until stage \( j \) for \( i > j \), we use a descending index.)

It has been shown in Theorem 1 of Chen [1998] that \( IP_i \overset{d}{=} V_i + U_i \) for \( i = 1, ..., N \). In other words, \( IP_i \) has the same distribution as in a single-stage \((R, nQ)\) system with reorder level \( V_i \) and base quantity \( Q_i \). This important result implies that due to a possible material unavailability at stage \( i + 1 \) (the immediate upstream stage), the actual reorder point becomes random though a fixed reorder level \( R_i \) is used.

### 3.2 Newsboy Characterizations

We are now prepared for new results. First, we develop an alternative average cost formula for \( C(R) \). Then, subsystem costs are introduced and the first order difference functions of these costs are derived. These are explicit difference functions that lead to the newsboy characterizations.
Alternative average cost formula

Define

\[ B_i = \text{shortfall of the effective reorder point from the reorder level at stage } i, \text{ i.e., } B_i = R_i - V_i \text{ for } i = 1, \ldots, N. \]

\[ B_0 = \text{backorder level at stage 1, i.e., } B_0 = B. \]

Due to the infinite stock at the supplier,

\[ B_N = 0. \tag{6} \]

For \( i = N - 1, \ldots, 1: \)

\[ B_i = R_i - V_i \]

\[ = R_i - \min\{R_i, V_{i+1} + Z_iQ_i - D_{i+1}^{-}\} \]

\[ = R_i + \max\{-R_i, -V_{i+1} - Z_iQ_i + D_{i+1}^{-}\} \]

\[ = \max\{0, R_i - V_{i+1} - Z_iQ_i + D_{i+1}^{-}\} \]

\[ = \max\{0, (R_{i+1} - V_{i+1}) - (R_{i+1} - R_i) - Z_iQ_i + D_{i+1}^{-}\} \]

\[ = [B_{i+1} + D_{i+1}^{-} - (R_{i+1} - R_i) - Z_iQ_i]^+, \tag{7} \]

where \([x]^+ = \max\{0, x\}\) for \( x \in \mathbb{R} \). Since \( B = [D_1 - IP_1]^+ \) and \( IP_1 \overset{d}{=} V_1 + U_1 \), \( B = [D_1 - V_1 - U_1]^+ \).

Substituting \( V_1 = R_1 - B_1 \) in the expression for \( B \) leads to

\[ B_0 = B = [B_1 + D_1 - R_1 - U_1]^+. \tag{8} \]

**Lemma 1** \( IP_i \overset{d}{=} R_i - B_i + U_i \) for \( i = 1, \ldots, N \) where the \( B_i \) are defined by (6)-(7).

**Proof:** By Theorem 1 of Chen [1998], \( IP_i \overset{d}{=} V_i + U_i \) for \( i = 1, \ldots, N \). By definition, \( B_i = R_i - V_i \) for \( i = 1, \ldots, N \). Substituting \( V_i = R_i - B_i \) into the result of Theorem 1 in Chen [1998] leads to the result given in the lemma. \( \square \)
The following lemma gives an alternative formula for the expected long-run average cost of the system.

**Lemma 2** Let \( R \in \mathbb{N}^N \). The long-run average expected cost under the echelon stock \((R, nQ)\) policy with reorder levels \( R \) is

\[
C(R) = \sum_{i=1}^{N} h_i \left( R_i - E[B_i] + \frac{Q_i + 1}{2} - (L_i + 1)\mu \right) + (p + H_1)E[B_0],
\]

where \( B_i \) for \( i = N, \ldots, 0 \) are defined by (6)-(8).

**Proof**: The result is obtained by the substitution of (4) and the result of Lemma 1 into (3):

\[
C(R) = E \left[ \sum_{i=1}^{N} h_i IL_i + (p + H_1)B \right] = E \left[ \sum_{i=1}^{N} h_i (IP_i - D_i) + (p + H_1)B_0 \right] = E \left[ \sum_{i=1}^{N} h_i (R_i - B_i + U_i - D_i) + (p + H_1)B_0 \right] = \sum_{i=1}^{N} h_i \left( R_i - E[B_i] + \frac{Q_i + 1}{2} - (L_i + 1)\mu \right) + (p + H_1)E[B_0]. \quad \square
\]

**Subsystem costs and forward difference functions**

Consider the following \( i \)-stage serial subsystem: for \( i \in \{1, \ldots, N\} \), stage \( k \) follows echelon stock \((R_k, nQ_k)\) policy for \( k = 1, \ldots, i \), and stage \( i + 1 \) has ample stock. A similar expected long-run average cost expression is obtained for this subsystem as for the full system in Lemma 2:

\[
C_i(R_1, \ldots, R_i) \overset{\text{def}}{=} \sum_{k=1}^{i} h_k \left( R_k - E[B_k^{(i)}] + \frac{Q_k + 1}{2} - (L_k + 1)\mu \right) + (p + H_1)E[B_0^{(i)}], \quad (9)
\]

where the \( B_k^{(i)} \) for \( k = i, \ldots, 0 \) are defined by

\[
B_i^{(i)} = 0, \quad (10)
\]

\[
B_k^{(i)} = \left[ B_{k+1}^{(i)} + D_{k+1}^\ast - (R_{k+1} - R_k) - Z_kQ_k \right]^+ \quad k = i - 1, \ldots, 1, \quad (11)
\]

\[
B_0^{(i)} = \left[ B_1^{(i)} + D_1 - R_1 - U_1 \right]^+. \quad (12)
\]
Remark 2: Note that $B_k^{(N)} = B_k$ for $k = 0, ..., N$. For $i = 1, ..., N - 1$: $B_k^{(i)} = (B_k | B_i = 0)$ for $k = 0, ..., i$. Further, $C_N(R) = C(R)$.

Define $c_i(R_1, ..., R_i) = C_i(R_1, ..., R_i) - C_i(R_1, ..., R_i)$, which is the first order forward difference function of $C_i(\cdot)$ in $R_i$.

Lemma 3 Let $i \in \{1, ..., N\}$, and $(R_1, ..., R_i) \in \mathcal{R}$. Then,

$$c_i(R_1, ..., R_i) = h_i + \sum_{k=1}^{i-1} h_k \Pr\{ B_j^{(i)} > 0 \text{ for } j = i - 1, ..., k \}$$

$$- (p + H_1) \Pr\{ B_j^{(i)} > 0 \text{ for } j = i - 1, ..., 0 \},$$

(13)

where the $B_k^{(i)}$ for $k = i - 1, ..., 0$ are defined by (10)-(12), and the sum on the righthand side is taken to be a null sum when $i = 1$.

Proof: Let $\hat{B}_k^{(i)}$ be defined by (10)-(12), but with $R_i$ replaced by $R_{i+1}$. Then, $\hat{B}_i^{(i)} = B_i^{(i)} = 0$. Note that $\hat{B}_{i-1}^{(i)} = [B_i^{(i)} + D_i^+ - (R_{i+1} - R_{i-1}) - Z_{i-1} Q_{i-1}]^+ = [B_{i-1}^{(i)} - 1]^+$. If $B_{i-1}^{(i)} = 0$ then $\hat{B}_{i-1}^{(i)} = 0$; else if $B_{i-1}^{(i)} > 0$ then $\hat{B}_{i-1}^{(i)} - B_{i-1}^{(i)} = -1$. Next, $\hat{B}_{i-2}^{(i)} = [\hat{B}_{i-1}^{(i)} + D_{i-1}^+ - (R_{i-1} - R_{i-2}) - Z_{i-2} Q_{i-2}]^+$. If $B_{i-1}^{(i)} > 0$ and $B_{i-2}^{(i)} > 0$, then $\hat{B}_{i-2}^{(i)} - B_{i-2}^{(i)} = -1$; if $B_{i-1}^{(i)} = 0$ or $B_{i-2}^{(i)} = 0$, then $\hat{B}_{i-2}^{(i)} = B_{i-2}^{(i)}$. Continuing in this fashion shows that $\mathbb{E}[\hat{B}_k^{(i)}] - \mathbb{E}[B_k^{(i)}] = \Pr\{ B_j^{(i)} > 0 \text{ for } j = i - 1, ..., k \}, k = i - 1, ..., 0$.

This result in combination with (9) leads to (13). □

We are able to rewrite (13) in a recursive way, which is given in the next lemma.

Lemma 4 For $i \in \{1, ..., N\}$, and $(R_1, ..., R_i) \in \mathcal{R}$:

$$c_i(R_1, ..., R_i) = \sum_{k=1}^{i} h_k - (p + H_1) \Pr\{ B_0^{(i)} > 0 \} - \sum_{k=1}^{i-1} \Pr\{ B_k^{(i)} = 0 \} c_k(R_1, ..., R_k),$$

(14)

where the $B_k^{(i)}$ for $k = i - 1, ..., 0$ are defined by (10)-(12), and the second sum on the righthand side is taken to be a null sum when $i = 1$. 

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Proof: For $i = 1$, (14) is read as $c_1(R_1) = h_1 - (p + H_1)\Pr\{B_{0}^{(i)} > 0\}$. This is the expression given in Lemma 3. In the rest of the proof, let $i \geq 2$, and any summation with the lower limit greater than the upper limit (e.g., $\sum_{m=i}^{i-1}$) be a null sum. For $k \in \{1, \ldots, i-1\}$, we may rewrite

$$\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1, \ldots, k\}$$

as

$$\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1, \ldots, k\} = 1 - \left( \Pr\{B_k^{(i)} = 0\} + \Pr\{B_{k+1}^{(i)} = 0, B_k^{(i)} > 0\} + \ldots + \Pr\{B_{i-1}^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = i-2, \ldots, k\} \right)$$

$$= 1 - \left( \Pr\{B_k^{(i)} = 0\} + \sum_{m=k+1}^{i-1} \Pr\{B_m^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, k\} \right)$$

$$= 1 - \left( \Pr\{B_k^{(i)} = 0\} + \sum_{m=k+1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, k|B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\} \right), \quad (15)$$

Similarly,

$$\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1, \ldots, 0\} = 1 - \left( \Pr\{B_0^{(i)} = 0\} + \Pr\{B_1^{(i)} = 0, B_0^{(i)} > 0\} + \ldots + \Pr\{B_{i-1}^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = i-2, \ldots, 0\} \right)$$

$$= \Pr\{B_0^{(i)} > 0\} - \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, 0|B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\}. \quad (16)$$

Substituting (15) and (16) into (13), and rearranging the terms results in

$$c_i(R_1, \ldots, R_i) = \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\}$$

$$- \left[ \sum_{k=1}^{i-1} h_k \sum_{m=k+1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, k|B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\} \right]$$

$$-(p + H_1) \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, 0|B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\} \right].$$

Changing the order of summation gives

$$c_i(R_1, \ldots, R_i) = \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\}$$

$$- \left[ \sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, k|B_m^{(i)} = 0\} \right]$$

$$-(p + H_1) \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, \ldots, 0|B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\} \right].$$
Next,
\[
c_i(R_1, \ldots, R_i) = \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\}
- \left[ \sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(m)} > 0\} \text{ for } j = m - 1, \ldots, k \right]
- (p + H_1) \sum_{m=2}^{i-1} \Pr\{B_j^{(m)} > 0\} \text{ for } j = m - 1, \ldots, 0 \Pr\{B_m^{(i)} = 0\}
\]
\[
= \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \Pr\{B_1^{(i)} = 0\} \left[ h_1 - (p + H_1)\Pr\{B_0^{(1)} > 0\} \right]
- \sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \left[ h_m + \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(m)} > 0\} \text{ for } j = m - 1, \ldots, k \right]
- (p + H_1)\Pr\{B_j^{(m)} > 0\} \text{ for } j = m - 1, \ldots, 0\right]
\]
\[
= \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(R_1, \ldots, R_k),
\]
where the first equality follows from the fact that \( \{B_j^{(i)} > 0\} \text{ for } j = m - 1, \ldots, k | B_m^{(i)} = 0 \} \equiv \{B_j^{(m)} > 0\} \text{ for } j = m - 1, \ldots, k \) for \( k \in \{m - 1, \ldots, 0\} \), second equality from rewriting some of the expressions under a single summation, and the last equality from the equivalence of (13) for \( i = 1 \) and \( i = m \) to the expressions within the first and the second brackets, respectively. \( \square \)

**Newsboy Inequalities**

Let \( \overline{Y}_1 \) be a minimizing point of \( C_1(R_1) \). For \( i = 2, \ldots, N \), define recursively \( \overline{Y}_i \in \mathcal{S} \) as a point that minimizes \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \). The function \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \) is equal to \( \overline{G}_i(R_i) \) of Chen [1998] for \( i = 1, \ldots, N \) and \( R_i \in \mathcal{S} \). As shown there in Lemma 3, \( \overline{G}_i(R_i) \) is convex in \( R_i \) (and thus also \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i)) \). Due to Theorem 1 of Chen [1998], the optimal reorder levels are found by minimizing \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \) for \( i = 1, \ldots, N \) recursively.

**Theorem 1** Let \( i \in \{1, \ldots, N\} \). An optimal reorder level \( \overline{Y}_i \in \mathcal{S} \) is an element of the set \( y_i \in \{y_i^l, y_i^l + 1, \ldots, y_i^u\} \) where
\[
y_i^l = \min \left\{ R_i | \Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, \ldots, \overline{Y}_k) \right\}, \quad (17)
\]
\[ y_i^u = \min \left\{ R_i | \Pr \{ B_0^{(i)} = 0 \} > \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr \{ B_k^{(i)} = 0 \} c_k(\overline{Y}_1, \ldots, \overline{Y}_k) \right\}, \quad \text{and} \]

\[ B_i^{(i)} = 0, \quad (18) \]

\[ B_{i-1}^{(i)} = \left[ B_i^{(i)} + D_i^r - (R_i - \overline{Y}_{i-1}) - Z_{i-1}Q_{i-1} \right]^+, \quad (19) \]

\[ B_k^{(i)} = \left[ B_{k+1}^{(i)} + D_{k+1}^r - (\overline{Y}_{k+1} - \overline{Y}_k) - Z_kQ_k \right]^+, \quad \text{for } k = i - 2, \ldots, 1, \quad (20) \]

\[ B_0^{(i)} = \left[ B_1^{(i)} + D_1 - (\overline{Y}_1 + U_1) \right]^+. \quad (21) \]

(For \( i = 1 \): \( B_1^{(1)} = 0, \ B_0^{(1)} = [B_1^{(1)} + D_1 - (R_1 + U_1)]^+ \), and the summations \( \sum_{k=1}^0 \) are taken to be null sums.)

The set \( \left\{ R_i | \Pr \{ B_0^{(i)} = 0 \} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr \{ B_k^{(i)} = 0 \} c_k(\overline{Y}_1, \ldots, \overline{Y}_k) \right\} \neq \emptyset \); thus, \( y_i^l \) is finite. The set \( \left\{ R_i | \Pr \{ B_0^{(i)} = 0 \} > \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr \{ B_k^{(i)} = 0 \} c_k(\overline{Y}_1, \ldots, \overline{Y}_k) \right\} \) may be empty; then, \( y_i^u = +\infty \).

**Proof:** From the convexity of \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \) in \( R_i \), \( \overline{Y}_i \in Y_i = \{ y_i^l, y_i^l + 1, \ldots, y_i^u \} \) where \( y_i^l \) is the minimum \( R_i \) satisfying \( c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \geq 0 \), and \( y_i^u \) is the minimum \( R_i \) satisfying \( c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) > 0 \). By Lemma (4),

\[ c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) = \sum_{k=1}^i h_k - (p + H_1)\Pr \{ B_0^{(i)} > 0 \} - \sum_{k=1}^{i-1} \Pr \{ B_k^{(i)} = 0 \} c_k(\overline{Y}_1, \ldots, \overline{Y}_k) \geq 0 \]

with the \( B_k^{(i)} \) defined by (18)-(21). By substituting \( \Pr \{ B_0^{(i)} > 0 \} = 1 - \Pr \{ B_0^{(i)} = 0 \} \) and rearranging the terms, this inequality may be rewritten as

\[ \Pr \{ B_0^{(i)} = 0 \} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr \{ B_k^{(i)} = 0 \} c_k(\overline{Y}_1, \ldots, \overline{Y}_k). \]

The expression for \( y_i^u \) can be derived similarly.

By Lemma 3, \( \lim_{R_i \to +\infty} c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) = h_i \) for \( i = 1, \ldots, N \). Further, due to the convexity of \( C_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \) in \( R_i \), \( c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \) is a nondecreasing function of \( R_i \). Since \( h_i > 0 \), there exists finite \( R_i \) values that satisfy \( c_i(\overline{Y}_1, \ldots, \overline{Y}_{i-1}, R_i) \geq 0 \). Hence, \( y_i^l \) is a finite point. □
The intuitive message of Theorem 1 is informative. Assume that \( N = 1 \), i.e., there is a single-stage system. Note that (17) becomes \( \Pr\{ B_0^{(1)} = 0 \} \geq \frac{p+H_2}{p+H_1} = \frac{p}{p+h_1} \). The optimal reorder level, \( \bar{Y}_1 \), is chosen such that the probability of having no-stockout (as a consequence of this reorder level) is greater than or equal to \( \frac{p}{p+h_1} \), which is the newsboy fractile in a single-stage inventory system. For a general \( N \)-stage system, an optimal reorder level at each stage \( i \geq 1 \) is chosen such that the probability of no-stockout at stage 1 in the \( i \)-stage subsystem is at least equal to \( \frac{p+H_{i+1}}{p+H_1} \) plus a term that depends on the extend the newsboy fractiles are met at the stages \( 1, \ldots, i - 1 \). The second term is nonnegative, by definition, since \( c_k(\bar{Y}_1, \ldots, \bar{Y}_k) \geq 0 \) for \( k = 1, \ldots, N \). This leads to the following corollary.

**Corollary 1** For \( i = 1, \ldots, N \): \( \bar{Y}_i \) satisfies

\[
\Pr\{ B_0^{(i)} = 0 \} \geq \frac{p+H_{i+1}}{p+H_1},
\]

where \( B_k^{(i)} \) for \( k = i, \ldots, 0 \) are defined in (18)-(21).

We can verify the following intuitive relationship between the holding cost parameters and the reorder levels under an optimal policy.

**Corollary 2** There exists an optimal \((R, nQ)\) policy under which no safety stock is held at stage \( i + 1 \) when \( h_i \downarrow 0 \), \( i = 1, \ldots, N - 1 \).

**Proof:** Let \( i \in \{1, \ldots, N-1\} \). Note that \( \lim_{R_i \to +\infty} c_i(\bar{Y}_1, \ldots, \bar{Y}_{i-1}, R_i) = h_i \) and \( c_i(\bar{Y}_1, \ldots, \bar{Y}_{i-1}, R_i) \) is a nondecreasing function of \( R_i \) due to Lemma 3 and the convexity of \( C_i(\bar{Y}_1, \ldots, \bar{Y}_{i-1}, R_i) \) in \( R_i \), respectively. If \( h_i \downarrow 0 \), then \( y^u_i \to +\infty \); thus, we may choose \( Y_i = +\infty \). This then implies that all goods arriving at stage \( i + 1 \) are immediately forwarded to stage \( i \) at the beginning of each period; resulting in no-stock-keeping at stage \( i + 1 \). \( \square \)

When \( Q_i = 1 \) for \( i = 1, \ldots, N \), the system reduces to the Clark-Scarf model for which the optimality of base stock policies has been known for more than four decades (see Clark and Scarf...
[1960], Federgruen and Zipkin [1984], and Chen and Zheng [1994b]). The results of Theorem 1 still apply for the calculation of optimal base stock levels, \( S^* = (\overline{S}_1, ..., \overline{S}_N) \), with \( \overline{S}_i = Y_i + 1 \) for \( i = 1, ..., N \).

For \( N = 2 \) with \( Q_1 = Q_2 = 1 \), our newsboy inequalities are equivalent to the findings of Doğru, De Kok, and Van Houtum [2004] when their one-warehouse multi-retailer system has a single retailer.

### 3.3 Continuous Demand

The newsboy characterizations in Theorem 1 can be further sharpened when the demand process is continuous. For the rest of this section, assume that the demand distribution \( F \) is continuous on \((0, \infty)\) with \( F(0) = 0 \). Then, \( C_i(R_1, ..., R_i) \) becomes a continuous function. Further, \( c_i(R_1, ..., R_i) \) is now defined as \( \frac{\partial C_i(R_1, ..., R_i)}{\partial R_i} \). The results of Lemmas 3 and 4 still hold.

Contrary to the discrete demand case, we now know for \( i = 1, ..., N \) that there is a \( \overline{Y}_i \in \mathbb{R} \) such that \( c_i(\overline{Y}_1, ..., \overline{Y}_i) = 0 \), because of the continuity and convexity of \( C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) \). Also, for \( (R_1, ..., R_{i-1}) = (\overline{Y}_1, ..., \overline{Y}_{i-1}) \), the second summation in (14) vanishes. Thus, we find \( \overline{Y}_i \) such that

\[
c_i(\overline{Y}_1, ..., \overline{Y}_i) = \sum_{k=1}^{i} h_k - (p + H_i) \Pr\{B_0^{(i)} > 0\} = 0.
\]

This leads to the following simplified newsboy characterization.

**Theorem 2** If \( F \) is a continuous cdf on \((0, \infty)\) with \( F(0) = 0 \), then the optimal reorder level \( \overline{Y}_i \in \mathbb{R} \) satisfies

\[
\Pr\{B_0^{(i)} = 0\} = \frac{p + H_{i+1}}{p + H_1}, \quad i = 1, ..., N,
\]

where the \( B_k^{(i)} \) for \( k = i, ..., 0 \) are defined by (18)-(21).

The interpretation of this theorem is that the optimal reorder level at some stage \( i \) (assuming ample stock at stage \( i + 1 \)) leads to a probability of no-stockout at stage 1 that is equal to \( \frac{p + H_{i+1}}{p + H_1} \).
This result is a generalization of the newsboy equalities shown by Van Houtum and Zijm [1991] for
N-echelon serial systems without batching.

**Remark 3: On the connection to the lower bounds for the optimal reorder levels as derived by Shang and Song [2005]**

For \( i \in \{2, ..., N\} \), define:

\[
\hat{B}_i^{(i)} = 0, \\
\hat{B}_{i-1}^{(i)} = \hat{B}_i^{(i)} + D_i^- - (R_i - \bar{Y}_{i-1}) - Z_{i-1}Q_{i-1}, \\
\hat{B}_k^{(i)} = \hat{B}_{k+1}^{(i)} + D_{k+1}^- - (\bar{Y}_{k+1} - \bar{Y}_k) - Z_kQ_k \quad \text{for} \quad k = i - 2, ..., 1, \\
\hat{B}_0^{(i)} = \left[ \hat{B}_1^{(i)} + D_1 - (\bar{Y}_1 + U_1) \right]^+.
\]

For \( j = i, ..., 0 \), one can show recursively that \( \Pr\{\hat{B}_j^{(i)} \leq a\} \geq \Pr\{B_j^{(i)} \leq a\} \quad \forall a \in \mathbb{Z} \); i.e., \( B_j^{(i)} \) is
stochastically larger than \( \hat{B}_j^{(i)} \), which is denoted by \( B_j^{(i)} \succeq \hat{B}_j^{(i)} \). In particular, \( B_0^{(i)} \succeq \hat{B}_0^{(i)} \). The
expression for \( \hat{B}_0^{(i)} \) may be rewritten as,

\[
\hat{B}_0^{(i)} = [D_1 + \sum_{j=2}^{i} D_j^- - R_i - (\sum_{j=1}^{i-1} Z_jQ_j + U_1)]^+ = [D_1 + \sum_{j=2}^{i} D_j^- - (R_i + U_i)]^+,
\]

where the second equality follows from (5). Define

\[
r_i^1 = \min \left\{ R_i | \Pr\{\hat{B}_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} \right\}.
\]

This \( r_i^1 \) is identical to the lower bound for the optimal reorder level \( \bar{Y}_i \) as defined by Shang and
Song [2005, §3.5]. By Corollary 1, \( \bar{Y}_i \) is such that \( \Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} \). Since \( B_0^{(i)} \succeq \hat{B}_0^{(i)} \), it
follows that \( \bar{Y}_i \geq r_i^1, i \in \{2, ..., N\} \). Note that \( r_1^1 \) defined by Shang and Song [2005] is equal to the
optimal reorder point \( \bar{Y}_1 \); see their Theorem 4.
4. CONCLUDING REMARKS

In this note, we studied a periodic review model with i.i.d. demand, but it is straightforward to extend the results to a continuous review model with compound Poisson demand. Further, Chen [2000] shows that a pure assembly system with fixed batch sizes can be transformed into an equivalent serial system under a specific integer ratio assumption, and, as a result of this, the optimality of echelon stock \((R, nQ)\) policies (with a slight modification) still holds. For the assembly systems that the serial transformation is possible, our results are also valid.

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6. REFERENCES


