Temperature distribution in a Newtonian fluid injected between two semi-infinite plates.

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Abstract

The analysis of the temperature distribution in time and place of a hot heat-conducting Newtonian fluid injected between two cooled parallel plates is presented. The 2-dimensional flow has a free flow front moving with constant velocity. The kernel of the fluid remains almost at the inlet temperature, but at the walls boundary layers occur with steeply descending temperature. The inner solutions inside these boundary layers are determined. To this end, the total region is divided into three distinct regions: the region G_I far behind the flow front, the flow front region G_{II} , and the intermediate region G_{III} between G_I and G_{II} . The asymptotics owing to each region are presented. The fundamental small parameter here is the thickness-to-length ratio of the 2-dimensional flow region. In most of the cases, similarity solutions are found. In the flow front region, for the formulation of the inner solution a Wiener-Hopf technique is used. Via matching procedures, the separate boundary layers are linked to each other to form one global boundary layer for the whole front region. All calculations in this paper are performed by analytical means, and all results are in analytical form. Comparison of our results with numerical solutions shows perfect agreement.

Keywords: injection moulding; free flow front; temperature distribution; boundary layers.

1 Introduction

During the filling phase of an injection moulding process several peculiar phenomena occur, especially close to the *flow front*, which is a free boundary for the flow. Hence, the shape of the flow front is an unknown of the flow problem. At the flow front, the flow is a so-called fountain flow. For the 2-dimensional problem of an injection flow between two parallel plates, the mould walls, and for a Newtonian fluid, this flow problem is solved completely by analytical means in [1]. In that paper, no thermal effects are considered. In the present paper, we shall use the results of [1] to calculate, again as much as possible by analytical means, the temperature distribution in the injected fluid, taking into account the effects inherent to the flow front and the fountain flow there, according to [1]. We assume that the fluid is injected at a high temperature, whereas the parallel mould walls are kept at a uniform and constant lower temperature. So, the mould walls will cool down the hot fluid, thus causing temperature gradients in the fluid, that are most strong near those walls. An analogous flow

process occurs for instance in the injection moulding of compact discs. In this kind of processes, thermal effects reveal themselves in, amongst others, influences on the viscosity which is strongly temperature dependent. The effects due to temperature-dependent viscosities on the injection moulding of polymers melts have been investigated by Mavridis et al. [2], Wall and Wilson [3], Attia [4], Pinarbasy and Ozalp [5], Estep et al. [6], Becker and M^cKinley [7], and Nguyen-Chung and Mennig [8]. In Mavridis et al. [2], a temperature profile at the end of the filling phase and far behind the flow front has been calculated; see their Fig. 7. Their result shows that the temperature in the kernel of the flow (approximately 2/3 of the width of the gap between the plates) is still equal to the hot inlet temperature, whereas the fluid cools down to the wall temperature in two thin layers near the walls. Comparable results can be found in [7, Fig. 6]. Also in the other references temperature profiles are calculated, but this is always done by numerical means, in which the influence of (the fountain flow at) the flow front is not taken into account (this in contrast to the present paper). Only in [8], it looks as if the development of the flow front is taken into account. However, nowhere in this paper boundary conditions for the temperature on the flow front are specified. Moreover since no results of the temperature simulations are presented, a comparison with our results is impossible. In some of the mentioned papers, [3], [5], and [6], the calculated temperature distribution and, more specifically, its influence on the temperature-dependent viscosity, are used to investigate the (in)stability of the channel flow; significant influences on the stability of the flow are found. One aspect of the stability behaviour is the so-called *flow front instability*: in experiments on injection between parallel plates, it is observed that the motion of the flow front becomes instationary and asymmetric; the front starts wobbling, [9] or [10, Sect. 10.3]. It is believed that this front instability is caused by either thermal or elastic effects, or by a combination of the two.

To get some insight into the mechanism, we study a simplified problem in which the polymer is considered to be an incompressible Newtonian fluid, where the viscosity η depends on the temperature T only. The total problem is split up into a temperature problem and a velocity problem, which are coupled. In case the flow front is symmetric and stationary, moving with constant speed, the shape of the flow front, the streamlines, and the complete velocity field are calculated in [1]. When the flow front is slightly perturbed to an asymmetric shape and the temperature dependence of the viscosity is taken into account, the evolution in time of the flow front can be analysed, leading to a statement about the (in)stability of the flow front motion; this is explained in somewhat more detail in [9] or [10], and will be the subject of a forthcoming thesis of Gramberg [11].

In this paper, we will look at the pure temperature problem, in which we calculate T near and behind the flow front. This leads to a convection-diffusion problem, in the convective term of which the velocity $\mathbf{v}$ is used as found in [1]. The problem is formulated for a 2-dimensional region between two parallel plates, the mould. The mould is partially filled with polymer melt. At the inlet, the hot fluid is injected. As the walls are cooled, the melt gets also cooled at the walls. Due to the short process time which we consider here (only the filling phase is considered), the kernel of the injected fluid does not cool much and, therefore, will remain (almost) at the inlet temperature. Consequently, there will occur boundary layers at the walls, in which the temperature is lower than the inlet temperature. In these narrow layers asymptotics will be used to formulate and solve the temperature problem. It turns out that the asymptotics in a layer very close to the flow front differ essentially from those in a layer far behind the flow front. For both cases the asymptotic or boundary layer problems will be solved here. The fluid model we consider here is simple: a Newtonian fluid with constant

viscosity, and in the heat equation the viscous dissipation term is neglected.

In Section 2, the mathematical formulation of the temperature problem is presented. Three different regions are introduced there:

1. The region far behind the flow front (G_I).
2. The flow front region (G_{II}).
3. The intermediate region (G_{III}) between G_I and G_{II} .

In each of these regions different asymptotics apply. However the fundamental parameter is everywhere the thickness-to-length ratio of the 2-dimensional flow region. In Sections 3, 4, and 5, successively the asymptotic solutions in the boundary layers in G_I , G_{II} and G_{III} are derived. In Sections 3 and 5, the solution is found via the introduction of a similarity solution, whereas in Section 4, a solution is obtained via a Wiener-Hopf technique. In Section 5, the different boundary layers are linked to each other by a matching procedure. A review of the main results and the conclusions is given in Section 6.

2 Mathematical formulation

The system we consider here consists of two semi-infinite parallel plates at a distance $2h$, occupying the two half-planes $\{(X, Y, Z) \mid X > 0 \wedge Y = \pm h\}$. Here, we have introduced a system of Cartesian coordinates, X , Y and Z , where Y is the coordinate in the direction perpendicular to the plates; see Figure 1. Through the cross-section $\{X = 0 \wedge -h < Y < h\}$ a fluid is injected and it starts to flow between the plates with a velocity $\mathbf{V}$. We consider the problem to be two-dimensional, meaning that $\mathbf{V}$ is independent of Z , and that the Z -component of $\mathbf{V}$ is zero: $\mathbf{V} = U(X, Y, t)\mathbf{e}_X + V(X, Y, t)\mathbf{e}_Y$.

Figure 1: The flow region with moving front between two plates, divided into three regions: the region far behind the flow front G_I , the flow front region G_{II} , and the intermediate region G_{III} .

The region between the plates that at a certain time t is filled with fluid, is denoted by G and is bounded by the walls at $Y = \pm h$, the inlet $X = 0$, and the flow front, curve $ABC = \Gamma$

as indicated in Figure 1. The region G is given by

$$G = \{(X, Y) \mid -h < Y < h \wedge 0 < X < X_f(t) + \xi_f(Y, t)\} , \quad (1)$$

and the flow front $\Gamma (\equiv ABC)$ by

$$\Gamma = \{(X, Y) \mid X = X_f(t) + \xi_f(Y, t) \wedge -h < Y < h\}, \quad (2)$$

where $X_f(t)$ is the X -position of point A (or C), and $\xi_f(Y, t)$ is the horizontal position with respect to the line AC and at time t of a point on the flow front Γ having vertical position Y . The normal on Γ in a certain point $(X, Y) = (X_f(t) + \xi_f(Y, t), Y)$ and at time t is given by

$$\mathbf{n} = \mathbf{n}(Y, t) = \frac{1}{\sqrt{1 + (\xi_{f,Y}(Y, t))^2}} (1, -\xi_{f,Y}(Y, t)) , \quad (3)$$

where $(.)_Y$ denotes differentiation with respect to Y .

For the calculations of the temperature in G , to be presented in this paper, it is convenient to divide G into three disjoint regions; see Figure 1. Here, region G_I is the region far behind the flow front, region G_{II} is the flow front region, and region G_{III} is the transition region between G_I and G_{II} (so, $G = G_I \cup G_{II} \cup G_{III}$). In this paper we will calculate the temperature in the regions G_I , G_{II} , G_{III} , separately.

Here, we assume that the flow front has a fixed shape moving with constant speed V_f in the X -direction, and that it is symmetric with respect to Y . This means that $X_f(t) = V_f t$ and $\xi_f = \xi_f(Y) = \xi_f(-Y)$.

For an incompressible Newtonian fluid, the velocity near the flow front can be calculated by analytical means; see [1], where it was found that the shape of the flow front looks very much like a semi-circle. We introduce a comoving frame $\{oxy\}$ with the dimensionless coordinates

$$x = \frac{X - V_f t}{h}, \quad y = \frac{Y}{h} . \quad (4)$$

In this co-moving frame, the flow front is given by (in dimensionless form)

$$x = x_f(y) = \frac{1}{h} \xi_f\left(\frac{Y}{h}\right), \quad -1 < y < 1. \quad (5)$$

Since the shape of the flow front resembles a semi-circle, the characteristic measure of length in the X -direction in G_{II} is of the same order as the characteristic length h in the Y -direction, meaning that both x and y are $O(1)$ in G_{II} .

In the flow front region, the velocity with respect to the flow front is independent of time t , meaning that the components of the velocity $\mathbf{V} = U \mathbf{e}_x + V \mathbf{e}_y$ may be written there as

$$U(X, Y, t) = V_f(1 + u(x, y)) , \quad V(X, Y, t) = V_f v(x, y) , \quad (6)$$

where $\mathbf{v} = u(x, y)\mathbf{e}_x + v(x, y)\mathbf{e}_y$ is the dimensionless velocity with respect to the co-moving frame.

In the sequel, we consider the problem for larger times, such that $X_f t \gg h$. This implies that far behind the flow front the flow is considered to be fully developed and independent of time. Accordingly the velocity far behind the flow front is given by

$$U = U_P(Y) := \frac{3}{4} V_f \left(1 - \left(\frac{Y}{h} \right)^2 \right) , \quad V = 0 . \quad (7)$$

We assume that the temperature $T(X, Y, t)$ in G is governed by the energy equation; see [12, eq. (4.64)]. To simplify the problem, we neglect the effects due to energy dissipation and thermal expansion, and we assume ρ , the density, c_p , the specific heat, and λ the thermal conductivity, to be constant, rendering the diffusion-convection equation for the temperature:

$$\rho c_p \left(\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} \right) = \lambda \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right). \quad (8)$$

This differential equation has to be supplemented with a set of boundary conditions, such as:

- The temperature at the inlet $X = 0$ of the mould is assumed to be constant and equal to T_i .
- The temperature of the fluid at the walls of the mould is taken to be constant and equal to $T_w (< T_i)$. We assume that the temperature of the fluid in contact with the walls is also equal to T_w ; hence, $T(X, Y) = T_w$, for $Y = \pm h$ and $X < V_f t$.
- The thermal conductivity of air is much smaller than the thermal conductivity of the fluid (e.g. a polymer). Therefore, we assume that the flow front may be taken to be thermally insulated, meaning that for $(X, Y) \in \Gamma$, $\partial T / \partial n = 0$.

Since the types of flow are quite different in G_I and in G_{II} , the respective convective terms in (8) are also quite different. Therefore, we have to distinguish between the behaviour of the temperature far behind the flow front (in G_I), and the behaviour near the flow front (in G_{II}):

- *Behaviour of T in G_I , far behind the flow front.*

In G_I , we assume the effect of the flow front negligible and consequently the temperature may be assumed stationary, yielding $T = T_I(X, Y)$, for $(X, Y) \in G_I$. Here, $T_I(X, Y)$ is governed by the equation (derived from (8) with use of (7))

$$\frac{3}{4} \rho c_p V_f \left(1 - \left(\frac{Y}{h} \right)^2 \right) \frac{\partial T_I}{\partial X} = \lambda \Delta T_I(X, Y). \quad (9)$$

In G_I , the following two boundary conditions for T_I are relevant (and as our further analysis will show, also sufficient):

- $T_I(0, Y) = T_i$, for $-h < Y < h$;
- $T_I(X, \pm h) = T_w$, for $(X, \pm h) \in G_I$.

- *Behaviour of T in G_{II} , the flow front region.*

In G_{II} , the temperature is assumed to be stationary with respect to the co-moving frame $\{oxy\}$, yielding $T(X, Y, t) = T_{II}(x, y)$. Transforming (8) to the (x, y) -coordinate system, with use of (6), we obtain

$$h V_f \rho c_p \left(u \frac{\partial T_{II}}{\partial x} + v \frac{\partial T_{II}}{\partial y} \right) = \lambda \left(\frac{\partial^2 T_{II}}{\partial x^2} + \frac{\partial^2 T_{II}}{\partial y^2} \right). \quad (10)$$

The boundary conditions for T_{II} are

- At the flow front $x = x_f(y)$, the normal derivative of T_{II} is equal to zero, so

$$\frac{\partial T_{II}}{\partial x} - x'_f(y) \frac{\partial T_{II}}{\partial y} = 0, \quad \text{on } \Gamma ; \quad (11)$$

- $T_{II}(x, \pm 1) = T_w$, for $x \geq 0$.

- *Behaviour of T in G_{III} , the transition region between G_I and G_{II} .*

This region is needed to match the solution for T in G_I to that in G_{II} . How this will be done, will be specified further on.

We proceed by making the equations derived thus far dimensionless. We have defined x , y and $\mathbf{v}$ such that they are already dimensionless and of order $O(1)$ in G_{II} . We introduce dimensionless variables $\hat{X}$, $\hat{Y}$, $\hat{\mathbf{V}}$, $\hat{T}_I$ and $\hat{T}_{II}$, by

$$\hat{X} = \frac{X}{R}, \quad \hat{Y} = \frac{Y}{h}, \quad \hat{\mathbf{V}} = \frac{4}{3V_f} \mathbf{V}, \quad \hat{T} = \frac{T - T_w}{T_i - T_w}. \quad (12)$$

Here, R is a characteristic length in the X -direction of region G_I . Since we are looking at the behaviour for larger times, and far behind the flow front, we assume that $R/h \gg 1$ (see also Table 1).

Substitution of these scalings into the differential equations and boundary conditions for T_I and T_{II} as derived previously, renders the following set of dimensionless equations for $\hat{T}_I$ and $\hat{T}_{II}$:

- The dimensionless temperature $\hat{T}_I = \hat{T}_I(\hat{X}, \hat{Y})$ satisfies the differential equation

$$(1 - \hat{Y}^2) \frac{\partial \hat{T}_I}{\partial \hat{X}} = \lambda_I \left(\delta^2 \frac{\partial^2 \hat{T}_I}{\partial \hat{X}^2} + \frac{\partial^2 \hat{T}_I}{\partial \hat{Y}^2} \right), \quad (13)$$

where the dimensionless group λ_I is given by

$$\lambda_I = \frac{4R\lambda}{3\rho c_p h^2 V_f}, \quad (14)$$

and $\delta = h/R$, with $0 < \delta \ll 1$. Therefore, we will neglect the term preceded by δ^2 in the diffusive term of (13)), which leads us to

$$(1 - \hat{Y}^2) \frac{\partial \hat{T}_I}{\partial \hat{X}} = \lambda_I \frac{\partial^2 \hat{T}_I}{\partial \hat{Y}^2}. \quad (15)$$

The boundary conditions for T_I transform to

- $\hat{T}_I(0, \hat{Y}) = 1$, for $-1 < \hat{Y} < 1$;
- $\hat{T}_I(\hat{X}, \pm 1) = 0$, for $\hat{X} > 0$.

- The differential equation for the dimensionless temperature $\hat{T}_{II} = \hat{T}_{II}(x, y)$ is

$$u \frac{\partial \hat{T}_{II}}{\partial x} + v \frac{\partial \hat{T}_{II}}{\partial y} = \lambda_{II} \left(\frac{\partial^2 \hat{T}_{II}}{\partial x^2} + \frac{\partial^2 \hat{T}_{II}}{\partial y^2} \right), \quad (16)$$

where λ_{II} is given by

$$\lambda_{II} = \delta \lambda_I = \frac{\lambda}{\rho c_p h V_f} . \quad (17)$$

The boundary conditions for $\hat{T}_{II}$ are

- At $x = x_f(y)$, and for $-1 \leq y \leq 1$:

$$\frac{\partial \hat{T}_{II}}{\partial x} = x'_f(y) \frac{\partial \hat{T}_{II}}{\partial y} ; \quad (18)$$

- for $x \leq 0$,

$$\hat{T}_{II}(x, \pm 1) = 0 . \quad (19)$$

Table 1: Numerical values for material coefficients

symbol	order of magnitude	unit
R	$6 \cdot 10^{-2}$	m
h	$6 \cdot 10^{-4}$	m
V_f	1	m s^{-1}
ρ	10^3	kg m^{-3}
c_p	10^3	$\text{J kg}^{-1} \text{K}^{-1}$
λ	10^{-1}	$\text{W m}^{-1} \text{K}^{-1}$

As a standard for explicit numerical calculations, we will use numerical values for λ , V_f , ρ , c_p , R and h as given in Table 1. Based on these values, we find that $\lambda_I \approx 2.2 \cdot 10^{-2}$ and $\lambda_{II} \approx 1.6 \cdot 10^{-4}$. Since $\delta = h/R$, we write λ_I and λ_{II} as

$$\lambda_I = \delta \hat{\lambda} , \quad \lambda_{II} = \delta^2 \hat{\lambda} , \quad \text{with} \quad \hat{\lambda} = \frac{\lambda R^2}{\rho c_p h^3 V_f} , \quad (20)$$

where $\hat{\lambda} = 5/3$. As $\delta = 10^{-2} \ll 1$, we say that $\hat{\lambda} = O(1)$, $\lambda_I = O(\delta)$, and $\lambda_{II} = O(\delta^2)$, all with respect to the small parameter δ . This means that both λ_I and λ_{II} are small parameters and that we may use asymptotic techniques to find solutions for the temperatures $\hat{T}_I$ and $\hat{T}_{II}$. In the zeroth-order approximation, we omit the terms containing λ_I and λ_{II} in (13) and (16), respectively. This essentially amounts to neglecting the diffusive term with respect to the convective term. By doing this, we find the following differential equations for the zeroth-order approximations $\hat{T}_I^0$ and $\hat{T}_{II}^0$ of $\hat{T}_I$ and $\hat{T}_{II}$, respectively,

$$(1 - \hat{Y}^2) \frac{\partial \hat{T}_I^0}{\partial \hat{X}} = 0, \quad (21)$$

$$u \frac{\partial \hat{T}_{II}^0}{\partial x} + v \frac{\partial \hat{T}_{II}^0}{\partial y} = 0. \quad (22)$$

These equations express that both $\hat{T}_I^0$ and $\hat{T}_{II}^0$ are constant along the streamlines. Since $\hat{T} = 1$ at the inlet of the mould and since all streamlines cross the inlet, this implies that

$\hat{T}_I^0 = \hat{T}_{II}^0 = 1$ everywhere in the mould (then, of course, also $\hat{T}_{III}^0 = 1$). This solution does satisfy the boundary conditions at the inlet of the mould, and those at the flow front, but it does not satisfy the boundary conditions at the walls of the mould, where $\hat{T}_I = \hat{T}_{II} = 0$. Therefore, we expect that there are boundary layers near the walls where diffusion cannot be neglected with respect to convection. In the next three sections we shall investigate these boundary layers in the regions G_I , G_{II} and G_{III} , respectively. The fundamental small parameter, responsible for these boundary layers, is everywhere δ . However, the size of the boundary layers is proportional to δ^n , where n will have different values in the three distinct regions.

3 Boundary layer in G_I

In this section, we consider the boundary layers near the walls $\hat{Y} = \pm 1$ within G_I . The fundamental small parameter, responsible for these boundary layers, is here $\lambda_I = O(\delta)$. It will turn out that here the size of the boundary layer is proportional to $\delta^{1/3}$.

3.1 First-order solution

Since the problem is symmetric with respect to $\hat{Y}$, we only have to consider the boundary layer near $\hat{Y} = 1$. We introduce a new coordinate ζ defined by

$$\hat{Y} = 1 - \epsilon\zeta. \quad (23)$$

Here, ϵ is the, still unknown, thickness of the boundary layer, and $\zeta = O(1)$ in the boundary layer. Replacement of $\hat{Y}$ by ζ in (15) yields (omitting the hat in $\hat{X}$)

$$\epsilon\zeta(2 - \epsilon\zeta) \frac{\partial T_I}{\partial X} = \frac{\lambda_I}{\epsilon^2} \frac{\partial^2 T_I}{\partial \zeta^2}. \quad (24)$$

Since we want the left-hand side of (24) to be of the same order as the right-hand side, ϵ must satisfy

$$\epsilon^3 = \lambda_I = O(\delta), \quad \implies \quad \epsilon = \sqrt[3]{\lambda_I} = O(\delta^{1/3}). \quad (25)$$

With this choice for ϵ , (24) can be rewritten as

$$2\zeta \frac{\partial T_I}{\partial X} - \frac{\partial^2 T_I}{\partial \zeta^2} = \epsilon\zeta^2 \frac{\partial T_I}{\partial X}. \quad (26)$$

After neglecting terms of order ϵ in (26), we arrive at the following differential equation for the first-order asymptotic boundary layer solution in G_I , denoted by $T_I^1 = T_I^1(X, \zeta)$,

$$2\zeta \frac{\partial T_I^1}{\partial X} - \frac{\partial^2 T_I^1}{\partial \zeta^2} = 0, \quad \text{for } X > 0, \zeta > 0. \quad (27)$$

The boundary conditions for T_I^1 are given by

$$T_I^1(0, \zeta) = 1, \quad T_I^1(X, 0) = 0. \quad (28)$$

In order that this boundary layer solution (or *inner solution*) matches with the zeroth-order solution (*outer solution*), we require that

$$\text{for } \zeta \rightarrow \epsilon^{-1} \approx \infty: \quad T_I^1 \rightarrow T_I^0 = 1. \quad (29)$$

To find the solution $T_1(X, \zeta)$ of (27) satisfying the conditions (28)-(29), we assume that there exists a similarity solution, which means that we assume that $T_I^1(X, \zeta)$ can be written as

$$T_I^1(X, \zeta) = T_I^1(\sigma), \quad (30)$$

with $\sigma = \zeta X^\alpha$, for a certain value for α . Written in terms of σ , the partial differential equation (27) transforms into the ordinary differential equation for $T_I^1(\sigma)$ (after multiplication by $X^{-2\alpha}$)

$$2\alpha\zeta^2 X^{-\alpha-1} \frac{dT_I^1}{d\sigma} - \frac{d^2 T_I^1}{d\sigma^2} = 0. \quad (31)$$

To be able to eliminate both ζ and X from (31), $\zeta^2 X^{-\alpha-1}$ must be a multiple of σ , implying that there must exist a $\beta \in \mathbb{R}$, such that

$$\zeta^2 X^{-\alpha-1} = \sigma^\beta = \zeta^\beta X^{\alpha\beta}. \quad (32)$$

Equation (32) has exactly one solution for α and β , namely $\beta = 2$ and $\alpha = -1/3$. Substitution of these values for α and β into (31) yields

$$\frac{2}{3}\sigma^2 \frac{dT_I^1}{d\sigma} + \frac{d^2 T_I^1}{d\sigma^2} = 0. \quad (33)$$

Since $\alpha < 0$, the boundary condition at $X = 0$ and the matching condition for $\zeta \rightarrow \infty$ both imply a boundary condition for T_I^1 for $\sigma \rightarrow \infty$, which might lead to a contradiction. However, since $T_I^1 \rightarrow 1$ for both $X = 0$ and $\zeta \rightarrow \infty$, this is not the case here, and the conditions (28) and (29) transform into the boundary conditions for $T_I^1(\sigma)$:

$$T_I^1(0) = 0, \quad \text{and} \quad T_I^1(\infty) = 1. \quad (34)$$

Equation (33) can be integrated twice to yield the solution

$$T_I^1(\sigma) = C \int_0^\sigma e^{-\frac{2}{9}s^3} ds + D, \quad (35)$$

where C and D are integration constants. From (34), it follows that C and D are equal to

$$C = \left(\int_0^\infty e^{-\frac{2}{9}s^3} ds \right)^{-1}, \quad D = 0, \quad (36)$$

yielding the final solution for T_I^1 , written in terms of X and ζ , in the form

$$T_I^1(X, \zeta) = C \int_0^{\frac{\zeta}{\sqrt[3]{X}}} e^{-\frac{2}{9}s^3} ds. \quad (37)$$

This result represents the first-order term (i.e. for $\epsilon \rightarrow 0$) of the inner solution in the boundary layer at $Y = 1$. With this result, we can derive an expression for the thickness of the boundary

layer in G_I . We define this thickness $h_{bl} = h_{bl}(X)$ as the value of $(h - Y)$ at which the scaled temperature T_I differs $O(\epsilon)$ from the scaled inlet temperature, that is

$$T_I(X, \zeta_{bl}) = 1 - \epsilon , \quad (38)$$

where $\zeta_{bl} = h_{bl}/(\epsilon h)$. For the determination (roughly in first order) of ζ_{bl} , we use the first-order solution (37), and we make use of the asymptotic expansion of the right-hand side of (37) for large values of $\zeta/\sqrt[3]{X}$, reading

$$T_I(X, \zeta_{bl}) = 1 - e^{-\frac{2\zeta^3}{9X}} , \quad (39)$$

Equating (38) and (39), and eliminating ζ_{bl} in favour of h_{bl} , we obtain

$$\frac{h_{bl}}{h} \approx \epsilon \sqrt[3]{\frac{9}{2} \hat{X} \log \left(\frac{1}{\epsilon} \right)} \approx \sqrt[3]{-2\hat{\lambda} \delta \hat{X} \log \delta} , \quad (40)$$

where we have used (25) for ϵ .

The second-order solution for T_I (of $O(\epsilon)$) will be calculated in the next subsection. A comparison with numerical results will be presented in Section 3.3.

3.2 Second-order solution

The solution (37) is a first-order asymptotic boundary layer solution of (15). We can find a second-order solution, by assuming that the solution of (15) can be written as

$$T_I(X, Y) = T_I(X, 1 - \epsilon \zeta) = T_I^1(X, \zeta) + \epsilon T_I^2(X, \zeta) + O(\epsilon^2), \quad (41)$$

where $T_I^2(X, \zeta) = O(1)$ for $\zeta = O(1)$. Substituting (41) into (17), using (37) and neglecting terms of order ϵ^2 , we obtain

$$2\zeta \frac{\partial T_I^2}{\partial X} - \frac{\partial^2 T_I^2}{\partial \zeta^2} = -C \frac{\zeta^3}{3X^{4/3}} e^{-\frac{2\zeta^3}{9X}}. \quad (42)$$

The boundary conditions for T_I^2 are

$$T_I^2(X, 0) = 0 , \quad T_I^2(0, \zeta) = 0 , \quad (43)$$

and

$$T_I^2(0, \zeta) \rightarrow 0 , \quad \text{for } \zeta \rightarrow \infty . \quad (44)$$

Analogous to the first-order solution, we will try to find a similarity solution of (42). We assume that T_I^2 can be written as $T_I^2(X, \zeta) = \zeta F(\sigma)$, where $\sigma = \zeta/\sqrt[3]{X}$. With this assumption, the partial differential equation (42) turns into an ordinary differential equation for $F(\sigma)$ (after multiplication by $-\sqrt[3]{X}$)

$$\left(\frac{2}{3} \sigma^3 + 2 \right) F'(\sigma) + \sigma F''(\sigma) = \frac{C}{3} \sigma^3 e^{-\frac{2}{9} \sigma^3}. \quad (45)$$

The differential equation (45) is a quasi-linear ordinary differential equation. Therefore, the general solution of (45) can be composed of the solution of the homogeneous system plus a

particular solution, where the latter can be found from the first by the method of variation of constants. This leads to

$$F(\sigma) = C_1 \int_{\sigma}^{\infty} \frac{1}{s^2} e^{-\frac{2}{9}s^3} ds + C_2 + \frac{C}{15} \int_0^{\sigma} s^3 e^{-\frac{2}{9}s^3} ds = -\frac{C}{15} \int_{\sigma}^{\infty} s^3 e^{-\frac{2}{9}s^3} ds, \quad (46)$$

because $F(0)$ must be finite ($\Rightarrow C_1 = 0$) and $F(\sigma)$ should go to zero at infinity (this determines C_2).

Thus, the second-order asymptotic boundary layer solution T_I^2 is given by

$$T_I^2(X, \zeta) = -\frac{C\zeta}{15} \int_{\frac{\zeta}{\sqrt[3]{X}}}^{\infty} s^3 e^{-\frac{2}{9}s^3} ds. \quad (47)$$

3.3 Numerical results and comparison of first-order and second-order solution

We compare the asymptotic solutions $T_I^1(X, \zeta)$ and $T_I^1(X, \zeta) + \epsilon T_I^2(X, \zeta)$, where T_I^1 and T_I^2 are given by (37) and (47), respectively, with a numerical solution of (15) calculated by a finite-difference method. In Table 2, the numerical results are presented for a scheme of 10 grid points in both X - and Y -direction. In this table, T_1 , T_2 , ΔT_1 and ΔT_2 correspond to

$$\begin{aligned} T_1(X, Y) &= T_I^1\left(X, \frac{1-Y}{\epsilon}\right), & T_2(X, Y) &= (T_I^1 + \epsilon T_I^2)\left(X, \frac{1-Y}{\epsilon}\right), \\ \Delta T_1 &= \frac{T_1 - T_{num}}{T_{num}}, & \Delta T_2 &= \frac{T_2 - T_{num}}{T_{num}}, \end{aligned} \quad (48)$$

where we have chosen for X the value $X = 1$. In dimensional variables, this corresponds with $X = 6 \cdot 10^{-2} \text{m}$. Moreover, we have taken $\lambda_I = 0.01$, corresponding to a value of $\epsilon = \sqrt[3]{\lambda_I} = 0.215$.

Table 2: Comparison between asymptotic and numerical results at $X = 1$

Y value	Asymptotic solution		Num. sol. T_{num}	Relative difference	
	T_1	T_2		ΔT_1	ΔT_2
0.2	1.	1.	1.	0	0
0.4	0.9990	0.9980	0.9974	0.0016	0.0006
0.6	0.9465	0.9322	0.9303	0.0174	0.0020
0.7	0.8241	0.8033	0.7989	0.0315	0.0054
0.8	0.6031	0.5846	0.5775	0.0442	0.0122
0.9	0.3131	0.3032	0.2974	0.0528	0.0193

From Table 2, it follows that the asymptotic solutions and the numerical solution are in good agreement with each other. This despite the fact that the parameter $\epsilon = \sqrt[3]{\lambda} = 0.215$ is not really very small. The relative differences between the numerical solution and the first-order and second-order asymptotic solutions are of the same order, and are in fact only of some

relevance very near to the walls at $Y = \pm 1$. We conclude that for practical purposes the first-order asymptotic solution suffices. This is confirmed by the graphs for T_I^1 and T_I^2 shown in Figure 2 a. and b., respectively, where for X the value $X = 0.5$ is chosen. From Figure 2 a. we see that the boundary layer here reaches to $\zeta \approx 2$; for $\zeta > 2$ the temperature is equal to the inlet temperature, $T_I = 1$. Moreover, Figure 2 b. reveals that the second-order correction $T_I^2 = (T_1 - T_2)/\epsilon$ is maximal somewhere in the middle of the boundary layer, but even there the total correction remains below 1% of the inlet temperature. Hence, for practical purposes the first-order approximation suffices.

Figure 2: a. The first-order solution T_I^1 as a function of ζ for $\hat{X} = 0.5$; b. the second-order solution T_I^2 as a function of ζ for $\hat{X} = 0.5$.

Figure 3: A set of isotherms for T_I^1 in the $(\hat{X}, \zeta)$ -plane in the region G_I far behind the flow front region.

In Figure 3, we have depicted a set of isotherms, ranging from $\hat{T} = 0.9$ to $\hat{T} = 0.1$, in the $\hat{X}$ - ζ -plane in the region G_I , so near the upper wall (where $\hat{T} = 0$). We observe that the isotherms initially, that is for small $\hat{X}$ -values, behave rather steeply; this is due to the large difference between the inlet temperature and the wall temperature. However, they strongly flatten for increasing $\hat{X}$. The major changes in $\hat{T}$ in the vertical or Y -direction take place within the $O(\epsilon)$ -range $0 < \zeta < 1$ (the boundary layer) near the wall.

4 Boundary layer in G_{II}

In this section, we assume boundary layers in the upper and lower parts of the flow front region G_{II} near the points $x = 0$, $y = \pm 1$. The fundamental small parameter, responsible for these boundary layers, is here $\lambda_{II} = \delta^2 \hat{\lambda} = O(\delta^2)$.

4.1 Temperature problem in the boundary layer in G_{II}

Since the problem is symmetric with respect to y , we only have to consider the boundary layer near $y = 1$. We introduce a new coordinate system (σ, τ) , where σ and τ are defined by

$$x = -\epsilon_1 \sigma, \quad y = 1 - \epsilon_2 \tau. \quad (49)$$

Here, ϵ_1 and ϵ_2 are, still unknown, characteristic measures for the thicknesses of the boundary layer in the x - and y -directions, respectively (see Figure 4), where $0 < \epsilon_1, \epsilon_2 \ll 1$. To determine ϵ_1 and ϵ_2 , we substitute (49) into (16), and choose ϵ_1 and ϵ_2 such that the leading order terms in the resulting differential equation include both diffusive and convective terms.

Figure 4: Global and detailed view of boundary layer near the point $(x, y) = (0, 1)$.

Substitution of (49) into (16) yields the following differential equation for the temperature in the boundary layer:

$$\frac{\bar{u}}{\epsilon_1} \frac{\partial \bar{T}}{\partial \sigma} + \frac{\bar{v}}{\epsilon_2} \frac{\partial \bar{T}}{\partial \tau} = \frac{\lambda_{II}}{\epsilon_1^2} \frac{\partial^2 \bar{T}}{\partial \sigma^2} + \frac{\lambda_{II}}{\epsilon_2^2} \frac{\partial^2 \bar{T}}{\partial \tau^2}, \quad (50)$$

where

$$\begin{aligned} \bar{T}(\sigma, \tau) &= \hat{T}_{II}(-\epsilon_1 \sigma, 1 - \epsilon_2 \tau), \\ \bar{u}(\sigma, \tau) &= -u(-\epsilon_1 \sigma, 1 - \epsilon_2 \tau), \\ \bar{v}(\sigma, \tau) &= -v(-\epsilon_1 \sigma, 1 - \epsilon_2 \tau). \end{aligned} \quad (51)$$

For $\sigma = O(1)$ and $\tau = O(1)$, $\bar{u}$ may be approximated using a Taylor series expansion around $(\sigma, \tau) = (0, 0)$ according to (here, h.o.t. stands for higher-order terms)

$$\begin{aligned} \bar{u}(\sigma, \tau) &= \bar{u}(0, 0) + \sigma \frac{\partial \bar{u}}{\partial \sigma}(0, 0) + \tau \frac{\partial \bar{u}}{\partial \tau}(0, 0) + h.o.t. \\ &= 1 + \epsilon_1 \sigma u_{,x}(0, 1) + \epsilon_2 \tau v_{,y}(0, 1) + h.o.t., \end{aligned} \quad (52)$$

while, analogously, $\bar{v}$ can be approximated as

$$\bar{v}(\sigma, \tau) = \epsilon_1 \sigma v_{,x}(0, 1) + \epsilon_2 \tau v_{,y}(0, 1) + h.o.t., \quad (53)$$

where we used the fact that $\mathbf{u}(0, 1) = (-1, 0)$. Assuming that $|\nabla \mathbf{u}(0, 1)| = O(1)$, we obtain the following approximation for $\bar{\mathbf{u}}$

$$\bar{u}(\sigma, \tau) = 1 + O(\epsilon_1, \epsilon_2), \quad \bar{v}(\sigma, \tau) = O(\epsilon_1, \epsilon_2). \quad (54)$$

After substitution of (54) into (50), we find the differential equation for the temperature in the boundary layer:

$$\underbrace{\frac{\partial \bar{T}}{\partial \sigma}}_I = \underbrace{\frac{\lambda_{II}}{\epsilon_1} \frac{\partial^2 \bar{T}}{\partial \sigma^2}}_{II} + \underbrace{\frac{\lambda_{II} \epsilon_1}{\epsilon_2^2} \frac{\partial^2 \bar{T}}{\partial \tau^2}}_{III} + \frac{\epsilon_1}{\epsilon_2} O(\epsilon_1, \epsilon_2), \quad (55)$$

In this differential equation, term I is of $O(1)$. Therefore, we get the 'richest' differential equation, if both terms II and III are of $O(1)$ too, resulting in the following choice for ϵ_1 and ϵ_2 :

$$\epsilon_1 = \epsilon_2 = 2\lambda_{II} = 2\delta^2 \hat{\lambda} =: \epsilon \ll 1, \quad (56)$$

so ϵ is proportional to δ^2 here.

This renders the following differential equation for $\bar{T}$, which is valid in the boundary layer at the upper part of the flow front

$$\frac{\partial^2 \bar{T}}{\partial \sigma^2} + \frac{\partial^2 \bar{T}}{\partial \tau^2} - 2 \frac{\partial \bar{T}}{\partial \sigma} = 0, \quad \text{for } \tau > 0. \quad (57)$$

For a complete formulation of the temperature problem for $\bar{T}$, we still need the boundary conditions for (57), which follow from (18)-(19). Moreover, we will need matching conditions to match the boundary layer solutions in G_{II} to the zeroth-order solution in G_{II} (i.e. $\bar{T} = 1$). We start with the boundary conditions.

- The boundary $y = 1$, $x < 0$ corresponds to $\tau = 0$, $\sigma > 0$. Hence, $\bar{T}(\sigma, 0) = 0$, for all $\sigma > 0$.
- The boundary $x = x_f(y)$, for $0 < y < 1$, $x > 0$, i.e. the flow front, corresponds to

$$-\epsilon \sigma = x_f(1 - \epsilon \tau) \quad \text{and} \quad 0 \leq \tau \leq \frac{1}{\epsilon}. \quad (58)$$

Since $x_f(\zeta)$ is a monotone decreasing function in ζ for $0 < \zeta < 1$, the equation for the upper half of the flow front may be rewritten as

$$\epsilon \tau = 1 - x_f^{-1}(-\epsilon \sigma), \quad \text{for } \epsilon \sigma < 0, \quad (59)$$

where x_f^{-1} is the inverse function of x_f . Using a Taylor series expansion for $x_f^{-1}(x)$ in the neighbourhood of $x = 0$, we derive the following implicit equation for the flow front within the boundary layer in terms of σ and τ

$$\begin{aligned} \tau &= \frac{1}{\epsilon} \left(1 - x_f^{-1}(0) + \left(x_f^{-1} \right)'(0) \epsilon \sigma - \frac{1}{2} \left(x_f^{-1} \right)''(0) \epsilon^2 \sigma^2 + O(\epsilon^3) \right) \\ &= \left(x_f^{-1} \right)'(0) \sigma - \frac{1}{2} \left(x_f^{-1} \right)''(0) \epsilon \sigma^2 + O(\epsilon^2), \end{aligned}$$

holding for all $\sigma < 0$, as long as $\sigma = O(1)$ for $\epsilon \rightarrow 0$. Here, we used the fact that $x_f^{-1}(0) = x_f^{-1}(x_f(1)) = 1$.

Since the flow front is tangent to the wall in the point $(\sigma, \tau) = (0, 0)$, $(x_f^{-1})'(0) = 0$.

Using this, (4.1) indicates that in the boundary layer, $\tau = O(\epsilon)$, for $\sigma < 0$.

The boundary condition at the flow front following from (18) can be written as

$$\frac{\partial \bar{T}}{\partial \tau} = \frac{1}{x'_f(1 - \epsilon\tau)} \frac{\partial \bar{T}}{\partial \sigma} . \quad (60)$$

Since

$$\frac{1}{x'_f(1 - \epsilon\tau)} = (x_f^{-1})'(-\epsilon\sigma) = O(\epsilon), \quad (61)$$

and since ϵ is chosen such that $\partial \bar{T} / \partial \sigma$ is finite, the boundary condition at the flow front reduces to

$$\frac{\partial \bar{T}}{\partial \tau} = O(\epsilon) . \quad (62)$$

- The solution in the boundary layer must match with the inner solution ($T \approx 1$). Therefore, we require that for all $\sigma \in \mathbb{R}$, $\bar{T}(\sigma, \tau) \rightarrow 1$ for $\tau \rightarrow 1/\epsilon$. Moreover, we assume that $\bar{T}(\sigma, \tau) = O(1)$ for all σ and for all $\tau \geq 0$.

4.2 First-order solution

In the previous sections, we derived the differential equation for the temperature in the boundary layer in G_{II} , together with the pertinent boundary conditions, for small ϵ . For the first-order solution, we let $\epsilon \rightarrow 0$, which leads us to the following problem for $\bar{T}(\sigma, \tau)$ in the half plane $\tau > 0$:

$$\frac{\partial^2 \bar{T}}{\partial \sigma^2} + \frac{\partial^2 \bar{T}}{\partial \tau^2} - 2 \frac{\partial \bar{T}}{\partial \sigma} = 0, \quad \tau > 0, \quad \sigma \in \mathbb{R} ; \quad (63)$$

$$\bar{T}(\sigma, 0) = 0, \quad \text{for } \sigma > 0 , \quad (64)$$

$$\frac{\partial \bar{T}}{\partial \tau}(\sigma, 0) = 0, \quad \text{for } \sigma < 0 , \quad (65)$$

$$\bar{T}(\sigma, \tau) \rightarrow 1, \quad \text{for } \tau \rightarrow \infty, \quad \sigma \in \mathbb{R} , \quad (66)$$

We replace $\bar{T}$ by a new function $\phi = \phi(\sigma, \tau)$, defined according to

$$\phi(\sigma, \tau) = e^{-\sigma} (1 - \bar{T}(\sigma, \tau)) , \quad (67)$$

which goes sufficiently fast to zero for $\zeta \rightarrow \pm\infty$, to assure that its Fourier transform

$$\Phi(w, \tau) = \mathfrak{F}(\phi(\sigma, \tau); \sigma \rightarrow w) = \int_{-\infty}^{\infty} \phi(\sigma, \tau) e^{i w \sigma} d\sigma, \quad (68)$$

for $w \in \mathbb{C}$ exists.

From this definition, it follows that $\phi(\sigma, \tau)$ satisfies the following differential equation, valid on the half-plane $\tau \geq 0$, $\sigma \in \mathbb{R}$

$$\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{\partial^2 \phi}{\partial \tau^2} = \phi, \quad (69)$$

together with the boundary conditions

$$\phi(\sigma, 0) = e^{-\sigma}, \quad \text{for } \sigma > 0, \quad (70)$$

$$\frac{\partial \phi}{\partial \tau}(\sigma, 0) = 0, \quad \text{for } \sigma < 0, \quad (71)$$

and

$$\phi(\sigma, \tau) \rightarrow 0, \quad \text{for } \tau \rightarrow \infty, \quad \sigma \in \mathbb{R}. \quad (72)$$

The system (69)-(72) can be solved by using a Wiener-Hopf technique along the same lines as presented by Carrier et al. [14, Section 8-1]. This results in the following expression for $\Phi(w, \tau)$ (for the details of the derivation, we refer to [11])

$$\Phi(w, \tau) = \frac{1+i}{(w+i)\sqrt{w-i}} e^{-\tau\sqrt{w^2+1}}, \quad (73)$$

which is valid for $\tau > 0$ and $-1 < \text{Im}(w) < 1$.

This result should be compared to the second equation of (8-12) in [14]; in fact, it can be obtained from the latter equation by letting a go to zero, and replacing y by 2σ and λ by $(\lambda+i)/2$ (the reason for the latter is that we have to replace our function ϕ by $e^\sigma \phi$ in order to get the same problem as in [14], and this leads to a shift in λ in the Fourier transform).

To find $\phi(\sigma, \tau)$, we apply an inverse Fourier transformation on $\Phi(w, \tau)$ with respect to w , i.e.

$$\phi(\sigma, \tau) = \frac{1}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} \Phi(w, \tau) e^{-i\sigma w} dw, \quad (74)$$

with $-1 < \delta < 1$. From (73) and the definitions of $\sqrt{w^2+1}$ and $\sqrt{w-i}$, it follows that for all $\tau \geq 0$, $\Phi(w, \tau)$ is analytic in $\mathbb{C}$ excluding cuts from i to $i\infty$ and from $-i$ to $-i\infty$ along the imaginary axis. Therefore, in order to find a closed form for $\phi(\sigma, \tau)$, we may use contour integration. Since $\text{Re}(-iw) > 0$ for $\text{Im}(w) > 0$, and vice versa, we choose a closed contour C in the upper half of the w -plane to calculate $\phi(\sigma, \tau)$ for $\sigma < 0$, and a closed contour D in the lower half of the w -plane to calculate $\phi(\sigma, \tau)$ for $\sigma > 0$, excluding the cuts from $\pm i$ to $\pm i\infty$ along the imaginary axis. More specific, we choose C and D to be the boundaries of G^+ and G^- , respectively, where $G^\pm$ are given by

$$G^+ := \{z \in \mathbb{C} \mid \text{Im}(z) > \delta_1 \wedge |z| < R \wedge |z-i| > \epsilon \wedge z \notin I^+\}, \quad C = \partial G^+, \quad (75)$$

$$G^- := \{z \in \mathbb{C} \mid \text{Im}(z) < -\delta_2 \wedge |z| < R \wedge |z+i| > \epsilon \wedge z \notin I^-\}, \quad D = \partial G^-, \quad (76)$$

with

$$I^+ := \{z \in \mathbb{C} \mid \text{Im}(z) \geq 1 \wedge \text{Re}(z) = 0\}, \quad (77)$$

$$I^- := \{z \in \mathbb{C} \mid \text{Im}(z) \leq -1 \wedge \text{Re}(z) = 0\}. \quad (78)$$

In (75) and (76), δ_1 and δ_2 are arbitrary constants between 0 and 1, while R and ϵ are positive constants such that $R \gg 1$ and $1 - \epsilon > (\delta_1, \delta_2)$. By taking limits for $R \rightarrow \infty$ and $\epsilon \rightarrow 0$, we will derive a closed-form expression for $\phi(\sigma, \tau)$. To calculate this expression, we split C and D into six disjoint parts:

$$C = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6, \quad D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6, \quad (79)$$

Figure 5: Contours C and D in w -plane.

as shown in Figure 5.

First, consider the case where $\sigma > 0$. Then, $\delta \rightarrow -\delta_2$ in (74), and thus

$$\phi(\sigma, \tau) = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{D_1} \Phi(w, \tau) e^{-i\sigma w} dw. \quad (80)$$

Since $\Phi(w, \tau)$ is analytic in G^- , the contour integral along D must be zero, so

$$\int_{D_1} \Phi(w, \tau) e^{-i\sigma w} dw = - \int_{D_2 \cup D_3 \cup \dots \cup D_6} \Phi(w, \tau) e^{-i\sigma w} dw. \quad (81)$$

Since $\text{Re}(\sqrt{w^2 + 1}) \geq 0$ for all w for which it is defined, it follows from (73) that for all $\tau > 0$, $\Phi(w, \tau) = O(R^{-3/2})$ for $|w| = R \rightarrow \infty$. Therefore, from Jordan's lemma (see [14, p. 81-82]) it follows that

$$\lim_{R \rightarrow \infty} \int_{D_2 \cup D_6} \Phi(w, \tau) e^{-i\sigma w} dw = 0. \quad (82)$$

In the half-plane $\text{Im } w < 0$, $\Phi(w, \tau)$ has one singular point, being a pole of order 1 in $z = -i$. Therefore,

$$\lim_{\epsilon \rightarrow 0} \int_{D_4} \Phi(w, \tau) e^{-i\sigma w} dw = 2\pi i \text{Res}_{w=-i} \Phi(w, \tau) e^{-i\sigma w} = -2\pi e^{-\sigma}. \quad (83)$$

Finally, we have to integrate Φ along D_3 and D_5 , where $w = -iv \pm i\epsilon$, respectively, with $v > 1$, and $0 < \epsilon \ll 1$. Using that $\text{Re}(\sqrt{w^2 + 1}) \geq 0$, for all w , we deduce that for $w \in D_3 \cup D_5$,

$$\sqrt{w^2 + 1} = \sqrt{(-iv \pm i\epsilon)^2 + 1} \approx \mp i \sqrt{v^2 - 1} \left(1 \mp \frac{v}{v^2 - 1} \epsilon \right), \quad (84)$$

yielding, for $\epsilon \rightarrow 0$,

$$\sqrt{w^2 + 1} = \mp i \sqrt{v^2 - 1}, \quad (85)$$

for $w \in D_3$ or $w \in D_5$, respectively. Analogously, we deduce

$$\sqrt{w - i} = \frac{1 - i}{\sqrt{2}} \sqrt{v + 1}, \quad (86)$$

for $w \in D_3 \cup D_5$.

From (85) and (86), and with Φ according to (73), it follows that

$$\begin{aligned} \int_{D_3} \Phi(w, \tau) e^{-i\sigma w} dw &= i\sqrt{2} \int_R^{1+\epsilon} \frac{e^{i\tau\sqrt{v^2-1}}}{(v-1)\sqrt{v+1}} e^{-\sigma v} dv = \\ &= -i\sqrt{2} \int_{\sqrt{2\epsilon+\epsilon^2}}^{\infty} \frac{e^{-\sigma\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{\sqrt{x^2+1}-1}} e^{i\tau x} dx, \end{aligned} \quad (87)$$

where we used the coordinate transformation $v = \sqrt{x^2 + 1}$, and where we let $R \rightarrow \infty$.

Similarly, the integral along D_5 is given by

$$\int_{D_5} \Phi(w, \tau) e^{-i\sigma w} dw = i\sqrt{2} \int_{\sqrt{2\epsilon+\epsilon^2}}^{\infty} \frac{e^{-\sigma\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{\sqrt{x^2+1}-1}} e^{-i\tau x} dx. \quad (88)$$

Summing up the integrals in (87) and (88), and letting $\epsilon \rightarrow 0$, we find

$$\int_{D_3 \cup D_5} \Phi(w, \tau) e^{-i\sigma w} dw = 2\sqrt{2} \int_0^{\infty} \frac{e^{-\sigma\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{\sqrt{x^2+1}-1}} \sin(\tau x) dx. \quad (89)$$

Note that although the integrals in (87) and (88) are divergent for $\epsilon \downarrow 0$, the integral in (89) does converge for $x \rightarrow 0$. Recollecting all the results derived above, we thus obtain for $\sigma > 0$,

$$\phi(\sigma, \tau) = e^{-\sigma} - \frac{\sqrt{2}}{\pi} \int_0^{\infty} \frac{e^{-\sigma\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{\sqrt{x^2+1}-1}} \sin(\tau x) dx. \quad (90)$$

Next, we consider the case where $\sigma < 0$. Analogous to the previous case, we can derive

$$\phi(\sigma, \tau) = \frac{\sqrt{2}}{\pi} \int_0^{\infty} \frac{e^{\sigma\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{\sqrt{x^2+1}+1}} \cos(\tau x) dx, \quad (91)$$

for $\sigma < 0$.

The integrals in (90) and (91) may be interpreted as Fourier transforms with respect to x . In [15] it can be found that $\phi(\sigma, \tau)$ for both $\sigma \geq 0$ and $\sigma < 0$ is given by

$$\phi(\sigma, \tau) = e^{-\sigma} \operatorname{Erfc} \left(\sqrt{\sqrt{\sigma^2 + \tau^2} - \sigma} \right). \quad (92)$$

From (67), it then follows that $\bar{T}(\sigma, \tau)$ is equal to

$$\bar{T}(\sigma, \tau) = 1 - e^{\sigma} \phi(\sigma, \tau) = \operatorname{Erf} \left(\sqrt{\sqrt{\sigma^2 + \tau^2} - \sigma} \right). \quad (93)$$

Here, $\text{Erf}(z)$ and $\text{Erfc}(z)$ are the error function and the complementary error function, defined as

$$\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds, \quad \text{Erfc}(z) = 1 - \text{Erf}(z), \quad (94)$$

respectively.

At this point, we have derived a solution for $\bar{T}(\sigma, \tau)$. It is easily checked that this solution indeed satisfies all the equations and conditions of system (63)-(66).

Similarity Solution

The specific form of the solution (93) suggests that, just as in (30), there exists a similarity solution in which $\bar{T}(\sigma, \tau)$ is only a function of $\rho = \rho(\sigma, \tau) = \sqrt{\sqrt{\sigma^2 + \tau^2} - \sigma}$. To show that this is indeed true, we start from (63), which we first transform to cylindrical coordinates by taking

$$\sigma = r \cos \varphi, \quad \tau = r \sin \varphi, \quad \varphi \in [0, \pi], \quad (95)$$

yielding for $T = \tilde{T}(r, \varphi)$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{T}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \tilde{T}}{\partial \varphi^2} - 2 \cos \varphi \frac{\partial \tilde{T}}{\partial r} + \frac{2 \sin \varphi}{r} \frac{\partial \tilde{T}}{\partial \varphi} = 0. \quad (96)$$

Expressing ρ in terms of r and φ , we obtain

$$\rho = \tilde{\rho}(r, \varphi) = \sqrt{r - r \cos \varphi} = \sqrt{2} r^{1/2} \sin(\varphi/2). \quad (97)$$

Assuming that $\tilde{T}(r, \varphi) = T(\rho)$, and using this in (96), we arrive after some elementary calculations at

$$\frac{1}{2r} [T''(\rho) + 2\rho T'(\rho)] = 0. \quad (98)$$

This ordinary differential equation has the general solution

$$T(\rho) = c_1 \int_0^\rho e^{-t^2} dt + c_2. \quad (99)$$

The boundary conditions (64) and (65) can be written as

- for $\sigma \geq 0$,

$$\bar{T}(\sigma, 0) = \tilde{T}(r, 0) = T(0) = c_2 = 0; \quad (100)$$

- for $\sigma < 0$,

$$\frac{\partial \bar{T}}{\partial \tau}(\sigma, 0) = -\frac{1}{r} \frac{\partial \tilde{T}}{\partial \varphi}(r, \pi) = 0, \quad (\text{identically}); \quad (101)$$

- for $\sigma \in \mathbb{R}$, $\tau \rightarrow \infty$,

$$\bar{T}(\sigma, \tau) \rightarrow \tilde{T}(r \rightarrow \infty, \varphi) \rightarrow T(\rho \rightarrow \infty) \rightarrow \frac{\sqrt{\pi}}{2} c_1 = 1. \quad (102)$$

Hence, $c_1 = 2/\sqrt{\pi}$ and $c_2 = 0$, rendering our previous result (93).

Thickness of boundary layer

We can use the here obtained similarity solution for the determination of the thickness of the boundary layer in G_{II} . Let ρ_{bl} be the value of ρ at the border of the boundary layer. Then, we find ρ_{bl} from $\text{Erfc}(\rho_{bl}) = \epsilon$, resulting in

$$\rho_{bl} = \sqrt{\log(1/\epsilon)} (1 + o(1)) \approx \sqrt{-2 \log \delta} . \quad (103)$$

Once ρ_{bl} is known, we find the dimensionless thickness τ_{bl} of the boundary layer from $\sqrt{\sigma^2 + \tau_{bl}^2} - \sigma = \rho_{bl}^2$, yielding with $h_{bl} = 2h\hat{\lambda}\delta^2\tau_{bl}$

$$\frac{h_{bl}}{h} = 2\hat{\lambda}\delta^2\rho_{bl}^2 \sqrt{1 + \frac{2\sigma}{\rho_{bl}^2}} \approx -4\hat{\lambda}\delta^2 \log \delta \sqrt{1 - \frac{\sigma}{\log \delta}} . \quad (104)$$

From this, we see that $h_{bl} \downarrow 0$ for $\sigma \rightarrow -\rho_{bl}^2/2 \approx \log \delta$. This σ -value corresponds to $x = -2\hat{\lambda}\delta^2 \log \delta$, and indicates the right end point of the boundary layer in G_{II} .

Moreover, if $\sigma \gg \rho_{bl}^2 \sim \log(1/\delta)$ then

$$\frac{h_{bl}}{h} \sim 2\hat{\lambda}\delta^2\rho_{bl} \sqrt{2\sigma} \approx 4\hat{\lambda}\delta^2 \sqrt{-\sigma \log \delta} = 2\sqrt{2\hat{\lambda}} \delta \sqrt{\frac{V_f t - X}{h}} \sqrt{-\log \delta} , \quad (105)$$

where the latter step follows from (49), (56) and (4).

4.3 Numerical results

Figure 6: Isotherms for T_{II} , based on $\bar{T}(\sigma, \tau)$ according to (93), in the boundary layer in the (σ, τ) -plane near the flow front region

We have calculated a set of isotherms of T_{II} in the (σ, τ) -plane inside the G_{II} -boundary layer. The results are depicted in Figure 6. From this figure, we see that only a small part

of the boundary layer enters the flow front region, $\sigma < 0$. Moreover, we observe that the temperatures near the wall in the flow front region G_{II} are much higher (very close to 1) than those in the upstream direction ($\sigma \gg 1$). This effect is caused by the convection due to the fountain flow in the front region. The same effect was found, by numerical simulations, by Kamal et al. [16, Fig. 10].

The boundary layer immediately behind the flow front, i.e. for $\sigma > 0$, has to match with that in G_{III} ; how this is done, will be shown in the next section.

5 Temperature problem in the boundary layer in G_{III}

5.1 Differential equation and boundary conditions

In Sections 3 and 4, we derived (stationary) solutions for the temperature in boundary layers near the upper and lower wall of the mould that are valid in G_I and G_{II} , respectively. We still need to find a solution for the temperature near the upper and lower walls in G_{III} , which couples the solution in G_I to that in G_{II} . Due to the symmetry of the problem, we only need to consider the solution near the upper wall, $Y = h$.

Written in dimensional coordinates, the temperature T_{III} in G_{III} satisfies the differential equation (8). We choose here the following scaling for t , X , Y , U , V , and T :

- For t , X , Y , we take

$$t = \frac{R}{V_f} \hat{t}, \quad X = R \hat{X}, \quad Y = h - \epsilon_3 h \eta, \quad (106)$$

where $0 < \epsilon_3 \ll 1$ but still unknown, and $\eta > 0$ and of $O(1)$ with respect to ϵ_3 .

- Since already at distances a few times h behind the flow front, the velocity resembles strongly that of a fully-developed flow, V is taken equal to $V = 0$ in G_{III} , and U to

$$U = \frac{3}{4} V_f \left(1 - \left(\frac{Y}{h} \right)^2 \right) = \frac{3}{4} V_f \epsilon_3 \eta (2 - \epsilon_3 \eta). \quad (107)$$

- Similar to the previous sections, we scale T according to

$$T_{III}(\hat{X}, \eta, \hat{t}) = \frac{T(R\hat{X}, h(1 - \epsilon_3 \eta), R\hat{t}/V_f) - T_w}{T_i - T_w}. \quad (108)$$

Substitution of (107)-(108) into (8) renders the differential equation for the temperature T_{III} in G_{III} :

$$\frac{\partial T_{III}}{\partial \hat{t}} + \frac{3}{4} \epsilon_3 \eta (2 - \epsilon_3 \eta) \frac{\partial T_{III}}{\partial \hat{X}} = \frac{\lambda_{III}}{\epsilon_3^2} \left(\epsilon_3^2 \delta^2 \frac{\partial^2 T_{III}}{\partial \hat{X}^2} + \frac{\partial^2 T_{III}}{\partial \eta^2} \right), \quad (109)$$

where

$$\lambda_{III} := \frac{\lambda R}{\rho c_p V_f h^2} = \delta \hat{\lambda} = O(\delta), \quad (110)$$

see (20).

Since the diffusion term in the $\hat{X}$ -direction has a factor $\epsilon_3^2 \delta^2$, diffusion in the $\hat{X}$ -direction may

be neglected. In (109), we assume that the diffusion term in the η -direction is of the same order as the instationary term. Therefore, we take ϵ_3 equal to $\epsilon_3 = \sqrt{\lambda_{III}} = O(\delta^{1/2}) \ll 1$, which results in

$$\frac{\partial T_{III}}{\partial \hat{t}} + \frac{3}{4}\epsilon_3\eta(2 - \epsilon_3\eta)\frac{\partial T_{III}}{\partial \hat{X}} = \frac{\partial^2 T_{III}}{\partial \eta^2}. \quad (111)$$

The boundary conditions are given by

- For $\eta = 0$, $0 < \hat{X} < \hat{t}$: $T_{III} = 0$.
- For $\eta \rightarrow 1/\epsilon_3$: $T_{III} \rightarrow 1$.
- For $\hat{X} \rightarrow \hat{t}$, $0 < \eta < 1/\epsilon_3$: $T_{III} \sim T_{II}$.
- For $\hat{X} \rightarrow 0$, $0 < \eta < 1/\epsilon_3$: $T_{III} \sim T_I$.

5.2 First-order solution

We will approximate the solution of (111) for T_{III} with the boundary conditions given above, by taking $\epsilon_3 \rightarrow 0$ (hence $1/\epsilon_3 \rightarrow \infty$). This leads us to the differential equation for the first-order approximation T_{III}^1 of T_{III} :

$$\frac{\partial T_{III}^1}{\partial \hat{t}} = \frac{\partial^2 T_{III}^1}{\partial \eta^2}, \quad (112)$$

with boundary conditions similar to the ones above.

Since (112) is independent of $\hat{X}$, we can solve the problem for each $\hat{X}$ separately. Note that (112) has a solution of the form

$$T_{III}^1(\hat{X}, \eta, \hat{t}) = C_1(\hat{X}) \operatorname{Erf} \left(\frac{\eta + C_2(\hat{X})}{2\sqrt{\hat{t} + C_3(\hat{X})}} \right) + C_4(\hat{X}). \quad (113)$$

For $\eta = 0$, we require for all $\hat{X}$ with $0 < \hat{X} < \hat{t}$, that

$$T_{III}^1(\hat{X}, 0, \hat{t}) = C_1(\hat{X}) \operatorname{Erf} \left(\frac{C_2(\hat{X})}{2\sqrt{\hat{t} + C_3(\hat{X})}} \right) + C_4(\hat{X}) = 0, \quad (114)$$

yielding, $C_2(\hat{X}) = C_4(\hat{X}) = 0$.

For $\eta \rightarrow \infty$, we require

$$T_{III}^1(\hat{X}, \eta \rightarrow \infty, \hat{t}) = C_1(\hat{X}) = 1, \quad (115)$$

from which it follows that $C_1(\hat{X}) = 1$.

To determine $C_3(\hat{X})$, we must look at the behaviour of the temperature in region G_{II} , where the temperature is given by (see (93)),

$$T = \bar{T}(\sigma, \tau) = \operatorname{Erf} \left(\sqrt{\sqrt{\sigma^2 + \tau^2} - \sigma} \right). \quad (116)$$

Here, σ and τ are given by (49) and ϵ_1 and ϵ_2 by $\epsilon_1 = \epsilon_2 = 2\lambda_{II} = 2\delta\lambda_{III}$ (see (56)). Hence, written in $\hat{X}$ and η coordinates, first using (4) and then (106) and (??), σ and τ are found to be equal to

$$\sigma = -\frac{R}{h} \cdot \frac{x}{2\epsilon_3^2} = \frac{1}{2\hat{\lambda}} \frac{\hat{t} - \hat{X}}{\delta^3}, \quad \tau = \frac{R}{h} \cdot \frac{1-y}{2\epsilon_3^2} = \frac{1}{2\sqrt{\hat{\lambda}}} \frac{\eta}{\delta^{3/2}}. \quad (117)$$

The temperature in G_{II} , expressed in terms of $\hat{X}$ and η , follows from (116) as ($\hat{X} < \hat{t}$, and $\hat{X}$ not in a δ -neighbourhood of $\hat{t}$)

$$\begin{aligned} T_{II}(\hat{X}, \eta, \hat{t}) &= \text{Erf} \left(\sqrt{\frac{1}{2\hat{\lambda}\delta^3} \left\{ \sqrt{(\hat{t} - \hat{X})^2 + \hat{\lambda}\delta^3\eta^2} - (\hat{t} - \hat{X}) \right\}} \right) \\ &= \text{Erf} \left(\frac{\eta}{2\sqrt{\hat{t} - \hat{X}}} (1 + O(\delta^3)) \right). \end{aligned} \quad (118)$$

Comparing (118) with (113), we infer that $C_3(\hat{X}) = -\hat{X}$. All this leads us to the following solution for T_{III}^1 :

$$T_{III}^1(\hat{X}, \eta, \hat{t}) = \text{Erf} \left(\frac{\eta}{2\sqrt{\hat{t} - \hat{X}}} \right) = \frac{2}{\sqrt{\pi}} \int_0^{\eta/2\sqrt{\hat{t} - \hat{X}}} e^{-s^2} ds. \quad (119)$$

The first-order approximation of the temperature T_I in G_I is given by (37) as (with $X \rightarrow \hat{X}$)

$$T_I(\hat{X}, \zeta) = C \int_0^{\zeta/\sqrt[3]{\hat{X}}} e^{-\frac{2}{9}s^3} ds, \quad (120)$$

with C given by (36).

We continue with the determination of the thickness of the boundary layer in G_{III} . Let η_{bl} be the value of η at the border of the boundary layer, where $T_{III} = 1 - \epsilon_3$. Using (119), we then find η_{bl} from $\text{Erfc}(\eta_{bl}/2\sqrt{\hat{t} - \hat{X}}) = \epsilon$, resulting in

$$\frac{\eta_{bl}}{2\sqrt{\hat{t} - \hat{X}}} \approx \sqrt{-\log \epsilon_3} \approx \sqrt{-\frac{1}{2} \log \delta}, \quad (121)$$

yielding for the thickness of the boundary layer

$$\frac{h_{bl}}{h} = \sqrt{2\hat{\lambda}\delta} \sqrt{\hat{t} - \hat{X}} \sqrt{-\log \delta} = \sqrt{2\hat{\lambda}} \delta \sqrt{\frac{V_f \hat{t} - \hat{X}}{h}} \sqrt{-\log \delta}. \quad (122)$$

The precise position of the interface between G_{III} and G_I is not so relevant for our further investigations; we are more interested in the part of G_{III} in the vicinity of the flow front (directly behind the flow front). Therefore, a rather rough estimate for the transition zone between G_{III} and G_I suffices for us. With $\zeta = (3/4)^{1/3}(\hat{\lambda}\delta)^{1/6} \eta$, as follows from (23), (25), and (??), we see from (120) and (123) that T_I and T_{III} become of the same order when $\hat{X}$ becomes so small with respect to $\hat{t}$, that $\zeta/\sqrt[3]{\hat{X}} = (3/4\hat{X})^{1/3}(\hat{\lambda}\delta)^{1/6} \eta \sim \eta/2\sqrt{\hat{t} - \hat{X}} \approx \eta/2\sqrt{\hat{t}}$, as is the case when $\hat{X} \sim 6(\hat{\lambda}\delta)^{1/2} \hat{t}^{3/2}$ (thus $\hat{X}/\hat{t} = O(\delta^{1/2}) \ll 1$, indeed). This indicates that the left boundary of G_{III} , i.e. the interface between G_{III} and G_I , must be somewhere around $\hat{X} \sim 6(\hat{\lambda}\delta)^{1/2} \hat{t}^{3/2}$.

5.3 Second-order solution

We can find a second-order solution of (111) by assuming that T_{III} can be written in the form $T_{III} = T_{III}^1 + \epsilon_3 T_{III}^2 + O(\epsilon_3^2)$, where T_{III}^2 is of order $O(1)$. Substituting this form into (111), and retaining only first order terms in ϵ_3 , we arrive at the differential equation for T_{III}^2 :

$$-\frac{\partial T_{III}^2}{\partial \hat{t}} + \frac{\partial^2 T_{III}^2}{\partial \eta^2} = \frac{3}{2} \eta \frac{\partial T_{III}^1}{\partial \hat{X}} = \frac{6}{\sqrt{\pi}} \frac{s^3}{\eta} e^{-s^2}, \quad (123)$$

according to (119), and with $s = \eta/2\sqrt{(\hat{t} - \hat{X})}$.

The boundary conditions for T_{III}^2 follow from the boundary conditions listed at the end of Section 5.1. However, the third and the fourth condition are only relevant for the matching of the three boundary layer solutions. Here, we discuss this matching only up to the first-order matching and, therefore, the latter two boundary conditions need not to be considered here. So, there remain:

- For $\eta = 0$, $0 < \hat{X} < \hat{t}$: $T_{III}^2 = 0$.
- For $\eta \rightarrow \infty$: $T_{III}^2 \rightarrow 0$.

Equation (123) has a solution of the form $T_{III}^2(\hat{X}, \eta, \hat{t}) = \eta \phi(s)$, where ϕ satisfies the differential equation:

$$\frac{1}{2} s \phi''(s) + (1 + s^2) \phi'(s) = \frac{3}{\sqrt{\pi}} s^2 e^{-s^2}. \quad (124)$$

The general solution of (124) is given by

$$\phi(s) = \frac{3}{8} \text{Erf}(s) - \frac{3}{4\sqrt{\pi}} s e^{-s^2} + C \left(\text{Erf}(s) + \frac{1}{\sqrt{\pi}s} e^{-s^2} \right) + D. \quad (125)$$

Hence, the general solution for T_{III}^2 in terms of s and η is given by

$$T_{III}^2 = \tilde{T}_{III}^2(s, \eta) = \eta \left(\frac{3}{8} \text{Erf}(s) - \frac{3}{4\sqrt{\pi}} s e^{-s^2} + C \left(\text{Erf}(s) + \frac{1}{\sqrt{\pi}s} e^{-s^2} \right) + D \right), \quad (126)$$

where C and D are to be determined by the two boundary conditions given above.

The first boundary condition says that $T_{III}^2 = 0$, for $\eta = 0$, but then also $s = 0$. Taking in (126) the limit for $\eta \downarrow 0$ and $s \downarrow 0$, we find that

$$\tilde{T}_{III}^2(0, 0) = \lim_{\eta \downarrow 0} \frac{C}{\sqrt{\pi}} \frac{\eta}{s} = \frac{C}{\sqrt{\pi}} \sqrt{\hat{t} - \hat{X}} = 0, \quad (127)$$

from which it follows that $C = 0$.

For the second boundary condition, that is for $\eta \rightarrow \infty$, we have $s \rightarrow \infty$, and $T_{III}^2 \rightarrow 0$. Since $\text{Erf}(s) \rightarrow 1$ for $s \rightarrow \infty$, it must hold that $3/8 + C + D = 0$, and hence $D = -3/8$. Thus, we arrive at the second order solution for the temperature in G_{III} :

$$T_{III}^2 = \tilde{T}_{III}^2(s, \eta) = \frac{3}{8} \eta \left(\text{Erf}(s) - 1 - \frac{2}{\sqrt{\pi}} s e^{-s^2} \right). \quad (128)$$

In the next section, the effect of T_{III}^2 on the total solution is numerically evaluated; it is shown that the effect is small but observable.

Figure 7: The first-order solution T_{III}^1 (dashed curve) and the second-order solution $T_{III}^1 + \epsilon_3 T_{III}^2$ (solid curve) for the temperature in G_{III} as a function of η for $\hat{t} - \hat{X} = 0.3$

5.4 Numerical results and comparison of first- and second-order solutions

We have calculated the temperature T_{III} in some cross-section in G_{III} , i.e. at $\hat{t} - \hat{X} = 0.3$, as a function of η , especially near the upper wall. We did this for both the first-order and the second-order solutions. The results are depicted in Figure 7. Comparing the two solutions, we conclude that the differences are small. We also did some numerical (FEM) computations and found corresponding results. From Figure 7, we observe that the temperature goes to its inlet value 1 for $\eta \approx 2$; for $\delta = 0.01$, this means that the upper boundary occupies about 20% of the cross-section at $\hat{t} - \hat{X} = 0.3$.

To illustrate the result of the matching of the inner solutions in the boundary layers of G_{II} and G_{III} at the interface between these two regions, we have depicted in Figure 8 two isothermals, both for $\hat{T} = 0.5$, in this interfacial region. This interface is closely behind the flow front region. In the left part of Figure 8, we depict a global view in a $((\hat{t} - \hat{X}), \eta)$ -plane on a length-scale of $(\delta)h$; this figure shows a perfect matching (in fact, the dashed curve and the solid curve almost completely overlap each other). To make the differences between T_{II} and T_{III} more clear, we also draw a detailed view in a (σ, τ) -plane on a length-scale of $(\delta^2)h$ in the right part of the figure (see (117) for the relationship between the respective length scales). Here, indeed some difference is observed (as it should be), but this figure also shows that these curves tend to overlap for increasing σ , as is confirmed by the left graph.

NOTE

Within a thin-layer approximation, often used for the simulation of injection moulding processes for narrow moulds, it is common practice to approximate the curved flow front Γ by a straight line; see e.g. [2]. In this way, the influence of the shape of the flow front on the velocity and temperature distribution immediately behind the flow front is neglected. As it are especially these effects we are interested in, we did not follow this path here, but we will now try to indicate the inaccuracies in the near-by temperature caused by a straight-front approximation.

In for example [2], a thin-layer or lubrication-like approximation was used, yielding dominant

Figure 8: The isotherms $T_{II} = 0.5$ (solid curve) and $T_{III} = 0.5$ (dashed curve), showing the matching of the boundary-layer temperatures T_{II} and T_{III} near the flow front interface; a. global view in the $(\hat{t} - \hat{X}, \eta)$ -plane (along the horizontal axis $(\hat{t} - \hat{X})/\delta^2 = (\hat{t} - \hat{X}) \times 10^4$ is plotted), b. detailed view in the (σ, τ) -plane.

convection in the flow direction and dominant conduction in the thickness direction, whereby the heat conduction was assumed to prevail in the region closely behind the (straight) flow front. Neglecting heat convection in this region, they came to a heat diffusion equation (see [2, Eq. (25)]), exactly equal to our equation (112) for T_{III} . This means that in this approach the specific distribution, inclusive boundary layer, in G_{II} is completely discarded. The most typical effect there, namely that the temperature near the wall is much higher in the flow front region than in a point at the same distance of the wall in upstream direction, i.e. in G_{II} , is then not found; see Section 4.3, or [16]. This is because this effect is due to the convection by the fountain flow in G_{III} , and it is precisely this convection that is neglected in the thin-layer approximation in e.g. [16]. It is evident that for an analysis of the temperature, and the associated viscous flow for a fluid of temperature-dependent viscosity, the above-mentioned effect is essential.

6 Conclusions

In the preceding sections, we have considered the heat flow in a fluid entering the narrow region between two fixed parallel plates. The fluid flow has a free surface, the flow front; the shape of the flow front (almost semi-circular) and the flow patterns in and behind the flow region (fountain flow and 2-D Poiseuille flow, respectively) have been calculated in a previous paper [1]. We found that in the bulk of the fluid the temperature remains (to first order) equal to the inlet temperature; this is due to the fact that the flow is convection dominated there. Only in boundary layers near the cooled walls the temperature drastically changes. We have found three different types of boundary layers referring to three distinct regions of the flow: G_I , G_{II} and G_{III} . We now review the results in these regions:

1. In G_I ; the region far behind the flow front.

In this region, the influence of the flow front can be neglected, and thus the flow here is a pure Poiseuille flow. The temperature in this region is stationary with respect to a fixed frame. The small parameter ϵ , which is a measure for the thickness of the boundary layer, is here related to the dimensionless heat conduction coefficient $\lambda_I = 4\hat{\lambda}\delta/3 = O(\delta) \ll 1$, according to $\epsilon = \sqrt[3]{4\hat{\lambda}/3} \delta^{1/3}$. The inner solution within the boundary layer is governed by a convection-diffusion equation for the temperature T , whereas the outer solution, outside the boundary layer, i.e. in the kernel of the flow, is convection dominated. The inner solution was found in terms of a similarity solution, and both a first order ($O(\epsilon)$) and a second order ($O(\epsilon^2)$) asymptotic solution was derived. Despite the fact that the parameter ϵ was not really very small ($\epsilon = 0.21$) the differences between the first- and second- order solutions were minimal (only of some relevance very near the walls). The asymptotic solution was compared with a numerical solution, and again the agreement was very good. That, for G_I , ϵ is not very small implies that the boundary layer in this region is relatively thick. If we define the border of the boundary layer as the place where the scaled temperature differs $O(\epsilon)$ from the scaled inlet temperature, we found for the thickness h_{bl} of the boundary layer in G_I

$$\frac{h_{bl}}{h} = \sqrt[3]{-2\hat{\lambda}\delta\hat{X}\log\delta} . \quad (129)$$

In Section 5.2, we derived that the values for X in the region G_I are restricted by $0 < \hat{X} < 6(\hat{\lambda}\delta)^{1/2} (V_f t/R)^{3/2}$. Hence, the region G_I is time dependent, and its area is increasing with time.

2. In G_{II} ; the flow front region.

The type of flow in this region is a fountain flow, which is stationary with respect to a comoving frame (moving with the constant speed of propagation of the flow front). Therefore, also the temperature in G_{II} is taken to be stationary with respect to the comoving frame. In the main part of G_{II} the heat flow is convection dominated, implying that $T = T_i$ in G_{II} , except for two small boundary layers in the upper and lower tips of the flow front region. These layers extend to narrow layers at the walls in the region immediately behind the flow front. The small parameter ϵ here is $\epsilon = \lambda_{II} = 2\delta^2\hat{\lambda}$, which is much smaller than the one in G_I , so the boundary layers here are much thinner than those in G_I . The inner solution, of convection-diffusion type, was found by means of a Wiener-Hopf technique. It turned out that this solution could be written as a similarity solution, i.e. as $T_{II}(\sigma, \tau) = \text{Erf}(\rho)$, where $\rho = \rho(\sigma, \tau) = \sqrt{\sqrt{\sigma^2 + \tau^2} - \sigma}$. We found that the thickness h_{bl} of the boundary layer as function of σ is given by

$$\frac{h_{bl}}{h} = 2\hat{\lambda}\delta^2\rho_{bl}^2 \sqrt{1 + \frac{2\sigma}{\rho_{bl}^2}} \approx -\hat{\lambda}\delta^2\log\delta \sqrt{1 - \frac{4\sigma}{\log\delta}} . \quad (130)$$

Moreover, the right end point of this layer, where $h_{bl} \downarrow 0$ was found for $\sigma \rightarrow \log\delta$.

3. In G_{III} ; the intermediate region between G_I and G_{II} .

Also in the intermediate region G_{III} a boundary layer exists, forming the connection between the boundary layers in G_I and G_{II} . The small ϵ -parameter here is

$\sqrt{\lambda_{III}} = O(\delta^{1/2})$, which is large compared to the ϵ -parameter in G_{II} and small compared to the one in G_I . Again, an inner solution was found as a similarity solution; see (119). Moreover, a second-order solution was derived, and the results of the first- and second-order solutions were compared to each other, and to a numerical solution; good agreement was found.

In Section 5.2, we showed that the solution in G_{III} matches perfectly with the inner solution in G_{II} , while a good matching with the solution in G_I is obtained when the interface between G_{III} and G_I is taken somewhere near to $\hat{X} = (\hat{\lambda}\delta)^{1/2} (V_f t/R)^{3/2}$. Hence the region G_{III} ranges from $\hat{X} = (\hat{\lambda}\delta)^{1/2} (V_f t/R)^{3/2}$ to $\hat{X} \approx V_f t/R$. Therefore, in the region which we refer to as "region (closely) behind the flow front" we must use for the temperature the solution found for G_{III} .

In Section 5.2, we have found for the thickness of the boundary layer

$$\frac{h_{bl}}{h} = \sqrt{2\hat{\lambda}} \delta \sqrt{\frac{V_f t - \hat{X}}{h}} \sqrt{-\log \delta}, \quad (131)$$

which is of the same order as the one found in (105) (the difference of the factor 2 is due to the two different ϵ -parameters used in G_{II} and G_{III}). Hence, we can conclude that the boundary layers from G_{II} and G_{III} match perfectly.

From (129) and (131), we see that the borders of the boundary layers in G_I and G_{III} coincide at (provided $\hat{X} \ll \hat{t}$)

$$\hat{X} = \frac{4\sqrt{2}}{3} \left(-\hat{\lambda}\delta \log \delta \right)^{1/2} (\hat{t})^{3/2}. \quad (132)$$

Note that this place is within the range of the interface between G_I and G_{III} as estimated at the end of Section 5.2 by matching.

To conclude, we like to explain in how far, in our view, the results obtained here can be used for an investigation of flow front instabilities. We found that the temperature distribution shows boundary layers near the walls of the mould. The hope that these results can be used for an explanation of the instability (wobbling) of the flow front is based on the expectation that the wobbling could be driven by asymmetric temperature distributions. This asymmetry influences the flow, and this is most manifest in the boundary layers. In this context, we notice that an asymmetric flow front causes differences in temperature, and thus also in viscosity.

To explain our view further, let us assume there is initially an asymmetric flow front (for instance due to an asymmetric inlet of the mould) and that at certain time t the contact point of the flow front with the upper wall is farther (in the X -direction) than the corresponding point at the lower wall, i.e. $\xi_f(+h, t) > \xi_f(-h, t)$. In that case, the fluid near the upper wall will cool more than the fluid near the lower wall, and so the fluid will become stiffer (larger viscosity) at the upper wall than at the lower one. Consequently, the fluid will be retarded at the upper wall with respect to the fluid at the lower wall and, thus, the asymmetry will flip. This means that at a somewhat later time the flow front is farther at $Y = -h$ than at $Y = +h$. This cycle will repeat itself and then the front is wobbling. Hence, the differences in viscosity can induce an ongoing asymmetry of the flow front, which can result into instability.

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