Analysis of the convergence behaviour of the local defect correction method for one-dimensional convection-diffusion problems

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1 Introduction

Many boundary value problems produce solutions that have highly localized properties. In this paper we consider boundary value problems with solutions that have one or a few small regions with high activity.

We study a method based on a combination of standard finite difference discretizations on several uniform grids with different grid sizes that cover different parts of the domain. At least one grid should cover the entire domain; the mesh size of this global coarse grid is chosen in agreement with the relatively smooth behaviour of the solution outside the high activity regions. Apart from this global coarse grid, one or several local grids are used which are also uniform. Each of these local grids covers only a (small) part of the domain and contains a high activity region. The mesh sizes of the local grids are chosen in agreement with the behaviour of the solution in the corresponding high activity region. In this way, every part of the domain can be covered by a (locally) uniform grid with a mesh size that is in agreement with the behaviour of the continuous solution in that part of the domain. This refinement strategy is known as locally uniform grid refinement. The solution is approximated on the composite grid, which consists of the uniform coarse grid and subgrid(s). Note that such composite grids are highly structured and hence very simple data structures can be used.

The boundary value problem is solved on the composite grid by the local defect correction method (LDC) (see [4], [7]). In this method, which is an iterative process, a basic global discretization is improved by local discretizations defined in the subdomains. This update of the coarse grid solution is achieved by putting a defect correction term in the right hand side of the coarse grid problem. At each iteration step, the process yields a discrete approximation of the continuous solution on the composite grid. The discrete problem that is actually being solved is an implicit result of the iterative process. Therefore, the LDC method is both an iterative discretization and solution method.

An analysis of the LDC technique in combination with finite difference discretizations is presented in [4], [5], [6]. The LDC method is combined with finite volume discretizations in [3], [1] and [10]. An application of the LDC algorithm in a finite volume context was presented in [8] for numerical simulations of the flow and heat transfer in a glass tank. Finally, LDC is studied in combination with finite element discretizations in [11].

It should be noted, however, that previously LDC was used mostly for diffusion problems. For flow problems we have either pure convection or both convection and diffusion, so we need additional testing for LDC on such problems.

In this paper the major objective is to extend the results found for pure diffusion problems to convection diffusion problems. The paper is built up as follows. In Section 2, we formulate a stationary convection diffusion problem and describe the Local Defect Correction algorithm. In Section 3, we present a method for estimating the convergence behaviour of the LDC algorithm.
We find theoretical estimates for the convergence rate. The following factors have influence on the convergence rate: coarse grid mesh size $H$, refinement factor $\sigma$ and ratio of convection and diffusion coefficients $\kappa$. We investigate the influence of each of these factors on the convergence rate for different combinations of the discretization schemes (central differences on both grids, upwind on both grids and central differences for fine grid and upwind for coarse grid). We list the cases where we have fast convergence and the requirements for them. In Section 7, we show results of a few numerical experiments, which are typical for the LDC technique. The results of the numerical simulation confirms the theoretical estimates. Moreover, results of the numerical simulation show that LDC technique can even combine “unstable” numerical schemes to a stable scheme on the composite grid.

2 Problem description and formulation of the LDC algorithm

The problem we study in this paper is given by

\[
\begin{align*}
\epsilon u''(x) + cu'(x) &= f(x), & c, \epsilon > 0 & x \in (0, 1), \\
  u(0) &= u_0, & u(1) &= u_1,
\end{align*}
\]

where the diffusion parameter $\epsilon$ may be (very) small. We also introduce a parameter $\kappa = c/\epsilon$. Previous papers studied convection-diffusion equations, where the convection and diffusion coefficients are of the same order of magnitude. However, since many flow problems are convection-dominated, we would like to study the properties of the LDC technique for such types of problems too. We specifically take the convection coefficient $c$ to be fixed and the diffusion coefficient $\epsilon$ to be variable. For small $\epsilon$ this approach could also be interpreted as a transformation of the pure convection problem into a two point boundary value problem by means of adding ‘small’ diffusion. The general setting and description of the Local Defect Correction technique can be found in [2]. For small $\epsilon$, the solution may have a boundary layer on the left; moreover it may contain more areas of high activity induced by $f$. In such regions we need more grid points to obtain a resolution comparable to that one in the smooth region (where a much coarser grid is sufficient).

For the global coarse grid, denoted by $\Omega^H$, i.e. grid covering the whole domain $\Omega$, we choose $H > 0$, define $N := 1/H$, $x_i := iH$, $i = 0, 1, \ldots, N$, and set $\Omega^H := \{x_i|i = 1, 2, \ldots, N - 1\}$. An initial approximation $u^H$ on $\Omega^H$ can be found by solving the system

\[
L^H u^H = f^H,
\]

which is a discretization of boundary value problem (1). In (1), the right hand side $f^H$ incorporates the source term $f$ as well as the Dirichlet boundary conditions $u_0$ and $u_1$. We assume $L^H$ to be invertible.

Next we choose $\Omega_l \subset \Omega$ such that the high activity region of $u$ is contained in $\Omega_l$. For ease of presentation, we will assume that $\Omega_l = (0, \gamma)$, where $0 < \gamma < 1$ coincides with a grid point of the coarse grid, so $\gamma = kH$ ($1 < k < N - 1$). In $\Omega_l$, we define a local fine grid (grid size $h$), by

\[
\Omega^h_l := \{ih \mid i = 1, 2, \ldots, n - 1\},
\]

in which $h := H/\sigma$, $n := \gamma/h$. The refinement factor $\sigma$ is a positive integer, so that $\Omega^H \cap \Omega_l \subset \Omega^h_l$, i.e., grid points of the global coarse grid that lie in the area of refinement belong to the local fine grid too.

The initial coarse grid approximation $u^H$ can be used to define a new boundary value problem on $\Omega_l$: we use the coarse grid approximation at $x = \gamma$ to set

\[
\begin{align*}
\epsilon u''(x) + cu'(x) &= f(x), & x \in \Omega_l &= (0, \gamma), \\
u(0) &= u_0, & u(\gamma) &= u^H(\gamma).
\end{align*}
\]

Discretization of (4) on the local fine grid $\Omega^h_l$ yields the following approximation $u^h_l$, $i = 0$:

\[
L^h_l u^h_l = f^h_l - B^h_{l,r}(u^H(\gamma)).
\]
Note that $u^h_l$ depends on the coarse grid approximation via the artificial Dirichlet boundary condition through the second term on the right hand side. In (5), the matrix $L^h_l$ incorporates a discrete approximation of the differential operator $L$ on the subdomain $\Omega_l$ and expressions for the boundary conditions. The term $f^h_l$ on the right hand side incorporates the source term $f$ as well as the Dirichlet boundary condition $u_0$ given in (1). We assume $L^h_l$ to be invertible as well. We will use the local fine grid solution to update the coarse grid approximation. This update can be achieved by estimating the local discretization error of the coarse grid discretization. The estimate is used to formulate a modified discrete problem on the coarse base grid.

The grid points of the coarse grid can be partitioned into the points inside the area of refinement $\Omega_l$, the interface point $\gamma$, and the points outside $\Omega_l$. This gives $\Omega^H = \Omega^H_l \cup \Gamma^H \cup \Omega^H_c$, with $\Omega^H_l := \Omega^H \cap \Omega_l = \{x_i \mid i = 1, 2, \ldots, k - 1\}$, $\Gamma^H := \{\gamma\}$ and $\Omega^H_c := \Omega^H \setminus (\Omega^H_l \cup \Gamma^H) = \{x_i \mid i = k + 1, k + 2, \ldots, N - 1\}$. Using this partitioning of $\Omega^H$, we set

$$u^H = \begin{pmatrix} u^H_l \\ u^H_\gamma \\ u^H_c \end{pmatrix}. \tag{6}$$

We apply the same partitioning for other vectors. Assuming that the stencil at grid point $x$ involves (at most) function values at $x - H$, $x$, $x + H$, we can partition the discrete operator $L^H$ as

$$L^H = \begin{pmatrix} L^H_l & B^H_{l,\gamma} & 0 \\ B^H_{\gamma,l} & L^H_\gamma & B^H_{\gamma,c} \\ 0 & B^H_{c,\gamma} & L^H_c \end{pmatrix}, \tag{7}$$

where $B^H_{l,\gamma}$ describes the dependence of the coarse grid points inside the area of refinement on the points on the interface $\Gamma^H$. The other off diagonal matrices describe similar dependencies. With this decomposition the coarse grid initialization (2) may also be written as

$$\begin{pmatrix} L^H_l & B^H_{l,\gamma} & 0 \\ B^H_{\gamma,l} & L^H_\gamma & B^H_{\gamma,c} \\ 0 & B^H_{c,\gamma} & L^H_c \end{pmatrix} \begin{pmatrix} u^H_l \\ u^H_\gamma \\ u^H_c \end{pmatrix} = \begin{pmatrix} f^H_l \\ f^H_\gamma \\ f^H_c \end{pmatrix}. \tag{8}$$

We try to improve the coarse grid approximation by using the solution of (5), $u^h_l$, calculated on the local fine grid to estimate the local discretization error of the coarse grid discretization. For the description of this step, we introduce the operator $R^{H,h}$ as the restriction from $\Omega^h_l$ onto $\Omega^H_l$, viz. $(R^{H,h}v^h_l)(x) := v^h_l(x)$ for all $v^h$ and $x \in \Omega^H_l$.

If we substitute the projection on $\Omega^H$ of the exact solution $u$ of boundary value problem (1) into the coarse grid discretization (2), we would find the local discretization error or defect $d^h$, given by $L^H(u|_{\Omega^H}) = f^H + d^h$. In particular, we would find the following local defect on $\Omega^H_l$:

$$L^H_l(u|_{\Omega^H_l}) + B^H_{l,\gamma}(u(\gamma)) = f^H_l + d^h_l. \tag{9}$$

If we would know the values of the defect $d^h_l$, we could use them to find a better approximation on the coarse grid. This could be achieved by putting the defect vector on the right hand side of (2). Because we do not know the exact solution of the boundary value problem, we can calculate neither $d^h_l$ nor $d^h_c$. We can, however, use the approximation $u^h_l$ calculated on the local fine grid to estimate $d^h_l$. Using (9), we find

$$d^h_l = L^H_l(u|_{\Omega^H_l}) + B^H_{l,\gamma}(u(\gamma)) - f^H_l \approx L^H_l R^{H,h}u^h_l + B^H_{l,\gamma}u^h_l - f^H_l := \tilde{d}^H_l. \tag{10}$$

Using (10), we find an estimate of the local discretization error of the coarse grid discretization at all points of $\Omega^H_l$. Therefore, we can update the coarse grid approximation by placing the
estimate (10) in the right hand side of the coarse grid equation (2) or (8). This leads to the coarse grid correction step to find $u^H_1$ on the coarse grid

$$L^H u^H_1 = \left( \begin{array}{c} f_i^H + d_i^H \\ f_i^H \\ f_c^H \end{array} \right) = \left( \begin{array}{c} L^H R^{H,h} u^h_i + B^{H}_i u^H_1 \\ L^H R^{H,h} u^h_i \\ f_i^H \\ f_c^H \end{array} \right). \quad (11)$$

Solving (11) produces a new solution $u^H_1$ on the coarse grid. Because (11) incorporates estimates of the local discretization error of the coarse grid discretization, the new solution $u^H_1$ is assumed to be more accurate than $u^H_i$. Hence, the new solution $u^H_1$ provides a better artificial boundary condition at the interface point $\gamma$. A better solution on the local fine grid can be found as before by solving

$$L_i^h u^h_{i,i} = f_i^h - B^h_{i,\Gamma}(u^H_i(\gamma)) \quad (12)$$

with $i = 1$.

To summarize, we have outlined the following iterative method.

**Algorithm 1.**

Two-grid LDC algorithm with area of refinement chosen a priori

**Initialization**

- Solve the basic coarse grid problem (2).
- Solve the local fine grid problem (5).

**Iteration, $i = 1, 2, \ldots$**

- Solve the updated coarse grid problem (11).
- Solve the local fine grid problem (12).

### 3 Theoretical estimation of the convergence rate

As one can see from Section 2, the LDC algorithm is an iterative process. According to [2], the LDC iteration can be formulated in terms of coarse grid points located at the interface only. For our one-dimensional problem, the interface between the global coarse and local fine grid consists of the single point $x = \gamma$. For this reason, the iteration matrix $M$ from [2] is a scalar quantity that we will denote by $M$. We would like to have fast convergence of our iterative process and therefore we want to study the influence of various parameters on the convergence rate. In the following sections we present first theoretical results for the convergence rate, which in the subsequent section are illustrated by numerical experiments.

We can write the following expression

$$M = M_1 M_2, \quad (13)$$

where

$$M_1 = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \left( L^H \right)^{-1} \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad (14)$$

$$M_2 = B^H_{i,\Gamma} - L_i^h R^{H,h} \left( L_i^h \right)^{-1} B^h_{i,\Gamma}. \quad (15)$$

In the following we look at $M_2$ since it has the major effect on the convergence rate. Indeed, we have

$$|M| = \|M\|_\infty \leq \|M_1\|_\infty \|M_2\|_\infty \leq \frac{1}{8} \|M_2\|_\infty.$$
The last inequality is a well known result, see, e.g., [7]. We would like to show that $|M| < 1$, which means that the LDC algorithm converges. For this reason we let $g \in \mathbb{R}$ and analyse $M_2g$. Consider

$$M_2g = \left( B_{l,\Gamma}^H - L_{l}^H R_{H,h} \right) g = B_{l,\Gamma}^H g - L_{l}^H R_{H,h} v_{l}^h,$$

where

$$v_{l}^h = \left( L_{l}^h \right)^{-1} B_{l,\Gamma}^h g.$$

Rewriting, we find

$$L_{l}^h v_{l}^h + B_{l,\Gamma}^h (g) = 0.$$

According to the definitions of the matrices $L_{l}^h, B_{l,\Gamma}^h$, this is a discretization on the local fine grid of the following boundary value problem

$$\begin{cases} \epsilon u''(x) + cu'(x) = 0, \\ u(0) = u_0, \quad u(\gamma) = -g. \end{cases}$$

In the following we study the convergence behaviour of the LDC algorithm and consider the following combinations of discretization schemes

- Central differences both for fine grid and coarse grid. Here (on the coarse grid e.g.) we approximate the second derivative $u_i''$ by the following expression

$$u_i'' = \frac{u_{i-1} - 2u_i + u_{i+1}}{H^2}$$

and the first derivative $u_i'$ by

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2H}.$$

- Upwind both for fine grid and coarse grid. We approximate the second derivative by (19) and the first derivative by

$$u_i' = \phi \frac{u_i - u_{i-1}}{H} + (1 - \phi) \frac{u_{i+1} - u_i}{H},$$

where

$$\phi = \begin{cases} 1, & c < 0 \\ 0, & c \geq 0 \end{cases}$$

and $c$ is the convection coefficient in (1).

- Upwind for coarse grid and central differences for fine grid. In this case we use (19) and (21) on the coarse grid and (19) and (20) on the fine grid.

### 4 Analysis for central differences on both grids

Assume we use the central difference scheme (see (19) and (20)) both for fine and coarse grids. Using the fact that we can get an explicit formula for the solution $v_{l}^h$ (see, for example [9]), we can write the following

$$v_{l}^h(i) = -g \frac{1 - \tau^i}{1 - \tau^n}, \quad i = 0, 1, \ldots, n,$$

with

$$\tau = \frac{1 + d}{1 - d}, \quad d = \frac{ch}{2\epsilon} = \frac{kh}{2}.$$
Now we analyse
\[
\mathbf{z} := \mathbf{M}_2 \mathbf{g} = \mathbf{B}_i^H \mathbf{g} - \mathbf{L}_i^H \mathbf{R}_i^{H,h} \mathbf{v}_i^h.
\] (23)

According to (15), \(z\) is the residual we find by substituting the local fine grid solution \(v_i^h\) into the coarse discretization.

Therefore we can write the following expression for \(z_i\), the \(i\)-th component of \(z\)
\[
z_i = \frac{1}{H^2} \left[ \left( -1 - \frac{\kappa H}{2} \right) (v_i^h)_{i-1} + 2(v_i^h)_i + \left( -1 + \frac{\kappa H}{2} \right) (v_i^h)_{i+1} \right]
\]
\[= -\frac{g}{H^2} \left[ \left( -1 - \frac{\kappa H}{2} \right) \frac{1 - \tau^{\sigma(i-1)}}{1 - \tau^n} + \frac{1 - \tau^{\sigma i}}{1 - \tau^n} + \left( -1 + \frac{\kappa H}{2} \right) \frac{1 - \tau^{\sigma(i+1)}}{1 - \tau^n} \right],
\] (24)

where \(\sigma := H/h\) is the refinement factor and the subscript \(i, i = 1, 2, \ldots, k - 1\) denotes the coarse grid point inside the fine grid region.

We find
\[
z_i = \frac{-g\tau^{\sigma(i-1)}}{1 - \tau^n} \frac{1}{(1 - \frac{\kappa}{\sigma}) \frac{1}{H^2} \left[ (1 + \sigma d)(1 - d)^\sigma - 2(1 + d)^\sigma + (1 - \sigma d)(1 + d)^{2\sigma} \right]}.
\] (25)

As one can see from (25), the convergence rate depends on the following factors: coarse grid mesh size \(H\), refinement factor \(\sigma\) (or which is the same, fine grid mesh size \(h\)), factor \(\kappa = c/\epsilon\). Below we study the influence of each of these parameters on \(z_i\) separately.

First we study the influence of the coarse grid mesh size \(H\) on the convergence rate, so we fix \(\kappa\) and \(\sigma\) and make a Taylor series expansion around \(d = 0\) (since \(d = ch/(2\epsilon)\), so \(d = cH/(2\epsilon\sigma) = \kappa H/(2\sigma)\) and \(d \rightarrow 0\) is equivalent with \(H \rightarrow 0\)). In order to study dependence of the convergence rate on the coarse grid mesh size \(H\) we fix \(k\) and \(\sigma\) and make Taylor series expansion around \(d = 0\)

\[
z_i = \frac{-g\tau^{\sigma(i-1)}}{1 - \frac{\kappa}{\sigma}} \frac{1}{H^2} \left[ -\frac{4}{3} \left( 1 - \frac{1}{\sigma^2} \right) (\sigma d)^4 + O(d^5) \right]
\]
\[= \left( 1 - \frac{1}{\sigma^2} \right) \frac{g\tau^{\sigma(i-1)}}{1 - \frac{\kappa}{\sigma}} \frac{1}{H^2} \frac{\kappa^4}{12} H^2 + O(H^3).
\] (26)

For fixed \(\kappa\) and \(\sigma\), \(\frac{1}{1 - \frac{\kappa}{\sigma}}\) converges to 1 for \(H \rightarrow 0\). The only terms that depend on \(H\) left are \(\frac{g\tau^{\sigma(i-1)}}{1 - \frac{\kappa}{\sigma}}\) and \(H^2\). We analyse the behaviour of the first for \(H, h \rightarrow 0\). Expansion gives:

\[
\tau = \frac{1 - \frac{\kappa h}{2}}{1 + \frac{\kappa h}{2}} = \left( 1 - \frac{\kappa h}{2} + O(h^2) \right)^2 = 1 - \kappa h + O(h^2),
\]
\[
\tau^n = \left( 1 + \kappa h + O(h^2) \right)^n = \left( 1 - \frac{\kappa h}{2} + O\left( \frac{1}{n^2} \right) \right)^n \rightarrow \exp(-\kappa \gamma) \quad (n \rightarrow \infty).
\]

From (26) we can see that our convergence rate depends on the coarse grid size \(H\) as \(O(H^2)\) and does not depend on the fine grid size \(h\). As one can see in Figure 1 (a), we have \(O(H^2)\) dependence of the convergence rate on coarse grid mesh size \(H\) indeed.

Next we study the influence of the refinement factor \(\sigma\) on the convergence rate. From (25) we expect to have a weak dependence on the refinement factor. To see if this is true, we fix all parameters for the fine grid except the fine grid step size \(h\), and therefore \(\sigma\), and calculate the convergence factor \(M\). As can be seen from Figure 1 (b), indeed we have minor dependence of the convergence factor \(M\) on the refinement factor.

Finally, we study the influence of the factor \(\kappa = c/\epsilon\) on the convergence rate. Since our problem of interest is convection dominated, using the central difference scheme for the coarse grid discretization is not desirable. Indeed, the stability of central differences requires the mesh size to be of order \(\epsilon\) to avoid oscillations; and that is the behaviour we see in Figure 1 (c). As long
as Péclet number is smaller than 2 (stability constraint for central differences), the convergence factor decreases as $\kappa$ increases. For $\kappa > 200$, the central difference scheme breaks down. This is too restrictive and in such a case we do not need LDC at all, since with such a small step size we will resolve the whole problem. However, the amount of computational work increases tremendously. For this reason the central difference approximation is not suitable as a coarse grid approximation in our LDC method for convection-dominated problems.

Nevertheless, note that one of the nice properties of the LDC technique is that it converges, even for unstable numerical schemes, both on fine and coarse grids. An illustration of this fact can be found in Figure 2. The Péclet number $Pe$, calculated for the coarse grid mesh size is 5000 and for the fine grid 1000, so our central difference scheme is completely unstable (see Figure 2 (a), solid line is exact solution). However, after a number of iterations the solution becomes better and better, and the LDC method slowly converges (see Figure 2 (b)). Because the local grid

![Figure 1: Dependence of the convergence factor $M$ on the coarse grid mesh size $H$ (a), refinement factor $\sigma$ (b) and factor $\kappa = c/\epsilon$. (c)](image)

![Figure 2: Central differences both for fine and coarse grids.](image)

5 Analysis for upwind scheme on both grids

Next we consider the upwind discretization scheme for the convection term in (1) (see (19)) and we use it both for the fine and coarse grid discretization. The first part of the analysis is the same
as in Section 3 up to (18). For the upwind scheme we get the following expression for $v_i^h$

$$v_i^h(i) = -\frac{1}{1 - \tau^n} \frac{1 - \tau^i}{1 - \tau^n}, \quad i = 0, 1, \ldots, n,$$

(27)

with

$$\tau = 1 + d, \quad d = \frac{-ch}{\epsilon} = -\kappa h.$$

Proceeding as in Section 4 (see (24)-(25)), the components of

$$z := M_2g = B^H_l g - L^H_l R^H_l v^h_l$$

(28)

can be written as

$$z_i = -\frac{g\tau^{(i-1)}}{1 - \tau^n} \frac{1}{H^2} \left[ 1 + \sigma d - 2(1 + d)^\sigma - \sigma d(1 + d)^\sigma + (1 + d)^{2\sigma} \right].$$

(29)

As one can see from (29), the convergence rate depends on the coarse grid mesh size $H$, the refinement factor $\sigma$ and the factor $\kappa = c/\epsilon$. We now analyse the effect of $H$, $\sigma$ and $\kappa$ more carefully.

First we study the influence of the coarse grid mesh size $H$ on the convergence rate. We can show that for fixed $\kappa$ and $\sigma$ we have

$$z_i = \frac{1}{2} g \frac{\tau^{(i-1)}}{1 - \tau^n} \frac{\kappa^3 \sigma - 1}{\sigma} H + O(H^2).$$

(30)

Now we can analyse each term in the relation (30). The factor $\frac{1}{2} g \frac{\tau^{(i-1)}}{1 - \tau^n} \frac{\kappa^3 \sigma - 1}{\sigma}$ is just a constant. The only term left is $\frac{\tau^{(i-1)}}{1 - \tau^n}$. We analyse its behaviour for $H, h \to 0$.

Expansion gives:

$$\tau^n = (1 - \kappa h + O(h^2))^n = \left(1 - \frac{\kappa \gamma}{n} + O\left(\frac{1}{n^2}\right)\right)^n \to \exp\left(-\kappa \gamma\right) \quad (n \to \infty).$$

From (30) we can see that our convergence rate depends on the coarse grid size $H$ as $O(H)$ and does not depend on the fine grid size $h$. This type of behaviour one can observe in Figure 3 (a). Next we study influence of the refinement factor $\sigma$ on the convergence rate. If we fix $H$ and $\sigma$, we expect the convergence factor to have a minor dependence on the refinement factor. To see if this is true, we fix all the parameters except fine grid step size $h$, and therefore $\sigma$, and calculate the convergence factor $M$. As can be seen from Figure 3 (b), indeed we have only weak dependence of the convergence factor $M$ on the refinement factor. Finally, we investigate the influence of the factor $\kappa = c/\epsilon$ on the convergence rate. If we fix $H$ and $\sigma$, we expect fast convergence for $\kappa \gg 1$, because in this case the factor $\frac{\tau^{(i-1)}}{1 - \tau^n}$ in (30) tends to zero. Indeed, as one can see in Figure 3 (c), with $\epsilon$ tending to zero (that is $\kappa = c/\epsilon \to \infty$) we have a dramatic improvement of the convergence factor $M$. 

Figure 3: Dependence of the convergence factor $M$ on the coarse grid mesh size $H$ (a), refinement factor $\sigma$ (b) and factor $\kappa = c/\epsilon$. (c)
6 Analysis for upwind scheme on the coarse grid and central difference scheme on the fine grid

As it was mentioned in Section 4, there is one more possibility to use central differences, namely for the fine grid approximation. In this case we have to use a small mesh size precisely in the region where we really need it. In order to better understand the convergence properties of the combination upwind for the coarse grid and central differences for the high activity region, we perform an analysis, similar to those in Sections 4, 5. For this discretization we obtain the following expression for $z_i, i = 1, 2, \ldots, k - 1$

$$z_i = \frac{1}{H^2}(-g)^{\tau^{\sigma(i-1)}} \left[ 1 + 2\sigma d - 2\left(\frac{1 + d}{1 - d}\right)^\sigma - 2\sigma d \left(\frac{1 + d}{1 - d}\right)^\sigma + \left(\frac{1 + d}{1 - d}\right)^{2\sigma} \right], \quad (31)$$

where

$$\sigma = H/h, \quad d = -\frac{ch}{\epsilon} = -\kappa h, \quad \tau = 1 + d. \quad (31)$$

For fixed $\kappa$ and $\sigma$ we get

$$z_i = (-g)^{\tau^{\sigma(i-1)}} H + O(H^2). \quad (32)$$

As can be seen from (32), for the combination of upwind for the coarse grid and central difference for the fine grid our convergence factor $M$ depends on the coarse grid size $H$ like $O(H)$ and this is the same behaviour as for the combination upwind plus upwind. This can be seen in Figure 4 (a). Again we want to study the influence of different factors on the convergence rate. First we study the influence of the refinement factor $\sigma$ on the convergence rate. We expect to have weak dependence on the refinement factor. To see if it is true, we fix all the parameters except the fine grid step size $h$, and therefore $\sigma$, and calculate the convergence factor $M$. As can be seen from Figure 4 (b), indeed we have minor dependence of the convergence factor $M$ on the refinement factor. Finally we study the influence of the refinement factor $\kappa = c/\epsilon$ on the convergence rate. For fixed $H$ and $\sigma$, we expect fast convergence in the case $\kappa \gg 1$. Indeed, as one can see in Figure 4 (c), with $\epsilon$ tending to zero (that is $\kappa = c/\epsilon$ tends to infinity) we have a dramatic improvement of the convergence factor $M$. This can be explained by the fact that in this case the factor $\frac{\tau^{\sigma(i-1)}H^3}{1 - \tau^h}$ in (32) tends to zero.

7 Numerical results

The first goal of the numerical simulations is to check whether our LDC algorithm works for convection-diffusion equations indeed and whether the convergence in the numerical simulations is as predicted in Sections 3-6. We will also compare the observed convergence behaviour.
with the predictions of the convergence rate according to [2, Theorem 2]. This theorem allows us to calculate the convergence rate of the LDC algorithm without performing actual calculations.

We apply the theory now to (1). The source term \( f \) and the boundary conditions \( u_0 \) and \( u_1 \) are chosen such that

\[
  u(x) = \frac{1}{2} \left( \tanh(50(x - 1/8)) + 1 \right).
\]

(33)

The following parameters were chosen for computations: \( \kappa = 100 \) (convection-dominated) or \( \kappa = 0.1 \) (diffusion-dominated), \( \gamma = 0.3 \). As discretization schemes we choose central differences or upwind.

We estimate the convergence rate \( M \) by \( M_i \) defined as

\[
  M_i := \frac{\| \Delta f_i^H \|_\infty}{\| \Delta f_{i-1}^H \|_\infty}, \quad i = 1, 2, \ldots
\]

(34)

where \( \Delta f_i^H = f_i^H - f_{i-1}^H \) and \( f_i^H \) is

\[
  f_i^H(x, y) := \begin{cases} 
  f^H(x, y) + d_i^H(x, y), & (x, y) \in \Omega_i^H, \\
  f^H(x, y), & (x, y) \in \Gamma^H \cup \Omega_0^H.
  \end{cases}
\]

(35)

Of course, as one can see from (34), our estimate becomes less reliable if we have fast convergence and thus small \( M_i \), say \( M_i < 10^{-8} \). In Section 3 we have found that the convergence in the numerical simulations depends on \( H, \sigma \) and \( \kappa \). To verify the dependence found in theory, we will first study convection-dominated problems (\( \kappa = 100 \)) and then diffusion-dominated problems (\( \kappa = 0.01 \)).

As for numerical experiments, we provide several examples, which represent typical results we get.

**Example 1. Convection-dominated problems. Central difference scheme.** First we would like to study the dependence of the convergence factor \( M \) when we use central difference discretizations on both the global coarse and local fine grids. To this end we vary the coarse grid and fine grid mesh sizes \( H \) and \( h \); if we fix \( H \) and vary \( h \), we can study the influence of the refinement factor \( \sigma = H/h \). In Figure 5 we show the dependence of the convergence rate on \( h \) and hence \( \sigma \) for several values of \( H \). Figure 5 (a) shows the theoretical convergence rate given by [2, Theorem 2]; Figure 5 (b) shows the estimate \( M_i \) from (34) for \( i = 2 \). Indeed, as it was predicted, there is only a weak dependence on \( \sigma \). As one can see from Figure 5 (b), we have a flat graph of the convergence rate as \( h \) tends to zero, which means we have no dependence of the convergence rate on \( \sigma \). The varying parameter is the fine grid mesh size \( h \) (and therefore the refinement factor \( \sigma \)). The values of \( h \) are decreasing from right to left, so we can say that from right to left \( h \) tends to zero (and consequently \( \sigma \) tends to infinity). From the theory it was predicted that we have no dependence of the convergence rate \( M \) on the fine grid mesh size \( h \) (and refinement factor \( \sigma \)), so from our test we want to have a flat line. This is indeed true in Figure 5. In Figure 6 we show the dependence of the convergence rate on \( H \) for several values of \( h \). Figure 6 (a) shows the theoretical convergence rate given by [2, Theorem 2]; Figure 6 (b) shows the estimate \( M_i \) from (34) for \( i = 2 \). In Figure 6 (b) one can see the typical dependence of the convergence rate on the coarse grid size \( H \). From the theory in Section 4 we expect our convergence rate \( M \) not to depend on the fine grid mesh size \( h \) and to depend on the coarse grid mesh size as \( O(H^2) \). This is the behaviour we find in Figures 5 and 6. The numerical results confirm the analytical results from Section 4. We should remark that these results correspond to the ones in [2, Theorem 2]. The insight given by theory about the dependence of the convergence factor \( M \) on the coarse grid mesh size \( H \) is confirmed by the numerical experiment. We are able to correctly predict the convergence rates.

**Example 2. Convection-dominated problems. Upwind scheme.** In this section we use the upwind discretization on both grids and carry out the same experiments as for central differences. First we study if the convergence factor \( M \) depends on the refinement factor \( \sigma \) and on the coarse grid mesh size \( H \). From the theory presented in Section 6 we expect to have no dependence of the convergence factor on the fine grid mesh size. As for coarse grid mesh size \( H \) we expect an \( O(H) \) dependence. This dependence is indeed observed in the numerical experiments. This can
Figure 5: Dependence of the convergence rate on $h$. Central differences, $\kappa = 100$ (Example 1).

Figure 6: Dependence of the convergence rate on coarse grid size $H$. Central differences, $\kappa = 100$ (Example 1).

be seen in Figure 7 in which we show the estimate $M_i$ for the convergence rate for varying $h$ and $H$. We have not provided the convergence rate predicted by theory as these figures are almost identical (just like in Figures 5 and 6 in the previous example).

**Example 3. Diffusion-dominated problems. Central difference scheme.** For diffusion-dominated problems (here we consider $\kappa = 0.01$) the convergence factor $M$ is typically of order $10^{-8}$. Therefore the LDC iteration reaches its fixed point in one or two steps and it is hard to estimate the convergence factor reliably (see (34)). It was known from previous articles (see [4], [5], [6]) that the LDC algorithm performs well for pure diffusion problems, so it is not surprising that in the case of diffusion-dominated problems (with presence of small convection) the LDC method gives fast convergence. Some results for the diffusion dominates problems one can find in Figure 8.

**Example 4. Pure convection problem.** By means of the LDC technique we are able to solve not only convection-diffusion problems and pure diffusion problems, but also convection prob-
lems like
\[
\begin{aligned}
&\frac{\partial u}{\partial x} = f, \quad c > 0, \quad x \in \Omega = (0, 1), \\
u(0) = u_0.
\end{aligned}
\] (36)

The idea is to transform this problem into a two point boundary value problem and to solve the latter one by the LDC technique. In order to do this, we first extend our domain to the right and introduce a diffusive term \( \epsilon \frac{\partial^2 u}{\partial x^2} \) into (36) as well as an extra boundary condition on the right side. Since the correct boundary value on the right is not known, we will have a boundary layer of order \( \epsilon \). So instead of problem (35) we solve the following one
\[
\begin{aligned}
&\epsilon \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x} = f, \quad c > 0, \quad x \in \tilde{\Omega} = (0, 1 + \epsilon) \\
u(0) = u_0, \quad u(1 + \epsilon) = u_{\text{guess}}.
\end{aligned}
\] (37)

The results of the numerical tests with Péclet number \( Pe \), equal to 100000 are presented below. We used the upwind scheme on all grids. Depending on the fine grid mesh size \( h \) it is also
possible to use central differences in the high activity region, even in the case when the scheme is not stable.

References


