An extension of the Dunford-Pettis theorem with applications to a Bocce-type oscillation restriction criterion by Erik J. Balder
An extension of the Dunford-Pettis theorem with applications to a Bocce-type oscillation restriction criterion

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The Dunford-Pettis theorem identifies uniform integrability as a sufficient (and necessary) condition for relative weak compactness in $L^1_E$. The usual extension of this result to Bochner-integrable functions involves weak convergence of averages of functions. Here it is shown that another extension of the Dunford-Pettis theorem is possible, which involves norm-convergence of those averages. This brings strong relative compactness results of Castaing [11] and Jalby [20, 8] in line with the Dunford-Pettis theorem. It is also very useful in connection with Bocce-type oscillation restriction criteria, as is demonstrated by a unification of the tightness-based results in section 4 of [8].

1 Extension of the Dunford-Pettis theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $(E, \|\cdot\|)$ be a separable Banach space (the nonseparable case is discussed in Remark 1.8 below). Unless the contrary is explicitly mentioned, $E$ is equipped with the norm-topology. By separability of $E$, a function $f: \Omega \to E$ is Bochner integrable if and only if it is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(E)$ and $\int_\Omega \|f(\omega)\| \mu(d\omega) < +\infty$ [17]. The set of all such functions $f$ is denoted by $L^1(\Omega, \mathcal{A}, \mu; E)$ or simply $L^1_E$. Let $\mathcal{A}^+$ denote the collection of all nonnull sets in $\mathcal{A}$. Denote for $\mathcal{A}$ in $\mathcal{A}^+$ and $f \in L^1_E$ the mean (or average) of $f$ over $\mathcal{A}$ by $m_\mathcal{A}(f)$, i.e.,

$$m_\mathcal{A}(f) := \frac{\int_\mathcal{A} f \, d\mu}{\mu(\mathcal{A})}.$$ 

Recall the following definition of uniform integrability: a set $K$ in $L^1_E$ is uniformly integrable if

$$\lim_{\gamma \to -\infty} \sup_{f \in K} \int_{\{\|f\| \geq \gamma\}} \|f\| \, d\mu = 0.$$

For $E = \mathbb{R}$ the following result is classical:

**Theorem 1.1 (Dunford-Pettis)** Let $K \subset L^1_{\mathbb{R}}$ be uniformly integrable. Then $K$ is relatively weakly compact $^1$; i.e., for every sequence $(f_n)$ in $K$ there exist a subsequence $(f_{n'})$ and $f_* \in L^1_{\mathbb{R}}$ such that

$$\lim_{n'} m_\mathcal{A}(f_{n'}) = m_\mathcal{A}(f_*) \text{ for all } \mathcal{A} \in \mathcal{A}^+.$$

Actually, in Theorem 1.1 there is equivalence: uniform integrability is also a consequence of relative weak compactness.

Typical extensions to infinite-dimensional $E$ of Theorem 1.1 in the literature use the weak topology on $L^1_E$ for this purpose; for the averages $m_\mathcal{A}(f_{n'})$ in the above statement of the Dunford-Pettis theorem this implies their weak convergence in $E$ to the corresponding average $m_\mathcal{A}(f_*)$ of $f_*$. For

$^1$By the Eberlein-Šmulian theorem, relative weak and relative weak sequential compactness are equivalent.
instance, see [15, 16] for such extensions. One of the main result of this paper replaces such weak convergence of averages by norm convergence in $E$. This results in Theorem 1.3 below, which is clearly another way to extend Theorem 1.1. It also turns out to be quite useful in connection with studying the difference between weak and norm convergence in $L^p_E$ by means of notions based on Girardi’s Bocce criterion [19], as we show in section 2. The first result related to Theorem 1.3 that involves the norm topology for the integrals seems due to Castaing [11]; this was generalized by Jalby in [20] and [8, Lemma 4.3], where the following result was proven:

**Theorem 1.2** Let $K \subset L^p_E$ be uniformly integrable and tight. Then for every $A \in \mathcal{A}$ the set \( \{m_A(f) : f \in K\} \) in $E$ is relatively norm compact.

To understand Theorem 1.2, recall that $K$ is said to be tight [8, Definitions 4.1, 4.1'] if for each $\epsilon > 0$ there is a compact-valued multifunction $\Gamma : \Omega \to \mathcal{B}^E$, with $\mathcal{A} \times \mathcal{B}(E)$-measurable graph, such that

\[
\sup_{f \in K} \mu(\{\omega \in \Omega : f(\omega) \notin \Gamma(\omega)\}) \leq \epsilon.
\]

Observe that in Theorem 1.1 tightness of $(f_n)$ holds automatically by Markov’s inequality. In the present context, where $E$ is a Polish space, it can be shown that tightness as in the above definition is equivalent to a more classical form of tightness, which arises by taking each multifunction $\Gamma$, constant-valued in addition [more precisely, it is then equivalent to the classical tightness of the set of image measures $\{\mu^f : f \in K\}$]; see Remark 2.6 in [2] and the remarks following Theorem 14 in [22].

We now state the announced abridgement of Theorems 1.1 and 1.2; it figures as Theorem 6.9 in the author’s lecture notes [7].

**Theorem 1.3** Let $K \subset L^p_E$ be uniformly integrable and tight. Then for every sequence $(f_n)$ in $K$ there exist a subsequence $(f_{n_s})$ and $f_* \in L^p_E$ such that

\[
\lim_{n_s} \|m_A(f_{n_s}) - m_A(f_*)\| = 0 \quad \text{for every } A \in \mathcal{A}.
\]

The proof of this theorem will be given in section 3; although the theorem seems to be new, it is a quite simple consequence of the principal lower closure theorem of Young measure theory.

Theorem 1.3 is particularly suitable to be used in combination with Girardi’s Bocce criterion [19], which is a device to suppress the oscillations of functions. Such oscillations can be defined in terms of absolute – i.e., norm – deviations from local averages, which arises by taking each multifunction $\Gamma$, constant-valued in addition [more precisely, it is then equivalent to the classical tightness of the set of image measures $\{\mu^f : f \in K\}$]; see Remark 2.6 in [2] and the remarks following Theorem 14 in [22].

We now state the announced abridgement of Theorems 1.1 and 1.2; it figures as Theorem 6.9 in the author’s lecture notes [7].

**Theorem 1.4** Let $K \subset L^p_E$ be uniformly bounded. Then $K$ is relatively compact for convergence in measure if and only if $K$ is tight and $K$ satisfies (ORC).

2 By working with outer measures and/or outer integration, it turns out that such measurability is not really needed; e.g., cf. [3, 7].
Of course, it is not surprising that via (ORC) oscillations should be suppressed in order to have relative compactness in measure (consider for $E = \mathbb{R}$ the Rademacher functions). The proof of this theorem, which is given in section 3, is also based upon Young measure theory. More precisely, the proof of the necessity part combines the proofs of Theorem 1.3 and [8, Lemma 4.4], whereas the sufficiency proof is based upon an idea in [6]. Inspection of that proof will allow the following extension, which also comes from [6, 8].

**Remark 1.5** Let $K \subset L^1_B$ be uniformly bounded and let $D^*$ be a countable subset of $E^*$ that separates the points of $E$ (since $E$ is Polish, whence Suslin, such a subset always exists [12, III.32]). Then $K$ is already relatively compact for convergence in measure if $K$ is tight and the set $\{< f, x^* > : f \in K \}$ satisfies (ORC) in the sense of $L^1_B$ for every $x^* \in D^*$. In view of the necessity part of Theorem 1.4, this shows that, given uniform boundedness and tightness of $K$, the scalarized version of (ORC) is equivalent to (ORC) itself.

Another extension of the sufficiency part is as follows:

**Remark 1.6** In [14] Diaz and Mayoral combined Remark 1.5 with the following version of (ORC): a set $K$ in $L^1_B$ satisfies the *measure oscillation restriction criterion* (MORC) if for every sequence $(f_n)$ in $K$ and every $A \in \mathcal{A}^+$, $\epsilon > 0$ there exist $B \in \mathcal{A}^+$, $B \subset A$, and a subsequence $(f_{n'})$ of $(f_n)$ such that

$$\int_B ||f_{n'} - m_B(f_{n'})||/(1 + ||f_{n'} - m_B(f_{n'})||) d\mu \leq \epsilon \mu(B)$$

for every $B \in \mathcal{A}^+$, $B \subset A$, and such that

$$\liminf_n \int_B ||f_n - m_B(f_n)||/(1 + ||f_n - m_B(f_n)||) d\mu \leq \epsilon \mu(B).$$

Equivalently, this means that $K$ satisfies (MORC) if for every sequence $(f_n)$ in $K$ and every $A \in \mathcal{A}^+$, $\epsilon > 0$ there exist $B \in \mathcal{A}^+$, $B \subset A$, and a sequence $(A_{n'})$ of subsets of $B$ such that for infinitely many indices $n'$

$$\mu(B \setminus A_{n'}) < \epsilon \mu(B) \text{ and diam}[u_{n'}(A_{n'})] \leq \epsilon.$$

In the Banach space context of the present paper, it is easy to check that (LFC) implies (MORC) (for sequences) [21, Proposition 5]. Since [21, Theorem 9] states that tightness and (LFC) imply relative compactness in measure, it follows from Theorem 1.4 that, in the present Banach space context (LFC) and (ORC) are equivalent; this point was already observed in [14].

**Remark 1.7** In [21], in a context with $E$ a general Polish space, the following criterion was introduced: a sequence $(f_n)$ of measurable functions from $\Omega$ into $E$ satisfies the *local sequential Fréchet criterion* (LFC) if for every $A \in \mathcal{A}^+$, $\epsilon > 0$ and every subsequence $(f_{n'})$ of $(f_n)$ there exist $B \in \mathcal{A}^+$, $B \subset A$, and a sequence $(A_{n'})$ of subsets of $B$ such that for infinitely many indices $n'$

$$\mu(B \setminus A_{n'}) < \epsilon \mu(B) \text{ and diam}[u_{n'}(A_{n'})] \leq \epsilon.$$

In the Banach space context of the present paper, it is easy to check that (LFC) implies (MORC) (for sequences) [21, Proposition 5]. Since [21, Theorem 9] states that tightness and (LFC) imply relative compactness in measure, it follows from Theorem 1.4 that, in the present Banach space context (LFC) and (ORC) are equivalent for sequences in $L^1_B$ that are both tight and uniformly bounded.

Finally, a more global option is to consider a nonseparable Banach space $E$ in this paper:

**Remark 1.8** If the Banach space $E$ is nonseparable, both Theorem 1.3 and 1.4 hold by well-known manipulations with the Pettis measurability theorem [17]: Given any sequence $(f_n)$ in the space $L^1_B$ of Bochner-integrable functions, there exist a null set $N$ and a separable Banach subspace $\tilde{E}$ of $E$ such that each $f_n := f_n|_{\Omega \setminus N}$ belongs to the space $L^1(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu}; \tilde{E})$, where $\tilde{\Omega}$ denotes $\Omega \setminus N$ and $\tilde{\mathcal{A}}$ and $\tilde{\mu}$ are the obvious traces of $\mathcal{A}$ and $\mu$. To derive Theorem 1.3 in the new setting from its original counterpart, let $(f_n)$ correspond to $(f_n)$ as above, and observe that $(f_n)$ is tight: simply use $\Gamma_r := \Gamma_r \cap \tilde{E}|_{\tilde{\Omega}}$ (alternatively, one could use [8, Lemma 4.4], as is done in [14]). So Theorem 1.3 applies to $(f_n)$, after which the desired extension to $(f_n)$ follows easily. For the extension of Theorem 1.4 (which can entirely be phrased in terms of sequences - see its proof) a similar argument can be given; its details are left to the reader.
2 Applications to Bocce criterion results

In this section we show that most of the results in section 4 of [8], including those involved with limited convergence, are brought together by Theorems 1.4 and 1.3. The following corollary is Theorem 4.5 in [8].

Corollary 2.1 Let $(f_n)$ be a sequence in $\mathcal{L}_b^1$. Then $(f_n)$ converges strongly to $f_0$ in $\mathcal{L}_b^1$ if and only if

1. $(f_n)$ converges weakly to $f_0$,
2. $\{f_n : n \in \mathbb{N}\}$ satisfies (ORC),
3. $\{f_n : n \in \mathbb{N}\}$ is tight.

**Proof.** The proof of necessity of the three conditions is straightforward; it can be found in [8] and need not be repeated here. To prove sufficiency, observe that (1) guarantees uniform integrability of $(f_n)$. In view of the Lebesgue-Vitali theorem, it is therefore enough to prove that an arbitrary subsequence $(f_{n''})$ of $(f_n)$ contains a further subsequence $(f_{n''})$ that converges in measure to $f_0$. In view of (1) we have $f_{n''} = f_0$ a.e. (apply [1, 2.5.3]). An application of the Lebesgue-Vitali theorem then gives strong convergence of $(f_{n''})$ to $f_0$. Q.E.D.

**Remark 2.2** Observe that in the above proof (1) can be replaced by the combination (1')-(1''), with

1. $(||f_n||)$ is uniformly integrable,
2. $\lim_n \int_A f_n < f_0, x^* > = \int_A f < f_0, x^* >$ for every $A \in \mathcal{A}, x^* \in E^*$.

Our next corollary involves the main result. Theorem 2.3 of [14], which in turn generalizes [8, Theorem 3.8 (revisited)].

Corollary 2.3 Let $K$ be a subset of $\mathcal{L}_b^1$. Then $K$ is relatively strongly compact if and only if

1. $K$ is uniformly integrable,
2. for every $x^* \in E^*$ the set $\{< f, x^* > : f \in K\} \subset \mathcal{L}_b^1$ satisfies (ORC),
3. $K$ is tight.

**Proof.** Again, the proof of necessity is simple and can be skipped; cf. [8]. To prove sufficiency, let $(f_n)$ be an arbitrary sequence in $K$. Observe already that, by (1), $(||f_n||)$ is uniformly integrable in $\mathcal{L}_b^1$. By (1)-(2), Theorem 1.3 applies: there exist a subsequence $(f_{n''})$ of $(f_n)$ and $f_* \in \mathcal{L}_b^1$ such that $\lim_{n''} ||m_A(f_{n''}) - m_A(f_*)|| = 0$. Certainly, then, the above (1')-(1'') are fulfilled, with $f_0 := f_*$. Hence, by Corollary 2.1 (which can obviously be modified in the sense of Remark 1.5) and Remark 2.2, the subsequence $(f_{n''})$ converges strongly to $f_*$. Q.E.D.

Recall from [4, 8] that a sequence $(f_n)$ in $\mathcal{L}_b^1$ is said to converge *limitedly* to $f_0$ in $\mathcal{L}_b^1$ if

$$\lim_n \int_\Omega g(\omega, f_n(\omega) - f_0(\omega))d\mu(\omega) = 0 \text{ for all } g \in \mathcal{G}.$$ 

Here $\mathcal{G}$ is the set of all functions $g : \Omega \times E \rightarrow \mathbb{R}$ for which

1. $g(\omega, 0) = 0$ for every $\omega \in \Omega$,
2. $g(\omega, \cdot)$ is weakly continuous on $E$,
3. there exist $\phi \in \mathcal{L}_b^1$ and $C > 0$ such that
4. $|g(\omega, x)| \leq C||x|| + \phi(\omega)$ for all $\omega \in \Omega, x \in E$.

(iv) $g(\cdot, f(\cdot))$ is $\mathcal{A}$-measurable for each $f \in \mathcal{L}_b^1$.

The following result, which goes back to V. Jalby [20], is Theorem 4.6 in [8].
Corollary 2.4 Let \((f_n)\) be a sequence in \(L^1_E\). Then \((f_n)\) converges strongly to \(f_0\) in \(L^1_E\) if and only if

1. \((f_n)\) converges limitedly to \(f_0\),
2. \(\{f_n : n \in \mathbb{N}\}\) is tight.

**Proof.** For the non-obvious implication (we refer to [8] for the other one) it is, in view of the Lebesgue-Vitali theorem, enough to prove that an arbitrary subsequence of \((f_n)\) contains a further subsequence which converges to \(f_0\) in measure (recall that limited convergence implies weak convergence). In view of condition (2) and Remark 1.5, it is thus actually enough to prove that for an arbitrary \(z \in D^*\) the sequence \(\langle f_{n,z} \rangle\) meets (ORC).

3. **Proofs of the main results**

In this section we present proofs of Theorems 1.3 and 1.4 by means of some simple Young measure theory. Actually, all we need about Young measures is the following lower closure result of Fatou-Vitali-Prohorov-type [2], where \(\mathbb{N}\) denotes the Alexandrov compactification of the set \(\mathbb{N}\) of natural numbers and \(L^1_E\) the set of all measurable functions from \(\Omega \rightarrow \mathbb{E}\).

**Theorem 3.1** Let \(K \subset L^1_E\) be tight. Then to every sequence \((f_n)\) in \(K\) there correspond a subsequence \((f_{n'})\) and a transition probability \(\delta : \Omega \rightarrow \text{Prob}(E)\) such that

\[
\liminf_{n'} \int_E \ell(\omega, n', f_{n'}(\omega)) \mu(\omega) \geq \int \int_E \ell(\omega, \infty, x) \delta_*(\omega)(dx) \mu(d\omega)
\]

for every \(A \times B(\mathbb{N}) \times B(E)\)-measurable function \(\ell : \Omega \times \mathbb{N} \times E \rightarrow (-\infty, +\infty)\) such that \(\ell(\cdot, \cdot, \cdot)\) is lower semicontinuous on \(E\) for every \(\omega \in \Omega\), and such that the sequence of functions \(\min(\ell(\cdot, n', f_{n'}(\cdot)), 0), n \in \mathbb{N}\), is uniformly integrable. Also, \(\delta_*\) has the following support property

\[
\delta_*(\omega)(Ls f_{n'}(\omega)) = 1 \text{ for a.e. } \omega,
\]

where \(Ls f_{n'}(\omega)\) stands for the (pointwise) Kuratowski limes superior, i.e., \(Ls f_{n'}(\omega)\) is the set of all the limit points of the sequence \((f_{n'}(\omega))\).

Recall here that a transition probability \(\delta\) is a function from \(\Omega\) into the set \(\text{Prob}(E)\) of all probability measures on \((E, B(E))\) such that \(\delta(\cdot)(B)\) is measurable for every \(B \in B(E)\). The above result is entirely contained in the statement (p. 573) and proof (pp. 592-594) of [2, Theorem 1]; see also [3, Theorem 2.2] (note that \(E\), which is a Polish space, is certainly metrizable Lusin). It also follows by combining Theorems 14 and 17 of [22] – see also comments 3), 4) following Theorem 14 in [22]. We refer the interested reader to [5] and [7] for more advanced versions of the result.
Remark 3.2 If in Theorem 3.1 \( K \) is a subset of \( L^1_E \) that is in addition bounded (in \( L^1 \)-seminorm), then the transition probability \( \delta_* \) has the following extra property

\[
\int_{\Omega} \left[ \int_E ||\delta_*(\omega)(dx)|| \mu(d\omega) \right] < +\infty,
\]

which follows immediately from applying (3.1) to \( \ell(\omega, k, x) := ||x|| \). Hence,

\[
f_\ast(\omega) := \int_E x \delta_*(\omega)(dx),
\]

i.e., the barycenter of the probability measure \( \delta_*(\omega) \), is well-defined for \( \omega \) outside a certain null set. By setting \( f_\ast(\omega) := 0 \) on the null set itself, we obtain \( f_\ast \in L^1_E \).

**Proof of Theorem 1.3.** Let \( (f_n) \) be as stated. By Theorem 3.1 there exist subsequence \( (f'_{n'}) \) and a transition probability \( \delta_* \) satisfying (3.1)-(3.2). Let \( f_\ast \in L^1_E \) be the barycentric function associated to \( \delta_* \), as discussed in Remark 3.2. Let \( A \in \mathcal{A}^+ \) be arbitrary. We claim that

\[
\lim_{n \to \infty} (f_n - f_\ast) d\mu = 0.
\]

To obtain \( \alpha \leq 0 \), note that

\[
\alpha := \limsup_{n'} ||f_{n'} - f_\ast||.
\]

By the Hahn-Banach theorem, for each \( n' \) there exists \( x_{n'} \in U^\ast \) in the closed unit ball \( U^\ast \) in \( E^\ast \) such that

\[
||f_{n'}(\omega) - f_\ast(\omega)|| = ||f_{n'} - f_\ast||. \quad \text{(3.3)}
\]

Without loss of generality we suppose that \( (x_{n'}) \) converges to some \( x^\ast \in U^\ast \) in the weak star topology (Alaoglu-Bourbaki theorem). Renumber \( (n'' \) as \( (k) \) and define \( \ell : \Omega \times N \times E \to \mathbf{R} \) by \( \ell(\omega, k, x) := 1_A(\omega) < f_\ast(\omega) - x, x^\ast > \). Then \( \ell \) is as required in Theorem 3.1 (in particular, \( \ell(\omega, \cdot, \cdot) \) is even continuous on \( N \times E \)). By (3.1) we find

\[
-\alpha \geq \int_A \left[ \int_E < f_\ast(\omega) - x, x^\ast > \delta_*(\omega)(dx) \right] \mu(d\omega) = \int_A (f_\ast - x^\ast > - f_\ast : x^\ast >) d\mu = 0,
\]

and this proves the claim. Q.E.D.

**Proof of Theorem 1.4** (necessity). First, we prove tightness of \( K \). Since \( K \) is supposed to be relatively compact for convergence in measure, it is also relatively sequentially compact for such convergence (recall that the convergence in measure topology is metrizable). Therefore, elementary considerations (see the proof of Lemma 4.4 in [8]) give that any sequence \( (f_n) \) in \( K \) contains a subsequence \( (f_{n'}) \) such that

\[
\lim_{n'} \sup ||f_{n'} - f_\ast|| = 0.
\]

Let \( (f_{n'}) \) and \( f_\ast \) correspond to \( (f_n) \) just as in Theorem 1.3 and Remark 3.2. By (3.2) and (3.3) the following is obviously true:

\[
\delta_*(\omega) \in \text{Prob}(E) \text{ is carried by the singleton } \{f_\ast(\omega)\} \text{ for a.e. } \omega. \quad \text{(3.4)}
\]
From the proof of Theorem 1.3 we already know that
\[ \| m_C(f_{n'}) - m_C(f_\infty) \| \to 0 \text{ for every } C \in A^+, \quad (3.5) \]
where \( f_\infty \) stands for \( f_* \). Since \( \sup_{n'} \int_A \| f_{n'} \| < +\infty \), an application of the biting lemma [18, Lemma C] (see [8, p. 255] for further references) gives the existence of a sequence of "bites" \((B_j)\) in \( A, \mu(B_j) \to 0 \), such that for each \( j \)

\[ (\| f_{n'} \|) \text{ is uniformly integrable over } \Omega \setminus B_j. \]

Fix \( p \) such that \( \mu(A \setminus B_j) > 0 \). By Lemma 2.5 in [8] (or Egorov's theorem) there is a subset \( B \) of \( A \setminus B_j \) such that

\[ \int_B \| f_* - m_B(f_*) \| d\mu \leq \epsilon \mu(B). \]

We can now apply (3.1) to both \( \ell : (\omega, n', x) \mapsto 1_B(\omega)\|x - m_B(f_{n'})\| \) and \( \ell : (\omega, n', x) \mapsto -1_B(\omega)\|x - m_B(f_{n'})\| \) (observe that in both cases \( \ell(\omega, \cdot, \cdot) \) is continuous on \( \mathbb{N} \times E \) for every \( \omega \in \Omega \), thanks to (3.5)). This results in the identity

\[ \lim_{n'} \int_B \| f_{n'}(\omega) - m_B(f_{n'}) \| d\mu(\omega) = \int_B \int_E \| x - m_B(f_*) \| \delta_*(\omega)(dx) d\mu(\omega) = \int_B \| f_*(\omega) - m_B(f_*) \| d\mu(\omega), \]

thanks to the degenerate structure of \( \delta_* \) exhibited in (3.4). Combining the previous two inequalities, we get

\[ \liminf_{n'} \int_B \| f_0 - m_B(f_0) \| d\mu \leq \lim_{n'} \int_B \| f_{n'} - m_B(f_{n'}) \| d\mu \leq \epsilon \mu(B), \]

which demonstrates (ORC). Q.E.D.

**Proof of Theorem 1.4 (Sufficiency).** Let \( (f_n) \) be an arbitrary sequence in \( K \). We apply Theorem 3.1 to it, and find \((f_0', \delta, f_*) \in \mathcal{L}_E^1 \), just as in Theorem 3.1 and Remark 3.2. In the proof of Theorem 1.3 this was already shown to give

\[ \lim_{n' \to -\infty} \| m_B(f_{n'}) - m_B(f_*) \| = 0 \quad (3.6) \]

for every set \( B \) in \( A \) over which \( (f_{n'}) \) is (locally) uniformly integrable. Adopting an idea from [6], we claim that for a.e. \( \omega \) in \( \Omega \) the first order moment

\[ \phi(\omega) := \int_E \| x - f_*(\omega) \| \delta_*(\omega)(dx) \]

of the probability measure \( \delta_*(\omega) \) is zero. By the elementary Lemma 2.6 of [8] this claim holds if we can show, for every \( A \in \mathcal{A}^+ \) and \( \epsilon > 0 \), that there exists \( B \in \mathcal{A}^+, B \subset A \), such that

\[ \int_B \phi \, d\mu \leq \epsilon \mu(B). \quad (3.7) \]

Suppose momentarily that our claim has been proven. Then it follows that for a.e. \( \omega \) the probability measure \( \delta_*(\omega) \) is carried by the singleton \( \{ f_*(\omega) \} \) (i.e., the transition probability \( \delta_* \) is degenerate). Hence, for any \( \eta > 0 \) the inequality (3.1) gives \( \lim_{n'} \mu(\{ \| f_{n'} - f_* \| \geq \eta \}) = 0 \), when it is applied to \( \ell : \Omega \times \mathbb{N} \times E \to (-\infty, +\infty) \), given by

\[ \ell(\omega, k, x) := \ell(\omega, x) := \begin{cases} -1 & \text{if } \| x - f_*(\omega) \| \geq \eta \\ 0 & \text{otherwise} \end{cases} \]

Hence, we will then have proven the convergence in measure of \( (f_{n'}) \) to \( f_* \). So it remains to prove (3.7). Let \( A \in \mathcal{A}^+ \) and \( \epsilon > 0 \) be arbitrary. Let \( A' \subset A \) be such that

\[ \int_B \| f_*(\omega) - m_B(f_*) \| d\mu(\omega) \leq \frac{\epsilon \mu(B)}{2} \quad (3.8) \]
for all $B \in A^+$ with $B \subset A'$. By [8, Lemma 2.5] (or Egorov's theorem) such $A'$ exists in $A^+$. As in the proof above of the necessity part, we can locally obtain uniform integrability by the biting lemma: there exists a sequence $(B_j)$ in $A$ such that $\mu(B_j) \to 0$ and for each $j$

\[
(\|f_{n'}\|) \text{ is uniformly integrable over } \Omega \setminus B_j. 
\]

Fix $j$ such that $\mu(A' \setminus B_j) > 0$. By (ORC), there exists $B \in A^+$, $B \subset A' \setminus B_j$, such that

\[
\liminf_{n'} \int_B \|f_{n'} - m_B(f_*)\| = \liminf_{n'} \int_B \|f_{n'} - m_B(f_{n'})\| d\mu \leq \frac{\epsilon \mu(B)}{2},
\]

where we use (3.6). At the same time, we have also

\[
\liminf_{n'} \int_B \|f_{n'} - m_B(f_*)\| d\mu \geq \int_B \left[ \int_E \|x - m_B(f_*)\| \delta_*(\omega)(d\omega) d\mu(d\omega) \right]
\]

by applying Theorem 3.1 to $\ell : (\omega, n', x) \mapsto 1_B(\omega)\|x - m_B(f_*)\|$. Hence, we obtain

\[
\int_B \left[ \int_E \|x - m_B(f_*)\| \delta_*(\omega)(d\omega) d\mu(d\omega) \right] \leq \frac{\epsilon \mu(B)}{2}, \tag{3.9}
\]

which, when combined with (3.8), leads to (3.7). This is all that remained to be done. Q.E.D.

**Remark 3.3** Instead of working with the norm on $E$, the above proof can be repeated for the countably many seminorms $|\cdot, x^*|_x$ on $E$, $x^* \in D^*$; cf. Remark 1.5. This gives that

\[
\int_E \left| x - f_*(\omega) , x^* > | \delta_*(\omega)(dx) \right| = 0 \text{ a.e. for each } x^* \in D^*.
\]

The conclusion is that for a.e. $\omega$ the probability measure $\delta_*(\omega)$ is carried by the intersection of all sets $\{x \in E : < x, x^* > = < f_*(\omega), x^* >\}$, $x^* \in D^*$, i.e., by the singleton $\{f_*(\omega)\}$. This validates Remark 1.5. Observe, moreover, that the proof of Theorem 1.4 also goes through entirely if (ORC) is replaced by (MORC): indeed, (3.8) continues to hold as stated, which certainly implies that

\[
\int_B \frac{||f_* - m_B(f_*)||}{1 + ||f_* - m_B(f_*)||} d\mu \leq \frac{\epsilon \mu(B)}{2},
\]

and (3.9) now becomes

\[
\int_B \left[ \int_E \frac{||x - m_B(f_*)||}{1 + ||x - m_B(f_*)||} \delta_*(\omega)(dx) d\mu(d\omega) \right] \leq \frac{\epsilon \mu(B)}{2}.
\]

Since $(x, y) \mapsto ||x - y||/(1 + ||x - y||)$ is a metric on $E$, we still obtain (3.7), since

\[
\int_B \phi_* d\mu \leq \int_B \left[ \int_E \frac{||x - f_*(\omega)||}{1 + ||x - f_*(\omega)||} \delta_*(\omega)(dx) d\mu(d\omega) \right] \leq \epsilon \mu(B)
\]

by the triangle inequality. So also Remark 1.6 has been justified.

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**References**


