EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

Memorandum COSOR 94-24

**Infinite divisibility
and the waiting-time paradox**

K. van Harn
F.W. Steutel

Eindhoven, August 1994
The Netherlands

INFINITE DIVISIBILITY AND THE WAITING-TIME PARADOX

K. VAN HARN AND F.W. STEUTEL

Abstract. It is shown that the interarrival time Z covering the point zero in a stationary renewal process generated by X has the form $Z \stackrel{d}{=} X + Y$ with Y nonnegative and independent of X , if and only if X is infinitely divisible. In the special case that X has a compound-exponential distribution there is a similar decomposition of the stationary waiting time. These results shed some new light on the waiting-time paradox.

1. Introduction and summary

The *waiting-time paradox* consists of the fact that a passenger arriving at a bus stop will probably have to wait considerably longer than about half the interarrival time, X say, of two buses. The paradox is resolved by the observation that a passenger is more likely to arrive in a long interval than in a short one. Alternatively, the length Z of the interval covering the arrival time of the passenger tends to be longer than X , i.e., we have

$$(1.1) \quad Z \geq X$$

in distribution. On an appropriate sample space (1.1) can be written as

$$(1.2) \quad Z \stackrel{d}{=} X + Y,$$

where Y is nonnegative and, in general, X and Y are dependent.

In this note we consider the case where X and Y in (1.2) are *independent*. It turns out that this happens if and only if X is *infinitely divisible*, and this result sheds some new light on the waiting-time paradox and on the behaviour of nonnegative processes with stationary, independent increments [sii-processes]. In the special case that X is *compound-exponential* [see Definition 2.5], also the (stationary) waiting-time W of a bus passenger admits of a decomposition similar to (1.2):

$$(1.3) \quad W \stackrel{d}{=} X + A,$$

where A is nonnegative and independent of X .

In Section 2 we collect some results about renewal theory and infinite divisibility. Section 3 contains a characterization of infinite divisibility for nonnegative random variables by means of (1.2), and some of its consequences. In Section 4 equation (1.3) is considered for non-lattice distributions. In Section 5 we briefly present the solutions of (1.2) and (1.3) for distributions on the non-negative integers. We shall mostly use the renewal (life-time) terminology rather than the waiting-time terminology.

2. Preliminaries

We need some information on renewal theory and on infinite divisibility for non-negative random variables. Some of the results hold only for non-lattice distributions, and have analogues for lattice distributions. We shall only consider lattice random variables with values in $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Sometimes it will be essential that these random variables have positive probability at 0.

2.1. Renewal theory

We consider a renewal process generated by a sequence $(X_n)_{n \in \mathbb{N}}$ of independent, non-negative random variables distributed as X with distribution function F and expectation $\mathbb{E}X = \mu \in (0, \infty)$; these conditions will be assumed to hold throughout unless otherwise stated. We write $S_n = \sum_{k=1}^n X_k$, and define the number N_t of renewals in $(0, t]$, the age V_t of the unit in service at time t , the remaining life-time W_t of this unit (the waiting time for the next bus), and the total life-time Z_t as follows:

$$\begin{aligned} N_t &:= \#(\{n \in \mathbb{N} : S_n \leq t\}), \\ V_t &:= t - S_{N_t}, \quad W_t := S_{N_t+1} - t, \\ Z_t &= X_{N_t+1} = V_t + W_t. \end{aligned}$$

The following result is well known; see Feller (1971).

Lemma 2.1. *If F is non-lattice, then*

$$V_t \xrightarrow{d} V, \quad W_t \xrightarrow{d} W \stackrel{d}{=} V, \quad Z_t \xrightarrow{d} Z \quad [t \rightarrow \infty],$$

where Z and W have distribution functions given by

$$(2.1) \quad F_Z(z) = \frac{1}{\mu} \int_{(0,z]} x dF(x) \quad [z \geq 0],$$

$$(2.2) \quad F_W(w) = \frac{1}{\mu} \int_0^w \{1 - F(x)\} dx \quad [w \geq 0].$$

Alternatively, V , W and Z can be regarded as quantities in a stationary renewal process (started at $-\infty$); then Z is the life time straddling 0, say, and V and W are the lengths of the parts into which Z is divided by 0. The random variables V , W and Z will be used in this sense, sometimes without comment. They satisfy

$$(2.3) \quad Z = V + W \quad \text{and} \quad V \stackrel{d}{=} W \stackrel{d}{=} U Z,$$

where U is uniformly distributed on $(0, 1)$ and independent of Z ; see Winter (1989).

There is an analogue to Lemma 2.1 for lattice distributions. Now X is distributed on $\mathbb{Z}_+$, and the quantities N_n, V_n, W_n and Z_n are defined for $n \in \mathbb{N}$ in the same way as N_t, V_t, W_t and Z_t . We then have the following well-known result; see Feller (1968). We use the notations V, W and Z also for the limits of V_n, W_n and Z_n .

Lemma 2.2. *If X has an aperiodic distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$, then*

$$V_n \xrightarrow{d} V, \quad W_n \xrightarrow{d} W \stackrel{d}{=} V + 1, \quad Z_n \xrightarrow{d} Z \quad [n \rightarrow \infty],$$

where Z and W have distributions given by

$$(2.4) \quad \mathbb{P}(Z = k) = \frac{1}{\mu} k p_k \quad [k \in \mathbb{N}],$$

$$(2.5) \quad \mathbb{P}(W = k) = \frac{1}{\mu} \sum_{j=k}^{\infty} p_j \quad [k \in \mathbb{N}].$$

Remark. Comparison with (2.4) learns that (2.1) also holds for distributions on $\mathbb{Z}_+$. Since the span of the lattice is of no real importance, (2.1) holds for all distributions on $\mathbb{R}_+$ with a finite moment. Since W in (2.2) is absolutely continuous, the non-lattice condition there is essential. $\square$

2.2. Infinite divisibility

We recall the definition of infinite divisibility: a random variable X (or its distribution function F) is said to be *infinitely divisible* if for any $n \in \mathbb{N}$ iid random variables $X_1^{(n)}, \dots, X_n^{(n)}$ exist such that

$$(2.6) \quad X \stackrel{d}{=} X_1^{(n)} + \dots + X_n^{(n)}.$$

In what follows the Laplace-Stieltjes transform [LSt] of a function H will be denoted by $\widehat{H}$. We take the following result from Feller (1971).

Lemma 2.3. *A distribution function F on $\mathbb{R}_+$ is infinitely divisible iff its LSt satisfies*

$$(2.7) \quad \log \widehat{F}(s) = \int_0^{\infty} (e^{-sx} - 1) x^{-1} dK(x),$$

where K is a nondecreasing function on $\mathbb{R}_+$ with (necessarily) $\int_1^{\infty} x^{-1} dK(x) < \infty$.

K will be called the *canonical function* of F , or of the corresponding random variable. We note that the random variables $X^{(n)}$ in (2.6) are infinitely divisible with distribution function $F^{*(1/n)}$, the n -th convolution root of F , and with canonical function $K^{(n)} = (1/n)K$. Differentiation of (2.7) yields

$$(2.8) \quad -\widehat{F}'(s)/\widehat{F}(s) = \widehat{K}(s),$$

and inversion of this leads to the following equivalent relation [see Steutel (1970)]; let $s \downarrow 0$ for the final statement.

Corollary 2.4. *A distribution function F on $\mathbb{R}_+$ is infinitely divisible iff F satisfies*

$$(2.9) \quad \int_{(0,x]} y \, dF(y) = (F * K)(x) \quad [x > 0],$$

where K is nondecreasing on $\mathbb{R}_+$ and $*$ denotes convolution. The, possibly infinite, first moment μ of F is given by

$$(2.10) \quad \mu = \int_{[0,\infty)} dK(x) = K(\infty).$$

For Section 4 we need the following subclass of infinitely divisible distributions.

Definition 2.5. *Let $S(\cdot)$ be a process with stationary, independent increments and let T be exponentially distributed with $\mathbb{E}T = 1$, independent of $S(\cdot)$. Then the random variable $X := S(T)$ is said to have a compound-exponential distribution.*

The following lemma is immediate if we denote the (infinitely divisible) distribution function of $S(1)$ by F_0 .

Lemma 2.6. *A distribution function F on $\mathbb{R}_+$ is compound-exponential iff $\widehat{F}$ has the form*

$$(2.11) \quad \widehat{F}(s) = \frac{1}{1 - \log \widehat{F}_0(s)},$$

where F_0 is an infinitely divisible distribution function on $\mathbb{R}_+$. The distribution functions F and F_0 have the same first moment.

We conclude this section with analogues (and also special cases) of Corollary 2.4 and Lemma 2.6 for infinitely divisible distributions on $\mathbb{Z}_+$. Here we use probability generating functions [pgf's] rather than LSt's. We stress that the factors $X_j^{(n)}$ in (2.3) are $\mathbb{Z}_+$ -valued iff $P(X = 0) > 0$; for details we refer to Steutel (1970).

Lemma 2.7. *A distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is infinitely divisible iff the quantities r_k with $k \in \mathbb{Z}_+$ defined by*

$$(2.12) \quad (k+1)p_{k+1} = \sum_{j=0}^k p_j r_{k-j} \quad [k \in \mathbb{Z}_+],$$

are nonnegative, i.e., iff the coefficients of $R(z) := P'(z)/P(z)$ are nonnegative.

Lemma 2.8. *A distribution on $\mathbb{Z}_+$ with pgf P is compound-exponential iff P has the form*

$$(2.13) \quad P(z) = \frac{1}{1 - \log Q(z)} = \frac{1}{1 - \alpha(G(z) - 1)},$$

where Q is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ with $Q'/Q = \alpha G'$, $\alpha > 0$ and G a pgf.

3. An extreme case of the waiting-time paradox

We return to the renewal process generated by a nonnegative random variable X as described in Section 2.1. It is well known [see e.g. Ross (1970)] that for the total life-time Z with distribution function given by (2.1) we have

$$Z \stackrel{d}{\geq} X,$$

i.e., $\mathbb{P}(Z > x) \geq \mathbb{P}(X > x)$ for $x \in \mathbb{R}_+$, or on a suitable sample space,

$$(3.1) \quad Z \stackrel{d}{=} X + Y,$$

where Y is nonnegative and, in general, not independent of X . Here we are interested in the situation where X and Y are *independent*. The following theorem can also be read as a characterization theorem for infinitely divisible random variables on $\mathbb{R}_+$ with a finite first moment [compare Remark following Lemma 2.2].

Theorem 3.1. *Let Z be the life time covering the point 0 in a stationary renewal process generated by X . Then Z can be written as in (3.1) with X and Y independent iff X is infinitely divisible. The distribution function of Y is given by $F_Y = (1/\mu)K$, where K is the canonical function of X .*

PROOF. Let Z satisfy (3.1) with X and Y independent. Then by (2.1) we have

$$(3.2) \quad F_Z(z) = \frac{1}{\mu} \int_{(0,z]} x \, dF(x) = (F * F_Y)(z) \quad [z \geq 0].$$

Multiplying by μ we see that F satisfies the functional equation (2.9) with $K = \mu F_Y$, hence by Corollary 2.4 F is infinitely divisible. Conversely, if F is infinitely divisible, then by (2.10) the functional equation (2.9) can be written in the form (3.2) for some distribution function F_Y , which means that in (3.1) X and Y can be taken independent. $\square$

Remark. By a suitable choice of K , for $F_Y = (1/\mu)K$ any distribution function is possible. If $F(0) > 0$, then X is compound-Poisson [cf. van Harn (1978)], i.e.,

$$X \stackrel{d}{=} \tilde{X}_1 + \cdots + \tilde{X}_N,$$

where $\tilde{X}_1, \tilde{X}_2, \dots$ are independent and distributed as $\tilde{X}$ with distribution function G , say, and N is Poisson distributed, independent of the $\tilde{X}_j$'s. Now F_Y takes the form

$$F_Y(y) = \frac{1}{\mathbb{E}\tilde{X}} \int_{(0,y]} u \, dG(u),$$

i.e., Y has the same distribution as the total life-time $\tilde{Z}$ in the renewal process generated by $\tilde{X}$. This can, roughly, be explained as follows. Since X is a Poisson sum

of $\tilde{X}_j$'s, the (positive) X -interval covering a fixed point can be regarded as the sum of the $\tilde{X}$ -interval covering this point and a (Poisson) number of independent $\tilde{X}_j$'s, i.e., as $X + \tilde{Z} \stackrel{d}{=} X + Y$. $\square$

The question remaining is: what is the meaning of the random variable Y in (3.2). An answer can be obtained by considering the renewal process generated by $X^{(n)}$ with distribution function $F^{*(1/n)}$. Denoting the corresponding Z -random variable by $Z^{(n)}$, and using the fact that $X^{(n)}$ is infinitely divisible with canonical function $K^{(n)} = (1/n)K$, by Theorem 3.1 we have

$$Z^{(n)} \stackrel{d}{=} X^{(n)} + Y^{(n)},$$

where the distribution function of $Y^{(n)}$ is given by

$$F_{Y^{(n)}} = \frac{1}{\mathbb{E}X^{(n)}} K^{(n)} = \frac{n}{\mu} K^{(n)} = \frac{1}{\mu} K = F_Y,$$

independent of n . More generally, considering the sii-process $X(\cdot)$ with $X(1) \stackrel{d}{=} X$ and $X(1/n) \stackrel{d}{=} X^{(n)}$, we obtain the following result.

Theorem 3.2. *Let X be a nonnegative infinitely divisible random variable with finite first moment μ and with canonical function K . Let $X(\cdot)$ denote the sii-process with $X(1) \stackrel{d}{=} X$, and for $t > 0$ let $Z(t)$ be the Z -random variable in a stationary renewal process generated by $X(t)$. Then*

$$(3.3) \quad Z(t) \stackrel{d}{=} X(t) + Y \quad [t > 0],$$

where $X(t)$ and Y are independent and $F_Y = (1/\mu)K$, independent of t .

Equation (3.3) tells us that, if we approach the point 0 with small steps, i.e., if t is small, then this point is passed with a non-small step of length at least Y , in distribution. Also, though $X(t) \xrightarrow{d} 0$, we have $Z(t) \xrightarrow{d} Y \neq 0$ as $t \downarrow 0$: we get an extreme case of the waiting-time paradox. For large t , on the other hand, the term $X(t)$ will dominate in the right-hand side of (3.3): the paradox, almost, disappears. Combining (2.3) with (3.3) yields the following result for the remaining life-time W .

Corollary 3.3. *Let $X(\cdot)$ be as in Theorem 3.2, and for $t > 0$ let $W(t)$ be the W -random variable in a stationary renewal process generated by $X(t)$. Then*

$$(3.4) \quad W(t) \stackrel{d}{=} UX(t) + UY \quad [t > 0],$$

where U , $X(t)$ and Y are independent, U is uniform on $(0,1)$, and Y is as in Theorem 3.2. In particular,

$$(3.5) \quad W(t) \xrightarrow{d} UY \quad [t \downarrow 0].$$

4. Compound-exponential life times

Since the compound-exponential distributions [cf. Definition 2.5] are a special subclass of the infinitely divisible distributions, one may expect the random variable Y in (3.1) to have special properties, if X is taken compound-exponential. Indeed we have the following result.

Theorem 4.1. *If X is a nonnegative, compound-exponential random variable with a finite first moment, then the life time Z covering 0 in a stationary renewal process generated by X satisfies*

$$(4.1) \quad Z \stackrel{d}{=} X + X' + Y_0,$$

where $X' \stackrel{d}{=} X$, Y_0 is nonnegative, and X , X' and Y_0 are independent. Equivalently, the random variable Y in (3.1) satisfies $Y \stackrel{d}{=} X' + Y_0$ with X' and Y_0 independent and independent of X .

PROOF. By Lemma 2.6 the LSt $\hat{F}$ of X has the form $\hat{F}(s) = 1/(1 - \log \hat{F}_0(s))$, where F_0 is an infinitely divisible distribution function on $\mathbb{R}_+$. From this equation and (2.8) it follows that the canonical function K of F is related to the canonical function K_0 of F_0 by $K = F * K_0$:

$$\hat{K}(s) = -\frac{d}{ds} \log \hat{F}(s) = \hat{F}(s) \left\{ -\frac{d}{ds} \log \hat{F}_0(s) \right\} = \hat{F}(s) \hat{K}_0(s).$$

Dividing by μ , the first moment of both F and F_0 , and letting Y_0 be a random variable with distribution function $(1/\mu) K_0$, we see that Y in (3.1) satisfies

$$F_Y = F * F_{Y_0},$$

so Y can be obtained as $Y \stackrel{d}{=} X' + Y_0$ with X' and Y_0 independent. $\square$

Remark. As in Theorem 3.2 one can consider the renewal process generated by $X(t)$. When $X = X(1)$ is compound-exponential, then so is $X(t)$ for $0 < t < 1$. Repeating the calculations for Theorem 4.1 with X replaced by $X(t)$, we obtain

$$(4.2) \quad Z(t) \stackrel{d}{=} X(t) + X'(t) + Y_0(t) \quad [0 < t \leq 1],$$

where $Y_0(t) \stackrel{d}{=} X(1-t) + Y_0$ with Y_0 as in (4.1) independent of $X(1-t)$. So, we recover (3.3) with $Y \stackrel{d}{=} X + Y_0$ and Y_0 as in (4.1). $\square$

Choosing a uniformly distributed random variable U independent of X , X' and Y_0 , we can rewrite (4.1) as

$$Z \stackrel{d}{=} (X + UY_0) + (X' + \bar{U}Y_0),$$

where $\bar{U} = 1 - U \stackrel{d}{=} U$. Since we also have $Z = V + W$ with $V \stackrel{d}{=} W$ [cf. Lemma 2.1 and (2.3)], this suggests that in this case

$$(4.3) \quad W \stackrel{d}{=} X + A,$$

where X and A are independent with A of the form $A = UY_0$. This is the content of the next theorem. Since apart from some mass at zero the random variable A , and hence W , has an absolutely continuous distribution, we consider the non-lattice case here; cf. Remark following Lemma 2.2.

Theorem 4.2. *If X is compound-exponential and non-lattice with finite first moment μ , then the stationary remaining life-time W satisfies (4.3) with X and A independent, and A of the form $A \stackrel{d}{=} UY_0$, where Y_0 is nonnegative, U is uniform on $(0, 1)$ and independent of Y_0 .*

PROOF. Use Lemmas 2.1, 2.3 and 2.6. Since X has LSt $\hat{F} = 1/(1 - \log \hat{F}_0)$ with F_0 infinitely divisible, the LSt of W can be written as

$$\hat{F}_W(s) = \frac{1 - \hat{F}(s)}{\mu s} = \hat{F}(s) \frac{-\log \hat{F}_0(s)}{\mu s} = \hat{F}(s) \int_0^\infty \frac{1 - e^{-sx}}{sx} dF_{Y_0}(x),$$

where, as in the proof of Theorem 4.1, $F_{Y_0} = (1/\mu) K_0$ with K_0 the canonical function of F_0 . The equality in the extreme members of this equation easily translates in (4.3) with A of the form $A = UY_0$ as desired. $\square$

It is unclear whether (4.3) with X and A independent implies that X has a compound-exponential distribution. The fact that A is of the form $A \stackrel{d}{=} UY_0$ implies that A has a density on $(0, \infty)$ which is nonincreasing.

Remark. Proceeding as in the Remark following Theorem 4.1, for $W(t)$ we obtain

$$(4.4) \quad W(t) \stackrel{d}{=} X(t) + UY_0(t) \quad [0 < t \leq 1],$$

which can be written as

$$(4.5) \quad W(t) \stackrel{d}{=} (1 - U)X(t) + U\{X(t) + Y_0(t)\},$$

where $X(t) + Y_0(t) \stackrel{d}{=} Y$, independent of t ; compare Corollary 3.3. $\square$

5. Analogues for lattice distributions

In this section we briefly present the analogues of Theorems 3.1, 4.1 and 4.2 for $\mathbb{Z}_+$ -valued random variables X . We consider the discrete-time renewal process generated by X with $\mathbb{P}(X = k) = p_k$ for $k \in \mathbb{Z}_+$ and $\mathbb{E}X = \mu \in (0, \infty)$, and use the

notation as established in Lemma 2.2. We restrict attention to X with $p_0 > 0$. This leads to the clearest analogues, and it is no real restriction: The results for Z hold for arbitrary X ; for the result on W the restriction $\mathbb{P}(X \leq 1) > 0$ is necessary, and the case $p_0 = 0, p_1 > 0$ can be reduced to the case $p_0 > 0$ by a shift. The proofs differ only slightly from those in Sections 3 and 4.

Theorem 5.1. *Let X have an aperiodic distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$. Then the random variable Z with distribution given by (2.4) satisfies*

$$(5.1) \quad Z \stackrel{d}{=} X + Y,$$

where Y is a (necessarily $\mathbb{N}$ -valued) random variable independent of X , iff X is infinitely divisible.

PROOF. By (2.4), saying that Z satisfies (5.1) with X and Y independent, is equivalent to the following assertion:

$$\mathbb{P}(Z - 1 = k) = \frac{1}{\mu} (k + 1) p_{k+1} = \sum_{j=0}^k p_j \mathbb{P}(Y - 1 = k - j) \quad [k \in \mathbb{Z}_+].$$

An appeal to Lemma 2.7 finishes the proof; note that necessarily $\sum_{k=0}^{\infty} r_k = \mu$. $\square$

From the proof it will be clear that Y in (5.1) has distribution $(r_{k-1}/\mu)_{k \in \mathbb{N}}$. Of course, also Theorem 3.2 has a lattice analogue; we don't spell it out.

The analogues of Theorems 4.1 and 4.2 are combined in the following theorem. Note that a compound-exponential distribution on $\mathbb{Z}_+$ necessarily has positive mass at 0.

Theorem 5.2. *Let X have an aperiodic, compound-exponential distribution (p_k) on $\mathbb{Z}_+$. Then the random variable Z with distribution given by (2.4) satisfies*

$$(5.2) \quad Z \stackrel{d}{=} X + X' + Y_0,$$

where $X' \stackrel{d}{=} X$, Y_0 is $\mathbb{N}$ -valued, and X , X' and Y_0 are independent. For the random variable W with distribution given by (2.5) one has

$$(5.3) \quad W \stackrel{d}{=} X + A,$$

where X and A are independent and A has a nonincreasing distribution on $\mathbb{N}$.

PROOF. By Lemma 2.8 the pgf P of X has the form $P = 1/(1 - \log Q)$ with Q an infinitely divisible pgf; hence differentiation of $1/P$ leads to the equation $P'/P^2 = Q'/Q$. Since the pgf P_{Z-1} of $Z - 1$ is given by $(1/\mu) P'$, it follows that

$$P_{Z-1}(z) = \{P(z)\}^2 (1/\mu) Q'(z)/Q(z).$$

Now, $(1/\mu)Q'/Q$ can be read as the pgf of $Y_0 - 1$ for some $\mathbb{N}$ -valued random variable Y_0 ; cf. the remark following Theorem 5.1. We conclude that Z satisfies (5.2). With regard to (5.3) it is sufficient to note that

$$P_{W-1}(z) = \frac{1 - P(z)}{\mu(1 - z)} = P(z) \frac{-\log Q(z)}{\mu(1 - z)} = P(z) \frac{1 - G(z)}{(\mu/\alpha)(1 - z)};$$

cf. the second representation of (2.13) in Lemma 2.8. $\square$

6. An example

For $X(\cdot)$ in Theorem 3.2 we take a *Gamma process*: $X(t)$ has a gamma distribution with parameters t and λ , so with density $x \mapsto \lambda^t x^{t-1} e^{-\lambda x} / \Gamma(t)$. The canonical function of $X(t)$ has density $x \mapsto t e^{-\lambda x}$, and it follows that for the total life-time $Z(t)$ in the stationary renewal process generated by $X(t)$ [$t > 0$, fixed] we have

$$Z(t) \stackrel{d}{=} X(t) + Y,$$

where $X(t)$ and Y are independent and Y has an *exponential* distribution with parameter λ , independent of t .

For $0 < t \leq 1$ the random variable $X(t)$ is compound-exponential: its LSt can be represented as $1/(1 - \log \hat{F}_0)$, where F_0 is infinitely divisible with canonical function K_0 satisfying

$$\hat{K}_0(s) = -\frac{\hat{F}'_0(s)}{\hat{F}_0(s)} = -\frac{d}{ds} \log \hat{F}_0(s) = \frac{t}{\lambda} \left(\frac{\lambda}{\lambda + s} \right)^{1-t}.$$

From Theorem 4.1 it follows that for $0 < t \leq 1$

$$Z(t) \stackrel{d}{=} X(t) + X'(t) + Y_0(t),$$

where $Y_0(t)$ has a gamma distribution with parameters $1 - t$ and λ . The special case $t = 1$, i.e., $X(1)$ exponential, gives $Y_0(1) = 0$ with probability 1, as is well known. In general, $Z(t)$ has a gamma distribution with shape parameter $t + t + (1 - t) = t + 1$; for $t = \frac{1}{2}$ the random variable $Z(t)$ is the sum of three iid random variables.

By Theorem 4.2, for the remaining life-time $W(t)$ in the renewal process generated by $X(t)$ we find

$$W(t) \stackrel{d}{=} X(t) + A(t),$$

with

$$\hat{F}_{A(t)}(s) = \frac{(1 + s/\lambda)^t - 1}{ts/\lambda}.$$

For $t = 1$ we obtain the well-known result that $W(1) \stackrel{d}{=} X(1)$, i.e., $\mathbb{P}(A(1)) = 0 = 1$.

References

- [1] FELLER, W. (1968). *An introduction to probability theory and its applications*, vol. 1, 3-rd ed., Wiley, New York.
- [2] FELLER, W. (1971). *An introduction to probability theory and its applications*, vol. 2, 2-nd ed., Wiley, New York.
- [3] HARN, K. VAN (1978). *Classifying infinitely divisible distributions by functional equations*, Math. Centre Tracts **103**, Math. Centre, Amsterdam.
- [4] ROSS, S.M. (1970). *Applied probability models with optimization applications*, Holden-Day, San Francisco.
- [5] STEUTEL, F.W. (1970). *Preservation of infinite divisibility under mixing, and related topics*, Math. Centre Tracts **33**, Math. Centre, Amsterdam.
- [6] WINTER, B.B. (1989). Joint simulation of backward and forward recurrence times in a renewal process, J. Appl. Prob. **26**, 404–407.

July 15, 1994