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Elicitation of Expert Knowledge and Assessment of Imprecise Prior Densities for Lifetime Distributions

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Abstract
This report presents methods for elicitation of expert knowledge, to be used as information to assess imprecise prior densities for parameters of a model that describes the lifetime of a component. Also results on combination of elicited opinions are presented.

1. Introduction

1.1 Imprecise Probabilities

The theory of imprecise probabilities (Walley [11,12]) provides a framework for handling uncertainty in expert systems. The essential difference between this concept and the classical (frequentist) or Bayesian concepts is that one does not need to assign one (precise) value to be the probability \( P(A) \) of occurrence of event \( A \), but that two values, called the lower and upper probability for \( A \) (denoted by \( P(A) \) and \( P(A) \) respectively), bounding an interval can be assigned. A possible interpretation of such an interval is that \( P(A) \) is unknown, but one is sure that it lies within the interval \( [P(A),P(A)] \). A logical restriction is \( 0 \leq P(A) \leq 1 \). Further, based on the available information, no further distinction between values within the interval is made. Coolen [2] discusses possible interpretations, where it should be remarked that the subjective interpretation used by Walley [11], seems to be the correct base for a theory of imprecise probabilities. Walley's interpretation of probability is a generalization of that advocated by De Finetti [5].

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If $\underline{P}(A) = \overline{P}(A)$ the exact value of $P(A)$ is known. This situation can only occur in case of perfect information about event $A$. On the other hand, if there is no information at all, the only sensible statement is $0 \leq P(A) \leq 1$. The length of the interval $[\underline{P}(A), \overline{P}(A)]$ relates to the available information about $A$, and a degree of imprecision is defined as $\Delta(A) := \overline{P}(A) - \underline{P}(A)$. If $\underline{P}(A) = \overline{P}(A)$ then $\Delta(A) = 0$, and the probability of $A$ is said to be precise. If $\underline{P}(A) = 0$ and $\overline{P}(A) = 1$ then $\Delta(A) = 1$, and the probability of $A$ said to be vacuous (this represents total absence of information about $A$). Within the standard Bayesian framework (De Finetti [5]) $\underline{P}(A) = \overline{P}(A)$ is assumed, leaving no possibility for taking the amount of available information into account. Walley [11] gives an extensive discussion of the importance of imprecision.

Generally $\Delta(A)$ will be a decreasing function of the amount of relevant information. However, if new information about $A$ becomes available that conflicts with older information, $\Delta(A)$ may increase. Nevertheless, for many applications $\Delta(A)$ can be chosen as a decreasing function of the amount of information.

As shown by Coolen [2], the axioms on which the theory of imprecise probabilities is based (Wolfenson and Fine [13]) follow straightforwardly from the well-known Kolmogorov axioms for $P(A)$. Let $\Omega$ be the set of all possible events of interest, then $\underline{P}$ and $\overline{P}$ are assumed to satisfy the following basic axioms:

(A1) For all $A \in \Omega$: $\underline{P}(A) \geq 0$.

(A2) $\underline{P}(\Omega) = 1$.

(A3) For all $A, B \in \Omega$, with $A \cap B = \emptyset$: $P(A) + P(B) \leq P(A \cup B)$ and $\overline{P}(A) + \overline{P}(B) \geq \overline{P}(A \cup B)$.

(A4) For all $A \in \Omega$: $\overline{P}(A) + \overline{P}(A^c) = 1$, where $A^c$ is the complement of $A$.

Elementary consequences of the axioms A1–A4 (proofs in Coolen [2]) are:

(B1) $P(\emptyset) = \underline{P}(\emptyset) = 0$.

(B2) $\overline{P}(\Omega) = 1$.

(B3) $\overline{P}(A) \geq P(A)$.

(B4) $A \cap B = \emptyset$ implies $P(B) \leq P(A)$ and $\overline{P}(B) \geq \overline{P}(A)$.

(B5) $A \cap B = \emptyset$ implies $\underline{P}(A \cup B) \leq P(A) + \overline{P}(A \cup B)$.
It is important to check coherence of assessments of the upper and lower probabilities (Walley [11, section 2.5]). In short, coherence means that the probability assessments are not contradictory.

Let $A_1, \ldots, A_n$ be a partition of $\Omega$. Then (Walley [11, section 4.6.1]) the assessments $P(A_i)$ and $\overline{P}(A_i)$ are coherent if and only if conditions C1 and C2 hold for $1 \leq i \leq n$:

**Coherence conditions:**

(C1) $0 \leq P(A_i) \leq \overline{P}(A_i) \leq 1$.

(C2) $\overline{P}(A_i) + \sum_{j=1 \atop j \neq i}^{n} P(A_j) \leq 1 \leq P(A_i) + \sum_{j=1 \atop j \neq i}^{n} \overline{P}(A_j)$.

(Walley also gives a third condition, $\sum_{j=1}^{n} P(A_j) \leq 1 \leq \sum_{j=1}^{n} \overline{P}(A_j)$, but this is implied by C1 and C2.)

Special cases of imprecise probabilities are lower and upper cumulative distribution functions (cdf) for a real variable $X$ (Coolen [2]). These are the lower and upper probabilities of the events $X \leq x$ for $x \in \mathbb{R}$, and are denoted by $F(x) = P(X \leq x)$ and $\overline{F}(x) = \overline{P}(X \leq x)$.

One useful model asks for two functions to be assigned, say $l$ and $u$ with $0 \leq l(x) \leq u(x)$ for all $x$, such that all functions $h$ between $l$ and $u$ can, after normalization, be regarded as probability density functions (pdf) for $X$. The functions $l$ and $u$ are called lower and upper density functions. We assume, in this report, that $l$ and $u$ are continuous and have positive finite integrals. Given $l$ and $u$, the lower and upper cdf's, defined by

$$F(x) = \frac{\int_{-\infty}^{x} l(\omega) \, d\omega}{\int_{-\infty}^{x} l(\omega) \, d\omega + \int_{x}^{\infty} u(\omega) \, d\omega} \quad \text{and} \quad \overline{F}(x) = \frac{\int_{-\infty}^{x} u(\omega) \, d\omega}{\int_{-\infty}^{x} u(\omega) \, d\omega + \int_{x}^{\infty} l(\omega) \, d\omega}$$

are the bounds of all cdf's that can be constructed from densities (after normalization) that lie between $l$ and $u$ (Coolen [2]).
Continuity of \( l \) and \( u \) implies that these lower and upper cdf’s have pdf’s \( f \) and \( \overline{f} \), respectively, where

\[
\overline{f}(x) = \frac{\left( \int_{-\infty}^{x} l(\omega) \, d\omega + u(x) \right)^2}{\int_{-\infty}^{\infty} u(\omega) \, d\omega} \quad \text{and} \quad \int_{-\infty}^{\infty} l(\omega) \, d\omega
\]

Within the Bayesian framework, let \( l \) and \( u \) be imprecise prior densities for parameter \( \theta \in \Theta \) (for some parameter space \( \Theta \)) of a sampling distribution with pdf \( f(x|\theta) \). The model for \( X \) is assumed to be known except for the value of the parameter \( \theta \), so the form of the likelihood function \( L(\theta|x) \) is known. After observing data \( x \), imprecise posterior densities are

\[
l(\theta|x) = L(\theta|x) l(\theta) \quad \text{and} \quad u(\theta|x) = L(\theta|x) u(\theta).
\]

The imprecise posterior cdf’s, \( F(\theta|x) \) and \( \overline{F}(\theta|x) \), and corresponding pdf’s are derived as above.

Lower and upper predictive densities, based on prior densities \( l(\theta) \) and \( u(\theta) \), are

\[
l_X(x) = \int f(x|\theta) l(\theta) \, d\theta \quad \text{and} \quad u_X(x) = \int f(x|\theta) u(\theta) \, d\theta,
\]

Again these are not pdf’s. The lower and upper predictive cdf’s for \( X \) are derived as above. The predictive densities for \( X \) are updated by replacing \( l(\theta) \) and \( u(\theta) \) by \( l(\theta|x) \) and \( u(\theta|x) \).

### 1.2 Lifetime Distributions

A non-negative random variable \( X \) represents the lifetime (time to failure) of a component (this may be a single component, but also a technical system, regarded as one unit with regard to its lifetime). The probability distribution of \( X \) is unknown, and one is interested in the estimation of this distribution, based on the available relevant information (e.g. historical data, experimental data or expert opinions).

For many practical decision problems (e.g. maintenance scheduling as in
Coolen e.a. [4]) restriction to some parametric family of distributions, where the available information is used to estimate the distribution parameters, offers good solutions. Typical families are the Weibull, gamma and lognormal distributions. A member of such family is uniquely determined by its parameter values, so all information must be used to estimate the parameters. This estimation problem, which is a fundamental problem of statistics, has solutions depending on the criteria chosen.

In this report the Bayesian interpretation of parameters is adopted (De Finetti [5]), so these are regarded as random variables, having subjective probability distributions that should reflect the ideas and knowledge about them, based on the available information. The Bayesian framework explicitly allows the use of information provided by experts.

1.3 The Use of Expert Opinion as Information

In practical decision problems, the decision maker often has to rely on judgements of experts (Cooke [1]). Often an expert is asked for a straightforward opinion, but many problems are too complex to be adequately solved this way, and the use of mathematical models can be helpful. For example, the decision maker can ask an expert about the lifetime distribution of component of interest. The decision maker may want to consult several experts, leading to problems of combining information, especially if the information provided is contradictory.

When uncertainty is present a language capable of dealing with it is needed. French [6] suggests that probability is the only concept that can be used. This language must be able to express to the decision maker how certain an expert feels about the information he provides. By using subjective probability, as in the standard Bayesian framework, the level of knowledge cannot adequately be reported, and conflict between experts cannot be expressed (Walley [11, chapter 5]). The use of imprecise probability offers the necessary generality.

It is important to emphasize the difference between elicitation of expert opinions, and assessment of probabilities, based on these opinions and perhaps on other information. Elicitation is the process by which beliefs are measured, through explicit judgements and choices. For assessment combination of information from several sources (experts) and coherence are important.
1.4 Elicitation and Combination of Expert Opinions

Before a decision maker can assess imprecise probability distributions for the random variables presenting lifetimes, based on information provided by experts, the opinions of the experts have to be elicited and combined. In the literature, much attention has been paid to the problem of eliciting human knowledge (e.g. Cooke [1, ch.8], O'Hagan [7], Spetzler and Staël von Holstein [8], Walley [10, 11 ch.4]). In this report, a method is discussed for elicitation of expert opinions about lifetime distributions, that relates to the histogram technique proposed by Van Dorp [9], but is generalized to the concept of imprecise probabilities. An advantage of this method is that experts can avoid considering the parameters of a model, to express their opinion directly about the lifetime variable X. In this way the opinion relates to something known concretely by the expert and not to what in many cases is an abstract concept not directly related to the problem at hand. In our view, this is a necessary condition for practical application. Nevertheless, only after practical tests of an elicitation method one can be sure whether or not the method works for the problems of interest. In this report two different interpretations of the results of an elicitation procedure by a histogram method are discussed, and some other possible elicitation methods are mentioned briefly. Combination of information from several experts is also very important, and is extensively discussed by Cooke [1]. Cooke does not discuss the generalization to imprecise probabilities, but Walley [10,11] treats it. In this report three possible methods of combination are discussed. Other possibilities for combination are mentioned briefly.

1.5 Outline of this Report

In sections 2 and 3 two different histogram methods for eliciting expert opinions within the framework of imprecise probabilities are presented, and combination of the results of the elicitation process is discussed. In section 4 some other methods for elicitation and combination are presented, and in section 5 assessment of prior distributions, based on information that results from the elicitation and combination processes, is discussed.
2. Elicitation and Combination of Expert Opinions by a Histogram Technique

2.1 Elicitation by a Histogram Technique

When using the histogram technique for elicitation (Van Dorp [9, §2.1]), an expert in the field of interest is asked to state his lower and upper probabilities that a lifetime lies in a certain interval. However, this question is not asked explicitly, as the concept of probability may be unknown to the expert. The elicitation process yields discretized versions of continuous lower and upper densities. Afterwards a member of any parametric family of densities can be chosen by determination of parameter values according to some estimation criterion.

The main advantage of this procedure is that the experts are often acquainted with the interval length after which inspection or maintenance is carried out, and to provide statements about the lifetime in terms of these intervals is more comfortable for them than having to provide information about a distribution in continuous time.

To obtain imprecise discrete lifetime densities of a component a question such as the following can be asked (for practical application more attention should be paid to the choice of a suitable question, which is a psychological problem):

"Suppose there are n components of the prescribed type, all starting to operate at the same time \( t_0 = 0 \), performing identical tasks and working independently of each other. Assume that they operate until failure (no maintenance is carried out). Give a lower and upper bound, \( l_i \) and \( u_i \), of the average number of these n components that will fail in the interval \( I_i \), where the average must be interpreted as if such an experiment with n components is repeated very many times."

In order to ask such a question, the time axis is divided into \( m \) intervals as follows: \( 0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = \omega \), \( \{0, \omega\} = \bigcup_{i=1}^{m} [t_{i-1}, t_i) \).

The interval \( [t_{i-1}, t_i) \) is referred to as interval \( I_i \).

Asking an expert the question above for all intervals yields \( l_i \) and \( u_i \), the bounds for the average number of components that are expected to fail in the \( i \)-th interval, according to the opinion of the expert. It is obvious that \( l_i \leq u_i \) for all \( i = 1, \ldots, m \), and \( \bigcup_{j=1}^{m} I_j = \bigcup_{j=1}^{m} [t_{j-1}, t_j) \) must hold. Instead of asking this
question for each interval, the expert can represent the numbers \( l_i \) and \( u_i \) by a histogram. The value of \( n \) can be chosen by the decision maker or the expert. For \( l_i \) and \( u_i \) it seems natural to restrict to integers, but this is not necessary.

For notation we introduce vectors \( L := (l_1, \ldots, l_m) \) and \( U := (u_1, \ldots, u_m) \) that contain all information (given \( n \) and the intervals). The random variable \( X \) represents the lifetime of the component.

**Definition 2.1.1**

The lower and upper probabilities of \( X \in I_i \) are defined by

\[
\underline{p}_i = P(X \in I_i) := \frac{l_i}{n} \quad \text{and} \quad \overline{p}_i = \overline{P}(X \in I_i) := \frac{u_i}{n},
\]

respectively.

Theorem 2.1.2 expresses the coherence conditions C1 and C2 (section 1.1) in terms of \( l_i \) and \( u_i \).

**Theorem 2.1.2**

\[
0 \leq \underline{p}_i \leq \overline{p}_i \leq 1 \quad \text{and} \quad \underline{p}_i + \sum_{j \neq i} \underline{p}_j \leq 1 + \sum_{j \neq i} \overline{p}_j \quad \text{for all } i.
\]

**Proof**

The equivalence of these statements follows straightforwardly from definition 2.1.1.

In practice it may be attractive to assume that \( t_0, t_1, \ldots, t_{m-1} \) are equidistant, so \( t_j = i \cdot T \) for some \( T > 0 \), and that the same \( T \) can be used for eliciting the opinions of all experts.

The value of \( m \) can be chosen by an expert. Here two problems might arise. The first problem is that the expert might not want to express any opinion about \( I = [t_0, \infty) \). The second problem occurs if several experts choose different values of \( m \) (this problem will be addressed in section 2.3).

For the first problem within the concept of imprecise probabilities the lack of any opinion about \( I_m \) is represented by \( \underline{p}_m = 0 \) and \( \overline{p}_m = 1 \), so by \( l_m = 0 \) and \( u_m = n \). However, suppose \( \sum_{j=1}^{m-1} u_j < n \), then the minimal number of components
expected to fail within $I_m$ should not be less than $n-\sum_{j=1}^{m-1} u_j$. With an analogous argument for the maximal number of components expected to fail within $I_m$, one can improve the definitions $l_m=0$ and $u_m=n$ by defining

$$l_m = \max\left(0, n-\sum_{j=1}^{m-1} u_j\right) \quad \text{and} \quad u_m = n-\sum_{j=1}^{m-1} l_j.$$

These definitions are better in the sense that they do not introduce more imprecision than is necessary and do not imply incoherence. The condition C2 is based on an analogous argument.

**Definition 2.1.3**

If, for some $t \in \{1, \ldots, m\}$, $l_t + \sum_{j \neq t} u_j < n$ or $u_t + \sum_{j \neq t} l_j > n$ (by theorem 2.1.2 these assessments are incoherent), then coherent values for $l_t$ and $u_t$ are derived by defining

$$\tilde{l}_t := \max(l_t, n-\sum_{j \neq t} u_j) \quad \text{and} \quad \tilde{u}_t := \min(u_t, n-\sum_{j \neq t} l_j).$$

(Hereafter, $\tilde{l}_t$ and $\tilde{u}_t$ replace $l_t$ and $u_t$, and the ‘$\sim$’ is dropped.)

In example 2.1.6 the idea of these redefinitions will become clear, but first it is necessary to show that the definitions $\tilde{l}_t$ and $\tilde{u}_t$ indeed make sense. Lemma 2.1.4 and theorem 2.1.5 imply that the use of $\tilde{l}_t$ and $\tilde{u}_t$ never leads to more imprecision than using $l_t$ and $u_t$. In example 2.1.6 it is shown that $\tilde{l}_t$ and $\tilde{u}_t$ can lead to less imprecision than $l_t$ and $u_t$ do.

**Lemma 2.1.4**

$$l_t \leq \tilde{l}_t \leq u_t \leq \tilde{u}_t.$$

**Proof**

These relations result from definition 2.1.3, where for the second relation also the logical relations $l_i \leq u_i$ and $\sum_{j=1}^{m} l_j \leq \sum_{j=1}^{m} u_j$ are used.  

**Theorem 2.1.5**

Let $p_i^1$ and $\bar{p}_i^1$ be the lower and upper probabilities corresponding to $\{l_1, \ldots, l_m\}$ and $\{u_1, \ldots, u_m\}$, and let $p_i^2$ and $\bar{p}_i^2$ be the lower and upper probabilities corresponding to the $l$ and $u$ values, with $l_t$ replaced by $\tilde{l}_t$ or $u_t$ replaced by $\tilde{u}_t$ as in definition 2.1.3 (both $l_t$ and $u_t$ may be redefined).

Then $p_i^1 \leq p_i^2 \leq \bar{p}_i^2 \leq \bar{p}_i^1$ for all $i$. 

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Proof
Since for $i \neq t$ the $l$ and $u$ values do not change, the statement is obvious.
For $i = t$ the result follows immediately from lemma 2.1.4. 

Example 2.1.6
In a reliability assessment the life distribution of a component must be estimated. The expert is asked to give $l_i$ and $u_i$ for $i=1,2,3$, and for $n=100$. Assume that he gives the following values:

$l_1 = 10, u_1 = 30, l_2 = 30, u_2 = 60, l_3 = 20, u_3 = 30$.

Accepting these values, the lower and upper probabilities are:

$p_1 = 1/10, \bar{p}_1 = 3/10, p_2 = 3/10, \bar{p}_2 = 6/10, p_3 = 2/10, \bar{p}_3 = 3/10$.

At first look the values seem reasonable. However, the conditions $l_i + \sum_{j \neq i} u_j \leq n$ and $u_i + \sum_{j \neq i} l_j \leq n$ (for all $i$), are not fulfilled, as $l_2 + u_1 + u_3 = 90$. The assessments $u_1 = 30$ and $u_3 = 30$ imply that this expert is sure that at most 30 components will fail in the first interval $I_1'$, and also at most 30 will fail in the third interval $I_3'$, so at most 60 components will fail in these two intervals together. This logically implies that, according to this expert, at least 40 components will fail in the second interval $I_2$. So the value $l_2 = 30$ is indeed a possible lower bound, but given the total information it is possible to take $l_2 = 40$ as a replacement for $l_2 = 30$. So, from the information provided by the expert, the following values can be used to calculate lower and upper probabilities:

$l_1 = 10, u_1 = 30, l_2 = 40, u_2 = 60, l_3 = 20, u_3 = 30$.

The resulting lower and upper probabilities are:

$p_1 = 1/10, \bar{p}_1 = 3/10, p_2 = 4/10, \bar{p}_2 = 6/10, p_3 = 2/10, \bar{p}_3 = 3/10$.

Suppose that this expert does not want to provide any statement about the third interval $[t_2, \infty)$, while the values of $l_i$ and $u_i$ remain the same for $i=1,2$ (with $l_2 = 30$). It would be possible to define $l_3 = 0$ and $u_3 = 100$, so

$l_1 = 10, u_1 = 30, l_2 = 30, u_2 = 60, l_3 = 0, u_3 = 100$, and

$p_1 = 1/10, \bar{p}_1 = 3/10, p_2 = 3/10, \bar{p}_2 = 6/10, p_3 = 0, \bar{p}_3 = 1$.

But $u_1 + u_2 = 90$ implies that this expert does surely not expect less than 10 components to fail within $I_3'$, and also $l_1 + l_2 = 40$ implies that he does surely not expect more than 60 components to fail within $I_3$. Following definition 2.1.3, this leads to $l_3 = 10$ and $u_3 = 60$, and these values imply

$p_1 = 1/10, \bar{p}_1 = 3/10, p_2 = 3/10, \bar{p}_2 = 6/10, p_3 = 1/10, \bar{p}_3 = 6/10$.

(End of example 2.1.6.)
Theorem 2.1.5 guarantees that the imprecision is not increased by redefining \( l_t \) and \( u_t \) as \( \tilde{l}_t \) and \( \tilde{u}_t \), thus \( p_{i}^{2} - p_{i}^{1} \leq p_{i}^{2} - p_{i}^{1} \). Corollary 2.1.7 shows that the imprecision always decreases if redefinition according to definition 2.1.3 is applied.

**Corollary 2.1.7**

\[ \forall i \in \{1, \ldots, m\} : p_{i}^{1} = p_{i}^{2} \text{ and } p_{i}^{1} = p_{i}^{2} \Rightarrow \forall i \in \{1, \ldots, m\} : u_{i} + \sum_{j \in I} l_{j} \leq n \leq l_{i} + \sum_{j \not\in I} u_{j} \]

(i.e. there are no values \( l \) or \( u \) redefined according to definition 2.1.3).

**Proof**
The '\( \leq \)'-statement is trivial. The '\( \geq \)'-statement results from 2.1.1 and 2.1.3, together with theorem 2.1.5. The equalities for the lower and the upper probabilities hold if and only if \( \tilde{l}_t = l_t \) and \( \tilde{u}_t = u_t \), the case in which there are no redefinitions at all.

If the expert has not assigned values to \( l_m \) and \( u_m \), then corollary 2.1.7 implies that in almost all situations we can do better than defining \( l_m = 0 \) and \( u_m = n \). This is stated in corollary 2.1.8.

**Corollary 2.1.8**

Suppose an expert has assessed values to \( l_i \) and \( u_i \) for \( i=1, \ldots, m-1 \), such that at least one \( l_i > 0 \), or \( \sum_{j=1}^{m} u_j < n \). Then the definitions \( \tilde{l}_m = \max(0, n - \sum_{j=1}^{m-1} u_j) \) and \( \tilde{u}_m = \min(n, n - \sum_{j=1}^{m-1} l_j) \) lead to less imprecision (and hence are better) than \( l_m = 0 \) and \( u_m = n \).

**Proof**
This follows straightforwardly from definition 2.1.3 and corollary 2.1.7.

In the following sections combination of opinions is discussed. In section 2.2 all experts are assumed to assess values for \( l_i \) and \( u_i \) according to the same intervals, whereas in section 2.3 some generalizations are shortly discussed.
2.2 Combination of Opinions

In this section, three different methods of combining the opinions of K experts (K≥2) about the lifetime of a component, are discussed. This topic is important for a decision maker, who wants to use the knowledge of different people as information to solve a decision problem. It is assumed that all experts have expressed their opinions by a procedure like that in section 2.1, leading to vectors \( L_k := (l_{k,1}, \ldots, l_{k,m}) \) and \( U_k := (u_{k,1}, \ldots, u_{k,m}) \) for expert \( k \) (\( k=1, \ldots, K \)). It is assumed that all experts have used the same partition of the time-axis (intervals \( I_1, \ldots, I_m \)), and that (for all experts \( k \)) \( l_{k,i} \) and \( u_{k,i} \) follow the coherence conditions (theorem 2.1.2), so:

\[
0 \leq l_{k,i} \leq u_{k,i} \leq n \quad \text{and} \quad u_{k,i} - \sum_{j \neq i} l_{k,j} \leq i - \sum_{j \neq i} u_{k,j} \quad \text{for all } i.
\]

In other words all necessary redefinitions have been carried out to obtain coherence. For each expert, the lower and upper probabilities resulting from \( l_{k,i} \) and \( u_{k,i} \), according to definition 2.1.1, are denoted by \( \underline{P}_{k,i} = l_{k,i}/n \) and \( \overline{P}_{k,i} = u_{k,i}/n \) (we also use \( P_k = (P_{k,1}, \ldots, P_{k,m}) \) and \( \overline{P}_k = (\overline{P}_{k,1}, \ldots, \overline{P}_{k,m}) \)).

2.2.1 Combination by the Unanimity Rule

The first method of combination is the so-called unanimity rule (see Walley [11, section 4.3]).

Definition 2.2.1

The unanimity rule for combination of the opinions of \( K \) experts leads to combined lower and upper probabilities for the event \( X \in I_i \) (\( i=1, \ldots, m \)):

\[
P_i := \min\{\underline{P}_{k,i} \mid k=1, \ldots, K\} \quad \text{and} \quad \overline{P}_i := \max\{\overline{P}_{k,i} \mid k=1, \ldots, K\},
\]

respectively. (In this subsection, these imprecise probabilities are also denoted by vectors \( \underline{P} := (\underline{P}_1, \ldots, \underline{P}_m) \) and \( \overline{P} := (\overline{P}_1, \ldots, \overline{P}_m) \)).

For this method of combination, the imprecise probabilities corresponding to \( l_{k,i} \) and \( u_{k,i} \) for each expert are first calculated, then these imprecise probabilities are combined. Another possible method is to combine the \( l_{k,i} \) and \( u_{k,i} \) first, followed by defining imprecise probabilities according to these combined \( l \) and \( u \) values. This is presented in definition 2.2.2.
Definition 2.2.2
Let $l_i := \min\{l_{k,i} \mid k = 1, \ldots, K\}$ and $u_i := \max\{u_{k,i} \mid k = 1, \ldots, K\}$. Then combined imprecise probabilities for the event $x \in I_i$ are $p'_i := l_i/n$ and $\bar{p}'_i := u_i/n$.

Theorem 2.2.3 states that definitions 2.2.1 and 2.2.2 lead to the same imprecise probabilities.

Theorem 2.2.3
The combined imprecise probabilities $p'_i$ and $\bar{p}'_i$, according to definition 2.2.2, are equal to the imprecise probabilities $p_i$ and $\bar{p}_i$, according to definition 2.2.1.

Proof
$p'_i = \min\{l_{k,i} \mid k = 1, \ldots, K\}/n = \min\{l_{k,i}/n \mid k = 1, \ldots, K\} = \min\{p_{k,i} \mid k = 1, \ldots, K\} = p_i$. For the upper probability the proof is analogous. $\square$

Theorem 2.2.3 may seem to be trivial, and analogous results will hold for the other methods of combination discussed in this section. However, in chapter 3 a result like theorem 2.2.3 does not hold.

Although definitions 2.2.1 and 2.2.2 lead to the same results, in this report definition 2.2.1 is used as the method of combination according to the unanimity rule (this choice relates to chapter 3). Definition 2.2.2 is suitable for the proof of theorem 2.2.4, that states an important advantage of combination by the unanimity rule.

Theorem 2.2.4
If, for each expert $k$, the assessments $L_k$ and $U_k$ are coherent (according to theorem 2.1.2), then the imprecise probabilities $\underline{p}$ and $\overline{p}$, after combination by the unanimity rule, are also coherent.

Proof
For all $k$ and $i$, $0 \leq l_{k_1,i} \leq u_{k_1,i} \leq n$ and $u_{k_1,i} + \sum_{j \neq i} l_{k_1,j} \leq n \leq l_{k_1,i} + \sum_{j \neq i} u_{k_1,j}$. This implies that $0 \leq l_{k_1,i} \leq u_{k_1,i} \leq n$, since $l_i = \min\{l_{k,i}\}$ and $u_i = \max\{u_{k,i}\}$. Further, $u_i = \max\{u_{k,i}\} = u_{k_i,i}$ for some $k_i \in \{1, \ldots, K\}$, so $u_i + \sum_{j \neq i} l_{k_i,j} \leq u_{k_i,i} + \sum_{j \neq i} l_{k_i,j} \leq n$. The relation $n \leq l_i + \sum_{j \neq i} u_j$ is proved analogously. Definition 2.2.2 and
theorems 2.1.2 and 2.2.3 complete the proof.

Example 2.2.5 shows how this method works.

**Example 2.2.5**

Consider a situation in which a decision maker needs information about the lifetime of a component, and wants to know the opinions of two experts. Three intervals are given that form a partition of the time-axis \([0, \infty)\), thus \(m=3\) and \(k=2\), and suppose that \(n=100\). The results of the elicitation process are \(L_1=(10, 30, 20)\), \(U_1=(30, 60, 40)\), \(L_2=(40, 20, 10)\) and \(U_2=(60, 30, 30)\), which fulfil the coherence conditions of 2.1.2.

To combine these values using the unanimity rule, first the imprecise probabilities per expert are calculated: \(\tilde{P}_1=(1/10, 3/10, 2/10)\), \(\tilde{P}_2=(3/10, 6/10, 4/10)\), \(P_1=(4/10, 2/10, 1/10)\) and \(P_2=(6/10, 3/10, 3/10)\), and then the combined imprecise probabilities are \(\tilde{P}=\left(1/10, 2/10, 1/10\right)\) and \(P=\left(6/10, 6/10, 4/10\right)\) from 2.2.1.

(End of example 2.2.5.)

As a result of this method of combination, for an event \(X \in I_1\), the final imprecision is never less than the imprecision according to each expert, so

\[\Delta(I_1) = \tilde{P}_1-P_1 \geq \tilde{P}_k-P_k =: \Delta_k(I_1)\].

The result of combination by the unanimity rule can be considered as the conjunction of all expert opinions, more specific as the conjunction of all imprecision.

If there is much disagreement between experts, this will cause much imprecision after combination. If experts disagree because they have access to different evidence or because they analyse the same evidence in different ways (this may be caused by different backgrounds of the experts), the resulting imprecision seems to be logical, and is useful information for the decision maker.

A drawback of this method is that the final result does not explicitly indicate disagreement. In example 2.2.5, both experts obviously disagree about the number of components that will fail in the first interval (between 10 and 30 according to the first, between 40 and 60 according to the second expert, so the intersection of their opinions about \(I_1\) is empty), but this is not reported by the final result.
2.2.2 Combination by the Conjunction Rule

The second method of combination is the so-called conjunction rule (see Walley [11, section 4.3]). The name 'conjunction rule' is adopted in this report, because it is known in literature for the method described here. However, the name 'intersection rule' may seem to be more appropriate for the method in this subsection, whereas the method of subsection 2.2.1 might have been called a 'conjunction rule'. To understand the names that are adopted here, one should consider the theory as presented by Walley, where imprecise probabilities are based on a theory of betting behaviour. In case of combination by the unanimity rule, a gamble is accepted if and only if all experts accept it, and in case of the conjunction rule a gamble is accepted if and only if at least one expert accepts it.

Definition 2.2.6

The conjunction rule for combination of the opinions of K experts leads to combined lower and upper probabilities for the event $X \in I_i$ ($i=1, \ldots, m$): $p_i := \max\{p_{k,i} | k=1, \ldots, K\}$ and $\bar{p}_i := \min\{\bar{p}_{k,i} | k=1, \ldots, K\}$, respectively. (In this subsection, these imprecise probabilities are also denoted by vectors $P := (p_1, \ldots, p_m)$ and $\bar{P} := (\bar{p}_1, \ldots, \bar{p}_m)$.)

For this method of combination it is again (cf. definition 2.2.2 and theorem 2.2.3) obvious that the results are equal if one first combines the $L_k$ and $U_k$ for all experts, and after that calculates the corresponding imprecise probabilities.

Whereas combination by the unanimity rule never leads to problems of incoherence if the individual assessments are coherent, combination by the conjunction rule cannot always be applied. Before this is discussed, three examples are given. Example 2.2.7 shows how this method works. Examples 2.2.8 and 2.2.9 show how problems related to coherence can arise.

Example 2.2.7

Let $n=100$, $m=3$ and $K=2$. Suppose the two experts have assessed $L_1 = (10, 30, 20)$, $U_1 = (30, 60, 40)$, $L_2 = (20, 40, 10)$ and $U_2 = (40, 60, 30)$, with resulting imprecise probabilities $\bar{p}_1 = (1/10, 3/10, 2/10)$, $\bar{P}_1 = (3/10, 6/10, 4/10)$, $P_2 = (2/10, 4/10, 1/10)$ and $\bar{P}_2 = (4/10, 6/10, 3/10)$. It is easy to check that for both experts opinions
are coherent.

The conjunction rule, according to definition 2.2.6, leads to \( P = (2/10, 4/10, 2/10) \) and \( \bar{P} = (3/10, 6/10, 3/10) \). Again these imprecise probabilities are coherent (check the coherence conditions of section 1.1).

{End of example 2.2.7.}

Example 2.2.8

Let \( n=100 \), \( m=3 \), \( K=2 \), and suppose \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(20,30,10) \) and \( U_2=(40,60,30) \). The only change from 2.2.7 is that \( L_2,2=30 \) instead of 40.

The imprecise probabilities are \( \underline{P}_1=(1/10,3/10,2/10) \), \( \overline{P}_1=(3/10,6/10,4/10) \), \( \underline{P}_2=(2/10,3/10,1/10) \) and \( \overline{P}_2=(4/10,6/10,3/10) \). It is easy to check that for both experts these values are coherent.

Combination by the conjunction rule leads to \( \underline{P}=(2/10,3/10,2/10) \) and \( \overline{P}=(3/10,6/10,3/10) \). The coherence conditions are no longer satisfied, since \( P_2+\overline{P}_1+\overline{P}_3=9/10<1 \), in conflict with condition C2.

The definition 2.1.3 can of course be used to redefine \( P_2' \), in which case \( P_2'=4/10 \) should be chosen.

{End of example 2.2.8.}

Example 2.2.9

Here the data of example 2.2.5 are used: \( n=100 \), \( m=3 \), \( K=2 \) and \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(40,20,10) \) and \( U_2=(60,30,30) \). These individual assessments are coherent. Combination by the conjunction rule leads to \( \underline{P}=(4/10,3/10,2/10) \) and \( \overline{P}=(3/10,3/10,3/10) \). Now a rather more serious form of incoherence occurs: \( P_1>P_1' \).

It is obvious that this method is not suited for these data.

{End of example 2.2.9.}

The first example above (2.2.7) shows that this method certainly can be applied in some situations. However, the other two examples (2.2.8 and 2.2.9) show that problems with regard to coherence of the combined imprecise probabilities can arise. Therefore, theorems like 2.2.4 do not hold for combination by the conjunction rule.

In example 2.2.9 there is an obvious conflict between the two experts (this has already been mentioned at the end of subsection 2.2.1), that leads to incoherent results. The assessments of example 2.2.8 do not indicate such a conflict at first sight, and this example makes clear that the coherence
conditions should always be checked after combination by the conjunction rule.

The result of combination by the conjunction rule can be considered as the intersection of all expert opinions, more specifically as the intersection of all imprecision (remember the remark about the name of this method, made at the beginning of this subsection). As shown in example 2.2.9, if this intersection is empty than the combined imprecise probabilities are incoherent. As shown in example 2.2.8, non-empty intersection of all expert opinions does not imply that the combined imprecise probabilities are coherent. This leads to

**Theorem 2.2.10**

Suppose that, for each expert $k$, the assessments $L_k$ and $U_k$ are coherent, the according $\bar{p}_k$ and $\overline{p}_k$ are calculated, and thereafter are combined by the conjunction rule. Then necessary conditions for coherence of the combined imprecise probabilities are $\max_{i=1, \ldots, m} \{p_{k,i} | k=1, \ldots, K\} \leq \min_{i=1, \ldots, m} \{\overline{p}_{k,i} | k=1, \ldots, K\}$ for all $i=1, \ldots, m$, and $\sum_{j=1}^{J} \max_{k=1, \ldots, K} \{p_{k,j} | k=1, \ldots, K\} \leq 1 \leq \sum_{j=1}^{J} \min_{k=1, \ldots, K} \{\overline{p}_{k,j} | k=1, \ldots, K\}$. These conditions are not sufficient for coherence.

**Proof**

The necessity of these conditions follows from definition 2.2.6 and the coherence conditions $C_1$ and $C_2$. The fact that these are not sufficient follows from example 2.2.8.

For this method of combination, for an event $X \in I_1$, the final imprecision is never greater than the imprecision of each expert, so $\Delta(I_1) = \overline{p}_i - \overline{p}_i \leq \overline{p}_i, i - \overline{p}_i = \Delta(I_1)$. This seems to be a very attractive property, because reducing imprecision is always a goal when information is gathered or combined. However, one should bear in mind that, when using the conjunction rule, combined imprecise probabilities can be highly precise, even when based on almost contradictory assessments. This is shown by the following example.

**Example 2.2.11**

Let $n=100$, $m=3$, $K=2$, and suppose that $L_1=(10, 40, 10)$, $U_1=(30, 60, 30)$, $L_2=(30, 20, 30)$ and $U_2=(50, 40, 50)$. The coherence of these individual assessments is again easily verified, and combination by the conjunction
rule leads to \( \overline{P} = (3/10, 4/10, 3/10) \) and \( \underline{P} = (3/10, 4/10, 3/10) \), removing all imprecision.

{End of example 2.2.11.}

The above examples make clear that this method certainly should not be applied without careful consideration. Nevertheless, the method can be useful to a decision maker, as it can indicate conflict between experts which the unanimity rule can not. So the decision maker can see as a result of combination whether or not there is a conflict between experts about the number of components that will fail in a certain interval.

A situation as in example 2.2.9, where the upper bound according to one expert is less than the lower bound according to another expert, may seem strange, but is not unlikely to appear in practice if the experts have different evidence or backgrounds. In such a situation one should try to compare and combine the bodies of evidence on which the different models are based. Discussion between the experts could perhaps indicate the reasons for the conflict. Here well-known procedures as the Delphi method can be of great use (see Cooke [1, chapter 1]).

There are situations in which combination by the conjunction rule seems to be very reasonable, for example when all experts have the same evidence (knowledge about history and technical aspects of the component and related fields) and analogous backgrounds, but some of them may have more experience or feel more certain about the problem of interest. Then the fact that the imprecision is never greater than the imprecision per expert seems to be logical.

2.2.3 Combination by the Weighted Average Method

The third method of combination is the so-called weighted average method, where a weight is assigned to each expert, and then the combined lower and upper probabilities are defined to be the weighted averages of the individual imprecise probabilities. In the standard Bayesian framework this plays an important role in combination of expert opinions. The two methods presented in subsections 2.2.1 and 2.2.2 cannot be applied within the standard Bayesian framework (except when all experts agree) for then precise assessments must be given, and imprecision cannot be used. The method
presented here can also be used within the standard Bayesian context, where much literature is available on the problem of determining weights for experts (Cooke [1]). In this report the determination of weights is not considered.

**Definition 2.2.12**

In case of $K \geq 2$ experts, let $W=(w_1, \ldots, w_K)$ be a vector of real numbers $w_k$, such that $0 < w_k < 1$ for all $k=1, \ldots, K$, and $\sum_{k=1}^{K} w_k = 1$. Then $w_k$ is called the weight assigned to expert $k$.

**Definition 2.2.13**

The weighted average method for combination of the opinions of $K$ experts leads to combined lower and upper probabilities for $X_{\in I_i}$ $(i=1, \ldots, m)$:

$$P_i := \sum_{k=1}^{K} w_k P_{k,i} \quad \text{and} \quad \overline{P}_i := \sum_{k=1}^{K} w_k \overline{P}_{k,i} \quad \text{respectively}.$$  
(In this subsection, these imprecise probabilities are also denoted by vectors $P := (P_1, \ldots, P_m)$ and $\overline{P} := (\overline{P}_1, \ldots, \overline{P}_m).$)

In this method it does not make any difference if one first combines the $L_k$ and $U_k$ for all experts, with the same weights for the experts, and after that calculates the corresponding imprecise probabilities (cf. definition 2.2.2 and theorem 2.2.3).

**Theorem 2.2.14**

If, for each expert $k$, the assessments $L_k$ and $U_k$ are coherent, then the imprecise probabilities $P$ and $\overline{P}$, after combination by the weighted average method, are also coherent.

**Proof**

Coherent individual assessments imply that for all $k$: $0 \leq P_{k,i} \leq 1$ and $0 \leq \overline{P}_{k,i} \leq 1$, and

$$P_{k,i} + \sum_{j \notin i} P_{k,j} \leq 1 \leq P_{k,i} + \sum_{j \notin i} \overline{P}_{k,j} \quad \text{for all } i.$$  
The first relation implies that $0 \leq P_i = \sum_{k} w_k P_{k,i} \leq \sum_{k} w_k \overline{P}_{k,i} = \overline{P}_i \leq 1$, while coherence condition C2 follows by

$$\overline{P}_i + \sum_{j \notin i} \overline{P}_j = \sum_{k} w_k \overline{P}_{k,i} + \sum_{k} \sum_{j \notin i} w_k P_{k,j} = \sum_{k} w_k \left( P_{k,i} + \sum_{j \notin i} P_{k,j} \right) \leq \sum_{k} w_k = 1$$

(the relation $1 \leq P_i + \sum_{j \notin i} \overline{P}_j$ is proved analogously).  \[\square\]
The assignment of weights is a task for the decision maker. A logical possibility, if he does not have any reason to rely more on one expert than on other experts, is to set all weights equal, so \( w_k = \frac{1}{K} \) for all \( k \). On the other hand, if the decision maker has more confidence in one expert than in another expert, which can, for example, be caused by the knowledge that the experts are not equally experienced, it is reasonable to choose the weights such that the expert who is regarded to be the best receives the most weight. Cooke [1] discusses many weighting methods for the standard Bayesian framework, and it is interesting to generalize these results to the context of imprecise probabilities.

This method of combination cannot indicate disagreement between experts to the decision maker, and the amount of imprecision in the combined imprecise probabilities is, for each interval \( I_i \), the weighted average of the individual amounts of imprecision, as is stated by the following corollary.

**Corollary 2.2.15**

Let \( \Delta_k(I_i) = \bar{P}_{k,i} - P_{k,i} \) be the imprecision on interval \( I_i \) according to expert \( k \), and \( \Delta(I_i) = \bar{P}_i - P_i \) the imprecision on that interval after combination by the weighted average method, then \( \Delta(I_i) = \sum_k w_k \Delta_k(I_i) \).

This implies that the imprecision after combination lies between the extremes of imprecision of the individual experts.

**Proof**

\[
\Delta(I_i) = \bar{P}_i - P_i = \sum_k w_k \left( \bar{P}_{k,i} - P_{k,i} \right) = \sum_k w_k \Delta_k(I_i). 
\]

Example 2.2.16 shows how this method works.

**Example 2.2.16**

Again the data of example 2.2.5 are used: \( n=100 \), \( m=3 \), \( K=2 \) and \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(40,20,10) \) and \( U_2=(60,30,30) \). Suppose the decision maker has no reason to assign different weights to both experts, and hence chooses \( w_1 = w_2 = 1/2 \). Combination by the method of weighted average leads to \( \bar{P}=(25/100,25/100,15/100) \) and \( \bar{P}=(45/100,45/100,35/100) \).

Suppose that the decision maker has more confidence in the knowledge of expert 1 than in the knowledge of expert 2 (there are several possible
reasons for such a situation), but he wants to use the opinions of both
experts, and chooses \( w_1 = 2/3 \) and \( w_2 = 1/3 \). These weights lead to combined
imprecise probabilities \( P = (60/300, 80/300, 50/300) \) and
\( \bar{P} = (120/300, 150/300, 110/300) \).
For both combinations it is easily verified that the final imprecise
probabilities satisfy the coherence conditions.
[End of example 2.2.16.]

2.3 Some Generalizations

The methods of elicitation and combination discussed in section 2.1 and 2.2
assume that all experts use the same partition of the time-axis (intervals
\( I_1, \ldots, I_m \), with \( I_i = [t_{i-1}, t_i] \) and \( 0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = \omega \)). All experts have
to express their opinion using the same number \( m \) of intervals. In this
section some possible generalizations are discussed briefly.
As remarked in section 2.1, within the concept of imprecise probabilities it
is no problem if an expert does not want to express any opinion about the
last interval \( I_m \). However, experts may want to use different partitions of
the time-axis. Suppose expert \( k \) wants to use \( m_k \) intervals, let \( I_{k,i} \) denote
the \( i \)-th interval according to expert \( k \), with \( I_{k,i} = [t_{k,i-1}, t_{k,i}] \) and
\( 0 = t_{k,0} < t_{k,1} < \ldots < t_{k,m_k-1} < t_{k,m_k} = \omega \).
It would be possible, in case of two experts, that \( m_2 > m_1 \) and \( t_{1,i} = t_{2,i} \) for
\( i = 0, \ldots, m_1 - 1 \), so \( I_{1,i} = \bigcup_{j=m_1}^{m_2} I_{2,j} \). A first solution to this problem is to
define \( I'_{1,i} = I_{2,i} \) and values \( p'_{1,i} \) and \( \bar{p}'_{1,i} \), for \( i = m_1, \ldots, m_2 \), such that
\[
\sum_{j=m_1}^{m_2} p'_{1,i,j} \leq p_{1,m_1} \quad \text{and} \quad \sum_{j=m_1}^{m_2} \bar{p}'_{1,i,j} \geq \bar{p}_{1,m_1}.
\]
A second solution is to define \( I'_{2,m_1,j} = \bigcup_{j=m_1}^{m_2} I_{2,j} \), together with
\( m_1 - 1 \)
\( m_2 - 1 \)
(\text{compare definition 2.1.3})
\[
P'_{2,m_1,m_1} = \max \left( \sum_{j=m_1}^{m_2} p_{2,j}, 1 - \sum_{j=m_1}^{m_1 - 1} \bar{p}_{2,j} \right)
\quad \text{and}
\bar{P}'_{2,m_1,m_1} = \min \left( \sum_{j=m_1}^{m_2} \bar{p}_{2,j}, 1 - \sum_{j=m_1}^{m_1 - 1} p_{2,j} \right).
\]
In the first suggested solution one large interval is divided, where some possibly arbitrary choices must be made (values for imprecise probabilities corresponding to redefined intervals for expert 1, such that these do not disagree with the information expert 1 has provided). The second solution seems to be more logical, but here some information provided by expert 2 is lost.

Another possible problem arises if the experts choose different $t_{k,i}$ values. This problem is, generally, hard to solve, as solutions must again be based on defining new partitions of the time-axis, where the definition of according imprecise probabilities, without losing too much information or making too many assumptions, is the biggest problem. In practice, as already remarked in section 2.1, agreement between experts and decision maker, before the actual elicitation process is started, about a partition of the time-axis that is to be used by all of them, is by far the most suitable procedure.
3. Elicitation and Combination of Expert Opinions using a Density Interpretation

3.1 Elicitation using a Density Interpretation

The elicitation process described in chapter 2 is attractive in practical application, and the resulting imprecise probabilities can be easily interpreted. However, example 3.1.1 shows that a problem might occur.

Example 3.1.1

Here we begin with n=100 components and a partition into m=3 intervals. Suppose an expert has assessed \( l_1 = 20, \ u_1 = 20, \ l_2 = 40, \ u_2 = 60, \ l_3 = 20 \) and \( u_3 = 40 \), after elicitation by a question as in section 2.1. This would imply, following section 2.1, that \( p_1 = \frac{2}{10} \), so the expert would be sure that the probability of failure during \( I_1 \) equals \( \frac{2}{10} \).

Now suppose he is asked the following question:
"If very many tests of 120 identical components, starting to function at \( t_0 = 0 \), were executed, would the event that 20 of these fail during \( I_1 \), 60 during \( I_2 \) and 40 during \( I_3 \) (so the \( u \) values are realized) represent one of the sets of average numbers of failures per interval that you expected?"

If his answer is 'no', then there is no reason to doubt the correctness of the method described in chapter 2. If his answer is 'yes', then the expert may have misunderstood the elicitation question, although that seems to be a simple one. Out of 100 components he felt sure that, on the average, 20 would fail during \( I_1 \), so this would imply that, on the average,

\( \frac{(20/100)*120}{120} = \frac{24}{120} \text{ components out of 120 would fail during } I_1. \)

Probably, in practical problems, some experts would answer this second question with 'yes', which indicates that the method of chapter 2 might not be correct to deal with their information.

(End of example 3.1.1.)

This example indicates a possible problem when one uses the method of chapter 2. If the experts completely understood the theory described in this report, and interpreted their own assessments correctly, then no problem would exist, and there would be no need to consider such a problem.

To solve this problem, another interpretation of the results of elicitation is necessary. As a matter of fact, the method that is discussed in this
chapter is completely different to the method of chapter 2, both theoretically and in interpretation.

Situations such as example 3.1.1 are not unlikely in practice. This chapter uses a density interpretation for the results of the elicitation process. If there were no problems in applying elicitation techniques in practice, this interpretation would only be valid for assessments resulting after the question:

"Give lower and upper bounds $l_i$ and $u_i$, for $i=1,\ldots,m$, such that all sets $\{x_1,\ldots,x_m\}$, with $l_i \leq x_i \leq u_i$, are possible averages of results (with $x_i$ representing the number of components that fail during $I_i$) in many tests where $\sum_{j=1}^{m} x_j$ components start functioning at time $t_0=0$.

It is obvious that the interpretation is in terms of ratios of average numbers of failing components per interval (Walley [11, section 4.6]).

Elicitation using this last question seems to be impossible in practice. The elicitation process should be started by the question of section 2.1, followed by a question like the one in example 3.1.1. If it can be concluded that the expert did not interpret the primary question correctly for the method of chapter 2, then he does not really accept the meaning of the number $n$ used for elicitation.

The alternative method in this chapter does not explicitly use $n$, although it is useful for an expert to suggest such a value (for reference). As stated above, the interpretation of the bounds $l_i$ and $u_i$ that is adopted in this chapter is:

All sets $\{x_j,\ldots,x_m\}$, with $l_i \leq x_j \leq u_i$, are possible average results of many tests where $\sum_{j=1}^{m} x_j$ components start functioning at time $t_0=0$.

This interpretation is the same as that of imprecise densities (Coolen [2]).

The starting point of this theory is again a set of values $l_i$ and $u_i$, provided by an expert but now with the above interpretation. Again the time-axis is partitioned as in section 2.1, and it is assumed that $0 \leq l_i \leq u_i$ for all $i=1,\ldots,m$, and $\sum_{j=1}^{m} u_j < \infty$. 

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Definition 3.1.2

The lower and upper probabilities of \( X \in I_i \) are defined (for \( i = 1, \ldots, m \)) by

\[
P_i = P(X \in I_i) := \left( 1 + \frac{\sum_{j \neq i} u_j - l_i}{l_i} \right)^{-1} = \frac{l_i}{\sum_{j \neq i} u_j + l_i}
\]

and

\[
\bar{P}_i = \overline{P}(X \in I_i) := \left( 1 + \frac{\sum_{j \neq i} l_j - u_i}{u_i} \right)^{-1} = \frac{u_i}{\sum_{j \neq i} l_j + u_i},
\]

respectively.

(If \( l_i = 0 \) and \( \forall j \neq i : u_j = 0 \), then define \( P_i := 0 \). If \( u_i = 0 \) and \( \forall j \neq i : l_j = 0 \), then define \( \bar{P}_i := 0 \). These situations are not discussed any further in this report.)

Walley [11, section 4.6] gives relations from which these definitions follow (see Coolen [2]). Example 3.1.3 is added to enable the interpretation of definition 3.1.2.

Example 3.1.3

The data of example 3.1.1 are used again. An expert has assessed (for \( m = 3 \), \( n \) does not play an explicit role anymore) \( l_1 = 20, u_1 = 20, l_2 = 40, u_2 = 60, l_3 = 20 \) and \( u_3 = 40 \). According to definition 3.1.2: \( P_1 = \frac{20}{120}, \bar{P}_1 = \frac{20}{80}, P_2 = \frac{40}{100}, \bar{P}_2 = \frac{60}{100}, P_3 = \frac{20}{100} \) and \( \bar{P}_3 = \frac{40}{100} \).

The lower probability \( P_1 = \frac{20}{120} = \frac{1}{6} \) corresponds to the average result \( \{20, 60, 40\} \), whereas the upper probability \( \bar{P}_1 = \frac{20}{80} = \frac{1}{4} \) corresponds to \( \{20, 40, 20\} \). Both these average results are possible according to the expert.

Remark that the resulting probability for the event \( X \in I_1 \) is not precise anymore.

{End of example 3.1.3.}

Theorem 3.1.5 states sufficient conditions on \( l_i \) and \( u_i \) for coherence of the corresponding imprecise probabilities (according to definition 3.1.2). To prove this theorem, the following lemma is needed.

Lemma 3.1.4

\[
\forall i = 1, \ldots, m : u_i + \sum_{j \neq i} l_j \leq l_i + \sum_{j \neq i} u_j \Rightarrow \max_i \left\{ u_i + \sum_{j \neq i} l_j \right\} \leq \min_i \left\{ l_i + \sum_{j \neq i} u_j \right\}
\]

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Proof

Let $i_0$ be such that $u_{i_0} + \sum_{j \neq i_0} l_j = \max \{u_{i_1} + \sum_{j \neq i_1} l_j\}$ and $i_1$ such that $l_{i_1} + \sum_{j \neq i_1} u_j = \min \{l_{i_1} + \sum_{j \neq i_1} u_j\}$. Suppose $i_0 \neq i_1$, then

$u_{i_0} + \sum_{j \neq i_0} l_j = u_{i_1} + l_{i_1} + \sum_{j \neq i_0} l_j \leq l_{i_1} + u_{i_1} + \sum_{j \neq i_1} u_j = l_{i_1} + \sum_{j \neq i_1} u_j$.

If $i_0 = i_1$, then the given relation immediately completes the proof. \(\Box\)

Theorem 3.1.5

$0 \leq l_{i_1} \leq u_{i_1}$ and $u_{i_1} + \sum_{j \neq i_1} l_j \leq l_{i_1} + \sum_{j \neq i_1} u_j$ for all $i$.

$\Rightarrow$

$0 \leq p_{i_1} = \bar{p}_{i_1} \leq 1$ and $\bar{p}_{i_1} + \sum_{j \neq i_1} p_j \leq 1 \leq p_{i_1} + \sum_{j \neq i_1} \bar{p}_j$ for all $i$.

Proof

The relations $0 \leq p_{i_1}$ and $\bar{p}_{i_1} \leq 1$ are trivial, while $p_{i_1} \leq \bar{p}_{i_1}$ follows from definition 3.1.2, together with $0 \leq l_{i_1} \leq u_{i_1}$ and the following argument (this is used several times in this chapter)

(D1) For $x \geq 0$ and $y \geq 0$ ($x = y = 0$ excluded) the function $\frac{x}{x+y}$ is strictly increasing as function of $x$, and strictly decreasing as function of $y$.

Further,

$$1 = \frac{\sum_{j \neq i_1} u_j + l_{i_1}}{\sum_{j \neq i_1} u_j + l_{i_1}} = p_{i_1} + \frac{\sum_{j \neq i_1} u_j}{\sum_{j \neq i_1} u_j + l_{i_1}} = p_{i_1} + \sum_{j \neq i_1} \left\{ \frac{u_j}{\sum_{s \neq j} l_s + u_j} \right\}$$

$$p_{i_1} + \sum_{j \neq i_1} \left\{ \frac{u_j}{\sum_{s \neq j} l_s + u_j} \right\} = p_{i_1} + \sum_{j \neq i_1} \bar{p}_j,$$

where lemma 3.1.4 is used for the relation $\sum_{s \neq j} l_s + u_j \leq \sum_{j \neq i_1} l_s + u_j$.

The relation $\bar{p}_{i_1} + \sum_{j \neq i_1} p_j \leq 1$ is proved analogously. \(\Box\)

If theorem 3.1.5 is compared to theorem 2.1.2, an obvious difference is that in theorem 2.1.2 an 'if and only if'-relation is proved.
Theorem 3.1.6 states that the sufficient conditions of theorem 3.1.5 are not necessary for coherent imprecise probabilities.

Theorem 3.1.6

\[ 0 \leq \overline{p}_i \leq \overline{p}_i \leq 1 \quad \text{and} \quad \overline{p}_i + \sum_{j \neq i} \overline{p}_j \leq 1 \leq \overline{p}_i + \sum_{j \neq i} \overline{p}_j \quad \text{for all } i \]

does not imply that

\[ 0 \leq \bar{\lambda}_i \leq u_i \quad \text{and} \quad \bar{\lambda}_i + \sum_{j \neq i} \bar{\lambda}_j \leq \bar{\lambda}_i + \sum_{j \neq i} u_j \quad \text{for all } i. \]

Proof

Let \( m = 4 \), and \( \bar{\lambda}_1 = 20 \), \( u_1 = 10 \), \( \lambda_2 = 10 \) and \( u_2 = 40 \) for \( i = 2, 3, 4 \). Definition 3.1.2 leads to \( \overline{p}_1 = 1/7 \), \( \overline{p}_2 = 1/4 \), \( \overline{p}_3 = 1/10 \) and \( \overline{p}_4 = 1/2 \) for \( i = 2, 3, 4 \). These imprecise probabilities satisfy the coherence conditions, although \( \bar{\lambda}_1 > u_1 \).

In practice, it is reasonable to reject an assessment with \( \bar{\lambda}_1 > u_1 \), as this may be an indication that the expert did not answer the questions seriously, or has made a mistake.

Suppose that a decision maker receives results of an elicitation process, while it is not clear whether these should be interpreted according to the methods of chapter 2 or 3. Then the result of theorem 3.1.7 is important, for it implies that a careful decision maker ought to use the method of chapter 3.

Theorem 3.1.7

Suppose values \( \bar{\lambda}_i \) and \( u_i \) (\( i = 1, \ldots, m \)) are given, with \( 0 \leq \bar{\lambda}_i \leq u_i \), but it is unknown whether these relate to the method of chapter 2 (no value of \( n \) is known) or chapter 3. Let \( \overline{p}_i^2 \) and \( \overline{p}_i^2 \) be the corresponding imprecise probabilities according to definition 2.1.1 (with \( n \) not specified), and \( \overline{p}_i^3 \) and \( \overline{p}_i^3 \) according to definition 3.1.2.

Then \( \overline{p}_i^3 \leq \overline{p}_i^2 \) and \( \overline{p}_i^3 \leq \overline{p}_i^2 \).

(This implies that the method of chapter 3 does not lead to less imprecision than the method of chapter 2.)
Proof

Although \( n \) is not specified, it is known that, if the method of chapter 2 was originally used, \( u_i + \sum_{j \notin \mathcal{I}_i} l_j \leq n \leq \sum_{j \notin \mathcal{I}_i} u_j \) for all \( i \).

Therefore, \( p_1^3 = \frac{l_1}{\sum_{j \notin \mathcal{I}_i} u_j + l_1} \leq \frac{l_1}{n} = p_2^2 \) and \( p_1^3 = \frac{u_1}{\sum_{j \notin \mathcal{I}_i} l_j + u_1} \geq \frac{u_1}{n} = p_2^2 \).

It is obvious that the use of this interpretation (definition 3.1.2) does not allow any \( l \) or \( u \) value to be unspecified (compare section 2.1). Because \( n \), the total number of components, is not used here, it is, in case of incoherence, also impossible to derive better definitions for some \( l \) and \( u \) values to achieve coherence (compare definition 2.1.3).

Next, as in chapter 2, the combination of opinions of several experts is discussed. There is no analogous result to theorem 2.2.3.

3.2 Combination of Opinions

In this section three methods of combining expert opinions are discussed. The methods are analogous to the methods presented in section 2.2. Again it is assumed that vectors of \( l \) and \( u \) values for each expert \( k \) (\( k=1, \ldots, K \)) are the results of an elicitation procedure (with interpretation as discussed in section 3.1), denoted by \( L_k := (l_{k,1}, \ldots, l_{k,m}) \) and \( U_k := (u_{k,1}, \ldots, u_{k,m}) \). It is further assumed that all experts have used identical partitions of the time-axis (intervals \( I_1, \ldots, I_m \)). It is further assumed that for each expert the values \( l_{k,i} \) and \( u_{k,i} \) satisfy the conditions of theorem 3.1.5, so

\[
0 \leq l_{k,i} \leq u_{k,i} \quad \text{and} \quad u_{k,i} + \sum_{j \notin \mathcal{I}_i} l_{k,j} \leq l_{k,i} + \sum_{j \notin \mathcal{I}_i} u_{k,j} \quad \text{for all} \quad i.
\]

For each expert, \( k \), the lower and upper probabilities resulting from \( l_{k,i} \) and \( u_{k,i} \) by definition 3.1.2, are denoted by

\[
P_k(i) = \frac{l_{k,i}}{\sum_{j \notin \mathcal{I}_i} u_{k,j} + l_{k,i}} \quad \text{and} \quad \overline{p}_k(i) = \frac{u_{k,i}}{\sum_{j \notin \mathcal{I}_i} l_{k,j} + u_{k,i}}
\]

\( \{P_k = (P_{k,1}, \ldots, P_{k,m}) \) and \( \overline{P}_k = (\overline{P}_{k,1}, \ldots, \overline{P}_{k,m}) \}. \)
3.2.1 Combination by the Unanimity Rule

Definition 3.2.1

The unanimity rule for combination of the opinions of R experts leads to combined lower and upper probabilities for the event $X \in I_i$ (i=1, ..., m):

$$p_i := \min\{p_{k,i} | k=1, ..., K\} \quad \text{and} \quad \bar{p}_i := \max\{\bar{p}_{k,i} | k=1, ..., K\},$$

respectively.

(In this subsection, these imprecise probabilities are also denoted by vectors $\underline{p} := (p_1, ..., p_m)$ and $\bar{p} := (\bar{p}_1, ..., \bar{p}_m).$)

This definition is identical to definition 2.2.1, although the individual imprecise probabilities are different.

An important advantage of this method of combination is (compare theorem 2.2.4)

Theorem 3.2.2

If, for each expert k, the assessments $L_k$ and $U_k$ satisfy the conditions of theorem 3.1.5, then the imprecise probabilities $\underline{p}$ and $\bar{p}$, after combination by the unanimity rule, are coherent.

Proof

By theorem 3.1.5, $0 \leq p_{k,i} \leq \bar{p}_{k,i} \leq 1$ and $\bar{p}_{k,i} + \sum_{j \not= i} p_{k,j} \leq 1 \leq \underline{p}_{k,i} + \sum_{j \not= i} \bar{p}_{k,j}$ for all $k=1, ..., K$ and $i=1, ..., m$.

The relation $0 \leq \sum_{i=1}^m \underline{p}_i \leq 1$ follows from definition 3.2.1. Further,

$$\bar{p}_i = \max\{\bar{p}_{k,i} | k \in \mathcal{K}\},$$

for some $k \in \{1, ..., K\}$, so $\bar{p}_i + \sum_{j \not= i} \underline{p}_{k,j} \leq \bar{p}_{k,i} + \sum_{j \not= i} \bar{p}_{k,j} \leq 1$.

The relation $1 \leq \bar{p}_i + \sum_{j \not= i} \underline{p}_{k,j}$ is proved analogously.

Another possible method of combination, intuitively based on the same idea, is to combine the l and u values of all experts first, and to calculate the imprecise probabilities according to these combined l and u values. After this method is defined (3.2.3) theorem 3.2.4 states that this also leads to coherent imprecise probabilities.

Theorem 3.2.5 shows that definition 3.2.1 never leads to more imprecision than definition 3.2.3, and an example (3.2.6) is given where definition 3.2.1 leads to less imprecision. Also an example is given to show that illogical imprecise probabilities might result from applying definition 3.2.3.
The problems that arise here are based on the fact that the assessed \( l \) and \( u \) values per expert do not have to relate to a value of \( n \) that is equal for all experts.

**Definition 3.2.3**

Let \( l_i := \min \{ l_{k,i} \mid k=1,\ldots,K \} \) and \( u_i := \max \{ u_{k,i} \mid k=1,\ldots,K \} \), also denoted (in this subsection) by vectors \( L \) and \( U \). Combined imprecise probabilities for the event \( x \in E \) are

\[
p'_i := \frac{l_i}{\sum_{j \not\in I_i} u_j + l_i} \quad \text{and} \quad p''_i := \frac{u_i}{\sum_{j \not\in I_i} u_j + u_i}.
\]

(In this subsection, these imprecise probabilities are also denoted by vectors \( P' \) and \( P'' \).)

(Compare definition 2.2.2.)

**Theorem 3.2.4**

If, for each expert, the assessments \( L_k \) and \( U_k \) satisfy the conditions of theorem 3.1.5, then the imprecise probabilities \( P' \) and \( P'' \) are coherent.

**Proof**

It is given that \( 0 \leq l_{k,i} \leq u_{k,i} \) and \( u_{k,i} + \sum_{j \not\in I_i} l_{k,j} \leq l_{k,i} + \sum_{j \not\in I_i} u_{k,j} \) for all \( i \) and \( k \), and the following relations must be proved: \( 0 \leq p'_i \leq p''_i \leq 1 \) and

\[
p'_i + \sum_{j \not\in I_i} p'_j \leq 1 \leq p''_i + \sum_{j \not\in I_i} p''_j \quad \text{for all} \quad i.
\]

The first relation follows from definition 3.2.3 (again \( 0 \leq l_{k,i} \leq u_{k,i} \) and the argument D1 of the proof of theorem 3.1.5 are used). Further,

\[
p'_i + \sum_{j \not\in I_i} p'_j = \frac{l_i}{\sum_{j \not\in I_i} u_j + l_i} + \sum_{j \not\in I_i} \left( \frac{u_j}{\sum_{k \not\in I_i} l_k + u_j} \right)
\]

\[
= \frac{\min\{l_{k,i}\}}{\sum_{j \not\in I_i} \max\{u_{k,j}\} + \min\{l_{k,i}\}} + \sum_{j \not\in I_i} \left( \frac{\max\{u_{k,j}\}}{\sum_{s \not\in I_i} \min\{l_{k,s}\} + \max\{u_{k,j}\}} \right)
\]

\[
\leq \sum_{j \not\in I_i} \frac{\max\{u_{k,j}\} + \min\{l_{k,i}\}}{\sum_{j \not\in I_i} \max\{u_{k,j}\} + \min\{l_{k,i}\}}
\]

\[
= \frac{\sum_{j \not\in I_i} \max\{u_{k,j}\} + \min\{l_{k,i}\}}{\sum_{j \not\in I_i} \max\{u_{k,j}\} + \min\{l_{k,i}\}} = 1.
\]
Here the following relation is used, for \( j \neq i \):

\[
\sum_{s \neq j} \min(l_{k,s}) + \max(u_{k,j}) \leq \min(l_{k,i}) + \sum_{s \neq i} \max(u_{k,s}).
\]

The relation \( \bar{p}_i' + \sum_{j \neq i} \bar{p}_j' \leq 1 \) can be proved analogously.

Theorem 3.2.5

\[ \forall i=1,..,m: \quad p_i' \leq p_i \quad \text{and} \quad \bar{p}_i \leq \bar{p}_i'. \]

Proof

\[
p_i' = \frac{l_i}{\sum_j u_j + l_i} = \frac{\min(l_{k,i})}{\sum_{j \neq i} \max(u_{k,j}) + \min(l_{k,i})} \leq \frac{l_{k,i}}{\sum_{j \neq i} u_{k,j} + l_{k,i}} = p_{k,i}
\]

for all \( k \) (again argument \( D1 \) of the proof of theorem 3.1.5 is used).

Hence \( p_i' \leq \min(p_{k,i}) = p_i \).

The second relation can be proved analogously.

Example 3.2.6

Again the data of example 2.2.5 are used. So \( m=3 \) and \( K=2 \) (\( n \) does not play a role anymore), and the experts have assessed \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(40,20,10) \) and \( U_2=(60,30,30) \).

The corresponding imprecise probabilities are \( P_1'=\left(\frac{1}{11},\frac{3}{10},\frac{2}{11}\right) \), \( P_1'\bar{=}\left(\frac{3}{8},\frac{2}{3},\frac{1}{2}\right) \), \( P_2'\left(\frac{2}{5},\frac{2}{11},\frac{1}{10}\right) \) and \( P_2'\left(\frac{2}{3},\frac{3}{8},\frac{1}{3}\right) \).

According to definition 3.2.1, the combined imprecise probabilities are \( P=(1/11,2/11,1/10) \) and \( \bar{P}= (2/3,2/3,1/2) \).

Following definition 3.2.3, the combined \( l \) and \( u \) values are \( L=(10,20,10) \) and \( U=(60,60,40) \), leading to imprecise probabilities \( P'=(1/11,1/6,1/13) \) and \( \bar{P}'=(2/3,3/4,4/7) \).

{End of example 3.2.6.}

These results lead to the conclusion that definition 3.2.1 should be adopted for combination by the unanimity rule. This conclusion is confirmed by the following example.

Example 3.2.7

Let \( m=3 \) and \( K=2 \), and suppose that the experts have assessed \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(1,3,2) \) and \( U_2=(3,6,4) \). Here it is obvious that \( n \) does not play a role, and it is easily verified that these assessments imply coherence (theorem 3.1.5).
The corresponding imprecise probabilities according to expert 1 are $P_1 = (1/11, 3/10, 2/11)$ and $\overline{P}_1 = (3/8, 2/3, 1/2)$, and the same values are derived for expert 2, $P_2 = P_1$ and $\overline{P}_2 = \overline{P}_1$, so definition 3.2.1 leads to $P = (1/11, 3/10, 2/11)$ and $\overline{P} = (3/8, 2/3, 1/2)$.

Following definition 3.2.3, the combined $l$ and $u$ values are $L = (1, 3, 2)$ and $U = (30, 60, 40)$, leading to imprecise probabilities $P' = (1/101, 3/73, 2/92)$ and $\overline{P}' = (30/35, 60/63, 40/44)$.

Under the density interpretation, both experts have expressed the same information, and the first method leads to logical results. The second method leads to almost vacuous probabilities, as the lower probabilities are close to 0, and the upper probabilities close to 1. This indeed shows that it is not sensible to use definition 3.2.3.

{End of example 3.2.7.}

In chapter 4 other methods of elicitation are discussed, and there the opinions of several experts can only be combined by using imprecise probabilities per expert. So the above conclusions fit perfectly in the general concept of combination.

3.2.2 Combination by the Conjunction Rule

Definition 3.2.8

The conjunction rule for combination of the opinions of $K$ experts leads to combined lower and upper probabilities for the event $X \in I_i$ ($i=1, \ldots, m$):

$p_i := \max_p(p_{k, i} \mid k=1, \ldots, K)$ and $\overline{p}_i := \min_{\overline{p}}(\overline{p}_{k, i} \mid k=1, \ldots, K)$, respectively.

(In this subsection, these imprecise probabilities are also denoted by vectors $P := (p_1, \ldots, p_m)$ and $\overline{P} := (\overline{p}_1, \ldots, \overline{p}_m)$.)

(Compare definition 3.2.1.)

This definition is identical to definition 2.2.6 (only different individual imprecise probabilities). Here the same problems, concerning coherence, arise as in subsection 2.2.2. The following example illustrates such problems. As before when using this method of combination, coherence of the resulting imprecise probabilities should always be checked.
Example 3.2.9

Again the data of example 2.2.5 are used. So $m=3$, $K=2$, $L_1=(10,30,20)$, $U_1=(30,60,40)$, $L_2=(40,20,10)$ and $U_2=(60,30,30)$. The corresponding imprecise probabilities are $\underline{P}_1=(1/11,3/10,2/11)$, $\bar{P}_1=(3/8,2/3,1/2)$, $\underline{P}_2=(2/5,2/11,1/10)$ and $\bar{P}_2=(2/3,3/8,1/3)$. For both experts these imprecise probabilities are coherent.

According to definition 3.2.8, the combined imprecise probabilities are $\underline{P}=(2/5,3/10,2/11)$ and $\bar{P}=(3/8,3/8,1/3)$, so $\underline{p}_1=2/5>3/8=\bar{p}_1$. This is in conflict with the coherence conditions. (End of example 3.2.9.)

An analogous result to theorem 2.2.10 holds again. Example 3.2.9 proves that the necessary conditions for coherence, according to this theorem, are again not sufficient for coherence.

As in subsection 3.2.1, one might consider to combine the $l$ and $u$ values per expert first, followed by calculation of the according imprecise probabilities.

Definition 3.2.10

Let $l_{i1}:=\max(l_{k,i} | k=1, \ldots, K)$ and $u_{i1}:=\min(u_{k,i} | k=1, \ldots, K)$, also denoted (in this subsection) by vectors $L$ and $U$. Then combined imprecise probabilities for the event $X \in L_1$ are $\underline{P}_1:=\frac{l_{i1}}{\sum j \neq i u_j + l_{i1}}$ and $\bar{P}_1:=\frac{u_{i1}}{\sum j \neq i l_j + u_{i1}}$.

(In this subsection, these imprecise probabilities are also denoted by vectors $\underline{P}'$ and $\bar{P}'$.)

(Compare definition 3.2.3.)

If definition 3.2.10 is used, coherence of the resulting imprecise probabilities should also be checked. Using theorem 3.1.5, sufficient conditions for coherence of $\underline{P}'$ and $\bar{P}'$ are $0 \leq l_{i1} \leq u_{i1}$ and $u_{i1} + \sum_{j \neq i} l_j \leq l_{i1} + \sum_{j \neq i} u_j$ for all $i$, that can be written in terms of $l_{k,i}$ and $u_{k,i}$. Coherence of individual assessments does not imply coherence of the combined imprecise probabilities, as is shown by example 3.2.11.
Example 3.2.11
For the situation of example 3.2.9, the use of definition 3.2.10 leads to $L = (40, 30, 20)$ and $U = (30, 30, 30)$, so $\bar{F}' = (2/5, 1/3, 1/4)$ and $\bar{F}' = (3/8, 1/3, 3/10)$. Here $p_1 = 2/5 > 3/8 = p_1'$, implying incoherence.

For combination by the unanimity rule (subsection 3.2.1) two possible definitions were suggested, and it was shown (theorem 3.2.5) that definition 3.2.1 (that relates to definition 3.2.8) should be preferred as it leads to no greater imprecision than definition 3.2.3 (that relates to definition 3.2.10). Theorem 3.2.12 states that the opposite result holds for combination according to the conjunction rule.

Theorem 3.2.12
\[ v_{i=1, \ldots, m}: p_i \preceq p_i' \text{ and } \bar{p}_i \preceq \bar{p}_i'. \]

Proof
By argument D1,
\[ p_i = \max \left\{ \frac{1_{k,i}}{\sum_{j \notin i} u_{k,j} + 1_{k,i}} \right\} \leq \frac{\max \{1_{k,i}\}}{\sum_{j \notin i} \min \{u_{k,j}\} + \max \{1_{k,i}\}} \]
\[ \bar{p}_i = \max \left\{ \frac{1_{k,i}}{\sum_{j \notin i} u_{k,j} + 1_{k,i}} \right\} \leq \frac{\max \{1_{k,i}\}}{\sum_{j \notin i} \min \{u_{k,j}\} + \max \{1_{k,i}\}} = p_i'. \]

The second relation is proved in the same way.

Intuitively, the above theorem would lead to the choice of definition 3.2.10 for the conjunction rule of combination, within this chapter. This would be a problem when the information must be combined with information resulting from other elicitation processes (as remarked at the end of subsection 3.2.1). However, as in the previous subsection, the second method can lead to illogical results. Example 3.2.13 shows a situation where the intersections of the individual imprecise probabilities are non-empty (so there is some agreement between the experts) and definition 3.2.8 leads to coherent combined imprecise probabilities, but combination according to definition 3.2.10 leads to incoherence (one could say that definition 3.2.10 'deletes too much imprecision').

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Example 3.2.13
Let \( m=4 \) and \( K=2 \), and suppose the two experts have assessed \( \underline{L}_1=(20,10,20,10) \), \( \overline{U}_1=(30,40,45,40) \), \( \underline{L}_2=(0,10,10,10) \) and \( \overline{U}_2=(5,45,45,40) \), leading to
\[
\frac{\underline{L}_1}{\overline{U}_1} = \frac{20}{145}, \frac{10}{125}, \frac{20}{130}, \frac{10}{125},
\]
\[
\frac{\underline{L}_2}{\overline{U}_2} = \frac{0}{100}, \frac{10}{100}, \frac{10}{100}, \frac{10}{105}.
\]
For expert 1, these values are coherent.

Following definition 3.2.8, \( \underline{P}_1=(20/145, 10/125, 20/130, 10/125) \) and \( \overline{P}_1=(5/35, 40/90, 45/85, 40/90) \), which are coherent (note that \( E_1=20/145 < \overline{P}_1 \)).

However, according to definition 3.2.10 these assessments lead to
\( \underline{L}=(20,10,20,10) \) and \( \overline{U}=(5,40,45,40) \), so \( \underline{P}'=(20/145, 10/100, 20/105, 10/100) \) and \( \overline{P}'=(5/45, 40/90, 45/85, 40/90) \). These imprecise probabilities are incoherent, as \( \underline{P}_1=20/145 > 5/45=\overline{P}_1' \).

(End of example 3.2.13.)

Problems as presented in example 3.2.13 cannot arise the other way around, as a result of theorem 3.2.12.

Definition 3.2.8 is preferred to 3.2.10, because of the more logical interpretation and because of problems of the kind indicated above.

3.2.3 Combination by the Weighted Average Method

According to definition 2.2.12 weight \( w_k \) is assigned to expert \( k \).

Definition 3.2.14

The weighted average method for combination of the opinions of \( K \) experts leads to combined lower and upper probabilities for the event \( X \in I_i \)
\[
(i=1, \ldots, m): \quad P_i := \sum_{k=1}^{K} w_k P_{k,i} \quad \text{and} \quad \overline{P}_i := \sum_{k=1}^{K} w_k \overline{P}_{k,i}.
\]

(In this subsection, these imprecise probabilities are also denoted by vectors \( \underline{P}=(P_1, \ldots, P_m) \) and \( \overline{P}=(\overline{P}_1, \ldots, \overline{P}_m) \).)

This definition is identical to definition 2.2.13 (the \( P_{k,i} \) and \( \overline{P}_{k,i} \) are different). Theorem 3.2.15 is an important result, as it implies that coherence of the individual imprecise probabilities leads to coherence after
combination. This theorem is almost identical to theorem 2.2.14, except for the relation between individual assessments \( L_k, U_k \) and \( P_k, \bar{P}_k \).

**Theorem 3.2.15**

If, for each expert \( k \), \( P_k \) and \( \bar{P}_k \) are coherent, then the imprecise probabilities \( P \) and \( \bar{P} \), after combination according to definition 3.2.14, are also coherent.

**Proof**

The proof of this theorem is identical to the proof of theorem 2.2.14. \( \Box \)

By theorem 3.1.5 coherence of \( P \) and \( \bar{P} \) is assured if for all \( k \): \( 0 \leq l_{k,i} \leq u_{k,i} \) and \( u_{k,i} + \sum_{j \neq i} l_{k,j} \leq l_{k,i} + \sum_{j \neq i} u_{k,j} \) for all \( i \).

Definition 3.2.16 proposes a different combination method using weighted averages.

**Definition 3.2.16**

Let \( l_i := \sum_{k=1}^{K} w_k l_{k,i} \) and \( u_i := \sum_{k=1}^{K} w_k u_{k,i} \) also denoted by vectors \( L \) and \( U \) (in this subsection). Then combined imprecise probabilities for the event \( X \in \Omega_i \) are \( P'_i := \frac{l_i}{\sum_{j \neq i} l_i + l_i} \) and \( \bar{P}'_i := \frac{u_i}{\sum_{j \neq i} l_i + u_i} \). (In this subsection, these imprecise probabilities are also denoted by vectors \( P' \) and \( \bar{P}' \).)

**Theorem 3.2.17**

If, for each expert \( k \), \( 0 \leq l_{k,i} \leq u_{k,i} \) and \( u_{k,i} + \sum_{j \neq i} l_{k,j} \leq l_{k,i} + \sum_{j \neq i} u_{k,j} \) for all \( i \) (so, by theorem 3.1.5, \( P_k \) and \( \bar{P}_k \) are coherent), then the imprecise probabilities \( P' \) and \( \bar{P}' \) are coherent.

**Proof**

\( 0 \leq l_{k,i} \leq u_{k,i} \) and \( u_{k,i} + \sum_{j \neq i} l_{k,j} \leq l_{k,i} + \sum_{j \neq i} u_{k,j} \), for all \( k \) and \( i \), implies \( 0 \leq l_i \leq u_i \) and \( u_i + \sum_{j \neq i} l_j \leq l_i + \sum_{j \neq i} u_j \) for all \( i \). Theorem 3.1.5 completes the proof. \( \Box \)
There is not one of the definitions 3.2.14 and 3.2.16 that always leads to no greater imprecision as the other one. This is shown by

**Example 3.2.18**

Again the data of example 2.2.5 are used. So \( m=3 \), \( K=2 \), \( L_1=(10,30,20) \), \( U_1=(30,60,40) \), \( L_2=(40,20,10) \), \( U_2=(60,30,30) \), \( P_1=\left(\frac{1}{11},\frac{3}{10},\frac{2}{11}\right) \), \( P_2=\left(\frac{2}{5},\frac{2}{11},\frac{1}{10}\right) \) and \( P_2=\left(\frac{2}{3},\frac{3}{8},\frac{1}{3}\right) \).

Suppose the weights for both experts are equal, so \( w_1=w_2=\frac{1}{2} \). Then definition 3.2.14 leads to \( P=\left(\frac{27}{110},\frac{53}{220},\frac{31}{220}\right) \) and \( \Delta=\overline{P}-\underline{P}=\left(0.2754,0.2799,0.2758\right) \).

According to definition 3.2.16, the following results are derived:

\[ L=(25,25,15), \quad U=(45,45,35), \quad P'=\left(\frac{5}{21},\frac{5}{21},\frac{3}{21}\right) \quad \text{and} \quad \overline{P}'=\left(\frac{9}{17},\frac{9}{17},\frac{7}{17}\right), \]

with imprecision \( \Delta'_{1}=\overline{P}'-\underline{P}'=\left(0.2913,0.2913,0.2689\right) \).

In this example, \( \Delta_1<\Delta_1' \) and \( \Delta_2<\Delta_2' \), while \( \Delta_3>\Delta_3' \).

(End of example 3.2.18.)

There is no reason to choose any one of these definitions, except that combination with information from other elicitation processes is only possible under definition 3.2.14. Therefore it is suggested that definition 3.2.14 is preferable to 3.2.16. This is in agreement with the results for the unanimity and conjunction rules.

**3.3 Some Generalizations**

Some generalizations of the above results are possible, with regard to the partition of the time-axis. The possible solutions are analogous to those presented in section 2.3, because only the individual imprecise probabilities are used. From the above discussion, and chapter 4, it will be clear that one ought to transfer information into imprecise probabilities before combination takes place, whether or not all experts have presented their information using the same partition \( I_1, \ldots, I_m \).
4. Other Methods for Elicitation and Combination of Expert Opinions

In this chapter some alternative methods for elicitation are discussed, together with methods of combination of information from several experts and possibly obtained from different elicitation procedures. As the methods of chapters 2 and 3 seem to be most promising in eliciting knowledge with regard to lifetimes of components, no detailed descriptions of the methods in this chapter are given. Generally, useful references on elicitation and combination (within the standard Bayesian framework) are Cooke [1], O'Hagan [7], and Spetzler and Staël von Holstein [8]. Walley [10, 11, 12] has also considered these problems, within the concept of imprecise probabilities.

4.1 Some Alternative Elicitation Methods

It is important to emphasize the difference between elicitation and assessment of probabilities: "Elicitation is the process by which beliefs (of experts) are measured, through explicit judgements and choices. Assessment is the process by which probabilities are constructed (in the context of this report, by the decision maker) from the available evidence. Both processes result in probability models, but the aim in elicitation is to model pre-existing beliefs, whereas the aim in assessment is to formulate rational beliefs" (Walley, [11, ch.4]). Elicitation and assessment are processes that can influence each other. Problems or ideas that arise when assessing imprecise probabilities (e.g. in relation to some chosen model) can lead to another step in the elicitation process. In practice, both processes must be carefully studied and executed.

Next we discuss some other methods for elicitation that have been proposed in literature.

A first method, that also seems to be useful in practice, relates to the most common forms of judgements in every-day life, being either classificatory (that an event is probable) or comparative (that one event is at least as probable as another. These judgements are easy to understand and to elicit. According to Walley [11], classifications of the probable events often generate highly imprecise models. Comparative probability orderings can produce much greater precision, especially when the orderings are
complete, or when the events of interest are compared with standard events whose probabilities are precise and known.

Judgements of these kinds, expressed in ordinary language, are hard to translate into probability statements. For example, interpretation of a statement like 'event A is very probable' in terms of probability, for example $P(A)=0.8$, is necessarily based on strong assumptions. Such statements can also be translated into lower and upper probabilities. Of course, some assumptions are still necessary, but within this concept the arbitrariness of the choice of a single number has less influence, as only a lower or upper bound of an interval of possible values for $P(A)$ is related to the expression. Therefore, within the concept of imprecise probabilities this method seems to be applicable, whereas it can hardly be defended within the framework of precise probability.

This is an interesting research area for psychologists. Walley [12] gives an example of a scheme of translations for event A:

- A is extremely probable $\Rightarrow P(A)=0.98$
- A is highly probable $\Rightarrow P(A)=0.85$
- A is very probable $\Rightarrow P(A)=0.75$
- A is quite probable $\Rightarrow P(A)=0.6$
- A is probable (likely) $\Rightarrow P(A)=0.5$
- A is improbable (unlikely) $\Rightarrow P(A)=0.5$
- A is very unlikely $\Rightarrow P(A)=0.25$
- A is extremely unlikely $\Rightarrow P(A)=0.02$
- A has a good chance $\Rightarrow P(A)=0.4$ and $P(A)=0.85$

For analogous comparative judgements (using two events) see Walley [11, section 4.1].

Although this kind of elicitation seems to be simple for the experts, it can be a problem if they ask how terms like 'highly probable' should be interpreted. The previously suggested methods (ch. 2 and 3) do also not require more from the experts than to think of the lifetimes of the components, and gives them a simple standardized language in which to express their thoughts. The above method leads straightforwardly to lower and upper probabilities, which is important with regard to combination when several different elicitation methods are used.

For a second method (Walley [11, section 4.6]), experts must assess upper and lower quantiles for pre-specified probability values. For our purpose
this does not seem to be attractive, as this would require experts to express themselves by using a continuous time-axis, and this may be hard if one is not trained to do so. Further, this is somehow an inverse way to ask the elementary questions, that does not correspond to the way most people will think when considering lifetimes of components (here questions like 'give the smallest point of time for which you are certain that 20 out of 100 components have failed' must be answered).

Within the literature elicitation methods of this kind are often proposed, without imprecision (so question like 'at what point of time 5% of all components have failed' are asked). According to arguments that have been stated before in this report it should be clear that this is a very doubtful procedure in practice.

A third method of elicitation (Cooke [1, ch.8], Walley [11, ch.4]) seems to be the most logical if probability is interpreted as betting behaviour (Walley [11], after De Finetti [5]). Here experts are straightforwardly asked to place bets on certain events. Such a method can be useful if one is interested in a single event, but is not helpful for lifetimes (too many 'bets' would be needed).

All the methods discussed so far lead to imprecise probabilities for the random variable of interest. This is based on the idea that an expert with regard to the problem of interest may not be an expert with regard to mathematical models (distributions), and therefore, if such a model is assumed, it will be hard, if not impossible, to get sensible results by asking questions directly about this model, for example by asking (some properties of) prior distributions for parameters. In chapter 5, it will be shown that a member of any assumed parametric family of distributions can be chosen (optimally according to some criterion) so that it relates well to the results of the elicitation process.

However, in some practical situations probability distributions with interpretable parameters can serve well as models (for example, the parameter of an exponential distribution can reasonably well be interpreted). In such cases it might be possible to ask questions about this parameter, and the method presented in chapter 5 would not be necessary. Nevertheless, for most models this will not be possible (if there are two parameters, interpretation of these is already often a problem, especially
for frequently used lifetime distributions like the Weibull and lognormal), and then asking questions about the random variable (lifetime) itself is the best alternative. Even if a parameter has an interpretation, this may be difficult for the experts, as it may demand another interpretation of the situation than that which they are used to, and then also methods like those presented in chapters 2 and 3 should be prefered.

Which elicitation method is most useful depends on the nature of the problem and on the experts (if they are experienced with some way of expressing their believes, this should be used). All methods that appear to be promising lead to imprecise probabilities for events, and these imprecise probabilities should be used for combination of expert opinions. Once elicitation yields lower and upper probabilities for a certain event, it is no longer important how these values were elicited.

4.2 Combination of Expert Opinions

In chapters 2 and 3 three possible methods for the combination of opinions in the form of imprecise probabilities are proposed. Any elicitation procedure that results in imprecise probabilities per expert for certain events allows these probabilities to be combined. The decision maker has to combine the probabilities if he wishes to use all the information. Another situation in which combination is necessary occurs when the opinion of one expert is elicited by several methods. This situation can be treated as if these opinions came from different experts, although, of course, here it is certain that the differences are caused by the elicitation method and not difference in background or experience. This problem may allow the comparison of several elicitation methods.

The three methods of the previous chapters seem to be logical. It is important to note that both combination by the unanimity rule and by the conjunction rule cannot be applied in a standard Bayesian context, so, if imprecision is not allowed, combination must, one way or another, be based on weighting of the experts. Cooke [1] concludes not only that the basic problem is to assign scores to the expert, but stresses the fact that often two different reasons for scoring the experts exist, leading to different scores per expert.
As Cooke shows, there is a difference between assigning scores to each expert (based on data that come available), indicating how good his previews were related to the evidence, and assigning scores that must serve as weights, such that the final combination leads to optimal assessment by the decision maker. This is a very interesting topic for research, and generalization of these ideas and results to the concept of imprecise probabilities would be very useful. Cooke does not restrict to weighted averages as combination method, with weights assigned to the experts, but by reasons of interpretation these were chosen to be discussed in the previous chapters, and, at first sight, these seem to be the most logical combinations of weighted expert opinions.

Within the concept of imprecise probabilities, there are strong arguments (based on resulting imprecision if several opinions are known, given the situation and the experience or backgrounds of the experts) to use the unanimity rule or the conjunction rule for combination. Generally, if an elicitation procedure has resulted in individual imprecise probabilities, these methods can be applied according to definitions 2.2.1 and 2.2.6 (in chapter 3 the analogous definitions were chosen). If desired, one can define other combination rules, based on the same ideas. For example, one could delete experts whose opinions contain much imprecision (useful if the unanimity rule is used, to avoid that the opinions of some good experts disappear entirely in the final result), or delete experts who seem to be too confident about their knowledge (useful if the conjunction rule is used). Other intermediate methods could be defined accordingly, and the quality of a combination method will again depend on the situation. Some nice topics for future research come into the picture, as well for mathematicians as for psychologists. For the moment, we think that the combination methods of sections 2.2 and 3.2 can serve for practical problems concerning lifetimes of components. For example, a decision maker could use both the unanimity and the conjunction rules after elicitation, to get an indication of possible (dis)agreement between the consulted experts. If the experts do not disagree (see the subsections on the conjunction rule, 2.2.2 and 3.2.2), it is useful to regard the individual assessments before combination by the unanimity rule is applied, to avoid too much imprecision after combination. In the previous chapters already some arguments were given for the final choice of a combination method.
5. Assessment of Imprecise Prior Distributions

In this chapter the problem of choosing imprecise prior distributions, given combined expert information, is discussed. In the first section a general method is presented, and in the second section this method is applied using a model proposed by Coolen [3]. The third and fourth section can be regarded as appendices, discussing respectively the use of metrics within the method of this chapter and some aspects of the optimization process.

5.1 The Choice of Imprecise Prior Distributions

As in the previous chapters, one is interested in a random variable, say a lifetime variable $X \geq 0$. As before, suppose a partition $\{I_1, \ldots, I_m\}$ is given, with $I_i = [t_{i-1}, t_i)$ and $0 = t_0 < t_1 \ldots < t_{m-1} < t_m = \infty$. The decision maker has lower and upper probabilities $p_i = P(X \in I_i)$ and $\overline{p}_i = \overline{P}(X \in I_i)$, that are the results of eliciting the opinions of experts and then combining them (for this chapter, neither the method of elicitation nor the method of combination is relevant). These imprecise probabilities are assumed to be coherent, so for all $i$: $0 \leq p_i, \overline{p}_i \leq 1$ and $\overline{p}_i + \sum_{j \neq i} p_j \leq 1 \leq \overline{p}_i + \sum_{j \neq i} \overline{p}_j$.

At the points $t_i$, the corresponding imprecise cdf’s are $\underline{F}(0) = \overline{F}(0) = 0$,

\[
\frac{\underline{F}(t_i)}{\sum_{j=1}^{i} \overline{p}_j + \sum_{j=1}^{i} p_j} \quad \text{and} \quad \frac{\overline{F}(t_i)}{\sum_{j=1}^{i} \overline{p}_j + \sum_{j=1}^{i} p_j} \quad \text{for } i = 1, \ldots, m \quad \text{(so}
\]

\[
\underline{F}(\infty) = \overline{F}(\infty) = 1, \quad 0 \leq \underline{F}(t_i) \leq F(t_i) \leq 1, \quad \underline{F}(t_{i-1}) \leq \overline{F}(t_i) \quad \text{and} \quad \underline{F}(t_{i-1}) \leq \underline{F}(t_i) \quad \text{for all}
\]

\[i = 1, \ldots, m \}\}.

The goal of this chapter is to present a method for determining imprecise prior distributions, for a parameter of an assumed lifetime distribution, such that this corresponds to the expert opinions represented by the above calculated imprecise cdf’s in the points $t_i$. A parametric distribution is assumed for lifetime $X$, say with pdf $f(x | \theta)$, with parameter $\theta \in \Theta$, where the parameter space $\Theta$ has finite dimension. Suppose that a lower and an upper prior density, $\lambda(\theta | \tau)$ and $u(\theta | \tau)$ respectively, are assumed (see Coolen [2, 3]), depending on a finite dimensional (hyper-)parameter $\tau \in \mathbb{T}$ (here it is, for example, possible that $I$ depends on a parameter $\tau_i$ and $u$ on another...
parameter $\tau^*_2$, with $\tau = (\tau_1^*, \tau_2)$, $T$ is the hyperparameter space. It is supposed that $0 \leq l(\theta | \tau) \leq u(\theta | \tau) < \infty$ holds for all $\theta$ and $\tau$.

The corresponding lower and upper prior predictive densities are ($x \geq 0$)

$$l_x(x | \tau) = \int_0^x f(x | \theta) l(\theta | \tau) d\theta \quad \text{and} \quad u_x(x | \tau) = \int_0^x f(x | \theta) u(\theta | \tau) d\theta$$

respectively.

To these the following imprecise prior predictive cdf's correspond:

$$F_x(x | \tau) = \frac{\int_0^x l_x(w | \tau) dw}{\int_0^\infty l_x(w | \tau) dw + \int_0^x u_x(w | \tau) dw} \quad \text{and} \quad \overline{F}_x(x | \tau) = \frac{\int_0^x u_x(w | \tau) dw}{\int_0^\infty l_x(w | \tau) dw + \int_0^x u_x(w | \tau) dw}.$$

To determine imprecise prior densities such that the final model corresponds reasonably well to the expert opinions, the lower cdf $F_x$ should be compared to $F$, and the upper cdf $\overline{F}_x$ to $\overline{F}$ in the points $t_i$ ($i=1, \ldots, m-1$, as for $t_0$ and $t_m$ the values are all equal). Here the final goal is to determine $\tau \in \mathcal{T}$ such that $F_x(t_i | \tau)$ and $\overline{F}_x(t_i | \tau)$ have minimum distances to $F(t_i)$ and $\overline{F}(t_i)$ respectively.

It is necessary to introduce a criterion for these distances, such that the best value for the hyperparameter $\tau$ follows by some optimization procedure. The criteria are based on a metric $d$ on the space of upper and lower cdf's.

The final goal is to choose a value $\tau \in \mathcal{T}$, such that $F_x$ is a good approximation to $F$ and $\overline{F}_x$ to $\overline{F}$. To this end (see section 5.3), the discretized pdf's (on the intervals $I_i$, $i=1, \ldots, m$) corresponding to these cdf's are used:

$$f_i := F(t_i) - F(t_{i-1}); \quad \overline{f}_i := \overline{F}(t_i) - \overline{F}(t_{i-1});$$

$$\underline{f}_{X,i}(\tau) := F_x(t_i | \tau) - F_x(t_{i-1} | \tau); \quad \overline{f}_{X,i}(\tau) := \overline{F}_x(t_i | \tau) - \overline{F}_x(t_{i-1} | \tau).$$

The criterion proposed to determine a suitable value for $\tau$ is the minimization of the Sum of Mean Distances (section 5.3):

$$\text{SMD}(\tau) := \min_{\tau \in \mathcal{T}} \left\{ \sum_{i=1}^m \left( \int_1^1 d(f_i, \underline{f}_{X,i}(\tau)) + \int_1^1 d(\overline{f}_i, \overline{f}_{X,i}(\tau)) \right) \right\}, \ \tau \in \mathcal{T}.$$
A solution of the assessment problem is determined by the value \( \hat{\tau} \in T \) such that 

\[ \text{SMD}(\hat{\tau}) = \inf\{\text{SMD}(\tau) | \tau \in T \} . \]

This \( \hat{\tau} \), if it exists, defines suitable upper and lower prior densities.

In the following section, this idea is shown for a simple model.

### 5.2 The Choice of Imprecise Prior Densities for Distributions that belong to the One-Parameter Exponential Family

We show how the method proposed in section 5.1 works out. Let \( f(x|\theta) \) be the pdf of a random variable \( X \), and suppose this pdf can be put into the form 

\[ f(x|\theta) = g(x)h(\theta)\exp(t(x)\psi(\theta)) . \]

Further, suppose 

\[ l(\theta|\nu,\tau) = h(\theta)^\nu\exp(\tau\psi(\theta)) \]

and 

\[ u(\theta|\nu,\tau) = c_0 l(\theta|\nu,\tau) . \]

This model is proposed by Coolen [3], and, following the notation of section 5.1, the pair \( (c_0,\nu,\tau) \) can be regarded as the hyperparameter. These imprecise prior densities are conjugate, enabling updating in a simple analytical way.

If data \( x=x_1, \ldots, x_n \) come available, the according imprecise posterior densities are derived by replacing \( (\nu,\tau) \) by \( (\nu+n,\tau+\sum_{i=1}^n t(x_i)) \), together with replacement of \( c_0 \) by 

\[ c_n = \frac{c_0^0+n/v}{1+n/v} , \]

where \( v>0 \) can be interpreted as the number of data that provides an equal amount of information as the prior information does.

The prior information, given by \( p_i \) and \( p_i \) for \( i=1, \ldots, m \), is used to estimate \( (c_0,\nu,\tau) \), according to the method of section 5.1.

This is illustrated by an example, where three different metrics are used.

#### Example 5.2.1

Suppose elicitation and combination have resulted in (compare example 2.2.5) 

\[ p = (1/10,2/10,1/10) \] and \( \bar{p} = (6/10,6/10,4/10) \), with intervals \( I_1=[0,1) \), \( I_2=[1,2) \) and \( I_3=[2,\infty) \). These values lead to \( F(1)=1/11, F(2)=3/7, \bar{F}(1)=6/9 \) and \( \bar{F}(2)=12/13 \).

The chosen parametric distribution is the Weibull distribution with shape parameter 2 \( \{W(\alpha,2)\} \), and a gamma distribution \( \{G(\nu+1,\tau)\} \) is assumed as lower prior for the scale parameter \( \alpha \) (Coolen [3, chapter 6]). The upper prior density is equal to \( c_0 \) times the lower prior density, according to the method of section 5.2. The corresponding imprecise prior predictive cdf's
are $F_X(x|c_0, \nu, \tau) = \left[ 1 + c_0 \left( \frac{1}{F_{X,1}(x|\nu, \tau)} - 1 \right) \right]^{-1}$ and

$\bar{F}_X(x|c_0, \nu, \tau) = \left[ 1 + c_0^{-1} \left( \frac{1}{F_{X,1}(x|\nu, \tau)} - 1 \right) \right]^{-1}$, with

$F_{X,1}(x|\nu, \tau) = 1 - \left( \frac{\tau}{\tau+x^2} \right)^{\nu+1}$.

We have calculated estimates of the hyperparameters $c_0$, $\nu$ and $\tau$ using three different metrics:

$\text{d}_1(x,y) = (x-y)^2; \text{d}_2(x,y) = |x-y|; \text{d}_3(x,y) = |\ln(x)-\ln(y)|$.

For all of these, unconstrained optimization has led to problems because the global optimum is reached for $\nu=\infty$. However, if the ratio $\tau/\nu$ is kept constant, the minimum value of the goal function for small $\nu$ (say about 10) does not differ much from the global minimum. This makes the result of the optimization process very sensitive for the initial values. However, this example showed us that, for all these metrics and initial values, $c_0$ can reasonably well be estimated by 4.3 (all results differ only slightly from this value). The estimated ratio $\tau/\nu$ is, for all situations, close to 2.9. For this example it would be best to put constraints on $\nu$, based on the possible interpretation, and possibly also on the effect this hyperparameter has in updating (see section 5.4). From this point of view, it is also attractive to restrict to $\nu \in \mathbb{N}$. The above method can, for this example, help us to find good estimates of $c_0$ and $\tau$.

(End of example 5.2.1)

### 5.3 The Use of Metrics to Determine Suitable Values of Hyperparameters

In the notation of 5.1, the problem of interest is to find a value for $\tau$, such that both discretized pdf’s $f_\tau$ and $\bar{f}_\tau$ are well approximated by the pdf’s $\tilde{f}_{X,\tau}(x)$ and $\tilde{f}_{X,\tau}(x)$ respectively. It is obvious that the distance between the pdf’s needs to be defined. To approximate $f_\tau$ well by $\tilde{f}_{X,\tau}(x)$ it is useful to use the mean distance between these two pdf’s, where the mean is taken with regard to pdf’s $f_{\tau-1}$. In fact, this implies that a distance between $f_{\tau-1}$ and $\tilde{f}_{X,\tau}(x)$ on a certain interval is weighted by the probability that a random value of $X$ belongs to that interval, where $X$ is assumed to have the distribution with pdf $f_{\tau-1}$. Because there is no reason to add more value to good estimation of either $f_{\tau-1}$ or $\tilde{f}_{\tau-1}$, the minimization criterion that is introduced in section 5.1 is assumed.
In the examples of section 5.2, three metrics were used:
\[ d_1(x,y) = (x-y)^2, \quad d_2(x,y) = |x-y|, \quad \text{and} \quad d_3(x,y) = |\ln(x)-\ln(y)|. \]
Of course, any metric can be chosen, and, in special cases a best metric follows immediately.
The metrics \( d_1 \) and \( d_2 \) are always intuitively attractive in approximation problems. The metric \( d_3 \) may seem arbitrary, but there are some resemblances to the Kullback-Leibler information.

### 5.4 Some Aspects of Optimization

In this section we propose an initial value for \( c \), to be used in a numerical optimization procedure, and we also discuss the possible interpretation of \( \nu \) that can be useful to define constraints for the optimization procedure.

The following relation (Coolen [3]) holds for the model of section 5.2:
\[ c_0 = \frac{1 + \Delta_X(x_m)}{1 - \Delta_X(x_m)}, \quad \text{with} \quad \Delta_X(x) = \bar{F}_X(x) - \underline{F}_X(x) \leq \Delta_X(x_m) \quad \text{for all} \ x, \text{where} \ x_m \ \text{is the median of the distribution with pdf equal to} \ l(\theta \mid \tau) \ \text{after normalizing.} \]

A suitable initial value for \( c_0 \) is derived by considering \( \Delta(t_i) = F(t_i) - \underline{F}(t_i) \) for all \( i=1, \ldots, m \). Although this is not necessary for approximation of \( F \) and \( \bar{F} \) by \( F_X \) and \( \bar{F}_X \), it seems to be quite logical that \( \Delta_X(x_m) \leq \Delta(t_i) \) for all \( i \).

This suggests to define \( \Delta := \max(\Delta(t_i) \mid i=1, \ldots, m) \), and to choose \( c_0^{\text{in}} := \frac{1 + \Delta}{1 - \Delta} \) as initial value of \( c_0 \) for the optimization procedure.

One could also use some characteristics of \( F_X \) and \( \bar{F}_X \) (such as the means or medians) in relation to the prior information to define reasonable initial values for \( \nu \) and \( \tau \), but we do not consider this possibility here.

We end this chapter with a remark concerning the possible interpretation of \( (\nu, \tau) \) as sufficient statistics of an imaginary sample (see section 5.2). If data \( x_1, \ldots, x_n \) come available, the prior densities can be updated by replacing \( (\nu, \tau) \) by \( (\nu+n, \tau+\sum_{i=1}^{n} t(x_i)) \), together with replacement of \( c_0 \) by \( c_0^{\text{in}} = \frac{c_0 + n/\xi}{1+n/\xi} \) (for \( \xi \) see section 5.2). The situation that \( n=\nu \) can be interpreted by saying that the prior (imaginary sample) and the experimental data have equal influence on the form of the prior densities. This
interpretation can logically pose constraints to values of $v$ that one wants to allow as results of the above proposed optimization procedure (this also explains why it is attractive to choose $v \in \mathbb{N}$), as large values of $v$ imply little influence of experimental data on the form of the distribution of the parameter of interest.

It is important to remark that, although $n$ seems to have an analogous relation to $v$ and $\xi$, these two hyperparameters have different meanings, and there is no obvious reason to choose $v=\xi$. In case of updating, $\xi$ determines the amount of imprecision that is lost, while $v$ relates to the form of the prior densities. If possible, a decision maker should choose both $\xi$ and $v$ himself, but this may be a problem, as the choice of $v$ can only be made if the notion of probability distributions is well understood. The influence of the number of data compared to $v$ will probably suggest choosing $v$ quite small. There may be some arguments for $v=\xi$, but here more research is needed (especially in practical situations, where it would also be very useful if the problem would be considered from a psychological point of view: is it possible to teach a decision maker to assess values of $\xi$ and $v$ that make sense?). If $v$ is chosen before the optimization procedure is executed the routine is simplified. As shown by the example in section 5.2, there may be problems if both $v$ and $\tau$ are totally free in the optimization procedure, depending on their role in the predictive distributions. Such problems should be regarded whenever a particular model is chosen for practical application.
References


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