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A collection of numerically reliable algorithms for the deadbeat control problem

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FOR THE DEADBEAT CONTROL PROBLEM

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Abstract

Numerically reliable algorithms for the deadbeat control problem, allowing for a trade-off between the size of the feedback gains and the order of nilpotency of the regulated system, are presented.

Introduction

In this paper we present a collection of numerically reliable algorithms for the deadbeat control problem. Any particular algorithm in this collection uses unitary matrices in order to transform the system under consideration (using state space transformations) into a form such that the actual construction of a feedback matrix, solving the deadbeat control problem, consists of solving a set of linear equations.

The problem is the following

- (1) Given a system (A, B) ; construct a matrix F such that
- $$(A + BF)^k = 0 \text{ for some } k.$$

A system theoretic interpretation of (1) may be found in [7]. For a given F , the minimum k such that $(A + BF)^k = 0$ will be called the order of nilpotency of $A + BF$.

Problem (1) (particularly the case where k is supposed to be minimal a priori) received some attention in recent years. See [7], [1], [4]. Also a number of algorithms appeared. Some of them have been constructed for the deadbeat control problem specifically, others deal with the more general pole assignment problem cf. [4], [8], [5], [8], [3], [6].

In [8] the deadbeat control problem for minimum k is solved in a numerically reliable way and the minimum norm feedback matrix F is obtained. Here the norm is the Frobenius norm. The fact that one wants to solve (1) for minimum k may lead to large feedback gains, whereas in the case where only a feedback matrix F is needed such that $(A + BF)^k = 0$ for some k , smaller gains may meet the requirements.

An example, showing that a solution to problem (1), with the additional requirement that k be minimal, may contain large gains, is the following

$$A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}.$$

The minimum time (k minimal) feedback matrix F is

$$F = -B^{-1}A = \begin{bmatrix} 0 & -1/\varepsilon \\ -\alpha & -\beta \end{bmatrix}.$$

Here $A + BF = 0$ ($k=1$).

If one allows $k = 2$ then the following feedback matrix F will do

$$F = \begin{bmatrix} 0 & 0 \\ -\alpha & -\beta \end{bmatrix}.$$

Here

$$(A + BF)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0.$$

For small ε the minimum some feedback matrix F may have undesirable properties.

Therefore a collection of algorithms, having the following properties, will be proposed in the next:

- any member of the collection solves (1) in a numerically reliable way;
- the collection contains an algorithm solving (1) for minimum k if the system (A,B) is controllable;
- a trade-off between k and F (" k larger": " F smaller") may be obtained by a proper tuning of the algorithm selection procedure.

Concerning the solvability of problem (1) for a system (A,B) , it is well-known that there exists a feedback matrix F such that $A + BF$ is nilpotent if and only if every non controllable eigenvalue of A is zero. Therefore we will suppose that (A,B) satisfies this condition.

Such a system is state space isomorphic to a system of the form

$$\left(\begin{bmatrix} A_u & 0 \\ A_f & A_c \end{bmatrix}, \begin{bmatrix} 0 \\ B_c \end{bmatrix} \right)$$

where A_u is nilpotent and (A_c, B_c) is controllable. The corresponding state space isomorphism may be chosen to be unitary. See [2]. We will restrict ourselves to the case where (A,B) is controllable. (If $A_c + B_c F_c$ is nilpotent then

$$\begin{bmatrix} A_u & 0 \\ A_f & A_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} [0 \quad F_c]$$

is also nilpotent).

A further reduction of the problem (with respect to the dimension) may be obtained as follows.

For a controllable system (A,B) , where A has some zero eigenvalues, it is possible to transform (A,B) , by means of a unitary state space isomorphism, into the form

$$(2) \quad \left(\begin{bmatrix} A_o & A_f \\ 0 & A_c \end{bmatrix}, \begin{bmatrix} B_o \\ B_c \end{bmatrix} \right)$$

where the matrix A_o is nilpotent and the matrix A_c is regular. Obviously, the system (A_c, B_c) is controllable.

Observe that a solution to problem (1) for the system (A_c, B_c) also provides a solution to (1) for (A, B) . However we are not able anymore to reduce the order of nilpotency of the regulated system beyond the order of nilpotency of A_0 if we restrict ourselves to solutions to (1) of the form $[0, F_c]$ where F_c is a solution to (1) for (A_c, B_c) .

In order to be able to construct a feedback matrix F , solving (1) for minimum k , we will not make use of (2).

Results

In the next we will present an algorithm for the construction of a feedback matrix F satisfying (1).

Let (A, B) be a controllable system where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Algorithm

$i := 1$, $n_i := n$, $A_i := A$, $B_i := B$.

while $n_i > 0$ do

begin

Step 1: Using a minor modification of the singular value decomposition we have (some of the zero matrices may be empty) ($\cdot^T$ denotes transposition)

$$B_i = U_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \\ 0 & 0 & D_{ig} \end{bmatrix} V_i^T$$

where U_i and V_i are unitary matrices. Here D_{ib} and D_{ig} are diagonal matrices, together containing the singular values of B_i , such that

D_{ib} contains b_i "bad" singular values and D_{ig} contains g_i "good" singular values (Think of "bad" as "too small" and "good" as "large enough"). We will assume that "good" implies positive.

Step 2: Perform the unitary state space transformation $(U_{i1}^T A_i U_i, U_{i1}^T B_i)$.

$$\left(\begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix}, \begin{bmatrix} B_{ib} \\ B_{ig} \end{bmatrix} \right) := U_{i1}^T A_i U_i, U_{i1}^T B_i$$

where $A_{ig} \in \mathbb{R}^{g_i \times g_i}$ and

$$B_{ib} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \end{bmatrix} V_i^T, \quad B_{ig} = [0 \quad 0 \quad D_{ig} V_i^T].$$

Step 3: Compute a unitary matrix W_i such that

$$\begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix} W_i = \begin{bmatrix} \bar{A}_{ia} & 0 \\ \bar{A}_{if} & \bar{A}_{ig} \end{bmatrix}, \quad \bar{A}_{ia} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)}.$$

$$F_i := [0 \quad F_{ig}] := V_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -D_{ig}^{-1} \bar{A}_{ig} \end{bmatrix}.$$

Now it is clear that

$$U_{i1}^T A_i U_i W_i + U_{i1}^T B_i F_i = \begin{bmatrix} \bar{A}_{ia} & 0 \\ \bar{A}_{if} & 0 \end{bmatrix}.$$

Therefore we have

$$W_{i1}^T U_{i1}^T A_i U_i W_i + W_{i1}^T U_{i1}^T B_i F_i = \begin{bmatrix} A_{i+1} & 0 \\ * & 0 \end{bmatrix}$$

for some $A_{i+1} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)}$.

$$\text{Step 4: } W_i^T U_i^T B_i = \begin{bmatrix} B_{i+1} \\ * \end{bmatrix}$$

for some $B_{i+1} \in \mathbb{R}^{(n_i - g_i) \times m}$.

Step 5: $n_{i+1} := n_i - g_i$, $i := i + 1$.

end.

Observe that in Step 3 and Step 4

- (A_{i+1}, B_{i+1}) does not depend on F_i ;
- (A_{i+1}, B_{i+1}) is controllable;
- F_i places g_i poles at zero for the system $(W_i^T U_i^T A_i U_i W_i, W_i^T U_i^T B_i)$;
- B_{i+1} may contain all "good" singular values of B_i ;
 $(B_{i+1}$ consists of the first n_{i+1} rows of $W_i^T U_i^T B_i)$;

for each cycle i of this part of the algorithm.

In order to guarantee termination of (this part of) the algorithm we will suppose that the meaning of "good" is such that, in each cycle, at least one "good" singular value can be found in Step 1. Therefore the maximum number of cycles is n .

In this way we have obtained

$$n_1, \dots, n_k; g_1, \dots, g_k; W_1, \dots, W_{k-1}; U_1, \dots, U_k; F_{1g}, \dots, F_{kg}$$

where it is assumed that $n_{k+1} = 0$ (termination). The "missing" W_k may be taken to be the identity matrix.

Here

$$W_i \in \mathbb{R}^{n_i \times n_i} \quad i = 1, \dots, k-1.$$

$$U_i \in \mathbb{R}^{n_i \times n_i} \quad i = 1, \dots, k.$$

$$F_{ig} \in \mathbb{R}^{m \times g_i} \quad i = 1, \dots, k.$$

Next we compute a unitary matrix $U \in \mathbb{R}^{n \times n}$

$$U := U_1 W_1 \begin{bmatrix} U_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} W_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \dots \begin{bmatrix} U_{k-1} & 0 \\ 0 & I_{m_{k-1}} \end{bmatrix} \begin{bmatrix} W_{k-1} & 0 \\ 0 & I_{m_{k-1}} \end{bmatrix} \begin{bmatrix} U_k & 0 \\ 0 & I_{m_k} \end{bmatrix}$$

where I_{m_i} is the $m_i \times m_i$ identity matrix; $m_i = n - n_i$, $i = 2, \dots, k$.

The matrix $F_g \in \mathbb{R}^{m \times n}$ is formed as follows

$$F_g := [F_{kg}, \dots, F_{1g}].$$

In the final step of the algorithm we compute $F \in \mathbb{R}^{m \times n}$

$$F := F_g U^T$$

end of the algorithm.

In order to prove that the matrix F is a solution to problem (1) we observe that

$$(3) \quad U^T A U + U^T B F_g = \begin{bmatrix} \boxed{0} & 0 & \dots & \dots & 0 & & \\ * & \boxed{0} & & & & & \\ \vdots & & \ddots & & & & \\ \vdots & & & \ddots & & & \\ \vdots & & & & \boxed{0} & 0 & \\ \vdots & & & & * & \boxed{0} & \\ \vdots & & & & & & \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{g_k} \qquad \qquad \underbrace{\hspace{5em}}_{g_2} \quad \underbrace{\hspace{5em}}_{g_1}$

This shows that $(A + BF)^k = 0$. □

Furthermore we may conclude that the order of nilpotency of $A + BF$ is at most k .

Remarks

- The singular value decomposition is not really needed for Step 1 because we only need the following structure for $U_1^T B_1 V_1$.

$$(4) \quad U_1^T B_1 V_1 = \begin{bmatrix} * & 0 \\ * & G_1 \end{bmatrix}$$

where the $g_1 \times g_1$ -matrix G_1 is "good". Then the matrix F_{ig} is defined as

$$F_{ig} = \begin{bmatrix} 0 \\ -G_1^{-1} \overline{A}_{ig} \end{bmatrix}.$$

A factorization as in (4) may be obtained (for instance) by applying the QR algorithm with column permutations to B_1^T .

- If our meaning of "good" could not imply that in each cycle of the while-loop at least one "good" singular could be found in Step 1, we could re-define "good" such that "good" satisfies

if A_i is not nilpotent then at least one "good" singular value can be found in Step 1.

Then we would again obtain termination of the while-loop if the condition " $n_i > 0$ " is replaced by " $n_i > 0$ or A_i is not nilpotent".

The term "collection" in "collection of numerically reliable algorithms" (see title) can be justified as follows.

The discrimination between "good" and "bad" in the selection procedure for the singular value may heavily depend upon the application the user has in mind. Each choice he makes, determines a matrix F solving (1). We mention one selection policy in particular: All nonzero singular values are "good".

The member of the collection of algorithms, corresponding to this selection policy, produces the same feedback matrix (with minimum Frobenius norm) as the algorithm in [8].

The claim, made at the beginning of this paper, that the actual construction of the feedback matrix, solving (1), can be performed by solving a set of linear equations, may be substantiated as follows.

Consider (3) and let

$$U^T A U = \begin{bmatrix} A_{kk} & \dots & A_{k1} \\ \vdots & & \vdots \\ A_{1k} & \dots & A_{11} \end{bmatrix}, \quad U^T B = \begin{bmatrix} B_k \\ \vdots \\ B_1 \end{bmatrix},$$

where

$$A_{ij} \in \mathbb{R}^{g_i \times g_j}, \quad i = 1, \dots, k; j = 1, \dots, k.$$

$$B_i \in \mathbb{R}^{g_i \times m}, \quad i = 1, \dots, k.$$

Then we have

$$\begin{bmatrix} B_k \\ \vdots \\ B_i \end{bmatrix} F_{ig} = - \begin{bmatrix} A_{ki} \\ \vdots \\ A_{ii} \end{bmatrix}, \quad i = 1, \dots, k.$$

This also shows that we can compute F after having computed the state space isomorphic system $(U^T A U, U^T B)$.

With respect to the numerical properties of this collection of algorithms we refer to [8]. The arguments given there, also apply to any member of our collection. We may conclude that the algorithms are numerically reliable, though a formal proof of backward stability is lacking.

We consider the example, given in the beginning of this paper, once more.

$$A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose that ε is "bad" and that 1 is "good". Then we obtain

$$U_1 = I, \quad W_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A W_1 = \begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix}, \quad U = W_1,$$

$$(U^T A U, U^T B) = \left(\begin{bmatrix} \beta & \alpha \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix} \right),$$

$$F_g = \begin{bmatrix} 0 & 0 \\ -\alpha & -\beta \end{bmatrix}, \quad F = F_g U^T = \begin{bmatrix} 0 & 0 \\ -\alpha & -\beta \end{bmatrix}.$$

The next example illustrates the effect of different choices for "good" and "bad"

$$A = \begin{bmatrix} 3 & 2 & 1 & 5 & 6 \\ 4 & 6 & 3 & 2 & 8 \\ 5 & 7 & 0 & 9 & 1 \\ 3 & 5 & 1 & 7 & 6 \\ 9 & 0 & 8 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 6 & 0 \\ 3 & 5 & 1 \\ 6 & 0 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$

If "good" means " > 0 ", then F becomes

$$F \approx \begin{bmatrix} 1.06 & -0.86 & 0.01 & 1.00 & -3.45 \\ -4.43 & 2.44 & -3.95 & 0.06 & 2.62 \\ 1.64 & -2.67 & 3.36 & -3.57 & -2.13 \end{bmatrix}$$

with norm (Frobenius) $\|F\| \approx 10$.

If "good" means " > 1 ", then F becomes

$$F \approx \begin{bmatrix} -0.71 & 0.05 & -1.35 & 0.70 & -1.17 \\ -0.78 & -0.09 & -1.19 & 1.07 & 0.07 \\ 1.02 & -2.10 & 2.84 & -3.66 & -0.72 \end{bmatrix}$$

with norm $\|F\| \approx 6$.

If "good" means " > 3 ", then F becomes

$$F \approx \begin{bmatrix} -0.23 & -0.45 & -0.67 & 0.11 & -1.42 \\ -0.23 & -0.63 & -0.45 & -0.16 & 0.01 \\ -0.06 & -0.09 & -0.15 & -0.01 & -0.48 \end{bmatrix}$$

with norm $\|F\| \approx 2$.

Conclusions

Based on the judgement "bad" or "good", with respect to the singular values of the "B-matrix", a numerically reliable algorithm is obtained from a collection of algorithms solving the deadbeat control problem. Solutions to this problem, satisfying additional requirements with respect to the size of the feedback gains or the nilpotency index of the regulated system, may be obtained using a proper tuning of the "good"/"bad" selection policy. Practical considerations should supply the right meaning of "good" and "bad".

The starting point for our construction is the system (A,B) without assuming a special structure of A and/or B ; cf. [8] where the starting point is a block Hessenberg form, obtained by the so called staircase algorithm.

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