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CONVERGENCE PROPERTIES OF
INDEFINITE LINEAR QUADRATIC
PROBLEMS WITH RECEDING HORIZON

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Abstract: In this paper we study the following question: given a finite dimensional linear system together with a finite horizon (possibly indefinite) quadratic cost functional, when does the corresponding optimal cost converge to the optimal cost of the corresponding infinite horizon problem, as the length of the horizon tends to infinity? For the case that the linear quadratic problems are regular we establish necessary and sufficient conditions for this convergence to hold.

Keywords: Linear quadratic optimal control, indefinite cost, large horizon, Riccati equation.

1. INTRODUCTION

The finite horizon linear quadratic control problem for the linear time-invariant system

$$(1.1) \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

is concerned with choosing a control function u such that the cost functional

$$(1.2) \quad J_T(x_0, u) := \int_0^T \omega(x(t), u(t)) dt$$

is minimized. Here, ω is a real quadratic form on $\mathbb{R}^n \times \mathbb{R}^m$ given by

$$(1.3) \quad \omega(x, u) = x^T Q x + 2u^T S x + u^T R u.$$

In the above expressions, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{m \times m}$. It is assumed that Q is symmetric and that R is positive definite. Apart from this definiteness assumption on R , we allow ω to be indefinite.

In order for the integral in (1.2) to be well-defined, we restrict the control functions to be elements of the class $L_2[0, T]$ of all $\mathbb{R}^m$ -valued functions that are square integrable over $[0, T]$. The optimal cost for the above problem is then defined as

$$(1.4) \quad V_T^+(x_0) := \inf \{ J_T(x_0, u) \mid u \in L_2[0, T] \}.$$

In addition to the above, the infinite horizon linear quadratic problem deals with minimizing the indefinite integral

$$(1.5) \quad J(x_0, u) := \lim_{T \rightarrow \infty} J_T(x_0, u).$$

Let $L_{2, \text{loc}}(\mathbb{R}^+)$ denote the space of all $\mathbb{R}^m$ -valued locally square integrable functions on $\mathbb{R}^+$. The functional $J(x_0, u)$ is defined for all functions $u \in L_{2, \text{loc}}(\mathbb{R}^+)$ for which the limit in (1.5) exists in the sense that it is finite or infinite. This class of functions is denoted by

$$(1.6) \quad U(x_0) := \{ u \in L_{2, \text{loc}}(\mathbb{R}^+) \mid \lim_{T \rightarrow \infty} J_T(x_0, u) \text{ exists in } \mathbb{R}^e := \mathbb{R} \cup \{-\infty, +\infty\} \}$$

The optimal cost for the above infinite horizon linear quadratic problem is

$$(1.7) \quad V_f^+(x_0) := \inf \{ J(x_0, u) \mid u \in U(x_0) \}.$$

In this paper we are interested in the question whether the optimal cost for the finite horizon problem (1.4) converges to the optimal cost for the infinite horizon problem (1.7) as T tends to infinity. Of course, if the quadratic form ω is positive semi-definite then it is well-known that indeed $V_T(x_0) \rightarrow V_f^+(x_0)$ ($T \rightarrow \infty$) for all x_0 (provided that (A, B) is stabilizable). It turns out however that if ω is indefinite then this convergence no longer holds in general. In this paper we establish necessary and sufficient conditions for convergence to hold.

Actually, we shall treat the above question of convergence in the following, more general context. Let $N \in \mathbb{R}^{n \times n}$ be a symmetric matrix, let $T > 0$ and consider the finite horizon problem with cost functional

$$(1.8) \quad J_{T,N}(x_0, u) := J_T(x_0, u) + x^T(T)Nx(T).$$

The second term in the above represents a penalty on the terminal state. The matrix N is allowed to be indefinite. The optimal cost associated with the latter problem is given by

$$(1.9) \quad V_{T,N}^+(x_0) := \inf \{ J_{T,N}(x_0, u) \mid u \in L_2[0, T] \}.$$

In addition to this finite horizon problem we consider the infinite horizon problem of infimizing (1.5) under the constraint that $Nx(t)$ converges to zero as t tends to infinity. More specifically, let

$$U_N(x_0) := \{ u \in U(x_0) \mid \lim_{T \rightarrow \infty} Nx(T) = 0 \}$$

and consider the problem of infimizing (1.5) over the class $U_N(x_0)$. The optimal cost for this problem is given by

$$(1.10) \quad V_N^+(x_0) := \inf \{ J(x_0, u) \mid u \in U_N(x_0) \}.$$

The latter optimization problem was studied in detail in [8]. Of course, the problem (1.7) can be reobtained from this formulation as a special case by taking $N = 0$ (see also [9]).

Another special case of (1.10) is obtained by taking $N = I$, the identity matrix. This special case was treated in [10].

Now, in this paper we shall ask ourselves the question: when does the optimal cost for the finite horizon problem (1.9) converge to the optimal cost for the infinite horizon problem (1.10) as $T \rightarrow \infty$?

We conclude this introduction by noting that the questions to be studied here have been studied before in [2] and [11]. However, in these references only the case that both $\omega \geq 0$ and $N \geq 0$ was considered, while we intend to treat the most general case that ω and N are allowed to be indefinite.

2 THE ALGEBRAIC RICCATI EQUATION

The characterization of the optimal costs for the infinite horizon problems (1.7) and (1.10) centers around the algebraic Riccati equation (ARE):

$$(2.1) \quad A^T K + KA + Q - (KB + S^T)R^{-1}(B^T K + S) = 0.$$

Let Γ be the set of all real symmetric solutions of (2.1). According to [10], if (A, B) is controllable and $\Gamma \neq \emptyset$ then there is exactly one $K \in \Gamma$ such that the matrix $A_K := A - BR^{-1}(B^T K + S)$ has all its eigenvalues in $\mathbb{C}^- \cup \mathbb{C}^0$ and exactly one $K \in \Gamma$ such that A_K has all its eigenvalues in $\mathbb{C}^+ \cup \mathbb{C}^0$. Here, we define $\mathbb{C}^-(\mathbb{C}^0, \mathbb{C}^+) := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\} \cup \{s \in \mathbb{C} \mid \operatorname{Re} s = 0, \operatorname{Im} s > 0\}$. These elements of Γ are denoted by K^+ and K^- , respectively. It can be shown that if $K \in \Gamma$ then $K^- \leq K \leq K^+$. We denote A_{K^-} by A^- and A_{K^+} by A^+ .

If $M \in \mathbb{R}^{n \times n}$ then $\mathcal{X}^-(M)$ ($\mathcal{X}^0(M)$, $\mathcal{X}^+(M)$) denotes the span of all generalized eigenvectors of M corresponding to its eigenvalues in $\mathbb{C}^-$ ($\mathbb{C}^0$, $\mathbb{C}^+$).

Let Ω denote the set of all A^- -invariant subspaces of $\mathcal{X}^+(A^-)$. Let $\Delta := K^+ - K^-$ (the 'gap' of the ARE). The following result states that there exists a bijection between Ω and Γ :

Theorem 2.1 ([10], [3], [7]). Let (A, B) be controllable and assume that $\Gamma \neq \emptyset$. If $V \in \Omega$ then $\mathbb{R}^n = V \oplus \Delta^{-1}V^\perp$. There exists a bijection $\gamma : \Omega \rightarrow \Gamma$ defined by

$$\gamma(V) := K^- P_V + K^+ (I - P_V),$$

where P_V is the projector along $\Delta^{-1}V^\perp := \{x \in \mathbb{R}^n \mid \Delta x \in V^\perp\}$. If $K = \gamma(V)$ then $\mathcal{X}^+(A_K) = V$, $\mathcal{X}^0(A_K) = \ker \Delta$ and $\mathcal{X}^-(A_K) = \mathcal{X}^-(A^+) \cap \Delta^{-1}V^\perp$. $\square$

If $K = \gamma(V)$ then K is said to be supported by V .

3 THE INFINITE HORIZON PROBLEM WITH ASYMPTOTIC CONSTRAINTS

In this section we briefly recall the results from [8] on the infimization problem (1.10). Let $\mathcal{L}$ be a subspace of $\mathbb{R}^n$. A symmetric matrix $K \in \mathbb{R}^{n \times n}$ is called negative semi-definite on $\mathcal{L}$ if the following two conditions hold: (i) $\forall x \in \mathcal{L} : x^T K x \leq 0$ (ii) $\forall x \in \mathcal{L} : x^T K x = 0 \Leftrightarrow Kx = 0$.

If $V \subseteq \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ then $\langle V | M \rangle$ will denote the largest M -invariant subspace of V . A key role in the characterization of the optimal cost $V_N^+(x_0)$ is played by the subspace

$$(3.1) \quad V_N := \langle \ker N \cap \ker K^- | A^- \rangle \cap \mathcal{X}^+(A^-).$$

Observe that $V_N \in \Omega$. Thus, with V_N there corresponds exactly one solution of the ARE. This solution $\gamma(V_N)$ is denoted by K_N^+ . The following theorem is the main result of [8]:

Theorem 3.1 Let (A, B) be controllable. Assume that $\Gamma \neq \emptyset$ and that K^- is negative semi-definite on $\ker N$. Then we have

- (i) $V_N^+(x_0) = x_0^T K_N^+ x_0$ for all $x_0 \in \mathbb{R}^n$.
- (ii) For all $x_0 \in \mathbb{R}^n$ there exists an optimal u^* if and only if $\ker \Delta \subseteq \ker N \cap \ker K^-$.
- (iii) If $\ker \Delta \subseteq \ker N \cap \ker K^-$ then there exists exactly one optimal input and, moreover, this input is given by the feedback control law $u = -R^{-1}(B^T K_N^+ + S)x$. $\square$

4 THE FINITE HORIZON PROBLEM WITH ENDPOINT PENALTY

In this section we consider the finite horizon problem of infimizing the cost functional (1.8). We note that for the case that both $\omega \geq 0$ and $N \geq 0$ this problem is quite standard and is treated, for example, in [6]. The general case however is slightly more complicated. The following result can be found in [1, p.131]:

Lemma 4.1 Suppose there exists on the interval $[0, \infty)$ a solution $K(t) = K_N(t)$ of the Riccati differential equation (RDE):

$$(4.1) \quad \begin{aligned} \dot{K}(t) &= A^T K(t) + K(t)A + Q - (K(t)B + S^T)R^{-1}(B^T K(t) + S), \\ K(0) &= N. \end{aligned}$$

Then for all $T > 0$ and for all $x_0 \in \mathbb{R}^n$ we have

$$V_{T,N}^+(x_0) = x_0^T K_N^+(T) x_0.$$

For all $T > 0$ and $x_0 \in \mathbb{R}^n$ there is exactly one $u^* \in L[0, T]$ such that $J_{T,N}^+(x_0, u^*) = V_{T,N}^+(x_0)$. This input is given by the feedback control law $u = -R^{-1}(B^T K(T-t) + S)x$; $t \in [0, T]$. $\square$

It is well-known that if $\omega \geq 0$ and $N \geq 0$ then (4.1) indeed has a unique solution on $[0, \infty)$. In the general case, the RDE need not have a solution on a given interval. We do have the following:

Lemma 4.2 Suppose that (A, B) is controllable and that $\Gamma \neq \emptyset$. If $N - K^- \geq 0$ then (4.1) has a unique solution $K_N(t)$ on $[0, \infty)$. In fact, $K_N(t) = K^- + D(t)$, where $D(t)$ is the unique solution on $[0, \infty)$ of

$$(4.2) \quad \begin{aligned} \dot{D}(t) &= A^{-T} D(t) + D(t)A^- - D(t)BR^{-1}B^T D(t), \\ D(0) &= N - K^-. \end{aligned}$$

(with $A^- = A_K^-$, see section 2).

Proof The fact that (4.2) has a unique solution $D(t)$ on $[0, \infty)$ is standard (see for example [4, cor. 2.4.4]). It is then a matter of straightforward calculation to show that $K^- + D(t)$ satisfies (4.1). Uniqueness of $K^- + D(t)$ then follows from the uniqueness of $D(t)$. $\square$

5 CONVERGENCE OF THE OPTIMAL COST

In this section we shall give a formulation of our main result. Before doing this, we state the following lemma:

Lemma 5.1 Let $\mathcal{L}$ be a subspace of $\mathbb{R}^n$ and let $K \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then K is negative semi-definite on $\mathcal{L}$ if and only if there exists a symmetric matrix N such that $\ker N = \mathcal{L}$ and $N - K \geq 0$.

Proof A proof of this can be given similar to the proof of [8, lemma 3.3]. $\square$

Consider the problems (1.8) and (1.10). In the remainder of this section we assume that (A, B) is controllable and that $\Gamma \neq \emptyset$. According to the previous lemma, if $N - K^- \geq 0$ then K^- is negative semi-definite on $\ker N$. Conversely, if K^- is negative semi-definite on $\ker N$ then one can always find a symmetric matrix N_1 such that $\ker N_1 = \ker N$ and $N_1 - K^- \geq 0$.

We now formulate our main result:

Theorem 5.2 Assume that (A, B) is controllable, $\Gamma \neq \emptyset$, $N - K^- \geq 0$ and $\ker \Delta \subseteq \ker N \cap \ker K^-$. Then $V_{T, N}^+(x_0) \rightarrow V_N^+(x_0)$ ($T \rightarrow \infty$) for all $x_0 \in \mathbb{R}^n$ if and only if $\ker N \cap \ker K^-$ is A^- -invariant.

A proof of this theorem is given in section 6. In the remainder of this section we shall discuss some important special cases of this theorem. First we consider the case that $N = 0$, that is, the question whether $V_T^+(x_0)$ given by (1.4) converges to the optimal cost $V_f^+(x_0)$ given by (1.7).

Corollary 5.3 Assume that (A, B) is controllable, $\Gamma \neq \emptyset$, $K^- \leq 0$ and $\ker \Delta \subseteq \ker K^-$. Then $V_T^+(x_0) \rightarrow V_f^+(x_0)$ ($T \rightarrow \infty$) for all x_0 if and only if $\ker K^-$ is A^- -invariant. $\square$

It is well-known that if, in addition, $\omega \geq 0$ then if (A, B) is controllable we always have convergence of $V_T^+(x_0)$ to $V_f^+(x_0)$. This result can be reobtained using the previous corollary in combination with the following lemma:

Lemma 5.4 Assume that $\omega \geq 0$. If (A, B) is controllable then $\Gamma \neq \emptyset$, $K^- \leq 0$, $\ker \Delta \subseteq \ker K^-$ and $\ker K^-$ is A^- -invariant.

Proof A proof of these claims can be found in [9]. $\square$

Next, we consider the special case that $N \geq 0$ and $\omega \geq 0$. It turns out that if $\omega \geq 0$ then the subspace $\ker K^-$ can be calculated in terms of the matrices A , B , Q , S and R . In fact, it can be shown that if (A, B) is controllable then

$$\ker K^- = \langle \ker (Q - S^T R^{-1} S) \mid A - B R^{-1} S \rangle \\ \cap \{ \mathcal{X}^+(A - B R^{-1} S) \oplus \mathcal{X}^0(A - B R^{-1} S) \},$$

(see [9]). The latter subspace is exactly the undetectable subspace of the pair $(Q - S^T R^{-1} S, A - B R^{-1} S)$ (with respect to the stability set $\mathcal{C}^-$). Denote this subspace by $\mathcal{ND}((Q - S^T R^{-1} S, A - B R^{-1} S))$.

Corollary 5.5 Assume $\omega \geq 0$ and $N \geq 0$. If (A, B) is controllable and $\ker \Delta \subseteq \ker N \cap \mathcal{ND}((Q - S^T R^{-1} S, A - B R^{-1} S))$. Then $V_{T, N}^+(x_0) \rightarrow V_f^+(x_0)$ ($T \rightarrow \infty$) for all x_0 if and only if $\ker N \cap \mathcal{ND}((Q - S^T R^{-1} S, A - B R^{-1} S))$ is $(A - B R^{-1} S)$ -invariant. $\square$

A similar result was obtained in [2], with ' (A, B) controllable' replaced by ' (A, B) stabilizable' and ' $\ker \Delta \subseteq \ker N \cap \mathcal{ND}((Q - S^T R^{-1} S, A - B R^{-1} S))$ ' replaced by ' $\Delta > 0$ '. The methods used in [11] (and [2]) to prove this result will however turn out to be completely different from ours.

Note that the results presented up to now all require $N - K^- \geq 0$. We now ask ourselves the question: what can we do if this is not the case? Again consider the problems (1.9) and (1.10) and assume that K^- is negative semi-definite on $\ker N$. (see th.3.1). Now, the latter does not imply that $N - K^- \geq 0$ but according to lemma 5.1 we can find a new weighting matrix N_1 such that $\ker N = \ker N_1$ and $N_1 - K^- \geq 0$. Clearly, $U_N(x_0) = U_{N_1}(x_0)$ for all x_0 and hence $V_{N_1}^+(x_0) = V_N^+(x_0)$ for all x_0 . We can then apply theorem 5.2 with N replaced by N_1 and study the question whether $V_{T, N_1}^+(x_0)$ converges to $V_{N_1}^+(x_0) = V_N^+(x_0)$. An example of this is following: suppose we consider the problem with $N = I$. There always exist a real number $k \neq 0$ such that $kI - K^- \geq 0$. Of course, $V_I^+(x_0) = V_{kI}^+(x_0)$ for all x_0 . We can then apply theorem 5.2 to obtain:

Corollary 5.6 Assume that (A, B) is controllable, $\Gamma \neq \emptyset$ and $\Delta > 0$. Then there exists $k \neq 0$ such that $V_{T, kI}^+(x_0) \rightarrow V_I^+(x_0)$ ($T \rightarrow \infty$) for all x_0 . In fact, convergence holds for all $k \neq 0$ such that $kI \geq K^-$. $\square$

6 PROOF OF THE MAIN RESULT

Our proof of theorem 5.2 runs along a series of lemmas that we consider to be interesting in their own right. Due to lack of space the proofs of these lemmas are deferred to a future paper. Our first lemma deals with an arbitrary system (A, B) and an arbitrary matrix $R > 0$, independent of the previous context. Consider the standard Riccati differential equation

$$(6.1) \quad \begin{aligned} \dot{P}(t) &= A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t), \\ P(0) &= P_0, \end{aligned}$$

together with the standard algebraic Riccati equation

$$(6.2) \quad A^T P + PA - PBR^{-1}B^T P = 0.$$

Recall that if $P_0 \geq 0$ and (A,B) is controllable, then (6.1) has a unique solution $P(t) \geq 0$ on $[0, \infty)$ (see [4]). Also, (6.2) has at least one solution ([5]). Let P^+ be the largest real symmetric solution of (6.2).

Lemma 6.1 Assume that (A,B) is controllable and $\sigma(A) \subseteq \mathbb{C}^+$. Then $P^+ > 0$. For any $P_0 > 0$ we have $\lim_{t \rightarrow \infty} P(t) = P^+$. $\square$

Our following result again deals with the Riccati differential equation (6.1). The result is, in a sense, the converse of the previous lemma:

Lemma 6.2 Assume $P_0 \geq 0$ and assume that $\lim_{t \rightarrow \infty} P(t) =: P \in \mathbb{R}^{n \times n}$ exists. If $P > 0$ then $P_0 > 0$. $\square$

We now return to the original context of this paper. Consider the system (1.1), together with the quadratic form ω given by (1.3). Recall that we denote A_K^- by A^- (see section 2). An important role is played by the following algebraic Riccati equation in the unknown D :

$$(6.3) \quad A^{-T}D + DA^- - DBR^{-1}B^T D = 0$$

We make the following observation:

Lemma 6.3 Assume (A,B) is controllable and $\Gamma \neq \emptyset$. Let $K \in \mathbb{R}^{n \times n}$ be symmetric. Then K is a solution of (2.1) if and only if $D = K - K^-$ is a solution of (6.3). In particular $D = 0$ and $D = \Delta (= K^+ - K^-)$ are solutions of (6.3). In fact, 0 and Δ are the extremal solutions of (6.3) in the sense that any solution of (6.3) satisfies $0 \leq D \leq \Delta$. $\square$

Finally, we shall need the following result:

Lemma 6.4 Assume that (A,B) is controllable and $\Gamma \neq \emptyset$. Assume that $N - K^- \geq 0$. Let $D(t)$ be the solution of (4.3). Then for all $t \geq 0$ we have:

$$\langle \ker N \cap \ker K^- | A^- \rangle \subseteq \ker D(t). \quad \square$$

We have now collected the most important ingredients that will be used in our proof of theorem 5.2. In order to give this proof we shall make a suitable decomposition of the state space. Let V_N be the subspace defined in (3.1). Define

$$\begin{aligned} \mathcal{X}_1 &:= V_N \\ \mathcal{X}_2 &:= \ker \Delta, \\ \mathcal{X}_3 &:= \mathcal{X}^-(A^+) \cap \Delta^{-1}V_N^\perp. \end{aligned}$$

Denote $A_N := A_{K_N}^+$. According to theorem 2.1 we have $\mathcal{X}_1 = \mathcal{X}^+(A_N)$, $\mathcal{X}_2 = \mathcal{X}^0(A_N)$ and $\mathcal{X}_3 = \mathcal{X}^-(A)$. Hence $\mathbb{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$. With respect to this decomposition we have

$$(6.4) \quad A^- = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

for given matrices A_{ij} . This follows from the fact that both $\mathcal{X}_1$ and $\mathcal{X}_2$ are A^- -invariant. Note that $\sigma(A_{33}) \subseteq \mathbb{C}^+$. According to theorem 2.1, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \Delta^{-1}\mathcal{X}_1^\perp$. Since also $\mathcal{X}_2 = \ker \Delta$, we have

$$(6.5) \quad \Delta = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix}$$

with $\Delta_{11} > 0$ and $\Delta_{33} > 0$. Finally, we partition

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

Proof of theorem 5.2 From the assumption $\ker \Delta \subseteq \ker N \cap \ker K^-$ it follows that $\mathcal{X}_1 \oplus \mathcal{X}_2 \subseteq \ker N \cap \ker K^-$. Hence N and K^- have the form

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{33} \end{pmatrix}, \quad K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33}^- \end{pmatrix},$$

with $N_{33} - K_{33}^- \geq 0$. Since $K^+ = K^- + \Delta$, we have

$$K_N^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33}^- + \Delta_{33} \end{pmatrix}.$$

(Recall that $K_N^+ = K^-P + K^+(I - P)$, where P is the projector onto $\mathcal{X}_1 = \mathcal{V}_N$ along $\Delta^{-1}\mathcal{V}_N^+ = \mathcal{X}_2 \oplus \mathcal{X}_3$). By combining the above we see that

$$K_N^+ - K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix}.$$

Let $D(t)$ be the solution of the Riccati differential equation (4.2). By lemma 6.4 and the fact that $\mathcal{X}_2 = \ker \Delta$ is A^- -invariant, we have

$$D(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{33}(t) \end{pmatrix}$$

for some $D_{33}(t)$. By writing out (4.2) in the decomposition employed, we see that $D_{33}(t)$ is the unique solution of

$$(6.6) \quad \begin{aligned} D_{33}(t) &= A_{33}^T D_{33}(t) + D_{33}(t) A_{33} - D_{33}(t) B_3 R^{-1} B_3^T D_{33}(t) \\ D_{33}(0) &= N_{33} - K_{33}^+. \end{aligned}$$

also, $K_N^+ - K^-$ is a solution of the algebraic Riccati equation (6.3) (see lemma (6.3)). This implies that Δ_{33} is a solution to

$$(6.7) \quad A_{33}^T D_{33} + D_{33} A_{33} - D_{33} B_3 R^{-1} B_3^T D_{33} = 0.$$

In fact, since Δ is the largest solution of (6.3), Δ_{33} is the largest solution of (6.7).

('⇒') Assume that $V_{T,N}^+(x_0) \rightarrow V_N^+(x_0)$ for all x_0 or, equivalently, $K_N(t) \rightarrow K_N^+(t \rightarrow \infty)$. Here, $K_N(t)$ is the unique solution of (4.1). Then we have $K_N(t) - K^- \rightarrow K_N^+ - K^-$ ($t \rightarrow \infty$). Now, the point is that $K_N(t) - K^- = D(t)$. Hence we find that $D_{33}(t) \rightarrow \Delta_{33}$ ($t \rightarrow \infty$). Since $\Delta_{33} > 0$ it follows from lemma 6.2 that $N_{33} - K_{33}^- > 0$. We now prove the A^- -invariance of $\ker N \cap \ker K^-$. Let $x \in \ker N \cap \ker K^-$, $x = (x_1, x_2, x_3)$. Then $(N - K^-)x = 0$ whence $(N_{33} - K_{33}^-)x_3 = 0$ so $x_3 = 0$. Thus $A^-x = (A_{11}x_1, A_{22}x_2, 0)$. Since $\mathcal{X}_1 \oplus \mathcal{X}_2 \subseteq \ker N \cap \ker K^-$, the claim follows.

('⇐') Assume $\ker N \cap \ker K^-$ invariant under A^- . We then claim that $\mathcal{X}_1 \oplus \mathcal{X}_2 = \ker N \cap \ker K^-$. Indeed,

$$\begin{aligned}
\mathcal{X}_1 \oplus \mathcal{X}_2 &= \{ \langle \ker N \cap \ker K^- | A^- \rangle \cap \mathcal{X}^+(A^-) \} \oplus \ker \Delta \\
&= \{ (\ker N \cap \ker K^-) \cap \mathcal{X}^+(A^-) \} \oplus \ker \Delta \\
&= (\ker N \cap \ker K^-) \cap (\ker \Delta \oplus \mathcal{X}^+(A^-)),
\end{aligned}$$

where the last equality again uses the assumption $\ker \Delta \subseteq \ker K^- \cap \ker N$. Now, $\ker \Delta \oplus \mathcal{X}^+(A^-) = \mathcal{X}^0(A^-) \oplus \mathcal{X}^+(A^-) = \mathbb{R}^n$. This proves the claim. It follows from this that $N_{33} - K_{33} > 0$. Hence, since $\sigma(A_{33}) \subseteq \mathbb{C}^+$, the solution $D_{33}(t)$ of (6.6) converges to Δ_{33} , the largest solution of (6.7) (see lemma 6.1). In turn this implies that $D_{33}(t) \rightarrow K_N^+ - K^-$ or, equivalently, that $K_N^+(t) \rightarrow K_N^+$. Thus $V_{T,N}^+(x_0) \rightarrow V_N^+(x_0)$ ($T \rightarrow \infty$) for all x_0 . This completes our proof of theorem 5.2. $\square$

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List of COSOR-memoranda - 1988

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M 88-01	January	F.W. Steutel, B.G. Hansen	Haight's distribution and busy periods.
M 88-02	January	J. ten Vregelaar	On estimating the parameters of a dynamics model from noisy input and output measurement.
M 88-03	January	B.G. Hansen, E. Willekens	The generalized logarithmic series distribution.
M 88-04	January	J. van Geldrop, C. Withagen	A general equilibrium model of international trade with exhaustible natural resource commodities.
M 88-05	February	A.H.W. Geerts	A note on "Families of linear-quadratic problems": continuity properties.
M 88-06	February	Siquan, Zhu	A continuity property of a parametric projection and an iterative process for solving linear variational inequalities.
M 88-07	February	J. Beirlant, Willekens	Rapid variation with remainder and rates of convergence.
M 88-08	April	Jan v. Doremalen, J. Wessels	A recursive aggregation-disaggregation method to approxi- mate large-scale closed queuing networks with multiple job types.

Number	Month	Author	Title
M 88-09	April	J. Hoogendoorn, R.C. Marcelis, A.P. de Grient Dreux, J. v.d. Wal, R.J. Wijbrands	The Vax/VMS Analysis and measurement packet (VAMP): a case study.
M 88-10	April	E. Omev, E. Willekens	Abelian and Tauberian theorems for the Laplace transform of functions in several variables.
M 88-11	April	E. Willekens, S.I. Resnick	Quantifying closeness of distributions of sums and maxima when tails are fat.
M 88-12	May	E.E.M. v. Berkum	Exact paired comparison designs for quadratic models.
M 88-13	May	J. ten Vregelaar	Parameter estimation from noisy observations of inputs and outputs.
M 88-14	May	L. Frijters, T. de Kok, J. Wessels	Lot-sizing and flow production in an MRP-environment.
M 88-15	June	J.M. Soethoudt, H.L. Trentelman	The regular indefinite linear quadratic problem with linear endpoint constraints.
M 88-16	July	J.C. Engwerda	Stabilizability and detectability of discrete-time time-varying systems.
M 88-17	August	A.H.W. Geerts	Continuity properties of one-parameter families of linear- quadratic problems without stability.
M 88-18	September	W.E.J.M. Bens	Design and implementation of a push-pull algorithm for manpower planning.
M 88-19	September	A.J.M. Driessens	Ontwikkeling van een informatie systeem voor het werken met Markov-modellen.
M 88-20	September	W.Z. Venema	Automatic generation of standard operations on data structures.

Number	Month	Author	Title
M 88-21	October	A. Dekkers E. Aarts	Global optimization and simulated annealing.
M 88-22	October	J. Hoogendoorn	Towards a DSS for performance evaluation of VAX/VMS-clusters.
M 88-23	October	R. de Veth	PET, a performance evaluation tool for flexible modeling and analysis of computer systems.
M 88-24	October	J. Thiemann	Stopping a peat-moor fire.
M 88-25	October	H.L. Trentelman J.M. Soethoudt	Convergence properties of indefinite linear quadratic problems with receding horizon.