Binary sequences with restricted repetitions

by

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Abstract

Generating functions are given for the number of binary sequences, consisting of \( r \) zeros and \( s \) ones, no bit occurring more than \( k \) times in succession. For \( k = 2 \) a function theoretic analysis is given for the number of sequences containing as many zeros as ones.

AMS (MOS) Subject Classifications 05A15, 40C15
I. Introduction.

Recently, in feedback communication theory the following coding scheme was considered:

Let \( k \) be a fixed integer \( \geq 2 \). A message sequence is supposed to be a binary sequence in which no \( k+1 \) successive bits are all of the same parity. This sequence is to be transmitted across a binary symmetric channel with a noiseless, delayless feedback link. The received digits are sent back via the feedback link, so that the transmitter is aware of the transmission errors. Every time a transmission error occurs, a block of \( k+1 \) repetitions of the correct bit is inserted in the message sequence immediately after the symbol that was wrongly received. Transmission of message sequence plus inserted correction bits is continued until a given part of the original message sequence is transmitted. The receiver has various decoding procedures at his disposal (cf. [1], [2]). Different message sequences turn out to have different sensitivities with respect to channel errors, sequences with (almost) as many zeros as ones being the least sensitive (balanced sequences). In this paper a recurrence is given for the number of message sequences with prescribed \((0,1)\) inventory. For \( k = 2 \) a function theoretic analysis is given for the number of balanced sequences.
II. Mathematical formulation. Elementary results.

Let $k$ be a fixed positive integer. Let $S = S_k$ be the set of all finite binary sequences that contain no $k + 1$ zeros in succession and no $k + 1$ ones in succession. More specifically, let $A = A_k$ and $B = B_k$ denote the (complementary) subsets of $S$ consisting of those binary sequences, that start with a zero and with a one, respectively. Finally, for all $r \geq 0$, $s \geq 0$ \((r,s) \neq (0,0)\), $a_{r,s}$ and $b_{r,s}$ are defined to be the number of sequences in $A$ and $B$ respectively, that contain $r$ zeros and $s$ ones. It is useful to define $a_{00} := b_{00} := 1$, and $a_{rs} := b_{rs} := 0$ if $r < 0$ or $s < 0$. Every sequence in $A$ can be split up uniquely in a starting block of, say $j$ \((1 \leq j \leq k)\) zeros and a (possibly empty) sequence in $B$. A similar argument holds for sequences in $B$, so that

\[
\begin{align*}
\left\{ \begin{array}{l}
a_{r,s} = b_{r-1,s} + b_{r-2,s} + \cdots + b_{r-k,s} \\
b_{r,s} = a_{r,s-1} + a_{r,s-2} + \cdots + a_{r,s-k}
\end{array} \right. \\
\text{for } r \geq 0, s \geq 0, \text{ and } (r,s) \neq (0,0).
\]

Define the generating functions $\alpha$ and $\beta$ by

\[
\begin{align*}
\alpha(x,y) := \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{r,s} x^r y^s \\
\beta(x,y) := \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} b_{r,s} x^r y^s.
\end{align*}
\]

Then (1) can be restated in the form

\[
\left\{ \begin{array}{l}
\alpha(x,y) - 1 = (x + x^2 + \cdots + x^k)\beta(x,y) \\
\beta(x,y) - 1 = (y + y^2 + \cdots + y^k)\alpha(x,y)
\end{array} \right.
\]

so that

\[
\begin{align*}
\alpha(x,y) &= \frac{1 + x + x^2 + \cdots + x^k}{1 - (x + x^2 + \cdots + x^k)(y + y^2 + \cdots + y^k)} \\
\beta(x,y) &= \frac{1 + y + y^2 + \cdots + y^k}{1 - (x + x^2 + \cdots + x^k)(y + y^2 + \cdots + y^k)}.
\end{align*}
\]
The apparent identity \( a(x,y) = b(y,x) \) is immediately clear from the fact, that replacing zeros by ones and ones by zeros transforms every sequence of \( A \) into a unique sequence of \( B \) and conversely, so that \( a_{r,s} = b_{s,r} \). This argument also enables us to give an explicit construction of the array \((a_{r,s})_{r,s}\) in a recurrent way, viz.

\[
\begin{align*}
\begin{cases}
a_{r,s} := 0 & (r < 0 \text{ or } s < 0) \\
a_0,0 := 1
\end{cases}
\end{align*}
\]

\( \begin{align*}
a_{r,s} &= a_{s,r-1} + a_{s,r-2} + \ldots + a_{s,r-k} & (r \geq 0, s \geq 0, \\
& & (r,s) \neq (0,0))
\end{align*} \tag{4} \]

A more symmetric recurrence, which also directly follows from (3) may be obtained by applying (4) twice, i.e.

\[
\begin{align*}
\begin{cases}
a_{r,s} := 0 & (r < 0 \text{ or } s < 0) \\
a_{r,0} := 1 & (0 \leq r \leq k) \\
a_{r,0} := 0 & (r > k) \\
a_0,s := 0 & (s \geq 1)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
a_{r,s} &= \sum_{i=1}^{k} \sum_{j=1}^{k} a_{r-i,s-j} & (r \geq 1, s \geq 1)
\end{align*} \tag{5}
\]

For \( k = 2 \), e.g. the array \((a_{r,s})_{r,s}\) reads as follows:

<table>
<thead>
<tr>
<th>( r )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>12</td>
<td>17</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>13</td>
<td>29</td>
<td>42</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td>33</td>
<td>71</td>
<td>104</td>
</tr>
</tbody>
</table>

The array \((a_{r,s})_{r,s}\) for \( k = 2 \)
Remark. The number of sequences in $A$ of length $n = r + s$ equals $F_n$, the $n$-th Fibonacci number (n-th Fibonacci number of order $k$). This is easily illustrated by the generating function $a(t,t) = (1 - t - t^2 - \ldots - t^k)^{-1}$.

An interesting subset of $A$ is formed by the balanced sequences, i.e. sequences for which $r = s$. Their number corresponds with the number of paths in an $s \times s$ square from the left bottom vertex to the right top vertex, that have minimal length, consist of only horizontal and vertical segments of integer length $\leq k$ and start in horizontal direction. For arbitrary $k$ the analysis of these numbers is hard. For $k = 2$, however, a generating function and a recurrence relation can be found explicitly. For $k = 2$ the numbers are found on the diagonal $(d_s)_{s=0}^\infty$ of the array $(a_{r,s})_{r,s}$ and read as follows:

$$(d_s)_{s=0}^\infty := (1, 1, 3, 7, 17, 42, 104, 259, 648, 1627, 4098, 10350, \ldots).$$
III. Function theoretic analysis for the case $k = 2$.

The double series

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{r,s} x^r y^s$$

(cf. (5))

is absolutely and uniformly convergent for complex $x$ and $y$, $|x| = 1$, $|y| \leq \frac{1}{4}(\sqrt{3} - 1)$, hence the integral

$$\int_{|w|=1} \frac{w^{-1} a(w, z)}{w - (1 + w)(zw + z^2)} \, dw =$$

$$\int_{|w|=1} \frac{1 + w + w^2}{w(zw + z^2)} \, dw =$$

$$\int_{|w|=1} \frac{1 + w + w^2}{-z(w - w_1)(w - w_2)} \, dw ,$$

where $w_1$ and $w_2$ are the roots of the quadratic equation

$$zw^2 + (z^2 + z - 1)w + z^2 = 0 ,$$

so

$$w_1 = (2z)^{-1} [- z^2 - z + 1 + (1 - 2z - z^2 - 2z^3 + z^4)^{\frac{1}{2}}]$$

$$w_2 = (2z)^{-1} [- z^2 - z + 1 - (1 - 2z - z^2 - 2z^3 + z^4)^{\frac{1}{2}}] .$$

For small $z$ the root $w_1$ is outside and $w_2$ is inside the unit circle, so that by the residue theorem out integral has the value

$$\frac{w_2^2 + w_2 - 1}{z(w_2 - w_1)} ,$$
which can be written in the form

\[
\sum_{s=0}^{\infty} \frac{d_s}{z^s} = \sum_{s=0}^{\infty} \frac{a_{ss}}{z^s} = 
\]

\[
= - \frac{1}{2z^2} + \frac{1}{2} + \frac{1}{2z^2} \left( \frac{1}{z} - \frac{1}{z} - \frac{1}{z} + z^2 \right) (1 - 2z - z^2 - 2z^3 + z^4)^{-1} = 
\]

\[
= - \frac{1}{2z^2} + \frac{1}{2} + \frac{1}{2z^2} (1 - z)^2 (1 + z + z^2)^{\frac{1}{2}} (1 - 3z + z^2)^{-\frac{1}{2}} = 
\]

\[
= - \frac{1}{2z^2} \frac{2}{z_1} + \frac{1}{2} + \frac{1}{2z^2} \left( 1 - z \right)^2 \left( 1 - \frac{z}{z_1} \right)^{\frac{1}{2}} \left( 1 - \frac{z}{z_2} \right)^{\frac{1}{2}} \left( 1 - \frac{z}{z_3} \right)^{-\frac{1}{2}} \left( 1 - \frac{z}{z_4} \right)^{-\frac{1}{2}},
\]

where

\[
z_1 = e^{\frac{\pi}{3}i}, \quad z_2 = e^{-\frac{\pi}{3}i}, \quad z_3 = \frac{1}{2} (3 + \sqrt{5}), \quad z_4 = \frac{1}{2} (3 - \sqrt{5}).
\]

**Corollary.** Since this function has \(z_4\) as branch point of smallest absolute value, it follows that \(d_s\) asymptotically behaves as the Taylor coefficients of \((1 - \frac{z}{z_4})^{-\frac{1}{2}}\). By Stirling's formula this yields \(d_s \sim D s^{-\frac{1}{2}} F_{2s}\), \(D\) being a constant, \(F_{2s}\) a Fibonacci number.

It is also possible to obtain a recurrence relation for \(d_s\) from (6). For this purpose we write

\[
G(z) := - \frac{1}{2z^2} + \frac{1}{2} + \frac{1}{2z^2} \left( 1 - z \right)^2 (1 + z + z^2)^{\frac{1}{2}} (1 - 3z + z^2)^{-\frac{1}{2}},
\]

so that

\[
2z^2 G(z) + 1 - z^2 = (1 - z)^2 (1 + z + z^2)^{\frac{1}{2}} (1 - 3z + z^2)^{-\frac{1}{2}},
\]

and, by logarithmic differentiation

\[
\frac{4z G + 2z^2 G' - 2z}{2z^2 G + 1 - z^2} = - \frac{2}{1 - z} + \frac{1 + 2z}{1 + z + z^2} + \frac{3 - 2z}{1 - 3z + z^2},
\]

\[
\frac{2G + zG' - 1}{2z^2 G + 1 - z^2} = \frac{1 + 3z^2 - z^3}{1 - 3z + z^2 - z^3 + 3z^4 - z^5},
\]
\[ G'(z - 3z^2 + z^3) + G(2 - 6z - 2z^3) = 2 - 3z + 3z^2 - 2z^3. \]

Substitution of \( G(z) = \sum_{s=0}^{\infty} d_s z^s \) yields, by identification of coefficients

\[
\begin{align*}
    d_0 &= 1, \quad d_1 = 1, \quad d_2 = 3, \quad d_3 = 7, \quad d_4 = 17, \\
    (n+2)d_n - 3(n+1)d_{n-1} + (n-2)d_{n-2} - (n-1)d_{n-3} + \\
    &\quad + 3(n-4)d_{n-4} - (n-5)d_{n-5} = 0 \quad (n \geq 5).
\end{align*}
\]

References.


Note added in proof:

In a recent paper (The Fibonacci Quarterly, Vol. 12, No 1, 1974, p. 1-10) L. Carlitz gives generating functions like (3) for the slightly more general situation where no \( k+1 \) successive ones and no \( \ell+1 \) successive zeros are allowed.