The graph with spectrum $14^1 2^{40} (-4)^{10} (-6)^9$

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Abstract We show that there is a unique graph with spectrum as in the title. It is a subgraph of the McLaughlin graph. The proof uses a strong form of the eigenvalue interlacing theorem to reduce the problem to one about root lattices.

Keywords Graph spectrum · Strongly regular graph · Root lattice

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1 The graph $\Delta$

It was shown in [5] that there is a unique graph $Z$ with spectrum $30^1 2^{90} (-10)^{21}$ (with multiplicities written as exponents), namely the collinearity graph of the unique generalized quadrangle with parameters $GQ(3, 9)$. It is strongly regular with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$. Its automorphism group is $U_4(3) \cong PG\tilde{O}_6^*(3)$ (of order $2^{10} \cdot 3^6 \cdot 5 \cdot 7$), where the * denotes that the form may be multiplied by a constant.

It was shown in [1] that there is a unique graph $Y$ with spectrum $20^1 2^{60} (-7)^{20}$. It is strongly regular with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$, and is the second subconstituent of $Z$, the subgraph induced on the set of vertices at distance 2 from a fixed

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vertex $a$ of $Z$. Its automorphism group is $3^4 : ((2 \times S_6) \cdot 2)$ acting rank 3, the point stabilizer in $\text{Aut}(Z)$. One construction of $Y$ is found by taking $1^4 / \{1\}$ (where $1$ denotes the all-1 vector) inside $F_3^6$, where two cosets are adjacent when they differ by a weight-3 vector.

Let $\Delta$ be the second subconstituent of $Y$, the subgraph induced on the set of vertices at distance 2 from a fixed vertex $b$ of $Y$. Then $\Delta$ has spectrum $14^{1} 2^{40} (-4)^{10} (-6)^{9}$ (apply Theorem 5.1 of [5]) and automorphism group $(2^2 \times S_6) \cdot 2$, the stabilizer of the unordered pair $[a, b]$ in $\text{Aut}(Z)$, twice as large as the point stabilizer of $\text{Aut}(Y)$. The above description of $Y$ leads to a description of $\Delta$ as the graph on the cosets in $F_3^6$ with coordinates (up to permutation) either $000012 + \langle 1 \rangle$ or $001122 + \langle 1 \rangle$, where two cosets are adjacent when they differ by a weight 3 vector.

In this note we show that the graph $\Delta$ is determined by its spectrum.

This is an interesting case. The uniqueness proof is elegant and quite different from the methods found in the literature (cf. [3,4]).

## 2 Interlacing

An important tool is the following lemma on interlacing eigenvalues ([6], Theorem 2.1 (i), (ii); see also [2], Theorem 3.3.1).

**Lemma 2.1** Let $\Gamma$ be a graph on $n$ vertices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$, and let $\{X_1, \ldots, X_m\}$ be a partition of the vertex set of $\Gamma$ into nonempty parts. Let $r_{ij}$ be the average number of neighbours in $X_j$ of a vertex in $X_i$. Then the matrix $R = (r_{ij})$ has real eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, which satisfy

(i) (interlacing) $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$;

(ii) if $\mu_i = \lambda_i$, or $\mu_i = \lambda_{n-m+i}$ for some $i \in \{1, \ldots, m\}$, then $R$ has a $\mu_i$-eigenvector $v = (v_1, \ldots, v_m)^\top$, such that the vector $w \in \mathbb{R}^n$ whose entries are equal to $v_j$ for all vertices in $X_j$ $(j = 1, \ldots, m)$ is a $\mu_i$-eigenvector of $\Gamma$.

For example if $m = 1$ it follows that the average valency $\bar{k}$ of $\Gamma$ is at most equal to $\lambda_1$, and equality implies that the all-1 vector is a $\lambda_1$-eigenvector of $\Gamma$. Since $n\bar{k} = \sum \lambda_i^2$ it follows that $\Gamma$ is regular of valency $\lambda_1$ if $n\lambda_1 = \sum \lambda_i^2$.

## 3 Graphs cospectral to $\Delta$

Let $\Gamma$ be a graph with the same spectrum $14^{1} 2^{40} (-4)^{10} (-6)^{9}$ as $\Delta$.

We shall write $x \sim y$ $(x \not\sim y)$ when $x$ is a (non)neighbour of $y$ in $\Gamma$, and denote the number of common neighbours of $x$ and $y$ by $\lambda(x, y)$ ($\mu(x, y)$).

(i) By Lemma 2.1 we know that $\Gamma$ is regular of valency 14. Moreover $\Gamma$ is connected, because the multiplicity of the eigenvalue 14 equals 1.

If $\Gamma$ has adjacency matrix $A$, then $(A - 2I)(A + 4I)(A + 6I) = 72J$ so that $(A^3)_{xx} = 8$, and it follows that each vertex is in four triangles.

(ii) For a vertex $x$, let $T_x$ be a set of eight neighbours of $x$ such that $\{x\} \cup T_x$ contains the four triangles on $x$. Let $S_x$ be the set of the remaining six neighbours of $x$, and let $N_x$ be the set of 45 nonneighbours of $x$. The matrix of average row sums of $A$, partitioned according to $\{\{x\}, T_x, S_x, N_x\}$ is
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\[
\begin{pmatrix}
0 & 8 & 6 & 0 \\
1 & 1 & 0 & 12 \\
1 & 0 & 0 & 13 \\
0 & \frac{96}{45} & \frac{78}{45} & \frac{456}{45}
\end{pmatrix}
\]

with eigenvalues $14, 2, 0.40, -5.27$. The 2-eigenspace is $\langle (15, 3, 1, -1)^T \rangle$. By Lemma 2.1 it follows that the vector that is constant $15, 3, 1, -1$ on $\{x\}, T_x, S_x, N_x$, respectively, is 2-eigenvector of $A$. Therefore each vertex in $T_x$ has precisely one neighbour in $T_x$, that is, two triangles on $x$ have only $x$ in common. It also follows that if $z$ is a non-neighbour of $x$ with $a$ neighbours in $T_x$ and $b$ neighbours in $S_x$, then $2a + b = 6$ while $a + b = \mu(x, z)$, so that $a = 6 - \mu(x, z)$. In particular, $\mu(x, z) = 3$ implies that $z$ has no neighbours in $S_x$.

(iii) The rank 10 matrix $B = 4J - (A - 2I)(A + 6I)$ is positive semi-definite and hence can be written $B = N^T N$ for a $10 \times 60$ matrix $N$.

Let $\bar{x}$ be column $x$ of $N$. Then $x \mapsto \bar{x}$ is a representation of $\Gamma$ in Euclidean 10-space, with

\[
(\bar{x}, \bar{y}) = \begin{cases} 
2 & \text{if } x = y \\
-\lambda(x, y) & \text{if } x \sim y \\
4 - \mu(x, y) & \text{if } x \not\sim y
\end{cases}
\]

It follows that for nonadjacent vertices $x, y$ one has $2 \leq \mu(x, y) \leq 6$.

If $\{x, y, z\}$ is a triangle, then $\bar{x} + \bar{y} + \bar{z} = 0$ (since this sum has squared norm 0).

The matrix $B$ satisfies $JB = 0$ and $AB = -4B$ and $B^2 = 12B$ so that the rows of $B$ are integral vectors with sum 0 and squared norm 24.

Row $x$ of $B$ has a 2 at the $x$-position, and a $-1$ at the 8 positions $z \in T_x$ (with $\lambda(x, z) = 1$). If $\bar{x} = \bar{y}$, so that rows $x$ and $y$ of $B$ are identical, then $\mu(x, y) = 2$ and we see two 2’s and at least fourteen $-1$’s in each row, and since there can be at most two more nonzero entries, the row sum is nonzero, contradiction. It follows that the representation is injective.

If $(\bar{x}, \bar{y}) = -2$, then $\bar{y} = -\bar{x}$. Given $x$, this happens for at most one $y$. It follows that a row of $B$ has entries either $2^1 1^8 0^{42} (-1)^8 (-2)^1$ or $2^1 1^9 0^{39} (-1)^{11}$ (with multiplicities written as exponents).

(iv) Let us call a triangle a line. If $\mu(x, y) = 3$ then each of the six edges connecting $x$ and $y$ with their common neighbours are in a line. Now there are 24 lines not on $x$ meeting $T_x$, and each $y$ with $\mu(x, y) = 3$ determines three such lines, so if there are 9 such points $y$ then some line is seen twice. We find a line $\{y, y', z\}$ with $x \sim z$. Now $0 = (\bar{x}, \bar{y}) + (\bar{x}, \bar{y}') + (\bar{x}, \bar{z}) = 1 + 1 + (-1) = 1$, contradiction. It follows that no row of $B$ has pattern $2^1 1^9 0^{39} (-1)^{11}$.

(v) A set of roots (vectors of squared norm 2) with integral inner products spans a root lattice ([2], §3.10), so $A = \{x \mid x \in \text{VT} \}$ is a 10-dimensional root lattice, orthogonal direct sum of summands of the form $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7, \text{or } E_8$.

(vi) The roots of the orthogonal direct sum of root lattices are the roots of the summands, so that an orthogonal direct sum decomposition of $A$ gives a partition of VT such that $(\bar{y}, \bar{z}) = 0$ if $y, z$ are vertices from different parts. It follows that the three vertices of a triangle belong to the same part.

Consider the graph $T$ with vertex set VT where two vertices $x, y$ are adjacent when $(\bar{x}, \bar{y}) = -1$, i.e., when $xy$ is an edge in a triangle of $\Gamma$. Given $x$, consider the five subsets $S_i = \{u \in \text{VT} \mid (\bar{x}, \bar{u}) = i \}$ for $i = 2, 1, 0, -1, -2$. We have $|S_2| = |S_{-2}| = 1, |S_{-1}| = |S_1| = 8, |S_0| = 42$. The graph $T$ is regular of valency 8. In $T$, any vertex $y \in S_{-1}$ has 1 neighbour $x$, 1 neighbour in $S_{-1}$, 3 neighbours in $S_1$, and hence 3 neighbours in $S_0$. A vertex $z \in S_0$ has 0 or 2 $\Gamma$-neighbours in $S_{-1}$, so at most 2 $T$-neighbours. We see that the connected component of $T$ containing $x$ has at least $1 + 8 + 8 + 1 + (8 \cdot 3)/2 = 30$ vertices.

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It follows that either the root lattice $\Lambda$ is indecomposable, i.e., is $A_{10}$ or $D_{10}$, or has precisely two summands. Since $A_n$ has $n(n+1)$ roots, and $D_n$ has $2n(n-1)$ roots, the possibilities in the latter case are $A_5 + A_5$, $A_5 + D_5$, $D_5 + D_5$.

(vii) Suppose $\Lambda$ has a direct summand $D_5$. The root system $D_5$ has 40 roots, and 30 occur as images of vertices in the corresponding connected component $C$ of $T$. Let $\Phi$ be the graph on the 40 roots of $D_5$, adjacent when they have inner product $-1$, and consider $C$ a subset of the vertex set of $\Phi$. Let $D$ be the set of 10 roots not in $C$. The graph $\Phi$ is regular of valency 12. The valency inside $C$ is 8, so each vertex in $C$ has 4 neighbours in $D$. This gives 120 edges meeting $D$, so there are no internal edges in $D$ and no two roots of $D$ have inner product $-1$. Both $\Phi$ and $C$ are closed under $u \mapsto -u$, so also $D$ is, and no two roots of $D$ have inner product 1. Consequently, $D$ has only inner products 2, 0, $-2$ and consists of five mutually orthogonal pairs of opposite roots. But $D_5$ does not contain 5 mutually orthogonal roots. Contradiction.

(viii) Consider the graph $\Pi$ with as vertices the 30 pairs $\pm \bar{x}$, adjacent when they have non-zero inner product. Then $\Pi$ has valency 8 and $\lambda = 4$. Using a Weetman argument (cf. [7]) we see that a connected component of $\Pi$ has fewer than 30 vertices. It will follow that $\Lambda \cong A_5 + A_5$.

As follows. For geodesics $x_0 \sim x_1 \sim x_2 \sim \ldots$ we find lower bounds $n_i$ for the number of common neighbours of $x_i$ and $x_{i+1}$ at distance $i$ from $x_0$. We can take $n_1 = 2$ since two nonadjacent vertices in a 4-regular graph on 8 vertices must have at least 2 common neighbours. We can take $n_2 = 3$ since the set of common neighbours of $x_2$ and $x_0$ has valency at least $n_1 = 2$, and hence size at least 3 (and an 8-vertex graph of degree 4 cannot have a cut set of size 2). Now the local graph at $x_3$ has at least 4 vertices at distance 2 from $x_0$, and hence cannot have any at distance 4 from $x_0$ and a connected component of $\Pi$ has diameter at most 3 and size at most $1 + 8 + (8 \cdot 3)/3 + (8 \cdot 2)/4 = 21$, as desired.

(ix) Thus far, we identified the 60 vertices of $\Gamma$ with the 60 roots of $A_5 + A_5$, and can recognize the triangles of $\Gamma$. It remains to find the edges of $\Gamma$ that are not in a triangle.

Let $C$ and $D$ be the two sets of vertices belonging to the two systems $A_5$. Given $x \in C$, the 12 vertices $y \in C$ with $\langle x, y \rangle = 0$ have common $T$-neighbours with $x$, so are nonadjacent to $x$ in $\Gamma$. That determines the induced subgraph on $C$ and on $D$, and we have to find the edges between $C$ and $D$.

Suppose $x \in C$. If $\bar{y} = -\bar{x}$, then $\mu(x, y) = 6$, and the six common neighbours of $x$ and $y$ live in $D$, and form all neighbours of $x$ in $D$. If $u$ is a common neighbour of $x$ and $y$, and $\bar{v} = -\bar{u}$, then also $v$ is a common neighbour of $x$ and $y$. This means that for the edges across we can identify pairs of opposite roots, and have a geometry with 15 points and 15 lines, where each point is on 3 lines and each line has 3 points. The points can be identified with the pairs from a 6-set. Then subgraph on the set of points is $T(6)$. The lines consist of three mutually disjoint pairs. This is the unique generalized quadrangle of order 2.

This proves that $\Gamma$ is uniquely determined by its spectrum, and hence must be isomorphic to $\Delta$.

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References

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