UPPER BOUND ON THE EXPECTED SIZE OF THE INTRINSIC BALL

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Abstract

We give a short proof of Theorem 1.2(i) from [5]. We show that the expected size of the intrinsic ball of radius r is at most Cr if the susceptibility exponent γ is at most 1. In particular, this result follows if the so-called triangle condition holds.

Let $G = (V, E)$ be an infinite connected graph. We consider independent bond percolation on $G$. For $p \in [0, 1]$, each edge of $G$ is open with probability $p$ and closed with probability $1 - p$ independently for distinct edges. The resulting product measure is denoted by $\mathbb{P}_p$. For two vertices $x, y \in V$ and an integer $n$, we write $x \rightarrow y$ if there is an open path from $x$ to $y$, and we write $x \xrightarrow{\leq n} y$ if there is an open path of at most $n$ edges from $x$ to $y$. Let $C(x)$ be the set of all $y \in V$ such that $x \rightarrow y$. For $x \in V$, the intrinsic ball of radius $n$ at $x$ is the set $B_I(x, n)$ of all $y \in V$ such that $x \xrightarrow{\leq n} y$. Let $p_c = \inf \{ p : \mathbb{P}_p(|C(x)| = \infty) > 0 \}$ be the critical percolation probability. Note that $p_c$ does not depend on a particular choice of $x \in V$, since $G$ is a connected graph. For general background on Bernoulli percolation we refer the reader to [2].

In this note we give a short proof of Theorem 1.2(i) from [5]. Our proof is robust and does not require particular structure of the graph.

Theorem 1. Let $x \in V$. If there exists a finite constant $C_1$ such that $\mathbb{E}_p|C(x)| \leq C_1(p_c - p)^{-1}$ for all $p < p_c$, then there exists a finite constant $C_2$ such that for all $n$,

$$\mathbb{E}_p|B_I(x, n)| \leq C_2 n.$$ 

Before we proceed with the proof of this theorem, we discuss examples of graphs for which the assumption of Theorem 1 is known to hold. It is believed that as $p \to p_c$, the expected size of $C(x)$ diverges like $(p_c - p)^{-\gamma}$. The assumption of Theorem 1 can be interpreted as the mean-field bound $\gamma \leq 1$. It is well known that for vertex-transitive graphs this bound is satisfied if the triangle condition holds at $p_c$ [1]: For $x \in V$,

$$\sum_{y, z \in V} \mathbb{P}_{p_c}(x \leftrightarrow y) \mathbb{P}_{p_c}(y \leftrightarrow z) \mathbb{P}_{p_c}(z \leftrightarrow x) < \infty.$$ 

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This condition holds on certain Euclidean lattices \([3, 4]\) including the nearest-neighbor lattice \(\mathbb{Z}^d\) with \(d \geq 19\) and sufficiently spread-out lattices with \(d > 6\). It also holds for a rather general class of non-amenable transitive graphs \([6, 8, 9, 10]\). It has been shown in \([7]\) that for vertex-transitive graphs, the triangle condition is equivalent to the so-called open triangle condition. The latter is often used instead of the triangle condition in studying the mean-field criticality.

**Proof of Theorem 1.** Let \(p < p_c\). We consider the following coupling of percolation with parameter \(p\) and with parameter \(p_c\). First delete edges independently with probability \(1 - p_c\), then every present edge is deleted independently with probability \(1 - \frac{p}{p_c}\). This construction implies that for \(x, y \in V\), \(p < p_c\), and an integer \(n\),

\[
P_{p_c}(x \leftrightarrow^n y) \geq \left( \frac{p}{p_c} \right)^n P_{p}(x \leftrightarrow^n y).
\]

Summing over \(y \in V\) and using the inequality \(P_{p}(x \leftrightarrow^n y) \leq P_{p}(x \leftrightarrow y)\), we obtain

\[
E_{p_c}|B_I(x, n)| \leq \left( \frac{p_c}{p} \right)^n E_p|C(x)|.
\]

The result follows by taking \(p = p_c(1 - \frac{1}{2n})\).

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**References**


