ON THE LOW-FREQUENCY SUPPRESSION PERFORMANCE OF DC-FREE RUNLENGTH-LIMITED MODULATION CODES

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Abstract—Basic trade-offs between the rate of combined DC-free runlength-limited (DCRLL) modulation codes and the amount of suppression of low-frequency components are presented. The main results are obtained by means of a numerical study of dependencies between statistical properties of ideal, ‘maxentropic’ DCRLL sequences. The numerical results are mathematically founded by proving the observed behavior of the Shannon capacity of the DCRLL constraints for asymptotically large values of digital sum variation. Presented characteristics of maxentropic DCRLL sequences comply with the corresponding properties of maxentropic pure DC-free sequences, as previously considered by Justesen and Immink. Knowledge of the maxentropic bounds enables us to evaluate the performances of implemented DCRLL codes with respect to their low-frequency suppression capability. Among the considered codes are the EFM code as applied in the Compact Disc system, and the EFMPPlus code which has been adopted as the coding format of the Digital Versatile Disc system.

I. INTRODUCTION

IN THE Compact Disc system (CD), a combined DC-free runlength-limited (DCRLL) modulation code is used to transform the digital user (audio) bit stream into a sequence of binary channel symbols which is suitable for both storage on the disc and retrieval from the disc. The runlength is known as the number of consecutive like symbols occurring in a sequence. Runlength-limited sequences are characterized by two parameters, \( T_{\min} = (d+1) \) and \( T_{\max} = (k+1) \), denoting the minimum and the maximum runlengths occurring in the sequence. The parameter \( d \) controls the highest transition frequency while the parameter \( k \) ensures adequate frequency of transitions for synchronization of the read clock [2]. Suppression of the low-frequency content of the runlength-limited modulation sequence is employed in the CD system primarily to circumvent or reduce interaction between the data written on the disc and the servo systems that follow the track. According to Immink [8], the servo system is the Achilles' heel of the player as error correction is totally useless if track or clock loss occurs. Therefore, efficient suppression of the low-frequency components is the crucial criterion for designing DCRLL codes. The main goal is to achieve sufficient low-frequency suppression capability at minimum cost in code rate, i.e., the ratio between user bit rate and channel bit rate. In this paper, we determine the basic trade-offs between the rate of combined DC-free runlength-limited codes and the amount of suppression of low-frequency components. Knowledge of these trade-offs enables us to evaluate the performances of several selected implemented DCRLL codes with respect to their low-frequency suppression capability. Among the considered codes are the EFM code as used in the CD system, and the EFMPPlus code which, supported by this research study, has been adopted as the coding format of the Digital Versatile Disc system (DVD). As indicated above, DCRLL codes are very suitable for application in optical storage systems. The wide use of optical storage systems in applications like audio, multimedia or software distribution consequently gives an impression of the great engineering relevance of combined DC-free runlength-limited codes. However, DCRLL codes are certainly not confined to optical recording practice.

The following sections relate to the study of ideal, ‘maxentropic’ DCRLL sequences. In the next section, we will give an intuitive description of maxentropic DCRLL sequences, and will indicate how these sequences are modeled. A detailed description of the mathematical background relevant for this research study is far beyond the scope of this paper. As sources for further reading, providing insight into both the underlying mathematical background and the relation of this study to previous work, we recommend the book by Immink [1] and the article by Kerpez et al. [4]. An overview of modulation and coding for recording systems has been presented by, for example, Siegel and Wolf [3].

II. PRELIMINARIES

We assume that binary user information with a bit rate of \( f_b = 1/T_b \) is translated into a coded channel sequence having the channel bit rate \( 1/T_c \), where \( R = T_c/T_b \) denotes the rate of the code. We define the Running Digital Sum (RDS) of the encoded sequence \( \{x_i\} = \{\ldots, x_{-1}, x_0, \ldots, x_i, \ldots\} \), \( x_i \in \{-1, 1\} \), as

[\[ z_i = \sum_{j=-\infty}^{i} x_j = z_{i-1} + x_i. \]

Pierobon [6] showed that a sequence is DC-free if, and only if, the RDS assumes a finite number of values. This number is called the Digital Sum Variation (DSV), denoted by \( N \). DCRLL sequences combine restrictions, or ‘constraints’, imposed on both the runlengths and the RDS. So, we characterize DCRLL sequences by three integer parameters \((d, k, N)\), where by definition \( 0 \leq d < k \leq N-2 \). In
sequences comply with the prior known relations of ideal DCRLL sequences. Thus all our results obtained for ideal DCRLL sequences satisfy the free sequences, often called 'charge-constrained sequences', implying a finite constraints in the form of a square matrix having a size smaller than $N$. On the basis of a run-length graph, we will construct a stationary Markov chain, i.e., a specific kind of stochastic process including statistical dependencies between the random variables. Such a Markov chain forms the mathematical model of a DCRLL sequence. The fact that a DCRLL sequence can be modeled by a Markov chain is essential since it allows the evaluation of the entropy of the sequence. A maxentropic DCRLL sequence is obtained by carefully defining the Markov chain probabilities, represented by the entries of the underlying square matrix. This maxentropic Markov chain model is briefly described in Appendix A. A detailed description of this model can be found in [4]. The Markov chain description of maxentropic DCRLL sequences allows the computation of all relevant statistical properties of these sequences.

Of these properties we will consider in particular the power spectral density function or power spectrum. We will denote the power spectrum of a maxentropic $(d,k,N)$ constrained sequence by $H(\omega)$, where $\omega = 2\pi f T_c$. We will use the method described by Kerpez et al. [4] to evaluate the power spectrum $H(\omega)$. For each point of the frequency axis, this method mainly involves inversion of two complex valued square matrices of sizes $N-d-1$. Another statistical property of maxentropic $(d,k,N)$ constrained sequences to be considered is the sum variance, denoted by $\sigma^2(d,k,N)$, i.e., the variance of the RDS. The reason for investigating the sum variance $\sigma^2(d,k,N)$ stems from the fact that for maxentropic pure charge-constrained sequences it is a criterion of the low-frequency characteristic [5]. The computation of the sum variance is briefly described in Appendix A.

The outline of the next sections is as follows. In Section III, we will consider the trade-off between low-frequency suppression and rate loss for maxentropic DCRLL sequences. In order to quantify the low-frequency characteristic of these sequences, we will introduce an appropriate 'cut-off frequency'. For a wide range of $(d,k,N)$ constraints, we will then numerically investigate the trade-off between cut-off frequency and extra redundancy. This trade-off turns out to be linear in good approximation. In Section IV, we will numerically investigate for a wide range of $(d,k,N)$ constraints whether the sum variance of maxentropic DCRLL sequences can be used as a criterion of their low-frequency characteristic. For large DSVs, our numerical results reveal simple expressions for the sum variance of maxentropic DCRLL sequences and for the relation between sum variance and cut-off frequency. In Section V, we will then investigate the asymptotic behavior of the extra redundancy, which we will prove in Appendix B for the case that the $k$ constraint remains finite as $N \to \infty$, i.e., for practically interesting $k$ constraints [1]. All the relations of maxentropic DCRLL sequences presented in Sections III - V have corresponding relations in the theory of maxentropic pure charge-constrained sequences, as previously considered by Justesen [5] and Immink [1]. In Section VI, we will evaluate the performances of several implemented DCRLL...
codes with respect to their low-frequency suppression capabilities. In particular, we will consider the EFM code [1] as applied in the CD system, EFMPlus [8] as applied in the DVD system, and the Miller Squared code [9]. In these performance evaluations, we will compare code properties obtained in computer simulations with the corresponding maxentropic performance bounds.

III. TRADE-OFF BETWEEN REDUNDANCY AND LOW-FREQUENCY SUPPRESSION OF MAXENTROPIC DCLRLL SEQUENCES

We will start by considering a few examples of power spectra of maxentropic DCLRLL sequences. As can be seen in Fig. 1, the spectral density of these sequences vanishes at zero frequency, and there is a region of frequencies, close to the zero frequency, where the spectral density is low. The width of this spectral notch can be quantified by a parameter called the cut-off frequency. Motivated by Fig. 1, we define the cut-off frequency of a maxentropic \((d, k, N)\) constrained sequence, denoted by \(\omega_0\), by

\[
H(\omega_0) = \frac{H_0(d, k)}{2}, \tag{2}
\]

where \(H_0(d, k)\) denotes the DC content \(H(\omega \to 0)\) of the corresponding maxentropic \((d, k)\) constrained sequence. An expression for computing \(H_0(d, k)\) has been presented by Immink [1], and an example of a cut-off frequency \(\omega_0\) is shown in Fig. 1.

![Figure 1: Examples of power spectra of maxentropic \((2,10, N)\) constrained sequences, together with the power spectrum of a maxentropic \((2,10)\) constrained sequence (dashed). By way of example, the cut-off frequency \(\omega_0\) is shown for a maxentropic \((2,10, 21)\) constrained sequence (dotted).](image)

From the power spectrum \(H(\omega)\), we extract the cut-off frequency \(\omega_0\) for a wide range of \((d, k, N)\) constraints, and relate it to the extra redundancy. It turns out that for maxentropic \((d, k, N)\) constrained sequences, there exists in good approximation a linear trade-off between cut-off frequency and extra redundancy, given by

\[
\omega_0 \approx \rho(d, k, N) \frac{2 \ln 2}{\pi^2/6 - 1}, \quad N \gg 1. \tag{3}
\]

Relation (3) is fundamental in the sense that the constant of proportionality between cut-off frequency and extra redundancy is independent of the \((d, k, N)\) constraints. This constant of proportionality has been found in accordance with the theory of maxentropic pure charge-constrained sequences [1], and will be mathematically justified in Section V and Appendix B. As an example, relation (3) is illustrated in Fig. 2 for maxentropic \((2, \infty, N)\) constrained sequences. Approximation (3) appears accurate to within 5\% for \(N > 20\) and, as Fig. 2 indicates, it is fairly reliable also for moderate DSVs. In the next section, we will focus on the sum variance of maxentropic DCLRLL sequences.

![Figure 2: Cut-off frequencies of several maxentropic \((2, \infty, N)\) constrained sequences versus extra redundancy. The dotted line indicates equality in (3).](image)

IV. THE SUM VARIANCE AS A CRITERION OF THE LOW-FREQUENCY CHARACTERISTIC OF MAXENTROPIC DCLRLL SEQUENCES

In the case of maxentropic pure charge-constrained sequences, the sum variance is related to the low-frequency characteristic [5]. In this section, we will investigate whether a corresponding relation also exists for maxentropic \((d, k, N)\) constrained sequences. We will start by analyzing the influence of the \((d, k, N)\) constraints on the sum variance of maxentropic DCLRLL sequences. Subsequently, we will consider the relation between sum variance and cut-off frequency of these sequences. We will proceed by performing a numerical analysis of these properties for a wide range of \((d, k, N)\) constraints.

A. The Influence of the Constraints on the Sum Variance

We will start by analyzing the sum variance of maxentropic DCLRLL sequences in the absence of a specific \(k\) constraint (i.e. \(k \to \infty\)). It turns out that the influence of the \(d\)
constraint on the sum variance $\sigma_z^2(d, \infty, N)$ is negligible. Hence, we can approximate $\sigma_z^2(d, \infty, N)$ by the sum variance of maxentropic pure charge-constrained sequences, i.e., $\sigma_z^2(d, \infty, N) \approx (1/12 - \pi^{-2}/2)(N + 1)^2$ [1]. This approximation appears accurate to within 3% for $N > 9$. In a next step, we analyze the influence of the $k$ constraint on the sum variance $\sigma_z^2(d, k, N)$. As an example, Fig. 3 shows

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{Sum variances of several maxentropic $(1, k, N)$ constrained sequences versus $k$. The DSV is denoted by $N$.}
\end{figure}

the sum variances of several maxentropic $(1, k, N)$ constrained sequences versus $k$. Apparently, the sum variance $\sigma_z^2(d, k, N)$ for fixed values of $d$ and $N$ decreases slightly with decreasing $k$. At a minimum $k$ constraint, i.e., $k = d + 1$, the sum variance can be relatively well approximated by $\sigma_z^2(d, d + 1, N) \approx (1/12 - \pi^{-2}/2)(N - d)^2$. This approximation appears accurate to within 5% for $N > 21$. At fixed $d$ and $k$ constraints, we conclude that for asymptotically large values of DSV, irrespective of the $d$ and $k$ constraints, the sum variance of maxentropic DCRLL sequences is obtained as

$$\sigma_z^2(d, k, N) \approx \left(\frac{1}{12} - \frac{1}{2\pi^2}\right)N^2, \quad N \gg 1. \quad (4)$$

Below, we will consider whether the sum variance is related to the low-frequency characteristic for maxentropic DCRLL sequences.

\subsection*{B. The Relation between Sum Variance and Low-frequency Characteristic}

We have investigated the relation between the sum variance $\sigma_z^2(d, k, N)$ and the cut-off frequency $\omega_0$ for a wide range of $(d, k, N)$ constraints. At fixed $d$ and $k$ constraints, we found that for large values of DSV there is a relation between sum variance and cut-off frequency of maxentropic DCRLL sequences according to

$$2\omega_0 \sigma_z^2(d, k, N) \approx H_0(d, k), \quad N \gg 1. \quad (5)$$

Approximation (5) appears accurate to within 10% for $N > 17$. For small values of DSV relation (5) is not reliable. We conclude that for large values of DSV, the sum variance of maxentropic DCRLL sequences can serve as a criterion of their low-frequency characteristic. At fixed $d$ and $k$ constraints, we obtain from (3) and (5) that maxentropic DCRLL sequences for large values of DSV exhibit a constant sum variance-extra redundancy product, i.e.,

$$\rho(d, k, N) \sigma_z^2(d, k, N) \approx \frac{\pi^2/6}{4\ln 2} H_0(d, k), \quad N \gg 1. \quad (6)$$

In the absence of a specific $k$ constraint, relation (6) is illustrated in Fig. 4. Approximation (6) appears accurate to within 10% for $N > 25$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Sum variances of several maxentropic $(d, \infty, N)$ constrained sequences versus extra redundancy, where $d = 0$ ($\circ$), 1 ($\times$), and 2 ($+$). For $d = 1$, the dashed line indicates equality in (6). The dotted curves represent a constant value of DSV, denoted by $N$.}
\end{figure}

\section*{V. Mathematical Foundation}

All the properties of maxentropic DCRLL sequences presented in the preceding sections were obtained by numerically analyzing dependencies between statistical properties of these sequences. We will give a mathematical foundation for these properties by briefly showing the observed asymptotic behavior of the extra redundancy. From (4) and (6) we obtain that at fixed $d$ and $k$ constraints, the extra redundancy for asymptotically large values of DSV is given by

$$\rho(d, k, N) \approx \frac{\pi^2}{2\ln 2} H_0(d, k), \quad N \gg 1. \quad (7)$$

In Appendix B, we will briefly present a proof of this relation for the case that $k$ remains finite as $N \rightarrow \infty$, i.e., for practically interesting $k$ constraints [1]. We have proved relation (7) elsewhere [14] in the absence of a specific $k$ constraint, but since this proof appears more involved than for bounded $k$ constraints, it has been omitted here. Actually, this proof provides a mathematical indication for interpreting the DCRLL constraints in the form $(d, \infty, N)$ in the absence of a specific $k$ constraint.
VI. PERFORMANCES OF IMPLEMENTED CODES

In this section, we will assess the performances of several selected DCRLL codes with respect to their low-frequency suppression capabilities. In order to obtain a fair comparison between different DCRLL codes, both axes of the power spectra of these codes will be normalized for a fixed user bit rate, i.e., we will consider $H^*(fT_b) = RH(2\pi fT_c/R)$. As a performance criterion, we then determine the power spectral density of the coded sequence at a small fraction of the user bit rate, for example at $f_b/1000$. Maxentropic DCRLL sequences, where $R = C(d,k,N)$, provide an upper bound in low-frequency suppression capability, given a certain $(d,k)$ constraint and redundancy. We will compare the low-frequency suppression capability of the coded sequence with the corresponding maxentropic performance bound. If there were no constraint imposed on the encoder/decoder complexity, the maxentropic bound could be reached arbitrarily closely. In practice however, a limited encoder/decoder complexity has an effect on the code rate and/or the low-frequency suppression capability of an implemented DCRLL code.

We will start by considering the EFM code [1] as applied in the CD player, EFMPlus [8] as applied in the DVD system, and two other EFM alternatives as described in [8]. These four codes all have finite values of DSV and satisfy the (2,10) runlength constraint. Computer simulations were performed by Immink [8] to evaluate the power spectra of these codes. The low-frequency suppression performance for each of these codes is illustrated in Fig. 5. Apparently, all four codes allow further improvement in the suppression of the low frequency components. A strategy for improving the EFMPlus low-frequency suppression performance by about 3 dB is presented in [8].

![Figure 5: Power spectral density $H^*(10^{-3})$ versus extra redundancy for maxentropic (2,10,N) constrained sequences, for the EFM code, for EFMPlus, and for two other EFM alternatives.](image)

As a further example, we will consider the Miller Squared code [9] satisfying the (1,5,7) constraint. The power spectrum of the Miller Squared code has been determined in a computer simulation. As obtained from Fig. 6, the Miller Squared code has excellent performance with respect to low-frequency suppression. A current application of the Miller Squared code is in the D-2 digital video tape recorder [10].

![Figure 6: Power spectral density $H^*(10^{-3})$ versus extra redundancy for maxentropic (1,5,N) constrained sequences and for the Miller Squared code.](image)

VII. CONCLUSION

We have investigated dependencies between statistical properties of maxentropic DCRLL sequences. For these sequences, we introduced the extra redundancy and the cut-off frequency as appropriate measures for the rate loss and the low-frequency characteristic. Between cut-off frequency and extra redundancy, we found in good approximation a linear trade-off. For large values of DSV, we observed that the cut-off frequency is related to the variance of the RDS. All these properties of maxentropic DCRLL sequences were found by numerically analyzing dependencies between statistical properties of these sequences. We have mathematically founded our numerical results by proving the observed behavior of the Shannon capacity of the DCRLL constraints for asymptotically large values of DSV. By interpreting pure charge constraints in the form $(0,\infty,N)$, the theory of maxentropic pure charge-constrained sequences is obtained as a special instance of expressions (1)-(7) presented here. Knowledge of maxentropic DCRLL sequences enabled us to evaluate the performances of implemented DCRLL codes with respect to their low-frequency suppression capability. We showed that the Miller Squared code has excellent performance with respect to low-frequency suppression, whereas EFM and EFMPlus as applied in the CD and DVD systems, respectively, allow further improvements.

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APPENDIX A
DERIVATION OF THE SUM VARIANCE

In this appendix, we will derive the sum variance \( \sigma^2(d, k, N) \) of maxentropic \((d, k, N)\) constrained sequences from the corresponding Markov chain description as presented by Kerpez et al. [4]. The \((d, k, N)\) constraint is represented by a runlength graph. Let \( \{s_m\}_{m \in I_M} \), \( I_M = \{1, 2, \ldots, M\} \), denote the set of states of this graph, and recall that the number of states equals \( M = N - d - 1 \). Each state \( s_m \) is associated with the value \( U_m = d + m - (N - 1)/2 \), and we draw an edge of length \( l_{m,n} = U_m + U_n \) leading from state \( s_m \) to state \( s_n \), if \( d + 1 \leq l_{m,n} \leq k + 1 \), where \( m, n \in I_M \). Let \( A_M = A_M(D) \) denote the adjacency matrix associated with this runlength graph, where \( A_m^n \) denotes the mth element of \( A_M \). The nonzero elements of \( A_M \) are given by \( A_{m,n} = D^{l_{m,n}} \), i.e., the adjacency matrix \( A_M \) can be expressed in the form

\[
A_{m,n} = D^{l+1} \begin{cases} D^l, & \text{if } m + n = M + 1 + l, \\ 0, & \text{else}, \end{cases}
\]

for \( m, n \in I_M \), and for \( l = 0, 1, \ldots, k - d \). Note that \( A_M \) is a Hankel matrix, i.e., \( A_M \) is constant on the antidiagonals. The Shannon capacity is given by \( C(d, k, N) = \log_2 \lambda \), where \( \lambda \) denotes the largest real root of the characteristic polynomial \( \det(A_M^{-1} - I) \), and \( I \) denotes the identity matrix. On the runlength graph there is based a stationary Markov chain which we describe by the state transition probability matrix \( P \) and the equilibrium distribution vector \( \pi \). We define the (nonzero) elements of \( P \) by \( P_{m,n} = \pi_m / \pi_n \lambda^{-l_{m,n}} \), where the vector \( \pi \) satisfies \( A_M(\lambda^{-1})v = v \). The equilibrium distribution vector is defined by \( \pi P = \pi \) together with \( \sum_{m=1}^{M-1} \pi_m = 1 \). The DCRL sequences generated from the Markov chain thus defined are maxentropic [4].

As shown in [4], to each state \( s_m \), with \( m \in I_M \), there is associated the number of the RDS according to \((d + m - (N - 1)/2)^2 \). Further, recall that a binary symbol can only increase or decrease the RDS by one. So, to each edge of length \( l \) of the runlength graph, we can uniquely assign \( l \) values representing the squares of the RDS along this edge. In this way the sum variance of a maxentropic \((d, k, N)\) constrained sequence can be evaluated according to

\[
\sigma^2(d, k, N) = \frac{1}{L} \sum_{l=d+1}^{k+1} \sum_{m=1}^{M} \pi_m P_{m,(M+l-d-m)} 2^{2m,l},
\]

where \( L \) denotes the average runlength, and

\[
2^{2m,l} = \sum_{i=0}^{l-1} (d + m - i - \frac{N - 1}{2})^2.
\]

APPENDIX B
ASYMPTOTIC BEHAVIOR OF THE EXTRA REDUNDANCY

In this appendix, we briefly prove the behavior of the extra redundancy for asymptotically large values of DSV. We consider relation (7) for small values of \( d \) in the case that \( k \) remains finite as \( N \to \infty \). Recall that the \((d, k, N)\) constraint is represented by the \( M \times M \) Hankel type adjacency matrix \( A_M = A_M(D) \) as defined in Appendix A, where \( M = N - d - 1 \) [4]. To be able to determine the Shannon capacity \( C(d, k, N) = -\log_2 D_{\min} \), we have to find the smallest \( D = D_{\min} > 0 \), such that the largest eigenvalue of \( A_M(D) \) equals unity, i.e., \( \lambda_{\max}(A_M) = 1 \), as found in, for example, Proposition 4 of [11]. There is a vast amount of literature on the asymptotics of the eigenvalues of Toeplitz matrices, for example [12]. Further, we observe that the matrix \( A^2_M \) almost has a Toeplitz structure. To be precise, the lower right \((M - p) \times (M - p)\) submatrix of \( A^2_M \) is a symmetric Toeplitz matrix, whose elements on the \( j \)th diagonal are given by

\[
c_j = D^{2d+2} \sum_{i=0}^{p-1} D^i D^{j+i+1},
\]

where \( c_j = c_j \) (\( j = 0, 1, \ldots, M - p - 1 \)) and \( p = k - d \). Note that \( \lambda_{\max}(A^2_M) = \lambda_{\max}(A_M) \), since \( A_M \) is symmetric. Further, \( A^2_M \) is irreducible, as found with the aid of, for example, Theorem 1.6. of [13]. We define a symmetric \( M \times M \) Toeplitz matrix by \( T_M = (c_{j-i}) \), where \( s, j = 0, 1, \ldots, M - 1 \). As \( A^2_M \leq T_M \) entry by entry, we obtain

\[
\lambda_{\max}(T_{M-p}) \leq \lambda_{\max}(A^2_M) \leq \lambda_{\max}(T_M),
\]

as found with the aid of, for example, Lemma 2.4. (left hand side) and Theorem 2.1.3. (right hand side) of [13]. For \( M \to \infty \), the maximum eigenvalue \( \lambda_{\max}(T_{M-p}) \approx \lambda_{\max}(T_M) \) is found with the aid of, for example, Theorem 2.1 of Widom [12]. From this theorem we obtain the maximum eigenvalue of \( A^2_M \) as \( M \to \infty \) according to

\[
\lambda_{\max}(A^2_M) \approx f(0) + \frac{\pi^2 f''(0)}{2 M^2},
\]

where

\[
f(\Theta) = \sum_{j=-\infty}^{\infty} c_j e^{ij\Theta} = D^{2d+2} \sum_{i=0}^{p} D^i e^{i\Theta} D^{j+i+1}.
\]

Using a Taylor expansion for \( f(\Theta) \), we obtain

\[
f''(0) = 2 D^{2d+2} \left( \sum_{i=0}^{p} D^i \right)^2 - \sum_{i=0}^{p} D^i D^j \sum_{i=0}^{p} D^i,
\]

where \( f''(0) < 0 \) (Cauchy-Schwarz inequality). Further, we introduce the definitions \( \epsilon = \pi/M \), \( \phi(D) = f(0) = D^{2d+2} \left( \sum_{i=0}^{p} D^i \right)^2 \), and \( \psi(D) = -f''(0)/2 \). Hence, we have to find the smallest root \( D = D_0 \) of

\[
\lambda_{\max}(A^2_M) \approx \phi(D) - \epsilon^2 \psi(D) = 1,
\]

as \( \epsilon \to 0 \). In the case of \( \epsilon = 0 \), i.e., in the case of pure \((d,k)\) constraints, we hence obtain \( D_0 = D_{00} \) as the smallest root of the known characteristic equation \( D^{2d+1} \sum_{i=0}^{p} D^i = 1 \), found in, for example, (5.15) of [1]. With a Taylor approximation, \( D_0 \) is approximately given by \( D_0 \approx D_0 + \frac{3}{2} D_{00} \).
\[ e^2 \psi(D_0)/\phi(1)(D_0) \text{ as } \epsilon \rightarrow 0. \] So, as \( N \rightarrow \infty \), the extra redundancy results in

\[ \rho(d, k, N) = \log_2 \frac{D_0}{D_0} \approx \frac{\pi^2}{\ln \frac{2}{D_0 \phi(1)(D_0)}} M^2. \]

Using the characteristic equation \( D_0^{d+1} \sum_{l=0}^p l^2 D_0^l = 1 \), we find \( \psi(D_0) = D_0^{d+1} \sum_{l=0}^p l^2 D_0^l - D_0^{d+2} \left( \sum_{l=0}^p l D_0^l \right)^2 \), and \( D_0 \phi(1)(D_0) = (2d + 2) + 2D_0^{d+1} \sum_{l=0}^p l D_0^l \). According to Immink [1], Chapter 5.4.3, the DC content of maxentropic \((d, k)\) constrained sequences is given by

\[ H_0(d, k) = \frac{\sum_{l=0}^{k+1} f(l) - \sum_{l=0}^{k+1} l} {\sum_{l=0}^{k+1} l^2 D_0^l}. \]

Using the characteristic equation \( D_0^{d+1} \sum_{l=0}^p l^2 D_0^l = 1 \) and the substitution \( l' = l - (d + 1) \), we obtain \( \sum_{l=0}^{k+1} l^2 D_0^l = (d + 1) + D_0^{d+1} \sum_{l'=0}^p l'^2 D_0^l' \), and \( \sum_{l=0}^{k+1} l D_0^l = (d + 1)^2 + 2(d + 1)D_0^{d+1} \sum_{l'=0}^p l'^2 D_0^l'. \) A comparison of the above expressions reveals that

\[ \psi(D_0) \left( \frac{D_0}{\phi(1)(D_0)} \right) = \frac{H_0(d, k)}{2}. \]

Q.E.D.

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