Symmetries of the massive Thirring model

H. M. M. Ten Elkelder

Department of Mathematics and Computing Science, Eindhoven University of Technology, P. O. Box 513, Eindhoven, The Netherlands

(Received 26 September 1985; accepted for publication 20 November 1985)

For a Hamiltonian system every non-Hamiltonian symmetry gives rise to a recursion operator for symmetries. Using this method two recursion operators for symmetries of the massive Thirring model are constructed. The structure of the Lie algebra of symmetries generated by these operators is given.

I. INTRODUCTION

The existence of infinite series of symmetries is a very special property of a dynamical or Hamiltonian system. These series are often constructed by using a recursion operator for symmetries (also called Lénard operator, or strong symmetry or squared eigenfunctions operator). In Sec. II of this paper we make some general remarks on symmetries and tensor symmetries of a dynamical system. In particular, we show that for a Hamiltonian system every non-Hamiltonian symmetry gives rise to a recursion operator for symmetries. This method is applied to the massive Thirring model in Sec. III. Using two symmetries found by Kersten and Martini,2,3 we construct two recursion operators for symmetries of the massive Thirring model. These operators turn out to be each others' inverses. With these recursion operators we generate two infinite series of symmetries. One of these series corresponds to an infinite series of constants of the motion in involution. The other series consists of non-Hamiltonian symmetries. The corresponding Lie algebra of symmetries is also described. In Secs. II and III we use the framework of differential geometry. In Appendix A we show how the, at first instance finite-dimensional, differential geometry can be introduced on the topological vector space in which the Thirring model is studied. Some long expressions are given in Appendix B. Similar results as given in this paper for the massive Thirring model can be obtained for several other equations, see Ten Elkelder.4

We now make some remarks on the notation and terminology. A tensor field with contravariant order p and covariant order q will be called a \( p,q \) tensor field. The set of vector fields \( \{ (1,0) \text{ tensor fields} \} \) and the set of one-forms \( \{ (0,1) \text{ tensor fields} \} \) on a manifold \( \mathcal{M} \) will be denoted by \( \mathcal{T}(\mathcal{M}) \) [resp. \( \mathcal{T}^*(\mathcal{M}) \)]. The contraction between a one-form and a vector field \( A \) will be written as \( \langle \alpha, A \rangle \). The Lie derivative in the direction of a vector field \( A \) will be denoted as \( \mathcal{L}_A \). Applied to a vector field \( B \) this Lie derivative equals the Lie bracket \( [A,B] \), i.e., \( \mathcal{L}_A B = [A,B] \). Further we use the operators \( \partial = \partial/\partial x \) and \( \partial^{-1} \), defined by

\[
(\partial^{-1} f)(x) = \int_{-\infty}^{x} f(y) \, dy - \frac{1}{2} \int_{-\infty}^{\infty} f(y) \, dy.
\]

Then \( \partial \) and \( \partial^{-1} \) are both skew symmetric with respect to the \( L_2 \) inner product. These operators are assumed to act on everything that follows them, except when otherwise indicated.

II. TENSOR SYMMETRIES OF A DYNAMICAL SYSTEM

In this section we make some general remarks on symmetries of dynamical and Hamiltonian systems. Let \( X \) be a vector field on a manifold \( \mathcal{M} \). With \( X \) the following dynamical system is associated:

\[
\dot{u}(t) = X(u(t)), \quad \dot{u}(t) = \frac{d}{dt} u(t) \tag{2.1}
\]

A, possibly \( t \)-parametrized, tensor field \( \Xi \) on \( \mathcal{M} \), which satisfies

\[
\dot{\Xi} + \mathcal{L}_X \Xi = 0 \quad \left( \Xi = \frac{\partial}{\partial t} \Xi, \quad t \in \mathbb{R} \right) \tag{2.2}
\]

on \( \mathcal{M} \), will be called a tensor symmetry of (2.1). It follows from Leibniz’ rule that the tensor product of two tensor symmetries is again a tensor symmetry. Also every possible contraction in a tensor symmetry (or contracted multiplication of two tensor symmetries) yields again a tensor symmetry. If \( \Xi \) is a completely skew-symmetric \((0,p)\) tensor field (i.e., a differential \( p \)-form), then a new tensor symmetry can be constructed by exterior differentiation.

A tensor symmetry of type \((0,0)\) (i.e., a function) is called a constant of the motion or first integral. A tensor symmetry of type \((1,0)\) (i.e., a vector field on \( \mathcal{M} \)) will be called a symmetry. Finally a tensor symmetry of type \((1,1)\) will be called a recursion operator for symmetries.

Let \( Z \) be a symmetry and \( \Xi \) be an arbitrary tensor symmetry. Then

\[
\mathcal{L}_X \mathcal{L}_Z \Xi + \frac{\partial}{\partial t} \mathcal{L}_Z \Xi
\]

\[
= \mathcal{L}_Z \mathcal{L}_X \Xi + \mathcal{L}_Z \mathcal{L}_{[X,Z]} \Xi + \mathcal{L}_Z \Xi + \mathcal{L}_Z \mathcal{L}_Z \Xi = \mathcal{L}_Z (\mathcal{L}_X \Xi + \mathcal{L}_{[X,Z]} \Xi) + \mathcal{L}_Z \mathcal{L}_Z \Xi = 0.
\]

So the Lie derivative of a tensor symmetry in the direction of a symmetry yields again a tensor symmetry (of the same type as the original one).

Suppose \( \Lambda \) is a \((1,1)\) tensor field and \( \Phi \) and \( \Psi \) are skew symmetric \((0,2)\) [resp. \((2,0)\)] tensor fields. With these tensor fields the following linear mappings are associated:

\[
\hat{\Lambda}: \mathcal{T}(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M}),
\]

\[
\hat{\Phi}: \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{T}^*(\mathcal{M}),
\]

\[
\hat{\Psi}: \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{T}^*(\mathcal{M}).
\]

To simplify the notation we shall drop the hat and identify the tensor fields with the corresponding mappings (see also Appendix A). This also enables us to speak of the Lie deriva-
tive of such a mapping. In particular a two-form \( \Omega \) is skew-symmetric (0,2) tensor field \( \Omega \) is identified with a skew-symmetric mapping \( \Omega: \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{A}^*(\mathcal{M}) \). If the two-form is not degenerate this mapping has an inverse \( \Omega^*: \mathcal{A}^*(\mathcal{M}) \rightarrow \mathcal{A}(\mathcal{M}) \).

Now suppose that \( X \) is a Hamiltonian vector field, i.e., there exist a Hamiltonian \( H \) and a symplectic form \( \Omega \) on \( \mathcal{M} \) such that
\[
X = \Omega^{-1} dH. \tag{2.3}
\]
The closedness of \( \Omega \) implies that \( \mathcal{L}_X \Omega = d(\Omega X) = d dH = 0 \). Since \( \Omega = 0 \) this means that \( X \) is a tensor symmetry of type (0,2). From \( \Omega^{-1} = I \) we obtain that \( \Omega^{-1} \) is a tensor symmetry of type (2,0). Suppose that \( F \) is a constant of the motion. Then the one-form \( dF \) is a tensor symmetry of type (0,1) and \( \Omega^{-1} dF \) is a tensor symmetry of type (1,0), i.e., a symmetry. So every constant of the motion gives rise to a symmetry. Note that all symmetries obtained in this way are (possibly \( t \)-parametrized) Hamiltonian vector fields on \( \mathcal{M} \).

Let \( Z \) be a symmetry. Then \( \mathcal{L}_Z \Omega \) is a tensor symmetry of type (0,2). The contracted multiplication of the tensor symmetries \( \Omega^{-1} \) and \( \mathcal{L}_Z \Omega \) (in terms of mappings: the composition of \( \mathcal{L}_Z \Omega \): \( \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{A}^*(\mathcal{M}) \) and \( \Omega^{-1}: \mathcal{A}^*(\mathcal{M}) \rightarrow \mathcal{A}(\mathcal{M}) \)) is a tensor symmetry of type (1,1). So for every symmetry \( Z \),
\[
\Lambda = \Omega^{-1} \mathcal{L}_Z \Omega \tag{2.4}
\]
is a recursion operator for symmetries. Since \( \Omega \) is closed we have \( \mathcal{L}_Z \Omega = d(\Omega Z) \). So if \( Z \) is a Hamiltonian vector field we obtain by (2.4) the trivial recursion operator \( \Lambda = 0 \). Only in the case where \( Z \) is a non-Hamiltonian symmetry (i.e., a symmetry with \( \Omega Z \) not closed), we obtain by (2.4) a nonvanishing recursion operator for symmetries. So every non-Hamiltonian symmetry of a Hamiltonian system gives rise to a recursion operator for symmetries.

If a system has a recursion operator for symmetries \( \Lambda \), an infinite series of symmetries can be constructed by repeated application of this recursion operator to some symmetry. An important concept for understanding the algebra of symmetries generated in this way is the Nijenhuis tensor of \( \Lambda \) (see Nijenhuis and Schouten). With every (1,1) tensor field \( \Lambda \) associated a (1,2) tensor field \( N_\Lambda \), called the Nijenhuis tensor field of \( \Lambda \), such that for all vector fields \( A \),
\[
\mathcal{L}_{A \Lambda} - \Lambda \mathcal{L}_A \Lambda = N_\Lambda A. \tag{2.5}
\]
The right-hand side of this expression is the contracted multiplication of the (1,2) tensor field \( N_\Lambda \) and the vector field \( A \). This results again in a (1,1) tensor field. The importance of recursion operators for symmetries with a vanishing Nijenhuis tensor field has already been noticed by Magri, Fuchssteiner, Fuchssteiner and Fokas, and Gel'fand and Dorfman. It is easily seen how this property can be used. Let \( A \) and \( B \) be vector fields such that \( \mathcal{L}_\Lambda A = aA \) and \( \mathcal{L}_\Lambda B = bB \) for \( a, b \in \mathbb{R} \). Define \( A_k = \Lambda^k A \) and \( B_k = \Lambda^k B \) for \( k = 0, 1, 2, \ldots \). Then
\[
[A_k, B_l] = \mathcal{L}_{A_k} (\Lambda^l B) = (\mathcal{L}_{A_k} \Lambda^l) B + \Lambda^l \mathcal{L}_{A_k} B
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A)
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A)
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A)
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A)
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A)
= (\mathcal{L}_{A_k} \Lambda^l) B - \Lambda^l \mathcal{L}_B (\Lambda^k A).
\]

If the Nijenhuis tensor field of \( \Lambda \) vanishes we have
\[
\mathcal{L}_{A_k} \Lambda = \Lambda^k \mathcal{L}_A \Lambda = a \Lambda^{k+1}. \tag{2.7}
\]
Substitution in (2.6) finally results in
\[
[A_k, B_l] = [a b A_{k+l} + \Lambda^{k+l+1} [A, B]]. \tag{2.8}
\]
If \( \Lambda \) is invertible we can also define \( A_k \) and \( B_k \), for \( k = -1, -2, -3, \ldots \). Using
\[
\mathcal{L}_C \Lambda^{-1} = - \Lambda^{-1} (\mathcal{L}_C \Lambda) \Lambda^{-1}
\]
for every vector field \( C \) it is easily shown that in this case (2.7) and (2.8) also hold for negative integers \( k \) and \( l \).

III. RECURSION OPERATORS FOR SYMMETRIES OF THE MASSIVE THIRRING MODEL

The massive Thirring model is the following system of partial differential equations for the functions \( u_1(x,t), u_2(x,t), v_1(x,t), \) and \( v_2(x,t) \):
\[
\begin{align*}
\dot{u}_1 &= u_{1x} + m v_2 - R_2 v_1, \\
\dot{u}_2 &= - u_{2x} + m v_1 - R_1 v_2, \\
\dot{v}_1 &= v_{1x} - m u_2 + R_3 v_1, \\
\dot{v}_2 &= - v_{2x} - m u_1 + R_1 u_2, \\
- \infty &< x < \infty, \quad t > 0,
\end{align*}
\]
where \( R_1 = u_1^2 + v_1^2 \) and \( R_2 = u_2^2 + v_2^2 \). We assume that \( u_1, u_2, v_1, \) and \( v_2 \) are smooth and, together with their \( x \)-derivatives, decay sufficiently fast for \( |x| \rightarrow \infty \). We shall study (3.1) in some reflexive topological vector space \( \mathcal{W} \), which is the Cartesian product of function spaces for \( u_1, u_2, v_1, \) and \( v_2 \). \( \mathcal{W} \) and \( \mathcal{W}^* \) are constructed in such a way that their duality map \( \langle \cdot, \cdot \rangle \) is just the \( L_2 \) inner product. In terms of \( \mathcal{W} \) we can write (3.1) as
\[
\dot{u} = X(u) \tag{3.2}
\]
The nonlinear mapping \( X \) can be considered as a vector field on \( \mathcal{W} \). In this section we shall continue to use the differential geometrical language of Sec. II. For a definition of the various differential geometrical objects in this infinite-dimensional case see Appendix A.

Define the function (functional) \( H \) on \( \mathcal{W} \) by
\[
H = \int_{-\infty}^{\infty} \langle v_{1x} - v_2, v_2 \rangle + m R \frac{1}{2} A R \rangle^2 \rangle dx,
\]
where \( R = u_1 u_2 + v_1 v_2 \). Moreover let the symplectic form \( \Omega \) be represented by the linear mapping \( \Omega: \mathcal{W} \rightarrow \mathcal{W}^* \)
\[
\Omega = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
Then it is easily verified that
\[
X = \Omega^- dH,
\]
\[\text{i.e., the massive Thirring model is a Hamiltonian system with Hamiltonian } H \text{ and symplectic form } \Omega.\]

Symmetries for the massive Thirring model have recently been studied by Kersten \(^1\) and Kersten and Martini. \(^2,3\)

Amongst others they give the following symmetries:

\[
X_0 = \begin{pmatrix}
  v_1 \\
  v_2 \\
  -u_2 \\
  -u_1
\end{pmatrix},
\]

\[
Z_0 = \begin{pmatrix}
  -xu_{1x} - mu_{x} - \frac{1}{2} v_1 + v_1 R_2 x \\
  x u_{2x} - mu_{x} + \frac{1}{2} u_2 + v_2 R_2 x \\
  -xv_{1x} + mu_{x} - \frac{1}{2} v_1 - u_1 R_2 x \\
  xv_{2x} + mu_{x} + \frac{1}{2} u_2 - v_2 R_2 x
\end{pmatrix} t
\]

The expressions for the symmetries \(X_0\) and \(Z_0\) are also symmetries of the type considered in this paper. Refs. 1 - 3. These four symmetries have been found by Kersten \(^1\), Kersten and Martini. \(^2,3\)

\[
Z_1 = (1/m) p_2 X_0 - \frac{1}{2} mx(X_2 + X_0)
+ \frac{1}{m} t(X_2 - X_0)
\]

\[
+ \frac{1}{m} \left( \begin{pmatrix}
  lmu_2 \\
  lm v_2
\end{pmatrix} - X_k \right),
\]

\[\text{with } p_2 = (\partial\frac{1}{2} (u_{2,2x} - u_{2x,2} - R_2 + m R)) \text{ and } p_1 = (\partial\frac{1}{2} (u_{1,2} + R_2 + m R))\]

By considering the highest derivatives with respect to \(x\) in \(X_0\) and \(X_1\) and the structure of \(A_1\) and \(A_{-1}\), it is easily seen that none of these symmetries vanishes. It follows from (3.11) and (3.12) that \(\hat{Z}_{-1} = -Z_{-1}\). The expressions for the symmetries \(X_0\) and \(X_{-1}\) are given in Appendix B. From these expressions we see that the vector field \(X\), which is trivially a symmetry, is given by

\[
X = \frac{1}{m} (X_1 + X_{-1}).
\]

Because \(X_0\) and \(A_1\) do not depend explicitly on \(t\) (i.e., \(X_0 = 0\) and \(A_1 = 0\)), the same holds for all symmetries \(X_k\). Similarly we see that all symmetries \(X_k\) do not depend explicitly on \(x\). The time derivative of \(Z_0\) is given by

\[
\hat{Z}_0 = \left( \begin{pmatrix}
  u_{1x} \\
  u_{2x}
\end{pmatrix}
\right) = \frac{1}{2} m (X_1 - X_{-1}).
\]

So the time derivatives of the symmetries \(Z_k\) are given by

\[
\hat{Z}_k = \frac{1}{m} (X_{k+1} - X_{k-1}).
\]

The symmetries \(X_0, X_1, X_{-1}, X_2, X_{-2}, X_3, X_{-3}, Z_0, Z_1, Z_{-1}\) have already been given by Kersten. \(^1\) In his notation they are called \(Y_4, -2 m^{-1} Y_5, 2 m^{-1} Y_2, 4 m^{-2} Y_5, -4 m^{-2} Y_0, 8 m^{-3} Y_7, -8 m^{-3} Y_3, -Y_3, m^{-1} Z_1, \) and \(-m^{-1} Z_2\).

After these elementary properties of the symmetries \(X_k\) and \(Z_k\) we now turn to the structure of the corresponding Lie algebra. Straightforward but long computations show that

\[
[Z_0, Z_1] = Z_1, \quad [X_0, Z_1] = 0, \quad \text{and } [X_0, Z_0] = 0.
\]

Hence

\[
\mathcal{L}_{Z_0} A_1 = \mathcal{L}_{Z_0} \Omega^- \mathcal{L}_{Z_0} \mathcal{L}_{Z}, \quad \Omega = \Omega^- \mathcal{L}_{Z_0} \mathcal{L}_{Z}, \Omega
= \Omega^- (\mathcal{L}_{Z_0} \mathcal{L}_{Z}, \Omega + \mathcal{L}_{Z}, \mathcal{L}_{Z_0} \Omega)
= \Omega^- \mathcal{L}_{Z_0} \mathcal{L}_{Z}, \Omega = A_1,
\]

where we used that \(Z_0\) is a Hamiltonian vector field (i.e., \(\mathcal{L}_{Z_0} \Omega = 0 \) and \(\mathcal{L}_{Z} \Omega^- = 0\)) and the formula

\[
\mathcal{L}_{[A,B]} = \mathcal{L}_{A} \mathcal{L}_{B} - \mathcal{L}_{B} \mathcal{L}_{A}
\]

for all vector fields \(A\) and \(B\).

Similarly
\[ \mathcal{L}_{x_k} \lambda_1 = \mathcal{L}_{x_k} (\Omega^{-1} \mathcal{L}_z \Omega) = \Omega^{-1} \mathcal{L}_{x_k} \mathcal{L}_z \Omega \]
\[ = \Omega^{-1} (\mathcal{L}_z x_k \Omega + \mathcal{L}_z \mathcal{L}_z \Omega) = 0. \tag{3.15} \]

The structure of the Lie algebra of symmetries generated by the \( x_k \) and \( Z_k \) can be found now from (2.8) if the Nijenhuis tensor field of \( \Lambda \) vanishes. A gigantic computation shows that this is indeed the case. From (2.8) we now obtain that

\[
\begin{align*}
[X_k, X_l] &= 0, \\
[Z_k, Z_l] &= (l - k)Z_{k+l}, \\
[Z_k, X_l] &= lX_{k+l}, \\
k, l &= 0, 1, 2, 3, ..., 
\end{align*}
\]

Also (2.7) yields

\[
\mathcal{L}_{x_k} \lambda_1 = 0, \quad \mathcal{L}_{z_k} \lambda_1 = \Lambda_k^{k+1},
\]

\[ k = 0, 1, 2, 3, ..., \tag{3.17} \]

The recursion operators for symmetries \( \lambda_1 \) and \( \lambda_{-1} \) have been found by substitution of \( Z_1 \) and \( Z_{-1} \) in (2.7). There are several other ways to construct recursion operators for symmetries. For instance, as explained in Sec. II, the Lie derivative of a recursion operator for symmetries in the direction of a symmetry yields again a recursion operator for symmetries. From (3.17) we see that in this way we only obtain powers of \( \lambda_1 \). Another possible method is to use higher derivatives of \( \Omega \), i.e., to construct recursion operators of the form

\[
\Omega^{-1} \mathcal{L}^p_z \Omega, \quad \Omega^{-1} \mathcal{L}^p_{z_{-1}} \Omega, \quad p = 1, 2, 3, ..., \tag{3.18} \]

It is easily shown that this method also yields only powers of \( \lambda_1 \). From (3.9) and \( \mathcal{Z}_{-1} = - Z_{-1} \) we obtain

\[
\mathcal{L}_{z_k} \Omega = \Omega \Lambda_1, \quad \mathcal{L}_{z_{-1}} \Omega = - \Omega \Lambda_{-1}.
\]

Using Leibniz' rule and (3.17) for \( k = 1 \) it is now easily shown by induction that

\[
\mathcal{L}^p_z \Omega = p! \Omega \Lambda_1^p, \quad \mathcal{L}^p_{z_{-1}} \Omega = (1 - p)! \Omega \Lambda_{-1}^p. \tag{3.19} \]

So the recursion operators for symmetries constructed by (3.18) are also powers of \( \lambda_1 \).

The Lie derivative commutes with exterior differentiation. So we obtain from (3.19) the nontrivial conclusion that the two-forms \( \Omega \Lambda_1^p \) (\( p = 0, 1, 2, ... \)) are all closed. This result is in fact a special property of recursion operators with a vanishing Nijenhuis tensor field, see, for instance, Fuchssteiner and Fokas. Using the closedness of \( \Omega \Lambda_1^p \) it is easily shown that the symmetries \( X_k \) are Hamiltonian vector fields while the symmetries \( Z_k \) (\( k \neq 0 \)) are non-Hamiltonian vector fields. This follows because the closedness of \( \Omega \Lambda_1^p \) implies that

\[
d(\Omega X_k) = d(\Omega \Lambda_1^p X_0) = \mathcal{L}_x (\Omega \Lambda_1^p) = 0, \tag{3.20} \]

\[
d(\Omega Z_k) = d(\Omega \Lambda_1^p Z_0) = \mathcal{L}_z (\Omega \Lambda_1^p) = k \Omega \Lambda_1^p \neq 0, \quad \text{for} \; k \neq 0, \]

where we used (3.14) and (3.15) and that \( X_0 \) and \( Z_0 \) are Hamiltonian vector fields. The non-Hamiltonian symmetries \( Z_k \) (\( k \neq 0 \)) again give rise to recursion operators for symmetries. From (3.20) and the closedness of \( \Omega \) we obtain

\[
\Omega^{-1} \mathcal{L}_{z_k} \Omega = \Omega^{-1} d(\Omega Z_k) = k \Lambda_1^p. \]

So also in this way we obtain only powers of \( \Lambda_1 \).

On the linear space \( \mathcal{W} \) the closed one-forms \( \Omega X_k \) are exact, so there exist constants of the motion \( F_k \) such that

\[
X_k = \Omega^{-1} dF_k, \quad k = 0, 1, 2, 3, ... . \]

The explicit form of \( F_1, F_{-1}, F_2 \) and \( F_{-2} \) is given in Appendix B. From (3.13) we obtain \( H = \pm (F_1 + F_{-1}) \). The proof that the constants of the motion \( \tilde{F} \) are in involution is standard. Using the skew symmetry of \( \Omega \) and of \( \mathcal{L}_z \), \( \Omega \) we obtain for the Poisson bracket

\[
\{F_k, F_l\} = \langle dF_k, \Omega^{-1} dF_l \rangle = 0.
\]

Thus we have constructed an infinite series of constants of the motion in involution for the massive Thirring model.

An infinite set of Hamiltonian forms of the massive Thirring model is now easily obtained. Some elementary manipulations lead to

\[
X = (\Omega \Lambda_1^k)^{-1} d(\pm (F_{k+1} + F_{k-1})),
\]

\[ k = 0, 1, 2, ... . \]

So we can consider \( X \) as the Hamiltonian vector field with Hamiltonian \( \pm (F_{k+1} + F_{k-1}) \) and symplectic form \( \Omega \Lambda_1^k \), for \( k = 0, 1, 2, ... . \) Note that the original Hamiltonian form of the Thirring model (3.3) is obtained for \( k = 0 \).

Finally we give a very simple recursion formula for the constants of the motion \( F_k \). The Hamiltonian vector field corresponding to \( \mathcal{L}_{z_k} F_k \) is given by

\[
\Omega^{-1} \mathcal{L}_{z_k} F_k = \Omega^{-1} \mathcal{L}_{z_k} dF_k = \Omega^{-1} \mathcal{L}_{z_k} (\Omega \Lambda_1^k)
\]

\[ = \Lambda_1 X_k + [Z_1, X_k] = (1 + k)X_{k+1} \]

\[ = (1 + k) \Omega^{-1} dF_{k+1}. \]

This yields the recursion formula

\[
F_{k+1} = \frac{1}{k + 1} \mathcal{L}_{z_k} F_k = \frac{1}{k + 1} \langle dF_k, Z_{k+1} \rangle, \quad k \neq -1.
\]

In a similar way we obtain

\[
F_{k-1} = \frac{1}{k - 1} \mathcal{L}_{z_k} F_{k-1} = \frac{1}{k - 1} \langle dF_k, Z_{k-1} \rangle, \quad k \neq 1.
\]

In terms of the operator implementation of the differential geometry (see Appendix A) these two expressions read

\[
F_{k+1} = \frac{1}{k + 1} \int_{-\infty}^{\infty} (\delta F_k \frac{\partial}{\partial u_1} Z_1 + \delta F_k \frac{\partial}{\partial u_2} Z_2 + \delta F_k \frac{\partial}{\partial u_1} Z_3 + \delta F_k \frac{\partial}{\partial u_2} Z_4) dx,
\]

\[ F_{k-1} = \frac{1}{k - 1} \int_{-\infty}^{\infty} (\delta F_k \frac{\partial}{\partial u_1} Z_{-1} + \delta F_k \frac{\partial}{\partial u_2} Z_{-2} + \delta F_k \frac{\partial}{\partial u_1} Z_{-3} + \delta F_k \frac{\partial}{\partial u_2} Z_{-4}) dx,
\]

where \( Z_1, Z_3, Z_2, Z_4 \) and \( Z_{-1}, Z_{-3}, Z_{-2}, Z_{-4} \) are the four components of the symmetries \( Z_1 \) (resp. \( Z_{-1} \)).
ACKNOWLEDGMENT

I thank Ms. M. Van Heijst for her assistance with several extremely long computations. In particular the computation of the Nijenhuis tensor field of $\mathcal{A}$, could never have been completed without her help. I also thank Professor J. de Graaf for stimulating this research.

APPENDIX A: DIFFERENTIAL GEOMETRY ON A TOPOLOGICAL VECTOR SPACE

In the preceding section we worked completely in the setting of differential geometry. The used differential geometrical methods have a sound foundation on finite-dimensional manifolds. However, the massive Thirring model is considered on an infinite-dimensional topological vector space $\mathcal{M} = \mathcal{W}$. In this Appendix we shortly describe how the necessary differential geometry can be introduced on the topological vector space $\mathcal{W}$. A more comprehensive treatment is given in Ten Eikelder. We assume that $\mathcal{W}$ is reflexive. The duality map between $\mathcal{W}$ and $\mathcal{W}^\ast$ will be denoted by $(.,.)$. Since $\mathcal{W}$ is a linear space, we can make the following identifications for its tangent bundle and cotangent bundle:

$$T\mathcal{W} = \mathcal{W} \times \mathcal{W}^\ast, \quad T^\ast \mathcal{W} = \mathcal{W} \times (\mathcal{W}^\ast)^\ast.$$

Using these identifications it is easy to introduce (objects similar to) vector fields, differential forms, and tensor fields on $\mathcal{W}$. A vector field $A$ on $\mathcal{W}$ is a mapping

$$A: \mathcal{W} \rightarrow \mathcal{W} \times \mathcal{W}^\ast: \ u \mapsto \{A(u), \},$$

where $\tilde{A}: \mathcal{W} \rightarrow \mathcal{W}$ is a possible nonlinear mapping. So we can identify the vector field $A$ with the mapping $\tilde{A}$. Therefore $\tilde{A}$ also will be called a vector field. To simplify the notation we shall drop the tilde and write $A$. To introduce one-forms and tensor fields of higher order. This results in the following “conversion table”:

<table>
<thead>
<tr>
<th>$A \in \mathcal{X}(\mathcal{W})$, vector field</th>
<th>$A: \mathcal{W} \rightarrow \tilde{\mathcal{W}}$, one-form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in \mathcal{X}(\mathcal{W}^\ast)$, one-form</td>
<td>$\alpha: \mathcal{W} \rightarrow \mathcal{W}^\ast$</td>
</tr>
<tr>
<td>$\Phi (0,2)$ tensor field</td>
<td>$\Phi: \mathcal{W} \rightarrow L(\mathcal{W}, \mathcal{W}^\ast)$, (A1)</td>
</tr>
<tr>
<td>$\Psi (0,2)$ tensor field</td>
<td>$\Psi: \mathcal{W} \rightarrow L(\mathcal{W}, \mathcal{W}^\ast)$</td>
</tr>
<tr>
<td>$\Lambda (1,1)$ tensor field</td>
<td>$\Lambda: \mathcal{W} \rightarrow L(\mathcal{W}^\ast, \mathcal{W}^\ast)$</td>
</tr>
</tbody>
</table>

where $L(\mathcal{W}^\ast, \mathcal{W}^\ast)$ denotes the linear continuous mappings from $\mathcal{W}^\ast$ to $\mathcal{W}$, to $\mathcal{W}$. For instance, the contracted multiplication between a $(0,2)$ tensor field $\Phi$ and a $(1,1)$ tensor field $\Lambda$ yields a $(0,2)$ tensor field represented by the mapping $\Phi \Lambda: \mathcal{W} \rightarrow L(\mathcal{W}^\ast, \mathcal{W})$. In a similar way we can introduce higher-order tensor fields on $\mathcal{W}$. For instance, a $(0,3)$ tensor field $\Xi$ on $\mathcal{W}$ can be represented by a mapping $\Xi: \mathcal{W} \rightarrow L(\mathcal{W}, L(\mathcal{W}, \mathcal{W}^\ast))$.

Next we introduce Lie derivatives and (for differential forms) exterior derivatives. First some remarks on differential calculus in a topological vector space. Suppose $\mathcal{W}$ is some topological vector space and $f$ is (nonlinear) mapping $f: \mathcal{W} \rightarrow \mathcal{W}$. Then $f$ is called Gateaux differentiable in $\mathcal{W}$ if there exists a mapping $f'(u) \in L(\mathcal{W}, \mathcal{W}^\ast)$ such that for all $v \in \mathcal{W}$

$$\lim_{\varepsilon \to 0} (1/\varepsilon)(f(u + \varepsilon v) - f(u) + \varepsilon f'(u)v) = 0.$$

If $f$ is Gateaux differentiable at all points $u \in \mathcal{W}$ we can consider the Gateaux derivative as a mapping $f'$:

$\mathcal{W} \rightarrow L(\mathcal{W}, \mathcal{W}^\ast)$. Suppose $f'$ is again Gateaux differentiable in $u \in \mathcal{W}$. The second derivative of $f$ in $u \in \mathcal{W}$ is then a mapping $f''(u) \in L(\mathcal{W}, L(\mathcal{W}, \mathcal{W}^\ast))$. This mapping can be considered as a bilinear mapping $f''(u): \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}^\ast$. Under certain conditions (see, for instance, Yamamuro) this mapping is symmetric: $f''(u)(v,w) = f''(u)(w,v)$, for all $v, w \in \mathcal{W}$.

Suppose $B: \mathcal{W} \rightarrow \mathcal{W}$ is (represents) a vector field. The Gateaux derivative in $u \in \mathcal{W}$ is a linear mapping $B'(u) \in L(\mathcal{W}, \mathcal{W}^\ast)$. The dual of this mapping is denoted by $B'(u)^\ast \in L(\mathcal{W}^\ast, \mathcal{W})$. The Lie derivatives in the direction of a vector field $B$ of a function $F: \mathcal{W} \rightarrow \mathcal{R}$ and of the various tensor fields (vector fields, one-forms) considered in (A1) are defined by

$$\mathcal{L}_B F(u) = F'(u)B = F'(u)B(u),$$

$$\mathcal{L}_B \alpha(u) = \alpha'(u)B(u) + B^\ast \alpha(u),$$

$$\mathcal{L}_B \Phi(u) = (\Phi'(u)B(u)) + \Phi(u)B'(u) + B^\ast \Phi(u),$$

$$\mathcal{L}_B \Psi(u) = (\Psi'(u)B(u)) - \Psi(u)B^\ast(u) - B'(u)\Psi(u).$$

First some remarks on the notation in these expressions. Consider the formula for $\mathcal{L}_B \Phi$. Since $\Phi: \mathcal{W} \rightarrow L(\mathcal{W}, \mathcal{W}^\ast)$ we have $\Phi(u) \in L(\mathcal{W}, L(\mathcal{W}, \mathcal{W}^\ast))$. So $((\Phi'(u)B)C) \in L(\mathcal{W}^\ast, \mathcal{W})$ and $((\Phi'(u)B)C) \in L(\mathcal{W}^\ast, \mathcal{W})$. By definition,

$$((\Phi'(u)B)C) = \lim_{\varepsilon \to 0} ((1/\varepsilon)(\Phi(u + \varepsilon C) - \Phi(u)C - \Phi(u)B - \varepsilon C)).$$

Of course, in general this expression is not symmetric in $B$ and $C$. Therefore we shall always insert brackets in expressions of this type. It is easily seen that the Lie derivative of an object yields again an object of the same type. Note that the expressions given in (A2) strongly resemble the formulas for Lie derivatives in terms of local coordinates on a finite-dimensional manifold.

Now we turn to exterior derivatives of differential forms. Two-forms will be identified with skew-symmetric $(0,2)$ tensor fields, i.e., $\Phi: \mathcal{W} \rightarrow L(\mathcal{W}, \mathcal{W}^\ast)$. We define skew-symmetric $(0,2)$ tensor fields $\Phi(u) \in L(\mathcal{W}, \mathcal{W}^\ast)$ and $(\Phi(u)B) \in L(\mathcal{W}, \mathcal{W}^\ast)$, and $\Phi(u)B = \Phi(u)B$, for $B, C \in \mathcal{W}$. By definition,

$$(\Phi'(u)B)C = \lim_{\varepsilon \to 0} ((1/\varepsilon)(\Phi(u + \varepsilon C) - \Phi(u)C - \Phi(u)B - \varepsilon C)).$$

Next we introduce Lie derivatives and (for differential forms) exterior derivatives. First some remarks on differential calculus in a topological vector space. Suppose $\mathcal{W}$ is some topological vector space and $f$ is a (nonlinear) mapping $f: \mathcal{W} \rightarrow \mathcal{W}$. Then $f$ is called Gateaux differentiable in $u \in \mathcal{W}$ if there exists a mapping $f'(u) \in L(\mathcal{W}, \mathcal{W}^\ast)$ such that for all $v \in \mathcal{W}$

$$\lim_{\varepsilon \to 0} (1/\varepsilon)(f(u + \varepsilon v) - f(u) + \varepsilon f'(u)v) = 0.$$
Also these definitions strongly resemble the expressions in local coordinates of exterior derivatives of differential forms on a finite-dimensional manifold.

Definitions as above can, of course, always be given. The important observation, however, is that all formulas from classical differential geometry on a finite-dimensional manifold also hold in this case. The proofs of all used formulas are identical to the proofs in terms of local coordinates of the corresponding formulas on a finite-dimensional manifold. In particular we often used that for a closed two-form \( \Phi \) and an arbitrary vector field \( A \), the identity \( \nabla_A \Phi = d(\Phi A) \) holds.

In the case of the massive Thirring model the duality map \( \langle \cdot, \cdot \rangle \) between \( \mathcal{W} \) and \( \mathcal{W}^\ast \) is the \( L_2 \) inner product. In that case the derivative \( F'(u) \) of a function (functional) \( F \) on \( \mathcal{W} \) is usually denoted as \( \delta F/\delta u \), the variational derivative of \( F \). In terms of partial derivatives this means

\[
dF(u) = \begin{pmatrix} \frac{\partial F}{\partial u_1} \\ \frac{\partial F}{\partial u_2} \end{pmatrix}.
\]

**APPENDIX B: SOME EXPLICIT FORMULAS**

The symmetries \( X_i \) and \( X_{-1} \) are given by

\[
X_i = \frac{1}{m} \begin{pmatrix} m v_2 - u_1 R_2 \\ -2u_{x_2} + m v_1 - v_2 R_1 \\ -2v_{x_2} - m u_1 + u_2 R_1 \end{pmatrix},
\]

\[
X_{-1} = \frac{1}{m} \begin{pmatrix} m v_2 - u_1 R_2 \\ 2v_{x_1} - m u_2 + u_1 R_2 \\ -u_1 + u_2 R_1 \end{pmatrix}.
\]

The constants of the motion \( F_1, F_{-1}, F_2 \), and \( F_{-2} \) are given by

\[
F_1 = \frac{1}{m} \int_{-\infty}^{\infty} \left( -\frac{1}{2} R_1 R_2 + m R - 2u_{x_2} v_1 \right) dx,
\]

\[
F_{-1} = \frac{1}{m} \int_{-\infty}^{\infty} \left( -\frac{1}{2} R_1 R_2 + m R + 2u_{x_1} v_1 \right) dx,
\]

\[
F_2 = \frac{1}{m^2} \int_{-\infty}^{\infty} \left( \frac{1}{2} R_1 R_2 (R_1 + R_2) - m R(R_1 + R_2) \right) + \frac{1}{2} m^2 (R_1 + R_2) - 2m (u_{x_2} v_1 + u_{x_1} v_2) + 2u_{x_2}^2 + 2v_{x_2}^2 + 2R_1 (u_{x_2} v_2 - v_{x_2} u_2) + 4u_2 v_2 (u_{x_2} u_2 - v_{x_2} v_2) dx,
\]

\[
F_{-2} = \frac{1}{m^2} \int_{-\infty}^{\infty} \left( \frac{1}{2} R_1 R_2 (R_1 + R_2) - m R(R_1 + R_2) \right) + \frac{1}{2} m^2 (R_1 + R_2) + 2m (u_{x_2} v_1 + u_{x_1} v_2) + 2u_{x_2}^2 + 2v_{x_2}^2 + 2R_1 (v_{x_1} u_1 - u_{x_1} v_1) + 4u_2 v_2 (v_{x_1} v_1 - u_{x_1} u_1) dx.
\]

To reduce the expressions for the recursion operators we introduce the following abbreviations:

\[
(ijkl) = u_i \partial^{-1} u_j R_k + u_i R_l \partial^{-1} u_j,
\]

\[
i, j, k, l = 1, 2, 3, 4,
\]

\[
(ijkl) = m u_i \partial^{-1} u_j + m u_k \partial^{-1} u_l,
\]

\[
i, j, k, l = 1, 2, 3, 4,
\]

\[
(ij) = 2u_{x_i} \partial^{-1} u_j,
\]

\[
i, j = 1, 2, 3, 4,
\]

\[
(ij) = 2u_i \partial^{-1} u_{x_j},
\]

\[
i, j = 1, 2, 3, 4.
\]

where \( u_3 = v_1 \) and \( u_4 = v_2 \). The recursion operators for symmetries of the massive Thirring model \( \Lambda_i \) and \( \Lambda_{-1} \) are now given by

\[
\Lambda_i = \frac{1}{m} 
\begin{pmatrix}
-(3122) + (3241) & -(3212) + (3142) & + (34) + m - 2u_{x_1} v_2 \\
-(4121) + (3142) & -(4211) + (3241) & + (44) - (22) - R_1 - R_2 \\
-(1122) - (1221) & (1212) - (1122) & - (14) + 2u_{x_2} v_2 \\
(2121) - (1122) & (2211) - (1221) & -(24) - (42) - 2\partial \\
-(41) - 2u_{x_2} v_2 & -(41) - 2u_{x_2} v_2 & -(43) + m - 2u_{x_1} v_2
\end{pmatrix}
\]

\[
\Lambda_{-1} = \frac{1}{m} 
\begin{pmatrix}
(3122) - (3241) + (33) - (11) - R_1 - R_2 & -(3212) - (3142) + (42) + m - 2u_{x_2} v_2 \\
(4211) - (3142) + (44) + m - 2u_{x_2} v_2 & -(3211) - (3241) + (42) + m - 2u_{x_2} v_2 \\
-(1122) + (1221) - (1212) + (1122) + (32) - 2u_{x_2} v_2 & -(2121) + (1122) - (1212) + (1122) + (32) - 2u_{x_2} v_2 \\
-(23) + 2u_{x_1} v_2 & -(22) - (44) - R_1 - R_2 & -(23) + 2u_{x_1} v_2
\end{pmatrix}
\]