Proof. For large enough $\text{Re} \lambda, \text{Re} \mu, \text{Re} \nu$ the formulas (2) and (3) yield—specialized to the case of $x \in \mathbb{R}^2$, i.e., $n = 2$—

$$I(\lambda, \mu, \nu, \alpha) = \frac{1}{2\pi^2} \left[ \frac{J_\lambda(\alpha|x|)}{|x|^{\mu - \lambda}} \cdot \frac{J_\mu(|x|)}{|x|^{\nu - \mu}} \right]$$

$$= \frac{1}{2\pi^2} \left[ \frac{J_\lambda(\alpha|x|)}{|x|^{\mu - \lambda}} \cdot \frac{J_\mu(|x|)}{|x|^{\nu - \mu}} \right]$$

$$= \frac{\Gamma(\mu + \nu - 1)}{\mu \Gamma(\nu) \Gamma(\mu)} \left[ \frac{J_\lambda(\alpha|x|)}{|x|^{\mu - \lambda}} \cdot \frac{J_\mu(\alpha|x|)}{|x|^{\nu - \mu}} \right]$$

$$= \frac{2\Gamma(\mu + \nu - 1)}{\Gamma(\mu) \Gamma(\nu) \alpha^\lambda} \int_0^\infty J_\lambda(\alpha t)J_{\mu + \nu - 1}(|x|) t^{2\lambda - \mu - \nu - 1} \, dt.$$  

From formula 2.12.31.1 in [3, p. 209, 210], we infer that

$$I(\lambda, \mu, \nu, \alpha) = \frac{1}{2^{\lambda + \nu + 3} \Gamma(\lambda + 1) \Gamma(\mu) \Gamma(\nu)} F_1(1, 2 - \mu - \nu, \lambda + 1; \alpha^2).$$

The general case is thus deduced by analytic continuation, since $I(\lambda, \mu, \nu, \alpha)$ is absolutely convergent under the stated conditions.

For $k \in \mathbb{N}$, we obtain from [4, formula 7.3.1.123, p. 462] the following corollary which contains the second equation of the problem as a special case (i.e., $\mu = \nu = m, k = 1$).

**Corollary 3.** Let $k \in \mathbb{N}, \mu, \nu \in \mathbb{C}$, $\text{Re} \mu > 1 - k$, $\text{Re} \nu > 1 - k$, $\text{Re}(\mu + \nu) > 1$, $\text{Re}(\mu + \nu) > \frac{3}{2} - k$. Then

$$I(k, \mu, \nu, \alpha) = \frac{\Gamma(\mu + \nu + k - 1)}{2^{k + \mu + \nu - 3} \Gamma(\mu) \Gamma(\nu) \Gamma(\mu + \nu + 1)} \left[ (1 - \alpha^2)^{k + \mu + \nu - 2} \right]$$

$$- \sum_{l=0}^{k-1} \left( \frac{(2 - \mu - \nu - l) \cdots (2 - \mu - \nu - 1)}{l!} \right) \alpha^{2l}.$$  

**References**


Also solved by J. BOERSMA (Eindhoven University of Technology, Eindhoven, the Netherlands), C. GEORGHIOU (University of Patras, Greece), M. L. GLASSER (Clarkson University), CARL C. GROSJEAN (University of Ghent, Ghent, Belgium), NORBERT ORTNER (University of Innsbruck, Austria) and the proposers.

**A Definite Integral**

**Problem 94-18**, by M. L. GLASSER (Clarkson University), J. BOERSMA, AND P. J. DE DOELDER (Technical University, Eindhoven, the Netherlands).
Evaluate the integral
\[ F_v(x) = \int_0^\infty \frac{1 - J_0(x\sqrt{2t})(\sqrt{1+t^2} - t)^v \nu}{t\sqrt{1+t^2}} \, dt \quad (v \geq -1, \quad x \geq 0). \]

This integral arises in studying the amplitude-amplitude correlation function for oscillations of a stack of lipid membranes [1].

REFERENCE


Dedicated to the memory of P. J. de Doelder.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, the Netherlands) and M. L. GLASSER (Clarkson University, Potsdam, NY).

We present a closed-form expression for \( F_v(x) \) in case \( v \) is an odd integer. For general \( v > -1 \), \( F_v(x) \) is expressed in terms of Meijer's \( G \)-functions.

First we evaluate \( F_v(0) \). Via the substitution \( t = \sinh(s/2) \) we find, by [1, form. 1.7.2(14)],

\[
F_v(0) = \int_0^\infty \frac{1 - (\sqrt{1+t^2} - t)^v}{t\sqrt{1+t^2}} \, dt = \frac{1}{2} \int_0^\infty \frac{1 - e^{-vs/2}}{\sinh(s/2)} \, ds \\
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \). Using standard recurrence relations for the Bessel function \( J \), we determine the derivative

\[
\frac{d}{dx} (xF_v'(x)) = 2x \int_0^\infty J_0(x\sqrt{2t}) \frac{(\sqrt{1+t^2} - t)^v}{\sqrt{1+t^2}} \, dt. \\
\]

In the latter integral we set [2, form. 4.14(1)]

\[ \frac{(\sqrt{1+t^2} - t)^v}{\sqrt{1+t^2}} = \int_0^\infty J_v(y) e^{-ty} \, dy \]

and the order of integration is interchanged. The resulting inner integral is evaluated by [2, form. 4.14(25)], viz.

\[ \int_0^\infty J_0(x\sqrt{2t})e^{-ty} \, dt = y^{-1} \exp[-\frac{1}{2}x^2/y]. \]

Consequently we find, by [2, form. 4.15(8)],

\[
\frac{d}{dx} (xF_v'(x)) = 2x \int_0^\infty J_v(y)y^{-1} \exp[-\frac{1}{2}x^2/y] \, dy \\
\frac{d}{dx} (xF_v'(x)) = 2x \int_0^\infty J_v(t^{-1})t^{-1} \exp[-\frac{1}{2}x^2 t] \, dt = 4xJ_v(x)K_v(x). \\
\]

By integration of (3) starting from the initial value (1), we are led to the representation

\[
F_v(x) = \psi(\frac{1}{2}v + \frac{1}{2}) - \psi(\frac{1}{2}) + 4 \int_0^x s^{-1} ds \int_0^s t J_v(t) K_v(t) \, dt \\
\]

\[ = \psi(\frac{1}{2}v + \frac{1}{2}) - \psi(\frac{1}{2}) + 4 \int_0^x t J_v(t) K_v(t) \log(x/t) \, dt, \quad v > -1. \]
From the Mellin transform [2, form. 6.8(41), corrected]

\[ \mathcal{M}[J_v(t)K_v(t)] = \int_0^\infty J_v(t)K_v(t)t^{s-1} \, dt = \frac{2^{s-3}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s + \frac{1}{2}v)}{\Gamma(1 - \frac{1}{4}s + \frac{1}{2}v)}, \]

valid for \( \text{Re} \ s > \max(0, -2v) \), we deduce the special values

\[ \int_0^\infty tJ_v(t)K_v(t) \, dt = \frac{1}{2}, \]
\[ \int_0^\infty tJ_v(t)K_v(t) \log t \, dt = \frac{1}{2} \left[ \log 2 + \frac{1}{2} \psi(1) + \frac{1}{2} \psi\left(\frac{1}{2}v + \frac{1}{2}\right) \right] \]
\[ = \frac{1}{4} \psi\left(\frac{1}{2}v + \frac{1}{2}\right) - \frac{1}{4} \psi\left(\frac{1}{2}\right) - \frac{1}{2} \gamma, \]

where \( \gamma \) is Euler’s constant. These values are now used to reduce the representation (4) to

\[ F_v(x) = 2 \log x + 2\gamma + 4 \int_x^\infty tJ_v(t)K_v(t) \log(t/x) \, dt. \]

The latter form is suitable to determine the asymptotic expansion of \( F_v(x) \) as \( x \to \infty \); the leading terms are given by

\[ F_v(x) = 2 \log x + 2\gamma - 2x^{-1}e^{-x}\left[ \sin(x - \frac{1}{2}v\pi - \frac{1}{4}\pi) + O(x^{-1}) \right], \quad (x \to \infty). \]

Starting from the differential equations for the Bessel functions \( J_v \) and \( K_v \), we determine the primitive function

\[ 2 \int tJ_v(t)K_v(t) \, dt = t[J_v(t)K'_v(t) - J'_v(t)K_v(t)] + C, \]

which is used to integrate by parts in (4) and (6). As a result we obtain the representations

\[ F_v(x) = \psi\left(\frac{1}{2}v + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + 2 \int_0^x \left[ J_v(t)K'_v(t) - J'_v(t)K_v(t) + \frac{1}{t} \right] \, dt, \quad v > -1, \]

\[ F_v(x) = 2 \log x + 2\gamma - 2 \int_x^\infty [J_v(t)K'_v(t) - J'_v(t)K_v(t)] \, dt \]
\[ = 2 \log x + 2\gamma + 2J_v(x)K_v(x) + 4 \int_x^\infty J'_v(t)K_v(t) \, dt \]
\[ = 2 \log x + 2\gamma - 2J_v(x)K_v(x) - 4 \int_x^\infty J'_v(t)K_v(t) \, dt. \]

Notice that the representations (6) and (8) may serve to define \( F_v(x) \) for any \( v \in \mathbb{R} \) and \( x > 0 \).

It does not seem possible to evaluate any of the integrals (7) and (8) in closed form, except if \( v \) is an odd integer. Setting \( v = \pm 1 \) in (8), we find

\[ F_1(x) = 2 \log x + 2\gamma - 2 \int_x^\infty [J_1(t)K'_1(t) - J'_1(t)K_1(t)] \, dt \]
\[ = 2 \log x + 2\gamma + 2 \int_x^\infty [J_1(t)K_0(t) + J_0(t)K_1(t)] \, dt \]
\[ = 2 \log x + 2\gamma + 2J_0(x)K_0(x), \]

\[ F_{-1}(x) = 2 \log x + 2\gamma - 2 \int_x^\infty [-J_1(t)K'_1(t) + J'_1(t)K_1(t)] \, dt \]
\[ = 2 \log x + 2\gamma - 2J_0(x)K_0(x). \]
Consider next the difference

\[
F_{v+1}(x) - F_{v-1}(x) = \int_0^\infty J_0(x \sqrt{2t}) \frac{(\sqrt{1+t^2} - t)^{v-1}}{t \sqrt{1+t^2}} \left[ 1 - (\sqrt{1+t^2} - t)^2 \right] dt
\]

which is identical to the integral in (2). Thus, in view of (3), we arrive at the recurrence relation

\[
(11) \quad F_{v+1}(x) - F_{v-1}(x) = 4J_v(x)K_v(x).
\]

By starting from (9) or (10), the relation (11) permits the closed-form evaluation of \( F_v(x) \) when \( v \) is an odd integer.

The Mellin transform of \( F_v(x) \) is easily determined from (3) and (5). By taking the inverse transform, we obtain the following representations by Mellin-Barnes integrals:

\[
(12) \quad F_v(x) = \psi\left(\frac{1}{2} + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \frac{\Gamma\left(\frac{1}{2} s\right) \Gamma\left(\frac{3}{4} + \frac{1}{4} s\right)}{s \Gamma\left(\frac{1}{2} v + \frac{1}{2} - \frac{1}{4} s\right)} x^{-s} ds,
\]

\[
\text{max}(-2, -2v - 2) < c < 0, \quad v > -1;
\]

\[
(13) \quad F_v(x) = 2 \log x + 2\gamma + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \frac{\Gamma\left(\frac{1}{2} s\right) \Gamma\left(\frac{1}{2} v + \frac{1}{2} + \frac{1}{4} s\right)}{s \Gamma\left(\frac{1}{2} v + \frac{1}{2} - \frac{1}{4} s\right)} x^{-s} ds,
\]

\[
c > 0, \quad v > -1.
\]

The two integrals differ by the residue at the double pole \( s = 0 \) of the integrand. By closing the integration contour in (12) to the left, the integral may be evaluated by residue calculus. As a result we find a power-series expansion of \( F_v(x) \) containing powers \( x^{2k} \) and \( x^{2v+2+4k} \), and, if \( v \) is an integer, logarithmic terms \( x^{2v+2+4k} \log x \), where \( k = 0, 1, 2, \ldots \). The Mellin-Barnes integrals in (12) and (13) can be identified with certain Meijer’s \( G \)-functions, as defined in [1, form. 5.3(1)]. Thus, (12) and (13) can be expressed as

\[
F_v(x) = \psi\left(\frac{1}{2} v + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + \frac{1}{2\sqrt{\pi}} G_{32}^{26} \left( \begin{array}{c} x^4 \\ 64 \end{array} \right) \left( \begin{array}{c} 1, 1 \\ \frac{1}{2}, 1, \frac{1}{2} + \frac{1}{2} v, 0, 0, \frac{1}{2} - \frac{1}{2} v \end{array} \right),
\]

\[
F_v(x) = 2 \log x + 2\gamma + \frac{1}{2\sqrt{\pi}} G_{15}^{40} \left( \begin{array}{c} x^4 \\ 64 \end{array} \right) \left( \begin{array}{c} 1 \\ 0, 0, \frac{1}{2}, \frac{1}{2} + \frac{1}{2} v, \frac{1}{2} - \frac{1}{2} v \end{array} \right).
\]

REFERENCES


Also solved by C. C. Grosjean (University of Ghent, Ghent, Belgium) and N. Ortner and P. Wagner (University of Innsbruck, Austria).