Ideal pulse-tube refrigerators with real gases

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The factor of 100 difference between experiments and theory in the coefficient of performance (COP) of pulse-tube refrigerators is always attributed to the nonideal behavior of these devices. We show that the thermodynamic properties of the nonideal working fluid have a profound influence on the COP, explaining one order of magnitude of the observed discrepancy and thus posing more realistic limitations for technological improvements. However, our analysis also shows that the real gas properties of the working fluid can even result in a higher value than predicted, opening prospects for improvements. © 2005 American Institute of Physics. [DOI: 10.1063/1.2011777]

I. INTRODUCTION

Cryocoolers, which have no moving parts at low temperatures, are studied intensively since they have many advantages over existing cooling technologies: there is no need for a liquid-helium bath cryostat, the cost can be low, the reliability is high, there are low mechanical vibrations, and small magnetic interferences. In recent years there have been impressive developments in these cryocoolers, especially in the field of pulse-tube refrigerators (PTR’s).\(^1\) Ingenious concepts led to improvements in cooling performances and in temperature range, which are reviewed, e.g., by Radebaugh,\(^2\) de Waele,\(^3\) and Ravex.\(^4\) However, until now many of its fundamental properties are not well understood. In particular, the very large discrepancy between the coefficient of performance (COP) (ratio between the cooling power \(\dot{Q}_L\) and the power \(\dot{W}\)), derived for ideal PTR’s with ideal gases, and the COP realized in actual practice has not been explained. The ideal COP is equal to the ratio of the low temperature \(T_L\) and room temperature \(T_H\):\(^5\)

\[
\xi_0 = \frac{\dot{Q}_L}{\dot{W}} = \frac{T_L}{T_H}.
\]  

Equation (1) implies that 0.5 W of cooling power at \(T_L = 4.2\) K with \(T_H = 300\) K requires at least 36 W compressor power. In reality, however, 5 kW are needed, about two orders of magnitude more. A factor of 4 is due to the dissipation in the room-temperature components (compressor, valves), leaving still a gap of 1.3 kW and 36 W, which is a factor of 29 discrepancy. It is tempting to attribute this large discrepancy to the many imperfections of the cooler such as flow resistances, bad heat transfer, turbulence, heat leaks, etc. However, we will explain that the largest part of the gap between the highly idealized case and the more realistic COP is due to the nonideality of the working gas. We calculate the COP of ideal pulse-tube refrigerators with nonideal gases. We will show that the pressure dependence of the enthalpy of the gas, which is zero in the case of an ideal gas, has a large influence on the COP. This results in a much lower COP, even if the imperfections of the system are minimized. It limits the technological innovations and has a similar scope of validity as the Carnot efficiency for cryocoolers, but on a more realistic basis. This result has tremendous conceptual and technological implications.

II. IDEAL PULSE-TUBE REFRIGERATOR

In this paper we discuss a so-called Stirling-type single-orifice PTR (Fig. 1). We assume that the system is ideal. This means that the flow resistances of all components, except the orifice, are zero; all heat exchangers are ideal, i.e., they have a constant temperature and the gas flowing out always has the temperature of the heat-exchanger body; the processes in the compressor space and in the tube are reversible and adiabatic; the heat capacity of the matrix material in the regenerator is very high; the thermal contact between the gas and the regenerator material is very good; and the pressure variations are small. The heat conduction and the thermal conductivity play a special role as will be discussed in detail later on. All these assumptions are the usual assumptions which hold for an ideal system. If, in addition, we assume that the working fluid is an ideal gas Eq. (1) can be derived.

The first law of thermodynamics is used in the form

\[
\dot{U} = \sum_k \dot{Q}_k + \sum_k n_k H_{mk} - \sum_k p_k \dot{V}_k + \dot{W},
\]

with \(\dot{U}\) the rate of change of the internal energy of the system under consideration, \(\dot{Q}_k\) the heat flows into the system at the

![FIG. 1. Schematic diagram of a Stirling-type single-orifice PTR. From left to right the system consists of a compressor with moving piston (piston); a heat exchanger at room temperature (X), a regenerator, a low-temperature heat exchanger (X), a tube (tube), a third heat exchanger (X), which is at room temperature, an orifice (O), and a reservoir (res). The space between the piston and \(T_H\) is called the compressor space. The cooling power is generated at low temperature \(T_L\). The room temperature is \(T_H\). The part in between \(X_H\) and \(X_L\) is isolated thermally from the surroundings, e.g., by a vacuum chamber.](Image)
various boundaries which are labeled with the index \( k \), \( n_k \) the molar flow of matter into the system, \( H_{mk} \) the molar enthalpy of this matter, \( V_k \) the rate of change of the volume of the system at various moving boundaries such as a piston, and \( p_k \) the corresponding pressure behind it. Finally, \( W \) takes into account all other forms of power applied to the system. The potential and kinetic energies are neglected. We use the notation \( Y \) for the flow of a thermodynamic state function \( Y \) and \( \dot{Y} \) for the rate of change of \( Y \). Even though the dimensions of \( Y \) and \( \dot{Y} \) are the same their physical meaning is distinctly different. In this case the properties which are not properties of state, this distinction is meaningless and we will use the dot notation to indicate flow rates.

The second law will be used in the form

\[
\dot{S} = \sum_k \frac{\dot{Q}_k}{T_k} + \sum_k n_k \dot{S}_{mk} + \sum_k \dot{S}_{bk}, \quad \text{with all } \dot{S}_{bk} = 0. \tag{3}
\]

Here \( \dot{S} \) is the rate of change of the entropy of the system and \( T_k \) are the temperatures at which the heat flows \( \dot{Q}_k \) enter the system. With \( \dot{S}_{mk} \) we represent the molar entropy of the matter flowing into the system. Finally, \( \dot{S}_{bk} \) represent the entropy production rates due to the various irreversible processes inside the system. We discuss a state in which all system parameters are perfectly periodic. As a result the time-averaged values \( \overline{U} \) and \( \overline{S} \) are zero.

In our analysis we assume that the motion of the piston is controlled in such a way that the pressure varies stepwise between \( p \) and \( p + \Delta p \) so that there are two time intervals each with constant pressure and constant flow. In this way we avoid complicated averaging formulas. As \( \Delta p \) is small the pressure difference, driving the gas through the orifice is small, and the amount of gas \( n_r \) flowing through the regenerator during a half cycle tend to become small as well. However, \( n_r \) can still be macroscopic if the orifice \( O \) is opened far enough or if the cycle time \( t_c \) is long enough. Therefore, we will treat \( n_r \) as a macroscopic quantity. In this situation the enthalpy flows, cooling power, etc., are first order in \( \Delta p \). During the change in pressure a small amount of gas flows through the regenerator to compensate for the compressibility of the gas in the tube, but this extra flow leads to second-order effects and will be neglected.

### III. IDEAL PULSE-TUBE REFRIGERATOR WITH REAL GAS

In order to derive a new expression for the COP we concentrate on the energy balance near the regenerator. The total energy flow \( \overline{E} \) in the regenerator is the sum of the heat flow

\[
\dot{Q}_L = - \kappa A_r \frac{dT}{dl}, \tag{4}
\]

and the enthalpy flow \( H_p \), with \( \kappa \) an effective thermal conductivity, \( A_r \) the area of the cross section of the regenerator, and \( l \) the length coordinate. Energy conservation requires that \( \overline{E} = \text{constant} \).

The temperature and pressure dependence of the molar enthalpy \( H_m(p, T) \) can be derived from the general relation

\[
dH_m = C_p dT + H_p dp, \quad \text{with } \quad C_p = \text{molar heat capacity at constant pressure} \quad \text{and} \quad H_p = \text{molar volume} \times \text{thermal-expansion coefficient at constant pressure} \quad \text{and} \quad V_m \text{ the molar volume}.
\]

Isothermal enthalpy changes, as in the regenerator, are equal to \( \Delta H_m = H_p \Delta p \), so the average energy flow in the regenerator is

\[
\overline{E} = \dot{Q}_L + n_r H_p \Delta p \quad [\text{with } n_r = n_r(t_c/2)].
\]

Using Eq. (4) and integrating over the regenerator length \( L_r \), we get the relation

\[
1 = \int_{T_L}^{T_H} \frac{\kappa A_r/L_r}{\overline{E} - n_r H_p \Delta p} dT
\]

which determines the value of \( \overline{E} \).

For the molar enthalpy also holds the general relation

\[
dH_m = T dS + V_m dp, \quad \text{so, for isentropic changes (as in the compressor space and in the tube), } \Delta H_m = V_m \Delta p.
\]

The average power \( W \) applied to the compressor is equal to the average energy flow in the compressor space, so

\[
\overline{W} = n_r V_L \Delta p \quad [\text{with } V_L = V_m(p, T_H)].
\]

(The quantities \( V_L, H_{pl}, \) and \( H_{plt} \), which will be used later on, are defined in a similar way.) The time-averaged enthalpy flow in the tube can be written as

\[
\overline{E} = n_r V_L \Delta p.
\]

With these relations the first law, applied to \( X_L \), gives for the cooling power

\[
\dot{Q}_L = n_r V_L \Delta p = \overline{E}.
\]

So the COP is

\[
\xi = \frac{n_r V_L \Delta p - \overline{E}}{n_r V_H \Delta p}.
\]

Equation (7) shows that the maximum value for COP is found for the minimum value for \( \overline{E} \). With Eq. (5) it can be shown\(^6\) that this is the case if

\[
\kappa A_r/L_r = 0.
\]

In this limit the integral can only give finite values if the denominator is equal to zero somewhere in the temperature interval \([T_L, T_H]\).

The only realistic working fluid for applications in low-temperature cryocoolers on a large scale is \(^4\)He. Figure 2 is a plot of \( H_p \) for \(^4\)He at 15 bars as a function of temperature.\(^7\) Inspection of this plot shows that the minimum energy flow is found if

\[
\overline{E} = n_r \Delta p H_{p0}, \quad \text{with } H_{p0} = \max[H_{pl}, H_{plt}].
\]

Equation (9) expresses that the total energy flow is equal to the enthalpy flow in one of the two ends of the regenerator. At this particular end the heat flow and the temperature gradient are both equal to zero. At the other end the heat flow is nonzero and, due to Eq. (8), the temperature gradient is in-
In order to compare Eq. 9 by value possible in the ends of the system. Equation the fluid at the low-temperature and at the high-temperature ideal nature of the working fluid in terms of the properties of which shows that \( T_x \) the intersections with the high-temperature branch determine the values of \( V_L \).

IV. DISCUSSION

Let us consider the case that \( H_{pl} > H_{ph} \). Now Eqs. (9) and (11) give

\[
\psi_{LT} = \frac{V_L - H_{pl}T_H}{V_H T_L}. \tag{12}
\]

This relation can be simplified to

\[
\psi_{LT} = \left( \frac{\partial V}{\partial T} \right)_{pl} V_H T_H. \tag{13}
\]

This is an elegant relation expressing the effect of the non-ideal nature of the working fluid in terms of the properties of the fluid at the low-temperature and at the high-temperature ends of the system. Equation (13) consistently explains why \( ^3\)He works better than \( ^4\)He as a working fluid in cryocoolers, since \( (\partial V/\partial T)_{pl} \) is higher for \( ^3\)He than for \( ^4\)He. For 2.5 K \( < T < 10 \) K the molar volume of \( ^4\)He can be approximated by

\[
V_m = V_0(1 + (T/T_0)^3). \tag{14}
\]

With this relation Eq. (13) gives

\[
\psi_{LT} = 3 \frac{V_0 T_H T_L^3}{V_H T_0^3}, \tag{15}
\]

which shows that \( \psi \) goes down very fast if \( T_L \) is reduced.

For combinations of \( T_L \) and \( T_H \) where \( H_{pl} < H_{ph} \) the nonideality factor is finitely large. With Eqs. (7) and (9) we arrive at the basic relation for the COP with a real gas

\[
\xi = \frac{V_L - H_{p0}}{V_H}. \tag{10}
\]

In order to compare Eq. (10) with the COP of an ideal gas [Eq. (1)] we introduce the nonideality factor \( \psi \)

\[
\psi = \frac{\xi}{\xi_0} = \frac{V_L - H_{p0}T_H}{V_H T_L}. \tag{11}
\]

From Eq. (9) it can be seen that \( \psi \) always assumes the lowest value possible in the \([T_L, T_H]\) interval. For an ideal gas \( H_{p0} = 0 \) and \( V_L/V_H = T_L/T_H \) so \( \psi = 1 \), as it should.

We have found no way to simplify this expression further.

Now we will have a closer look at Fig. 2. The highest values of \( H_L \) are found in the low-temperature region. In fact, if \( T_L < 6.6 \) K, \( H_{pl} > H_{ph} \) for all values of \( T_H \), so Eq. (12) holds over the whole temperature range. On the other hand, for \( T_L > 12.3 \) K (which is the temperature where \( H_p \) has its minimum) \( H_{pl} < H_{ph} \) for all values of \( T_H \). Now Eq. (15) holds over the whole range. In the intermediate temperature range there is a crossover value \( T_x \) for \( T_H \). For \( T_H > T_x \) Eq. (12) holds, while for \( T_H < T_x \) Eq. (15) should be used.

Figure 3 gives plots of \( \psi_{LT} \) for combinations of six values of \( T_L \). In the majority of cases \( \psi < 1 \), but there are also some cases where \( \psi > 1 \) if \( H_{pl} \) and \( H_{ph} \) are both negative (hence \( H_{p0} < 0 \)). This means that the COP can be enhanced due to the nonideality of the gas. This is a surprising result since it is generally believed that Eq. (1) gives an upper limit for the COP of pulse-tube refrigerators. One may wonder whether our result is in conflict with the second law. The entropy production rate in the regenerator is given by

\[
\dot{S}_r = \int_{T_L}^{T_H} \frac{\dot{Q}_r}{T^2} dT = \dot{n}_r \Delta p \int_{T_L}^{T_H} \frac{H_{p0} - H_p}{T^2} dT. \tag{16}
\]

Since \( H_{p0} - H_p > 0 \) (even if \( H_{ph} < 0 \)) and all other quantities in Eq. (16) are positive as well, we may conclude that \( \dot{S}_r > 0 \) and that there is no conflict with the second law. The enhanced cooling power can be considered as coming from Joule-Thomson-like effects in the PTR.

The enhancement of the COP for helium at 15 bars takes place only in the temperature range between 8.2 and 36.4 K. So it can only be used in the second stage of the pulse-tube refrigerator if it operates in that temperature range. However, one might consider using other substances (such as nitrogen or methane) which show cooling-power enhancement in different temperature ranges. Now we look at the performance of a PTR operating at 50 bars from room temperature to 130 K. Such devices have economic prospects, e.g., cooling machines for the process industry. A PTR working with two
stages, one with nitrogen and one with methane, has a COP which is 24% higher than that of a single-stage machine with helium.

Figure 3 shows that, for low $T_L$ values, $\psi$ is significantly lower than one, indicating a strong reduction of the COP due to the nonideality of the gas. If $T_L=4.2$ K and $T_H=300$ K we get for helium at 15 bars $\psi=0.163$. With this value 0.5 W of cooling power at 4.2 K requires at least 0.22 kW compressor power. With a real power of 1.3 kW this leaves only a factor of 6 discrepancy instead of the factor of 29 mentioned earlier. The remaining discrepancy of a factor of 6 can be attributed to imperfections in the system such as flow resistance, turbulence, bad heat transfer, etc., which are not unusual.

V. CONCLUSIONS

In conclusion, in this paper we have shown that the limits of technological innovation in this field are not set by Eq. (1), but by Eq. (10) which has a similar scope of validity as the Carnot efficiency of cryocoolers but on a much more realistic basis. This is due to the thermodynamic properties of the working fluid which determine the performance of pulse-tube refrigerators. At very low temperatures the COP, calculated with real gas parameters, is very much lower than for an ideal gas. This explains to a large extent the gap between the ideal and the real performance. On the other hand we have shown that there is an intermediate temperature range where the COP is enhanced.

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