\[ S^* = \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_j^2 + \cdots \right) \left( \frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_i^2 + \cdots \right) \left[ (\epsilon_i - \epsilon_j)^2 + \cdots \right] \]

\[ + \frac{1}{8} \sum_{i=1}^{k} \sum_{j=k+1}^{n} \left( \frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_j^2 + \cdots \right) \left( \frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_i^2 + \cdots \right) \left[ \frac{3}{2} + (\epsilon_j - \epsilon_i) \sqrt{3} - (\epsilon_j - \epsilon_i)^2 + \cdots \right] \]

\[ + \frac{1}{8} \sum_{i=k+1}^{n} \sum_{j=1}^{k} \left( \frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_j^2 + \cdots \right) \left( \frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_i^2 + \cdots \right) \left[ \frac{3}{2} + (\epsilon_i - \epsilon_j) \sqrt{3} - (\epsilon_i - \epsilon_j)^2 + \cdots \right] \]

\[ \frac{1}{4} \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \left( \frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_j^2 + \cdots \right) \left( \frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_i^2 + \cdots \right) \left[ (\epsilon_i - \epsilon_j)^2 + \cdots \right] \]

\[ = \frac{27}{32} k(n - k) - \frac{9}{8} (2n - 3k) \sum_{j=1}^{k} \epsilon_j^2 - \frac{9}{8} (3k - n) \sum_{j=k+1}^{n} \epsilon_j^2 - \frac{9}{8} \left( \sum_{j=1}^{k} \epsilon_j - \sum_{j=k+1}^{n} \epsilon_j \right)^2 + \cdots, \]

in which \( 27k(n - k)/32 = 27[n/2](n - [n/2])/32 \) is the value of the given \( S \) when solution (2) is inserted into it and the result neglects only fourth and higher degree terms in the \( \epsilon \)'s.

Note that

\[ 2n - 3 \left\lfloor \frac{n}{2} \right\rfloor > 0 \quad \text{for} \quad n > 2 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n > 0 \quad \text{for} \quad n \geq 2, \quad n \neq 3, \]

both expressions being equal to \( n/2 \) when \( n \) is even. For \( n \) odd,

\[ 2n - 3 \left\lfloor \frac{n}{2} \right\rfloor = \frac{n + 3}{2} > 0 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n = \frac{n - 3}{2} \begin{cases} > 0, & n \geq 5, \\ = 0, & n = 3. \end{cases} \]

The obtained formula proves that \( S = 27k(n - k)/32 \) is a local maximum, since there exists a neighborhood around (2) in which \( S \) can take on only smaller values than at (2).

**A Function Arising in One-Dimensional Percolation**

*Problem* 95-16, by M. L. Glasser (Clarkson University).

There has been controversy over the function

\[ G(t) = \int_{0}^{\infty} e^{-t/x} e^{-1/2} x^2 \, dx, \]

which arises in studies of hopping transport for one-dimensional percolation, particularly concerning its behavior for large and small \( t \) [1]. Find computationally effective expansions for this function.

**REFERENCE**


We shall derive two expansions for the function \( G(t) \), namely, a series expansion in powers of \( t \) and \( \log t \) that is convergent for \( t \geq 0 \), and an asymptotic expansion valid for \( t \to \infty \).

First we determine the Mellin transform of \( G(t) \):

\[
\mathcal{M}\{G(t)\} = \int_0^{\infty} G(t) t^{z-1} dt = \int_0^{\infty} e^{-x^2/2} dx \int_0^{\infty} e^{-t/x} t^{z-1} dt
\]

\[
= \Gamma(z) \int_0^{\infty} e^{-x^2/2} x^z dx = 2^{(z-1)/2} \Gamma(z) \Gamma \left( \frac{1}{2} z + \frac{1}{2} \right),
\]

valid for \( \Re z > 0 \). Then, inversely, \( G(t) \) can be represented by the Mellin–Barnes integral

\[
G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z) \Gamma \left( \frac{1}{2} z + \frac{1}{2} \right) 2^{(z-1)/2} t^{-z} dz, \quad c > 0.
\]

Here the integrand has simple poles at \( z = -2n \) and double poles at \( z = -2n - 1 \), where \( n = 0, 1, 2, \ldots \). By closing the contour to the left, the integral is evaluated by standard residue calculus with the result

\[
G(t) = 2^{-1/2} \pi \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)! \Gamma(n + \frac{1}{2})} t^{2n} - 2^{1/2} \pi \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} n! \left[ \psi(2n + 2) + \frac{1}{2} \psi(n + 1) - \log(2^{-1/2} t) \right] t^{2n+1}
\]

\[
= \left( \frac{\pi}{2} \right)^{1/2} \phantom{0} F_2 \left( \frac{1}{2}, \frac{1}{2}; -t^2/8 \right) - \left[ 1 - \frac{3}{2} \gamma - \log(2^{-1/2} t) \right] t_0 F_2 \left( 1, \frac{3}{2}; -t^2/8 \right)
\]

\[
- \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n! (\frac{1}{2})_n n!} \left( \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{2k+1} \right) \left( \frac{t^2}{8} \right)^n.
\]

Here, \( \psi(x) = \Gamma'(x)/\Gamma(x) \), while \( \gamma \) denotes Euler’s constant and the notation \( 0 F_2 \) stands for the generalized hypergeometric function. Obviously, the series expansion for \( G(t) \) is convergent for \( t \geq 0 \).

Next we turn to the asymptotics of \( G(t) \) as \( t \to \infty \). Since the exponent \( t/x + x^2/2 \) is stationary at \( x = t^{1/3} \), we substitute \( x = t^{1/3} u \), whereupon the integral passes into

\[
G(t) = t^{1/3} \int_0^{\infty} \exp \left[ -t^{2/3} (u^{-1} + \frac{1}{2} u^2) \right] du.
\]

The exponent \( u^{-1} + \frac{1}{2} u^2 \) attains its absolute minimum \( 3/2 \) at \( u = 1 \). Following Laplace’s method \([1, \S 5.1]\), we apply the transformation of integration variables \( x = x(u) \) or, inversely, \( u = u(x) \), specified by

\[
u^{-1} + \frac{1}{2} u^2 = \frac{3}{2} (1 + x^2), \quad \text{sgn}(u - 1) = \text{sgn} x.
\]

Then the integral for \( G(t) \) transforms into

\[
G(t) = t^{1/3} \exp \left[ -\frac{3}{2} t^{2/3} \right] \int_{-\infty}^{\infty} \exp \left[ -\frac{3}{2} t^{2/3} x^2 \right] \frac{du(x)}{dx} dx.
\]
Around $x = 0$ the inverse function $u = u(x)$ has the Taylor expansion

$$u = u(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

in which the coefficients $a_n$ are readily determined by means of a computer algebra program package. The asymptotic expansion of $G(t)$ now follows by the use of Watson's lemma [1, §4.1]. Thus we replace $du(x)/dx$ by its Taylor expansion and integrate term by term, with the result

$$G(t) \sim t^{1/3} \exp \left[ -\frac{3}{2} t^{2/3} \right] \sum_{n=1}^{\infty} n a_n \int_{-\infty}^{\infty} \exp \left[ -\frac{3}{2} t^{2/3} x^2 \right] x^{n-1} dx$$

$$= \exp \left[ -\frac{3}{2} t^{2/3} \right] \sum_{n=0}^{\infty} (2n + 1) \Gamma \left( n + \frac{1}{2} \right) a_{2n+1} \left( \frac{2}{3} \right)^{n+\frac{1}{2}} t^{-2n/3} \quad (t \to \infty).$$

By the use of MAPLE we have evaluated the first seven coefficients $a_n$. The corresponding four-term asymptotic expansion is found to be

$$G(t) = \left( \frac{2\pi}{3} \right)^{1/2} \exp \left[ -\frac{3}{2} t^{2/3} \right] \left\{ 1 - \frac{1}{18} t^{-2/3} + \frac{25}{648} t^{-4/3} - \frac{1225}{34992} t^{-2} + O(t^{-8/3}) \right\} \quad (t \to \infty).$$

It has been verified that for $t \geq 3$ the present expansion reproduces $G(t)$ with a relative error less than 0.0013. Over the range $0 < t < 3$, the function $G(t)$ may be calculated from the power series expansion presented above.

Addition remarks. (1) The Mellin–Barnes integral representation of $G(t)$ can be rewritten in terms of the definition [2, Form. 5.3(1)] of Meijer's $G$-function, thus leading to

$$G(t) = (2\pi)^{-1/2} G_{30}^{03} \left( \frac{t^2}{8} \mid 0, \frac{1}{2}, \frac{1}{2} \right).$$

(2) It is easily verified that $y = G(t)$ is a solution of the differential equation

$$(*) \quad ty''' + y'' + y = 0,$$

which has a regular singularity at $t = 0$. Using the method of Frobenius [3, Chap. XVI], one may construct three independent series solutions viz.

$$y_1(t) = {}_0 F_2 \left( \frac{1}{2}, \frac{1}{2}; -t^2/8 \right), \quad y_2(t) = t {}_0 F_2 \left( 1, \frac{3}{2}; -t^2/8 \right),$$

$$y_3(t) = \frac{d}{d\sigma} \left[ t^\sigma F_3 \left( 1; \frac{1}{2} + \frac{1}{2} \sigma, \frac{1}{2} + \frac{1}{2} \sigma, 1 + \frac{1}{2} \sigma; -t^2/8 \right) \right]_{\sigma=1}\$$

$$= t \log t {}_0 F_2 \left( 1, \frac{3}{2}; -t^2/8 \right) - t \sum_{n=1}^{\infty} \frac{(-1)^n}{n! (\frac{3}{2})_n n!} \left( \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{2k+1} \right) \left( \frac{t^2}{8} \right)^n.$$

Clearly,

$$G(t) = \left( \frac{\pi}{2} \right)^{1/2} y_1(t) - \left( 1 - \frac{3}{2} y + \frac{1}{2} \log 2 \right) y_2(t) + y_3(t).$$
The differential equation (\(\ast\)) may also be solved by the method of Laplace transformation [3, Chap. XVIII], leading to contour-integral solutions of the form

\[ y(t) = \int_C e^{-t/z} W(z) dz. \]

This function \(y(t)\) satisfies the differential equation provided that

\[ \frac{d}{dz} (z^{-1} W) + (z^{-2} + 1) W = 0, \quad \text{hence} \quad W(z) = e^{-z^2/2}, \]

and

\[ [z^{-1} e^{-t/z} W(z)]_C = [z^{-1} e^{-t/z - z^2/2}]_C = 0. \]

The latter condition is satisfied if the contour \(C\) is taken along the positive real axis, whereupon the original integral representation for \(G(t)\) is recovered. Other possible choices for \(C\) are a simple closed contour encircling \(z = 0\) in the positive direction, or a contour along the real axis with an indentation above or below \(z = 0\). The corresponding contour-integral solutions can be shown to be related to the series solutions \(y_1(t)\) and \(y_2(t)\) by

\[ \int_{(0^+)} e^{-t/z} e^{-z^2/2} dz = -2\pi i y_2(t), \]

\[ \int_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} dz = (2\pi)^{1/2} y_1(t) + \pi i y_2(t), \]

\[ \int_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} dz = (2\pi)^{1/2} y_1(t) - \pi i y_2(t). \]

The transformation of variables \(x = x(u)\) is given by

\[ x^2 = \frac{2}{3} \left( u^{-1} + \frac{1}{2} u^2 \right) - 1 = \frac{(u - 1)^2(u + 2)}{3u}, \quad \text{sgn} x = \text{sgn}(u - 1); \]

hence

\[ x = x(u) = (u - 1) \left( \frac{u + 2}{3u} \right)^{1/2}. \]

The Taylor coefficients \(a_n\) of the inverse function \(u = u(x)\) can also be determined by means of Lagrange’s inversion formula [4, §7.32] viz.

\[ a_n = \frac{1}{n!} \frac{d^{n-1}}{d u^{n-1}} \left[ \left( \frac{3u}{u + 2} \right)^{n/2} \right]_{u=1} = \frac{1}{n!} \frac{d^{n-1}}{d u^{n-1}} \left[ \left( \frac{1 + u}{1 + u/3} \right)^{n/2} \right]_{u=0}. \]

As for the coefficients \(a_{2n+1}\) of odd index appearing in the asymptotic expansion of \(G(t)\), the 2nth derivative may be evaluated via the expansion of \((1 + u)^{n+1/2}\) and \((1 + u/3)^{-n-1/2}\) in binomial series. Thus it is found that \(a_{2n+1}\) is expressible in terms of a hypergeometric function \(F\) of argument \(1/3\) viz.

\[ a_{2n+1} = \frac{(-1)^n \left( -\frac{1}{2} \right)^n}{2^{2n} n!} F \left( -2n, n + \frac{1}{2}; \frac{3}{2} - n; \frac{1}{3} \right). \]
By an application of Petkovsek's algorithm, Koepf [5] has proven that there is no closed-form expression for the present $F(\frac{1}{3})$. An application of the computer algebra procedure "sumrecursion," due to Koepf [6], leads to the recurrence relation

$$6(n + 1)(2n + 3)a_{2n+1} + (12n^2 + 12n + 1)a_{2n+1} + 2(n - 1)(2n + 1)a_{2n-1} = 0,$$

valid for $n = 0, 1, 2, \ldots$, with initial values $a_{-1} = 0$ and $a_1 = 1$. This relation is most useful for the calculation of the coefficients $a_{2n+1}$.

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Also solved by Paul Bracken (University of Florida), Brian Bradie (Christopher Newport University), Hongwei Chen (Christopher Newport University), S. L. Cole and J. J. Lewandowski (Rensselaer Polytechnic Institute), Carl C. Grosjean (University of Ghent, Belgium), W. B. Jordan (Scotia, NY), Kee-Wai Lau (Hong Kong), Allen R. Miller (Washington, DC), and the proposer.

Expansions for Elementary Functions

Problem 95-19, by Joe Keane (Union City, CA).

Consider this series:

$$S = \sum_{j=0}^{\infty} (A + jB) \left\{ \frac{18k(2k - 1)}{(6k + 1)(6k + 5)} X \right\}.$$

Show that we obtain $S = \arctan t$ with the following values:

$$A = 15 + 25t^2 + 8t^4,$$
$$B = 18 + 30t^2 + 8t^4,$$
$$M = t/(15(1 + t^2)^2),$$
$$X = -4t^6/(27(1 + t^2)^2).$$

Also show that we obtain $S = \log z$ with the following values:

$$A = 10z(z^2 + 10z + 1) - (z - 1)^4,$$
$$B = 12z(z^2 + 10z + 1) - 2(z - 1)^4,$$
$$M = (z - 1)/(60z^2(z + 1)),$$
$$X = (z - 1)^6/[108z^2(z + 1)^2].$$