INTEGRAL TRANSFORMATIONS
AND
SPACES OF TYPE S

PROEFSCHRIFT

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CORNELIS ADRIANUS MARIA VAN BERKEL

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Dit proefschrift is goedgekeurd door de promotoren
prof. dr. ir. J. de Graaf
en
prof. dr. J. Boersma

Copromotor:
dr. ir. S.J.L. van Eijndhoven
Aan mijn ouders
CONTENTS

Introduction 1

1. Functions of rapid decrease 9

1.1 The Schwartz space $S(\mathbb{R})$ 9
1.2 The subspace $S_{\text{even}}(\mathbb{R})$ and the Hankel transformation $\mathcal{H}_c$ 14
1.3 The space $S(\mathbb{R}^+)$ and the Hankel-Clifford transformation $\mathcal{H}_c$ 23
1.4 The Schwartz space $S(\mathbb{R}^n)$ 29
1.5 Spherical harmonics 34
1.6 The space $S(\mathbb{R}^n)$ and expansions in spherical harmonics 41
1.7 The Radon transformation on $S(\mathbb{R}^n)$ 50

2. Gel'fand-Shilov spaces 61

2.1 The Gel'fand-Shilov spaces $S_0^0(\mathbb{R})$ 61
2.2 The subspaces $S_{0,\text{even}}^0(\mathbb{R})$ and the Hankel transformation $\mathcal{H}_c$ 74
2.3 The spaces $G_0^0(\mathbb{R}^+)$ and the Hankel-Clifford transformation $\mathcal{H}_c$ 78
2.4 The Gel'fand-Shilov spaces $S_0^0(\mathbb{R}^n)$ 84
2.5 The spaces $S_0^0(\mathbb{R}^n)$ and expansions in spherical harmonics 93
2.6 The Radon transformation on $S_0^0(\mathbb{R}^n)$ 99

3. Fractional calculus and Gegenbauer transformations 105

3.1 The Weyl fractional operators 105
3.2 The Riemann-Liouville fractional operators 111
3.3 Rodrigues-type formulas for Gegenbauer functions 115
3.4 The Weyl-Gegenbauer transformations 119
3.5 The Riemann-Liouville-Gegenbauer transformations 122
3.6 The inverse Gegenbauer transformations 124
3.7 Integral equations involving Gegenbauer functions 128

Bibliography 135

Glossary of symbols 138

Index 141

Samenvatting 142

Curriculum vitae 143
INTRODUCTION

Generalized functions are of great interest in several branches of mathematics and physics. Laurent Schwartz [45] has contributed importantly to the development of the theory of generalized functions (distributions). Schwartz's construction of a Fourier invariant theory of generalized functions is based on the test-function space $S$ consisting of all infinitely differentiable functions $f(x) = f(x_1, \ldots, x_q)$ with the property

$$\sup_{x \in \mathbb{R}^q} \left| x_1^{k_1} \ldots x_q^{k_q} \frac{\partial^{i_1+\ldots+i_q} f(x)}{\partial x_1^{i_1} \ldots \partial x_q^{i_q}} \right| < \infty, \quad k_j, i_j = 0, 1, 2, \ldots, \quad j = 1, \ldots, q.$$  

Inspired by Schwartz's work, Gel'fand and Shilov [22] introduced and investigated some other types of test-function spaces, which they called "Spaces of type $S^\alpha$". Their definition is the following.

For non-negative numbers $\alpha_1, \ldots, \alpha_q$, the space $S_{\alpha} = S_{\alpha_1, \alpha_2, \ldots, \alpha_q}$ consists of all infinitely differentiable functions $f$ on $\mathbb{R}^q$ with the property

$$\sup_{x \in \mathbb{R}^q} \left| x_1^{k_1} \ldots x_q^{k_q} \frac{\partial^{i_1+\ldots+i_q} f(x)}{\partial x_1^{i_1} \ldots \partial x_q^{i_q}} \right| \leq B_{i_1, \ldots, i_q} A_1^{k_1} \ldots A_q^{k_q} \alpha_1^{\alpha_1} \ldots \alpha_q^{\alpha_q}, \quad k_j, i_j = 0, 1, 2, \ldots, \quad j = 1, \ldots, q,$$

where the constants $B_{i_1, \ldots, i_q}, A_1, \ldots, A_q$ depend on the function $f$.

For non-negative numbers $\beta_1, \ldots, \beta_q$, the space $S^\beta = S^{\beta_1, \ldots, \beta_q}$ consists of all infinitely differentiable functions $f$ on $\mathbb{R}^q$ with the property

$$\sup_{x \in \mathbb{R}^q} \left| x_1^{k_1} \ldots x_q^{k_q} \frac{\partial^{i_1+\ldots+i_q} f(x)}{\partial x_1^{i_1} \ldots \partial x_q^{i_q}} \right| \leq A_{i_1, \ldots, i_q} B_1^{i_1} \ldots B_q^{i_q} \beta_1^{i_1} \ldots \beta_q^{i_q}, \quad k_j, i_j = 0, 1, 2, \ldots, \quad j = 1, \ldots, q,$$

where the constants $A_{i_1, \ldots, i_q}, B_1, \ldots, B_q$ depend on the function $f$.

The space $S^\alpha = S^{\alpha_1, \ldots, \alpha_q}$ consists of all infinitely differentiable functions $f$ on $\mathbb{R}^q$ with the property

$$\sup_{x \in \mathbb{R}^q} \left| x_1^{k_1} \ldots x_q^{k_q} \frac{\partial^{i_1+\ldots+i_q} f(x)}{\partial x_1^{i_1} \ldots \partial x_q^{i_q}} \right| \leq C A_1^{k_1} \ldots A_q^{k_q} B_1^{i_1} \ldots B_q^{i_q} \alpha_1^{i_1} \ldots \alpha_q^{i_q}, \quad k_j, i_j = 0, 1, 2, \ldots, \quad j = 1, \ldots, q,$$

where the constants $C, A_1, \ldots, A_q, B_1, \ldots, B_q$ depend on the function $f$.

Obviously, the spaces $S_{\alpha}, S^\alpha$ and $S^\beta$ are subspaces of $S$. In the definition of the space $S_{\alpha}$ there are mainly constraints on the decrease of the functions at infinity, while in the definition of $S^\beta$ there are mainly constraints on the growth of the partial derivatives $(\partial^{i_1+\ldots+i_q} f(x))$ as $i_1 + \ldots + i_q \to \infty$.

If $\alpha_j \leq \beta_j \leq \beta_j$ for $j = 1, \ldots, q$, then we have the inclusions

$$S_{\alpha} \subset S_{\beta}, \quad S^\alpha \subset S^\beta, \quad S^\beta \subset S^\alpha.$$
For arbitrary $\alpha_i, \beta_j \geq 0$, the spaces $S_\alpha$ and $S^0$ are nontrivial, i.e., contain not only the null function. The space $S^0_\alpha$ is trivial if and only if for $j = 1, \ldots, q$,

$$\alpha_j + \beta_j < 1 \text{ or } (\alpha_j, \beta_j) = (1, 0) \text{ or } (\alpha_j, \beta_j) = (0, 1).$$

From a topological point of view, Gel'fand and Shilov [22] described the spaces $S_\alpha$, $S^0$ and $S^0_\alpha$ as a union of countably normed spaces. Thus they were able to define sequential convergence in the spaces $S_\alpha$, $S^0$ and $S^0_\alpha$, such that these spaces became sequentially complete.

Gel'fand and Shilov investigated their spaces of type $S$ for the sake of solving the Cauchy problem for partial differential equations. For the functional analyst these spaces are interesting in themselves because of their rich structure. In each of the definitions of $S$, $S_\alpha$, $S^0$ and $S^0_\alpha$, the supremum norm can be replaced by the norm in $L_2(\mathbb{R}^n)$. Therefore, the theory of unbounded linear operators in Hilbert space can be invoked to give alternative descriptions. In the Hilbert-space approach the spaces of type $S$ can be expressed as the intersection of so-called Gevrey spaces brought about by the self-adjoint operators $Q_j$ of multiplication by $x_j$, and the self-adjoint operators $P_j = i \partial / \partial x_j$ of partial differentiation, $j = 1, \ldots, q$, in the Hilbert space $L_2(\mathbb{R}^n)$; see Van Eijndhoven [12] and Ter Elst [17]. Now the theory of locally convex topological vector spaces can be applied. Indeed, the Gevrey spaces are equipped with a well-described topology and so it is natural to endow the spaces of type $S$ with the corresponding intersection topology. It can be shown that sequential convergence in the sense of this intersection topology agrees with Gel'fand and Shilov's definition of sequential convergence.

Let us mention some nice properties of the spaces of type $S$.

(i) The space $S_{\alpha_0, \ldots, \alpha_q}$ equals the space of all infinitely differentiable functions on $\mathbb{R}^q$ with compact support.

(ii) For $\alpha_j > 0, \ j = 1, \ldots, q$, the functions $f$ in $S_\alpha$ decrease exponentially: There exists $t > 0$, depending on $f$, such that

$$\sup_{x \in \mathbb{R}^q} |\exp(t |x_1|^{\alpha_1} + t |x_2|^{\alpha_2} + \ldots + t |x_q|^{\alpha_q}) f(x)| < \infty.$$

(iii) For $0 \leq \beta_j \leq 1, \ j = 1, \ldots, q$, the functions in $S^0$ have an analytic continuation into the complex space $\mathbb{C}^q$.

(iv) The class of spaces of type $S$ remains invariant under the Fourier transformation $F$ on $L_2(\mathbb{R}^n)$,

$$F(S_\alpha) = S^\alpha, \quad F(S^0) = S^0, \quad F(S^0_\alpha) = S^0_\alpha.$$

(v) Evidently, the space $S^0_\alpha$ is contained in $S_\alpha \cap S^0$. Kashpirovskii [33] showed that

$$S^0_\alpha = S_\alpha \cap S^0.$$

(vi) The Hermite functions

$$\psi_n(x) = (x^{1/2} e^{n!})^{-1/2} (-1)^n \exp\left(\frac{x^2}{2}\right) \left(\frac{d}{dx}\right)^n \exp(-x^2), \ n \in \mathbb{N}_0, \ x \in \mathbb{R},$$

constitute an orthonormal basis in $L_2(\mathbb{R})$. So the products $\Psi_n(x) = \prod_{j=1}^q \psi_n(x_j), \ n \in \mathbb{N}_0^q$, $x \in \mathbb{R}^q$, constitute an orthonormal basis in $L_2(\mathbb{R}^q)$. Simon [46] has proved that the space $S$ can be characterized in terms of the expansion coefficients of its elements, with respect to the basis $\{\Psi_n : n \in \mathbb{N}_0^q\}$:
The analogue for the space \( S^q \) with \( \alpha_j \geq \frac{1}{2}, j = 1, \ldots, q \), reads
\[
S^q = \{ f \in L_2(\mathbb{R}^q) : \exists \beta \geq 0 \}
\]
\[
\left( \exp(t n_1^{1/(2\alpha_1)} + t n_2^{1/(2\alpha_2)} + \ldots + t n_q^{1/(2\alpha_q)}) (f, \psi_0)_{L_2(\mathbb{R}^q)} \right) \in L_\infty \}
\]
which was proved by Zhang [51] in the case \( q = 1 \). Such a characterization cannot exist for the other spaces of type \( S \), because they are not Fourier invariant (see property (iv)).

The space \( S^q \) is the analyticity space corresponding to the positive self-adjoint operator \( \sum_{\nu=1}^{\infty} \left( -a^2/dx^2 + x^2\right)^{1/2(2\nu)} \) in \( L_2(\mathbb{R}^q) \), cf. De Graaf [24], De Bruijn [3].

In this thesis we go into a further study of the spaces \( S, S_0, S^0 \) and \( S^0_q \). The space \( S \) is called the Schwartz space and the spaces \( S_0, S^0, S^0_q \) are called Gel'fand-Shilov spaces.

We only consider the case \( \alpha_1 = \alpha_2 = \ldots = \alpha_q = \alpha, \beta_1 = \beta_2 = \ldots = \beta_q = \beta \). We deal with several integral transformations (Fourier, Hankel, Radon, Weyl, Riemann-Liouville, Erdélyi-Kober). In particular, we are interested in finding new characterizations of the Gel'fand-Shilov spaces by means of these integral transformations.

Example 1: A function \( f \) on \( \mathbb{R}^q \) belongs to \( S_\alpha \) if and only if there exist \( A, B > 0 \) such that
\[
\| x_1^{\alpha_1/2} \ldots x_q^{\alpha_q/2} f \|_{L_2(\mathbb{R}^q)} \leq B A^k k^{\alpha_0}, \quad k \in \mathbb{N}_0
\]
and
\[
\| x_1^{\alpha_1/2} \ldots x_q^{\alpha_q/2} f \|_{L_2(\mathbb{R}^q)} \leq \infty, \quad l \in \mathbb{N}_0
\]
From the properties (iv) and (v) of the Gel'fand-Shilov spaces, similar characterizations for the spaces \( S^0 \) and \( S^0_q \) follow, see Chapter 2, Theorem 2.42.

Starting point of our investigation was the well-known result that a function \( f \in L_2(\mathbb{R}^q) \) has an expansion with respect to spherical harmonics,
\[
f(x) = f(\rho \omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} r^m f_{m,j}(r) Y_{m,j}(\omega),
\]
where \( r \) is the length of \( x \), \( r = |x| = (x_1^2 + \ldots + x_q^2)^{1/2} \), and \( \omega \) its direction, \( \omega = x/r \in S^{q-1} = \{ \eta \in \mathbb{R}^q : |\eta| = 1 \} \). The function \( Y_{m,j} \) is a spherical harmonic of degree \( m \). That is, the function \( x \mapsto |x|^m Y_{m,j}(x/|x|) \) is a homogeneous harmonic polynomial of degree \( m \) on \( \mathbb{R}^q \). \( N(q, m) \) is the dimension of the vector space of all spherical harmonics of degree \( m \) on \( \mathbb{R}^q \). The set \( \{ Y_{m,j} : m \in \mathbb{N}_0, j = 1, \ldots, N(q, m) \} \) is an orthonormal basis in the Hilbert space \( L_2(S^{q-1}, d\sigma^{q-1}) \) where \( d\sigma^{q-1} \) is the surface measure of \( S^{q-1} \).

Thus we have
\[
r^m f_{m,j}(r) = \int_{S^{q-1}} f(\rho \omega) Y_{m,j}(\omega) d\sigma^{q-1}(\omega)
\]
and
\[
\| f \|_{L_2(\mathbb{R}^q)}^2 = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} \| r^m f_{m,j} \|_{L_2(\mathbb{R}^q, d\sigma^{q-1})}^2.
\]
Now an application of the Hecke-Bochner theorem (see Theorem 1.33) yields that the Fourier transform \( \mathcal{F} f \) is given by

\[
(\mathcal{F} f)(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{N(r,m)} (-i)^m r^m (H_{m+\nu/2-1} f_{m,j})(r) Y_{m,j}(\omega), \quad r > 0, \quad \omega \in S^{d-1}.
\]

(4)

Here for \( \nu \geq -\frac{1}{2} \), \( H_\nu \) denotes the Hankel transformation

\[
(H_\nu g)(x) = \int_0^\infty (xy)^{-\nu} J_\nu(xy) g(y) y^{2\nu+1} \, dy, \quad x > 0,
\]

where \( J_\nu \) is the Ressel function of the first kind and of order \( \nu \).

By means of the relations (1), (3) and (4), the characterization of \( S_\nu \) as given in Example 1 can be reformulated as follows: A function \( f \) on \( \mathbb{R}^d \) belongs to \( S_\nu \) if and only if there exist \( A, B > 0 \) such that

\[
\sum_{m=0}^{\infty} N(r,m) \sum_{j=1}^{N(r,m)} \|r^{k+m} f_{m,j}\|_{L^2(\mathbb{R}^{d+\nu/2})}^2 \leq B^2 A^{2k} k^{2k\nu}, \quad k \in \mathbb{N}_0,
\]

and

\[
\sum_{m=0}^{\infty} N(r,m) \sum_{j=1}^{N(r,m)} \|r^{k+m} H_{m+\nu/2-1} f_{m,j}\|_{L^2(\mathbb{R}^{d+\nu/2})}^2 < \infty, \quad l \in \mathbb{N}_0.
\]

Similar characterizations follow for the spaces \( S^0 \) and \( S^0_\nu \), see Chapter 2, Theorem 2.48.

So for a function \( f \) in a Gelfand-Shilov space, the corresponding \( f_{m,j} \) and \( H_{m+\nu/2-1} f_{m,j} \) are rapidly decreasing at infinity. It is shown that these properties are necessary and sufficient for the function \( f_{m,j} \) to belong to a Gelfand-Shilov space again (see Chapter 2, Theorem 2.21).

Considerations of this type have led us to a study of the Hankel transformation \( H_\nu \) on \( S_{\text{even}}, S^0_{\text{even}} \), \( S^0_{\nu,\text{even}} \), \( S^0_{\text{even}}, S^0_{\nu,\text{even}} \), consisting of the even functions in \( S, S_\nu, S^0, S^0_\nu \), respectively. Here it is understood that the definition (5) is supplemented by \( (H_\nu g)(x) = (H_\nu g)(-x) \) for \( x < 0 \), which makes \( H_\nu g \) into an even function.

Let us review some properties of the Hankel transformation.

(i) For \( \nu \geq -\frac{1}{2} \), the Laguerre functions

\[
L_\nu(x) = \left( \frac{2\Gamma(n+1)}{\Gamma(n+\nu+1)} \right)^{1/2} \exp(-\frac{1}{2}x^2) L_n^\nu(x^2), \quad n \in \mathbb{N}_0,
\]

constitute an orthonormal basis in the Hilbert space \( X_{2\nu+1} = L_2(\mathbb{R}^d; x^{2\nu+1} \, dx) \). Here \( L_n^\nu \) denotes the generalized Laguerre polynomial,

\[
L_n^\nu(x) = \frac{x^{-\nu}}{n!} \left( \frac{d}{dx} \right)^n [e^{-x} x^{\nu+n}], \quad n \in \mathbb{N}_0.
\]

Now the Hankel transformation \( H_\nu \) can be extended to a unitary operator on \( X_{2\nu+1} \) with the property \( H_\nu^{-1} = H_{-\nu} \), because

\[
H_\nu L_n^\nu = (-1)^n L_n^\nu, \quad n \in \mathbb{N}_0.
\]
The spaces $S_{\text{even}}$ and $S_{\alpha,\text{even}}$ with $\alpha \geq \frac{1}{2}$, can be characterized in terms of the expansion coefficients of their elements with respect to the basis $\{ L_n^\alpha : n \in \mathbb{N}_0 \}$.

Let $\nu \geq -\frac{1}{2}$. Then

$$S_{\text{even}} = \{ f \in X_{2\nu+1} : (n^k (f, L_n^\alpha)_{X_{2\nu+1}}) \in l_\infty, \ k \in \mathbb{N}_0 \},$$

$$S_{\alpha,\text{even}} = \{ f \in X_{2\nu+1} : \exists C > 0 \text{ such that } (f, L_n^\alpha)_{X_{2\nu+1}} \in l_\infty \},$$

see Chapter 1, Theorem 1.8 and Chapter 2, Theorem 2.18.

From these characterizations it follows that

$$B \nu (S_{\text{even}}) = S_{\text{even}}, \quad B \nu (S_{\alpha,\text{even}}) = S_{\alpha,\text{even}}.$$

(ii) For the remaining Gel'fand-Shilov spaces it is proved in this thesis that

$$B \nu (S_{\alpha,\text{even}}) = S_{\alpha,\text{even}}, \quad B \nu (S_{\delta,\text{even}}) = S_{\delta,\text{even}}, \quad B \nu (S_{\mu,\text{even}}) = S_{\mu,\text{even}},$$

see Chapter 2, Theorem 2.22.

(iii) A function $f \in X_{2\nu+1}$ can be extended to a function in $S_{\alpha,\text{even}}$, if and only if there exist $A, B > 0$ such that

$$\| x^k f \|_{L_2(\mathbb{R}^n)} \leq B A^k k!,$$

and

$$\| x^k B \nu (f) \|_{L_2(\mathbb{R}^n)} < \infty, \quad k, l \in \mathbb{N}_0.$$

From the preceding property (ii) and Kashpirovskii's intersection result $S_{\alpha,\text{even}} = S_{\alpha,\text{even}} \cap S_{\delta,\text{even}}$, similar characterizations for the spaces $S_{\alpha,\text{even}}$ and $S_{\delta,\text{even}}$ follow, see Chapter 2, Theorem 2.21.

(iv) Products of Hankel transformations have the following properties:

$$B \nu \circ B \nu (f) (x) = \frac{2^{\nu+1}}{\Gamma (\nu - \mu)} \int f(y) (y^2 - x^2)^{\nu-\mu} \frac{dy}{y}, \quad -\frac{1}{2} \leq \mu < \nu,$$

$$B \nu \circ B \nu (f) (x) = \left( \frac{d}{dx} \right)^k f(x), \quad \nu \geq -\frac{1}{2}, \quad k \in \mathbb{N}_0.$$

These relations suggested the idea of characterizing the spaces $S_{\text{even}}, S_{\alpha,\text{even}}, S_{\delta,\text{even}}, S_{\mu,\text{even}}$, by means of the operators $x$ and $x^{-1} \frac{d}{dx}$, cf. Chapter 1, Corollary 1.21, and Chapter 2, Corollary 2.29.

Returning to the expansion (1), we come to the conclusion that the basic function $(n^k f) \mapsto \sum_{m,j} (\omega) Y_{m,j} (\omega)$ belongs to $S_{\alpha,\text{even}}, S_{\delta,\text{even}}, S_{\mu,\text{even}}$, if and only if $f_{m,j} \in S_{\text{even}}, S_{\alpha,\text{even}}, S_{\delta,\text{even}}, S_{\mu,\text{even}}$, see Chapter 1, Theorem 1.37 and Chapter 2, Theorem 2.45. In particular, by taking $m = 0$, we obtain a characterization of the radially symmetric functions: A radially symmetric function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $S_{\text{even}}, S_{\alpha,\text{even}}, S_{\delta,\text{even}}$ if and only if there exists $g \in S_{\text{even}}, S_{\alpha,\text{even}}, S_{\delta,\text{even}}, S_{\mu,\text{even}}$, such that $f(z) = g(|z|)$; see Chapter 1, Theorem 1.36 and Chapter 2, Theorem 2.46.

The Radon transform of a function on $\mathbb{R}^n$ is defined as the set of its integrals over the hyperplanes in $\mathbb{R}^n$. A known result is the so-called projection theorem (see Theorem 1.46), which provides the fundamental relationship between the Radon transformation $\mathcal{R}$ and the Fourier transformation $\hat{f}$.
\( (\mathcal{R}f)(p, \omega) = (2\pi)^{3/2-1} \int_{\mathbb{R}} \langle \mathcal{F}f \rangle(t\omega) e^{ipt} \, dt, \quad p \in \mathbb{R}, \quad \omega \in S^{3-1}. \) \hspace{1cm} (6)

Helgason [27] and Ludwig [37] characterized the image of the Schwartz space \( S \) under the Radon transformation \( \mathcal{R} \) and they presented several formulas for \( \mathcal{R}^{-1} \). Since \( \mathcal{F}(S) = S \), the characterization of \( \mathcal{R}(S) \) comes down to the characterization of all functions

\[ (p, \omega) \mapsto \int_{\mathbb{R}} f(t\omega) e^{ipt} \, dt, \quad f \in S. \]

We give a new proof of the characterization of \( \mathcal{R}(S) \), see Chapter 1, Theorem 1.47. The proof is based on the identification of \( S \) with a subspace of \( L_2(\mathbb{R} \times S^{3-1}) \), see Chapter 1, Theorem 1.43. As a side result, we come to the solution of the following factorization problem: For which functions \( g \) and \( h \) does the function \( (rw) \mapsto g(r) h(\omega) \) belong to \( S \)? (see Chapter 1, Theorem 1.44).

In this thesis we start an investigation of the Radon transformation on the Gel'fand-Shilov spaces, see Section 2.6. We are mainly interested in the action of the Radon transformation on the basic functions \( (rw) \mapsto r^m f_{m,j}(r) Y_{m,j}(\omega) \) in the Gel'fand-Shilov spaces. We recall that such a basic function belongs to \( S_{d, S_0}, S_0^0 \), if and only if \( f_{m,j} \in S_{d, even}, S_0^0, S_0^{d, even} \), respectively.

Let us consider such a basic function,

\[ f(r\omega) = r^m f_{m,j}(r) Y_{m,j}(\omega), \quad r \geq 0, \quad \omega \in S^{3-1}. \] \hspace{1cm} (7)

By means of the projection theorem (6) and the Hecke-Bochner theorem (cf. (4)), we have

\[ (\mathcal{R}f)(p, \omega) = (2\pi)^{3/2-1} (-i)^m \int_{\mathbb{R}} r^m (\mathcal{H}_{m+q/2-1} f_{m,j})(t) e^{ipt} \, dt \cdot Y_{m,j}(\omega), \]

\[ p \in \mathbb{R}, \quad \omega \in S^{3-1}. \] \hspace{1cm} (8)

From the definition of the Hankel transformation in (5) with \( \nu = -\frac{1}{2} \), it follows that \( \mathcal{H}_{-1/2} \) equals the Fourier-cosine transformation. Hence, the expression (8) can be written as

\[ (\mathcal{R}f)(p, \omega) = g_{m,j}(p) Y_{m,j}(\omega), \quad p \in \mathbb{R}, \quad \omega \in S^{3-1}, \]

with

\[ g_{m,j}(p) = (2\pi)^{3/2-1} \left( \frac{d}{dp} \right)^m (\mathcal{M}_{-1/2} \mathcal{H}_{m+q/2-1} f_{m,j})(p), \quad p \in \mathbb{R}. \] \hspace{1cm} (9)

Due to the property (ii) of the Hankel transformation it follows that \( f_{m,j} \in S_{d, even}, S_0^{d, even} \), if and only if \( g_{m,j} \in (d/dp)^m(S_{d, even}), (d/dp)^m(S_0^{d, even}) \), \( (d/dp)^m(S_{d, even}), (d/dp)^m(S_0^{d, even}) \).

For the evaluation of (10), we apply property (iv) of the Hankel transformation and the Rodrigues' formula for the Gegenbauer polynomial \( C_n^{q/2-1} \), to obtain

\[ g_{m,j}(p) = \lambda(q,m) \int_{r=0}^{\infty} (r^2 - p^2)^{(q-3)/2} C_n^{q/2-1}(p/r) r^m f_{m,j}(r) \, dr, \quad p > 0, \] \hspace{1cm} (11)

where \( \lambda(q,m) \) is a multiplicative constant. This formula can also be found in Ludwig [37].

Deans [7], [8] found the inversion of (11), viz.
\[ r^n f_{m,j}(r) = \mu(q,m) r^{2-q} \int \frac{(p^2 - r^2)^{(q-3)/2} C_{m/2-1}^2(p/r) g_{m,j}^{(q-1)}(p)}{p} \, dp, \quad r > 0, \quad (12) \]

where \( \mu(q,m) \) is a multiplicative constant.

This led to our study of integral equations of Mellin-convolution type, on which there is a vast amount of literature. In the literature there are two common solution methods for this class of integral equations. The first method seems to be rather ad hoc: By clever guessing, sometimes supported by a formal application of the Mellin transformation, a solution of the integral equation is proposed. Then correctness of the proposed solution is shown by substitution into the integral equation, which requires the evaluation of a convolution integral.

The second method is based on fractional integral/differential operators. It is shown that the integral operator can be factorized as a product of multiplication operators and fractional integral/differential operators. From this, the existence and uniqueness of the solution is inferred. However, in these approach solutions in a manageable form are not always available.

In Chapter 3 we have worked out the latter method for integral equations of Mellin-convolution type where the convolutor contains a Gegenbauer polynomial or more generally, a Gegenbauer function. After extending the Rodrigues' formula for Gegenbauer polynomials to Rodrigues-type formulas for Gegenbauer functions, we show that products of fractional integral/differential operators can be reduced to integral operators with Gegenbauer function in their kernels, see Sections 3.4, 3.5. In Section 3.6 we show that the inverse of such an integral operator can be factorized as the product of a fractional differential operator and an integral operator involving again a Gegenbauer function. This result follows from an intertwining relation for fractional integral/differential operators which in its simplest form reads

\[ \left( \frac{d}{dx} \right)^{2n+1} = \left( \frac{1}{x} \frac{d}{dx} \right)^n x^{2n+1} \left( \frac{1}{x} \frac{d}{dx} \right)^{n+1}, \quad n \in \mathbb{N}_0. \]

To finish this introduction we briefly describe the contents of the separate chapters. Chapter 1 contains a detailed study of the Schwartz space \( S \). In order to get a good understanding of the functions of \( q \) variables, much attention is first paid to the functions of one variable, see Sections 1.1, 1.2 and 1.3. In particular, the space \( S_{even}(R) \) is studied extensively in Section 1.2. In the analysis integral operations such as the Fourier transformation and the Hankel transformation play an important role, but also fractional integral/differential operators are involved. In Section 1.5 we give an introduction to the theory of spherical harmonics. Sections 1.4 and 1.6 provide results on the Schwarts space \( S(R^d) \). Of special interest is the characterization of the radially symmetric functions and of the basic functions \( r^m f_{m,j}(r) Y_{m,j}(\Theta) \), as elements of \( S(R^d) \). Then the characterization of a general \( f \in S(R^d) \) can be found by expanding \( f \) into a series with respect to spherical harmonics. Furthermore, in Section 1.6 the space \( S(R^d) \) is embedded in \( L_1(R \times S^{d-1}) \) in such a way that the image of \( S(R^d) \) under the Radon transformation \( R \) becomes obvious, cf. Section 1.7. The evaluation of the image \( R\{r^m f_{m,j}(r) Y_{m,j}(\Theta)\} \) leads to special Gegenbauer transformations which are studied separately and in a broader context in Chapter 3.

The main objective of Chapter 2 is to carry over the results of Chapter 1 to the Gel'fand-Shilov spaces \( S_{\alpha}, S^\theta \) and \( S_{\alpha}^\theta \). The set-up of Chapter 2 is very similar to that of Chapter 1.
on the Schwartz space. Because the Gel'fand-Shilov spaces are subspaces of the Schwartz space, we are able to strengthen and extend many of the results of Chapter 1. Only the factorization problem in $S^\beta$ and $S^\alpha_\omega$ with $\beta > 1$, and the characterization of the images $\mathcal{R}(S_\omega)$ and $\mathcal{R}(S_\alpha^\beta)$ with $\alpha > 0$, are still open and require further investigation.

The plan of Chapter 3 is the following. In the first two sections we gather some relevant material on fractional calculus based on Weyl operators and on Riemann-Liouville operators. To avoid technical complications, we introduce the spaces $S_\omega$ and $S_\omega^\beta$, which consist of $C^\omega$-functions on $(0, \infty)$; the functions in $S_\omega$ have a specified behaviour at $\infty$ and the functions in $S_\omega^\beta$ have a specified behaviour at 0. All statements are in terms of these spaces. Section 3.3 is devoted to the derivation of Rodrigues-type formulas for Gegenbauer functions, which extend the usual Rodrigues' formula for the Gegenbauer polynomials. In Sections 3.4 and 3.5 we introduce the so-called Gegenbauer transformations and in Section 3.6 we present several formulas for their inverses. Here we like to mention already that the inverse of a Gegenbauer transformation is again a Gegenbauer transformation. In Section 3.7 the theory of Gegenbauer transformations is utilized to solve certain integral equations of Mellin-convolution type. The solutions obtained are compared to results from the literature.
CHAPTER 1

Functions of rapid decrease

1.1 The Schwartz space $S(\mathbb{R})$

In his treatise on the theory of distributions [45, Section VII.3] Schwartz introduced the space $(S)$ of infinitely differentiable functions of rapid decrease on $\mathbb{R}^n$. In this section we gather some properties of this space for $q = 1$.

Definition 1.1 The Schwartz space $S(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R})$ for which 

$$x^k f^{(l)} \in L_\infty(\mathbb{R}), \quad k, l \in \mathbb{N}_0.$$ 

The topology on $S(\mathbb{R})$ is defined by the countable set of seminorms 

$$f \mapsto \|x^k f^{(l)}\|_{L_\infty(\mathbb{R})}, \quad k, l \in \mathbb{N}_0.$$ 

This topology is metrizable and gives $S(\mathbb{R})$ the structure of a complete metric space.

The multiplication operator $\times$ and the differentiation operator $d/dx$ map $S(\mathbb{R})$ continuously onto itself. For $f \in S(\mathbb{R})$ we have $(1 + x^2) f \in L_\infty(\mathbb{R})$, whence we obtain $S(\mathbb{R}) \subset L_p(\mathbb{R})$, $p \geq 1$. In particular, we can define the Fourier transformation $\mathcal{F}$ on $S(\mathbb{R})$ by 

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(y) e^{-ixy} \, dy, \quad f \in S(\mathbb{R}), \quad x \in \mathbb{R}. \quad (1.1)$$

In [45, Section VII.6, Theorem XII], Schwartz has proved that the Fourier transformation $\mathcal{F}$ maps $S(\mathbb{R})$ continuously onto itself. We give some characterizations of $S(\mathbb{R})$.

Theorem 1.2 Let $f \in C^\infty(\mathbb{R})$. The following assertions are equivalent:

(i) \quad $f \in S(\mathbb{R})$;

(ii) \quad $x^k f^{(l)} \in L_2(\mathbb{R}), \quad k, l \in \mathbb{N}_0$;

(iii) \quad $x^k f \in L_2(\mathbb{R})$ and $f^{(l)} \in L_2(\mathbb{R}), \quad k, l \in \mathbb{N}_0$;

(iv) \quad $x^k f \in L_\infty(\mathbb{R})$ and $f^{(l)} \in L_\infty(\mathbb{R}), \quad k, l \in \mathbb{N}_0$.

Proof. We follow the scheme (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Suppose $f \in S(\mathbb{R})$. Then $x^k f^{(l)} \in S(\mathbb{R}) \subset L_2(\mathbb{R}), \quad k, l \in \mathbb{N}_0$. So (i) $\Rightarrow$ (ii). Obviously (ii) $\Rightarrow$ (iii).

Suppose $f$ satisfies assertion (iii). Then for $k, l \in \mathbb{N}$ and $x \in \mathbb{R}$ we estimate 

$$|x^k f(x)|^2 = 2 \Re \int_0^\infty e^{ixt} \bar{f}(t) \left( x^k f(t) \right)' \, dt$$

$$\leq 2k \|x^k f\|_{L_2(\mathbb{R})} \|x^{k-1} f\|_{L_2(\mathbb{R})} + 2 \|x^k f\|_{L_\infty(\mathbb{R})} \|f'\|_{L_2(\mathbb{R})},$$

which together with the Cauchy-Schwarz inequality gives (iv) $\Rightarrow$ (i).

Thus, by Theorem 1.2, $S(\mathbb{R})$ is a complete metric space.
\[|f^{(1)}(x)|^2 - |f^{(1)}(0)|^2 = 2 \text{ Re} \int_0^x \frac{f^{(1)}(t)}{t} f^{(n+1)}(t) \, dt \leq 2 \|f^{(1)}\|_{L_2(\mathbb{R})} \|f^{(n+1)}\|_{L_2(\mathbb{R})}.
\]

This proves assertion (iv).

Finally, suppose \( f \) satisfies assertion (iv). Then for \( k \in \mathbb{N} \) and \( x \in \mathbb{R} \) we have, through integration by parts,

\[
|z^k f'(z)|^2 = 2 \text{ Re} \int_0^x t^{k-1} f'(t) (t^k f'(t))' \, dt = 2 \text{ Re} \left[ z^{2k-1} \int_0^x (k f'(x) + z f''(x)) \right.
\]

\[
- \int_0^x t^{2k-2} f'(t) \left( k(2k-1) f'(t) + 3k t f''(t) + t^2 f'''(t) \right) \, dt \right].
\]

From this result we conclude that \( z^k f' \) is bounded on \( \mathbb{R} \). Now replace \( f \) by \( f' \), then the argument can be repeated and an induction procedure yields \( f \in S(\mathbb{R}) \). \( \Box \)

Next we want to present some functional analytic characterizations of \( S(\mathbb{R}) \). For that purpose we introduce some terminology.

Let \( T \) be a self-adjoint operator in a Hilbert space \( H \). According to Nelson [41], a vector \( f \in H \) is said to be a \( C^\infty \)-vector for \( T \) if \( f \in D(T^n), n \in \mathbb{N} \). The set of \( C^\infty \)-vectors for \( T \) is called the \( C^\infty \)-domain of \( T \) and is denoted by \( D^{\infty}(T) \):

\[
D^{\infty}(T) = \bigcap_{n=1}^{\infty} D(T^n). \tag{1.2}
\]

The topology on \( D^{\infty}(T) \) is defined by the countable set of seminorms

\[
f \mapsto \|T^n f\|_{\mathcal{H}} \quad n \in \mathbb{N}_0.
\]

This topology is metrizable and gives \( D^{\infty}(T) \) the structure of a complete metric space. For a unitary operator \( U \) on \( H \) we have

\[
D^{\infty}(UTU^*) = U(D^{\infty}(T)). \tag{1.3}
\]

We introduce some notations:

\( \mathcal{F} \) denotes the Fourier transformation on \( L_1(\mathbb{R}) \), i.e. \( \mathcal{F} \) is the unique unitary operator on \( L_1(\mathbb{R}) \) such that \( \mathcal{F} f = \mathcal{F} f, f \in S(\mathbb{R}) \).

\( P \) denotes the differentiation operator in \( L_2(\mathbb{R}) \) defined on the domain

\[
D(P) = \{ f \in L_2(\mathbb{R}) : f \text{ is absolutely continuous }, \ f' \in L_2(\mathbb{R}) \}
\]

by \( Pf = if' \).

\( Q \) denotes the multiplication operator in \( L_2(\mathbb{R}) \) defined on the domain

\[
D(Q) = \{ f \in L_2(\mathbb{R}) : zf \in L_2(\mathbb{R}) \}
\]

by \( Qf = zf \).

The operators \( \mathcal{F}, P \) and \( Q \) are discussed at length in Helmburg [29, Sections 16, 18, 19].

We quote some useful results. For \( f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \) we have for almost all \( x \in \mathbb{R} \),

\[
(\mathcal{F} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-iyx} \, dy, \quad (\mathcal{F}^n f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{iyx} \, dy. \tag{1.4}
\]
The operators $P$ and $Q$ are unbounded and self-adjoint. For $f \in S(\mathcal{R})$ we have by interchanging differentiation and integration, $PFf = FQf$ and $PF^*f = -F^*Qf$. Even more is true,

$$ P = FQF^* , \quad Q = -FPF^* . $$

Thus it follows by means of (1.3) that

$$ D^{\infty}(P) = F(D^{\infty}(Q)) , \quad D^{\infty}(Q) = F(D^{\infty}(P)) . $$

From Theorem 1.2 we obtain the following functional analytic characterization of $S(\mathcal{R})$:

**Theorem 1.3**

$$ S(\mathcal{R}) = D^{\infty}(Q) \cap D^{\infty}(P) . $$

The topology on $S(\mathcal{R})$ equals the intersection topology on $D^{\infty}(Q) \cap D^{\infty}(P)$.

The operators $F$, $P$ and $Q$ map $S(\mathcal{R})$ continuously into itself.

**Theorem 1.4**

(i) $F(S(\mathcal{R})) = S(\mathcal{R}) ,$

(ii) $Q^n(S(\mathcal{R})) = \{ f \in S(\mathcal{R}) : f^{(k)}(0) = 0 , \; k = 0, \ldots, n-1 \} , \; n \in \mathbb{N} ,$

(iii) $P^n(S(\mathcal{R})) = \{ f \in S(\mathcal{R}) : \int_\mathcal{R} f(y) y^n dy = 0 , \; k = 0, \ldots, n-1 \} , \; n \in \mathbb{N} .$

**Proof.** (i) From (1.6) and Theorem 1.3 we have

$$ F(S(\mathcal{R})) = F(D^{\infty}(Q)) \cap F(D^{\infty}(P)) = D^{\infty}(P) \cap D^{\infty}(Q) = S(\mathcal{R}) . $$

(ii) Let $n \in \mathbb{N}$ and let $f \in S(\mathcal{R})$ with $f^{(k)}(0) = 0, \; k = 0, \ldots, n-1$. We define the function $g$ on $\mathcal{R}$ by

$$ g(x) = \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(xt) \, dt , \; x \in \mathcal{R} . $$

Then $f(x) = x^n g(x), \; x \in \mathcal{R}$, and for $k, l \in \mathbb{N}_0$ and $x \in \mathcal{R}$ we estimate

$$ |x^k g(x)| \leq \sup_{|\xi| \leq 1} |g(\xi)| + \sup_{|\xi| > 1} |\xi^{k+n} g(\xi)| \leq \frac{1}{n!} \| f^{(n)} \|_{L_w(\mathcal{R})} + \| x^k f \|_{L_w(\mathcal{R})} , $$

$$ |g^{(l)}(x)| = \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} t^l f^{(n+l)}(xt) \, dt \leq \frac{l!}{(n+l)!} \| f^{(n+l)} \|_{L_w(\mathcal{R})} . $$

So $g \in S(\mathcal{R})$ by Theorem 1.2, and therefore, $f \in Q^n(S(\mathcal{R}))$.

Conversely, suppose $f \in Q^n(S(\mathcal{R}))$. Then obviously $f \in S(\mathcal{R})$ with $f^{(k)}(0) = 0, \; k = 0, \ldots, n-1$.

(iii) Observe that for $f \in S(\mathcal{R})$ and $k \in \mathbb{N}_0$,
\[(P^k F^* f)(0) = (-1)^k (F^* Q^k f)(0) = \frac{(-1)^k}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) y^k dy.\]

Now the result to be proved follows from (ii) and the fact that for \(n \in \mathbb{N},\)
\[P^n(S(\mathbb{R})) = F^* Q^n(S(\mathbb{R})) = \{f \in S(\mathbb{R}) : F^* f \in Q^n(S(\mathbb{R}))\}.\][\hfill \Box]

The space \(S(\mathbb{R})\) can be characterized by means of the Fourier transformation:

**Theorem 1.5** Let \(f \in L_2(\mathbb{R}).\) The following assertions are equivalent:

(i) \(f \in S(\mathbb{R});\)

(ii) \(x^k f \in L_\infty(\mathbb{R})\) and \(x^l F f \in L_\infty(\mathbb{R}), \ k, l \in \mathbb{N}_0;\)

(iii) \(x^k f \in L_2(\mathbb{R})\) and \(x^l F f \in L_2(\mathbb{R}), \ k, l \in \mathbb{N}_0.\)

**Proof.** We follow the scheme (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

Since \(F(S(\mathbb{R})) = S(\mathbb{R}),\) (i) \(\Rightarrow\) (ii). Obviously (ii) \(\Rightarrow\) (iii). By use of (1.6) and Theorem 1.3, (iii) \(\Rightarrow\) (i). \[\hfill \Box\]

The space \(S(\mathbb{R})\) can be characterized in terms of the Hermite expansion coefficients of its elements. The Hermite functions,
\[
\psi_n(x) = (\pi^{-1/2} 2^n n!)^{-1/2} \exp(-\frac{1}{2} x^2) H_n(x) , \ n \in \mathbb{N}_0 ,
\]
constitute an orthonormal basis in \(L_2(\mathbb{R}).\) Here \(H_n\) denotes the Hermite polynomial,
\[
H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2) , \ n \in \mathbb{N}_0 ,
\]
see Magnus, Oberhettinger and Soni [38, pp. 249, 252]. By means of [38, pp. 252, 254] we obtain the relations
\[
Q \psi_n = \sqrt{(n + 1)/2} \psi_{n+1} + \sqrt{n/2} \psi_{n-1} , \ n \in \mathbb{N} ,
\]
\[
P \psi_n = -i \sqrt{(n + 1)/2} \psi_{n+1} + i \sqrt{n/2} \psi_{n-1} , \ n \in \mathbb{N} ,
\]
\[
F \psi_n = (-i)^n \psi_n , \ n \in \mathbb{N}_0 .
\]

For later use we need an \(L_\infty\)-estimate for \(x^k \psi_n^0(x).\) From [18, 10.18(19)] we quote the inequality
\[
|\psi_n(x)| < 1.086435 \pi^{-1/4} < 1 , \ x \in \mathbb{R} ,
\]
established by Cramér and Charlier. Next it can be shown by an induction argument based on (1.8a,b) that
\[
|x^k \psi_n^0(x)| \leq (2^{k+1}(n + k + l)/n!)^{1/2} , \ x \in \mathbb{R} , \ k, l \in \mathbb{N}_0 .
\]

It also follows from (1.8a,b) that the Hermite functions constitute an orthonormal basis of the eigenfunctions of the operator \(P^2 + Q^2,\)
\[(P^2 + Q^2) \psi_n = (2n + 1) \psi_n, \quad n \in \mathbb{N}_0.\]  

(1.11)

Now we arrive at the announced characterization of the space \(S(\mathbb{R})\):

**Theorem 1.6**

(i) \(S(\mathbb{R}) = D^\infty(Q) \cap D^\infty(P) = D^\infty(P^2 + Q^2)\)

\[= \{ f \in L_2(\mathbb{R}) : (n^k(f, \psi_n)_{L_2(\mathbb{R})}) \in l_\infty, \quad k \in \mathbb{N}_0 \}. \]

The equalities hold as topological vector spaces.

(ii) For \(f \in S(\mathbb{R})\) we have

\[f(x) = \sum_{n=0}^{\infty} (f, \psi_n)_{L_2(\mathbb{R})} \psi_n(x), \quad x \in \mathbb{R}, \]

where the series converges in \(S(\mathbb{R})\). The series converges absolutely with respect to the \(L_\infty(\mathbb{R})\)-norm and the series converges uniformly on \(\mathbb{R}\).

**Proof.** (i) The first equality has been established in Theorem 1.3. Since the operators \(P\) and \(Q\) map \(S(\mathbb{R})\) continuously into itself, we have \(S(\mathbb{R}) \subseteq D^\infty(P^2 + Q^2)\). We shall show that

\[D^\infty(P^2 + Q^2) \subseteq \{ f \in L_2(\mathbb{R}) : (n^k(f, \psi_n)_{L_2(\mathbb{R})}) \in l_\infty, \quad k \in \mathbb{N}_0 \}\]

\[\subseteq (D^\infty(Q) \cap D^\infty(P)). \]

Let \(f \in D^\infty(P^2 + Q^2)\). By applying (1.11) we derive for \(k \in \mathbb{N}_0\),

\[
\sup_{n \in \mathbb{N}_0} |n^k(f, \psi_n)_{L_2(\mathbb{R})}| \leq \sum_{n=0}^{\infty} |n^k(f, \psi_n)_{L_2(\mathbb{R})}| \leq \sum_{n=0}^{\infty} |(2n + 1)^k (f, \psi_n)_{L_2(\mathbb{R})}|^2
\]

\[= \sum_{n=0}^{\infty} |((P^2 + Q^2))^k f, \psi_n)_{L_2(\mathbb{R})}|^2 = \|(P^2 + Q^2)^k f\|_{L_2(\mathbb{R})}^2 < \infty. \]

Next suppose \(f \in L_2(\mathbb{R})\) such that \((n^k(f, \psi_n)_{L_2(\mathbb{R})}) \in l_\infty, \quad k \in \mathbb{N}_0\). Let \(k \in \mathbb{N}_0\). By (1.8a) there exist coefficients \(a_{j,k}(n), j = 0, \ldots, k\), such that

\[x^k \psi_n(x) = \sum_{j=0}^{k} a_{j,k}(n) \psi_{n+k-j}(x), \quad n \in \mathbb{N}_0, \]

and, by (1.8a), it can be proved that

\[|a_{j,k}(n)|^2 \leq 2^k(n+k)^k, \quad j = 0, \ldots, k, \quad n \in \mathbb{N}_0. \]

Using the latter two results we deduce

\[
\sum_{n=0}^{\infty} \left| \int_{\mathbb{R}} x^k f(x) \psi_n(x) \, dx \right|^2 = \sum_{n=0}^{\infty} \left| \sum_{j=0}^{k} a_{j,k}(n) (f, \psi_{n+k-j})_{L_2(\mathbb{R})} \right|^2
\]
\[ \leq 2^k \left( \sum_{j=0}^{k} \sup_{n \in \mathbb{N}_0} \left| (n+1)(n+k) \psi_{n+k-2}(f, \psi_{n+k-2}) \mathcal{Q}_n(f) \right|^2 \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (n+1)^{-2} < \infty. \]

Hence \( f \in \mathcal{D}^\infty(Q) \). By (1.9) it follows that \( (n^k \mathcal{F} f, \psi_n)_{\mathcal{Q}_n} \in l_\infty \), \( k \in \mathbb{N}_0 \). So, by replacing \( f \) by \( \mathcal{F} f \) in the above argument, we infer \( \mathcal{F} f \in \mathcal{D}^\infty(Q) \). That is, \( f \in \mathcal{D}^\infty(F) \), by (1.6).

Now it is not difficult to see that the equalities hold as topological vector spaces.

(ii) Let \( f \in S(\mathbb{R}) \) and let \( f_N = \sum_{n=0}^{N-1} (f, \psi_n)_{\mathcal{Q}_n} \psi_n \). Then \( f - f_N \to 0 \) \( (N \to \infty) \) in \( \mathcal{D}^\infty(P^2 + Q^2) \). For \( k \in \mathbb{N}_0 \),

\[ \|(P^2 + Q^2)^k (f - f_N)\|_{\mathcal{Q}_n}^2 = \sum_{n=N}^{\infty} |(2n + 1)^k (f, \psi_n)_{\mathcal{Q}_n}|^2 \to 0 \quad (N \to \infty), \]

by assertion (i). That is \( f - f_N \to 0 \) \( (N \to \infty) \) in \( \mathcal{D}^\infty(P^2 + Q^2) = S(\mathbb{R}) \).

The remaining part of assertion (ii) follows by means of the estimate \( |\psi_n(x)| \leq 1 \) from (1.10).

Theorem 1.6 can also be found in Simon [46, Theorem 1] and in Jones [32, Section 5.2]. A more general result has been proved by Goodman [23, Section 6].

### 1.2 The subspace \( S_{\text{even}}(\mathbb{R}) \) and the Hankel transformation \( H_v \)

In this section we consider the subspace \( S_{\text{even}}(\mathbb{R}) \) of even functions in \( S(\mathbb{R}) \). The topology on \( S_{\text{even}}(\mathbb{R}) \) is the induced topology. Likewise, \( S_{\text{odd}}(\mathbb{R}) \) is the subspace of odd functions in \( S(\mathbb{R}) \). By use of Laguerre expansions and properties of the Hankel transformation we come to the main result: An even function \( f \in L_2(\mathbb{R}) \) belongs to \( S_{\text{even}}(\mathbb{R}) \) if and only if both \( f \) and its Hankel transform \( H_v f \) decrease at \( +\infty \) more rapidly than any negative power of \( x \) (Theorem 1.12). Furthermore, the operators \( H_v \), \( H_{v, \mu} \), \( \mu, \nu \geq -\frac{1}{2} \), yield a fractional calculus for the operator \( -x^{-1} d/dx \) on \( S_{\text{even}}(\mathbb{R}) \).

We start with a straightforward consequence of Theorem 1.4.

**Theorem 1.6a** Let \( n \in \mathbb{N} \). Then

(i) \( \mathcal{F}(S_{\text{even}}(\mathbb{R})) = S_{\text{even}}(\mathbb{R}) \),

(ii) \( Q(S_{\text{even}}(\mathbb{R})) = S_{\text{odd}}(\mathbb{R}) \),

\( Q^{2n}(S_{\text{even}}(\mathbb{R})) = \{ f \in S_{\text{even}}(\mathbb{R}) : f^{(2k)}(0) = 0, \ k = 0, \ldots, n-1 \} \),

\( Q^{2n+1}(S_{\text{even}}(\mathbb{R})) = \{ f \in S_{\text{odd}}(\mathbb{R}) : f^{(2k+1)}(0) = 0, \ k = 0, \ldots, n-1 \} \),

(iii) \( P(S_{\text{even}}(\mathbb{R})) = S_{\text{odd}}(\mathbb{R}) \),

\( P^{2n}(S_{\text{even}}(\mathbb{R})) = \{ f \in S_{\text{even}}(\mathbb{R}) : \int f(y) y^{2k} dy = 0, \ k = 0, \ldots, n-1 \} \),

\( P^{2n+1}(S_{\text{even}}(\mathbb{R})) = \{ f \in S_{\text{odd}}(\mathbb{R}) : \int f(y) y^{2k+1} dy = 0, \ k = 0, \ldots, n-1 \} \).
On the basis of Theorem 1.6 the space $S_{\text{even}}(\mathcal{R})$ can be characterized in terms of the Laguerre expansion coefficients of its elements. For $\nu \geq -\frac{1}{2}$, the Laguerre functions

$$L_\nu^\nu(x) = \left( \frac{2\Gamma(n + 1)}{\Gamma(n + \nu + 1)} \right)^{1/2} \exp(-\frac{1}{2}x^2) \int_0^\infty \exp(-x) x^\nu \, dx,$$  \hspace{1cm} n \in \mathbb{N}_0,$$  \hspace{1cm} (1.12)$$

constitute an orthonormal basis in the Hilbert space $X_{2\nu+1} = L_2(\mathbb{R}^2; y^{2\nu+1} \, dy)$. Here $L_\nu^\nu$ denotes the generalized Laguerre polynomial,

$$L_\nu^\nu(x) = e^{-x} \frac{e^x}{n!} \left( \frac{d}{dx} \right)^n [e^{-x} x^{\nu+n}], \quad n \in \mathbb{N}_0,$$

see [38, p. 241].

Lemma 1.7 Let $\nu, \mu \geq -\frac{1}{2}$ with $\nu \neq \mu$. Then

(i) \quad $E_\nu^\nu = \sum_{m=0}^{n} A_{m,n}^{\nu,\mu} L_m^\mu, \quad n \in \mathbb{N}_0,$$

where $A_{m,n}^{\nu,\mu}$ is the upper triangular matrix with entries

$$A_{m,n}^{\nu,\mu} = \left( \frac{\Gamma(n + 1)}{\Gamma(n + \nu + 1)} \right)^{1/2} \frac{\Gamma(n - m + \mu + 1)}{\Gamma(n - m + 1)} \left( \frac{n!}{m!} \right) \Gamma(m + \mu + 1), \quad 0 \leq m \leq n.$$

(ii) There exist constants $C_{\nu,\mu} > 0$ and $l_{\nu,\mu} \in \mathbb{N}$ such that

$$|A_{m,n}^{\nu,\mu}| \leq C_{\nu,\mu} n^{l_{\nu,\mu}}, \quad 1 \leq m \leq n.$$ 

Proof. (i) The assertion follows from the formula [38, p. 249]

$$E_\nu^\nu = \sum_{m=0}^{n} \frac{(\nu - \mu)_{n-m}}{(n-m)!} L_m^\mu, \quad n \in \mathbb{N}_0.$$

(ii) From Van Eijndhoven and De Graaf [14, Lemma a.11] we quote: For $c, d > 0$ there exists a constant $K_{c,d} > 0$ such that

$$\Gamma(m + c) / \Gamma(m + d) \leq K_{c,d} m^{-d}, \quad m \in \mathbb{N}.$$ 

By means of this inequality and by observing that $|(\nu - \mu)_{n-m}| \leq (|\nu - \mu|)_{n-m} = \Gamma(n - m + |\nu - \mu|) / \Gamma(n + |\nu - \mu|)$, we estimate

$$|A_{m,n}^{\nu,\mu}| \leq K_{\nu,\mu} K_{\nu+1,\mu+1} n^{\nu-\nu} K_{\nu+\mu+2}^{-} n^{\nu} m^{\nu}(n-m)^{2\nu-\mu-2}, \quad 1 \leq m < n.$$ 

Now the assertion is readily obtained. \hfill \Box

Theorem 1.8 Let $\nu \geq -\frac{1}{2}$. Then

(i) $S_{\text{even}}(\mathcal{R}) = \{ f \in X_{2\nu+1} : (n^\nu(f, L_n^\nu)_{X_{2\nu+1}}) \in l_\infty, \quad k \in \mathbb{N}_0 \}.$

(ii) For $f \in S_{\text{even}}(\mathcal{R})$ we have

$$f(x) = \sum_{n=0}^{\infty} (f, E_n^\nu)_{X_{2\nu+1}} E_n^\nu(x), \quad x \in \mathbb{R}^2.$$
where the series converges in \( S_{\text{even}}(R) \). The series converges absolutely with respect to the \( L_\infty(R^+) \)-norm and the series converges uniformly on \( R^+ \).

**Proof.** The Hermite polynomial \( H_{2n} \) can be expressed in terms of the Laguerre polynomial \( L_n^{-1/2} \), [38, p. 240].

\[
H_{2n}(x) = (-1)^n 2^n n! L_n^{-1/2}(x^2), \quad n \in \mathbb{N}_0.
\]

Hence the Hermite function \( \psi_{2n} \) can be expressed in terms of the Laguerre function \( L_n^{-1/2} \),

\[
\psi_{2n}(x) = \frac{(-1)^n}{\sqrt{2}} L_n^{-1/2}(x), \quad n \in \mathbb{N}_0.
\]  \( (1.13) \)

Thus, for \( \nu = -\frac{1}{2} \) assertion (i) is a corollary of Theorem 1.6.

We next introduce the notation

\[ V_\nu = \{ f \in X_{2\nu+1} : (n^k (f, L_n^\nu)_{X_{2\nu+1}}) \leq l_\infty, \quad k \in \mathbb{N}_0 \}. \]

Let \( \nu, \mu \geq -\frac{1}{2} \) with \( \nu \neq \mu \). We show that \( V_\nu \subset V_\mu \). Let \( f \in V_\nu \) and let \( a_n = (f, L_n^\nu)_{X_{2\nu+1}} \).

According to Lemma 1.7(ii) there exist constants \( C_{\nu, \mu} > 0 \) and \( l_{\infty} \in \mathbb{N} \), such that for \( k, m \in \mathbb{N} \),

\[
m^k \sum_{n=m}^{\infty} |a_n A_{m,n}^\nu| \leq C_{\nu, \mu} \sum_{n=1}^{\infty} |a_n n^{k+l_{\infty}}| \leq C_{\nu, \mu} \frac{\pi^2}{6} \sup_{n \in \mathbb{N}_0} |a_n n^{k+l_{\infty}+2}|. \]

\( (1.14) \)

Hence, the double series

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n A_n^\nu B_m^\mu
\]

converges absolutely in \( X_{2\nu+1} \). Therefore, the double series converges in \( X_{2\nu+1} \) with

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n A_n B_m^\nu = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} A_n B_m^\nu = \sum_{n=0}^{\infty} a_n L_n^\nu = f .
\]

Thus we have

\[
f \in X_{2\nu+1} \quad \text{with} \quad (f, L_n^\nu)_{X_{2\nu+1}} = \sum_{n=0}^{\infty} a_n A_n B_n^\nu.
\]

(1.15)

By using (1.14) we infer \( f \in V_\mu \).

Now the embedding \( S_{\text{even}}(R) = V_{-\frac{1}{2}} \subset V_\nu \subset V_{-\frac{1}{2}} = S_{\text{even}}(R) \) proves assertion (i).

(ii) Let \( f \in S_{\text{even}}(R) \), let \( a_n = (f, L_n^\nu)_{X_{2\nu+1}} \) and let \( f_N = \sum_{n=0}^{N-1} a_n L_n^\nu \). We show that \( f - f_N \to 0 \) \( (N \to \infty) \) in \( S_{\text{even}}(R) \). By using (1.11), (1.13) and (1.15) with \( f \) replaced by \( f - f_N \) and \( \mu \) by \(-\frac{1}{2}\), we have for \( k \in \mathbb{N} \),

\[
((f^2 + Q^2)^k (f - f_N), \psi_{2m})_{L_2(R^+)} = (4m + 1)^k (f - f_N, \psi_{2m})_{L_2(R^+)}
\]

\[
= (-1)^m \sqrt{2} (4m + 1)^k (f - f_N, L_m^{-1/2})_{L_2(R^+)}
\]

\[
= (-1)^m \sqrt{2} (4m + 1)^k \sum_{n=\max(n,N)}^{\infty} a_n A_{m,n}^{-1/2}.
\]
According to Lemma 1.7 (ii) there exist constants \( C = C_{\nu, -1/2} > 0 \) and \( l = l_{\nu, -1/2} \in \mathbb{N} \), such that for \( k \in \mathbb{N} \),

\[
\frac{1}{2} \| (P^2 + Q^2)^k (f - f_N) \|_{L^2(R)}^2
\]

\[
= \sum_{m=0}^{N} (4m + 1)^{2k} \left( \sum_{n=N}^{\infty} a_n A_{m,n}^{\nu, -1/2} \right)^2 + \sum_{m=N+1}^{\infty} (4m + 1)^{2k} \left( \sum_{n=m+1}^{\infty} a_n A_{m,n}^{\nu, -1/2} \right)^2
\]

\[
\leq C^2 \sum_{m=0}^{N} \left( \sum_{n=N}^{\infty} |a_n| (4n + 1)^k n^{1/2} \right)^2 + C^2 \sum_{m=N+1}^{\infty} m^{-2} \left( \sum_{n=m}^{\infty} |a_n| (4n + 1)^k n^{1/2} \right)^2
\]

\[
\leq (C \sup_{n \in \mathbb{N}} (|a_n| (4n + 1)^k n^{1/2})) \left( \sum_{n=N}^{\infty} n^{-2} \right)^2
\]

\[
+ \frac{C}{k} \left( \sup_{n \in \mathbb{N}} (|a_n| (4n + 1)^k n^{1/2}) \right) \sum_{n=N+1}^{\infty} m^{-2} .
\]

Now, by use of assertion (i), it follows that \( \| (P^2 + Q^2)^k (f - f_N) \|_{L^2(R)} \to 0 \) \( (N \to \infty) \), \( k \in \mathbb{N} \). That is \( f - f_N \to 0 \) \( (N \to \infty) \) in \( S_{\text{sum}}(R) \).

From Lemma 1.7 with \( \mu = -\frac{1}{2} \), relation (1.13) and the estimate \( |\psi_n(x)| \leq 1 \) from (1.10), it follows that there exist \( k_\nu, l_\nu \in \mathbb{N} \) such that for \( x \in R^+ \) and \( \nu \in \mathbb{N} \),

\[
|L_\nu(x)| \leq k_\nu n^{\nu} .
\]

The remaining part of assertion (ii) follows by means of this estimate. \( \square \)

Let us introduce the Hankel transformation \( H_\nu \) for \( \nu \geq -\frac{1}{2} \) on \( L_1(R^+; y^{\nu+1/2} dy) \), defined by

\[
(H_\nu f)(x) = \int_0^\infty (xy)^{-\nu} J_\nu(xy) f(y) y^{\nu+1/2} \, dy , \quad f \in L_1(R^+; y^{\nu+1/2} dy) , \quad x > 0 , \quad (1.16)
\]

and \( (H_\nu f)(x) = (H_\nu f)(-x) \) for \( x < 0 \).

Here \( J_\nu \) denotes the Bessel function of the first kind and of order \( \nu \). Observe that the integral converges absolutely, due to the asymptotic behaviour of the Bessel function, \( J_\nu(t) = O(t^\nu) \) as \( t \downarrow 0 \) and \( J_\nu(t) = O(t^{-1/2}) \) as \( t \to \infty \). Since \( \left(\frac{1}{2} \pi t\right)^{1/2} J_{\nu}(t) = \cos t \), the Hankel transformation \( H_{-\frac{1}{2}} \) equals the Fourier-cosine transformation,

\[
(H_{-\frac{1}{2}} f)(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty f(y) \cos(xy) \, dy , \quad f \in L_1(R^+), \quad x > 0 . \quad (1.17)
\]

From [38, p. 244] we quote the formula

\[
L_\nu(x) = \frac{1}{2} (-1)^n x^{-\nu/2} e^{x/2} \int_0^\infty e^{-y/2} y^{\nu/2} L_n(y) J_\nu(\sqrt{xy}) \, dy .
\]

By replacing \( x \) by \( x^2 \) and \( y \) by \( y^2 \) it follows that

\[
H_\nu L_\nu(x) = (-1)^n E_\nu(x) , \quad n \in \mathbb{N}_0 . \quad (1.18)
\]
So \( H_\nu \) can be extended to a unitary operator on \( X_{2\nu+1} \),
\[
H_\nu f = \sum_{n=0}^{\infty} (-1)^n \langle f, \pi_n \rangle_{X_{2\nu+1}} \pi_n, \quad f \in X_{2\nu+1},
\]  
with the property
\[
H_\nu^* = I.
\]

**Theorem 1.9** Let \( \nu \geq -\frac{1}{2} \). Then the Hankel transformation \( H_\nu \) maps \( S_{\text{even}}(\mathbb{R}) \) continuously onto itself.

**Proof.** The theorem is a consequence of (1.18) and Theorem 1.8.

In order to characterize \( S_{\text{even}}(\mathbb{R}) \) by means of the Hankel transformation \( H_\nu \) we present two auxiliary results. These results are interesting in themselves and will be applied once again in Section 2.2.

**Lemma 1.10** Let \( -\frac{1}{2} \leq \mu < \nu \), let \( \alpha > \nu + 1 \) and let \( f \in L_2(\mathbb{R}^+; (1+y^2)^\alpha y^{2\nu+1} \, dy) \). Then \( H_\nu f \in X_{2\nu+1} \) and for \( x > 0 \),
\[
(H_\mu(H_\nu f))(x) = \frac{2^{-\nu+1}}{\Gamma(\nu - \mu)} \int_0^\infty f(y) (y^2 - x^2)^{\nu-\mu-1} y \, dy.
\]

**Proof.** For convenience we write \( X_{2\nu+1}^2 = L_2(\mathbb{R}^+; (1+y^2)^\alpha y^{2\nu+1} \, dy) \). The proof is divided into three steps.

*Step 1.* By means of the Cauchy-Schwarz inequality we have
\[
\int_0^\infty |f(y)| y^{2\nu+1/2} \, dy \leq \left( \int_0^\infty |f(y)|^2 (1+y^2)^\nu y^{2\nu+1} \, dy \right)^{1/2} \left( \int_0^\infty (1+y^2)^{-\alpha} \, dy \right)^{1/2}.
\]

So \( f \in L_1(\mathbb{R}^+; y^{\nu+1/2} \, dy) \). Now, by use of (1.18) and the inequality (cf. Watson [49, 3.31(1)])
\[
|J_\nu(z)| \leq \frac{|z/2|^\nu}{\Gamma(\nu+1)} \exp(|\text{Im} \, z|) / \Gamma(\nu + 1), \quad z \in C,
\]
we estimate for \( x > 0 \),
\[
|H_\nu f(x)| \leq \int_0^\infty |(xy)^{-\nu} J_\nu(xy) f(y) y^{2\nu+1} \, dy \leq \sup_{\xi > 0} |\xi^{-\nu} J_\nu(\xi)| \int_0^\infty |f(y)| y^{2\nu+1} \, dy
\]
\[
\leq \frac{2^{-\nu}}{\Gamma(\nu + 1)} \|f\|_{X_{2\nu+1}} \left( \int_0^\infty y^{2\nu+1} (1+y^2)^{-\alpha} \, dy \right)^{1/2} = \left( \frac{2^{-2\nu-1} \Gamma(\alpha - \nu - 1)}{\Gamma(\alpha) \Gamma(\nu + 1)} \right)^{1/2} \|f\|_{X_{2\nu+1}}.
\]

Hence
\[
\int_0^1 |(H_\nu f)(x)|^2 x^{2\nu+1} \, dx \leq \frac{2^{-2\nu-2} \Gamma(\alpha - \nu - 1)}{(\mu + 1) \Gamma(\alpha) \Gamma(\nu + 1)} \|f\|^2_{X_{2\nu+1}}.
\]

Since \(H_\nu\) is a unitary operator on \(X_{2\nu+1}\) and since \(X_{2\nu+1}^2 \subset X_{2\nu+1}\), we have
\[
\int_0^\infty |(H_\nu f)(x)|^2 x^{2\nu+1} \, dx \leq \int_0^\infty |(H_\nu f)(x)|^2 x^{2\nu+1} \, dx \leq \|f\|^2_{X_{2\nu+1}} \leq \|f\|^2_{X_{2\nu+1}}.
\]

By combining the latter two results it follows that \(H_\nu\) is a bounded operator from \(X_{2\nu+1}\) into \(X_{2\nu+1}\).

**Step 2.** By means of the Cauchy-Schwarz inequality we have
\[
\int_0^\infty \int_0^\infty f(y) (y^2 - x^2)^{-\nu-1} y \, dy \, dx \leq \int_0^\infty \int_0^\infty |f(y)|^2 (y^2 - x^2)^{-\nu-1} (1 + y^2) \alpha y \, dy \, dx
\]
\[
\leq \int_0^\infty (y^2 - x^2)^{-\nu-1} (1 + y^2)^{-\alpha} y \, dy \int_0^\infty (y^2 - x^2)^{-\nu-1} \alpha (1 + y^2)^{-\alpha} \, dx.
\]

By substituting \(y^2 - x^2 = (x^2 + 1) t\) we obtain for \(\alpha > 0\),
\[
\int_0^\infty (y^2 - x^2)^{-\nu-1} (1 + y^2)^{-\alpha} y \, dy = \frac{1}{2} (x^2 + 1)^{-\nu-\alpha} \int_0^\infty t^{-\mu-\alpha} (1 + t)^{-\alpha} \, dt
\]
\[
\leq \frac{\Gamma(\nu - \mu) \Gamma(\alpha - \nu + \mu)}{2 \Gamma(\alpha)}.
\]

Moreover,
\[
\int_0^\infty \int_0^\infty |f(y)|^2 (y^2 - x^2)^{-\nu-1} (1 + y^2)^\alpha y \, dy \, dx \leq \int_0^\infty |f(y)|^2 (1 + y^2)^\alpha y \int_0^\infty (y^2 - x^2)^{-\nu-1} x^{2\nu+1} \, dx \, dy
\]
\[
= \frac{\Gamma(\nu - \mu) \Gamma(\mu + 1)}{2 \Gamma(\nu + 1)} \|f\|^2_{X_{2\nu+1}}.
\]

Let us introduce the operator \(F_{\alpha, \nu}\) on \(X_{2\nu+1}^2\), defined by
\[
(F_{\alpha, \nu} f)(x) = \frac{2^{2\nu+1} \Gamma(\nu - \mu)}{\Gamma(\nu + 1)} \int_0^\infty f(y) (y^2 - x^2)^{-\nu-1} y \, dy, \quad x > 0.
\]
Then it follows that $F_{\mu,\nu}$ is a bounded operator from $X_{3\nu+1}^2$ into $X_{2\nu+1}$.

**Step 3.** First suppose $\nu - \mu > 1$. Then the double integral

$$
\int_0^\infty \int_0^\infty (zt)^{-\nu} J_\nu(zt) (ty)^{-\nu} J_\nu(ty) f(y) y^{2\nu+1} t^{2\nu+1} dy \, dt
$$

converges absolutely. Hence, by means of the integral formula [38, p. 100],

$$
\int_0^\infty J_\nu(zt) J_\nu(bt) t^{\nu-1} \, dt = \left\{ \begin{array}{ll}
2^{\nu-1} a^\nu b^{-\nu} (b^2 - a^2)^{\nu-a-1} \Gamma(\nu-\mu)^{-1}, & b > a, \\
0, & b < a,
\end{array} \right.
$$

we infer

$$(\mathcal{H}_{\nu}(\mathcal{H}_\nu f))(x) = \int_0^\infty (zt)^{-\nu} J_\nu(zt) \left\{ \int_0^\infty (ty)^{-\nu} J_\nu(ty) f(y) y^{2\nu+1} dy \right\} t^{2\nu+1} \, dt$$

$$= x^{-\nu} \int_0^\infty \left\{ \int_0^\infty J_\nu(zt) J_\nu(ty) t^{\nu-1} \, dt \right\} f(y) y^{\nu+1} dy$$

$$= \frac{2^{\nu-1}}{\Gamma(\nu-\mu + 1)} \int x f(y) (y^2 - x^2)^{\nu-\mu} \, dy \, . \quad (1.22)$$

Next suppose $0 < \nu - \mu \leq 1$. Let $g \in C_0^\infty(\mathbb{R}^+)$, the space of $C^\infty$-functions on $\mathbb{R}^+$ with compact support. Since $(d/dx) [x^{\nu+1} J_{\nu+1}(x)] = x^{\nu+1} J_{\nu}(x)$, see [38, p. 67], we have through integration by parts

$$\mathcal{H}_{\nu+1}((x^{-1} d/dx) g) = -\mathcal{H}_\nu g \, .$$

Then, by use of (1.22) with $\nu$ replaced by $\nu + 1$ and $f$ replaced by $(x^{-1} d/dx) g$, we deduce

$$(\mathcal{H}_{\nu}(\mathcal{H}_\nu g))(x) = -(\mathcal{H}_{\nu}(\mathcal{H}_{\nu+1}((x^{-1} d/dx) g)))(x)$$

$$= \frac{-2^{\nu-\nu}}{\Gamma(\nu-\mu + 1)} \int x g'(y) (y^2 - x^2)^{\nu-\mu} \, dy \, , \quad x > 0 \, .$$

Integration by parts yields $$(\mathcal{H}_{\nu}(\mathcal{H}_\nu g))(x) = (F_{\mu,\nu} g)(x), \quad x > 0 \, .$$ Since the space $C_0^\infty(\mathbb{R}^+)$ is dense in $X_{2\nu+1}^2$ and since both $\mathcal{H}_\nu \mathcal{H}_\nu$ and $F_{\mu,\nu}$ are bounded operators from $X_{2\nu+1}^2$ into $X_{2\nu+1}$, the theorem follows. \hfill \Box

In Section 3.1, formulas (3.7) and (3.8), we define the Erdélyi-Kober operators $\mathcal{W}_\lambda, \lambda \in \mathbb{R}$, on the space $S_- (\mathbb{R}^+) = \{ f \in C^\infty(\mathbb{R}^+) : x^k f^{(k)} \in L^\infty([1, \infty)), \quad k, l, \in \mathbb{N}_0 \}$ by

$$(\mathcal{W}_\nu f)(x) = \frac{2^{1-\lambda}}{\Gamma(\lambda)} \int x f(y) (y^2 - x^2)^{\lambda-1} \, dy \, , \quad f \in S_- (\mathbb{R}^+), \quad x > 0, \quad \lambda > 0 \, ,$$

$$\mathcal{W}_0 = I \, , \quad \mathcal{W}_{-\lambda} = \mathcal{W}_{1-\lambda} (x^{-1} d/dx)^{\lambda}, \quad \lambda > 0 \, . \quad (1.23)$$
Furthermore, we show that the operators $B_{\lambda}$, $\lambda \in \mathcal{R}$, constitute a group on $S_{\nu}(\mathcal{R}^+)$; cf. Theorem 3.6(i). By using this property and the relations (1.20), (1.23) and Lemma 1.10, it follows that

$$B_{\mu}B_{\nu}f = B_{\mu-\nu}f, \quad f \in S_{\nu}(\mathcal{R}^+), \quad \mu, \nu \geq -\frac{1}{2}. \quad (1.24)$$

So, due to Theorem 1.9:

The operators $B_{\lambda}$, $\lambda \in \mathcal{R}$, constitute a group of continuous operators on $S_{\nu}(\mathcal{R})$. \quad (1.25)

In particular, the operator $x^{-1} d/dx$ maps $S_{\nu}(\mathcal{R})$ continuously onto itself.

**Lemma 1.11** Let $-\frac{1}{2} \leq \mu < \nu$ and let $W$ be a nonnegative function on $\mathcal{R}^+$, such that $(1 + x^2)^{\nu-\frac{1}{2}} W(x)$ is nondecreasing on $[b, \infty)$ for some $b > 0$. Let $f \in L^2[0, \infty)$ such that

$$z^k f \in L^2(\mathcal{R}^+) \quad \text{and} \quad z^l B_{\mu} f \in L^2(\mathcal{R}^+), \quad k, l \in \mathbb{N}_0.$$ 

Suppose in addition that $z^{\nu-\mu} W(x) B_{\mu} f \in L^2(\mathcal{R}^+)$. Then

$$(1 + z^2)^{\nu-\frac{1}{2}} W(x) B_{\mu} f \in L^2(\mathcal{R}^+).$$

**Proof.** Since $z^k f \in L^2(\mathcal{R}^+), k \in \mathbb{N}_0$, it follows that $f \in L^2(\mathcal{R}^+; y^{\mu+1/2} dy)$. So $B_{\mu} f$ is continuous by (1.16). Therefore, $(1 + x^2)^{\nu-\frac{1}{2}} W(x) B_{\mu} f \in L^2([0, \infty])$. Since $B_{\mu}^2 = I$ and $B_{\mu} f \in L^2(\mathcal{R}^+; (1 + y^2)^{\mu+2} y^{\mu+1} dy)$ we have by Lemma 1.10 with $\alpha = \nu + 2$ and $f$ replaced by $B_{\mu} f$,

$$\frac{\Gamma^2(\nu - \mu)}{4^{\nu-\frac{1}{2}}} \int_0^\infty \left| (1 + z^2)^{\nu-\frac{1}{2}} W(x) \left( B_{\mu} f \right)(x) \right|^2 dx$$

$$\leq \int_0^\infty \left( \int_0^\infty (1 + y^2)^{\nu-\frac{1}{2}} W(y) \left| (B_{\mu} f)(y) \right| (y^2 - x^2)^{\nu-\frac{1}{2}} y \, dy \right)^2 dx$$

$$\leq \int_0^\infty \left( \int_0^\infty |W(y) (B_{\mu} f)(y)|^2 (y^2 - x^2)^{\nu-\frac{1}{2}} y \, dy \right)$$

$$\left( \int_0^\infty (1 + y^2)^{2\nu-\frac{3}{2}} (y^2 - x^2)^{\nu-\frac{1}{2}} y \, dy \right) dx$$

$$= \frac{1}{2} B(\nu - \mu, \nu - \mu + 3/2)$$

$$\int_0^\infty \int_0^\infty |W(y) (B_{\mu} f)(y)|^2 (y^2 - x^2)^{\nu-\frac{1}{2}} y \, dy \left( 1 + x^2 \right)^{\mu-\frac{3}{2}} dx$$

$$= \frac{1}{2} B(\nu - \mu, \nu - \mu + 3/2)$$

$$\int_0^\infty \int_0^\infty \left( y^2 - x^2 \right)^{\nu-\frac{1}{2}} (1 + x^2)^{\mu-\frac{3}{2}} |W(y) (B_{\mu} f)(y)|^2 \, dy dx.$$
Since the inner integral admits the estimate

\[
\int_0^y (y^2 - z^2)^{\alpha-1} (1 + z^2)^{\alpha-3/2} \, dz \leq \frac{1}{b} \int_0^y (y^2 - z^2)^{\alpha-1} \, dx 
\]

\[
\leq \frac{1}{2b} \int_0^y (y^2 - x)^{\alpha-1} \, dx = \frac{y^{2\alpha-2\mu}}{2b(\alpha-\mu)},
\]

we obtain the wanted result because of the assumption \( x^{\nu-\mu} W(x) H_\nu f \in L^3(\mathbb{R}^+) \). □

We now come to the main result of this section.

**Theorem 1.12** Let \( \nu \geq -\frac{1}{2} \) and let \( f \in X_{2\nu+1} \). The following assertions are equivalent:

(i) \( f \in S_{\text{even}}(\mathbb{R}) \);

(ii) \( x^kf \in L^1(\mathbb{R}^+) \) and \( x^l H_\nu f \in L^1(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 

(iii) \( x^kf \in L^3(\mathbb{R}^+) \) and \( x^l H_\nu f \in L^3(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \).

**Proof.** We follow the scheme (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i).

Since \( H_\nu(S_{\text{even}}(\mathbb{R})) = S_{\text{even}}(\mathbb{R}) \), (i) \( \Rightarrow \) (ii). Obviously (ii) \( \Rightarrow \) (iii).

Suppose \( f \) satisfies assertion (iii). If \( \nu = -\frac{1}{2} \), then \( f \in S_{\text{even}}(\mathbb{R}) \) by (1.17) and Theorem 1.5. Next suppose \( \nu > -\frac{1}{2} \). For \( l \in \mathbb{N}_0 \) we introduce the function \( W(x) = (1 + x^2)^{3\nu+3/2} \).

Then the functions \( f \) and \( W \) satisfy the assumptions of Lemma 1.11 with \( \mu = -\frac{1}{2} \). Therefore, \( (1 + x^2)^{3\nu} H_{-\frac{1}{2}} f \in L^3(\mathbb{R}^+) \). So \( f \in S_{\text{even}}(\mathbb{R}) \) as in the case \( \nu = -\frac{1}{2} \). □

As a consequence we obtain a functional analytic characterization of \( S_{\text{even}}(\mathbb{R}) \).

**Corollary 1.13** Let \( \nu \geq -1/2 \). Then

(i) \( S_{\text{even}}(\mathbb{R}) = D^\infty(\mathbb{R}) \cap D^\infty(H_\nu Q^2 H_\nu) \),

where \( Q \) is the self-adjoint operator defined on \( \{ f \in X_{2\nu+1} : xf \in X_{2\nu+1} \} \) by \( Qf = xf \).

(ii) \( (H_\nu Q^2 H_\nu) L^\infty_n = (-d^2/dx^2 - (2\nu + 1) x^{-1} d/dx) L^\infty_\nu \)

\[
= (4n + 2\nu + 2 - x^2) L^\infty_\nu, \quad n \in \mathbb{N}_0.
\]

**Proof.** (i) Let \( f \in X_{2\nu+1} \). From (1.3) with \( T \) replaced by \( Q^2 \) and \( U \) by \( H_\nu \) it follows that \( f \in D^\infty(H_\nu Q^2 H_\nu) \) if and only if \( H_\nu f \in D^\infty(\mathbb{R}) \). So we have to show that the following assertions are equivalent.

(1) \( f \in S_{\text{even}}(\mathbb{R}) \);

(2) \( x^kf \in X_{2\nu+1} \) and \( x^l H_\nu f \in X_{2\nu+1} \), \( k, l \in \mathbb{N}_0 \).

...
Suppose \( f \) satisfies (2). Then \( f \in L_1(\mathbb{R}^+; y^{n+1/2} \, dy) \) and so \( \mathcal{H}_\nu f \) is continuous by (1.16). Similarly, \( \mathcal{H}_\nu f \in L_1(\mathbb{R}^+; y^{n+1/2} \, dy) \) and so \( f = \mathcal{H}_\nu (\mathcal{H}_\nu f) \) is continuous by (1.16). Hence \( f \) satisfies assertion (iii) of Theorem 1.12 and therefore, \( f \in S_{\text{even}}(\mathcal{H}) \).

The converse, (1) \( \Rightarrow \) (2), follows from the fact that \( \mathcal{H}_\nu (S_{\text{even}}(\mathcal{H})) = S_{\text{even}}(\mathcal{H}) \).

(ii) From [38, pp. 241, 243] we deduce
\[
x^2 L^\ast_n = -((n+1) (n+\nu +1))^{1/2} L^\ast_{n+1} + (2(n+1) \nu + 1) L^\ast_n - (n(n+\nu))^{1/2} L^\ast_{n-1},
\]
\[
(\partial^2 / \partial x^2 + (2\nu + 1) x^{-1} \partial / \partial x + (4n + 2\nu + 2 - x^2)) L^\ast_n = 0, \quad n \in \mathbb{N}_0.
\]

By means of these relations and (1.18) the assertion follows.

\[\square\]

1.3 The space \( S(\mathbb{R}^+) \) and the Hankel-Clifford transformation \( \mathcal{H}_\nu \)

In this section we introduce \( S(\mathbb{R}^+) \), the space of rapidly decreasing functions on \( \mathbb{R}^+ \). We present the connection with each of the spaces \( S(\mathcal{H}) \) and \( S_{\text{even}}(\mathcal{H}) \). Because of these connections we are able to establish characterizations of \( S(\mathbb{R}^+) \) which are similar to those of \( S(\mathcal{H}) \) and \( S_{\text{even}}(\mathcal{H}) \) as given in Sections 1.1 and 1.2. Furthermore, it follows that \( S_{\text{even}}(\mathcal{H}) \) can be characterized by means of the operators \( z \) and \( z^{-1} \partial / \partial z \) (see Corollaries 1.18 and 1.21).

Definition 1.14 The space \( S(\mathbb{R}^+) \) consists of all functions \( g \in C^\infty(\mathbb{R}^+) \) for which
\[
z^k g^{(l)} \in L_\infty(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0.
\]
The topology on \( S(\mathbb{R}^+) \) is defined by the countable set of seminorms
\[
g \mapsto \| z^k g^{(l)} \|_{L_\infty(\mathbb{R}^+)}, \quad k, l \in \mathbb{N}_0.
\]

This topology is metrizable and gives \( S(\mathbb{R}^+) \) the structure of a complete metric space.

We present the connection between \( S(\mathbb{R}^+) \) and \( S(\mathcal{H}) \), and the connection between \( S(\mathbb{R}^+) \) and \( S_{\text{even}}(\mathcal{H}) \):

Theorem 1.15

(i) \( S(\mathbb{R}^+) = \{ g \in C^\infty(\mathbb{R}^+) : g(z) = f(x), \quad x \in \mathbb{R}^+, \text{ for some } f \in S(\mathcal{H}) \} \)

(ii) \( S_{\text{even}}(\mathcal{H}) = \{ f \in C^\infty(\mathcal{H}) : f(x) = g(x^2), \quad x \in \mathbb{R}, \text{ for some } g \in S(\mathbb{R}^+) \} \).

Proof. (i) Let \( g \in S(\mathbb{R}^+) \). By means of Borel’s theorem, see [52, Chapter 1, Exercise 1], there exists a function \( f \in C^\infty(\mathcal{H}) \) such that \( f(x) = g(x), \quad x \in \mathbb{R}^+ \), and \( f(x) = 0, \quad x \leq -1 \). Clearly \( f \in S(\mathcal{H}) \). The converse is obvious.

(ii) Let \( f \in S_{\text{even}}(\mathcal{H}) \). Introduce the function \( g : \mathbb{R}^+ \to C \) by \( g(z) = f(\sqrt{x}), \quad x \in \mathbb{R}^+ \). By (1.25) the operator \( z^{-1} \partial / \partial z \) maps \( S_{\text{even}}(\mathcal{H}) \) onto itself. So for \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x \to 0^+} |z^k \partial / \partial x|^l f(\sqrt{x}) = \sup_{x \to 0^+} |z^k \partial / \partial z|^l f(z) < \infty.
\]
Hence \( g \in S(\mathbb{R}^+) \) and \( f(x) = g(x^2), \ x \in \mathbb{R} \). The converse is obvious. \( \square \)

As a straightforward consequence of Theorems 1.2 and 1.15 we obtain the following characterizations of \( S(\mathbb{R}^+) \) and of \( S_{\text{even}}(\mathbb{R}) \):

**Corollary 1.16** Let \( g \in C^{\omega}(\mathbb{R}^+) \). The following assertions are equivalent:

(i) \( g \in S(\mathbb{R}^+) \); 
(ii) \( x^k g^{(l)} \in L_1(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 
(iii) \( x^k g \in L_2(\mathbb{R}^+) \) and \( g^{(l)} \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 
(iv) \( x^k g \in L_2(\mathbb{R}^+) \) and \( g^{(l)} \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \).

Let \( f \in C^{\omega}(\mathbb{R}^+) \). The following assertions are equivalent:

(i) \( f \in S_{\text{even}}(\mathbb{R}) \); 
(ii) \( x^k (x^{-1/2} d/dx)^l f \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 
(iii) \( x^k (x^{-1/2} d/dx)^l f \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 
(iv) \( x^k f \in L_2(\mathbb{R}^+) \) and \( (x^{-1/2} d/dx)^l f \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \); 
(v) \( x^k f \in L_2(\mathbb{R}^+) \) and \( (x^{-1/2} d/dx)^l f \in L_2(\mathbb{R}^+) \), \( k, l \in \mathbb{N}_0 \).

For \( \nu \geq -\frac{1}{2} \) we define the Fourier-Laguerre functions \( \mathcal{L}^\nu_n, n \in \mathbb{N}_0 \), by

\[
\mathcal{L}^\nu_n(x) = \left( \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)} \right)^{1/2} \exp(-\frac{1}{2}x) L_n^\nu(x), \ n \in \mathbb{N}_0, \tag{1.26}
\]

and the Hankel-Clifford transformation \( \mathcal{H}_\nu \) on \( L_1(\mathbb{R}^+; y^{\nu/2-1/4} dy) \) by

\[
(\mathcal{H}_\nu g)(x) = \frac{1}{2} \int_0^\infty (xy)^{-\nu/2} J_n(\sqrt{xy}) g(y) y^\nu dy, \ g \in L_1(\mathbb{R}^+; y^{\nu/2-1/4} dy), \ x > 0. \tag{1.27}
\]

By means of the operator \( T_p \), defined by

\[
(T_p g)(x) = g(x^p), \ x > 0, \ p \in \mathbb{R}, \tag{1.28}
\]

it follows from (1.12), (1.16), (1.26) and (1.27) that

\[
\mathcal{L}^\nu_n = \frac{1}{\sqrt{2}} T_{1/2} \mathcal{H}^\nu_n, \ n \in \mathbb{N}_0, \ \mathcal{H} = T_{1/2} \mathcal{H}_\nu T_{1/2}. \tag{1.29}
\]

The Fourier-Laguerre functions \( \mathcal{L}^\nu_n, n \in \mathbb{N}_0, \) constitute an orthonormal basis in the Hilbert space \( X_\nu = L_2(\mathbb{R}^+; y^\nu dy) \). By use of (1.18) and (1.29) we have.
\( \mathcal{H}_\nu \mathcal{L}_n^\nu = (-1)^n \mathcal{L}_n^\nu , \quad n \in \mathbb{N}_0 . \) \hspace{1cm} (1.30)

So \( \mathcal{H}_\nu \) can be extended to a unitary operator on \( X_\nu , \)

\[ \mathcal{H}_\nu g = \sum_{n=0}^{\infty} (-1)^n (g, \mathcal{L}_n^\nu) x_n \mathcal{L}_n^\nu , \quad g \in X_\nu , \] \hspace{1cm} (1.31)

with the property

\[ \mathcal{H}_\nu = I . \] \hspace{1cm} (1.32)

Due to Theorem 1.15(ii) the results for \( S_{\text{even}}(\mathbb{R}) \) established in Section 1.2 can be translated into results for \( S(\mathbb{R}^+) \). Let us gather these results.

**Theorem 1.17** Let \( \nu \geq -\frac{1}{2} \). Then

(i) \( S(\mathbb{R}^+) = \{ f \in X_\nu : (n^\nu (g, \mathcal{L}_n^\nu) x_n) \in L_\infty , \quad k \in \mathbb{N}_0 \} \).

(ii) For \( g \in S(\mathbb{R}^+) \) we have

\[ g(x) = \sum_{n=0}^{\infty} (g, \mathcal{L}_n^\nu) x_n \mathcal{L}_n^\nu (x) , \quad x \in \mathbb{R}^+ , \]

where the series converges in \( S(\mathbb{R}^+) \). The series converges absolutely with respect to the \( L_\infty(\mathbb{R}^+) \)-norm and the series converges uniformly on \( \mathbb{R}^+ \).

(iii) The Hankel-Clifford transformation \( \mathcal{H}_\nu \) maps \( S(\mathbb{R}^+) \) continuously onto itself.

(iv) Let \( -\frac{1}{2} \leq \mu < \nu \), let \( \alpha > \nu + 1 \) and let \( g \in L_2(\mathbb{R}^+; (1+y)^{\alpha+y} dy) \). Then \( \mathcal{H}_\nu g \in X_\mu \) and for \( x > 0 \),

\[ (\mathcal{H}_\mu(\mathcal{H}_\nu g))(x) = \frac{2^{\mu-\nu}}{\Gamma(\nu - \mu)} \int_{x}^{\infty} g(y) (y-x)^{\nu-\mu-1} dy . \] \hspace{1cm} (1.33)

In Section 3.1, formulas (3.3) and (3.5), we define the Weyl operators \( W_\lambda , \lambda \in \mathbb{R} \), on the space \( S_- = \{ g \in C^\infty(\mathbb{R}^+) : x^k g^{(k)} \in L_\infty([1, \infty)) , \quad k, l \in \mathbb{N}_0 \} \) by

\[ (W_\lambda g)(x) = \frac{1}{\Gamma(\lambda)} \int_{x}^{\infty} g(y) (y-x)^{\lambda-1} dy , \quad g \in S_- , \quad x > 0 , \quad \lambda > 0 , \]

\[ W_0 = I , \quad W_{-\lambda} = W_{[\lambda]-1} (-d/dx)^{[\lambda]} , \quad \lambda > 0 . \] \hspace{1cm} (1.34)

Furthermore, we show in Theorem 3.3 that the operators \( W_\lambda , \lambda \in \mathbb{R} \), constitute a group on \( S_- (\mathbb{R}^+) \). By using this property and the relations (1.32), (1.33) and (1.34), it follows that

\[ \mathcal{H}_\mu \mathcal{H}_\nu g = 2^{\nu-\mu} W_{\nu-\mu} g , \quad g \in S(\mathbb{R}^+) , \quad \mu, \nu \geq -\frac{1}{2} . \] \hspace{1cm} (1.35)

So, due to Theorem 1.17(iii):

The operators \( W_\lambda , \lambda \in \mathbb{R} \), constitute a group of continuous operators on \( S(\mathbb{R}^+) \).
In particular, the operator $d/dx$ maps $S(\mathbb{R}^+)$ continuously onto itself. The space $S(\mathbb{R}^+)$ can be characterized by means of the Hankel-Clifford transformation $\mathcal{H}_\nu$:

**Theorem 1.18** Let $\nu \geq -\frac{1}{2}$ and let $g \in X_\nu$. The following assertions are equivalent:

(i) $g \in S(\mathbb{R}^+)$;

(ii) $x^k g \in L_\infty(\mathbb{R}^+)$ and $x^l \mathcal{H}_\nu g \in L_\infty(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0$;

(iii) $x^k g \in L_2(\mathbb{R}^+)$ and $x^l \mathcal{H}_\nu g \in L_2(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0$.

As a consequence we obtain a functional analytic characterization of $S(\mathbb{R}^+)$.

**Corollary 1.19** Let $\nu \geq -\frac{1}{2}$. Then

(i) $S(\mathbb{R}^+) = D^\infty(Q) \cap D^\infty(\mathcal{H}_\nu Q \mathcal{H}_\nu)$,

where $Q$ is the self-adjoint operator defined on $g \in X_\nu : xg \in X_\nu$ by $Qg = xg$.

(ii) $(\mathcal{H}_\nu Q \mathcal{H}_\nu) \mathcal{L}_\nu^\nu = (-2\sqrt{\pi} d/dx)^2 - 2(2\nu + 1) d/dx) \mathcal{L}_\nu^\nu$

$$= (4n + 2\nu + 2 - x) \mathcal{L}_\nu^\nu, \quad n \in \mathbb{N}_0.$$

The subspace $S_-(\mathbb{R}^+) = \{g \in S(\mathbb{R}^+) : g(0^+) = 0, \quad l \in \mathbb{N}_0\}$ admits the following characterization.

**Theorem 1.19a** Let $\nu \geq -\frac{1}{2}$. Then a function $g \in S(\mathbb{R}^+)$ belongs to $S_-(\mathbb{R}^+)$ if and only if

$$\sum_{n=0}^{\infty} (g, \mathcal{L}_\nu^\nu) X_n \left( \frac{\Gamma(n + \nu + 1)}{\Gamma(n + 1)} \right)^{1/2} n^k = 0, \quad k \in \mathbb{N}_0.$$

**Proof.** Let $g \in S(\mathbb{R}^+)$. Since the operator $d/dx$ is continuous on $S(\mathbb{R}^+)$ by (1.36), we have

$$(d/dx)^k g(x) = \sum_{n=0}^{\infty} (g, \mathcal{L}_n^\nu) X_n (d/dx)^k \mathcal{L}_n^\nu(x), \quad x \in \mathbb{R}^+, \quad k \in \mathbb{N}_0,$$

where the series converges in $S(\mathbb{R}^+)$, see Theorem 1.17(ii). Now, by using the properties [38, pp. 240, 241],

$$\mathcal{L}_n^\nu(0) = \Gamma(n + \nu + 1) / \Gamma(n + 1) \Gamma(\nu + 1), \quad (d/dx) \mathcal{L}_n^\nu(x) = -\mathcal{L}_{n+1}^\nu(x), \quad n \in \mathbb{N},$$

the function $g$ belongs to $S_-(\mathbb{R}^+)$ if and only if

$$0 = \lim_{x \to 0} ((d/dx)^k [e^{x/2} g(x)])$$

$$= \lim_{x \to 0} \sum_{n=k}^{\infty} (g, \mathcal{L}_n^\nu) X_n (d/dx)^k [e^{x/2} \mathcal{L}_n^\nu(x)]$$

$$= \frac{(-1)^k}{\Gamma(n + k + 1)} \sum_{n=k}^{\infty} (g, \mathcal{L}_n^\nu) X_n \left( \frac{\Gamma(n + \nu + 1)}{\Gamma(n + 1)} \right)^{1/2} \frac{\Gamma(n + 1)}{\Gamma(n - k + 1)} \quad k \in \mathbb{N}_0.$$
This proves the theorem because the latter series converges absolutely by Theorem 1.17(i) and because \( \Gamma(n+1)/\Gamma(n-k+1) \) can be expressed as a linear combination of \( n^j \), \( j = 1, \ldots, k \).

\[ \Box \]

Theorem 1.19a can also be found in Duran [10, Theorem 3.2]. At the end of this section we prove a characterization of \( S'(R^+;\nu) \), inspired by Theorem 1.18 and Duran’s paper [11]. Here is the idea: By using Theorem 1.17(iii), (1.35) and the fact that \( H \) is a unitary operator on \( X \), we have for \( g \in S(R^+) \) and \( l \in \mathbb{N}_0 \),

\[ \|x^{l/2}Hg\|_{L_2(R^+)} = \|Hx^{l/2}g\|_{L_2(R^+)} = 2^l \|x^{l/2}g^{(l)}\|_{L_2(R^+)} . \]

So by taking \( g \in S(R^+) \), \( \nu = 0 \) and by replacing \( k \) by \( k/2 \) and \( l \) by \( l/2 \), Theorem 1.18(iii) admits the reformulation

\[ x^{k/2}g \in L_2(R^+) \quad \text{and} \quad x^{l/2}g^{(l)} \in L_2(R^+) , \quad k, l \in \mathbb{N}_0 \ . \]

We show that these conditions on \( g \in C^\infty(R^+) \) imply \( g \in S(R^+) \).

**Theorem 1.20** Let \( g \in C^\infty(R^+) \). The following assertions are equivalent:

(i) \( g \in S(R^+) \);

(ii) \( x^{(k+l)/2}g^{(l)} \in L_\infty(R^+) \) \( , \quad k, l \in \mathbb{N}_0 \);

(iii) \( x^{(k+l)/2}g^{(l)} \in L_2(R^+) \) \( , \quad k, l \in \mathbb{N}_0 \);

(iv) \( x^{k/2}g \in L_2(R^+) \) and \( x^{l/2}g^{(l)} \in L_2(R^+) \) \( , \quad k, l \in \mathbb{N}_0 \);

(v) \( x^{k/2}g \in L_\infty(R^+) \) and \( x^{l/2}g^{(l)} \in L_\infty(R^+) \) \( , \quad k, l \in \mathbb{N}_0 \).

**Proof.** We follow the scheme (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i) \( \Rightarrow \) (v) \( \Rightarrow \) (iv).

The implications (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) and (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (v) need no explanation.

Suppose \( g \) satisfies assertion (iv). Introduce the function \( f \) by \( f(x) = g(x+1) \) \( , \quad x > 0 \).

Then for \( x > 0 \),

\[ |x^{k/2}f^{(l)}(x)| \leq |(x+1)^{(k+l)/2}g^{(l)}(x+1)| \] \( , \quad k, l \in \mathbb{N}_0 \) \( . \) \hspace{1cm} (1.37)

By using (1.37) with \( l = 0 \) or \( k = 0 \), and Corollary 1.16, it follows that \( f \in S(R^+) \) \( . \)

For \( l \in \mathbb{N}_0 \) we have

\[ \left( \int_0^\infty |g^{(l)}(y)| y^{1/2-1/4} \, dy \right)^2 \leq \left( \int_0^\infty |g^{(l)}(y)|^2 y^1 \, dy \right) \left( \int_0^\infty y^{-1/2} \, dy \right) < \infty . \]

Since \( f \in S(R^+) \), we infer \( g^{(l)} \in L_1(R^+; y^{1/2-1/4} \, dy) \). By (1.27), \( Hg^{(l)} \) is continuous with

\[ (\mathcal{H}_1g^{(l)})(x) = \frac{1}{2} \int_0^\infty (2y)^{-1/2} J_l(\sqrt{2y}) \, g^{(l)}(y) \, y^1 \, dy , \quad x > 0 \] \hspace{1cm} (1.38)

Next we want to apply integration by parts. For that purpose, we first show that
\[ x^{l/2}g^{(l-1)} \in L_{\infty}(\mathbb{R}^+) , \quad l \in \mathbb{N} . \] (1.39)

For \( 0 < x < 1 \) and \( l \in \mathbb{N} \) we have

\[ \| x^{l/2}g^{(l-1)}(x) \|^2 - |g^{(l-1)}(1)|^2 = \]
\[ = 2 \Re \int_0^1 x^{l/2}g^{(l-1)}(y) \left( \frac{1}{2} x^{l/2-1}g^{(l-1)}(y) + x^{l/2}g^{(l)}(y) \right) \, dy \]
\[ \leq 2 \int_0^1 \left( \frac{1}{2} y^{l-1} \| g^{(l-1)}(y) \|^2 + |y^{l-1/2}g^{(l-1)}(y)y^{l/2}g^{(l)}(y)| \right) \, dy \]
\[ \leq l \int_0^\infty |y|^{l-1/2} \| g^{(l-1)}(y) \|^2 \, dy + 2 \left( \int_0^\infty |y|^{l-1} |y|^{l-1/2} \, dy \right)^{1/2} \left( \int_0^\infty |y|^{l/2} |g^{(l)}(y)|^2 \, dy \right)^{1/2} . \]

Hence \( x^{l/2}g^{(l-1)} \in L_{\infty}([0, 1]) \). Since \( f \in S(\mathbb{R}^+) \), we infer \( x^{l/2}g^{(l-1)} \in L_{\infty}(\mathbb{R}^+) \).

By using (1.21), (1.39) and the fact that \( f \in S(\mathbb{R}^+) \), we have for \( x > 0 \) and \( l \in \mathbb{N} \),

\[ (xy)^{-l/2} J_l(xy g^{(l-1)}(y)) \rightarrow 0 \quad \text{for} \quad y \downarrow 0 \quad \text{and for} \quad y \rightarrow \infty . \]

Hence by means of the relation \( (d/dz) [z^\nu J_\nu(z)] = \nu J_{\nu-1}(z) \), see [38, p. 67], we obtain through integration by parts in (1.38),

\[ (\mathcal{H}_l g^{(l)})(x) = -\frac{1}{\nu} \int_0^\infty (xy)^{-(l+1)/2} J_{l-1}(\sqrt{xy}) g^{(l-1)}(y) \frac{d^2 y}{y^{1/2}} = -\frac{1}{\nu} (\mathcal{H}_{l-1} g^{(l-1)})(x) . \]

By repeating this procedure we have

\[ \mathcal{H}_l g^{(l)} = (-2)^{-l} \mathcal{H}_0 g , \quad l \in \mathbb{N}_0 . \]

Now, since \( \mathcal{H}_l \) is a unitary operator on \( X_l \), we deduce for \( l \in \mathbb{N}_0 \),

\[ \int_{\mathbb{R}^+} x^l \| (\mathcal{H}_l g)(x) \|^2 \, dx = \frac{2^{2l}}{l} \int_{\mathbb{R}^+} x^l \| (\mathcal{H}_0 g^{(l)})(x) \|^2 \, dx = \frac{2^{2l}}{l} \int_{\mathbb{R}^+} x^l |g^{(l)}(x)|^2 \, dx < \infty . \]

Hence \( g \in S(\mathbb{R}^+) \), due to Theorem 1.18.

Finally, suppose \( g \) satisfies assertion \( (v) \). Introduce the function \( f \) by \( f(x) = g(x + 1) \), \( x > 0 \). By use of (1.37), with \( k = 0 \) or \( l = 0 \), and Corollary 1.16 it follows that \( f \in S(\mathbb{R}^+) \), as before. Since \( x^{l/2}g^{(l)} \in L_{\infty}(\mathbb{R}^+) \), \( l \in \mathbb{N}_0 \), and \( f \in S(\mathbb{R}^+) \), we estimate for \( k, l \in \mathbb{N}_0 \),

\[ \int_{\mathbb{R}^+} x^k |g(x)|^2 \, dx \leq \sup_{0 \leq r \leq 1} |g(x)|^2 + \int_{\mathbb{R}^+} x^k |f(x-1)|^2 \, dx < \infty , \]

\[ \int_{\mathbb{R}^+} x^l |g^{(l)}(x)|^2 \, dx \leq \sup_{0 \leq r \leq 1} |x^{l/2}g^{(l)}(x)|^2 + \int_{\mathbb{R}^+} x^l |f^{(l)}(x-1)|^2 \, dx < \infty . \]
That is, $g$ satisfies assertion (iv).

As a straightforward corollary of Theorem 1.15(ii) and Theorem 1.20 we obtain the following characterization of $S_{\text{even}}(\mathbb{R})$:

**Corollary 1.21** Let $f \in C^\infty(\mathbb{R}^+)$, The following assertions are equivalent:

(i) \[ f \in S_{\text{even}}(\mathbb{R}) \; ; \]

(ii) \[ x^{k+l}(x^{-1} \frac{d}{dx})^l f \in L_\infty(\mathbb{R}^+) \; , \quad k,l \in \mathbb{N}_0 \; ; \]

(iii) \[ x^{k+l}(x^{-1} \frac{d}{dx})^l f \in L_2(\mathbb{R}^+) \; , \quad k,l \in \mathbb{N}_0 \; ; \]

(iv) \[ x^k f \in L_2(\mathbb{R}^+) \quad \text{and} \quad x^l(x^{-1} \frac{d}{dx})^l f \in L_2(\mathbb{R}^+) \; , \quad k,l \in \mathbb{N}_0 \; ; \]

(v) \[ x^k f \in L_\infty(\mathbb{R}^+) \quad \text{and} \quad x^l(x^{-1} \frac{d}{dx})^l f \in L_\infty(\mathbb{R}^+) \; , \quad k,l \in \mathbb{N}_0 \; . \]

1.4 The Schwartz space $S(\mathbb{R}^q)$

The Schwartz space $S(\mathbb{R}^q)$ is the space of infinitely differentiable functions of rapid decrease on $\mathbb{R}^q$. In this section we gather some properties of this space for $q \geq 2$.

**Definition 1.22** The Schwartz space $S(\mathbb{R}^q)$ consists of all functions $f \in C^\infty(\mathbb{R}^q)$ for which

\[ \left\| x^k \partial_j^l f \right\|_{L_\infty(\mathbb{R}^q)} \; , \quad k,l \in \mathbb{N}_0^q \; . \]

Here we use the multi-index notation $x^k = x_1^{k_1} \cdots x_q^{k_q}$ and $\partial_j^l = \partial_j^{l_1} \cdots \partial_j^{l_q}$ with $\partial_j = \partial / \partial x_j$, $j = 1, \ldots, q$. The topology on $S(\mathbb{R}^q)$ is defined by the countable set of seminorms

\[ f \mapsto \left\| x^k \partial_j^l f \right\|_{L_\infty(\mathbb{R}^q)} \; , \quad k,l \in \mathbb{N}_0^q \; . \]

This topology is metrizable and gives $S(\mathbb{R}^q)$ the structure of a complete metric space.

The multiplication operators $x_j$ and the differentiation operators $\partial_j$, $j = 1, \ldots, q$, map $S(\mathbb{R}^q)$ continuously into itself. For $f \in S(\mathbb{R}^q)$ we have $(1 + |x|^q)^{p} f = (1 + x_1^q + \ldots + x_q^q)^{p} f \in L_\infty(\mathbb{R}^q)$, whence we obtain $S(\mathbb{R}^q) \subset L_p(\mathbb{R}^q)$, $p \geq 1$. In particular, we can define the partial Fourier transformations $\mathcal{F}_j$, $j = 1, \ldots, q$, and the Fourier transformation $\mathcal{F}$ on $S(\mathbb{R}^q)$ as follows. For $f \in S(\mathbb{R}^q)$ and $z \in \mathbb{R}^q$,

\[ (\mathcal{F}_j f) (z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_q) e^{-ix_j y} dy_j \; , \quad (1.40) \]

\[ (\mathcal{F} f) (z) = ((\mathcal{F}_1 \cdots \mathcal{F}_q) f) (z) = (2\pi)^{-q/2} \int_{\mathbb{R}^q} f(y) e^{-i(x \cdot y)} dy \; . \quad (1.41) \]
In [45, Section VII.6, Theorem XII], Schwartz has proved that the (partial) Fourier transformation(s) \( F, F_1, \ldots, F_q \) map \( S(\mathbb{R}^q) \) continuously onto itself.

Next we want to present some (functional analytic) characterizations of \( S(\mathbb{R}^q) \). For that purpose we introduce some terminology.

Let \( T_1, \ldots, T_q \) be mutually strongly commuting self-adjoint operators in a Hilbert space \( H \).

Then we write
\[
T = (T_1, \ldots, T_q),
\]
\[
T_k = T_1^{k_1} \cdot \ldots \cdot T_q^{k_q}, \quad |T| = (T \cdot T)^{1/2},
\]
\[
T^k = T_1^{k_1} \ldots T_q^{k_q}, \quad k \in \mathbb{N}_0^q,
\]
\[
BTC = (BT_1C, \ldots, BT_qC), \quad B, C \text{ operators in } H.
\]

The joint \( C^\infty \)-domain of \( T \), denoted by \( D^\infty(T) \), is defined by
\[
D^\infty(T) = \bigcap_{k \in \mathbb{N}_0^q} D(T^k).
\]  \hspace{1cm} (1.42)

The topology on \( D^\infty(T) \) is defined by the countable set of seminorms
\[
f \mapsto \|T^k f\|_H, \quad k \in \mathbb{N}_0^q.
\]

This topology is metrizable and gives \( D^\infty(T) \) the structure of a complete metric space.

For a unitary operator \( U \) on \( H \) we have
\[
D^\infty(U'TU'^{-1}) = U(D^\infty(T)).
\]  \hspace{1cm} (1.43)

We introduce some notations: Let \( j \in \{1, \ldots, q\} \).

\( F_j \) denotes the \( j \)-th partial Fourier transformation on \( L_2(\mathbb{R}^q) \), i.e. \( F_j \) is the unique unitary operator on \( L_2(\mathbb{R}^q) \) such that \( F_j f = \mathcal{F}_j f, \quad f \in S(\mathbb{R}^q) \).

\( \mathcal{F} \) denotes the Fourier transformation on \( L_2(\mathbb{R}^q) \), i.e. \( \mathcal{F} = F_1 \ldots F_q \).

\( P_j \) denotes the partial differentiation operator in \( L_2(\mathbb{R}^q) \) defined on the domain
\[
D(P_j) = \{ f \in L_2(\mathbb{R}^q) : f \text{ is absolutely continuous with respect to the } j\text{-th variable}, \partial_j f \in L_2(\mathbb{R}^q) \}
\]

by \( P_j f = i\partial_j f \).

\( Q_j \) denotes the multiplication operator in \( L_2(\mathbb{R}^q) \) defined on the domain
\[
D(Q_j) = \{ f \in L_2(\mathbb{R}^q) : z_j f \in L_2(\mathbb{R}^q) \}
\]

by \( Q_j f = z_j f \).

For \( f \in L_1(\mathbb{R}^q) \cap L_2(\mathbb{R}^q) \) we have for almost all \( x \in \mathbb{R}^q \),
\[
(\mathcal{F}_j f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^q} f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_q) e^{-ix_j y_j} dy_j,
\]
\[
(\mathcal{F} f)(x) = (2\pi)^{-q/2} \int_{\mathbb{R}^q} f(y) e^{-i(x-y) z} dy.
\]  \hspace{1cm} (1.44)
\[(F^*_j f) (x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_q) e^{i y x_j} dy, \]

\[(F^* f) (x) = (2\pi)^{-q/2} \int_{\mathbb{R}^q} f(y) e^{i x y} dy. \quad (1.45)\]

The operators \(P_j\) and \(Q_j\) are unbounded and self-adjoint and they satisfy the relations

\[P_j = F_j Q_j F^*_j = F Q_j F^*, \quad P = F Q F^*, \quad |P| = F |Q| F^*, \quad (1.46)\]

\[Q_j = -F_j P_j F^*_j = -F P_j F^*, \quad Q = -F P F^*, \quad |Q| = -F |P| F^*. \quad (1.47)\]

Thus it follows by means of (1.33) and (1.43) that

\[D^\infty (P_j) = F_j (D^\infty (Q_j)) = F (D^\infty (Q_j)), \quad D^\infty (P) = F (D^\infty (Q)), \]

\[D^\infty (|P|) = F (D^\infty (|Q|)), \quad (1.48)\]

\[D^\infty (Q_j) = F_j (D^\infty (P_j)) = F (D^\infty (P_j)), \quad D^\infty (Q) = F (D^\infty (P)), \]

\[D^\infty (|Q|) = F (D^\infty (|P|)). \quad (1.49)\]

It is not difficult to show that

\[D^\infty (Q) = D^\infty (|Q|) = \bigcap_{j=1}^q D^\infty (Q_j). \quad (1.50)\]

Hence, by use of (1.48), we also have

\[D^\infty (P) = D^\infty (|P|) = \bigcap_{j=1}^q D^\infty (P_j). \quad (1.51)\]

In the sequel \(\Delta\) denotes the Laplacian: \(\Delta = -|P|^2\). We give some classical analytic characterizations of \(\mathcal{S}(\mathbb{R}^q)\).

**Theorem 1.23** Let \(f \in C^\infty (\mathbb{R}^q)\). The following assertions are equivalent:

(i) \(f \in \mathcal{S}(\mathbb{R}^q)\);

(ii) \(x^k \partial^l f \in L_2 (\mathbb{R}^q), \quad k, l \in \mathbb{N}_0^q\);

(iii) \(x^k f \in L_2 (\mathbb{R}^q)\) and \(\partial^l f \in L_2 (\mathbb{R}^q), \quad k, l \in \mathbb{N}_0^q\);

(iv) \( |x|^k f \in L_2 (\mathbb{R}^q)\) and \( \Delta^l f \in L_2 (\mathbb{R}^q), \quad k, l \in \mathbb{N}_0\);

(v) \(x^k f \in L_2 (\mathbb{R}^q)\) and \(\partial^j f \in L_2 (\mathbb{R}^q), \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0\);

(vi) \(x^k f \in L_\infty (\mathbb{R}^q)\) and \(\partial^j f \in L_\infty (\mathbb{R}^q), \quad k, l \in \mathbb{N}_0^q\);

(vii) \(x^k f \in L_\infty (\mathbb{R}^q)\) and \(\partial^j f \in L_\infty (\mathbb{R}^q), \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0\).
Proof. We follow the scheme: (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v), (i) $\implies$ (vi) $\implies$ (vii) $\iff$ (v).

The assertions (iii), (iv), and (v) are mutually equivalent by (1.50) and (1.51); the implications (iii) $\implies$ (ii), (i) $\implies$ (vi) need no explanation. Thus it remains to be shown that (i) $\iff$ (ii), (iii) $\iff$ (ii) and (vii) $\iff$ (v).

Suppose $f \in S(\mathbb{R}^n)$. Then $x^k \partial^k f \in S(\mathbb{R}^n) \subset L_2(\mathbb{R}^n)$, $k, l \in \mathbb{N}_0^n$. So (i) $\iff$ (ii). Conversely, suppose $f$ satisfies assertion (ii). For $k, l \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ we have

$$(2\pi)^{n/2} |x^k \partial^k f(x)| \leq \int_{\mathbb{R}^n} |(\mathcal{F}(x^k \partial^k f))(y)| \, dy$$

$$= \int_{\mathbb{R}^n} \left| (\mathcal{F}((I - \Delta)^s (x^k \partial^k f))) (y) (1 + |y|^2)^{-s} \right| \, dy$$

$$\leq \|(I - \Delta)^s (x^k \partial^k f)\|_{L_2(\mathbb{R}^n)} \|1 + |y|^2\|^{-s}_{L_2(\mathbb{R}^n)}$$

That is $f \in S(\mathbb{R}^n)$.

Next suppose $f$ satisfies assertion (iii). For $k, l \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}} x^k_1 |f(x_1, y)|^2 \, dx_1 \right) \, dy < \infty \quad \text{and} \quad \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}} |(\partial_1 f)(x_1, y)|^2 \, dx_1 \right) \, dy < \infty.$$

So for almost all $y \in \mathbb{R}^{n+1}$,

$$\int_{\mathbb{R}} x^k_1 |f(x_1, y)|^2 \, dx_1 < \infty \quad \text{and} \quad \int_{\mathbb{R}} |(\partial_1 f)(x_1, y)|^2 \, dx_1 < \infty, \quad k, l \in \mathbb{N}_0.$$

That is, according to Theorem 1.2:

The function $x_1 \mapsto f(x_1, y) \in S(\mathbb{R})$ for almost all $y \in \mathbb{R}^{n+1}$.

Let $k \in \mathbb{N}_0^n$ and let $l = (k_2, \ldots, k_q)$. By applying integration by parts and the Cauchy-Schwarz inequality we deduce

$$\int_{\mathbb{R}^n} |x^k (\partial_1 f)(x)|^2 \, dx = \int_{\mathbb{R}^{n+1}} y^{2l} \left( \int_{\mathbb{R}} x^{2k_1} (\partial_1 f)(x_1, y) \frac{(\partial_1 f)}{(x_1, y)} \, dx_1 \right) \, dy$$

$$= \int_{\mathbb{R}^{n+1}} y^{2l} \left( \int_{\mathbb{R}} f(x_1, y) \left(2k_1 x_1^{2k_1-1} (\partial_1 f)(x_1, y) + x_1^{2k_1} (\partial_1^2 f)(x_1, y) \right) \, dx_1 \right) \, dy$$

$$\leq 2k_1 \|x_1^{-1} x^{2k} f\|_{L_2(\mathbb{R}^n)} \|\partial_1 f\|_{L_2(\mathbb{R}^n)} + \|x^{2k} f\|_{L_2(\mathbb{R}^n)} \|\partial_1^2 f\|_{L_2(\mathbb{R}^n)}.$$

Hence $\partial_1 f$ satisfies assertion (iii). Similarly, $\partial_j f$ satisfies assertion (iii), $j = 2, \ldots, q$.

Now an inductive procedure yields that $f$ satisfies assertion (ii).

Finally, suppose $f$ satisfies assertion (vii). For $y \in \mathbb{R}^{n+1}$ and $k, l \in \mathbb{N}_0$ we have

$$\sup_{x \in \mathbb{R}} |x^k f(x_1, y)| \leq \sup_{x \in \mathbb{R}^n} |x^k f(x)| < \infty,$$

$$\sup_{x \in \mathbb{R}} |(\partial_1 f)(x_1, y)| \leq \sup_{x \in \mathbb{R}^n} |(\partial_1 f)(x)| < \infty.$$

That is, according to Theorem 1.2:
The function \( x_1 \mapsto f(x_1, y) \in S(\mathbb{R}) \) for all \( y \in \mathbb{R}^{n-1} \).

Furthermore, we have \( f \in L_1(\mathbb{R}^n) \) because

\[
\int_{\mathbb{R}^n} |f(x)| \, dx \leq \sup_{x \in \mathbb{R}^n} |(1 + |x|^2)^{\eta} f(x)| \int_{\mathbb{R}^n} (1 + |x|^2)^{-\eta} \, dx.
\]

By use of the latter two results we estimate for \( k, l \in \mathbb{N}_0 \),

\[
\int_{\mathbb{R}^n} |x_1^k f(x)|^2 \, dx \leq \sup_{x \in \mathbb{R}^n} |x_1^k f(x)| \int_{\mathbb{R}^n} |f(x)| \, dx,
\]

\[
\int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \left( \left( \frac{\partial_{x_k}^l f}{x_1} \right)(x_1, y) \right)^2 \, dx_1 \right) \, dy = (-1)^l \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \left( \frac{\partial_{x_k}^l f}{x_1} \right)(x_1, y) \, dx_1 \right) \, dy
\]

\[
\leq \sup_{x \in \mathbb{R}^n} |(\partial_{x_k}^l f)(x)| \int_{\mathbb{R}^n} |f(x)| \, dx.
\]

Hence \( x_1^k f \in L_2(\mathbb{R}^n) \) and \( \partial_{x_k}^l f \in L_2(\mathbb{R}^n) \). Similarly \( x_1^k f \in L_2(\mathbb{R}^n) \) and \( \partial_{x_k}^l f \in L_2(\mathbb{R}^n) \), \( j = 2, \ldots, q \). Therefore, \( f \) satisfies assertion (v) and the proof is complete. \( \Box \)

As a straightforward consequence of Theorem 1.23 we obtain the following functional analytic characterization of \( S(\mathbb{R}^n) \):

**Theorem 1.24**

\[ S(\mathbb{R}^n) = D^\infty(\mathbb{Q}) \cap D^\infty(\mathbb{P}) = D^\infty(|\mathbb{Q}|) \cap D^\infty(|\mathbb{P}|) = \bigcap_{j=1}^q [D^\infty(Q_j) \cap D^\infty(P_j)] . \]

The equalities hold as topological vector spaces.

The operators \( F, F_j, P_j \) and \( Q_j \) map \( S(\mathbb{R}^n) \) continuously into itself. By Theorem 1.24 and the relations (1.48), (1.49) we have

\[ F_j(S(\mathbb{R}^n)) = S(\mathbb{R}^n) \]

\[ F_j(S(\mathbb{R}^n)) = S(\mathbb{R}^n) , \quad j = 1, \ldots, q . \]  \( (1.52) \)

The space \( S(\mathbb{R}^n) \) can be characterized by means of the (partial) Fourier transformation(s) (the proof runs similar to the proof of Theorem 1.5):

**Theorem 1.25** Let \( f \in L_2(\mathbb{R}^n) \) and let \( p \in \{2, \infty\} \). The following assertions are equivalent:

(i) \( f \in S(\mathbb{R}^n) \);

(ii) \( x_1^k f \in L_p(\mathbb{R}^n) \) and \( x_1^l F_j f \in L_p(\mathbb{R}^n) , \quad k, l \in \mathbb{N}_0^\circ \);

(iii) \( x_1^k f \in L_p(\mathbb{R}^n) \) and \( x_1^l F_j f \in L_p(\mathbb{R}^n) \), \( k, l \in \mathbb{N}_0 \);

(iv) \( x_1^k f \in L_p(\mathbb{R}^n) \) and \( x_1^j F_j f \in L_p(\mathbb{R}^n) \), \( j = 1, \ldots, q \), \( k, l \in \mathbb{N}_0 \).
The space $S(\mathbb{R}^o)$ can be characterized in terms of the Hermite expansion coefficients of its elements. The Hermite functions on $\mathbb{R}^o$,

$$\Psi_n(x) = \psi_{n_1}(x_1) \cdots \psi_{n_q}(x_q), \quad n \in \mathbb{N}_0^q$$  \hfill (1.53)

constitute an orthonormal basis in $L^2(\mathbb{R}^o)$ and they are eigenfunctions of the operator $|P|^2 + |Q|^2$, cf. (1.11),

$$|P|^2 + |Q|^2 \Psi_n = (2|n| + q) \Psi_n, \quad n \in \mathbb{N}_0^q, \quad |n| = n_1 + \ldots + n_q.$$  \hfill (1.54)

By applying the same techniques as in the proof of Theorem 1.6 we arrive at the announced characterization of the space $S(\mathbb{R}^o)$:

**Theorem 1.26**

(i) $S(\mathbb{R}^o) = \bigcap_{j=1}^q [D^\infty(Q_j) \cap D^\infty(P_j)] = \bigcap_{j=1}^q D^\infty(P_j^2 + Q_j^2) = D^\infty(|P|^2 + |Q|^2)$

$$= \{ f \in L^2(\mathbb{R}^o) : \langle n_j^k, f, \Psi_n \rangle_{L^2(\mathbb{R}^o)} \in l_\infty, \ k \in \mathbb{N}_0^0, \ j = 1, \ldots, q \},$$

$$= \{ f \in L^2(\mathbb{R}^o) : \langle |n|^k, f, \Psi_n \rangle_{L^2(\mathbb{R}^o)} \in l_\infty, \ k \in \mathbb{N}_0^0 \}.$$

The equalities hold as topological vector spaces.

(ii) For $f \in S(\mathbb{R}^o)$ we have

$$f(x) = \sum_{n \in \mathbb{N}_0^q} \langle f, \Psi_n \rangle_{L^2(\mathbb{R}^o)} \Psi_n(x), \quad x \in \mathbb{R}^o,$$

where the series converges in $S(\mathbb{R}^o)$. The series converges absolutely with respect to the $L^\infty(\mathbb{R}^o)$-norm and the series converges uniformly on $\mathbb{R}^o$.

1.5 **Spherical harmonics**

In this section we present an introduction to spherical harmonics. The results will be used in subsequent sections where we use expansions in spherical harmonics in $S(\mathbb{R}^o)$. We equip the vector space $\mathbb{R}^q$ with the Euclidean inner product $x \cdot y = x_1y_1 + \ldots + x_qy_q$, $x, y \in \mathbb{R}^q$. Let $e_1, \ldots, e_q$ be the standard basis in $\mathbb{R}^q$, that is, $(e_j)_k = \delta_{j,k}$, $j, k = 1, \ldots, q$. The unit sphere in $\mathbb{R}^q$ is denoted by $S^{q-1}$ and its elements by Greek symbols $\xi, \eta, \omega, \ldots$. Each $x \in \mathbb{R}^q \setminus \{0\}$ can be uniquely written as $x = r\omega$ with $r$ the length of $x$, $r = |x|$, and $\omega$ its direction, $\omega = x/r \in S^{q-1}$. Usually $r$ is called the radial part of $x$ and $\omega$ the spherical part of $x$.

There exists a measure $\sigma^{q-1}$ on the unit sphere $S^{q-1}$ such that

$$\int_{\mathbb{R}^q} f(x) dx = \int_{S^{q-1}} \int_{\mathbb{R}^q} f(r\omega) \sigma^{q-1}(d\omega) dr, \quad f \in L^1(\mathbb{R}^q), \quad q \geq 2.$$  \hfill (1.55)

We gather some well-known properties of the measure $\sigma^{q-1}$.

**Theorem 1.27** Let $q \geq 2$.

(i) Let $g \in L^1(S^{q-1})$. Then the following recursive integration formula holds for $q \geq 3$:
\[
\int_{S^{q-1}} g(\omega) \, d\sigma^{q-1}(\omega) = \int_{0}^{2\pi} \int_{0}^{\pi} g(\cos \varphi, \xi \sin \varphi) \sin^{q-2} \varphi \, d\varphi \, d\sigma^{q-2}(\xi),
\]

with
\[
\int_{S^{1}} g(\omega) \, d\sigma^{1}(\omega) = \int_{0}^{2\pi} g(\cos \varphi, \sin \varphi) \, d\varphi.
\]

(ii) The measure \(\sigma^{q-1}\) is a finite measure with
\[
\Omega_{q} = \sigma^{q-1}(S^{q-1}) = 2\pi^{q/2}/\Gamma(q/2).
\]

(iii) Let \(g \in L_{1}(S^{q-1})\) and let \(A \in \mathbb{R}^{n\times n}\) be an orthogonal matrix. Then
\[
\int_{S^{q-1}} g(A\omega) \, d\sigma^{q-1}(\omega) = \int_{S^{q-1}} g(\omega) \, d\sigma^{q-1}(\omega).
\]

(iv) Let \(h\) be a continuous function on \([-1, 1]\), let \(\xi \in S^{q-1}\) and let \(A \in \mathbb{R}^{n\times n}\) be an orthogonal matrix such that \(Ae_{1} = \xi\). Then
\[
\int_{S^{q-1}} h(\omega - \xi) \, d\sigma^{q-1}(\omega) = \Omega_{q-1} \int_{0}^{\pi} h(\cos \varphi) \sin^{q-2} \varphi \, d\varphi.
\]

**Proof.** (i) Let \(f(x) = \exp(-|x|)\) \(g(x/|x|)\), \(x \in \mathbb{R}^{n}\setminus\{0\}\). Then \(f \in L_{1}(\mathbb{R}^{n})\) and by (1.55)
\[
\int_{\mathbb{R}^{n}} f(x) \, dx = \Gamma(q) \int_{S^{q-1}} g(\omega) \, d\sigma^{q-1}(\omega).
\]

By using (1.55), with \(q\) replaced by \(q - 1\) and \(f\) by \(y \mapsto f(z_{1}, y)\), and by introducing polar coordinates in the \((z_{1}, r)\)-plane thereafter, we deduce
\[
\begin{align*}
\int_{\mathbb{R}^{n}} f(x) \, dx &= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} f(x_{1}, y) \, dy \right) \, dx_{1} \\
&= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \exp(-\sqrt{z_{1}^{2} + r^{2}}) \, g((z_{1}, r\omega)/\sqrt{z_{1}^{2} + r^{2}}) \, r^{q-2} \, dr \, d\sigma^{q-2}(\omega) \right) \, dx_{1} \\
&= \Gamma(q) \int_{0}^{\pi} \int_{S^{q-2}} g(\cos \varphi, \omega \sin \varphi) \sin^{q-2} \varphi \, d\varphi \, d\sigma^{q-2}(\omega).
\end{align*}
\]

By identifying these two results, we obtain the stated recursive integration formula. For \(g \in L_{1}(S^{1})\) we have, by (1.55) with \(q = 2\) and by using polar coordinates in \(\mathbb{R}^{2}\),
\[
\int_{S^{1}} g(\omega) \, d\sigma^{1}(\omega) = \int_{\mathbb{R}^{2}} \exp(-|x|) \, g(x/|x|) \, dx = \int_{0}^{2\pi} g(\cos \varphi, \sin \varphi) \, d\varphi.
\]

(ii) From assertion (i), with \(g(\omega) \equiv 1\), we obtain
\[ \Omega_q = \Omega_{q-1} \int_0^1 \sin^{q-2}(\varphi) \, d\varphi = \Omega_{q-1} \sqrt{\pi} \Gamma((q-1)/2) / \Gamma(q/2) \quad \text{with} \quad \Omega_2 = 2\pi. \]

Now an induction procedure yields the wanted result.

(iii) The assertion is a consequence of the property of the Lebesgue measure

\[ \int_{\mathbb{R}^q} f(Ax) \, dx = \int_{\mathbb{R}^q} f(x) \, dx, \quad f \in L_1(\mathbb{R}^q), \quad A \text{ an orthogonal matrix in } \mathbb{R}^{q \times q}. \]

(iv) Observe that \( \mathbf{A} \cdot \xi = \omega \cdot \mathbf{A} e_1 = \omega \cdot e_1, \omega \in S^{q-1} \). So by means of assertion (iii) with \( g(\omega) = h(\omega \cdot \xi) \), assertion (i) with \( g(\omega) = h(\omega \cdot e_1) \), and assertion (ii), we deduce

\[ \int_{S^{q-1}} h(\omega \cdot \xi) \, d\sigma^{q-1}(\omega) = \int_{S^{q-1}} h(\omega \cdot e_1) \, d\sigma^{q-1}(\omega) = \Omega_{q-1} \int_0^1 h(\cos \varphi) \sin^{q-2} \varphi \, d\varphi. \]

\[ \square \]

A polynomial \( p \) on \( \mathbb{R}^q \) is called harmonic if \( \Delta p = 0 \), and homogeneous of degree \( m \) (or \( m \)-homogeneous) if \( p(\lambda x) = \lambda^m p(x), \quad \lambda > 0, \quad x \in \mathbb{R}^q \).

From Müller [39, Definition 1] we quote the definition of spherical harmonics.

**Definition 1.28** A spherical harmonic of degree \( m \) in \( q \) dimensions is the restriction to \( S^{q-1} \) of an \( m \)-homogeneous harmonic polynomial on \( \mathbb{R}^q \).

By \( \mathcal{Y}_m^q \) we denote the space of all spherical harmonics of degree \( m \) in \( q \) dimensions. A spherical harmonic \( Y_m \in \mathcal{Y}_m^q \) has the properties

\[ Y_m(-\omega) = (-1)^m Y_m(\omega), \quad \omega \in S^{q-1}, \]

\[ Y_m \text{ has a unique harmonic extension } \bar{Y}_m \text{ defined by } \]

\[ \bar{Y}_m(r\omega) = r^m Y_m(\omega), \quad r > 0, \quad \omega \in S^{q-1}. \]

We equip \( \mathcal{Y}_m^q \) with the \( L_2(S^{q-1}) \)-inner product. In Faraut [21, pp. 89] the following well-known results have been proved.

**Theorem 1.29**

(i) The space \( \mathcal{Y}_m^q \) is a finite-dimensional Hilbert space of dimension \( N(q,m) \) given by

\[ N(q,m) = (2m + q - 2) \Gamma(m + q - 2) / (\Gamma(q - 1) \Gamma(m + 1)). \]

(ii) For \( k \neq l \), the spaces \( \mathcal{Y}_m^q \) and \( \mathcal{Y}_m^q \) are orthogonal subspaces of \( L_2(S^{q-1}) \) and the linear span of the spherical harmonics is dense in \( L_2(S^{q-1}) \):

\[ L_2(S^{q-1}) = \bigoplus_{m=0}^\infty \mathcal{Y}_m^q. \]

In the remainder of this section it is understood that \( q \geq 3 \).

A spherical harmonic \( Y_m \in \mathcal{Y}_m^q \) is called zonal with pole \( \xi \in S^{q-1} \) if for all orthogonal matrices \( A \in \mathbb{R}^{q \times q} \) with the property \( A \xi = \xi \) we have
\[
Y_m(\omega) = Y_m(\omega), \quad \omega \in S^{q-1}.
\]

It turns out that a zonal spherical harmonic can be expressed in terms of the Gegenbauer polynomial \( C_m^{q/2-1} \) defined by, cf. Erdélyi et al. [18, 11.1.2 (25)],

\[
C_m^{q/2-1}(x) = \frac{(-2)^m \Gamma(m + q/2 - 1) \Gamma(m + q - 2)}{m! \Gamma(q/2 - 1) \Gamma(2m + q - 2)} (1 - x^2)^{(q-3)/2} (d/dx)^{m} [(1 - x^2)^{(q-3)/2}], \quad |x| \leq 1.
\]

Indeed, in [18, 11.2 Lemma 1] the following well-known result has been proved:

**Lemma 1.30** Let \( Y_m \in \mathcal{Y}_m^\omega \) be a zonal spherical harmonic with pole \( \xi \in S^{r-1} \). Then there exists \( \lambda \in C \) such that

\[
Y_m(\omega) = \lambda C_m^{q/2-1}(\omega \cdot \xi), \quad \omega \in S^{r-1}.
\]

Of special interest is the Funk-Hecke theorem, see [18, p. 247]:

**Theorem 1.31** (Funk-Hecke) Let \( f \) be a continuous function on \([-1, 1]\) and let \( Y_m \in \mathcal{Y}_m^\omega \). Then

\[
\int \limits_{S^{q-1}} f(\omega \cdot \xi) Y_m(\omega) \, d\sigma^{q-1}(\omega) = \lambda_m Y_m(\xi), \quad \xi \in S^{r-1},
\]

with

\[
\lambda_m = \left( \Omega_{q-1}/C_m^{q/2-1}(1) \right) \int \limits_{-1}^{1} f(t) C_m^{q/2-1}(t) \left( 1 - t^2 \right)^{(q-3)/2} \, dt.
\]

We mention two applications of the Funk-Hecke theorem:

**Theorem 1.32** Let \( Y_m \in \mathcal{Y}_m^\omega \), let \( r > 0 \) and let \( \xi \in S^{r-1} \). Then

(i) \[
\int \limits_{S^{q-1}} C_m^{q/2-1}(\omega \cdot \xi) Y_m(\omega) \, d\sigma^{q-1}(\omega) = \left( \Omega_q C_m^{q/2-1}(1) / N(q,m) \right) Y_m(\xi),
\]

(ii) \[
\int \limits_{S^{r-1}} \exp(-ir(\omega \cdot \xi)) Y_m(\omega) \, d\sigma^{q-1}(\omega) = \left( -i \right)^m (2\pi)^{q/2} r^{q/2+n+1} J_{m+n/2-1}(r) Y_m(\xi).
\]

**Proof.** (i) Take \( f = C_m^{q/2-1} \) in Theorem 1.31. Then we obtain the stated integral formula by means of Theorem 1.29 (ii), Theorem 1.29 (i) and the formulas [18, 11.1.2 (26) and (28)]

\[
\int \limits_{-1}^{1} \left( C_m^{q/2-1}(t) \right)^2 \left( 1 - t^2 \right)^{(q-3)/2} \, dt = \frac{2^{q+r} \pi \Gamma(m + q - 2)}{m! (2m + q - 2) \left[ \Gamma(q/2 - 1) \right]^2}, \quad (1.57)
\]

\[
C_m^{q/2-1}(1) = \frac{\Gamma(m + q - 2)}{m! \Gamma(q - 2)}.
\]

(1.58)
(ii) Take \( f(t) = \exp(-irt) \) in Theorem 1.31. Then we obtain the stated integral formula by means of Theorem 1.27 (ii), relation (1.58) and the integral formula [38, p. 221]

\[
\int_0^1 \exp(iz \cos \theta) C_n^{m/2-1}(\cos \theta) \sin^{m/2-1} \, d\theta
\]

\[
= \frac{i^n \sqrt{\pi} \Gamma(q/2 - 1/2) \Gamma(m + q - 2)}{\Gamma(q/2) \Gamma(m + 1)} \left( \frac{2}{z} \right)^{q/2-1} J_{m+q/2-1}(z). \tag{1.59}
\]

\[ \square \]

Since \( \mathcal{Y}_m^\omega \) is a finite-dimensional Hilbert space, there exists a reproducing kernel \( K_m^\omega : S^{q-1} \times S^{q-1} \to C \) of \( \mathcal{Y}_m^\omega \), uniquely defined by the conditions:

1. For \( \xi \in S^{q-1} \) the function \( \omega \mapsto K_m^\omega(\omega, \xi) \) belongs to \( \mathcal{Y}_m^\omega \);
2. for \( \xi \in S^{q-1} \) and \( Y_m \in \mathcal{Y}_m^\omega \) we have \( Y_m(\xi) = \int_{S^{q-1}} Y_m(\omega) K_m^\omega(\omega, \xi) \, d\sigma^{q-1}(\omega) \);

see Aronszajn [1, pp. 343, 346]. From Lemma 1.30 and Theorem 1.32 (i) we deduce the explicit expression for the reproducing kernel \( K_m^\omega \) of \( \mathcal{Y}_m^\omega \):

\[
K_m^\omega(\omega, \xi) = (N(q, m) / \Omega_q) \, C_m^{q/2-1}(\omega, \xi), \quad \omega, \xi \in S^{q-1}. \tag{1.60}
\]

As a consequence we obtain the \( L_\infty \)-estimate for \( Y_m \in \mathcal{Y}_m^\omega \):

\[
|Y_m(\xi)|^2 \leq \|Y_m\|^2_{L_2(S^{q-1})} \|K_m^\omega(\cdot, \xi)\|_{L_2(S^{q-1})} = \left( \frac{N(q, m)}{\Omega_q} \right) \|Y_m\|^2_{L_2(S^{q-1})}, \quad \xi \in S^{q-1}. \tag{1.61}
\]

In the paper [2] by Van Berkel and Van Eijndhoven, various \( L_\infty \) and \( L_1 \) estimates for (partial derivatives of) spherical harmonics have been established. For later use, we mention one of them [2, Theorem 8(ii)]: For \( I \in \mathbb{N}_0^q \) and \( Y_m \in \mathcal{Y}_m^\omega \),

\[
|(|\partial I| Y_m)(\xi)|^2 \leq \frac{2^{||I||} N(q, m - |I|) \, m! \Gamma(m + q/2)}{\Omega_q (m - |I|)! \Gamma(m + q/2 - |I|)} \|Y_m\|^2_{L_2(S^{q-1})}, \quad \xi \in S^{q-1}. \tag{1.62}
\]

From Theorem 1.32 (ii) we obtain the Hecke-Bochner theorem, cf. [21, Théorème II.10]:

**Theorem 1.33 (Hecke-Bochner)** Let \( g : [0, \infty) \to C \) such that \( g \in L_1(\mathbb{R}^+; r^{m+q-1} \, dr) \). Let \( Y_m \in \mathcal{Y}_m^\omega \) and define \( f : \mathbb{R}^+ \to C \) by

\[
f(r\omega) = g(r) \hat{Y}_m(r\omega), \quad r \geq 0, \quad \omega \in S^{q-1}.
\]

Then the Fourier transform \( \mathcal{F} f \) is given by

\[
(\mathcal{F} f)(r\omega) = (-i)^m \left( \mathcal{H}_{m+q/2-1} g \right) (r) \hat{Y}_m(r\omega), \quad r \geq 0, \quad \omega \in S^{q-1}.
\]

Thus we see that the Fourier transform of a radially symmetric function \( f(r\omega) = g(r) \), with \( g \in L_1(\mathbb{R}^+; r^{q-1} \, dr) \), is also radially symmetric and can be expressed as a Hankel transform,

\[
(\mathcal{H} f)(r\omega) = \left( \mathcal{H}_{q/2-1} g \right) (r), \quad r \geq 0, \quad \omega \in S^{q-1}. \tag{1.63}
\]
The latter formula can also be derived without using the Funk-Hecke theorem. Indeed, it readily follows by use of (1.55), Theorem 1.27 (ii) and (iv), and (1.59) with \( m = 0 \).

Next we shall introduce an orthonormal basis in \( L_2(\mathbb{R}^q) \) of eigenfunctions of the operator \( |P|^2 + |Q|^2 \), which differs from the Hermite basis \( \Psi_n, n \in \mathbb{N}_0^q \), cf. (1.54). For that purpose we establish some auxiliary results. We define the differential operator \( x \cdot \nabla \) and the momentum operator \( M \) by

\[
  x \cdot \nabla = \sum_{j=1}^q x_j \partial_j \quad \text{and} \quad M = \sum_{1 \leq j < k \leq q} (x_j \partial_k - x_k \partial_j)^2.
\]

(1.64)

Lemma 1.34

(i) \( |x|^2 \Delta = (x \cdot \nabla)^2 + (q - 2) (x \cdot \nabla) + M \);

(ii) \( M \hat{Y}_m = -m(m + q - 2) \hat{Y}_m, \quad Y_m \in \mathcal{Y}_m^q \);

(iii) Let \( g \in C^2(\mathbb{R}) \), let \( h \in C^2(\mathbb{R}^q) \) and define \( f \in C^2(\mathbb{R}^q \setminus \{0\}) \) by \( f(x) = g(|x|) \cdot h(x) \), \( x \in \mathbb{R}^q \). Then for \( x \in \mathbb{R}^q \setminus \{0\} \),

\[
  (Mf)(x) = g(|x|) \cdot (Mh)(x).
\]

Proof. (i) The identity follows by a straightforward calculation.

(ii) Let \( Y_m \in \mathcal{Y}_m^q \). Since the polynomial \( Y_m \) is homogeneous of degree \( m \), we have \( (x \cdot \nabla) \hat{Y}_m = m \hat{Y}_m \). Now, since \( \hat{Y}_m \) is harmonic, it follows from assertion (i) that

\[
  0 = |x|^2 \Delta \hat{Y}_m = (m^2 + (q - 2)m + M) \hat{Y}_m.
\]

(iii) For \( j \in \{1, \ldots, q\} \) we have

\[
  (\partial_j f)(x) = (x_j/|x|) g'(|x|) \cdot h(x) + g(|x|) \cdot (\partial_j h)(x), \quad x \in \mathbb{R}^q \setminus \{0\}.
\]

Hence for \( 1 \leq j < k \leq q \),

\[
  ((x_j \partial_k - x_k \partial_j) f)(x) = g(|x|) \cdot ((x_j \partial_k - x_k \partial_j) h)(x), \quad x \in \mathbb{R}^q \setminus \{0\}.
\]

The wanted identity is now obvious. \( \Box \)

Let \( \{ Y_{m,j} : j = 1, \ldots, N(q, m) \} \) be an orthonormal basis in \( \mathcal{Y}_m^q \), fixed throughout the thesis.

Theorem 1.35 Define the functions \( u_{n,m,j} : \mathbb{R}^q \rightarrow \mathbb{C} \) by

\[
  u_{n,m,j}(r\omega) = \mathcal{F}^{n+q/2-1}(r) \hat{Y}_{m,j}(r\omega), \quad n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m), \quad r \geq 0, \quad \omega \in S^{q-1}.
\]

Then the functions \( u_{n,m,j} \) constitute an orthonormal basis in \( L_2(\mathbb{R}^q) \) and
\( M_{n,m,j} = -m(m + q - 2) \psi_{n,m,j}, \quad n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m) \)

(iii) \( \|u_{n,m,j}\|_{L^2(\mathbb{R}^d)}^2 \leq \left( \frac{2n + m + q - 1}{q - 1} \right)^2, \quad n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m) \).

**Proof.** The spherical harmonics \( Y_{n,j}, \quad m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m) \), constitute an orthonormal basis in \( L^2(\mathbb{S}^{d-1}) \), by Theorem 1.29 (ii). Likewise, for fixed \( m \in \mathbb{N}_0 \), the functions \( r \mapsto r^m \mathcal{L}_n^{m+i/2-1}(r) \), \( n \in \mathbb{N}_0 \), constitute an orthonormal basis in \( X_{q-1} \), by (1.12). By means of Parseval’s identity and Fubini’s theorem we have for \( f \in L^2(\mathbb{R}^d) \),

\[
\|f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left( \int_{\mathbb{S}^{d-1}} |f(r\omega)|^2 \, d\sigma^{d-1}(\omega) \right) r^{d-1} \, dr
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{N(q,m)} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(r\omega) \overline{Y_{n,j}(\omega)} \, d\sigma^{d-1}(\omega) r^{d-1} \, dr
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{N(q,m)} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(r\omega) \overline{Y_{n,j}(\omega)} \, d\sigma^{d-1}(\omega) r^m \mathcal{L}_n^{m+i/2-1}(r) \, dr
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{N(q,m)} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(r\omega) \mathcal{L}_n^{m+i/2-1}(r) \overline{Y_{n,j}(\omega)} \, dr \, d\sigma^{d-1}(\omega)
\]

Hence the functions \( u_{n,m,j}, \quad n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m) \) constitute an orthonormal basis in \( L^2(\mathbb{R}^d) \).

(i) This identity is a consequence of Lemma 1.34 (ii) and (iii).

(ii) Observe that \( x \cdot \nabla = r \partial_r \partial_r \). By means of Lemma 1.34 (i), identity (i) and Corollary 1.13 (ii), we deduce for \( n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m), \quad r > 0, \quad \omega \in \mathbb{S}^{d-1}, \)

\[
(\|P\|^2 + |Q|^2) u_{n,m,j} (r\omega)
\]

\[
= (-\partial^2 / \partial r^2 - (q - 1) r^{-1} \partial / \partial r - r^{-2} M + \partial^2) \mathcal{L}_n^{m+i/2-1}(r) \tilde{Y}_{n,j}(r\omega)
\]

\[
= \tilde{Y}_{n,j}(r\omega) r^{-m} (-\partial^2 / \partial r^2 - (q - 1) r^{-1} d/dr + m(m + q - 2) r^{-2} + r^2)
\]

\[
r^m \mathcal{L}_n^{m+i/2-1}(r)
\]

\[
= \tilde{Y}_{n,j}(r\omega) (-\partial^2 / \partial r^2 - (2m + q - 1) r^{-1} d/dr + r^2) \mathcal{L}_n^{m+i/2-1}(r)
\]

\[
= (4n + 2m + q) \mathcal{L}_n^{m+i/2-1}(r) \tilde{Y}_{n,j}(r\omega) = (4n + 2m + q) u_{n,m,j}(r\omega)
\]

(iii) Since both \( \{\Psi_k : k \in \mathbb{N}_0\} \) and \( \{u_{n,m,j} : n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m)\} \)

are orthonormal bases in \( L^2(\mathbb{R}^d) \) of eigenfunctions of the operator \( |P|^2 + |Q|^2 \), cf. (1.54) and identity (ii), we have for \( n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m) \),

\[
u_{n,m,j} = \sum_{k \in \mathbb{N}_0} (u_{n,m,j}, \Psi_k)_{L^2(\mathbb{R}^d)} \Psi_k
\]
and therefore, by the Cauchy-Schwarz inequality and the estimate (1.10),
\[ |u_{m,j}(x)|^2 \leq \|u_{m,j}\|_{L^2(\mathbb{R}^q)}^2 \sum_{k \in \mathbb{N}_0^q, |k| = 2n+m} |\Psi_k(x)|^2 \]
\[ \leq \sum_{k \in \mathbb{N}_0^q, |k| = 2n+m} 1 = \binom{2n + m + q - 1}{q - 1}, \quad x \in \mathbb{R}^q. \]

Finally, we introduce the Laplace-Beltrami operator $\Delta_{LB}$ in $L_2(S^{q-1})$, defined on the domain
\[ D(\Delta_{LB}) = \{ g \in L_2(S^{q-1}) : (m(m + q - 2) (g, Y_{m,j})_{L_2(S^{q-1})})_{m,j} \in L_2 \} \]
(1.67)
by
\[ \Delta_{LB} g = -\sum_{m=0}^{\infty} \sum_{j=1}^{N(q,m)} m(m + q - 2) (g, Y_{m,j})_{L_2(S^{q-1})} Y_{m,j}. \]
(1.68)

The operator $\Delta_{LB}$ is self-adjoint and due to Lemma 1.34 (ii) we have for $m \in \mathbb{N}_0$, $j = 1, \ldots, N(q,m)$,
\[ (\Delta_{LB} Y_{m,j})(\omega) = -m(m + q - 2) Y_{m,j}(\omega) = (M Y_{m,j})(\omega), \quad \omega \in S^{q-1}. \]
(1.69)

### 1.6 The space $S(\mathbb{R}^q)$ and expansions in spherical harmonics

We continue our discussion of the space $S(\mathbb{R}^q)$. First we characterize the radially symmetric functions in $S(\mathbb{R}^q)$. Thereafter, we consider products of radially symmetric functions and homogeneous harmonic polynomials in $S(\mathbb{R}^q)$. As an important result we show that a function $f \in S(\mathbb{R}^q)$ admits the following type of expansion,

\[ f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{N(q,m)} f_{m,j}(r) \tilde{Y}_{m,j}(r\omega), \quad r \geq 0, \quad \omega \in S^{q-1}, \]

where the functions $f_{m,j}$, $m \in \mathbb{N}_0$, $j = 1, \ldots, N(q,m)$, belong to $S_{even}(\mathbb{R})$ and where \{ $Y_{m,j} : j = 1, \ldots, N(q,m)$ \} is the orthonormal basis in $\mathcal{Y}_m$ introduced in Section 1.5; see Theorem 1.39.

In the second part of this section, we introduce the space $S(Z_q)$ and we identify $S(\mathbb{R}^q)$ with a subspace of $S(Z_q)$, see Theorem 1.43. By means of this characterization we can solve the following factorization problem in Theorem 1.44: For which functions $g$ and $h$ does the function $(r\omega) \mapsto g(r) h(\omega)$ belong to $S(\mathbb{R}^q)$?
The space $S(Z_q)$ reappears in the next section, where we study the Radon transformation on $S(\mathbb{R}^q)$.

**Theorem 1.36** Let $g : [0, \infty) \to C$ such that $g \in X_{q-1}$ and define the radially symmetric function $f : \mathbb{R}^q \to C$ by

\[ f(r\omega) = g(r), \quad r \geq 0, \quad \omega \in S^{q-1}. \]

Then $f \in S(\mathbb{R}^q)$ if and only if $g \in S_{even}(\mathbb{R})$. 
Proof. According to Theorem 1.25, the function \( f \) belongs to \( S(\mathbb{R}^d) \) if and only if
\[
|x|^k f \in L^p_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad |x|^l \mathcal{H}^l f \in L^p_{\text{loc}}(\mathbb{R}^d), \quad k, l \in \mathbb{N}_0.
\]
That is, by (1.63), \( f \in S(\mathbb{R}^d) \) if and only if
\[
r^k g \in L^p_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad r^l \mathcal{H}_{d/2-l}^l g \in L^p_{\text{loc}}(\mathbb{R}^d), \quad k, l \in \mathbb{N}_0.
\]
Now the wanted equivalence is obtained from the characterization of \( S_{\text{even}}(\mathbb{R}) \) in Theorem 1.12.

Theorem 1.37 Let \( g : [0, \infty) \to \mathbb{C} \) such that \( g \in X_{2m+q=1} \), let \( Y_m \in \mathcal{Y}_m \) and define \( f : \mathbb{R}^d \to \mathbb{C} \) by
\[
f(rw) = g(r) \hat{Y}_m(rw), \quad r \geq 0, \quad \omega \in S^{d-1}.
\]
Then \( f \in S(\mathbb{R}^d) \) if and only if \( g \in S_{\text{even}}(\mathbb{R}) \).

Proof. Suppose \( g \in S_{\text{even}}(\mathbb{R}) \). Define the radially symmetric function \( \hat{g} : \mathbb{R}^d \to \mathbb{C} \) by
\[
\hat{g}(rw) = g(r), \quad r \geq 0, \quad \omega \in S^{d-1}.
\]
Then \( f(x) = \hat{g}(x) Y_m(x), x \in \mathbb{R}^d \). Hence \( f \in S(\mathbb{R}^d) \), because \( Y_m \) is a polynomial and \( \hat{g} \in S(\mathbb{R}^d) \) by Theorem 1.36.

Conversely, suppose \( f \in S(\mathbb{R}^d) \). Since \( Y_m \) is bounded on \( S^{d-1} \), it follows from Theorem 1.25 and Theorem 1.33 (Hecke-Bochner), as in the proof of Theorem 1.36, that
\[
r^{k+m} g \in L^p_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad r^{k+m} \mathcal{H}_{d/2-l}^l g \in L^p_{\text{loc}}(\mathbb{R}^d), \quad k, l \in \mathbb{N}_0.
\]
Then we estimate for \( \rho > 0 \), by means of (1.16) and (1.21),
\[
|\mathcal{H}_{d/2-l}^l g(\rho)| = \int_0^\infty \rho^{-(m+q/2-1)} J_{m+q/2-1}(\rho r) g(r) r^{2m+q-1} dr 
\]
\[
\leq \sup_{\xi > 0} |\xi^{-(m+q/2-1)} J_{m+q/2-1}(\xi)| \int_0^\infty |g(r)| (1 + r^2) r^{2m+q-1} (1 + r^2)^{-1} dr 
\]
\[
\leq \frac{2^{-(m+q/2-1)}}{\Gamma(m + q/2)} \frac{\pi}{2} \sup_{r > 0} |g(r)| (1 + r^2) r^{2m+q-1}.
\]
That is, \( \mathcal{H}_{d/2-l}^l g \in L^p_{\text{loc}}(\mathbb{R}^d) \).

Observe that \( \mathcal{H}_{m+q/2-1}^l g = g \), by (1.20). So by replacing \( g \) by \( \mathcal{H}_{m+q/2-1}^l g \) in the above estimation, we also have \( g \in L^p_{\text{loc}}(\mathbb{R}^d) \). Thus it follows that
\[
r^k g \in L^p_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad r^l \mathcal{H}_{m+q/2-1}^l g \in L^p_{\text{loc}}(\mathbb{R}^d), \quad k, l \in \mathbb{N}_0.
\]
Then the characterization of \( S_{\text{even}}(\mathbb{R}) \) in Theorem 1.12 yields \( g \in S_{\text{even}}(\mathbb{R}) \). 

The space \( S(\mathbb{R}^d) \) can be characterized in terms of the expansion coefficients of its elements, with respect to the basis \( \{ u_{n,m,j}, \quad n, m \in \mathbb{N}_0, \quad j = 1, \ldots, \mathcal{N}(q, m) \} \), introduced in Theorem 1.35.
Theorem 1.38

(i) \( S(\mathbb{R}^q) = \{ f \in L_2(\mathbb{R}^q) : (n + m)^k (f, u_{n,m,j})_{L_2(\mathbb{R}^q)} u_{n,m,j} \in L_\infty, \ k \in \mathbb{N}_0 \} \).

(ii) For \( f \in S(\mathbb{R}^q) \) we have

\[
f(x) = \sum_{n,m=0}^{\infty} \sum_{j=1}^{N_{(n,m)}} (f, u_{n,m,j})_{L_2(\mathbb{R}^q)} u_{n,m,j}(x), \quad x \in \mathbb{R}^q,
\]

where the series converges in \( S(\mathbb{R}^q) \). The series converges absolutely with respect to the \( L_\infty(\mathbb{R}^q) \)-norm and the series converges uniformly on \( \mathbb{R}^q \).

Proof. (i) This assertion follows from the characterization of \( S(\mathbb{R}^q) \) in Theorem 1.26, viz.

\[ S(\mathbb{R}^q) = D^\infty(\mathbb{R}^q) \]

and the fact that the functions \( u_{n,m,j} \) constitute an orthonormal basis in \( L_2(\mathbb{R}^q) \) such that

\[
(\mathbb{R}^q) u_{n,m,j} = (m + 2n + q) u_{n,m,j}, \quad n, m \in \mathbb{N}_0, \ j = 1, \ldots, N(q, m),
\]

cf. Theorem 1.35 (ii).

(ii) By applying the same techniques as in the proof of Theorem 1.6 (ii), it follows that the mentioned series converges to \( f(x) \) in \( D^\infty(\mathbb{R}^q) \) = \( S(\mathbb{R}^q) \).

The remaining part of assertion (ii) readily follows from the estimate in Theorem 1.35 (iii): For \( n, m \in \mathbb{N}_0, \ j = 1, \ldots, N(q, m), \)

\[
\|u_{n,m,j}\|_{L_\infty(\mathbb{R}^q)} \leq \left( \frac{2n + m + q - 1}{q - 1} \right) \leq \frac{(2n + m + q - 1)^{r-1}}{(q - 1)!}.
\]

\[ \square \]

Theorem 1.39 Let \( f \in S(\mathbb{R}^q) \) and define the functions \( f_{m,j} : [0, \infty) \rightarrow C, m \in \mathbb{N}_0, \ j = 1, \ldots, N(q, m) \) by

\[
f_{m,j}(r) = \sum_{n=0}^{\infty} (f, u_{n,m,j})_{L_2(\mathbb{R}^q)} E_m^{r-1/2}(r), \quad r \geq 0.
\]

(1.70)

Then

(i) \( f_{m,j} \in \bar{S}_{\text{even}}(\mathbb{R}^q), \ m \in \mathbb{N}_0, \ j = 1, \ldots, N(q, m), \) and the series in (1.70) converges in \( \bar{S}_{\text{even}}(\mathbb{R}^q) \). The series converges absolutely with respect to the \( L_\infty(\mathbb{R}^q) \)-norm and the series converges uniformly on \( \mathbb{R}^q \);

(ii) \( f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{N_{(m,j)}} f_{m,j}(r) \bar{Y}_{m,j}(r\omega), \quad r \geq 0, \ \omega \in S^q \),

where the series converges in \( S(\mathbb{R}^q) \). The series converges absolutely with respect to the \( L_\infty(\mathbb{R}^q) \)-norm and the series converges uniformly on \( \mathbb{R}^q \);

(iii) \( r^m f_{m,j}(r) = \int_{S^{q-1}} f(r\omega) \bar{Y}_{m,j}(\omega) d\sigma^{q-1}(\omega), \quad r > 0, \ m \in \mathbb{N}_0, \ j = 1, \ldots, N(q, m). \)
Proof. (i) The assertion readily follows from Theorem 1.38 (i) and the characterization of $S_{even}(\mathcal{R})$ in Theorem 1.8.  
(ii) The assertion follows from (i) and Theorem 1.38 (ii).  
(iii) The assertion follows from (i) and the orthogonality relations for the functions $Y_{m,j}$.  

The remainder of this section deals with some preliminaries to the next section, where we study the Radon transformation on $S(\mathbb{R}^3)$.  
We introduce some notations.  
$Z_q$ denotes the unit cylinder in $\mathbb{R}^{q+1}$, defined by  
$$Z_q = \mathbb{R} \times S^{q-1}.$$  
$L_2(Z_q)$ denotes the Hilbert space of square integrable functions on $Z_q$ with inner product  
$$(f,g)_{L_2(Z_q)} = \int \int_{\mathbb{R} \times S^{q-1}} f(p,\omega) \overline{g(p,\omega)} \, dp \, d\omega.$$  
For $f \in L_2(\mathcal{R})$ and $g \in L_2(S^{q-1})$ we define $f \otimes g \in L_2(Z_q)$ by  
$$(f \otimes g)(p,\omega) = f(p) \, g(\omega), \quad (p,\omega) \in Z_q.$$  
For self-adjoint (unitary) operators $T_1$ and $T_2$ on $L_2(\mathcal{R})$ and $L_2(S^{q-1})$, respectively, we introduce the tensor product $T_1 \otimes T_2$ on $L_2(Z_q)$ as the unique self-adjoint (unitary) extension of the operator $T$, defined on the domain span$\{f \otimes g : f \in D(T_1), \ g \in D(T_2)\}$ by  
$$T(\sum_{j=1}^{n} c_j (f_j \otimes g_j)) = \sum_{j=1}^{n} c_j (T_1 f_j) \otimes (T_2 g_j),$$  
$n \in \mathbb{N}, \ c_j \in \mathbb{C}, \ f_j \in D(T_1), \ g_j \in D(T_2),$  
cf. Weidmann [50, Section 8.5].  
Since the Hermite functions $\psi_n$, $n \in \mathbb{N}_0$, constitute an orthonormal basis in $L_2(\mathcal{R})$ and since the spherical harmonics $Y_{m,j}$, $m \in \mathbb{N}_0$, $j = 1, \ldots, N(q,m)$, constitute an orthonormal basis in $L_2(S^{q-1})$, the functions $\psi_n \otimes Y_{m,j}$, $n,m \in \mathbb{N}_0$, $j = 1, \ldots, N(q,m)$, constitute an orthonormal basis in $L_2(Z_q)$.  
For the remainder of this section, the following operators are of special interest:  
$$\mathcal{F} \otimes I, \ Q \otimes I, \ \mathcal{P} \otimes I, \ (\mathcal{P}^2 + Q^2) \otimes I \ \text{and} \ I \otimes \Delta_{LB}.$$  
Let $f \in S(\mathbb{R}^{q+1})$ and let $g$ be the restriction of $f$ to $Z_q$. Then we observe that for $(p,\omega) \in Z_q$,  
$$(\mathcal{F} \otimes I) g)(p,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t,\omega) e^{-ipt} \, dt,$$  
$$(Q \otimes I) g)(p,\omega) = pg(p,\omega),$$  
$$(\mathcal{P} \otimes I) g)(p,\omega) = i(\partial/\partial p) \, g(p,\omega),$$  
$$(\mathcal{P}^2 + Q^2) \otimes I) g)(p,\omega) = (-\partial^2/\partial p^2 + p^2) \, g(p,\omega),$$  
$$(I \otimes \Delta_{LB}) g)(p,\omega) = (\Delta_{LB} g(p,\cdot)) (\omega).$$
Definition 1.40

\[ S(Z_t) = D^\infty(Q \otimes I) \cap D^\infty(P \otimes I) \cap D^\infty(I \otimes \Delta_{LB}) . \]

Let us give a motivation for this definition. Functions \( g \in S(Z_t) \) have the following properties:

1. For fixed \( \omega \in S^{n-1} \), the function \( p \mapsto g(p, \omega) \) belongs to \( S(\mathbb{R}) \) (because of Theorem 1.3 and \( g \in D^\infty(Q \otimes I) \cap D^\infty(P \otimes I) \));
2. For fixed \( p \in \mathbb{R} \), the function \( \omega \mapsto g(p, \omega) \) belongs to \( C^\infty(S^{n-1}) \) (because \( g \in D^\infty(I \otimes \Delta_{LB}) \)).

In fact, \( S(Z_t) \) consists of the restrictions of functions in \( S(\mathbb{R}^{n+1}) \) to \( Z_t \).

**Theorem 1.41**

(i) \( S(Z_t) = \bigcap_{k,l,m \in \mathbb{N}_0} D(Q^k P^l \otimes \Delta_{LB}^m) = D^\infty((P^2 + Q^2) \otimes I) \cap D^\infty(I \otimes \Delta_{LB}) \)

\[ = \{ g \in L_2(Z_t) : (n^m (g, \psi_n \otimes Y_{m,j}) L_2(z_t))_{n,m,j} \in l_\infty , \ k,l \in \mathbb{N}_0 \} . \]

The equalities hold as topological vector spaces.

(ii) For \( g \in S(Z_t) \) we have

\[ g(p, \omega) = \sum_{n,m=0}^{\infty} \sum_{j=1}^{N_{(p,j)}} (g, \psi_n \otimes Y_{m,j}) L_2(Z_t) \psi_n(p) Y_{m,j}(\omega) , \ (p, \omega) \in Z_t , \]

where the series converges in \( S(Z_t) \). The series converges absolutely with respect to the \( L_\infty(Z_t) \)-norm and the series converges uniformly on \( Z_t \).

**Proof.** (i) By using the characterizations of \( S(\mathbb{R}) \) in Theorem 1.2 and 1.6 (i) we derive

\[ \bigcap_{k,l,m \in \mathbb{N}_0} D(Q^k P^l \otimes \Delta_{LB}^m) = \bigcap_{k,l,m \in \mathbb{N}_0} (D(Q^k P^l \otimes I) \cap D(I \otimes \Delta_{LB}^m)) \]

\[ = \bigcap_{k,l,m \in \mathbb{N}_0} (D(Q^k \otimes I) \cap D(P^l \otimes I) \cap D(I \otimes \Delta_{LB}^m)) \]

\[ = D^\infty(Q \otimes I) \cap D^\infty(P \otimes I) \cap D^\infty(I \otimes \Delta_{LB}) = S(Z_t) \]

\[ = D^\infty((P^2 + Q^2) \otimes I) \cap D^\infty(I \otimes \Delta_{LB}) \]

\[ = \{ g \in L_2(Z_t) : (n^m (g, \psi_n \otimes Y_{m,j}) L_2(z_t))_{n,m,j} \in l_\infty , \ k,l \in \mathbb{N}_0 \} . \]

\[ = \{ g \in L_2(Z_t) : (n^m (g, \psi_n \otimes Y_{m,j}) L_2(z_t))_{n,m,j} \in l_\infty , \ k,l \in \mathbb{N}_0 \} . \]
(ii) By applying the same techniques as in the proof of Theorem 1.6 (ii), it follows that the mentioned series converges to \( g(p, \omega) \) in \( D(c) \) and \( \wedge D(L) \otimes \wedge D(L) = S(Z_\gamma) \).

By the inequalities (1.10) and (1.61) and by Theorem 1.29 (i) we have

\[
\| Y_{m-j} \|_{L_m(S^{r-1})}^2 \leq N(q, m)/\Omega_q
\]

\[
= \left(2m + q - 2\right) \Gamma(m + q - 2) / \left(\Omega_q \Gamma(q - 1) \Gamma(m + 1)\right)
\]

\[
\leq 2(m + q/2 - 1)^{r-2} / \left(\Omega_q \Gamma(q - 1)\right), \quad m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m).
\]

The remaining part of assertion (ii) readily follows from these estimates. \( \square \)

A function \( g \) is even on \( Z_\gamma \) if \( g(p, \omega) = g(-p, -\omega) \), \( (p, \omega) \in Z_\gamma \). The space \( S_{\text{even}}(Z_\gamma) \) is the subspace of even functions in \( S(Z_\gamma) \).

Let \( f \) be a function on \( \mathbb{R}^n \). Then \( Vf \) is the function on \( Z_\gamma \) defined by

\[
(V f) (p, \omega) = f(p\omega), \quad (p, \omega) \in Z_\gamma.
\]

(1.71)

Obviously, \( Vf \) is even on \( Z_\gamma \).

In the next theorem we show that the space \( S(\mathbb{R}^n) \) can be regarded as a subspace of \( S_{\text{even}}(Z_\gamma) \). Moreover, we establish the connection between the momentum operator \( M \) and the Laplace-Beltrami operator \( I \otimes \Delta_{LB} \), cf. (1.69).

**Theorem 1.42** Let \( f \) be a function on \( \mathbb{R}^n \).

(i) If \( f \in S(\mathbb{R}^n) \), then \( Vf \in S_{\text{even}}(Z_\gamma) \) and for \( l \in \mathbb{N}_0 \),

\[
(M^l f) (r\omega) = ((I \otimes \Delta_{LB}^l) V f) (r, \omega), \quad r > 0, \quad \omega \in S^{n-1}.
\]

(ii) If \( Vf \in S_{\text{even}}(Z_\gamma) \), then \( f \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and for \( l \in \mathbb{N}_0 \),

\[
(M^l f) (r\omega) = ((I \otimes \Delta_{LB}^l) V f) (r, \omega), \quad r > 0, \quad \omega \in S^{n-1}.
\]

**Proof.** (i) Suppose \( f \in S(\mathbb{R}^n) \). Then for \( k \in \mathbb{N} \) we have

\[
\int_{\mathbb{R}^n} \int_{S^{n-1}} |p^k f(p\omega)|^2 dp \, d\sigma^{n-1}(\omega)
\]

\[
= \int_{\mathbb{R}^n} \int_{S^{n-1}} (1 + p^2)^{-1} (1 + p^2)^{1/2} |f(p\omega)|^2 dp \, d\sigma^{n-1}(\omega)
\]

\[
\leq \pi \Omega_q \sup_{x \in \mathbb{R}^n} ((1 + |x|^2) |x|^2/f(x)|^2) < \infty.
\]

That is, \( Vf \in D^\infty(\mathbb{R}^n \setminus \{0\}) \).

Since \( ((\mathcal{P} \otimes I) V f)(p, \omega) = \omega_\gamma(\partial_\gamma f)(p\omega) + \ldots + \omega_q(\partial_q f)(p\omega) \), \( (p, \omega) \in Z_\gamma \), there exist constants \( a_{k,l} \in \mathbb{R} \), \( l \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) with \( |k| = l \), such that for \( l \in \mathbb{N} \),

\[
((\mathcal{P} \otimes I) V f)(p, \omega) = \sum_{k \in \mathbb{N}_0, |k| = l} a_{k,l} \omega^k(\partial_k f)(p\omega), \quad (p, \omega) \in Z_\gamma.
\]

(1.72)
For $m, n \in \mathbb{N}_0^2$ we have
\[
\int \int_{\mathbb{R}^2} |w^m(\partial^m f)(p\omega) \omega^n (\partial^n f)(p\omega)| \, dp \, d\sigma^{-1}(\omega) \leq \pi \Omega_q \sup_{\xi \in \mathbb{R}^n} ((1 + |\xi|^2) \, |(\partial^m f)(\xi) (\partial^n f)(\xi)|) < \infty.
\]
From the latter two results we conclude that $Vf \in D^\infty(P \otimes I)$. Thus we have shown that $Vf \in D^\infty(\mathbb{Q} \otimes I) \cap D^\infty(P \otimes I)$. This implies (by Theorem 1.3) that for fixed $\omega \in S^{r-1}$, the function $p \mapsto (Vf)(p, \omega)$ belongs to $S(\mathbb{K})$. By using Theorem 1.6 (ii), we then deduce for $r > 0, m \in \mathbb{N}_0, j = 1, \ldots, N(q, m),$
\[
\int_{S^{r-1}} f(\tau \xi) \, Y_{m,j}(\overline{\xi}) \, d\sigma^{-1}(\xi) = \int_{S^{r-1}} \left( \sum_{n=0}^{\infty} \left( \int_{\mathbb{R}} f(t \xi) \psi_n(t) \, dt \right) \psi_n(r) \right) \, Y_{m,j}(\overline{\xi}) \, d\sigma^{-1}(\xi)
\]
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \int_{S^{r-1}} f(t \xi) \psi_n(t) \, Y_{m,j}(\overline{\xi}) \, dt \, d\sigma^{-1}(\xi) \, \psi_n(r) = \sum_{n=0}^{\infty} (Vf, \psi_n \otimes Y_{m,j})_{L^2(S^{r-1})} \psi_n(r).
\]
The momentum operator $M$ maps $S(\mathbb{K})$ continuously into itself. Let $r > 0, \omega \in S^{r-1}$ and $l \in \mathbb{N}_0$. By means of Theorem 1.39 (ii), (iii), Lemma 1.34 (ii) and the previous result it follows that $(M^l f)(r \omega)$ equals
\[
(-1)^l \sum_{n=0}^{\infty} \sum_{m=0}^{N(q, m)} (m(m + q - 2))^l \int_{S^{r-1}} f(\tau \xi) \, Y_{m,j}(\overline{\xi}) \, d\sigma^{-1}(\xi) \, Y_{m,j}(\omega)
\]
\[
= (-1)^l \sum_{n=0}^{\infty} \sum_{m=0}^{N(q, m)} (m(m + q - 2))^l (Vf, \psi_n \otimes Y_{m,j})_{L^2(S^{r-1})} \psi_n(r) \, Y_{m,j}(\omega),
\]
where the series converges absolutely with respect to the $L^2(S^{r-1})$-norm.
Hence $Vf \in D^\infty(P \otimes \Delta_{LB})$ and
\[
(M^l f)(r \omega) = ((I \otimes \Delta_{LB}) Vf)(r, \omega), \quad r > 0, \quad \omega \in S^{r-1}.
\]
This completes the proof that $Vf \in S_{even}(Z_q)$.

(ii) Suppose $Vf \in S_{even}(Z_q)$. According to Theorem 1.41 (ii) we have
\[
f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{N(q, m)} (Vf, \psi_n \otimes Y_{m,j})_{L^2(S^{r-1})} \, |x|^{-m} \psi_n(|x|) \, Y_{m,j}(x), \quad x \in \mathbb{K} \setminus \{0\}.
\]
By means of the estimates (1.10) and (1.62), we have for $k, n, p \in \mathbb{N}_0, l \in \mathbb{N}_0$, $m \geq |l|, j = 1, \ldots, N(q, m),$
\[
\|r^k \psi_n^{(l)}\|_{L^2(R^n)} \leq 2^{k+p}(n + k + p)! / n! \leq 2^{k+p}(n + k + p)^{k+p} \cdot
\]
\[
\|r^k \hat{Y}_{m,j}\|_{L^2(S^{r-1})} \leq 2^{k+l} N(q, m - |l|) \, m! \, \Gamma(m + q/2) / \Omega_q (m - |l|) \, \Gamma(m + q/2 - |l|) \leq 2^{k+l}(m - |l| + q/2 - 1)^{l-1} \, m! \, (m + q/2 - 1)^{l-1} / (\Omega_q (q - 1)).
\]
Let $\varepsilon > 0$ and let $G_\varepsilon = \{ z \in \mathbb{R}^n : |z| > \varepsilon \}$. Then it follows from the previous estimates and Theorem 1.41 (i) that for $l \in \mathbb{N}_0^n$,

\[
\sum_{n,m=0}^{\infty} N_{(m,n)} \sum_{j=1}^{N_{(j,m)}} \| (Vf, \psi_n \otimes Y_{m,j})_{L^2(Z_4)} \partial_l^j (|z|^{-m} \psi_n(|z|) Y_{m,j}(z)) \|_{L^2(G_\varepsilon)} < \infty . \tag{1.73}
\]

Hence $f \in C^\infty(\mathbb{R}^n \setminus \{ 0 \})$ and, by Lemma 1.34, $(M^l f)(r \omega)$ equals

\[
(-1)^l \sum_{n,m=0}^{\infty} N_{(m,n)} \sum_{j=1}^{N_{(j,m)}} (m(m + q - 2))^l (Vf, \psi_n \otimes Y_{m,j})_{L^2(Z_4)} \psi_n(r) Y_{m,j}(\omega)
= (\{(l \otimes \Delta^l_{L^2}) Vf\}(r, \omega), \ r > 0, \ \omega \in S^{s-1}, \ l \in \mathbb{N}_0^n . \]

\[
\square
\]

Let $f$ be a function on $\mathbb{R}^n$ such that $Vf \in S_{\text{even}}(Z_4)$. Then we have proved in Theorem 1.42 (ii) that $f \in C^\infty(\mathbb{R}^n \setminus \{ 0 \})$. From the proof it is not hard to see that $x^k f$, $k \in \mathbb{N}_0^n$, is bounded on $\mathbb{R}^n$, while it follows from (1.73) that $\partial_l^j f$, $l \in \mathbb{N}_0^n$, is bounded on $G_\varepsilon = \{ z \in \mathbb{R}^n : |z| > \varepsilon \}$, $\varepsilon > 0$. Hence, we may conclude from Theorem 1.23 that $f \in S(\mathbb{R}^n)$ if and only if $f$ is infinitely differentiable at $0$. An additional condition on $Vf$ is needed to ensure that $f$ is infinitely differentiable at $0$.

**Theorem 1.43** Let $f$ be a function on $\mathbb{R}^n$. Then $f \in S(\mathbb{R}^n)$ if and only if

(i) $Vf \in S_{\text{even}}(Z_4)$ and

(ii) for $l \in \mathbb{N}_0^n$, the function $\omega \mapsto ((F^l \otimes I) Vf)(0, \omega)$ is the restriction to $S^{s-1}$ of a homogeneous polynomial of degree $l$ on $\mathbb{R}^n$.

**Proof.** Suppose $f \in S(\mathbb{R}^n)$. Then $f$ satisfies the conditions (i) and (ii) by Theorem 1.42 (i) and the identity (1.72).

Conversely, suppose $Vf$ satisfies the conditions (i) and (ii). We shall prove that

$$
|x|^k f \in L_2(\mathbb{R}^n) \quad \text{and} \quad \Delta^l f \in L_2(\mathbb{R}^n), \quad k, l \in \mathbb{N}_0^n .
$$

Since $Vf \in S(Z_4) \subset D^\infty(Q \otimes I)$, it follows that $|x|^k f \in L_2(\mathbb{R}^n)$, $k \in \mathbb{N}_0^n$.

Let $l \in \mathbb{N}_0^n$. We show that $\Delta^l f \in L_2(\mathbb{R}^n)$ by a separate investigation of the two integrals

\[
\int_1^{\infty} \int_{S^{s-1}} |(\Delta^l f)(r \omega)|^2 r^{s-1} \, dr \, d\sigma^{s-1}(\omega) ,
\]

\[
\int_0^{1} \int_{S^{s-1}} |(\Delta^l f)(r \omega)|^2 r^{s-1} \, dr \, d\sigma^{s-1}(\omega) . \tag{1.74}
\]

From the proof of Theorem 1.42 (ii) it follows that $|x|^{(s+1)/2} \Delta^l f$, $l \in \mathbb{N}_0^n$, is bounded on $\{ x \in \mathbb{R}^n : |x| > 1 \}$. Therefore, the first integral of (1.74) is finite.

Next we consider the second integral of (1.74). Inspired by Helgason's paper [27, p. 162], we introduce the functions $e_m$, $m \in \mathbb{N}_0^n$, defined by

$$
e_m(t) = \sum_{k=m}^{\infty} (-it)^k / k! , \quad t \in \mathbb{R} , \quad m \in \mathbb{N}_0^n .$$
These functions have the obvious properties
\[ c_m'(t) = -i c_{m-1}(t), \quad t \in \mathbb{R}, \quad m \geq 1; \] (1.75)
\[ c_m(t) = e^{-\omega t} - \sum_{k=0}^{m-1} \frac{(-it)^k}{k!}, \quad t \in \mathbb{R}, \quad m \in \mathbb{N}_0; \] (1.76)
\[ |t^{-m} c_m(t)| \leq e + 1, \quad t \in \mathbb{R}, \quad m \in \mathbb{N}_0. \] (1.77)

By (1.76) we have for \( r > 0, \omega \in S^{l-1}, \)
\[ f(r\omega) = (Vf)(r,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(t\omega + \omega r)} ((P^* \otimes I) Vf)(t,\omega) \, dt \]
\[ = \sum_{k=0}^{l-1} \frac{(-ir)^k}{k!}((P^* \otimes I) Vf)(0,\omega) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e_{2l}(rt) ((P^* \otimes I) Vf)(t,\omega) \, dt. \]

By condition (ii), the sum in the latter expression forms a polynomial \( p \) on \( \mathbb{R}^d \) of degree \( 2l - 1 \), so that \( \Delta p = 0. \) Hence, by means of Lemma 1.34 (i) (with \( z \cdot \nabla = r \partial / \partial r \)) and Theorem 1.42 (ii), we find
\[ (\Delta^l f)(r\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \partial^2 / \partial r^2 + (q - 1)r^{-2}\partial / \partial r + r^{-2}\Delta_{LB} \right)^l e_{2l}(rt) ((P^* \otimes I) Vf)(t,\omega) \, dt. \]

By (1.75) and (1.77), there exist coefficients \( b_{j,k,l} \in C, j = 0, \ldots, l, k = 0, \ldots, 2l, \) such that for \( r > 0, \omega \in S^{l-1}, \)
\[ ||(\Delta^l f)(r\omega)|| \leq \sum_{j=0}^{l} \sum_{k=0}^{2l-j} b_{j,k,l} \int_{\mathbb{R}} t^2 (\partial^2 / \partial r^2) e_{2l-k}(rt) ((P^* \otimes \Delta_{LB}^l) Vf)(t,\omega) \, dt \]
\[ \leq (e + 1) \sum_{j=0}^{l} \sum_{k=0}^{2l-j} |b_{j,k,l}| \int_{\mathbb{R}} \left| ((Q^{2l} P^* \otimes \Delta_{LB}^l) Vf)(t,\omega) \right| \, dt. \]

Since \( (Q^{2l} P^* \otimes \Delta_{LB}^l) Vf \in S(Q^l), \quad j = 0, \ldots, l, \) it follows that the second integral of (1.74) is finite. Hence \( \Delta^l f \in L_2(\mathbb{R}^d), \quad l \in \mathbb{N}_0. \)

We conclude that \( f \in S(\mathbb{R}^d), \) by Theorem 1.23. \( \square \)

Let \( \mathcal{E}_m \) be the orthogonal projection of \( L_2(S^{l-1}) \) onto \( Y^m_0, \) that is
\[ \mathcal{E}_m h = \sum_{j=1}^{N(m,\omega)} (h, Y_{m,j})_{L_2(S^{l-1})} Y_{m,j}, \quad h \in L_2(S^{l-1}). \] (1.78)

We recall the definition of the space \( S_-(\mathbb{R}^d), \)
\[ S_-(\mathbb{R}^d) = \{ g \in S(\mathbb{R}^d) : g(0+)(0+) = 0, \quad l \in \mathbb{N}_0 \}. \]

We come to the following factorization result in \( S(\mathbb{R}^d). \)
Theorem 1.44 Let $g : [0, \infty) \to C$ such that $g \in X_{\nu-1}$, let $h : S^{\nu-1} \to C$ such that $h \in L_2(S^{\nu-1})$ and define $f : R^d \to C$ by

$$f(r \omega) = g(r \cdot h(\omega), \ r \geq 0, \ \omega \in S^{\nu-1}.$$ (i) If $h = \sum_{m=0}^{M} E_{2m} h$ with $E_{2m} h \neq 0$,
then $f \in S(S^{\nu})$ if and only if $g \in Q^{2M}(S_{even}(R))$.

(ii) If $h = \sum_{m=0}^{M} E_{2m+1} h$ with $E_{2m+1} h \neq 0$,
then $f \in S(S^{\nu})$ if and only if $g \in Q^{2M+1}(S_{even}(R))$.

(iii) If $h$ cannot be represented by a finite expansion as in (i) or (ii),
then $f \in S(S^{\nu})$ if and only if $g \in S_{m}(R^n)$ and $h \in D_m(\Delta_{LB})$.

Proof. By means of Corollary 1.16 and Theorem 1.43 we have $f \in S(S^{\nu})$ if and only if $g \in S(S^{\nu})$, $h \in D_m(\Delta_{LB})$ and, for $l \in N_0$, the function $\omega \mapsto g^{l}(0+) h(\omega)$ is the restriction to $S^{\nu-1}$ of a homogeneous polynomial of degree $l$ on $R^d$.

Now the theorem follows by using the characterization of $Q^{n}(S_{even}(R))$ in Theorem 1.6a (ii) and by observing that $Y_{m,l}$ can be extended to a homogeneous polynomial of degree $m + 2k$, $k \in N_0$, on $R^d$.

1.7 The Radon transformation on $S(R^d)$

The Radon transform of a function on $R^d$ is defined as the set of its integrals over the hyperplanes in $R^d$. This transform has its origin in Radon's paper [44] where he determines a function on $R^d$ from its line integrals. Nowadays, the Radon transformation has found spectacular applications in medical imaging.

We shall study the Radon transformation from a theoretical point of view. As general references we recommend Ludwig [37], Helgason [28], Deans [9] and Natterer [40].

We introduce some notations.

The notation $H = H(p, \omega)$ with $(p, \omega) \in Z_\nu$ stands for the hyperplane in $R^d$ given by

$$H = H(p, \omega) = \{z \in R^d : z \cdot \omega = p\}.$$ (1.79)

Let $f$ be a function on $R^d$, integrable over each hyperplane in $R^d$. For $(p, \omega) \in Z_\nu$, the Radon transform $(R_f)(p, \omega)$ is the integral of $f$ over the hyperplane $H(p, \omega)$,

$$(R_f)(p, \omega) = \int_{H(p,\omega)} f(z) \, d\mu(z),$$ (1.79)

where $\mu$ is the Euclidean measure on the hyperplane $H(p, \omega)$.

The Radon transform $R_f$ is even on $Z_\nu$ because $H(-p, -\omega) = H(p, \omega)$, $(p, \omega) \in Z_\nu$. For $\omega \in S^{\nu-1}$, let $A_\omega \in R^{d \times d}$ be an orthogonal matrix such that $A_\omega e_1 = \omega$. We then have

$$(R_f)(p, \omega) = \int_{R^{d}} f(A_\omega(p, y)) \, dy, \quad (p, \omega) \in Z_\nu,$$ (1.80)
Lemma 1.45 Let $f \in S(\mathbb{R}^n)$ and let $h : \mathbb{R} \to C$ such that $h \in L_\infty(\mathbb{R})$. Then for $\omega \in S^{n-1},$
\[ \int_{\mathbb{R}} h(p) (\mathcal{R}f)(p, \omega) \, dp = \int_{\mathbb{R}^n} h(x \cdot \omega) f(x) \, dx. \]

Proof. For $\omega \in S^{n-1}$ we have by (1.80),
\[ \int_{\mathbb{R}} h(p) (\mathcal{R}f)(p, \omega) \, dp = \int_{\mathbb{R}} h(p) \left( \int_{\mathbb{R}^{n-1}} f(A_\omega(p, y)) \, dy \right) \, dp. \]

Now we obtain the wanted identity by the substitution $x = A_\omega(p, y), p = x \cdot \omega, dp \, dy = dx$. \qed

An important result is the so-called projection theorem, which provides the fundamental relationship between the Radon transformation and the Fourier transformation, cf. Ludwig [37, (1.3)].

Theorem 1.46 Let $f \in S(\mathbb{R}^n)$. Then
\[ \mathcal{R}f = (2\pi)^{(n-1)/2} (\mathcal{F}^* \otimes I) \mathcal{V}F f. \]

Proof. For $(t, \omega) \in Z_0$ we deduce by means of Lemma 1.45 with $h(p) = \exp(-ipt),$
\[ \langle (\mathcal{F} \otimes I) \mathcal{R}f \rangle (t, \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}f)(p, \omega) \exp(-ipt) \, dp \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f(x) \exp(-it(x \cdot \omega)) \, dx = (2\pi)^{(n-1)/2} (\mathcal{F}f)(t, \omega) \]
\[ = (2\pi)^{(n-1)/2} (\mathcal{V}F f)(t, \omega) \]
where the operator $\mathcal{V}$ is defined by (1.71). \qed

The range of the Radon transformation when applied to $S(\mathbb{R}^n)$ has been characterized by Helgason [27, Theorem 4.1] by means of differential forms. Independently, Ludwig derived the same characterization by means of expansions in spherical harmonics, see [37, Theorem 2.1]. However, as observed by Helgason [28, p. 72], a crucial point in Ludwig's proof is missing and seems difficult to be settled in Ludwig's context.

The characterization of $\mathcal{V}(S(\mathbb{R}^n))$ as a subspace of $L_0(Z_0)$, implicit in Theorem 1.43, together with Theorem 1.46 immediately lead to the following characterization of $\mathcal{R}(S(\mathbb{R}^n)).$

Theorem 1.47 Let $g$ be a function on $Z_0$. Then $g \in \mathcal{R}(S(\mathbb{R}^n))$ if and only if
(i) $g \in S_{even}(Z_0)$ and
(ii) $g$ obeys the so-called Helgason-Ludwig consistency conditions: For $l \in \mathcal{N}_0$, the function
\[ \omega \mapsto \int_{\mathbb{R}} t^l g(t, \omega) \, dt \]
is the restriction to $S_0^{-1}$ of a homogeneous polynomial of degree $l$ on $\mathbb{R}^n$.\]
Proof. Since $F(S(R^n)) = S(R^n)$, see (1.52), we deduce by means of Theorems 1.46 and 1.48,

$$\mathcal{R}(S(R^n)) = \left( F^* \otimes I \right) V(S(R^n)) = \left( F^* \otimes I \right) \left\{ \{ h \in S_{\text{even}}(Z) : \text{for } l \in \mathbb{N}_0, \text{the function } \omega \mapsto ((P^l \otimes I) h) (0, \omega) \text{ is the restriction to } S^{r-1} \text{ of a homogeneous polynomial of degree } l \text{ on } R^n \} \right\}$$

$$= \{ g \in S_{\text{even}}(Z) : \text{for } l \in \mathbb{N}_0, \text{the function } \omega \mapsto ((P^l F^* \otimes I) g) (0, \omega) \text{ is the restriction to } S^{r-1} \text{ of a homogeneous polynomial of degree } l \text{ on } R^n \} .$$

The following observation completes the proof,

$$((P^l F^* \otimes I) g) (0, \omega) = ((FQ^l \otimes I) g) (0, \omega) = \frac{1}{\sqrt{2\pi}} \int_R t^l g(t, \omega) dt .$$

Our next aim is to derive Radon inversion formulas. To that end we introduce some operators. $M_{\text{sgn}}$ denotes the operator of multiplication by $\text{sgn}(x)$ on $L_2(R)$ defined by

$$(M_{\text{sgn}} f) (x) = \text{sgn}(x) f(x), \ x \in R, \ f \in L_2(R) , \ (1.81)$$

where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 , \\ -1 & \text{if } x < 0 . \end{cases} \ (1.82)$$

$\mathcal{H}$ denotes the Hilbert transformation on $L_2(R)$ defined by

$$\mathcal{H} = -i F^* M_{\text{sgn}} F . \ (1.83)$$

Lemma 1.48

(i) $(s\mathcal{H})^2 = I$ ,

(ii) $P\mathcal{H} = \mathcal{H}P$ ,

(iii) for $f \in S(R)$ we have

$$\left( \mathcal{H} f \right) (x) = \frac{1}{2\pi} \int_R p^{-1} (f(x - p) - f(x + p)) dp = \frac{1}{\pi} \int_R \left( x - p \right)^{-1} f(p) dp , \ x \in R ,$$

where $\int$ denotes that the Cauchy principal value is to be taken.

Proof. (i), (ii) The equalities are evident.

(iii) As a preliminary we present the well-known integral formula

$$\frac{1}{\pi} \int_{-\infty}^{\infty} p^{-1} \sin(\xi p) dp = \text{sgn}(\xi) , \ \xi \in R \setminus \{0\} .$$

For $f \in S(R)$ we derive, by using Lebesgue's dominated convergence theorem and Fubini's theorem,
\[ (\mathcal{H} f) (x) = \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}} \text{sgn}(\xi) \langle \mathcal{F} f \rangle (\xi) e^{ix\xi} d\xi \]

\[ = \frac{-i}{\pi \sqrt{2\pi}} \lim_{N \to \infty} \int_{-N}^{N} \left( \int_{-N}^{N} p^{-1} \sin(\xi p) \, dp \right) \langle \mathcal{F} f \rangle (\xi) e^{ix\xi} d\xi \]

\[ = \frac{1}{2\pi \sqrt{2\pi}} \lim_{N \to \infty} \int_{-N}^{N} p^{-1} \left( \int_{\mathbb{R}} \langle \mathcal{F} f \rangle (\xi) \left( e^{i(x-s)\xi} - e^{i(x+s)\xi} \right) d\xi \right) dp \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} p^{-1} (f(x-p) - f(x+p)) \, dp = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\mathbb{R}} p^{-1} (f(x-p) - f(x+p)) \, dp \]

\[ = \frac{1}{\pi} \lim_{\epsilon \to 0} \left( \int_{-\infty}^{\epsilon} (x-s)^{-1} f(s) \, ds + \int_{\epsilon}^{\infty} (x-s)^{-1} f(s) \, ds \right) \]

\[ = \frac{1}{\pi} \int_{\mathbb{R}} (x-p)^{-1} f(p) \, dp, \quad x \in \mathbb{R}. \]

We regard \( \mathcal{R} \) as a densely defined operator from \( L_2(\mathbb{R}^d) \) into \( L_2(\mathbb{R}) \), with domain \( S(\mathbb{R}^d) \). For the adjoint operator \( \mathcal{R}^* \) we have the following result.

**Lemma 1.49** Let \( g : \mathbb{Z} \to \mathbb{C} \) such that \( g \in L_2(\mathbb{Z}) \) and, for \( \omega \in S^{-1} \), the function \( p \mapsto g(p, \omega) \in L_{\infty}(\mathbb{R}) \). Then \( g \in D(\mathcal{R}^*) \) and

\[ (\mathcal{R}^* g)(x) = \int_{S^{-1}} g(x \cdot \omega, \omega) \, d\sigma^{-1} (\omega), \quad x \in \mathbb{R}^d. \]

**Proof.** Let \( f \in S(\mathbb{R}^d) \). By Lemma 1.45 with \( h(p) = g(p, \omega) \), we have

\[ \int_{S^{-1}} \left( \int_{\mathbb{R}} (\mathcal{R} f)(p, \omega) g(p, \omega) \, dp \right) d\sigma^{-1} (\omega) = \int_{\mathbb{R}^d} f(x) \left( \int_{S^{-1}} g(x \cdot \omega, \omega) \, d\sigma^{-1} (\omega) \right) dx. \]

Now we come to the following Radon inversion formula, cf. [37, Theorem 1.1].

**Theorem 1.50** Let \( f \in S(\mathbb{R}^d) \). Then

(i) \( f = \frac{1}{2(2\pi)^{1-d}} \mathcal{R}^* (i\mathcal{H})^{-1} (\mathcal{R})^{-1} (I \mathcal{R} f) \)

\[ = \frac{1}{2(2\pi)^{1-d}} \begin{cases} \mathcal{R}^* (\mathcal{H}^{-1} (\mathcal{R})^{-1} (I \mathcal{R} f) \mathcal{R} f) & \text{for } q \text{ odd}, \\ \mathcal{R}^* (i\mathcal{H}^{-1} (\mathcal{R})^{-1} (I \mathcal{R} f) \mathcal{R} f) & \text{for } q \text{ even}. \end{cases} \]
For \( q \) odd we have
\[
f(x) = \frac{1}{2} (2\pi r)^{1-q} \int_{S^{q-1}} ((\partial/\partial p)^{q-1} \mathcal{R} f)(x \cdot \omega, \omega) \, d\sigma^{q-1}(\omega), \quad x \in \mathbb{R}^q.
\]
For \( q \) even we have
\[
f(x) = 2(2\pi r)^{-q} \int_{\mathbb{R}^q} p^{-1} (\partial/\partial p)^{q-1} \left( \int_{S^{q-1}} (\mathcal{R} f)(x \cdot \omega + p, \omega) \, d\sigma^{q-1}(\omega) \right) \, dp, \quad x \in \mathbb{R}^q.
\]

**Proof.** (i) By means of Lemma 1.49 we deduce for \( x \in \mathbb{R}^q \),
\[
f(x) = (2\pi)^{-q/2} \int_{S^{q-1}} \left( \int_{\mathbb{R}^q} r^{q-1}(\mathcal{R} f)(r, \omega) \exp(ir(x \cdot \omega)) \, dr \right) \, d\sigma^{q-1}(\omega)
\]
\[
= \frac{1}{2} (2\pi)^{-q/2} \int_{S^{q-1}} \left( \int_{\mathbb{R}^q} \text{sgn}(r) r^{q-1}(\mathcal{V} \mathcal{F} f)(r, \omega) \exp(ir(x \cdot \omega)) \, dr \right) \, d\sigma^{q-1}(\omega)
\]
\[
= \frac{1}{2} (2\pi)^{-q/2} (\mathcal{R}^* (\tilde{F}^* M_{\text{sgn}}^{-1} Q^{q-1} \otimes I) \mathcal{V} \mathcal{F} f)(x).
\]
The present result can be rewritten as
\[
f = \frac{1}{2} (2\pi)^{-(q-1)/2} \mathcal{R}^* (\tilde{F}^* M_{\text{sgn}}^{-1} E) (\tilde{F}^* Q^{q-1} \mathcal{E}) \otimes I) (\tilde{F}^* \otimes I) \mathcal{V} \mathcal{F} f.
\]
Since \( P = -\tilde{F}^* Q \mathcal{F} \) and \( \mathcal{H} = -i \tilde{F}^* M_{\text{sgn}} \mathcal{F} \), cf. (1.5) and (1.83), we obtain the wanted equalities from Theorem 1.46 and Lemma 1.48 (i).

(ii) For \( q \) odd, the integral expression follows from assertion (i) and Lemma 1.49.
For \( q \) even, the function \( g := ((-P)^{q-1} \otimes I) \mathcal{R} f \) is odd on \( Z_q \). That is, \( g(-p, -\omega) = -g(p, \omega), (p, \omega) \in Z_q \). By means of Lemmas 1.48 and 1.49 we deduce for \( x \in \mathbb{R}^q \),
\[
(\mathcal{R}^* i \mathcal{H} ((-P)^{q-1} \otimes I) \mathcal{R} f)(x)
\]
\[
= \frac{i}{2\pi} \int_{S^{q-1}} \left( \int_{\mathbb{R}^q} p^{-1} (g(x \cdot \omega - p, \omega) - g(x \cdot \omega + p, \omega)) \, dp \right) \, d\sigma^{q-1}(\omega)
\]
\[
= \frac{-i}{\pi} \int_{\mathbb{R}^q} p^{-1} \left( \int_{S^{q-1}} g(x \cdot \omega + p, \omega) \, d\sigma^{q-1}(\omega) \right) \, dp
\]
\[
= \frac{2(-i)^q}{\pi} \int_{\mathbb{R}^q} p^{-1} (\partial/\partial p)^{q-1} \left( \int_{S^{q-1}} (\mathcal{R} f)(x \cdot \omega + p, \omega) \, d\sigma^{q-1}(\omega) \right) \, dp.
\]
The wanted integral expression then follows from assertion (i). \( \square \)

Observe that Theorem 1.50 (ii) with \( q = 2 \) equals Radon's original inversion formula, see [44, A. Satz II].
Lemma 1.51

(i) \[ \mathcal{R} |P|^2 = (P^2 \otimes I) \mathcal{R}, \]

(ii) \[ |P|^2 \mathcal{R}^* = \mathcal{R}^* (P^2 \otimes I), \]

(iii) \[ \mathcal{R}^* \mathcal{R} |P|^2 = |P|^2 \mathcal{R}^* \mathcal{R}. \]

Proof. (i) By means of Theorem 1.46 and the relations (1.47) and (1.5) we derive

\[
\mathcal{R} |P|^2 = (2\pi)^{(q-1)/2} \left( \mathcal{F}^* \otimes I \right) V |P|^2 \mathcal{F} = (2\pi)^{(q-1)/2} \left( \mathcal{F}^* \otimes I \right) V |Q|^2 \mathcal{F} = (2\pi)^{(q-1)/2} \left( \mathcal{F}^* Q^2 \otimes I \right) V \mathcal{F} = (2\pi)^{(q-1)/2} \left( \mathcal{F}^* \otimes I \right) V \mathcal{F} = (P^2 \otimes I) \mathcal{R}.
\]

(ii) The equality is obtained by taking adjoints in (i).

(iii) The equality follows from (i) and (ii). \(\square\)

Now we can prove the Radon inversion formula, as established by John [31, (1.14), (1.15)].

Theorem 1.52 Let \( f \in S(\mathbb{R}^q). \) Then

(i) \( f = \frac{1}{2} (2\pi)^{1-q} \left\{ \begin{array}{ll}
\Delta^{(q-1)/2} \mathcal{R}^* \mathcal{R}_f & \text{for } q \text{ odd,}\\
\Delta^{(q-2)/2} \mathcal{R}^* (\mathcal{H} \otimes I) \mathcal{R}_f & \text{for } q \text{ even.}
\end{array} \right. \)

(ii) For \( q \) odd we have

\[
f(x) = \frac{1}{2} (2\pi)^{1-q} \Delta^{(q-1)/2} \int_{S^{q-1}} (\mathcal{R}f) (x \cdot \omega, \omega) \, d\sigma^{q-1}(\omega), \quad x \in \mathbb{R}^q.
\]

For \( q \) even we have

\[
f(x) = (2\pi)^{-q} \Delta^{(q-2)/2} \int_{S^{q-1}} \int_{\mathbb{R}} \left( (p - (x \cdot \omega))^{-1} \left( \frac{\partial}{\partial p} \mathcal{R}f \right) (p, \omega) \, dp \right) \, d\sigma^{q-1}(\omega), \quad x \in \mathbb{R}^q.
\]

Proof. (i) The formula is obtained from Theorem 1.50 (i) and Lemmas 1.51 (ii) and 1.48 (ii).

(ii) The integral expressions follow from assertion (i) and Lemmas 1.49 and 1.48 (iii). \(\square\)

As an application we consider the Cauchy problem for the wave equation in \( \mathbb{R}^q, \) with odd \( q \geq 3, \)

\[
\begin{cases}
\Delta u = u_{tt}, & x \in \mathbb{R}^q, \ t \in \mathbb{R}, \\
u(x; 0) = \psi(x), \ u_t(x; 0) = \varphi(x), & x \in \mathbb{R}^q,
\end{cases}
\] (1.84)
with initial values \( \varphi, \psi \in C_{0}^{\infty}(\mathbb{R}^{q}) \).
We solve the Cauchy problem (1.84) by means of the Radon transformation and we derive the strong Huygens' principle for the solution, cf. Lax and Phillips [34, Chapter IV] and Petkov [43, Chapter II].

**Theorem 1.53** Let \( q \geq 3 \) be odd and let \( \varphi, \psi \in C_{0}^{\infty}(\mathbb{R}^{q}) \). Then the Cauchy problem (1.84) has a unique solution given by

\[
\begin{align*}
 u(x; t) &= \frac{1}{2}(2\pi i)^{1-q} \Delta^{(q-1)/2} \int_{S^{q-1}} (\mathcal{R}\varphi) (x \cdot \omega - t, \omega) \, d\sigma^{q-1}(\omega) \\
      &= \frac{1}{2}(2\pi i)^{1-q} \Delta^{(q-3)/2} \int_{S^{q-1}} ((\partial/\partial p) \mathcal{R}\psi) (x \cdot \omega - t, \omega) \, d\sigma^{q-1}(\omega) \quad (x; t) \in \mathbb{R}^{q} \times \mathbb{R}.
\end{align*}
\]

**Proof.** The Cauchy problem (1.84) has a unique solution by [34, IV Theorem 1.1]. We introduce the Radon transforms

\[
\begin{align*}
 \hat{u}(p, \omega; t) &= (\mathcal{R}u(\cdot; t))(p, \omega), \quad (p, \omega) \in \mathbb{Z}_{q}, \quad t \in \mathbb{R}, \\
 \hat{\varphi} &= \mathcal{R}\varphi, \quad \hat{\psi} = \mathcal{R}\psi, \quad \hat{\psi}_{p} = (-iP \otimes I) \hat{\psi}.
\end{align*}
\]

By applying the Radon transformation to (1.84) and by using Lemma 1.51 (i), we are led to the following Cauchy problem for the one-dimensional wave equation:

\[
\begin{align*}
 \begin{cases}
 \hat{u}_{pp} = \hat{u}_{tt}, & (p, \omega) \in \mathbb{Z}_{q}, \quad t \in \mathbb{R}, \\
 \hat{u}(p, \omega; 0) = \hat{\varphi}(p, \omega), \quad \hat{u}_{t}(p, \omega; 0) = \hat{\psi}(p, \omega), & (p, \omega) \in \mathbb{Z}_{q}.
\end{cases}
\end{align*}
\]

The solution of this Cauchy problem is known to be

\[
\begin{align*}
 \hat{u}(p, \omega; t) &= \frac{1}{2} \hat{\varphi}(p + t, \omega) + \frac{1}{2} \hat{\psi}(p - t, \omega) + \frac{1}{2} \int_{-t}^{t} \hat{\psi}(s, \omega) \, ds, \quad (p, \omega) \in \mathbb{Z}_{q}, \quad t \in \mathbb{R}.
\end{align*}
\]

By means of the Radon inversion formula of Theorem 1.52 and Lemma 1.51 (ii) we obtain the solution of the Cauchy problem (1.84),

\[
\begin{align*}
 u(x; t) &= \frac{1}{2}(2\pi i)^{1-q} \Delta^{(q-1)/2} \mathcal{R}^{*}(\frac{1}{2} \hat{\varphi}(p + t, \omega) + \frac{1}{2} \hat{\psi}(p - t, \omega)) \\
      & \quad - \Delta^{(q-3)/2} \mathcal{R}^{*}(P^{2} \otimes I) (\frac{1}{2} \int_{-t}^{t} \hat{\psi}(s, \omega) \, ds)
\end{align*}
\]

\[
\begin{align*}
      &= \frac{1}{2}(2\pi i)^{1-q} \Delta^{(q-1)/2} \int_{S^{q-1}} (\hat{\varphi}(x \cdot \omega + t, \omega) + \hat{\psi}(x \cdot \omega - t, \omega)) \, d\sigma^{q-1}(\omega) \\
      & \quad + \frac{1}{2}(2\pi i)^{1-q} \Delta^{(q-3)/2} \int_{S^{q-1}} (\hat{\psi}_{p}(x \cdot \omega + t, \omega) - \hat{\psi}_{p}(x \cdot \omega - t, \omega)) \, d\sigma^{q-1}(\omega), \quad (x; t) \in \mathbb{R}^{q} \times \mathbb{R}.
\end{align*}
\]
Since \( \phi \) is even on \( \mathbb{Z}_q \) and since \( \hat{\psi}_p \) is odd on \( \mathbb{Z}_q \), we have

\[
\int_{S^{q-1}} \hat{\phi}(x \cdot \omega + t, \omega) \, d\sigma^{q-1}(\omega) = \int_{S^{q-1}} \hat{\phi}(x \cdot \omega - t, \omega) \, d\sigma^{q-1}(\omega) ,
\]

\[
\int_{S^{q-1}} \hat{\psi}_p(x \cdot \omega + t, \omega) \, d\sigma^{q-1}(\omega) = - \int_{S^{q-1}} \hat{\psi}_p(x \cdot \omega - t, \omega) \, d\sigma^{q-1}(\omega) .
\]

Combination of the latter three results yields the stated solution for \( u(x; t) \). \( \square \)

Now we can give a short proof of the strong Huygens' principle, cf. [43, Corollary 2.3.4].

**Corollary 1.54 (strong Huygens' principle)** Let \( q \geq 3 \) be odd, let \( B > 0 \) and let \( \phi, \psi \in C_0^\infty(\mathbb{R}^q) \) such that \( \phi(x) = \psi(x) = 0 \), \( |x| \geq B \).

Then for \( |t| > B \), the solution \( u(x; t) \) of the Cauchy problem (1.8a) vanishes for \( |x| < |t| - B \).

**Proof.** Let \( (x; t) \in \mathbb{R}^q \times R \) such that \( |t| > B \) and \( |x| < |t| - B \). Then for \( \omega \in S^{q-1} \) one has

\[
| x \cdot \omega - t | \geq |t| - |x| \geq |t| - |x| \geq B ,
\]

and consequently,

\[
(\mathcal{R}_\mathbb{F}) (x \cdot \omega - t, \omega) = 0 , \quad ((\partial / \partial t) \mathcal{R}_\mathbb{F}) (x \cdot \omega - t, \omega) = 0 .
\]

Thus the solution \( u(x; t) \), as given in Theorem 1.53, vanishes. \( \square \)

In the next theorem we characterize the range of the Radon transformation when applied to the radially symmetric functions in \( S(\mathbb{R}^q) \). In this characterization the Erdélyi-Kober operators \( \mathcal{W}_\lambda, \lambda \in \mathbb{R} \), defined in (1.23), appear. We recall that the operators \( \mathcal{W}_\lambda, \lambda \in \mathbb{R} \), constitute a group of continuous operators on \( S_{\text{even}}(\mathbb{R}) \), see (1.25).

**Theorem 1.55**

(i) Let \( f_0 \in S_{\text{even}}(\mathbb{R}) \) and define the radially symmetric function \( f : \mathbb{R}^q \to C \) by

\[
f(r\omega) = f_0(r) , \quad r \geq 0 , \quad \omega \in S^{q-1} .
\]

Then \( f \in S(\mathbb{R}^q) \) and its Radon transform is given by

\[
(\mathcal{R} f)(p, \omega) = \frac{2\pi}{\Gamma((q - 1)/2)} \int_0^\infty f_0(r) \left( r^2 - p^2 \right)^{(q-3)/2} r \, dr , \quad p \geq 0 , \quad \omega \in S^{q-1} .
\]

The Erdélyi-Kober transform \( \mathcal{W}_{(q-1)/2} f_0 \) belongs to \( S_{\text{even}}(\mathbb{R}) \).

(ii) Let \( g_0 \in S_{\text{even}}(\mathbb{R}) \) and define the function \( g : Z_q \to C \) by

\[
g(p, \omega) = g_0(p) , \quad (p, \omega) \in Z_q .
\]
Then \( g \) is the Radon transform of the function \( R^{-1} g \) given by
\[
(R^{-1} g)(r) = (2\pi)^{-(s-1)/2} \langle BV_{-(s-1)/2} g_0 \rangle (r)
\]
\[
= \begin{cases} 
(2\pi)^{-(s-1)/2} (-r^{-1}d/dr)^{(s-1)/2} g_0(r) & \text{for } q \text{ odd}, \\
2(2\pi)^{-s/2} (-r^{-1}d/dr)^{s/2} \int_0^\infty g_0(p) \left( p^2 - r^2 \right)^{-s/2} p \, dp & \text{for } q \text{ even},
\end{cases}
\]
\[
r \geq 0, \quad \omega \in S_{s-1}.
\]

The Erdélyi-Kober transform \( BV_{-(s-1)/2} g_0 \) belongs to \( S_{even}(R) \).

**Proof.** (i) According to Theorem 1.36 we have \( f \in S(R^s) \).

By means of Theorem 1.46 and 1.33 (Hecke-Bochner) and the relations (1.24) and (1.23) we deduce for \( p \geq 0, \omega \in S_{s-1} \),
\[
(2\pi)^{-(s-1)/2} \langle R f \rangle (p, \omega) = ((R^* \otimes f) V P f)(p, \omega) = (\mathcal{H}_{-1/2} \mathcal{H}_{q/2-1} f_0)(p)
\]
\[
= \langle BV_{-(s-1)/2} f_0 \rangle (p) = \frac{2^{1-(s-1)/2}}{\Gamma((q-1)/2)} \int_0^\infty f_0(r) \left( r^2 - p^2 \right)^{(s-3)/2} r \, dr.
\]

As observed, we have \( BV_{-(s-1)/2} f_0 \in S_{even}(R) \) by (1.25).

(ii) The assertion readily follows by inversion of the result from assertion (i). \( \square \)

At the end of this section we want to characterize the range of the Radon transformation when applied to functions \( f \in S(R^s) \) of the form
\[
f(r\omega) = f_m(r) Y_m(\omega) \quad \text{with} \quad f_m \in Q^m(S_{even}(R)), \quad Y_m \in Y_m^m.
\]

In this characterization Weyl-Gegenbauer transformations are involved. In Chapter 3 we study these transformations at length. Here we need their definition and some of their integral representations only; see (3.37) and Theorems 3.22 and 3.38.

Let \( m, q \in \mathbb{N}_0, q \geq 2 \). The Weyl-Gegenbauer transformation \( W_{m,q/2-1} \) is defined on the space \( S_{even}(R^s) = \{ f \in C^\infty(R^s) : x^{\ell} f^{(q)} \in L_{m,s}(1, \infty) \}, \ k, l \in \mathbb{N}_0 \) by
\[
W_{m,q/2-1} f = W_m W_{m+q/2-1} Q^{-m}.
\]

Here the operators \( W_\lambda, \lambda \in R \), are the Weyl operators defined in (1.34), while the operators \( W_{m, q} \), \( m \in R \), are the Erdélyi-Kober operators defined in (1.23).

The operator \( W_{m,q/2-1} \) is a bijection on \( S_{even}(R^s) \) with inverse
\[
W_{m,q/2-1}^* = Q^m W_{m-(q-1)/2} W_m.
\]

Since the Erdélyi-Kober operator \( BV_{m+(q-1)/2} \) maps \( S_{even}(R) \) onto itself, by (1.25), we have
\[
W_{m,q/2-1}(Q^m(S_{even}(R))) = \mathcal{P}^m(S_{even}(R)).
\]

In this connection we refer to Theorem 1.6a where we have characterized the spaces \( Q^m(S_{even}(R)) \) and \( \mathcal{P}^m(S_{even}(R)) \).

Let \( f \in S_{even}(R^s) \). Both the function \( W_{m,q/2-1} f \) and the function \( W_{m,q/2-1}^{-1} f \) have an integral representation involving the Gegenbauer polynomial \( C_m^{q/2-1} \). By setting (cf. (3.32))
\[ C_{m}^{2-1} = \frac{2^{(q-1)/2}}{\Gamma(q/2 - 1/2)} \frac{\Gamma(q - 2) \ m!}{\Gamma(m + q - 2) \Gamma(q/2 - 1/2)} C_{m}^{(2-1)} \]  

we have

\[ (\mathcal{W}G_{m,q/2-1} f)(x) = \frac{1}{x} \int_{0}^{\infty} \frac{1}{t} \left( t^2 - x^2 \right)^{(q-1)/2} C_{m/2}^{(2-1)}(t/x) f(t) t \ dt , \]

\[ (\mathcal{W}G_{m,q/2-1}^{-1} f)(x) = \frac{1}{x} \int_{0}^{\infty} \frac{1}{t} \left( t^2 - x^2 \right)^{(q-1)/2} C_{m/2}^{(2-1)}(t/x) f(x/t) \ dt . \]

**Theorem 1.56**

(i) Let \( f_{m} \in Q^{m}(S_{\text{even}}(\mathbb{R})), \) let \( Y_{m} \in \mathcal{U}_{m} \) and define \( f : \mathbb{R}^{d} \to \mathbb{C} \) by

\[ f(r \omega) = f_{m}(r) \ Y_{m}(\omega) , \ r \geq 0 , \ \omega \in S^{d-1} . \]

Then \( f \in S(\mathbb{R}^{d}) \) and its Radon transform is given by

\[ (\mathcal{R} f)(p, \omega) = (2\pi)^{(d-1)/2} (\mathcal{W}G_{m,q/2-1} f_{m})(p) Y_{m}(\omega) \]

\[ = (2\pi)^{(d-1)/2} \int_{0}^{\infty} \left( r^2 - p^2 \right)^{(d-1)/2} C_{m/2}^{(2-1)}(r/p) f_{m}(r) r \ dr \ Y_{m}(\omega) , \]

\[ p \geq 0 , \ \omega \in S^{d-1} . \]

The Weyl-Gegenbauer transform \( \mathcal{W}G_{m,q/2-1} f_{m} \) belongs to \( P^{m}(S_{\text{even}}(\mathbb{R})). \)

(ii) Let \( g_{m} \in P^{m}(S_{\text{even}}(\mathbb{R})), \) let \( Y_{m} \in \mathcal{U}_{m} \) and define \( g : \mathbb{Z}_{q} \to \mathbb{C} \) by

\[ g(p, \omega) = g_{m}(p) Y_{m}(\omega) , \ (p, \omega) \in \mathbb{Z}_{q} . \]

Then \( g \) is the Radon transform of the function \( \mathcal{R}^{-1} g \) given by

\[ (\mathcal{R}^{-1} g)(r \omega) = (2\pi)^{(d-1)/2} (\mathcal{W}G_{m,q/2-1}^{-1} g_{m})(r) \ Y_{m}(\omega) \]

\[ = \frac{(-\sqrt{2\tau})^{1-d}}{\tau^{d/2}} \int_{0}^{\infty} \left( \tau^2 - r^2 \right)^{(d-1)/2} C_{m/2}^{(2-1)}(\tau/r) g_{m}(\tau) \ d\tau \ Y_{m}(\omega) , \]

\[ r \geq 0 , \ \omega \in S^{d-1} . \]

The inverse Weyl-Gegenbauer transform \( \mathcal{W}G_{m,q/2-1}^{-1} g_{m} \) belongs to \( Q^{m}(S_{\text{even}}(\mathbb{R})). \)

**Proof.** (i) According to Theorem 1.37 we have \( f \in S(\mathbb{R}^{d}). \)

By means of Theorems 1.46 and 1.39 (Hecke-Bochner) and relations (1.5) and (1.24) we deduce for \( p \geq 0 , \ \omega \in S^{d-1} ; \)

\[ (2\pi)^{(d-1)/2} (\mathcal{R} f)(p, \omega) = ((F^{*} \otimes 1) V F f)(p, \omega) \]

\[ = (\hat{f}^{*} \hat{m}^{*} \mathcal{H}_{m}\hat{q}/2-1 \ Q^{*} \hat{m} \hat{f}_{m})(p) \ Y_{m}(\omega) \]

\[ = (\widehat{W_{m}H_{m+q/2-1}^{-1} Q^{*} \hat{m} \hat{f}_{m}})(p) \ Y_{m}(\omega) \]

\[ = (\widehat{W_{m}H_{m+q/2-1}^{-1} Q^{*} \hat{m} \hat{f}_{m}})(p) \ Y_{m}(\omega) \]

The proof is completed by using the relations (1.85), (1.89) and (1.87).

(ii) The assertion readily follows by inversion of the result from assertion (i). \( \square \)
CHAPTER 2

Gel'fand-Shilov spaces

2.1 The Gel'fand-Shilov spaces $S^\alpha_\beta(\mathbb{R})$

In their celebrated treatise on generalized functions [22, Chapter IV] Gel'fand and Shilov introduced some new subspaces of the Schwartz space $S(\mathbb{R}^n)$, which they called spaces of type $S$. In this section we gather some properties of these spaces for $q = 1$.

Definition 2.1 Let $\alpha, \beta \geq 0$.
(i) The space $S^\alpha_\beta(\mathbb{R})$ consists of all functions $f \in S(\mathbb{R})$ for which their exist $A, B, C > 0$ $(k \in \mathbb{N}_0)$, such that

$$\|x^k f^{(l)}\|_{L_\infty(\mathbb{R})} \leq B (A^k (k!))^{\alpha} \ , \ k, l \in \mathbb{N}_0 .$$

(ii) The space $S^\alpha_{\infty}(\mathbb{R})$ consists of all functions $f \in S(\mathbb{R})$ for which their exist $A, B > 0$ $(k \in \mathbb{N}_0)$, such that

$$\|x^k f^{(l)}\|_{L_\infty(\mathbb{R})} \leq A (B^l (l!))^{\alpha} \ , \ k, l \in \mathbb{N}_0 .$$

(iii) The space $S^\alpha_\beta(\mathbb{R})$ consists of all functions $f \in S(\mathbb{R})$ for which their exist $A, B, C > 0$, such that

$$\|x^k f^{(l)}\|_{L_\infty(\mathbb{R})} \leq C A^k (B^l (l!))^{\alpha} \beta \ , \ k, l \in \mathbb{N}_0 .$$

Because of the inequality $e^{-m} m^m \leq m! \leq m^m$, cf. Lemma 2.4 (i), the factorials $k!$ and $l!$ in Definition 2.1 may be replaced by $k^\alpha$ and $l^\alpha$, respectively.

Let $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{R}_0^\infty = \mathbb{R}^+ \cup \{\infty\}$ and $\mathbb{R}_0^{0, \infty} = \mathbb{R}^+ \cup \{0, \infty\}$. To make some of the subsequent results also valid for $(\alpha, \beta) = (\infty, \infty)$, we define $S^\infty_{\infty}(\mathbb{R}) = S(\mathbb{R})$. We shall call the spaces $S^\alpha_\beta(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}^{0, \infty}$, $(\alpha, \beta) \neq (\infty, \infty)$, Gel'fand-Shilov spaces. We observe that Gel'fand and Shilov used the notations $S_{\alpha, \beta}$, $S^\alpha_\beta$ and $S^\alpha_{\infty}$ for $S^\alpha_\beta(\mathbb{R})$, $S^\alpha_\beta(\mathbb{R})$ and $S^\alpha_{\infty}(\mathbb{R})$, respectively. We do not adopt their notation because our notation does not discriminate between $S_{\alpha, \beta}$, $S^\alpha_\beta$ and $S^\alpha_{\infty}$, as a result of which many properties of the Gel'fand-Shilov spaces can be formulated concisely.

The multiplication operator $x$ and the differentiation operator $d/dx$ map $S^\alpha_{\beta}(\mathbb{R})$ into itself. The reader may observe that the smaller the numbers $\alpha$ and $\beta$, the stronger are the constraints on the functions in $S^\alpha_\beta(\mathbb{R})$. So the question arises whether there exist numbers $\alpha$ and $\beta$, such that the constraints are so strong that $S^\alpha_\beta(\mathbb{R})$ becomes trivial, i.e. contains the null function only. In this connection we present the following theorem, see [22, Chapter IV, Sections 2.1, 8.1, 8.2].

Theorem 2.2 Let $\alpha, \beta \in \mathbb{R}^+$. Then

(i) $C^\infty_{\beta}(\mathbb{R}) = S^\alpha_{\infty}(\mathbb{R}) \subset S^\alpha_\beta(\mathbb{R})$;

(ii) $F'(C^\infty_{\beta}(\mathbb{R})) = S^\alpha_{\infty}(\mathbb{R}) \subset S^\alpha_\beta(\mathbb{R})$;

(iii) $S^\alpha_{\beta}(\mathbb{R})$ is nontrivial if and only if $\alpha > 1$. 

(iv) \( S_\alpha^1(\mathbb{R}) \) is nontrivial if and only if \( \beta > 1 \);

(v) \( S_\alpha^0(\mathbb{R}) \) is nontrivial if and only if \( \alpha + \beta \geq 1 \).

For \( \beta \leq 1 \), the functions in \( S_\alpha^0(\mathbb{R}) \) have an analytic continuation in the complex plane with a well-specified growth behaviour, see [22, Chapter IV, Sections 2.2, 2.3, 7, 8].

**Theorem 2.3** Let \( \alpha \in \mathbb{R}^+ \). Then
(i) a function \( f \in S_\alpha^0(\mathbb{R}) \) can be extended to an analytic function on the strip \( |y| < 1/\beta \) of the complex plane \( z = x + iy \);
(ii) a function \( f \in S_\alpha^0(\mathbb{R}) \) with \( \beta < 1 \) can be extended to an entire analytic function.

In the next two lemmas we present several estimations which we shall use frequently in the sequel.

**Lemma 2.4** Let \( k, l \in \mathbb{N}_0 \), let \( x, t, \alpha \in \mathbb{R}^+ \). Then
(i) \( e^{-x} k^l \leq k! \leq k^l \);
(ii) \( (k + l)! \leq 2^{k+l} k! l! \);
(iii) \( (k - l)! \leq k!/l! \), provided that \( k \geq l \);
(iv) \( (k/\alpha)! \leq (e^2(1 + 1/\alpha))^{k} k! \);
(v) \( x^k \exp(-t x^{1/\alpha}) \leq (k \alpha / (\alpha t)^{k/\alpha}) \leq (\alpha / e x^{1/\alpha})^k \exp(\alpha/e) \exp(-\alpha/e x^{1/\alpha}) \).

**Proof.** (i) \( e^k = \sum_{n=0}^{\infty} k^n/k! \geq k^k/k! \geq 1 \);
(ii) \( (k + l)! = \binom{k+l}{k} k! l! \leq 2^{k+l} k! l! \);
(iii) \( (k - l)! = k!/l!(k/l)! \leq k!/l! \), provided that \( k \geq l \);
(iv) By means of assertion (i) we estimate

\[
(k/\alpha)! \leq \left(\frac{k}{\alpha}\right)^{\alpha} k^\alpha \leq (k/\alpha + 1)^{\alpha} (k/\alpha + 1)^k \\
\leq e^{(k/\alpha + 1)^k} = (e(1 + 1/\alpha))^k k! \leq (e^2(1 + 1/\alpha))^{k} k! .
\]

(v) On \( \mathbb{R}^+ \), the function \( f(x) = x^k \exp(-t x^{1/\alpha}) \) attains its maximum at \( x = (k \alpha / t)^{\alpha} \).
(vi) The inequality is taken from [22, p. 171].

**Lemma 2.5** Let \( f \in S(\mathbb{R}) \). Then
(i) \( \|f\|_{L^1(\mathbb{R})} \leq \sqrt{2} (\|f\|_{L^\infty(\mathbb{R})} + \|f\|_{L^1(\mathbb{R})}) \);
(ii) \( \|f\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2}} (\|f\|_{L^1(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}) \).
(iii) \( \|Q^k P^l f\|_{L^2(\mathbb{R})} \leq \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j! \|Q^{2k-j} f\|_{L^2(\mathbb{R})} \|P^{2l-j} f\|_{L^2(\mathbb{R})}, \quad k, l \in \mathbb{N}_0. \)

Proof. (i) \( \|f\|_{L^2(\mathbb{R})}^2 \leq \sup_{x \in \mathbb{R}} (1+x^2)|f(x)|^2 \int_\mathbb{R} (1+x^2)^{-1} \, dx \leq \pi (\|f\|_{L^2(\mathbb{R})} + \|xf\|_{L^2(\mathbb{R})})^2; \)

(ii) \( |f(x)| \leq \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} |(P^2 f)(y)| \, dy \leq \frac{1}{\sqrt{2}} \left( \int_\mathbb{R} (1+y^2)^{-1} |(P^2 f)(y)|^2 \, dy \right)^{1/2} \)

\[ \leq \frac{1}{\sqrt{2}} (\|f\|_{L^2(\mathbb{R})} + \|P^2 f\|_{L^2(\mathbb{R})}), \quad x \in \mathbb{R}; \]

(iii) since the operators \( P \) and \( Q \) are self-adjoint we derive by means of Leibniz's differentiation rule for \( k, l \in \mathbb{N}_0, \)

\[ \|Q^k P^l f\|_{L^2(\mathbb{R})}^2 = (P^2 Q^{2k} P^l f, f)_{L^2(\mathbb{R})} = \sum_{j=0}^{\min(2k,l)} i^j \binom{2k}{j} \binom{l}{j} j! \|Q^{2k-j} f, Q^{2l-j} f\|_{L^2(\mathbb{R})}. \]

Now we obtain the wanted result by the Cauchy-Schwarz inequality. \( \Box \)

We give several characterizations of the Gel'fand-Shilov spaces. For the corresponding characterization of \( S(\mathbb{R}) \) we refer to Theorem 1.2.

Theorem 2.6 Let \( f \in S(\mathbb{R}). \)

I. Let \( \alpha \in \mathbb{R}_0^+. \) Then the following assertions are equivalent:

(i) \( f \in S^\alpha_{\mathbb{R}}(\mathbb{R}); \)

(ii) there exist \( A, B > 0 \) such that \( \|x^k f^0\|_{L^2(\mathbb{R})} \leq B(A^k(k!)^\alpha), \quad k \in \mathbb{N}_0; \)

(iii) there exist \( A, B > 0 \) such that \( \|x^k f\|_{L^2(\mathbb{R})} \leq BA^k(k!)^\alpha, \quad k \in \mathbb{N}_0; \)

(iv) there exist \( A, B > 0 \) such that \( \|x^k f\|_{L^2(\mathbb{R})} \leq BA^k(k!)^\alpha, \quad k \in \mathbb{N}_0. \)

II. Let \( \beta \in \mathbb{R}_0^+. \) Then the following assertions are equivalent:

(i) \( f \in S^\beta_{\mathbb{R}}(\mathbb{R}); \)

(ii) there exist \( A, B > 0 \) such that \( \|x^k f^0\|_{L^2(\mathbb{R})} \leq A_k B^k(l!)^\beta, \quad k \in \mathbb{N}_0; \)

(iii) there exist \( A, B > 0 \) such that \( \|f^0\|_{L^2(\mathbb{R})} \leq AB^k(l!)^\beta, \quad l \in \mathbb{N}_0; \)

(iv) there exist \( A, B > 0 \) such that \( \|f^0\|_{L^2(\mathbb{R})} \leq AB^k(l!)^\beta, \quad l \in \mathbb{N}_0. \)

III. Let \( \alpha, \beta \in \mathbb{R}_0^+ \) with \( \alpha + \beta \geq 1. \) Then the following assertions are equivalent:

(i) \( f \in S^\alpha_{\mathbb{R}}(\mathbb{R}); \)

(ii) there exist \( A, B, C > 0 \) such that \( \|x^k f^0\|_{L^2(\mathbb{R})} \leq CA^k B^k(l!)^\alpha (l!)^\beta, \quad k, l \in \mathbb{N}_0; \)

(ii) there exist \( A, B, C > 0 \) such that \( \|x^k f^0\|_{L^2(\mathbb{R})} \leq CA^k B^k(l!)^\alpha (l!)^\beta, \quad k, l \in \mathbb{N}_0; \)
(iii) there exist $A, B > 0$ such that
\[ \| x^k f \|_{L^q(B)} \leq BA^k(k!)^\alpha, \quad \| f^{(l)} \|_{L^q(B)} \leq AB^l(l!)^\beta, \quad k, l \in \mathbb{N}_0; \]

(iv) there exist $A, B > 0$ such that
\[ \| x^k f \|_{L^q(B)} \leq BA^k(k!)^\alpha, \quad \| f^{(l)} \|_{L^q(B)} \leq AB^l(l!)^\beta, \quad k, l \in \mathbb{N}_0. \]

**Proof.** I. We follow the scheme (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Obviously (i) $\Rightarrow$ (iv). For $k \in \mathbb{N}_0$ we have
\[ \| Q^k P^l f \|_{L^q(B)} \leq \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j! \| Q^{2k-j} f \|_{L^q(B)} A^{2k-j} \| f \|_{L^q(B)} \leq (B \sum_{j=0}^{l} \binom{l}{j} j! \| Q^{2l-j} f \|_{L^q(B)} (2^{l+1}(A + 1))^{2k}(k!)^{2\alpha}. \]

That is, $f$ satisfies (ii).

By means of Lemma 2.5 (ii) it follows that (ii) $\Rightarrow$ (i).

II. We follow the scheme (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Obviously (i) $\Rightarrow$ (iv). For $l \in \mathbb{N}_0$ we have
\[ \| P^l f \|_{L^q(B)} = \| Q^{2l} f \|_{L^q(B)} \leq \| f \|_{L^q(B)} \| f \|_{L^q(B)} \leq \| f \|_{L^q(B)} \| f \|_{L^q(B)} \leq (A \sum_{j=0}^{l} \binom{l}{j} j! \| Q^{2l-j} f \|_{L^q(B)} (2^{l+1}(B + 1))^{2k}(l!)^{2\beta}. \]

That is, $f$ satisfies (ii).

By means of Lemma 2.5 (ii) it follows that (ii) $\Rightarrow$ (i).

III. We follow the scheme (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Obviously (i) $\Rightarrow$ (iv). From the implication (iv) $\Rightarrow$ (iii) in I and in II we obtain (iv) $\Rightarrow$ (iii).

Suppose $f$ satisfies (iii). By means of Lemma 2.5 (iii) and Lemma 2.4 (ii), (iii) and by the assumption that $\alpha + \beta \geq 1$, we estimate for $k, l \in \mathbb{N}_0$.\[ \| Q^k P^l f \|_{L^q(B)} \leq \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j! \| Q^{2k-j} f \|_{L^q(B)} A^{2k-j} \| f \|_{L^q(B)} \leq (A \sum_{j=0}^{l} \binom{l}{j} j! \| Q^{2l-j} f \|_{L^q(B)} (2^{l+1}(B + 1))^{2k}(l!)^{2\beta}. \]

That is, $f$ satisfies (ii).

By means of Lemma 2.5 (ii) it follows that (ii) $\Rightarrow$ (i).
\[
\|Q^k P^l f\|_{L^2(R)}^2 \leq \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \frac{i!}{j!} B A^{2k-j-1} \left(2k-j\right)!^\alpha A B^{2l-j-1} \left(2l-j\right)!^\beta \\
\leq A B (A+1)^{2k} (B+1)^{2l} \left(2k\right)!^\alpha \left(2l\right)!^\beta \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \frac{i!}{j!} (j!)^{1-(\alpha+\beta)} \\
\leq A B (A+1)^{2k} (B+1)^{2l} \left(2k\right)!^\alpha \left(2l\right)!^\beta \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \frac{i!}{j!} (j!)^{1-(\alpha+\beta)} \\
\leq A B (2^{n+1} (A+1))^k \left(2^{n+1} (B+1)^2\right) \left(2k\right)!^\alpha \left(2l\right)!^\beta.
\]

That is, \(f\) satisfies (ii).

By means of Lemma 2.5 (ii) it follows that \((ii) \Rightarrow (i)\).

From Theorem 2.6 we immediately obtain for \(\alpha + \beta \geq 1\),

\[
S_{\alpha}^{\beta}(R) \cap S_{\alpha}^{\beta}(R) = S_{\alpha}^{\beta}(R).
\]

(2.1)

This intersection result was first proved by Khashpsovskii [33]. He states that this result is also true for \(\alpha + \beta < 1\): "In the case when \(\alpha + \beta < 1\) \((S_{\alpha}^{\beta}\) consists of the unique function \(f(x) \equiv 0\), Eq. (2.1) follows from the Phragmén-Lindelöf and Liouville theorems." Since this argument is not quite obvious, we present another proof here which is inspired by Duran’s paper [11].

**Theorem 2.7** Let \(\alpha, \beta \in \mathbb{R}^+\). Then

\[
S_{\alpha}^{\beta}(R) \cap S_{\alpha}^{\beta}(R) = S_{\alpha}^{\beta}(R).
\]

**Proof.** As observed, only the case \(0 \leq \alpha + \beta < 1\) remains to be considered.

Let \(f \in S_{\alpha}^{\beta}(R) \cap S_{\alpha}^{\beta}(R)\). By Theorem 2.6 there exist \(A, B > 0\) such that

\[
\|Q^k P^l f\|_{L^2(R)} \leq B A^k (k!)^\alpha, \quad \|P^l f\|_{L^2(R)} \leq A B^l (l!)^\beta, \quad k, l \in \mathbb{N}_0.
\]

Hence there exist \(C, D > 0\) such that

\[
\|Q^m P^k f\|_{L^2(R)}^2 \leq C D^{2k} (k!)^{1+\alpha+\beta}, \quad k \in \mathbb{N}_0, \quad m \in \{k,k-1\}.
\]

Indeed, by means of Lemma 2.5 (iii) and Lemma 2.4 (ii), (iii) we estimate for \(k \in \mathbb{N}_0, m \in \{k,k-1\},\)

\[
\|Q^m P^k f\|_{L^2(R)}^2 \leq \sum_{j=0}^{\min(2m,k)} \binom{2m}{j} \frac{k!}{j!} A 2m-j-1 \left(2m-j\right)!^\alpha A B^{2k-j-1} \left(2k-j\right)!^\beta \\
\leq A B (2(A+1) (B+1))^{2k} \left(2k\right)!^\alpha \left(2k\right)!^\beta \sum_{j=0}^{k} \binom{k}{j} \frac{i!}{j!} (j!)^{1-(\alpha+\beta)} \\
\leq A B (2(A+1) (B+1))^{2k} \left(2k\right)!^\alpha \left(2k\right)!^\beta \sum_{j=0}^{k} \binom{k}{j} \frac{i!}{j!} (j!)^{1-(\alpha+\beta)}.
\]
\[ \leq AB^{2/2+\alpha+\beta}(A+1)(B+1)k^{1+\alpha+\beta}. \]

Next we derive for \( k \in \mathbb{N} \), by means of Lemma 2.5 (ii),
\[ \|Q^k P^k f\|_{L_\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \left( \|Q^k P^k f\|_{L_2(\mathbb{R})} + k \|Q^{k-1} P^k f\|_{L_2(\mathbb{R})} + \|Q^k P^{k+1} f\|_{L_2(\mathbb{R})} \right) \]
\[ \leq \frac{C}{\sqrt{2}} D^k(k)^{(1+\alpha+\beta)/2} (1 + k + D(k+1)^{(1+\alpha+\beta)/2}) \]
\[ \leq \frac{C}{\sqrt{2}} D^k(k)^{(1+\alpha+\beta)/2} (2D)^k(k+1)^{(1+\alpha+\beta)/2}. \]

Since \( f \in S_0^\omega(R) \) with \( \beta < 1 \), the function \( f \) can be extended to an entire analytic function, by Theorem 2.3 (ii). Let \( x \in C \). Then there exist \( x \geq 0 \) and \( 0 \leq \psi \leq 2\pi \) such that
\[ z = x + 2xe^{i\psi}. \]

By applying the \( L_\infty \)-estimate for \( Q^k P^k f \), we deduce
\[ |f(z)| = |f(x + 2xe^{i\psi})| = |\sum_{k=0}^{\infty} f^{(k)}(x)(k!)^{\alpha} (2xe^{i\psi})^k| \leq \sum_{k=0}^{\infty} 2^k(k!)^{1-\alpha} |x|^k f^{(k)}(x) | \]
\[ \leq \|f\|_{L_\infty(\mathbb{R})} + \frac{C}{\sqrt{2}} (D + 1) \sum_{k=1}^{\infty} (4D)^k(k!)^{(\alpha-\beta)-1/2}. \]

Hence \( f \) is bounded on \( C \). Then \( f \) must be constant, due to Liouville's theorem. As \( f(x) \to 0 \) for \( |x| \to \infty \) we have \( f(z) = 0 \), \( z \in \mathbb{R} \).

Thus we have shown that \( S_0^\omega(R) \cap S_0^\omega(R) = \{0\} \). Since, by Definition 2.1, \( S_0^\omega(R) \subset S_0^\omega(R) \cap S_0^\omega(R) \), we are done. In this manner we have also proved that \( S_0^\omega(R) \) is trivial if \( 0 \leq \alpha + \beta < 1 \). \( \square \)

The action of the Fourier transformation on a Gel'fand-Shilov space swaps the indices \( \alpha \) and \( \beta \), while the operators \( P \) and \( Q \) map \( S_0^\omega(R) \) continuously onto itself; compare Theorem 1.4.

**Theorem 2.8** Let \( \alpha, \beta \in \mathbb{R}_{\geq 0} \). Then

(i) \( \mathcal{F}(S_0^\omega(R)) = S_0^\omega(R) \),

(ii) \( Q^\alpha S_0^\omega(R) = \{ f \in S_0^\omega(R) : f^{(k)}(0) = 0, \ k = 0, \ldots, n-1 \}, \ n \in \mathbb{N} \),

(iii) \( P^\alpha S_0^\omega(R) = \{ f \in S_0^\omega(R) : \int_R f(y)^n \ dy = 0, \ k = 0, \ldots, n-1 \}, \ n \in \mathbb{N} \).

**Proof.** (i) For \( f \in S(R) \) we have
\[ \|Q^k P^l \mathcal{F} f\|_{L_2(\mathbb{R})} = \|P^l Q^k f\|_{L_2(\mathbb{R})}, \quad k, l \in \mathbb{N}_0. \]

By using this equality with \( l = 0 \) or \( k = 0 \), the result follows from Theorem 2.6.

(ii) Let \( n \in \mathbb{N} \) and let \( f \in S_0^\omega(R) \) with \( f^{(k)}(0) = 0, \ k = 0, \ldots, n-1 \). According to the proof of Theorem 1.4 (ii), there exists a function \( g \in S(R) \) such that \( g(x) = x^n f(x) \) and
\[ \|z^k g\|_{L^m(\mathbb{R})} \leq \frac{1}{n!} \|f^{(n)}\|_{L^m(\mathbb{R})} + \|z^k f\|_{L^m(\mathbb{R})}, \quad k \in \mathbb{N}_0. \]

\[ \|g^{(i)}\|_{L^m(\mathbb{R})} \leq \frac{n}{(n+i)!} \|f^{(n+i)}\|_{L^m(\mathbb{R})}, \quad i \in \mathbb{N}_0. \]

By Theorem 2.6, it readily follows that \( g \in S_0^\alpha(\mathbb{R}) \) and therefore, \( f \in Q^\alpha(S_0^\alpha(\mathbb{R})). \) Conversely, suppose \( f \in Q^\alpha(S_0^\alpha(\mathbb{R})). \) Then obviously \( f \in S_0^\alpha(\mathbb{R}) \) with \( f^{(k)}(0) = 0, \quad k = 0, \ldots, n - 1. \)

(iii) The proof runs along the same lines as the proof of Theorem 1.4 (iii). \( \square \)

**Corollary 2.9** Let \( \alpha, \beta \in (0, \infty) \) and let \( f : \mathbb{R} \to \mathbb{C}. \) Then the following assertions are equivalent:

(i) \( f \in S_0^\alpha(\mathbb{R}) \);

(ii) \( f \in S_0^\beta(\mathbb{R}) \) and \( Ff \in S_0^\beta(\mathbb{R}) \).

**Proof.** The equivalence is a straightforward consequence of Theorems 2.7 and 2.8 (i). \( \square \)

We are going to relate the Gel'fand-Shilov spaces to Gevrey spaces brought about by the operators \( P \) and \( Q. \) First we introduce some notations taken from Ter Elst's thesis [17].

Let \( T \) be a self-adjoint operator in a Hilbert space \( H. \) For \( \lambda \geq 0 \) and \( A > 0 \) the normed space \( S_{\lambda, A}(T) \) is defined by

\[ S_{\lambda, A}(T) = \{ f \in D^\infty(T) : \|f\|_{T, \lambda, A} = \sup_{n \in \mathbb{N}_0} (\|T^n f\|_H A^{-n(n+1)} - \lambda) < \infty \}. \quad (2.2) \]

The Gevrey space \( S_\lambda(T), \) brought about by \( T, \) is taken to be the union

\[ S_\lambda(T) = \bigcup_{A > 0} S_{\lambda, A}(T). \quad (2.3) \]

Clearly \( S_\lambda(T) \subset S_\mu(T) \) if \( \lambda < \mu. \) The topology for \( S_\lambda(T) \) is the inductive limit topology generated by the normed spaces \( S_{\lambda, A}(T), \ A > 0. \) Since the topology of \( S_\lambda(T) \) is regular, see [17, Theorem 1.11], we obtain the following characterization of sequential convergence in \( S_\lambda(T). \) For a sequence \( (f_k)_{k \in \mathbb{N}} \subset D^\infty(T) \) we have

\[ f_k \to 0 \ (k \to \infty) \text{ in } S_\lambda(T) \text{ if and only if there exists } A > 0 \text{ such that } \]

\[ f_k \in S_{\lambda, A}(T), \quad k \in \mathbb{N}, \text{ and } \|f_k\|_{T, \lambda, A} \to 0 \ (k \to \infty). \quad (2.4) \]

Consistent with (2.2) and (2.3), we define \( S_\infty(T) = D^\infty(T). \)

For a unitary operator \( U \) on \( H \) we have, cf. (1.3),

\[ S_\lambda(UTU^*) = U(S_\lambda(T)), \quad \lambda \in \mathbb{R}_{0, \infty}. \quad (2.5) \]

In particular we have for the Gevrey spaces \( S_\lambda(P) \) and \( S_\lambda(Q), \) brought about by the operators \( P \) and \( Q, \) respectively, via the Fourier transformation,

\[ S_\lambda(P) = \mathcal{F}(S_\lambda(Q)), \quad S_\lambda(Q) = \mathcal{F}(S_\lambda(P)), \quad \lambda \in \mathbb{R}_{0, \infty}. \quad (2.6) \]
cf. (1.6). The space $S_\lambda(Q), \lambda \in R^+$, admits the following characterization.

**Lemma 2.10** Let $\lambda \in R^+$. Then

$$S_\lambda(Q) = \{ f \in L_2(R) : \exists_{t > 0} \exp(t |x|^{1/\lambda}) f \in L_2(R) \}$$

**Proof.** Let $f \in S_\lambda(Q)$. Then there exist $A, B > 0$ such that

$$\|Q^k f\|_{L_2(R)}^2 \leq B A^k (k!)^{2\lambda}, \quad k \in N_0.$$  

Let $t = (4e^2 (1 + 1/(2\lambda)) (A + 1)^{1/(2\lambda)})^{-1}$. Then, by means of Lemma 2.4 (iv) we have

$$\int_R |\exp(t |x|^{1/\lambda}) f(x)|^2 dx = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \int_R |x|^{k/\lambda} |f(x)|^2 dx$$

$$\leq \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} (\|Q^{k/(2\lambda)} f\|_{L_2(R)}^2 + \|f\|_{L_2(R)}^2)$$

$$\leq B e^{2t} + \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} B A^k (k/(2\lambda))!^{2\lambda}$$

$$\leq B e^{2t} + \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} B (A + 1)^{k/(2\lambda) + 1} (e^2 (1 + 1/(2\lambda)))^k k!$$

$$= B e^{2t} + B(A + 1) \sum_{k=0}^{\infty} 2^{-k} < \infty.$$  

Hence, $\exp(t |x|^{1/\lambda}) f \in L_2(R)$ for $t$ as indicated.

For the converse, let $f \in L_2(R)$ and suppose there exists $t > 0$ such that

$$\exp(t |x|^{1/\lambda}) f \in L_2(R).$$

Then for $k \in N_0$ we have

$$\int_R |x^k f(x)|^2 dx \leq \sup_{x \in R} |x^k \exp(-t |x|^{1/\lambda})|^2 \int_R |\exp(t |x|^{1/\lambda}) f(x)|^2 dx$$

$$\leq (\lambda/t)^{2\lambda} (k!)^{2\lambda} \int_R |\exp(t |x|^{1/\lambda}) f(x)|^2 dx$$

where we used Lemma 2.4 (v) for the latter inequality. Hence, $f \in S_\lambda(Q)$. \qed

In Theorem 1.3 we have seen that $S(R) = S_\infty(Q) \cap S_\infty(P)$. The Gel'fand-Shilov spaces admit a similar functional analytic characterization in terms of Gevrey spaces, cf. Van Eijndhoven [12, Theorem 4.5].

**Theorem 2.11** Let $\alpha, \beta \in R^+_\infty$. Then

$$S_{\alpha}^\beta(R) = S_\alpha(Q) \cap S_\beta(P).$$
Proof. For $\alpha \in \mathcal{B}_0^+$ we have, by Theorem 2.6 (i) and Theorem 1.3,
\[ S_0^\omega (R) = S_0(Q) \cap S(R) = S_0(Q) \cap S_0(P). \]
For $\beta \in \mathcal{B}_0^+$ we have, by Theorem 2.8 (i), the previous result and relation (2.6),
\[ S_0^\omega (R) = \mathcal{F}(S_0^\omega (R)) = \mathcal{F}(S_0(P)) \cap \mathcal{F}(S_0(Q)) = S_0(P) \cap S_0(Q). \]
For $\alpha, \beta \in \mathcal{B}_0^+$ we have, by Theorem 2.7 and the previous two results,
\[ S_0^\omega (R) = S_0^\omega (R) \cap S_0^\omega (R) = (S_0(Q) \cap S_0(P)) \cap (S_0(P) \cap S_0(Q)) = S_0(Q) \cap S_0(P). \]
\[ \square \]

Remarks. (1) The equality in Theorem 2.11 was first proved by Van Eijndhoven [12, Theorem 4.5] in the case $\alpha + \beta \geq 1$ with $\alpha, \beta \in \mathcal{B}_0^+$, and later by Ter Elst [17, Theorem 3.2] in the case $\alpha + \beta \geq 1$ with $\alpha, \beta \in \mathcal{B}_0^+$.
(2) Since the Gel'fand-Shilov space $S_0^\omega (R)$ can thus be written as the intersection of the well-defined topological spaces $S_0(Q)$ and $S_0(P)$, we endow $S_0^\omega (R)$ with the intersection topology. It can be shown that sequential convergence in the intersection topology of $S_0^\omega (R)$ agrees with Gel'fand and Shilov's definition of sequential convergence in $S_0^\omega (R)$, cf. [22, Chapter IV, Section 3]. The space $S_0^\omega (R)$ is complete.
(3) The operators $P$ and $Q$ map $S_0^\omega (R)$ continuously into itself. The operator $\mathcal{F}$ maps $S_0^\omega (R)$ continuously onto $S_0^\omega (R)$.

The next theorem states that functions in $S_0^\omega (R)$ decrease exponentially at infinity.

**Theorem 2.12** Let $\alpha \in \mathcal{B}_0^+$ and let $f \in S(R)$. Then the following assertions are equivalent:

(i) $f \in S_0^\omega (R)$;

(ii) there exists $t > 0$ such that $\exp(t|z|^{1/\alpha}) f \in L_\omega (R)$;

(iii) there exists $t > 0$ such that $\exp(t|z|^{1/\alpha}) f \in L_\omega (R)$.

Proof. By Lemma 2.10 and Theorem 2.11 we have (i) $\iff$ (ii). Obviously, (iii) $\implies$ (ii).
We show that (i) $\implies$ (iii). Suppose $f \in S_0^\omega (R)$. By Definition 2.1 (i) there exist $A, C > 0$ such that
\[ |f(x)| \leq C(|x|^A)^{-k} k^{k_0}, \quad x \in R, \quad k \in N_0. \]
Hence, by Lemma 2.4 (vi),
\[ |f(x)| \leq C \inf_{k \in N_0} \left( \frac{(|x|^A)^{-k} k^{k_0}}{(2^{(\alpha \epsilon/k_b)} \exp(-\alpha \epsilon/2) \exp((\gamma/\alpha) (|x|^A)^{1/\alpha}))} \right), \quad x \in R. \]
\[ \square \]

The characterizations of $S_0^\omega (R)$ established in Theorems 2.6 (i) and 2.12 lead, by Corollary 2.9, to analogous characterizations of $S_0^\omega (R)$ and $S_0^\omega (R)$, $\alpha, \beta \in \mathcal{B}_0^+$. 
Recalling the characterization of $S(\mathbb{R})$ by means of the Fourier transformation in Theorem 1.5, we obtain the following characterizations of the Gel'fand-Shilov spaces.

**Theorem 2.13** Let $f \in L^p_2(\mathbb{R})$ and let $p \in (2, \infty)$.
1. Let $\alpha \in \mathbb{R}^+$. Then the following assertions are equivalent:
   (i) $f \in S_\alpha^p(\mathbb{R})$;
   (ii) there exist $A, B > 0$ such that
       \[ \|x^k f\|_{L^p(\mathbb{R})} \leq BA^k(k!)^{\alpha} \text{ and } x^l \mathcal{F} f \in L_p(\mathbb{R}), \quad k, l \in \mathbb{N}_0; \]
   (iii) there exists $t > 0$ such that
       \[ \exp(t|x|^{1/\alpha}) f \in L_p(\mathbb{R}) \text{ and } x^l \mathcal{F} f \in L_p(\mathbb{R}), \quad l \in \mathbb{N}_0. \]

11. Let $\beta \in \mathbb{R}^+$. Then the following assertions are equivalent:
   (i) $f \in S_\beta^p(\mathbb{R})$;
   (ii) there exist $A, B > 0$ such that
       \[ x^k f \in L_p(\mathbb{R}) \text{ and } \|x^l \mathcal{F} f\|_{L^p(\mathbb{R})} \leq AB^l(l!)^{\beta}, \quad k, l \in \mathbb{N}_0; \]
   (iii) there exists $t > 0$ such that
       \[ x^k f \in L_p(\mathbb{R}) \text{ and } \exp(t|x|^{1/\beta}) \mathcal{F} f \in L_p(\mathbb{R}), \quad k \in \mathbb{N}_0. \]

111. Let $\alpha, \beta \in \mathbb{R}^+$. Then the following assertions are equivalent:
   (i) $f \in S_\alpha^p(\mathbb{R})$;
   (ii) there exist $A, B > 0$ such that
       \[ \|x^k f\|_{L^p(\mathbb{R})} \leq BA^k(k!)^{\alpha} \text{ and } \|x^l \mathcal{F} f\|_{L^p(\mathbb{R})} \leq AB^l(l!)^{\beta}, \quad k, l \in \mathbb{N}_0; \]
   (iii) there exists $t > 0$ such that
       \[ \exp(t|x|^{1/\alpha}) f \in L_p(\mathbb{R}) \text{ and } \exp(t|x|^{1/\beta}) \mathcal{F} f \in L_p(\mathbb{R}). \]

As we have seen in Theorem 1.6, the functions in $S(\mathbb{R})$ can be characterized in terms of their Hermite expansion coefficients:

\[ S(\mathbb{R}) = \mathcal{D}_N^\infty(p^2 + q^2) = \{ f \in L^2_2(\mathbb{R}) : (n^k(f, \psi_n)_{L^2_2(\mathbb{R})}) \in l_\infty, \quad k \in \mathbb{N}_0 \}. \]

Here $\{ \psi_n : n \in \mathbb{N}_0 \}$ denotes the Hermite basis in $L^2_2(\mathbb{R})$ as introduced in (1.7). Zhang has proved the analogue for the functions in $S_\alpha^p(\mathbb{R})$ with $\alpha \geq \frac{1}{2}$, see [51, Theorem 2 and Lemma 3]:
Let $\alpha \geq \frac{1}{2}$ and let $f \in L_{2}(\mathbb{R})$. Then $f \in S_{\alpha}^{\infty}(\mathbb{R})$ if and only if
\[
\exists_{\lambda>0} (f, \psi_{n})_{L_{2}(\mathbb{R})} = O(\exp(-t^{n/\lambda(2n)})), \quad n \to \infty.
\]
This characterization of $S_{\alpha}^{\infty}(\mathbb{R})$ is contained in the following theorem, which we provide with a new proof. First we present an auxiliary result.

**Lemma 2.14** Let $\lambda \in \mathbb{R}^{+}$. Then
\[
S_{\lambda}(P^{2} + Q^{2}) = \{ f \in L_{2}(\mathbb{R}) : \exists_{\lambda>0} (\exp(t^{n/\lambda}) (f, \psi_{n})_{L_{2}(\mathbb{R})} \in l_{\infty}) \}.
\]

**Proof.** Let $f \in L_{2}(\mathbb{R})$ and let $a_{n} = |(f, \psi_{n})_{L_{2}(\mathbb{R})}|$, $n \in \mathbb{N}_{0}$.
Suppose $f \in S_{\lambda}(P^{2} + Q^{2})$. Then there exist $A, B > 0$ such that
\[
\sum_{n=0}^{\infty} n^{2k} a_{n}^{2} \leq \sum_{n=0}^{\infty} (2n + 1)^{2k} a_{n}^{2} = \|(P^{2} + Q^{2}) f\|_{L_{2}(\mathbb{R})}^{2} \leq B A^{k} k!^{2k}, \quad k \in \mathbb{N}_{0}.
\]
Let $t = (4e^{2}(1 + 1/(2\lambda))) (A + 1)^{-1/(2\lambda)}$. Then, by means of Lemma 2.4 (iv) we have
\[
sup_{n \in \mathbb{N}_{0}} |\exp(t^{n/\lambda}) a_{n}|^{2} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2n^{1/\lambda})^{k}}{k!} a_{n}^{2} \leq \sum_{k=0}^{\infty} \frac{(2t)^{k}}{k!} B A^{k} k!^{2k} \leq \sum_{k=0}^{\infty} \frac{(2t)^{k}}{k!} B (A + 1)^{k/(2\lambda)} e^{2} (1 + 1/(2\lambda))^{k} k! = B (A + 1)^{2} \sum_{k=0}^{\infty} 2^{-k} < \infty.
\]
Hence, $\exp(t^{n/\lambda}) a_{n} \in l_{\infty}$ for $t$ as indicated.
For the converse, suppose there exists $t > 0$ such that $\exp(t^{n/\lambda}) a_{n} \in l_{\infty}$.
Let $0 < \tau < t$. Then $\exp(\tau^{1/\lambda}) a_{n} \in l_{\tau}$.
For $k \in \mathbb{N}_{0}$ we have
\[
\sum_{n=0}^{\infty} (2n + 1)^{2k} a_{n}^{2} \leq \sup_{n \in \mathbb{N}_{0}} |(2n + 1)^{k} \exp(-\tau^{n/\lambda}) f |^{2} \sum_{n=0}^{\infty} |\exp(\tau^{n/\lambda}) a_{n}|^{2} \leq (1 + 3(\lambda/\tau)^{2k} (k!)^{2k} C \sum_{n=0}^{\infty} |\exp(\tau^{n/\lambda}) a_{n}|^{2}
\]
where we used Lemma 2.4 (v) for the latter inequality. Hence, $f \in S_{\lambda}(P^{2} + Q^{2})$. \hfill \Box

We arrive at the announced characterization of the space $S_{\alpha}^{\infty}(\mathbb{R})$, $\alpha \geq \frac{1}{2}$.

**Theorem 2.15** Let $\alpha \geq \frac{1}{2}$. Then
\[
S_{\lambda}^{\infty}(\mathbb{R}) = S_{\alpha}(Q) \cap S_{\alpha}(P) = S_{2\alpha}(P^{2} + Q^{2}) = \{ f \in L_{2}(\mathbb{R}) : \exists_{\lambda>0} (\exp(t^{n/\lambda}) (f, \psi_{n})_{L_{2}(\mathbb{R})} \in l_{\infty}) \}.
\]
The equalities hold as topological vector spaces.

(ii) For \( f \in S^0_\infty(\mathcal{R}) \) we have

\[
 f(x) = \sum_{n=0}^{\infty} (f, \psi_n)_{l_2(\mathcal{R})} \psi_n(x), \quad x \in \mathcal{R},
\]

where the series converges in \( S^0_\infty(\mathcal{R}) \).

Proof. (i) The first and the third equality have been established in Theorem 2.11 and Lemma 2.14, respectively. We shall show that

\[ S^0_\infty(\mathcal{R}) \subset S_{2^n}(P^2 + Q^2) \]

and that

\[ \{ f \in L_2(\mathcal{R}) : \exists_{\gamma_{20}} \exp(\gamma_{1/2\alpha}) (f, \psi_n)_{l_2(\mathcal{R})} \in l_\infty \} \subset S^0_\infty(\mathcal{R}) \].

Let \( f \in S^0_\infty(\mathcal{R}) \). For \( n \in \mathcal{N} \) let \( V_n \) denote the set of operators

\[ Q^{k_1} P^{l_1} \ldots Q^{k_n} P^{l_n}, \quad k, l \in \mathcal{N}_0^n \text{ with } |k| + |l| = n. \]

According to [12, Theorem 4.3 (iii)], there exist \( A, B > 0 \) such that for \( n \in \mathcal{N} \) and \( T_n \in V_n \),

\[ \|T_n f\|_{l_2(\mathcal{R})} \leq BA^n(n!)^n. \]

Let \( n \in \mathcal{N} \). There exist operators \( T_{2^n} \in V_{2^n}, \quad j = 1, \ldots, 2^n \), such that

\[ (P^2 + Q^2)^n f = \sum_{j=1}^{2^n} T_{2^n} f. \]

Therefore, by the preceding estimate we have

\[ \|(P^2 + Q^2)^n f\|_{l_2(\mathcal{R})} \leq 2^n BA^{2^n}((2n)!)^n \leq B(2^{2a+1}A^2)^n (n!)^{2n}. \]

That is, \( f \in S_{2^n}(P^2 + Q^2) \).

Next, let \( f \in L_2(\mathcal{R}) \) and let there exist \( t > 0 \) such that \( (\exp(tn^{1/2\alpha})) (f, \psi_n)_{l_2(\mathcal{R})} \in l_\infty \).

Then \( f \in S(\mathcal{R}) \) by Theorem 1.6. By means of (1.10) we estimate for \( k \in \mathcal{N}_0, \quad x \in \mathcal{R}, \)

\[
|z^k f(x)| = \left| \sum_{n=0}^{\infty} (f, \psi_n)_{l_2(\mathcal{R})} z^k \psi_n(x) \right|
\leq \sup_{n \in \mathcal{N}_0} \left| \exp(tn^{1/2\alpha}) (f, \psi_n)_{l_2(\mathcal{R})} \right| 2^{k/2} \sum_{n=0}^{\infty} \frac{((n+k)!/n!)^{1/2} \exp(-tn^{1/2\alpha})}{(n+k)!}.\]

By using Lemma 2.4, we estimate

\[
\sum_{n=0}^{\infty} \frac{((n+k)!/n!)^{1/2} \exp(-tn^{1/2\alpha})}{(n+k)!}
\leq \sum_{n=0}^{k} \frac{(2k)!/k!^{1/2}}{\sum_{n=0}^{k} (n+k)!/k!} \exp(-tn^{1/2\alpha})
\]

\[
\leq \sum_{n=0}^{k} \frac{(2k)!/k!^{1/2}}{\sum_{n=0}^{k} (n+k)!/k!} \exp(-tn^{1/2\alpha})
\]
\[ (k+1)(2^{k+1})^{1/2} + \sum_{n=k+1}^{\infty} (2n)^{k/2} \exp(-tn^{1/(2a)}) \]
\[ \leq 2^{k+1}(k)^{2/2} + 2^{2k/2} \sum_{n=1}^{\infty} n^{-2} (\sqrt{n})^{k+4} \exp(-t(\sqrt{n})^{1/4}) \]
\[ \leq 2^{k+1}(k)^{2/2} + 2^{2k}(\pi^2/6)(\alpha/\gamma)^{2k+4} ((k+4))^\alpha \]
\[ \leq 2^{k+1}(k)^{2/2} + (\pi^2/6)(4)^{k} (2\alpha/\gamma)^{2k} (4\alpha/\gamma)^k (k)^\alpha \]

By combining the previous results, we infer that there exist \( A, B > 0 \) such that
\[ \|z^k f\|_{L_\infty(R^1)} \leq BA^k(k)^\alpha, \quad k \in \mathbb{N}_0. \]

That is, \( f \in S_{\infty}^a(R^1) \), by Theorem 2.61 (iv).

Since \( f \in L_2(R^1) \) and \( (\exp(\text{tn}^{1/(2a)}) (f, \psi_n)_{L_2(R^1)}) \in L_\infty \) we also have \( f \in L_2(R^1) \) and by (1.9), \( (\exp(-\text{tn}^{1/(2a)}) (\mathcal{F}f, \psi_n)_{L_2(R^1)}) \in L_\infty \). By replacing \( f \) by \( \mathcal{F}f \) in the preceding derivation, we conclude that also \( \mathcal{F}f \in S_{\infty}^a(R^1) \).

Hence, \( f \in S_{\infty}^a(R^1) \) by Corollary 2.9. Now it is not difficult to see that the equalities hold as topological vector spaces.

(ii) Let \( f \in S_{\infty}^a(R^1) \) and let \( f_N = \sum_{n=0}^{N-1} (f, \psi_n)_{L_2(R^1)} \psi_n. \) We show that \( f - f_N \to 0 \) \((N \to \infty)\) in \( S_{2a}(P^2 + Q^2) \). By assertion (i), there exists \( t > 0 \) such that
\[ \sup_{n \in \mathbb{N}_0} |\exp(\text{tn}^{1/(2a)}) (f, \psi_n)_{L_2(R^1)}| = B < \infty. \]

For \( N \in \mathbb{N} \) we estimate, by using Lemma 2.4 (ii), (v),
\[ \| (P^2 + Q^2)^k (f - f_N) \|_{L_2(R^1)}^2 = \sum_{n=N}^{\infty} (2n+1)^{2k} \| (f, \psi_n)_{L_2(R^1)} \|^2 \]
\[ \leq B^2 \sum_{n=N}^{\infty} \exp(-2\text{tn}^{1/(2a)}) (3n)^{2k} \]
\[ \leq B^2 3^{2k} \sum_{n=N}^{\infty} \exp(-\text{tn}^{1/(2a)}) n^{2k} \sum_{n=N}^{\infty} \exp(-\text{tn}^{1/(2a)}) \]
\[ \leq B^2 (12a/\gamma)^{4k} (k)^{4k} \sum_{n=N}^{\infty} \exp(-\text{tn}^{1/(2a)}). \]

Hence \( f - f_N \in S_{2a(12a/\gamma)^2} (P^2 + Q^2), \) \( N \in \mathbb{N} \) and
\[ \| f - f_N \|_{P^2 + Q^2} \to 0 \quad (N \to \infty). \]

That is, by (2.4), \( f - f_N \to 0 \) \((N \to \infty)\) in \( S_{2a}(P^2 + Q^2) \).

\[ \square \]

**Theorem 2.16** Let \( \alpha \in \mathbb{R}_{\alpha_{\infty}}^a \). Then
\[ S_{\alpha}(Q) \cap S_{\alpha}(P) = S_{2a}(P^2 + Q^2) \text{ if and only if } \alpha \geq \frac{1}{2}. \]

**Proof.** For \( \alpha \geq \frac{1}{2} \) we have \( S_{\alpha}(Q) \cap S_{\alpha}(P) = S_{2a}(P^2 + Q^2) \), by Theorem 2.15 (i). For \( \alpha < \frac{1}{2} \) the space \( S_{\alpha}(Q) \cap S_{\alpha}(P) \) is trivial, by Theorems 2.11 and 2.2 (v), but the space \( S_{2a}(P^2 + Q^2) \) contains the Hermite functions \( \psi_n, n \in \mathbb{N}_0 \), by Lemma 2.14.

\[ \square \]
2.2 The subspaces $S^\beta_{\alpha, \text{even}}(IR)$ and the Hankel transformation $H_\nu$

In this section we consider the subspace $S^\beta_{\alpha, \text{even}}(IR)$ of even functions in $S^\beta_{\alpha}(IR)$. The topology on $S^\beta_{\alpha, \text{even}}(IR)$ is the induced topology. Likewise, $S^\beta_{\alpha, \text{odd}}(IR)$ is the subspace of odd functions in $S^\beta_{\alpha}(IR)$. The set-up of this section is similar to that of Section 1.2 where we discussed the space $S_{\text{even}}(IR)$.

Let us start with a straightforward consequence of Theorem 2.8 (cf. Theorem 1.6a).

**Theorem 2.17** Let $\alpha, \beta \in IR^+$ and let $n \in NR$. Then

(i) $F(S^\beta_{\alpha, \text{even}}(IR)) = S^\beta_{\alpha, \text{even}}(IR)$,

(ii) $Q(S^\beta_{\alpha, \text{even}}(IR)) = S^\beta_{\alpha, \text{odd}}(IR)$,

$$Q^2(S^\beta_{\alpha, \text{even}}(IR)) = \{ f \in S^\beta_{\alpha, \text{even}}(IR) : f^{(2k)}(0) = 0, \ k = 0, \ldots, n-1 \},$$

$$Q^2(S^\beta_{\alpha, \text{even}}(IR)) = \{ f \in S^\beta_{\alpha, \text{odd}}(IR) : f^{(2k+1)}(0) = 0, \ k = 0, \ldots, n-1 \},$$

(iii) $P(S^\beta_{\alpha, \text{even}}(IR)) = S^\beta_{\alpha, \text{odd}}(IR)$,

$$P^2(S^\beta_{\alpha, \text{even}}(IR)) = \{ f \in S^\beta_{\alpha, \text{even}}(IR) : \int_R f(y) y^{2k} dy = 0, \ k = 0, \ldots, n-1 \},$$

$$P^2(S^\beta_{\alpha, \text{even}}(IR)) = \{ f \in S^\beta_{\alpha, \text{odd}}(IR) : \int_R f(y) y^{2k+1} dy = 0, \ k = 0, \ldots, n-1 \}. $$

In Theorem 1.8 we have characterized the functions in $S_{\text{even}}(IR)$ in terms of their expansion coefficients with respect to the Laguerre basis $\{ L_n^\alpha : n \in NR \}$ in $X_{2+1}$, as introduced in (1.12): For $\nu \geq -\frac{1}{2}$,

$$S_{\text{even}}(IR) = \{ f \in X_{2+1} : (n^k (f, L_n^\alpha)_{X_{2+1}}) \in IR \}.$$ 

Starting from Theorem 2.15 and using the same techniques as in the proof of Theorem 1.8, we establish the analogous characterization of the functions in $S^\beta_{\alpha, \text{even}}(IR)$ with $\alpha \geq \frac{1}{2}$.

**Theorem 2.18** Let $\alpha \geq \frac{1}{2}$ and let $\nu \geq -\frac{1}{2}$. Then

(i) $S^\beta_{\alpha, \text{even}}(IR) = \{ f \in X_{2+1} : \exists_{\nu \geq 0} \{ (tn^{1/(2\alpha)} (f, L_n^\alpha)_{X_{2+1}}) \in IR \} \}$.

(ii) For $f \in S^\beta_{\alpha, \text{even}}(IR)$ we have

$$f(x) = \sum_{n=0}^{\infty} (f, L_n^\alpha)_{X_{2+1}} L_n^\alpha(x), \ x \in IR^+,$$

where the series converges in $S^\beta_{\alpha, \text{even}}(IR)$.

Next we shall study the Hankel transformation $H_\nu$ and the Erdélyi-Kober operator $W_\gamma$, defined in (1.16) and (1.23), respectively, on $S^\beta_{\alpha, \text{even}}(IR)$. As we have seen in Theorem 1.9 and in (1.25), these operators are bijections on $S_{\text{even}}(IR)$. By (1.17) we have for $f \in S_{\text{even}}(IR)$,
\[(\mathcal{F}f)(x) = (\mathcal{H}_{-1/2}f)(x), \quad x > 0.\]

So it follows from Corollary 2.9 that a function \(f\) belongs to \(S_{\alpha,\text{even}}^0(\mathcal{R})\) if and only if

\[f \in S_{\alpha,\text{even}}^0(\mathcal{R}) \quad \text{and} \quad \mathcal{H}_{-1/2}f \in S_{\beta,\text{even}}^0(\mathcal{R}).\]

Now the natural question arises whether in this characterization the role of \(\mathcal{H}_{-1/2}\) can be taken over by \(\mathcal{H}_\nu\) with arbitrary \(\nu \geq -1/2\). In Theorem 2.20 an affirmative answer to this question is given.

As a consequence we can characterize the functions \(f \in S_{\alpha,\text{even}}^0(\mathcal{R})\) in terms of conditions of decrease at \(+\infty\) of the function \(f\) and of its Hankel transform \(\mathcal{H}_\nu f\) (see Theorem 2.21). Moreover, it then follows in a straightforward manner that (see Theorem 2.22)

\[\mathcal{B}_\nu(S_{\alpha,\text{even}}^0(\mathcal{R})) = S_{\alpha,\text{even}}^0(\mathcal{R}), \quad \mathcal{H}_\nu(S_{\alpha,\text{even}}^0(\mathcal{R})) = S_{\beta,\text{even}}^0(\mathcal{R}).\]

For \(\lambda = m \in \mathbb{Z}\) and \(\nu + 1/2 = n \in \mathbb{N}_0\), these identities can easily be verified:

**Lemma 2.19** Let \(\alpha, \beta \in \mathbb{R}^+_\infty\). Then

(i) \(\mathcal{B}_m(S_{\alpha,\text{even}}^0(\mathcal{R})) = S_{\alpha,\text{even}}^0(\mathcal{R}), \quad m \in \mathbb{Z}\),

(ii) \(\mathcal{H}_{-1/2+m}(S_{\alpha,\text{even}}^0(\mathcal{R})) = S_{\beta,\text{even}}^0(\mathcal{R}), \quad n \in \mathbb{N}_0\).

**Proof.** (i) By (1.23) and Theorem 2.17 we have

\[\mathcal{B}_{-1}(S_{\alpha,\text{even}}^0(\mathcal{R})) = (z^{-1} d/dz)(S_{\alpha,\text{even}}^0(\mathcal{R})) = S_{\alpha,\text{even}}^0(\mathcal{R}).\]

The identity to be proved is now obvious, since the operators \(\mathcal{B}_{\lambda}, \lambda \in \mathbb{R}\), constitute a group of continuous operators on \(S_{\alpha,\text{even}}^0(\mathcal{R})\), by (1.25).

(ii) Let \(n \in \mathbb{N}_0\). By using (1.24) and the equality \(\mathcal{H}_{-1/2}^2 = I\) we have

\[\mathcal{H}_{-1/2+n}(S_{\alpha,\text{even}}^0(\mathcal{R})) = \mathcal{B}_n(\mathcal{H}_{-1/2}(S_{\alpha,\text{even}}^0(\mathcal{R}))).\]

The result to be proved then follows from Theorem 2.17 (i) and assertion (i). \(\square\)

We establish a generalization of Corollary 2.9.

**Theorem 2.20** Let \(\alpha, \beta \in \mathbb{R}^+_\infty\), let \(\nu \geq -1/2\) and let \(f : \mathcal{R} \to \mathbb{C}\). Then the following assertions are equivalent:

(i) \(f \in S_{\alpha,\text{even}}^0(\mathcal{R})\);

(ii) \(f \in S_{\alpha,\text{even}}^\infty(\mathcal{R})\) and \(\mathcal{H}_\nu f \in S_{\beta,\text{even}}^\infty(\mathcal{R})\).

**Proof.** First we consider the case \(\alpha = \infty, \beta = 0\).

From Theorem 2.2 (i) we know that \(S_{\alpha,\text{even}}^\infty(\mathcal{R}) = C_{\alpha}^\infty(\mathcal{R})\), the subspace of even functions in \(C_0^\infty(\mathcal{R})\). The operators \(\mathcal{B}_{\lambda}, \lambda \in \mathbb{R}\), map \(S_{\alpha,\text{even}}^\infty(\mathcal{R})\) onto itself, by (1.25). Hence it follows immediately from the definition of \(\mathcal{B}_{\lambda}\) in (1.23), that \(\mathcal{B}_{\lambda}(\lambda \in \mathbb{R})\) maps \(S_{\alpha,\text{even}}^\infty(\mathcal{R})\) onto itself. Then, by using (1.24) and Theorem 2.17 (i) we have

\[\mathcal{H}_\nu(S_{\alpha,\text{even}}^\infty(\mathcal{R})) = \mathcal{H}_{-1/2}(\mathcal{B}_{\nu+1/2}(S_{\alpha,\text{even}}^\infty(\mathcal{R}))) = S_{\infty,\text{even}}^\infty(\mathcal{R}).\]
Since $H^2 = I$, the theorem holds true for $\alpha = \infty$, $\beta = 0$.
Next we consider the case $\alpha = \infty$, $\beta \in \mathbb{R}^+$.
Suppose $f \in S^0_{\omega, \text{even}}(\mathbb{R})$. Choose $n \in \mathbb{N}$ such that $-1/2 + n > \nu$. By Lemma 2.19 (ii) we have $H_{-1/2+n} f \in S^\infty_{\omega, \text{even}}(\mathbb{R})$. Hence, by Theorem 2.12, there exists $t > 0$ such that
$$\exp(tx^{1/\beta}) H_{-1/2+n} f \in L_2(\mathbb{R}^+) .$$
By applying Lemma 1.11 with $\nu$ replaced by $-1/2 + n$ and $\mu$ replaced by $\nu$ and $W(x) = \exp(x^{1/\beta})$, we obtain
$$(1 + x^2)^{\nu - n - 1/2} \exp(\frac{1}{2}t x^{1/\beta}) H_{\omega} f \in L_2(\mathbb{R}^+) .$$
Hence
$$\exp(t x^{1/\beta}) H_{\omega} f \in L_2(\mathbb{R}^+) .$$
That is, $H_{\omega} f \in S^\infty_{\omega, \text{even}}(\mathbb{R})$ by Theorem 2.12.
Conversely, suppose $f \in S^\infty_{\omega, \text{even}}(\mathbb{R}) = S_{\text{even}}(\mathbb{R})$ and $H_{\omega} f \in S^\infty_{\omega, \text{even}}(\mathbb{R})$. By Theorem 2.12 there exists $t > 0$ such that
$$\exp(t x^{1/\beta}) H_{\omega} f \in L_2(\mathbb{R}^+) .$$
By Lemma 1.11 with $\mu = -1/2$ and $W(x) = \exp(x^{1/\beta})$ it follows that
$$(1 + x^2)^{-\nu - 1/2} \exp(\frac{1}{2}t x^{1/\beta}) H_{-1/2} f \in L_2(\mathbb{R}^+) .$$
Hence
$$\exp(t x^{1/\beta}) H_{-1/2} f \in L_2(\mathbb{R}^+) .$$
That is, $H_{-1/2} f \in S^\infty_{\omega, \text{even}}(\mathbb{R})$ by Theorem 2.12. Now Theorem 2.17 (i) yields $f \in S^\infty_{\omega, \text{even}}(\mathbb{R})$. So the theorem holds true for $\alpha = \infty$, $\beta \in \mathbb{R}^+$.
In case $\alpha \in \mathbb{R}^+, \beta = \infty$, the result of the theorem is obvious in view of Theorem 1.9.
In case $\alpha, \beta \in \mathbb{R}^+$, the theorem follows from the preceding results for $\alpha = \infty$ or $\beta = \infty$ and Kashpirovskii’s intersection result in Theorem 2.7.

Recalling the characterization of $S_{\text{even}}(\mathbb{R})$ by means of the Hankel transformation $H_{\omega}$
in Theorem 1.12, we present the following characterization of the subspaces of even functions in the Gelfand-Shilov spaces, obtainable by means of Theorems 2.20 and 2.13.

**Theorem 2.21.** Let $\nu \geq -\frac{1}{2}$, let $f \in \mathcal{X}_{\nu+1}$ and let $p \in \{2, \infty\}$.
1. Let $\alpha \in \mathbb{R}^+$. Then the following assertions are equivalent:
   (i) $f \in S^\infty_{\alpha, \text{even}}(\mathbb{R})$;
   (ii) there exist $A, B > 0$ such that
   $$\|x^k f\|_{L_p(\mathbb{R}^+)} \leq BA^k(k!)^\nu$$ and
   $x^l H_{\omega} f \in L_p(\mathbb{R}^+) , \quad k, l \in \mathbb{N}_0 ;$
   (iii) there exists $t > 0$ such that
   $$\exp(t x^{1/\alpha}) f \in L_p(\mathbb{R}^+)$$ and
   $x^l H_{\omega} f \in L_p(\mathbb{R}^+) , \quad l \in \mathbb{N}_0 .$$
II. Let $\beta \in \mathbb{R}^+$. Then the following assertions are equivalent:

(i) $f \in S^0_{\alpha, even}(\mathbb{R})$ ;

(ii) there exist $A, B > 0$ such that

$$z^k f \in L_p(\mathbb{R}^+), \text{ and } ||z^l \mathcal{H}_s f||_{L_p(\mathbb{R}^+)} \leq AB^l(l!)^\beta, \quad k, l \in \mathbb{N}_0 ;$$

(iii) there exists $t > 0$ such that

$$z^k f \in L_p(\mathbb{R}^+) \text{ and } \exp(t z^{1/\beta}) \mathcal{H}_s f \in L_p(\mathbb{R}^+), \quad k \in \mathbb{N}_0 .$$

III. Let $\alpha, \beta \in \mathbb{R}^+$. Then the following assertions are equivalent:

(i) $f \in S^0_{\alpha, even}(\mathbb{R})$ ;

(ii) there exist $A, B > 0$ such that

$$||z^k f||_{L_p(\mathbb{R}^+)} \leq BA^k(k!)^\alpha \text{ and } ||z^l \mathcal{H}_s f||_{L_p(\mathbb{R}^+)} \leq AB^l(l!)^\beta, \quad k, l \in \mathbb{N}_0 ;$$

(iii) there exists $t > 0$ such that

$$\exp(t z^{1/\alpha}) f \in L_p(\mathbb{R}^+) \text{ and } \exp(t z^{1/\beta}) \mathcal{H}_s f \in L_p(\mathbb{R}^+) .$$

Theorem 2.22 Let $\alpha, \beta \in \mathbb{R}^+$, let $\nu \geq -1/2$ and let $\lambda \in \mathbb{R}$. Then

(i) $\mathcal{H}_s(S^0_{\alpha, even}(\mathbb{R})) = S^0_{\alpha, even}(\mathbb{R})$ ,

(ii) $\mathcal{W}_\nu(S^0_{\beta, even}(\mathbb{R})) = S^0_{\beta, even}(\mathbb{R})$ .

Proof. (i) From Theorem 2.20 with $\alpha = \infty$ it follows that $\mathcal{H}_s(S^0_{\beta, even}(\mathbb{R})) = S^0_{\beta, even}(\mathbb{R})$. Now the wanted identity follows from the equality $\mathcal{H}_s^2 = I$ in (1.20) and Kashpirovskii’s intersection result in Theorem 2.7.

(ii) The identity follows from assertion (i) and (1.24), viz.

$$\mathcal{H}_s \mathcal{H}_s f = \mathcal{W}_{-\nu} f \text{ for } f \in S_{even}(\mathbb{R}), \mu, \nu \geq -1/2 .$$

The study of the Hankel transformation $\mathcal{H}_s$ on $S^0_{\alpha, even}(\mathbb{R})$ has been the subject of a number of papers. Pathak [42] states that $\mathcal{H}_s(S^0_{\alpha, even}(\mathbb{R})) = S^0_{\alpha, even}(\mathbb{R})$ in case $0 < \alpha, \beta < 1$. However, the part of his proof, that is based on contour integration in the complex plane, is incorrect. A corrected proof has been given by Van Eijndhoven and Kerkhof [16]. Van Eijndhoven and de Graaf [15] have proved that $\mathcal{H}_s(S^0_{\alpha, even}(\mathbb{R})) = S^0_{\alpha, even}(\mathbb{R})$ in case $1/2 \leq \alpha \leq 1$, by using properties of the Laguerre polynomials. The paper [13] by Van Eijndhoven and Van Berkel, contains part of the results of Theorems 2.20 and 2.21. Finally, we present a functional analytic characterization of $S^0_{\alpha, even}(\mathbb{R})$. 
Corollary 2.23 Let $\alpha, \beta \in \mathbb{R}^+_\text{even}$ and let $\nu \geq -\frac{1}{2}$. Then
\[ S^\beta_{\alpha, \text{even}}(\mathbb{R}) = S_{2\nu}(Q^2) \cap S_{2\lambda}(\mathcal{H}_e Q^2 \mathcal{H}_v) , \]
where $Q$ is the self-adjoint operator defined on $\{ f \in X_{2\nu+1} : zf \in X_{2\nu+1} \}$ by $Qf = zf$.

Proof. First we consider the case $\beta = \infty$, $\alpha \neq \infty$. Let $f \in X_{2\nu+1}$. Observe that $f \in S_{2\nu}(\mathcal{H}_e Q^2 \mathcal{H}_v)$ if and only if $\mathcal{H}_e f \in S_{\infty}(Q^2)$. So we have to show that the following assertions are equivalent:

1. $f \in S^\infty_{\alpha, \text{even}}(\mathbb{R})$;
2. there exist $A, B > 0$ such that
\[ \| z^{2k} f \|_{X_{2\nu+1}} \leq B A^k (|k|)^{2\alpha} \quad \text{and} \quad z^{2l} \mathcal{H}_e f \in X_{2\nu+1}, \quad k, l \in \mathbb{N}_0 . \]

Suppose $f$ satisfies (2). Then $f \in S_{\infty}(\mathbb{R})$ by Corollary 1.13 and
\[ \| z^{2k} f \|_{L^2(R^+)} \leq \| f \|_{L^\infty(R^+)} + \| z^{2l} f \|_{L_{2\nu+1}} , \quad k \in \mathbb{N}_0 . \]

Hence it follows that $f \in S^\infty_{\alpha, \text{even}}(\mathbb{R})$ by Theorem 2.61 (iii). The converse, (1) $\Rightarrow$ (2), follows from the definition of $S^\infty_{\alpha}(\mathbb{R})$ and Theorem 1.9.

Next we consider the case $\alpha = \infty$, $\beta \neq \infty$. By observing that $S_{2\nu}(\mathcal{H}_e Q^2 \mathcal{H}_v) = \mathcal{H}_e (S_{2\nu}(Q^2))$, $\gamma \in \mathbb{R}^+_\text{even}$ and that $\mathcal{H}_e^2 = I$, we deduce by means of Theorem 2.22 (i) and the previous result,
\[ S^\beta_{\infty, \text{even}}(\mathbb{R}) = \mathcal{H}_e (S^\infty_{\beta, \text{even}}(\mathbb{R})) = \mathcal{H}_e (S_{2\alpha}(Q^2) \cap S_{\infty}(\mathcal{H}_e Q^2 \mathcal{H}_v)) \]
\[ = S_{2\beta}(\mathcal{H}_e Q^2 \mathcal{H}_v) \cap S_{\infty}(Q^2) . \]

Finally, for $\alpha \neq \infty$ and $\beta \neq \infty$ the wanted identity follows from the previous results and Kashpirovskii's intersection result in Theorem 2.7.

\[ \square \]

2.3 The spaces $G^\alpha(\mathbb{R}^*)$ and the Hankel-Clifford transformation $\mathcal{H}_\nu$

In Theorem 1.15 (ii) we have shown that $S(\mathbb{R}^*)$ is equal to the space
\[ \{ g \in C^\infty(\mathbb{R}^*) : \text{the function } z \mapsto g(z^2) \text{ belongs to } S_{\text{even}}(\mathbb{R}) \} . \]

The present section deals with the spaces $G^\alpha(\mathbb{R}^*)$ defined as follows:

Definition 2.24 Let $\alpha, \beta \in \mathbb{R}^+_\text{even}$. Then
\[ G^\alpha(\mathbb{R}^*) = \{ g \in C^\infty(\mathbb{R}^*) : \text{the function } z \mapsto g(z^2) \text{ belongs to } S^\beta_{\alpha, \text{even}}(\mathbb{R}) \} . \]

The spaces $G^\alpha(\mathbb{R}^*)$ are related to the spaces $G^\alpha$, $G^\beta$ and $G^\alpha_\nu$, introduced by Duran [11]. In [11, Definitions 2.1, 2.2 and 2.3], the latter spaces are defined by means of certain conditions on the norms.
\[ \|z^{(k+1)/2}g^{(l)}\|_{L^2(\mathbb{R}^+)} \quad k, l \in \mathbb{N}_0, \quad \|z^{l}g^{(l)}\|_{L^2(\mathbb{R}^+)} \quad 0 \leq k \leq l/2. \]

Furthermore, Duran shows in [11, Corollary 3.8] that a function \( g \) belongs to the space \( G_{\alpha} \), \( G_{\alpha}^0 \) or \( G_{\alpha}^{0/2} \), if and only if the function \( x \mapsto g(x^2) \) belongs to the space \( S_{\alpha/2, \text{even}}, S_{\alpha/2, \text{even}}^{0/2} \) or \( S_{\alpha/2, \text{even}}^{0/2} \), respectively. Thus we have the connection

\[ G_{2\alpha} = G_{\alpha}^{0/2}(\mathbb{R}^+) \quad G_{2\alpha} = G_{\alpha}^{0/2}(\mathbb{R}^+) \quad G_{2\alpha} = G_{\alpha}^{0}(\mathbb{R}^+) \]

Our treatment of the spaces \( G_{\alpha}^{0}(\mathbb{R}^+) \) is reverse to Duran's. We start from Definition 2.24

and then we characterize the spaces \( G_{\alpha}^{0}(\mathbb{R}^+) \) by means of the operators \( x \) and \( d/dz \). As a consequence, we deduce new characterizations of the spaces \( S_{\alpha, \text{even}}^{0}(\mathbb{R}) \) by means of the operators \( x \) and \( x^{-1}d/dx \).

The results for \( S_0^{0}(\mathbb{R}) \) and \( S_{\alpha, \text{even}}^{0}(\mathbb{R}) \) established in Sections 2.1 and 2.2 can be translated into results for \( G_{\alpha}^{0}(\mathbb{R}^+) \). We recall the definition of the Hankel-Clifford transformation \( \mathcal{H}_v \) in (1.27) and of the Weyl operator \( W_\gamma \) in (1.34). We also recall the formulas

\[ \mathcal{H}_v = T_{1/2} B H \mathcal{T}_1 \quad \text{and} \quad \mathcal{H}_v \mathcal{H}_u = 2^{v-u} W_{v-u} \]

from (1.29) and (1.35). By means of Theorem 1.15 (ii) and various results from Sections 2.1 and 2.2 we gather the following properties of the spaces \( G_{\alpha}^{0}(\mathbb{R}^+) \).

**Theorem 2.25** Let \( \alpha, \beta \in \mathbb{R}^+_{\text{even}} \). Then

(i) \( G_{\alpha}^{0}(\mathbb{R}^+) = S(\mathbb{R}^+) \);

(ii) \( G_{\alpha}^{0}(\mathbb{R}^+) \) is nontrivial if and only if \( S_{\alpha}(\mathbb{R}) \) is nontrivial;

(iii) for \( \beta \leq 1 \), the functions in \( G_{\alpha}^{0}(\mathbb{R}^+) \) have an analytic continuation in the complex plane;

(iv) \( G_{\alpha}^{0}(\mathbb{R}^+) \cap G_{\alpha}^{0}(\mathbb{R}^+) = G_{\alpha}^{0}(\mathbb{R}^+) \);

(v) let \( \nu \geq -1/2 \) and let \( g : \mathbb{R}^+ \rightarrow \mathbb{C} \), then \( g \in G_{\alpha}^{0}(\mathbb{R}^+) \) if and only if \( g \in G_{\alpha}^{0}(\mathbb{R}^+) \) and \( \mathcal{H}_v g \in G_{\alpha}^{0}(\mathbb{R}^+) \);

(vi) \( W_\mu(G^{0}(\mathbb{R}^+)) = G^{0}(\mathbb{R}^+) \), \( \nu \geq -1/2 \);

(vii) \( W_\mu(G^{0}(\mathbb{R}^+)) = G^{0}(\mathbb{R}^+) \), \( \lambda \in \mathbb{R} \);

(viii) \( G_{\alpha}^{0}(\mathbb{R}^+) = S_{2\alpha}(Q) \cap S_{2\alpha}(\mathcal{H}_v Q H_\alpha) \), \( \nu \geq -1/2 \), where \( Q \) is the self-adjoint operator defined on \( \{ g \in X_\alpha : xg \in X_\alpha \} \) by \( Qg = xg \);

(ix) \( G_{\alpha}^{0}(\mathbb{R}^+) = \{ g \in X_\alpha : \exists_{\nu>0} (\exp(tn^{1/(2\alpha)}) (g, L_{\alpha}^2)_{X_\alpha}) \in l_\infty \}, \quad \alpha \geq 1/2, \quad \nu \geq -1/2 \), and for \( g \in G_{\alpha}^{0}(\mathbb{R}^+) \) we have

\[ g(z) = \sum_{n=0}^{\infty} (g, L_{\alpha}^2)_{X_\alpha} z^n, \quad z \in \mathbb{R}^+ \]

where the series converges in \( G_{\alpha}^{0}(\mathbb{R}^+) \).
Next we shall derive several characterizations of the spaces $G^k_\alpha(\mathbb{R}^+)$ by means of the operators $x$ and $d/dx$. We start with a characterization of $G^\alpha_\infty(\mathbb{R}^+)$; by applying the Hankel-Clifford transformation $\mathcal{H}_\alpha$, we obtain a characterization of $G^0_\infty(\mathbb{R}^+)$; then by means of the intersection result $G^0_\infty(\mathbb{R}^+) \cap G^\alpha_\infty(\mathbb{R}^+) = G^0_\alpha(\mathbb{R}^+)$ we arrive at a characterization of $G^0_\alpha(\mathbb{R}^+)$. 

**Theorem 2.26** Let $\alpha, \beta \in \mathbb{R}^+_0$ and let $g \in C^\infty(\mathbb{R}^+)$. Then 

(i) $g \in G^\alpha_\infty(\mathbb{R}^+)$ if and only if there exist $A, B > 0$ such that 
\[
\|x^{k/2}g\|_{L^2(\mathbb{R}^+)} \leq BA^k(k!)^\alpha, \quad x^{l/2}g^{(l)} \in L^2(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0;
\]

(ii) $g \in G^\alpha_\infty(\mathbb{R}^+)$ if and only if there exist $A, B > 0$ such that 
\[
x^{k/2}g \in L^2(\mathbb{R}^+), \quad \|x^{l/2}g^{(l)}\|_{L^2(\mathbb{R}^+)} \leq AB^l(l!)^\alpha, \quad k, l \in \mathbb{N}_0;
\]

(iii) $g \in G^\alpha_\infty(\mathbb{R}^+)$ if and only if there exist $A, B > 0$ such that 
\[
\|x^{k/2}g\|_{L^2(\mathbb{R}^+)} \leq BA^k(k!)^\alpha, \quad \|x^{l/2}g^{(l)}\|_{L^2(\mathbb{R}^+)} \leq AB^l(l!)^\alpha, \quad k, l \in \mathbb{N}_0.
\]

**Proof.** (i) The equivalence is a straightforward consequence of Theorems 1.20 and 2.6.2.5. 

(ii) By Theorem 2.25 (v) we have 
\[
g \in G^\alpha_\infty(\mathbb{R}^+) \text{ if and only if } \mathcal{H}_\alpha g \in G^\alpha_\infty(\mathbb{R}^+)
\]

Since $\mathcal{H}_\alpha$ is unitary on $X_k, k \in \mathbb{N}_0$, we have for $h \in S(\mathbb{R}^+)$, 
\[
\|x^{l/2}\mathcal{H}_\alpha h\|_{L^2(\mathbb{R}^+)} = \|\mathcal{H}_\alpha h\|_{X_0} = 2^l \|x^{l/2}h^{(l)}\|_{L^2(\mathbb{R}^+)}, \quad l \in \mathbb{N}_0.
\]

Noting that $\mathcal{H}^2_\alpha = I$, we obtain the wanted equivalence from (i). 

(iii) The equivalence is obtained from (i) and (ii) because of Theorem 2.25 (iv). \hfill \Box

In the above characterizations we want 

1. to replace $L^2(\mathbb{R}^+)$ and the $L^2(\mathbb{R}^+)$-norm by $L^\infty(\mathbb{R}^+)$ and the $L^\infty(\mathbb{R}^+)$-norm; 
2. to replace the conditions on $x^{k/2}g$ and $x^{l/2}g^{(l)}$ by conditions on $x^{(k+1)/2}g^{(k)}$ and 
   $x^{(l+1)/2}g^{(l)}$, 
   cf. the characterizations of $S(\mathbb{R}^+)$ in Theorem 1.20. 

As a preliminary we establish the following lemma.

**Lemma 2.27** Let $g \in S(\mathbb{R}^+)$. Then 

(i) $\|x^{(k+1)/2}g^{(k)}\|_{L^2(\mathbb{R}^+)} = 2^{k-1} \|x^{(k+1)/2}(\mathcal{H}_\alpha g)^{(k)}\|_{L^2(\mathbb{R}^+)}$, $k, l \in \mathbb{N}_0$; 

(ii) $\|x^{(k+1)/2}g^{(k)}\|_{L^2(\mathbb{R}^+)} \leq \sum_{j=0}^{\min(k,l)} \binom{k}{j} \binom{l}{j} j!$ 

\[
\|x^{(k+1)/2}g\|_{L^2(\mathbb{R}^+)} \|x^{(l+1)/2}g^{(l)}\|_{L^2(\mathbb{R}^+)} \quad k, l \in \mathbb{N}_0.
\]

**Proof.** (i) By means of (1.35), viz. $\mathcal{H}_\alpha \mathcal{H}_\alpha = 2^\nu W_{\nu-n}$, and the property that $\mathcal{H}_\alpha$ is a unitary operator on $X_{k+l}$, we derive
\[ \|x^{(k+l)/2}g^{(l)}\|_{L^2(R^+)} = 2^{-l} \|H_x H_0 g\|_{L^2(R^+)} = 2^{-l} \|H_x H_0 H_0 g\|_{L^2(R^+)} = 2^{-l} \|x^{(k+l)/2}(H_0 g)^{(l)}\|_{L^2(R^+)} . \]

(ii) First suppose \( k \geq l \). Then, through integration by parts we have

\[ \|x^{(k+l)/2}g^{(l)}\|^2_{L^2(R^+)} = \int x^{(k+l)/2}(x)g^{(l)}(x)dx \]

\[ = \left(-1\right)^l \sum_{j=0}^l \binom{l}{j} \binom{k+l}{j} j! \int x^{k+l-j} g^{(2l-j)}(x) \frac{\partial^j g(x)}{\partial x^j} dx \]

\[ \leq \sum_{j=0}^l \binom{l}{j} \binom{k+l}{j} j! \|x^{(2l-j)/2}g^{(2l-j)}\|_{L^2(R^+)} \|x^{(k+l)/2}g\|_{L^2(R^+)} . \]

Next suppose \( k < l \). Then, by (i) and the previous estimate,

\[ \|x^{(k+l)/2}g^{(l)}\|^2_{L^2(R^+)} = 2^{2k-2} \|x^{(k+l)/2}(H_0 g)^{(l)}\|^2_{L^2(R^+)} \]

\[ \leq 2^{2k-2} \sum_{j=0}^k \binom{k}{j} \binom{k+l}{j} j! \|x^{(2l-j)/2}(H_0 g)^{(2l-j)}\|_{L^2(R^+)} \|x^{(2l-j)/2}g\|_{L^2(R^+)} \]

\[ = \sum_{j=0}^k \binom{k}{j} \binom{k+l}{j} j! \|x^{(2l-j)/2}g\|_{L^2(R^+)} \|x^{(2l-j)/2}g\|_{L^2(R^+)} . \] \[ \square \]

**Theorem 2.28** Let \( g \in \mathcal{C}^{\infty}(R^+) \).

1. Let \( x \in R_0^* \). Then the following assertions are equivalent:

(i) \( g \in \mathcal{C}^{\infty}(R^+) \);

(ii) there exist \( A, B > 0 \) (\( l \in \mathbb{N}_0 \)) such that

\[ \|x^{(k+l)/2}g^{(l)}\|_{L^2(R^+)} \leq B_l A^k(k!)^l \] \( k, l \in \mathbb{N}_0 \);

(iii) there exist \( A, B > 0 \) (\( l \in \mathbb{N}_0 \)) such that

\[ \|x^{(k+l)/2}g^{(l)}\|_{L^1(R^+)} \leq B_l A^k(k!)^l \] \( k, l \in \mathbb{N}_0 \);

(iv) there exist \( A, B > 0 \) such that

\[ \|x^{l/2}g\|_{L^1(R^+)} \leq B A^l(k!)^l \] \( k, l \in \mathbb{N}_0 \);

(v) there exist \( A, B > 0 \) such that

\[ \|x^{l/2}g\|_{L^2(R^+)} \leq B A^l(k!)^l \] \( k, l \in \mathbb{N}_0 \);
II. Let $\beta \in \mathbb{R}_2^+$. Then the following assertions are equivalent:

(i) $g \in G_\infty^\beta(\mathbb{R}^+)$;

(ii) there exist $A, B > 0$ ($k \in \mathbb{N}_0$) such that

$$\|x^{(k+1)/2}g^{(0)}\|_{L_\infty(\mathbb{R}^+)} \leq A_k B^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(iii) there exist $A, B > 0$ ($k \in \mathbb{N}_0$) such that

$$\|x^{(k+1)/2}g^{(0)}\|_{L_2(\mathbb{R}^+)} \leq A_k B^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(iv) there exist $A, B > 0$ such that

$$x^{k/2}g \in L_2(\mathbb{R}^+), \quad \|x^{l/2}g^{(0)}\|_{L_2(\mathbb{R}^+)} \leq AB^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(v) there exist $A, B > 0$ such that

$$x^{k/2}g \in L_\infty(\mathbb{R}^+), \quad \|(1 + z)x^{l/2}g^{(0)}\|_{L_\infty(\mathbb{R}^+)} \leq AB^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0.$$

III. Let $\alpha, \beta \in \mathbb{R}_2^+$ with $\alpha + \beta \geq 1$. Then the following assertions are equivalent:

(i) $g \in G_\infty^\beta(\mathbb{R}^+)$;

(ii) there exist $A, B, C > 0$ such that

$$\|x^{(k+1)/2}g^{(0)}\|_{L_\infty(\mathbb{R}^+)} \leq CA_k B^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(iii) there exist $A, B, C > 0$ such that

$$\|x^{(k+1)/2}g^{(0)}\|_{L_2(\mathbb{R}^+)} \leq CA_k B^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(iv) there exist $A, B > 0$ such that

$$\|x^{k/2}g\|_{L_2(\mathbb{R}^+)} \leq BA_k^\alpha(\pi)^\alpha, \quad \|x^{l/2}g^{(0)}\|_{L_2(\mathbb{R}^+)} \leq AB^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0;$$

(v) there exist $A, B > 0$ such that

$$\|x^{k/2}g\|_{L_\infty(\mathbb{R}^+)} \leq BA_k^\alpha(\pi)^\alpha, \quad \|(1 + x)x^{l/2}g^{(0)}\|_{L_\infty(\mathbb{R}^+)} \leq AB^\beta(\pi)^\beta, \quad k, l \in \mathbb{N}_0.$$

**Proof.** In each of the cases I, II and III, we follow the scheme (i) $\Leftrightarrow$ (iv), (ii) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

The equivalence (i) $\Leftrightarrow$ (iv) is taken from Theorem 2.26. Obviously (ii) $\Rightarrow$ (v). By using Theorem 1.20 it follows that (v) $\Rightarrow$ (iv) in case I; in cases II and III the implication (v) $\Rightarrow$ (iv) is obvious. By Lemma 2.27 (ii) we have (iv) $\Rightarrow$ (iii).

Finally, the implication (iii) $\Rightarrow$ (ii) follows by means of the following estimates for $g \in \mathcal{S}(\mathbb{R}^+)$. 

\[ \left| \frac{(k+1)^2}{2} g^{(l)}(x) \right| = 2 \text{Re} \int_{0}^{\infty} t^{(k+1)/2} g^{(l)}(t) \left( t^{(k+1)/2} g^{(l)}(t) \right)' \, dt \]

\[ \leq (k + l) \left\| x^{(k+1)/2} g^{(l)} \right\|_{L_2(\mathbb{R}^+)} + 2 \left\| x^{(k+1)/2} g^{(l)} \right\|_{L_2(\mathbb{R}^+)} \left\| x^{(k+1)/2} g^{(l+1)} \right\|_{L_2(\mathbb{R}^+)}, \]

\[ k \in \mathbb{N}, \quad l \in \mathbb{N}_0, \]

and

\[ 2^{l+1} \left| x^{l/2} g^{(0)}(x) \right| = 2 \left| x^{l/2} (\mathcal{H}_0 g)(x) \right| = \left| \int_{0}^{\infty} J_l(\sqrt{2} y) \left( \mathcal{H}_0 g \right)(y) y^{l/2} \, dy \right| \]

\[ \leq \left\| (1 + y) y^{l/2} \mathcal{H}_0 g \right\|_{L_2(\mathbb{R}^+)} \leq \left\| y^{l/2} \mathcal{H}_0 g \right\|_{L_2(\mathbb{R}^+)} + \left\| y^{(l+1)/2} g \right\|_{L_2(\mathbb{R}^+)} \]

\[ = 2 \left\| y^{l/2} g^{(0)} \right\|_{L_2(\mathbb{R}^+)} + 2^{l+2} \left\| y^{(l+3)/2} g^{(l+2)} \right\|_{L_2(\mathbb{R}^+)}, \quad l \in \mathbb{N}_0. \]

In the latter derivation we used the inequality \( |J_l(t)| \leq 1 \) for \( t \geq 0 \), and Lemma 2.27 (i). \( \square \)

**Remark.**

From Theorem 2.28 it follows that the condition

\[ \left\| x^m f^{(l)}(t) \right\|_2 \leq C_{m,p}, \quad 0 \leq m \leq p/2 \]

in [11, Definitions 2.1, 2.2 and 2.3] is superfluous.

As a straightforward corollary of Theorem 2.28 we present the following characterizations of the spaces \( S^s_{0, even}(\mathbb{R}). \)

**Corollary 2.29** Let \( f \in C^\infty(\mathbb{R}^+). \)

1. Let \( \alpha \in \mathbb{R}^+_0. \) Then the following assertions are equivalent:

(i) \( f \in S^\infty_{0, even}(\mathbb{R}); \)

(ii) there exist \( A, B_1 > 0 \) (\( l \in \mathbb{N}_0 \)) such that

\[ \left\| x^k (x^{-1} \, d/dx)^l f \right\|_{L_2(\mathbb{R}^+)} \leq B_1 A^k (k!)^\alpha, \quad k, l \in \mathbb{N}_0; \]

(iii) there exist \( A, B_1 > 0 \) (\( l \in \mathbb{N}_0 \)) such that

\[ \left\| x^k (x^{-1} \, d/dx)^l f \right\|_{L_2(\mathbb{R}^+)} \leq B_1 A^k (k!)^\alpha, \quad k, l \in \mathbb{N}_0; \]

(iv) there exist \( A, B > 0 \) such that

\[ \left\| x^k f \right\|_{L_2(\mathbb{R}^+)} \leq B A^k (k!)^\alpha, \quad x^l (x^{-1} \, d/dx)^l f \in L_2(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0; \]

(v) there exist \( A, B > 0 \) such that

\[ \left\| x^k f \right\|_{L_\infty(\mathbb{R}^+)} \leq B A^k (k!)^\alpha, \quad x^l (x^{-1} \, d/dx)^l f \in L_\infty(\mathbb{R}^+), \quad k, l \in \mathbb{N}_0. \]
II. Let $\beta \in \mathbb{R}_0^+$. Then the following assertions are equivalent:

(i) $f \in S^p_{\infty, \text{aven}}(\mathbb{R})$;

(ii) there exist $A_k, B > 0$ ($k \in \mathbb{N}_0$) such that

$$
\|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A_k B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(iii) there exist $A, B > 0$ ($k \in \mathbb{N}_0$) such that

$$
\|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A_k B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(iv) there exist $A, B > 0$ such that

$$
x^k f \in L^p(\mathbb{R}^+), \quad \|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(v) there exist $A, B > 0$ such that

$$
x^k f \in L^p(\mathbb{R}^+), \quad \|(1 + x^2) x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0.
$$

III. Let $\alpha, \beta \in \mathbb{R}_0^+$ with $\alpha + \beta \geq 1$. Then the following assertions are equivalent:

(i) $f \in S^\alpha_{\infty, \text{aven}}(\mathbb{R})$;

(ii) there exist $A, B, C > 0$ such that

$$
\|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq C A^k B^l(l!)^\alpha (l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(iii) there exist $A, B, C > 0$ such that

$$
\|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq C A^k B^l(l!)^\alpha (l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(iv) there exist $A, B > 0$ such that

$$
x^k f \in L^p(\mathbb{R}^+), \quad \|x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0;
$$

(v) there exist $A, B > 0$ such that

$$
x^k f \in L^p(\mathbb{R}^+), \quad \|(1 + x^2) x^{k+1} (x^{-1} d/dx)^{k+1} f\|_{L^p(\mathbb{R}^+)} \leq A B^l(l!)^\beta, \quad k, l \in \mathbb{N}_0.
$$

2.4 The Gelfand-Shilov spaces $S^p_{\alpha}(\mathbb{R}^d)$

In this section we consider the Gelfand-Shilov spaces $S^p_{\alpha}(\mathbb{R}^d)$ and we gather some properties of these spaces for $q \geq 2$, cf. [22, Chapter IV, Section 9].

Definition 2.30 Let $\alpha, \beta \geq 0$.

(i) The space $S^p_{\alpha}(\mathbb{R}^d)$ consists of all functions $f \in S(\mathbb{R}^d)$ for which there exist $A, B_j > 0$ ($j \in \mathbb{N}_0$), such that

$$
\|x^{k+1} f\|_{L^p(\mathbb{R}^d)} \leq B_j |A^{(k)}(||k||!)|^\alpha, \quad k, l \in \mathbb{N}_0.
$$
(ii) The space $S^0_\infty(\mathbb{R}^n)$ consists of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ for which there exist $A_j, B > 0$ ($j \in \mathbb{N}_0$), such that

$$\|x^k \partial^l f\|_{L^\infty(\mathbb{R}^n)} \leq A_k B^l (\|f\|)^\beta, \quad k, l \in \mathbb{N}_0^n.$$  

(iii) The space $S^0_\infty(\mathbb{R}^n)$ consists of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ for which there exist $A, B, C > 0$, such that

$$\|x^k \partial^l f\|_{L^\infty(\mathbb{R}^n)} \leq C A^l B^j (\|f\|)^\alpha (\|\partial f\|)^\beta, \quad k, l \in \mathbb{N}_0^n.$$  

We recall the multi-index notations $x^k = x_1^{k_1} \cdots x_n^{k_n}$ and $\partial^l = \partial_1^{l_1} \cdots \partial_n^{l_n}$ with $\partial_j = \partial/\partial x_j$, $j = 1, \ldots, n$; furthermore, $|k| = k_1 + \cdots + k_n, k \in \mathbb{N}_0^n$.

To make some of the subsequent results also valid for $(\alpha, \beta) = (\infty, \infty)$, we define $S^\infty_\infty(\mathbb{R}^n) = S(\mathbb{R}^n)$.

The multiplication operators $x_j$ and the differentiation operators $\partial_j$, $j = 1, \ldots, n$, map $S^0_\infty(\mathbb{R}^n)$ into itself. The space $S^0_\infty(\mathbb{R}^n)$ is nontrivial if and only if $S^0_\infty(\mathbb{R}^n)$ is nontrivial, see [22, p. 243].

Later on in this section we will present several characterizations of $S^0_\infty(\mathbb{R}^n)$. For that purpose we introduce some notations from Ter Elst's thesis [17].

Let $T = (T_1, \ldots, T_q)$ be a $q$-tuple of mutually strongly commuting self-adjoint operators in a Hilbert space $H$. For $\lambda \geq 0$ and $A > 0$ the normed space $S_{\lambda, A}(T)$ is defined by

$$S_{\lambda, A}(T) = \{ f \in D^\infty(T) : \|f\|_{T, \lambda, A} = \sup_{n \in \mathbb{N}} (\|T^n f\|_H A^{-\lambda n} (\|f\|_H)^{-1}) < \infty \}.$$  

(2.7)

The Gevrey space $S_\lambda(T)$, brought about by $T$, is taken to be the union

$$S_\lambda(T) = \bigcup_{A > 0} S_{\lambda, A}(T).$$  

(2.8)

Clearly $S_\lambda(T) \subset S_\mu(T)$ if $\lambda < \mu$. The topology for $S_\lambda(T)$ is the inductive limit topology generated by the normed spaces $S_{\lambda, A}(T), \lambda > 0$. Since the topology of $S_\lambda(T)$ is regular, see [17, Theorem 1.11], we obtain the following characterization of sequential convergence in $S_\lambda(T)$. For a sequence $(f_k)_{k \in \mathbb{N}} \subset D^\infty(T)$ we have

$$f_k \to 0 \quad (k \to \infty) \quad \text{in} \ S_\lambda(T) \quad \text{if and only if there exists} \ A > 0 \ \text{such that}$$  

$$f_k \in S_{\lambda, A}(T), \quad k \in \mathbb{N}, \quad \|f_k\|_{T, \lambda, A} \to 0 \quad (k \to \infty).$$  

(2.9)

Consistent with (2.7) and (2.8) we define $S_\lambda(\mathbb{T}) = D^\infty(T)$.

For a unitary operator $U$ on $H$ we have, cf. (1.43),

$$S_\lambda(UTU^*) = U(S_\lambda(T)), \quad \lambda \in \mathbb{R}_0^{+\infty}.$$  

(2.10)

In particular we have for the Gevrey spaces $S_\lambda(\mathbb{P})$ and $S_\lambda(\mathbb{Q})$, brought about by the operator tuples $\mathbb{P}$ and $\mathbb{Q}$, via the Fourier transformation,

$$S_\lambda(\mathbb{P}) = \mathcal{F}(S_\lambda(\mathbb{Q})), \quad S_\lambda(\mathbb{Q}) = \mathcal{F}(S_\lambda(\mathbb{P})), \quad \lambda \in \mathbb{R}_0^{+\infty},$$  

(2.11)

cf. (1.48), (1.49). Moreover, for $j = 1, \ldots, q$ and $\lambda \in \mathbb{R}_0^{+\infty}$ we have

$$S_\lambda(\mathbb{P}_j) = \mathcal{F}_j(S_\lambda(Q_j)) = \mathcal{F}_j(S_\lambda(Q_j)), \quad S_\lambda(|\mathbb{P}|) = \mathcal{F}(S_\lambda(|\mathbb{Q}|)).$$  

(2.12)
\[ S_\lambda(Q_j) = \mathcal{F}(S_\lambda(P_j)) = \mathcal{F}(S_\lambda(Q)) , \quad S_\lambda(|Q|) = \mathcal{F}(S_\lambda(|P|)) . \tag{2.13} \]

The results of the following lemma are known. We give an elementary proof; for \( \lambda = \infty \), we refer to (1.50), (1.51).

**Lemma 2.31** Let \( \lambda \in \mathbb{R}_{0}^{+} \). Then

(i) \[ S_\lambda(Q) = S_\lambda(|Q|) = \bigcap_{j=1}^{q} S_\lambda(Q_j) , \]

(ii) \[ S_\lambda(P) = S_\lambda(|P|) = \bigcap_{j=1}^{q} S_\lambda(P_j) . \]

**Proof.** (i) We prove the inclusions

\[ S_\lambda(Q) \subset \bigcap_{j=1}^{q} S_\lambda(Q_j) \subset S_\lambda(|Q|) \subset S_\lambda(Q) . \]

The first inclusion is obvious. Suppose \( f \in S_\lambda(Q_j) \), \( j = 1, \ldots, q \). By using Newton’s binomial and the Cauchy-Schwarz inequality we have for \( n = 2, \ldots, q \) and \( k \in \mathbb{N}_0 \),

\[ \| (x_1^2 + \ldots + x_n^2)^{k/2} f \|_{L^2(\mathbb{R}^n)} \leq \sum_{j=0}^{k} \binom{k}{j} \| (x_1^2 + \ldots + x_{n+1}^2)^{j/2} f \|_{L^2(\mathbb{R}^n)} \| x_n^{2k-2j} f \|_{L^2(\mathbb{R}^n)} . \]

Then it follows by induction that

\[ f \in S_\lambda((Q_1^2 + \ldots + Q_n^2)^{1/2}) , \quad n = 2, \ldots, q . \]

Thus we have shown the second inclusion.

The third inclusion follows from the inequality \( |x^k| \leq |x|^{|k|} \), \( x \in \mathbb{R}^n \), \( k \in \mathbb{N}_0 \).

(ii) By applying Fourier transformation the assertion follows from (i). \( \square \)

The proof of the following lemma is similar to that of Lemma 2.10.

**Lemma 2.32** Let \( \lambda \in \mathbb{R}^{+} \) and \( j = 1, \ldots, q \). Then

\[ S_\lambda(|Q|) = \{ f \in L^2(\mathbb{R}^n) : \exists_{\omega \in \mathbb{R}^n} \exp(t |x|^{1/\lambda}) f \in L^2(\mathbb{R}^n) \} , \]

\[ S_\lambda(Q_j) = \{ f \in L^2(\mathbb{R}^n) : \exists_{\omega \in \mathbb{R}^n} \exp(t |x_j|^{1/\lambda}) f \in L^2(\mathbb{R}^n) \} . \]

The proof of the following lemma is similar to that of Lemma 2.14; it is recalled that \( \{ \Psi_n : n \in \mathbb{N}_0 \} \) is the Hermite basis in \( L^2(\mathbb{R}^n) \), as introduced in (1.53), (1.7).

**Lemma 2.33** Let \( \lambda \in \mathbb{R}^{+} \) and \( j = 1, \ldots, q \). Then

\[ S_\lambda(|Q|^2 + |Q|) = \{ f \in L^2(\mathbb{R}^n) : \exists_{\omega \in \mathbb{R}^n} \exp(t |n|^{1/\lambda}) (f, \Psi_n)_{L^2(\mathbb{R}^n)} \in L^\infty \} , \]

\[ S_\lambda(Q_j^2 + Q_j) = \{ f \in L^2(\mathbb{R}^n) : \exists_{\omega \in \mathbb{R}^n} \exp(t |n_j|^{1/\lambda}) (f, \Psi_n)_{L^2(\mathbb{R}^n)} \in L^\infty \} . \]
Corollary 2.34 Let \( \lambda \in \mathbb{R}^+ \). Then
\[
S_\lambda(\|P\|^2 + \|Q\|^2) = \bigcap_{j=1}^{q} S_\lambda(P_j^2 + Q_j^2).
\]

Proof. Suppose \( f \in S_\lambda(P_j^2 + Q_j^2), j = 1, \ldots, q \). Due to the inequality
\[
(m_1 + m_2)^{1/\lambda} \leq \max(1, 2^{(0/\lambda)-1})(m_{1/\lambda} + m_{2/\lambda}), \quad m_1, m_2 \in \mathbb{R}_0^+,
\]
and due to Lemma 2.33, it follows by induction that there exists \( t > 0 \) such that
\[
(\exp(t(n_1 + \ldots + n_k)^{1/\lambda}) (f, \psi_n)_{L_2(\mathbb{R}^n)}) \in l_\infty, \quad k = 2, \ldots, q.
\]
Hence, \( f \in S_\lambda(\|P\|^2 + \|Q\|^2) \) by Lemma 2.33.

Obviously, \( S_\lambda(\|P\|^2 + \|Q\|^2) \subset S_\lambda(P_j^2 + Q_j^2), \quad j = 1, \ldots, q \).
\( \square \)

We present the analogue of Lemma 2.5.

Lemma 2.35 Let \( f \in S(\mathbb{R}^n) \). Then
(i) there exists \( A_\epsilon > 0 \) such that \( \|f\|_{L_2(\mathbb{R}^n)} \leq A_\epsilon (1 + |x|)^{\epsilon} \|f\|_{L_\infty(\mathbb{R}^n)} \);
(ii) there exists \( B_\epsilon > 0 \) such that \( \|f\|_{L_\infty(\mathbb{R}^n)} \leq B_\epsilon (1 - \Delta)^{\epsilon} \|f\|_{L_2(\mathbb{R}^n)} \);
(iii) \( \|Q^k P^l f\|_{L_2(\mathbb{R}^n)} \leq \min(2^{ka}, 1) \sum_{|\alpha| = 0}^{\infty} \left( \begin{array}{c} 2k_a \\ j_a \end{array} \right) \left( \begin{array}{c} l_n \\ j_n \end{array} \right) \|Q^{2\alpha} f\|_{L_2(\mathbb{R}^n)} \|P^{2\beta} f\|_{L_2(\mathbb{R}^n)}, \quad k, l \in \mathbb{N}_0^n.
\]

Proof. (i) \( \|f\|_{L_2(\mathbb{R}^n)}^2 \leq \sup_{x \in \mathbb{R}^n} ((1 + |x|)^{\epsilon} f(x))^2 \int_{\mathbb{R}^n} (1 + |x|)^{-2\epsilon} dx \)
(ii) \( (2\pi)^{n/2} |f(x)| \leq \int_{\mathbb{R}^n} |(F f)(y)| dy = \int_{\mathbb{R}^n} |(F(I - \Delta)^{\epsilon} f)(y)| (1 + |y|^2)^{-\epsilon} dy \leq (1 + |y|^2)^{-\epsilon} \|f\|_{L_2(\mathbb{R}^n)} \)
(iii) since the self-adjoint operators \( P_j \) and \( Q_j \) commute with each of the operators \( P_m \) and \( Q_m \) if \( j \neq m \), we derive by means of Leibniz' differentiation rule for \( k, l \in \mathbb{N}_0^n,
\]
\[
\|Q^k P^l f\|_{L_2(\mathbb{R}^n)} = \left( \sum_{\alpha} (\mathbf{Q}^{\alpha} \mathbf{P}^{\beta} f)(y) \right)_{L_2(\mathbb{R}^n)}
\]
\[
= \sum_{|\alpha| = 0}^{\infty} \left( \begin{array}{c} 2k_a \\ j_a \end{array} \right) \left( \begin{array}{c} l_n \\ j_n \end{array} \right) \|Q^{2\alpha} f\|_{L_2(\mathbb{R}^n)} \|P^{2\beta} f\|_{L_2(\mathbb{R}^n)}.
\]

Now we obtain the wanted result by the Cauchy-Schwarz inequality. \( \square \)

We give some classical analytic characterizations of the spaces \( S_\varphi^0(\mathbb{R}^n) \). For the corresponding characterizations of \( S(\mathbb{R}^n) \) and \( S_\varphi^0(\mathbb{R}) \) we refer to Theorem 1.23 and Theorem 2.6.
Theorem 2.36 Let $f \in S(R^n)$. Then the following assertions are equivalent:

(i) $f \in S_0^\infty(R^n)$;

(ii) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq A(k!)^{\alpha}$, $k, l \in \mathbb{N}_0$;

(iii) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $k, l \in \mathbb{N}_0$;

(iv) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $k \in \mathbb{N}_0$;

(v) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $j = 1, \ldots, q$, $k \in \mathbb{N}_0$;

(vi) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $k \in \mathbb{N}_0$;

(vii) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $j = 1, \ldots, q$, $k \in \mathbb{N}_0$.

II. Let $\beta \in R_0^\infty$. Then the following assertions are equivalent:

(i) $f \in S_0^\infty(R^n)$;

(ii) there exist $A, B > 0$ such that $\| x^k \partial^l f \| \leq A(k!)^{\beta}$, $k, l \in \mathbb{N}_0$;

(iii) there exist $A, B > 0$ such that $\| \partial^l f \| \leq A(k!)^{\beta}$, $l \in \mathbb{N}_0$;

(iv) there exist $A, B > 0$ such that $\| \Delta^l f \| \leq A(k!)^{\beta}$, $l \in \mathbb{N}_0$;

(v) there exist $A, B > 0$ such that $\| \partial^l f \| \leq A(k!)^{\beta}$, $j = 1, \ldots, q$, $l \in \mathbb{N}_0$;

(vi) there exist $A, B > 0$ such that $\| \partial^l f \| \leq A(k!)^{\beta}$, $j = 1, \ldots, q$, $l \in \mathbb{N}_0$;

(vii) there exist $A, B > 0$ such that $\| \partial^l f \| \leq A(k!)^{\beta}$, $j = 1, \ldots, q$, $l \in \mathbb{N}_0$.

III. Let $\alpha, \beta \in R_0^\infty$ with $\alpha + \beta > 1$. Then the following assertions are equivalent:

(i) $f \in S_0^\infty(R^n)$;

(ii) there exist $A, B, C > 0$ such that

$\| x^k \partial^l f \| \leq CA^k B^\beta(k!)^{\alpha}$, $k, l \in \mathbb{N}_0$;

(iii) there exist $A, B > 0$ such that

$\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $\| \partial^l f \| \leq A(k!)^{\beta}$, $k, l \in \mathbb{N}_0$;

(iv) there exist $A, B > 0$ such that

$\| x^k \partial^l f \| \leq B(k!)^{\alpha}$, $\| \Delta^l f \| \leq A(k!)^{\beta}$, $k, l \in \mathbb{N}_0$;
(v) there exist $A, B > 0$ such that
\[ \|x^k f\|_{L^q(\mathbb{R}^n)} \leq BA^k(k)^{\alpha}, \quad \| \partial_j f \|_{L^q(\mathbb{R}^n)} \leq AB^j(j)^{\beta}, \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0; \]

(vi) there exist $A, B > 0$ such that
\[ \|x^k f\|_{L^q(\mathbb{R}^n)} \leq BA^k(k)^{\alpha}, \quad \| \partial_j f \|_{L^q(\mathbb{R}^n)} \leq AB^j(j)^{\beta}, \quad k, l \in \mathbb{N}_0; \]

(vii) there exist $A, B > 0$ such that
\[ \|x^k f\|_{L^q(\mathbb{R}^n)} \leq BA^k(k)^{\alpha}, \quad \| \partial_j f \|_{L^q(\mathbb{R}^n)} \leq AB^j(j)^{\beta}, \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0. \]

**Proof.** In each of the cases I, II and III, we follow the scheme (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). (i) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (v).

The assertions (iii), (iv) and (v) are mutually equivalent by Lemma 2.31; the implications (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (vi) $\Rightarrow$ (vii) need no explanation. The equivalence (i) $\Leftrightarrow$ (ii) readily follows by means of Lemma 2.35 (i) and (ii). The implication (iii) $\Rightarrow$ (ii) can be proved in the same manner as the implication (iii) $\Rightarrow$ (ii) in Theorem 2.6, using Lemma 2.35 (iii).

Finally, the implication (vii) $\Rightarrow$ (v) follows from the estimates
\[ \|Q_k f\|_{L^q(\mathbb{R}^n)} = \langle Q_k^2 f, f \rangle_{L^2(\mathbb{R}^n)} \leq \|Q_k^2 f\|_{L^q(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}, \quad j = 1, \ldots, q, \quad k \in \mathbb{N}_0, \]
\[ \|P_l f\|_{L^q(\mathbb{R}^n)} = \langle P_l^2 f, f \rangle_{L^2(\mathbb{R}^n)} \leq \|P_l^2 f\|_{L^q(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}, \quad j = 1, \ldots, q, \quad l \in \mathbb{N}_0. \]

From Theorem 2.36 we obtain several results for the spaces $S^\alpha_\beta(\mathbb{R}^n)$. We start with the analogue of Theorem 2.7 (Kashpirovskii's intersection result).

**Theorem 2.37** Let $\alpha, \beta \in \mathbb{R}_+^0$. Then
\[ S^\alpha_\beta(\mathbb{R}^n) \cap S^\alpha_\beta(\mathbb{R}^n) = S^\alpha_\beta(\mathbb{R}^n). \]

**Proof.** For $\alpha + \beta \geq 1$ the identity immediately follows from Theorem 2.36.

Next suppose $0 \leq \alpha + \beta < 1$. Let $f \in S^\alpha_\beta(\mathbb{R}^n) \cap S^\alpha_\beta(\mathbb{R}^n)$. Then for fixed $y \in \mathbb{R}^{n-1}$ the function $x_1 \rightarrow f(x_1, y)$ belongs to $S^\alpha_\beta(\mathbb{R}) \cap S^\alpha_\beta(\mathbb{R})$ by Theorem 2.36 (vi), (vii) and Theorem 2.6. Hence $f(x_1, y) = 0, x_1 \in \mathbb{R}_+$, by Theorems 2.7 and 2.2. Since $y \in \mathbb{R}^{n-1}$ is arbitrary, $f$ must be the null function. Thus we have shown that $S^\alpha_\beta(\mathbb{R}^n) \cap S^\alpha_\beta(\mathbb{R}^n)$ is the trivial space. Since, by Definition 2.30, $S^\alpha_\beta(\mathbb{R}^n) \subset S^\alpha_\beta(\mathbb{R}^n) \cap S^\alpha_\beta(\mathbb{R}^n)$, we are done. \[ \square \]

The action of the Fourier transformation $\mathcal{F}$ on $S^\alpha_\beta(\mathbb{R}^n)$ swaps the indices $\alpha$ and $\beta$, compare Theorem 2.8 and Corollary 2.9 (the proofs carry over).

**Theorem 2.38** Let $\alpha, \beta \in \mathbb{R}_{0,0}^+$. Then
\[ \mathcal{F}(S^\alpha_\beta(\mathbb{R}^n)) = S^\beta_\alpha(\mathbb{R}^n). \]
Corollary 2.39 Let $\alpha, \beta \in \mathbb{R}_+^*$ and let $f : \mathbb{R}^n \to \mathbb{C}$. Then the following assertions are equivalent:

(i) $f \in S_0^\alpha(\mathbb{R}^n)$;

(ii) $f \in S_0^\alpha(\mathbb{R}^n)$ and $\mathcal{F}f \in S_0^\beta(\mathbb{R}^n)$.

In Theorem 2.11 we proved the functional analytic characterization

$$S_0^\alpha(\mathbb{R}^n) = S_\alpha(Q) \cap S_\beta(P), \quad \alpha, \beta \in \mathbb{R}_+^*.$$ 

By using the same techniques as in the proof of Theorem 2.11 and by means of Lemma 2.31 we obtain the following functional analytic characterization of $S_0^\alpha(\mathbb{R}^n)$ in terms of Gevrey spaces.

Theorem 2.40 Let $\alpha, \beta \in \mathbb{R}_+^*$. Then

$$S_0^\alpha(\mathbb{R}^n) = S_\alpha(Q) \cap S_\beta(P) = S_\alpha(|Q|) \cap S_\beta(|P|) = \bigcap_{j=1}^q \left[ S_\alpha(Q_j) \cap S_\beta(P_j) \right].$$

As a consequence of Lemma 2.32 and Theorem 2.40, the functions in $S_0^\alpha(\mathbb{R}^n)$ decrease exponentially at infinity; cf. Theorem 2.12.

Theorem 2.41 Let $\alpha \in \mathbb{R}^+$, let $p \in \{2, \infty\}$ and let $f \in S(\mathbb{R}^n)$. Then the following assertions are equivalent:

(i) $f \in S_0^\alpha(\mathbb{R}^n)$;

(ii) there exists $t > 0$ such that $\exp(t \chi \pi^{1/\alpha}) f \in L_p(\mathbb{R}^n)$;

(iii) there exists $t > 0$ such that $\exp(t \chi \pi^{1/\alpha}) f \in L_p(\mathbb{R}^n)$, $j = 1, \ldots, q$.

The characterizations of $S_0^\alpha(\mathbb{R}^n)$ established in Theorems 2.36 and 2.41 lead, by Corollary 2.39, to analogous characterizations of $S_0^\alpha(\mathbb{R}^n)$ and $S_0^\beta(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}^*$. Recalling the characterization of $S(\mathbb{R}^n)$ by means of the Fourier transformation in Theorem 1.25, we obtain the following characterizations of the spaces $S_0^\alpha(\mathbb{R}^n)$.

Theorem 2.42 Let $f \in L_2(\mathbb{R}^n)$ and let $p \in \{2, \infty\}$.

1. Let $\alpha \in \mathbb{R}^+$. Then the following assertions are equivalent:

(i) $f \in S_0^\alpha(\mathbb{R}^n)$;

(ii) there exist $A, B > 0$ such that

$$\|z^k \mathcal{F} f\|_{L_p(\mathbb{R}^n)} \leq BA^{k\alpha}(|k|)^\alpha \quad \text{and} \quad z^l \mathcal{F} f \in L_p(\mathbb{R}^n), \quad k, l \in \mathbb{N}_0^*;$$

(iii) there exist $A, B > 0$ such that

$$\|z^k f\|_{L_p(\mathbb{R}^n)} \leq BA^{k\alpha}(|k|)^\alpha \quad \text{and} \quad |z|^l \mathcal{F} f \in L_p(\mathbb{R}^n), \quad k, l \in \mathbb{N}_0;$$
(iv) there exist $A, B > 0$ such that
$$\|x_j^k f\|_{L^p(\mathbb{R}^d)} \leq BA^{|k|} \|f\|_{L^p(\mathbb{R}^d)}, \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0;$$
(v) there exists $t > 0$ such that
$$\exp(t \|x\|^2/\alpha) f \in L^p(\mathbb{R}^d) \text{ and } |x|^l Ff \in L^p(\mathbb{R}^d), \quad l \in \mathbb{N}_0;$$
(vi) there exists $t > 0$ such that
$$\exp(t \|x_j\|^2/\alpha) f \in L^p(\mathbb{R}^d) \text{ and } x_j^l Ff \in L^p(\mathbb{R}^d), \quad j = 1, \ldots, q, \quad l \in \mathbb{N}_0.$$

II. Let $\beta \in \mathbb{R}^+$. Then the following assertions are equivalent:
(i) $f \in S_0^\beta(\mathbb{R}^d)$;
(ii) there exist $A, B > 0$ such that
$$x^k f \in L^p(\mathbb{R}^d) \text{ and } \|x^l Ff\|_{L^p(\mathbb{R}^d)} \leq AB^{|l|} \|f\|_{L^p(\mathbb{R}^d)}, \quad k, l \in \mathbb{N}_0^d;$$
(iii) there exist $A, B > 0$ such that
$$|x|^k f \in L^p(\mathbb{R}^d) \text{ and } \|x^l Ff\|_{L^p(\mathbb{R}^d)} \leq AB^{|l|} \|f\|_{L^p(\mathbb{R}^d)}, \quad k, l \in \mathbb{N}_0;$$
(iv) there exist $A, B > 0$ such that
$$x_j^k f \in L^p(\mathbb{R}^d) \text{ and } \|x_j^l Ff\|_{L^p(\mathbb{R}^d)} \leq AB^{|l|} \|f\|_{L^p(\mathbb{R}^d)}, \quad j = 1, \ldots, q, \quad k, l \in \mathbb{N}_0;$$
(v) there exists $t > 0$ such that
$$|x|^k f \in L^p(\mathbb{R}^d) \text{ and } \exp(t \|x\|^2) Ff \in L^p(\mathbb{R}^d), \quad k \in \mathbb{N}_0;$$
(vi) there exists $t > 0$ such that
$$x_j^k f \in L^p(\mathbb{R}^d) \text{ and } \exp(t \|x_j\|^2) Ff \in L^p(\mathbb{R}^d), \quad j = 1, \ldots, q, \quad k \in \mathbb{N}_0.$$

III. Let $\alpha, \beta \in \mathbb{R}^+$. Then the following assertions are equivalent:
(i) $f \in S_0^\alpha(\mathbb{R}^d)$;
(ii) there exist $A, B > 0$ such that
$$\|x^k f\|_{L^p(\mathbb{R}^d)} \leq BA^{|k|} \|f\|_{L^p(\mathbb{R}^d)} \text{ and } \|x^l Ff\|_{L^p(\mathbb{R}^d)} \leq AB^{|l|} \|f\|_{L^p(\mathbb{R}^d)}, \quad k, l \in \mathbb{N}_0^d;$$
(iii) there exist $A, B > 0$ such that
$$\|x^k f\|_{L^p(\mathbb{R}^d)} \leq BA^{|k|} \|f\|_{L^p(\mathbb{R}^d)} \text{ and } \|x^l Ff\|_{L^p(\mathbb{R}^d)} \leq AB^{|l|} \|f\|_{L^p(\mathbb{R}^d)}, \quad k, l \in \mathbb{N}_0;$$
(iv) there exist $A, B > 0$ such that
\[ \| x_j^k f \|_{L_p(\mathbb{R}^d)} \leq B A^k (k) \alpha \] and
\[ \| x_j^i f \|_{L_p(\mathbb{R}^d)} \leq AB^i (i) \alpha \] \quad j = 1, \ldots, q, \quad k, i \in \mathbb{N}_0 ;
(v) there exists $t > 0$ such that
\[ \exp(t |x|^\alpha f) \in L_p(\mathbb{R}^d) \] and
\[ \exp(t |x|^\beta f) \in L_p(\mathbb{R}^d) ;
(vi) there exists $t > 0$ such that
\[ \exp(t |x_j^1|^{1/\alpha} f) \in L_p(\mathbb{R}^d) \] and
\[ \exp(t |x_j^1|^{1/\beta} f) \in L_p(\mathbb{R}^d) , \quad j = 1, \ldots, q .

As we have seen in Theorem 2.15, the functions in $S_0^0 (\mathbb{R}^d)$ with $\alpha \geq \frac{1}{2}$ can be characterized in terms of their Hermite expansion coefficients:
\[ S_0^0 (\mathbb{R}^d) = S_0^0 (P^2 + Q^2) = \{ f \in L_2 (\mathbb{R}^d) : \exists_{n>0} \left( \exp(t |x|^\alpha f) (x, \psi_n)_{L_2(\mathbb{R}^d)} \right) \in L_\infty \} .
Here $\{ \psi_n : n \in \mathbb{N}_0 \}$ denotes the Hermite basis in $L_2(\mathbb{R})$ as introduced in (1.7). By a similar reasoning we obtain the analogue for $S_0^0 (\mathbb{R}^d)$ with $\alpha \geq \frac{1}{2}$.

**Theorem 2.43** Let $\alpha \geq \frac{1}{2}$. Then
(i) \[ S_0^0 (\mathbb{R}^d) = \bigcap_{j=1}^2 \left( S_0 (Q_j) \cap S_0 (P_j) \right) = \bigcap_{j=1}^2 S_0 (P_j^2 + Q_j^2) = S_0 (|P|^2 + |Q|^2) \]
\[ = \{ f \in L_2 (\mathbb{R}^d) : \exists_{n>0} \left( \exp(t |x|^\alpha f) (x, \psi_n)_{L_2(\mathbb{R}^d)} \right) \in L_\infty , \quad j = 1, \ldots, q \}
\[ = \{ f \in L_2 (\mathbb{R}^d) : \exists_{n>0} \left( \exp(t |x|^\alpha f) (x, \psi_n)_{L_2(\mathbb{R}^d)} \right) \in L_\infty \} .
(ii) For $f \in S_0^0 (\mathbb{R}^d)$ we have
\[ f(x) = \sum_{n \in \mathbb{N}^d_0} (f, \psi_n)_{L_2(\mathbb{R}^d)} \psi_n(x) , \quad x \in \mathbb{R}^d ,
where the series converges in $S_0^0 (\mathbb{R}^d)$.

**Proof.** (i) The first equality is taken from Theorem 2.40. By using the same techniques as in the proof of Theorem 2.15 we have $S_0 (Q_j) \cap S_0 (P_j) = S_0 (P_j^2 + Q_j^2) , \quad j = 1, \ldots, q$. The proof is completed by means of Lemma 2.33 and Corollary 2.34.

(iii) The proof is similar to the proof of Theorem 2.15 (ii). \qed

Finally, we present the analogue of Theorem 2.16 (the proof is the same).

**Theorem 2.44** Let $\alpha \in \mathbb{R}^{>0}$. Then
\[ S_0 (|Q|) \cap S_0 (|P|) = S_0 (|P|^2 + |Q|^2) \] if and only if $\alpha \geq \frac{1}{2}$.
2.5 The spaces $S^\alpha_0(B^m)$ and expansions in spherical harmonics

In this section we want to carry over the results for the Schwartz space $S(B^m)$ established in Section 2.6, to the Gel'fand-Shilov spaces $S^\alpha_0(B^m)$.

We start with the characterization of the products of radially symmetric functions and homogeneous harmonic polynomials in $S^\alpha_0(B^m)$. For the corresponding characterization of such products in $S(B^m)$ we refer to Theorem 1.37.

**Theorem 2.45** Let $\alpha, \beta \in R_{\nu, \omega}$. Let $g \in S_{even}(B)$, let $Y_m \in Y_m^n$ and define $f : R^m \rightarrow C$

$$f(rw) = g(r) \hat{Y}_m(rw), \quad r \geq 0, \quad \omega \in S^{m-1}.$$ 

Then $f \in S^\alpha_0(B^m)$ if and only if $g \in S^\beta_{\alpha, even}(B)$.

**Proof.** First we deal with the case of $S^\alpha_{\nu}(B^m)$, $0 \leq \alpha < \infty$. For $\alpha = 0$, the equivalence follows from Theorems 2.36 i (iv) and 2.6 i (iii). Suppose $0 < \alpha < \infty$. Since $g \in S_{even}(B)$, the following assertions are equivalent by Theorems 2.41 and 2.12:

- $f \in S^\alpha_{\nu}(B^m)$;
- there exists $t > 0$ such that $r^{m+\nu-1} \exp(tr^{1/\alpha}) g \in L_2(B^m)$;
- there exists $r > 0$ such that $\exp(tr^{1/\alpha}) g \in L_2(B^m)$;
- $g \in S^\alpha_{\alpha, even}(B)$.

Next we deal with the case of $S^\alpha_0(B^m)$, $0 \leq \beta < \infty$. By Theorem 1.33 (Hecke-Bochner) we have $(Ff)(rw) = (-i)^m (H_{m+\nu/2-1}g)(r) \hat{Y}_m(rw), \quad r > 0, \quad \omega \in S^{m-1}$. Since $g \in S_{even}(B)$, the following assertions are equivalent by Corollary 2.39 and Theorems 2.41, 2.21 II:

- $f \in S^\beta_{\nu}(B^m)$;
- there exists $t > 0$ such that $r^{m+\nu-1} \exp(tr^{1/\beta}) \beta_{m+\nu/2-1}g \in L_2(B^m)$;
- there exists $r > 0$ such that $\exp(tr^{1/\beta}) \beta_{m+\nu/2-1}g \in L_2(B^m)$;
- $g \in S^\beta_{\beta, even}(B)$.

By means of the intersection result $S^\alpha_{\nu}(B^m) \cap S^\beta_{\nu}(B^m) = S^\nu_{\nu}(B^m)$, in Theorem 2.37, we obtain the wanted equivalence in the case of $S^\nu_0(B^m)$, $0 \leq \alpha, \beta < \infty$. The case of $S^\nu_0(B^m) = S(B^m)$ is covered by Theorem 1.37.

By taking $m = 0$ in Theorem 2.45, we obtain the following characterization of the radially symmetric functions in $S^\nu_0(B^m)$. For the corresponding characterization of the radially symmetric functions in $S(B^m)$ we refer to Theorem 1.36.

**Theorem 2.48** Let $\alpha, \beta \in R_{\nu, \omega}$. Let $g \in S_{even}(B)$ and define the radially symmetric function $f : B^m \rightarrow C$ by
\[ f(r\omega) = g(r), \quad r \geq 0, \quad \omega \in S^{s-1}. \]

Then \( f \in S^{(\alpha)}_0(\mathbb{R}^d) \) if and only if \( g \in S^{(\alpha)}_{0,\text{even}}(I^d) \).

As we have seen in Theorem 1.38, the space \( S(\mathbb{R}^d) \) can be characterized in terms of the expansion coefficients of its elements with respect to the basis \( \{ u_{n,m,j} : n, m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q,m) \} \), introduced in Theorem 1.35, namely
\[
S(\mathbb{R}^d) = \{ f \in L_2(\mathbb{R}^d) : \langle (n+m)^{\alpha}(f, u_{n,m,j}), \omega \rangle_{L_2(\mathbb{R}^d)} \in I^{(s)} \quad k \in \mathbb{N}_0 \}. 
\]

Here is the analogue for the space \( S^{(\alpha)}_0(\mathbb{R}^d) \) with \( \alpha \geq \frac{1}{2} \) (the proof is similar to the proofs of Theorems 2.15 and 2.43).

**Theorem 2.47** Let \( \alpha \geq \frac{1}{2} \). Then
\begin{enumerate}
  \item \( S^{(\alpha)}_0(\mathbb{R}^d) = \{ f \in L_2(\mathbb{R}^d) : \exists_{\beta > 0} (\exp(t(n+m)^{2\alpha}) \langle f, u_{n,m,j}, \omega \rangle_{L_2(\mathbb{R}^d)} \in I^{(s)} \quad k \in \mathbb{N}_0 \} \}
  
  \item For \( f \in S^{(\alpha)}_0(\mathbb{R}^d) \) we have
  \[ f(x) = \sum_{n,m=0}^{\infty} \sum_{j=1}^{N(q,m)} (f, u_{n,m,j})_{L_2(\mathbb{R}^d)} u_{n,m,j}(x), \quad x \in \mathbb{R}^d, \]
  where the series converges in \( S^{(\alpha)}_0(\mathbb{R}^d) \).
\end{enumerate}

As we have seen in Theorem 1.39, a function \( f \in S(\mathbb{R}^d) \) admits the expansion
\[
f(r\omega) = \sum_{n,m=0}^{\infty} \sum_{j=1}^{N(q,m)} f_{n,m,j}(r) \tilde{Y}_{n,j}(r\omega), \quad r \geq 0, \quad \omega \in S^{s-1},
\]
where the series converges in \( S(\mathbb{R}^d) \) and where \( f_{n,m,j} \in S_{\text{even}}(I) \) is given by
\[
r^m f_{n,m,j}(r) = \int_{S^{s-1}} f(r\omega) \tilde{Y}_{n,j}(\omega) \, d\sigma^{s-1}(\omega), \quad r > 0.
\]

For functions \( f \) in a Gel'fand-Shilov space we prove the following theorem.

**Theorem 2.48** Let \( f \in S(\mathbb{R}^d) \).
1. Let \( \alpha \in \mathbb{R}^d \). Then the following assertions are equivalent:
   \begin{enumerate}
   \item \( f \in S^{(\alpha)}_0(\mathbb{R}^d) \);
   \item \( \exists_{\beta > 0} \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,m)} \| \exp(t^{1/\alpha}) r^m f_{n,m,j} \|_{L^2}^2 < \infty \).
   \end{enumerate}

2. Let \( \beta \in \mathbb{R}^d \). Then the following assertions are equivalent:
   \begin{enumerate}
   \item \( f \in S^{(\beta)}_0(\mathbb{R}^d) \);
   \item \( \exists_{\gamma > 0} \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,m)} \| \exp(t^{1/\gamma}) r^m f_{n,m,j} \|_{L^2}^2 < \infty \).
   \end{enumerate}
III. Let $\alpha, \beta \in \mathbb{R}^+$. Then the following assertions are equivalent:

(i) $f \in S_0^2(\mathbb{R}^q)$;

(ii) $\exists t > 0 \sum_{m=0}^{\infty} \sum_{j=1}^{N(q,m)} \| \exp(t|z|^{1/\alpha}) r^m f_{m,j} \|_{L^2_{q+1}} < \infty$ and

$$\sum_{m=0}^{\infty} \sum_{j=1}^{N(q,m)} \| \exp(t|z|^{1/\beta}) r^m R_{m+1/2-1} f_{m,j} \|_{L^2_{q+1}}^2 < \infty.$$ 

IV. Let $\alpha, \beta \in \mathbb{R}^+_\infty$ and let $f \in S_0^2(\mathbb{R}^q)$. Then

$$f_{m,j} \in S_{q,\text{even}}^2(\mathbb{R}^q), \quad m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q,m),$$

and the series (2.14) converges in $S_0^2(\mathbb{R}^q)$.

Proof. I. By Theorem 2.41 we have $f \in S_0^\infty(\mathbb{R}^q)$ if and only if there exists $t > 0$ such that $\exp(t|z|^{1/\alpha}) f \in L^2(\mathbb{R}^q)$, whence we obtain the wanted equivalence.

II. The equivalence follows from I and Theorems 2.38, 1.33 (Hecke-Bochner).

III. The equivalence follows from I, II and Theorem 2.37.

IV. The result follows from I, II, III and Theorems 2.12, 2.20 and 2.7. The convergence of the series (2.14) in $S_0^2(\mathbb{R}^q)$ is obvious from I (ii), II (ii), III (ii).

The remainder of this section deals with some preliminaries to the next section, where we study the Radon transformation on $S_0^2(\mathbb{R}^q)$.

In (1.71) we introduced the operator $V$ as follows. Let $f$ be a function on $\mathbb{R}^q$. Then $V f$ is the function on $Z_q = \mathbb{R} \times S^{q-1}$ defined by

$$(V f)(p, \omega) = f(p \omega), \quad (p, \omega) \in Z_q.$$ 

We recall the characterization of the image $V(S(\mathbb{R}^q))$ by means of the operators $Q \otimes I$, $P \otimes I$ and $I \otimes \Delta_{LB}$ (see Theorem 1.49 and Definition 1.40):

Let $f$ be a function on $\mathbb{R}^q$. Then $f \in S(\mathbb{R}^q)$ if and only if

$$V f \in D^\infty(Q \otimes I) \cap D^\infty(P \otimes I) \cap D^\infty(I \otimes \Delta_{LB})$$

(2.16)

and $V f$ satisfies the homogeneity conditions:

For $l \in \mathbb{N}_0$, the function $\omega \mapsto ((P^l \otimes I) V f)(0, \omega)$ is the restriction to $S^{q-1}$ of a homogeneous polynomial of degree $l$ on $\mathbb{R}^q$.

(2.17)

In the next theorem we present a characterization of the image $V(S_0^2(\mathbb{R}^q))$ by means of the operators $Q \otimes I$, $P \otimes I$ and $I \otimes \Delta_{LB}$. For the image $V(S_0^2(\mathbb{R}^q))$ with $\beta \in \mathbb{R}^+_\infty$, we have not been able to establish such a characterization, cf. Theorem 2.30.

Theorem 2.49 Let $\alpha \in \mathbb{R}^+_\infty$ and let $f$ be a function on $\mathbb{R}^q$. Then $f \in S_0^2(\mathbb{R}^q)$ if and only if

$$V f \in S_0(Q \otimes I) \cap D^\infty(P \otimes I) \cap D^\infty(I \otimes \Delta_{LB})$$

(2.18)

and $V f$ satisfies the homogeneity conditions (2.17).
Proof. First we observe that the case $\alpha = \infty$ is covered by Theorem 1.43.
Next, suppose $f \in S^\alpha_c(\mathcal{R}^q)$ with $\alpha \in \mathbb{R}^q$. Since $S^\alpha_c(\mathcal{R}^q) \subset S(\mathcal{R}^q)$, the function $V f$ satisfies (2.16) and (2.17). We show that $V f \in S_\alpha(Q \otimes I)$. Due to Theorem 2.36 (iv) there exist $A, B > 0$ such that for $k \in \mathbb{N}_0$,

$$
\int_{S^{t-1}} \left( \int_{\mathcal{R}} |p^{k} f(p\omega)|^2 \, dp \right) \, d\sigma^{t-1}(\omega)
$$

$$
\leq 2 \int_{S^{t-1}} \left( \int_{0}^{\infty} |f(p\omega)|^2 \, dp + \int_{1}^{\infty} |p^{k} f(p\omega)|^2 \, dp \right) \, d\sigma^{t-1}(\omega)
$$

$$
\leq 2 \Omega_q \|f\|_{L^2(\mathcal{R}^q)}^2 + 2 \|z^k f\|_{L^2(\mathcal{R}^q)}^2 \leq BA^k(k!)^{2\alpha}.
$$

That is, $V f \in S_\alpha(Q \otimes I)$.

Conversely, suppose $V f$ satisfies (2.18) and (2.17). We show that $f \in S^\alpha_c(\mathcal{R}^q)$. Since $S_\alpha(Q \otimes 1) \subset \mathcal{D}(Q \otimes 1)$, the function $V f$ satisfies (2.16). Hence $f \in S(\mathcal{R}^q)$. According to the definition of $S_\alpha(Q \otimes 1)$ there exist $A, B > 0$ such that for $k \in \mathbb{N}_0$,

$$
\int_{\mathcal{R}^q} \|z^k f(z)\|^2 \, dx = \int_{S^{t-1}} \left( \int_{0}^{\infty} |r^k (V f)(r, \omega)|^2 \, dr \right) \, d\sigma^{t-1}(\omega)
$$

$$
\leq \int_{S^{t-1}} \left( \int_{0}^{1} |r(f(r, \omega)|^2 \, dr + \int_{1}^{\infty} |r^{k+1} (V f)(r, \omega)|^2 \, dr \right) \, d\sigma^{t-1}(\omega)
$$

$$
\leq \Omega_q \|f\|_{L^2(\mathcal{R}^q)}^2 + BA^{k+1}(k+q-1)!^{2\alpha}.
$$

Since $(k+q-1)! \leq 2^{k+q-1} k! (q-1)!$, it follows from Theorem 2.36 (iv) that $f \in S^\alpha_c(\mathcal{R}^q)$. □

Theorem 2.50 Let $\alpha, \beta \in \mathbb{R}^{+}_{0,\infty}$ and let $f \in S^\alpha(\mathcal{R}^q)$. Then

$$V f \in S_\alpha(Q \otimes I) \cap S_\beta(P \otimes I) \cap S_{\alpha+\beta}(I \otimes \Delta_L L)
$$

and $V f$ satisfies the homogeneity conditions (2.17).

Proof. If $\beta = \infty$, the result follows from Theorem 2.49.

Suppose $\alpha = \infty$, $\beta \neq \infty$. Since $S^\alpha(\mathcal{R}^q) \subset S(\mathcal{R}^q)$, the function $V f$ satisfies (2.16) and (2.17). We show that $V f \in S_\beta(P \otimes I)$.

For $(p, \omega) \in Z_q$ we have $(\partial / \partial p V f)(p, \omega) = \omega_1(\partial_1 f)(p\omega) + \ldots + \omega_q(\partial_q f)(p\omega)$. Hence for $l \in \mathbb{N}_0$ it follows that $(\partial / \partial p^l V f)(p, \omega)$ is the sum of $q^l$ derivatives of the form $\omega_j(\partial_j f)(p\omega)$ with $j \in \mathbb{N}_0^q, |j| = l$. So, due to Definition 2.30 (ii) there exist $A, B > 0$ such that for $l \in \mathbb{N}_0$,

$$
\int_{S^{t-1}} \left( \int_{\mathcal{R}} \left| (\partial^l / \partial p^l V f)(p, \omega) \right|^2 \, dp \right) \, d\sigma^{t-1}(\omega)
$$

$$
\leq \pi \Omega_q \sup_{(p, \omega) \in Z_q} (1 + p^2) \left| (\partial^l / \partial p^l V f)(p, \omega) \right|^2 \leq \pi \Omega_q \Lambda B^l(k!)^{2\beta}.
$$
That is, \( V f \in S_p(\mathbb{R} \otimes I) \).

Finally, suppose \( \alpha \neq \infty \) and \( \beta \neq \infty \). Since \( S^p_\alpha(\mathbb{R}^q) = S_{\alpha,\infty}(\mathbb{R}^q) \cap S^p_\infty(\mathbb{R}^q) \), it follows from the preceding result and Theorem 2.49 that \( V f \) satisfies (2.17) and \( V f \in S_{\alpha,\infty}(\mathbb{R}^q) \cap S_p(\mathbb{R} \otimes I) \).

We show that \( V f \in S_{2\alpha+2\beta}(I \otimes \Delta_{LB}) \). For that purpose, we use the connection with the momentum operator \( M \), see Theorem 1.42 (i),

\[
(M^k f)(r\omega) = \left( (I \otimes \Delta_{LB}^k) V f \right)(r\omega), \quad r > 0, \quad \omega \in S^{p-1};
\]

we recall the definition of the momentum operator \( M \) from (1.64), namely

\[
M = \sum_{1 \leq i, j \leq N} (x_i \partial_j - x_j \partial_i)^2.
\]

By evaluation of the squares, it follows that \( M \) is the sum of \( 2q(q - 1) \) operators of the form \( x_i \partial_j x_i \partial_j \), with \( i, i, j, j \in \{1, 2, \ldots, q\} \). Hence, for \( k \in \mathbb{N} \), it follows that \( M^k \) is the sum of \( (2q(q - 1))^k \) operators of the form \( \prod_{m=1}^{k} x_{i_m} \partial_{j_m} \) with \( i_m, j_m \in \{1, 2, \ldots, q\} \) for \( m = 1, 2, \ldots, 2k \). By [17, Theorem 3.2] it follows that there exist \( A, B > 0 \) such that for \( k \in \mathbb{N}_0 \),

\[
\|M^k f\|_{L_2(\mathbb{R}^q)} \leq BA^k((2k)!)^{q+3} \leq B(2^{2q+2\beta}A^k)(k!)^{2q+2\beta}.
\]

Similarly, it follows that there exist \( A, B > 0 \) such that for \( k \in \mathbb{N}_0 \),

\[
\|(I - \Delta)^k M^k f\|_{L_2(\mathbb{R}^q)} \leq BA^k(k!)^{2q+2\beta}.
\]

By means of Theorem 1.42 (i) and Lemma 2.35 (ii) we estimate for \( k \in \mathbb{N}_0 \),

\[
\int_{S^{p-1}} \left( \int_{\mathbb{R}} \left| \left( (I \otimes \Delta_{LB}^k) V f \right)(p, \omega) \right|^2 \, dp \right) \, d\sigma^{p-1}(\omega)
\]

\[
\leq 2 \int_{S^{p-1}} \left( \int_{\mathbb{R}} \left| (M^k f)(p\omega) \right|^2 \, dp \right) \, d\sigma^{p-1}(\omega)
\]

\[
\leq 2\Omega_q \|M^k f\|_{L_2(\mathbb{R}^q)}^2 + 2 \|M^k f\|_{L_2(\mathbb{R}^q)}^2
\]

\[
\leq 2\Omega_q B_q^2 \|(I - \Delta)^k M^k f\|_{L_2(\mathbb{R}^q)}^2 + 2 \|M^k f\|_{L_2(\mathbb{R}^q)}^2.
\]

Combination of the latter three results yields \( V f \in S_{2\alpha+2\beta}(I \otimes \Delta_{LB}) \). \( \Box \)

For the characterization of \( V(S_{\alpha,\infty}(\mathbb{R}^q)) \) with \( \alpha > 1 \), we refer to Corollary 2.58.

By means of Theorems 2.49 and 2.50 we tackle the following factorization problem: For which functions \( g \) and \( h \) does the function \( (r\omega) \to g(r) h(\omega) \) belong to \( S_{\alpha,\infty}(\mathbb{R}^q) \)? We have not been able to completely solve this problem. The case \( 1 < \beta < \infty \) still requires further investigation. The solution of the above factorization problem in \( S(\mathbb{R}^q) \) can be found in Theorem 1.44. We recall that \( L_m \) is the orthogonal projection of \( L_2(S^{p-1}) \) onto \( \mathcal{Y}_m \), see (1.78), and we observe that the spaces \( Q^\alpha(S_{\alpha,\infty}(\mathbb{R}^q)), k \in \mathbb{N} \), are described in Theorem 2.17 (ii).

**Theorem 2.51** Let \( \alpha, \beta \in \mathbb{R}_0^+, \) let \( g : [0, \infty) \to C \) such that \( g \in X_{\alpha-1} \), let \( h : S^{p-1} \to C \) such that \( h \in L_2(S^{p-1}) \) and define \( f : \mathbb{R}^q \to C \) by
\[ f(r\omega) = g(r) h(\omega), \quad r \geq 0, \quad \omega \in S^{d-1}. \]

(i) If \( h = \sum_{m=0}^M E_{2m} h \) with \( E_{2m} h \neq 0 \),
then \( f \in S_{\infty}^0 (\mathbb{R}^d) \) if and only if \( g \in Q^M(S_{\infty, \text{even}}^0(\mathbb{R})). \)

(ii) If \( h = \sum_{m=0}^M E_{2m+1} h \) with \( E_{2m+1} h \neq 0 \),
then \( f \in S_{\infty}^0 (\mathbb{R}^d) \) if and only if \( g \in Q^{M+1}(S_{\infty, \text{even}}^0(\mathbb{R})). \)

(iii) If \( h \) cannot be represented by a finite expansion as in (i) or (ii),
then (1) \( f \in S_{\infty}^0 (\mathbb{R}^d) \) if and only if
\[
 g \in S_{\infty, \text{even}}^0(\mathbb{R}) \quad \text{with} \quad g^{(0)}(0) = 0, \quad l \in \mathbb{N}_0, \quad \text{and} \quad h \in D^{\infty}(\Delta_{LB});
\]
(2) \( f \in S_{\infty}^0 (\mathbb{R}^d) \) with \( 0 \leq \beta \leq 1, \) if and only if \( g \equiv 0; \)
(3) \( f \in S_{\infty}^0 (\mathbb{R}^d) \) with \( 1 < \beta < \infty, \) implies
\[
 g \in S_{\infty, \text{even}}^0(\mathbb{R}) \quad \text{with} \quad g^{(0)}(0) = 0, \quad l \in \mathbb{N}_0, \quad \text{and} \quad h \in S_{2\beta}(\Delta_{LB}).
\]

**Proof.** Suppose \( f \in S_{\infty}^0 (\mathbb{R}^d). \) Since \( S_{\infty}^0 (\mathbb{R}^d) \subseteq S(\mathbb{R}^d), \) it follows from Theorem 1.44 that \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) or \( g \in S_{\infty, \text{odd}}(\mathbb{R}). \) Now, from Theorems 2.50 and 2.11 it follows that
\[
g \in S_{\infty}^0 (\mathbb{R}^d) \quad \text{and} \quad h \in S_{2\beta}(\Delta_{LB}).
\]

(i), (ii) The equivalences follow from (2.19), Theorems 1.44 (i), (ii) and 2.45.

(iii) (1) Suppose \( f \in S_{\infty}^0 (\mathbb{R}^d). \) Then it follows from (2.19) and Theorem 1.44 (iii) that \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) with \( g^{(0)}(0) = 0, \) \( l \in \mathbb{N}_0, \) and \( h \in D^{\infty}(\Delta_{LB}). \) Conversely, suppose \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) with \( g^{(0)}(0) = 0, \) \( l \in \mathbb{N}_0, \) and \( h \in D^{\infty}(\Delta_{LB}). \) Then \( f \in S(\mathbb{R}^d), \) by Theorem 1.44 (iii). According to Theorem 2.61 (iii), there exist \( A, B > 0 \) such that for \( k \in \mathbb{N}_0, \)
\[
\int_{\mathbb{R}^d} |x|^k |f(x)|^2 \, dx = \int_{S^{d-1}} |h(\omega)|^2 \, d\sigma^{d-1}(\omega) \int_0^\infty \int_0^\infty |r^k g(r)|^2 \, r^{d-1} \, dr \leq BA^2(k^2)^\omega.
\]

Hence \( f \in S_{\infty}^0 (\mathbb{R}^d), \) by Theorem 2.36 (i) (iv).

(iii) (2) Suppose \( f \in S_{\infty}^0 (\mathbb{R}^d) \) with \( 0 \leq \beta \leq 1. \) Then it follows from (2.19) and Theorem 1.44 (iii) that \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) with \( g^{(0)}(0) = 0, \) \( l \in \mathbb{N}_0. \) Since \( g \) can be extended to an analytic function on a strip \( |y| < 1/B \) of the complex plane \( z = x + iy, \) by Theorem 2.3, we have \( g \equiv 0 \) in that strip. This yields the wanted equivalence.

(iii) (3) Suppose \( f \in S_{\infty}^0 (\mathbb{R}^d) \) with \( 1 < \beta < \infty. \) Then it follows from (2.19) and Theorem 1.44 (iii) that \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) with \( g^{(0)}(0) = 0, \) \( l \in \mathbb{N}_0. \) Let \( g \in S_{\infty, \text{even}}^0(\mathbb{R}) \) and define \( f : \mathbb{R} \to C \) by \( f(x) = g(x), x \in \mathbb{R}. \) Then \( f \in S_{\infty}^0 (\mathbb{R}^d) \) by Theorem 2.46.

Consider now the product \( f \cdot f, \) and write \( S_{\infty}^0 (\mathbb{R}^d) = S_{\infty}^0 (\mathbb{R}^d) \cap S_{\infty}^0 (\mathbb{R}^d). \) Obviously, \( f \in S_{\infty}^0 (\mathbb{R}^d) \) implies also \( f \cdot f \in S_{\infty}^0 (\mathbb{R}^d). \) Starting from \( f \in S_{\infty}^0 (\mathbb{R}^d) \subset S_{\infty}^0 (\mathbb{R}^d), \)
\( f \in S_{\infty}^0 (\mathbb{R}^d), \) it follows by means of Theorem 2.36 (vii) that \( f : f \in S_{\infty}^0 (\mathbb{R}^d). \) Thus we conclude that \( f \cdot f \in S_{\infty}^0 (\mathbb{R}^d), \) that is, the function \( r \omega \rightarrow g(r) h(\omega) \) belongs to \( S_{\infty}^0 (\mathbb{R}^d). \) From (2.19) we then infer \( h \in S_{2\beta}(\Delta_{LB}) \).

The spherical harmonics \( \{ Y_{m,j} : m \in \mathbb{N}_0, \quad j = 1, \ldots, N(g, m) \} \) constitute an orthonormal basis in \( L^2(S^{d-1}) \) of eigenfunctions of the Laplace-Beltrami operator \( \Delta_{LB}. \)
\[ \Delta_{LB} Y_{m,j} = -m(m + q - 2) Y_{m,j}, \quad m \in \mathbb{N}_0, \quad j = 1, \ldots, N(q, m), \]

see (1.69). Therefore, the Gevrey space \( S_{\lambda}^\alpha(\Delta_{LB}) \), brought about by \( \Delta_{LB} \), admits the following characterization:

**Lemma 2.52** Let \( \lambda \in \mathbb{R}^* \). Then
\[ S_{\lambda}(\Delta_{LB}) = \{ h \in L_2(S^{q-1}) : \exists \omega_0 \ (\exp(t m^{1/2}) (h, Y_{m,j})_{L_2(S^{q-1})}) m,j \in \omega_0 \} . \]

The proof of Lemma 2.52 runs along the same lines as the proof of Lemma 2.14. For the factorization problem in \( S_{\alpha}^\alpha(\mathbb{R}^d) \) with \( \alpha > 1 \), we present the following result.

**Theorem 2.53** Let \( \alpha > 1 \), let \( g \in S(\mathbb{R}) \) and let \( h \in D^{\alpha}(\Delta_{LB}) \). Then the factorized function \( (r \omega) \mapsto g(r) h(\omega) \) belongs to \( S_{\alpha}^\alpha(\mathbb{R}^d) \) if and only if there exists \( t > 0 \) such that
\[ (\exp(t n^{1/2}) + t m^{1/2}) (g, r^n \mathbb{E}_n^{m+q/2-1})_{L_2(S^{q-1})} (h, Y_{m,j})_{L_2(S^{q-1})} m,j \in \omega_0 . \]

Here \( \mathbb{E}_n^{m+q/2-1} \) is the Laguerre function defined in (1.12).

**Proof.** The equivalence follows immediately from Theorem 2.47 (i) by observing that \( u_{n,m}(r \omega) = r^n \mathbb{E}_n^{m+q/2-1}(r) Y_{m,j}(\omega) \), \( n, m \in \mathbb{N}_0, j = 1, \ldots, N(q, m), r \geq 0, \omega \in S^{q-1} \), see Theorem 1.35, and by using the inequality
\[ n^3 + m^3 \leq 2(n + m)^3 \leq \max(2, 2^3) (n^3 + m^3), \quad n, m \in \mathbb{N}_0, \quad \lambda > 0 . \]

\[ \square \]

### 2.6 The Radon transformation on \( S_{\alpha}^\alpha(\mathbb{R}^d) \)

In Section 1.7 we studied the Radon transformation on \( S(\mathbb{R}^d) \). We recall the relationship between the Radon transformation \( \mathcal{R} \) and the Fourier transformation \( \mathcal{F} \), see Theorem 1.46,
\[ \mathcal{R} f = (2\pi)^{(q-1)/2} (\mathcal{F}^* \otimes I) V \mathcal{F} f , \quad f \in S(\mathbb{R}^d) . \]

(2.20)

The Fourier invariance of \( S(\mathbb{R}^d) \) and the characterization of \( V(S(\mathbb{R}^d)) \) given in Theorem 1.43, led to the characterization of \( \mathcal{R}(S(\mathbb{R}^d)) \) in Theorem 1.47:

Let \( g \) be an even function on \( Z_q \). Then \( g \in \mathcal{R}(S(\mathbb{R}^d)) \) if and only if
\[ g \in D^{\alpha}(Q \otimes I) \cap D^{\alpha}(P \otimes I) \cap D^{\alpha}(I \otimes \Delta_{LB}) \]

(2.21)

and \( g \) obeys the Helgason-Ludwig consistency conditions:

For \( l \in \mathbb{N}_0 \), the function \( \omega \mapsto \int_{\mathbb{R}} l^j g(t, \omega) dt \) is the restriction
\[ \text{to } S^{q-1} \text{ of a homogeneous polynomial of degree } l \text{ on } \mathbb{R}^d . \]

(2.22)

For the Gel’fand-Shilov spaces we have \( \mathcal{F}(S_{\alpha}^\alpha(\mathbb{R}^d)) = S_{\alpha}^\alpha(\mathbb{R}^d) \), see Theorem 2.38, so that by (2.20) we obtain
\[ \mathcal{R}(S_{\alpha}^\alpha(\mathbb{R}^d)) = (\mathcal{F}^* \otimes I) V(S_{\alpha}^\alpha(\mathbb{R}^d)) , \quad \alpha, \beta \in \mathbb{R}^{+}_{0,\infty} . \]

(2.23)
In Theorem 2.49 we have been able to characterize the image $V(S_\infty^\alpha(\mathbb{R}^n))$ only for $\alpha = \infty$. Accordingly, we can characterize the image $\mathcal{R}(S_\infty^\beta(\mathbb{R}^n))$, only.

**Theorem 2.54** Let $\beta \in \mathbb{R}_{*,0}^+$ and let $g$ be an even function on $Z_\beta$. Then $g \in \mathcal{R}(\mathbb{R}^n)$ if and only if

$$g \in D_\infty^\alpha(Q \otimes I) \cap S_\beta(P \otimes I) \cap D_\infty^\alpha(I \otimes \Delta_{LB})$$

and $g$ obeys the Helgason-Ludwig consistency conditions (2.22).

**Proof.** According to Theorem 2.49 and (2.16), (2.17), we have

$$V(S_\infty^\alpha(\mathbb{R}^n)) = V(S_\infty^\alpha(\mathbb{R}^n)) \cap S_\beta(Q \otimes I).$$

Hence, by means of (2.23) and (2.6) we deduce

$$\mathcal{R}(S_\infty^\beta(\mathbb{R}^n)) = (F^* \otimes I) \left(V(S_\infty^\beta(\mathbb{R}^n)) \cap S_\beta(Q \otimes I)\right) = \mathcal{R}(S_\infty^\beta(\mathbb{R}^n)) \cap S_\beta(P \otimes I).$$

Now the theorem follows from (2.21), (2.22).

**Theorem 2.55** Let $\alpha, \beta \in \mathbb{R}_{*,0}^+$ and let $g \in \mathcal{R}(S_\infty^\beta(\mathbb{R}^n))$. Then

$$g \in S_\alpha(Q \otimes I) \cap S_\beta(P \otimes I) \cap S_{2\alpha+2\beta}(I \otimes \Delta_{LB})$$

and $g$ obeys the Helgason-Ludwig consistency conditions (2.22).

**Proof.** Since $S_\infty^\beta(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $g$ obeys the Helgason-Ludwig consistency conditions (2.22) by Theorem 1.47. Further, by means of (2.23), Theorem 2.50 and (2.6), we deduce

$$\mathcal{R}(S_\infty^\beta(\mathbb{R}^n)) = (F^* \otimes I) V(S_\infty^\beta(\mathbb{R}^n))$$

$$\subset (F^* \otimes I) \left(S_\beta(Q \otimes I) \cap S_\alpha(P \otimes I) \cap S_{2\alpha+2\beta}(I \otimes \Delta_{LB})\right)$$

$$= S_\beta(P \otimes I) \cap S_\alpha(Q \otimes I) \cap S_{2\alpha+2\beta}(I \otimes \Delta_{LB}).$$

Helgason [19, Corollary 4.3] characterized the image $\mathcal{R}(D(\mathbb{R}^n))$, where $D(\mathbb{R}^n) = S_\infty^\alpha(\mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. In our notation, Helgason's result reads as follows.

**Theorem 2.56** Let $g$ be an even function on $Z_\beta$. Then $g \in \mathcal{R}(S_\infty^\alpha(\mathbb{R}^n))$ if and only if

$$g \in S_\alpha(Q \otimes I) \cap D_\infty^\alpha(P \otimes I) \cap D_\infty^\alpha(I \otimes \Delta_{LB})$$

and $g$ obeys the Helgason-Ludwig consistency conditions (2.22).

**Corollary 2.57** Let $\beta > 1$ and let $g$ be an even function on $Z_\beta$. Then $g \in \mathcal{R}(S_\infty^\beta(\mathbb{R}^n))$ if and only if

$$g \in S_\alpha(Q \otimes I) \cap S_\beta(P \otimes I) \cap D_\infty^\alpha(I \otimes \Delta_{LB})$$

and $g$ obeys the Helgason-Ludwig consistency conditions (2.22).
Proof. Since \( S^0(\mathbb{R}^q) = S^0_S(\mathbb{R}^q) \cap S^0_u(\mathbb{R}^q) \), the result follows from Theorems 2.54 and 2.56. \( \square \)

As a continuation of Theorems 2.49 and 2.50, we are now able to characterize the image \( V(S^0_a(\mathbb{R}^q)) \) with \( a > 1 \).

**Corollary 2.58** Let \( a > 1 \) and let \( f \) be a function on \( \mathbb{R}^q \). Then \( f \in S^0_a(\mathbb{R}^q) \) if and only if

\[
Vf \in S_a(Q \otimes I) \cap S_a(P \otimes I) \cap D^{\infty}(I \otimes \Delta_{LB})
\]

and \( Vf \) satisfies the homogeneity conditions (2.17).

Proof. The result follows from (2.23), Corollary 2.57 and (2.6). \( \square \)

In the next theorem we characterize the range of the Radon transformation when applied to the radially symmetric functions in \( S^0_a(\mathbb{R}^q) \). In this characterization the Erdélyi-Kober operators \( \mathcal{W}_\lambda \), \( \lambda \in \mathbb{R} \), defined in (1.23), appear. From Theorem 2.22 (ii) we recall that

\[
\mathcal{W}_\lambda(S^0_{a,\text{even}}(\mathbb{R})) = S^0_{a,\text{even}}(\mathbb{R}), \quad \alpha, \beta \in \mathbb{R}^*_+ \quad \lambda \in \mathbb{R}.
\]

(2.24)

For the corresponding characterization of the range of the Radon transformation when applied to the radially symmetric functions in \( S(\mathbb{R}^q) \) we refer to Theorem 1.55.

**Theorem 2.59** Let \( a, \beta \in \mathbb{R}^*_+ \).

(i) Let \( f_0 \in S^0_{a,\text{even}}(\mathbb{R}) \) and define the radially symmetric function \( f : \mathbb{R}^q \to C \) by

\[
f(r, \omega) = f_0(r), \quad r \geq 0, \quad \omega \in S^{q-1}.
\]

Then \( f \in S^0_a(\mathbb{R}^q) \) and its Radon transform is given by

\[
(Rf)(p, \omega) = (2\pi)^{(q-1)/2} (\mathcal{W}_{(q-1)/2} f_0)(p)
\]

\[
= \frac{2\pi^{(q-1)/2}}{\Gamma((q-1)/2)} \int_0^\infty f_0(r) (r^2 - p^2)^{(q-3)/2} r \, dr, \quad p \geq 0, \quad \omega \in S^{q-1}.
\]

The Erdélyi-Kober transform \( \mathcal{W}_{(q-1)/2} f_0 \) belongs to \( S^0_{a,\text{even}}(\mathbb{R}) \).

(ii) Let \( g_0 \in S^0_{a,\text{even}}(\mathbb{R}) \) and define the function \( g : Z_q \to C \) by

\[
g(p, \omega) = g_0(p), \quad (p, \omega) \in Z_q.
\]

Then \( g \) is the Radon transform of the function \( R^{-1}g \) given by

\[
(R^{-1}g)(r, \omega) = (2\pi)^{-{(q-1)/2}} (\mathcal{W}_{(q-1)/2} g_0)(r)
\]

\[
= \begin{cases} 
(2\pi)^{-{(q-1)/2}} ((-t^{-1}d/dr)^{(q-1)/2} g_0(r) \text{ for } q \text{ odd}, \\
2(2\pi)^{-q/2} ((-t^{-1}d/dr)^{q/2} \int_0^\infty g_0(p) (p^2 - r^2)^{-q/2} p \, dp \text{ for } q \text{ even}, 
\end{cases}
\]

\[
r \geq 0, \quad \omega \in S^{q-1}.
\]
The Erdélyi-Kober transform \( \mathcal{W}_{-(q-1)/2} \) belongs to \( S^0_{n,\text{even}}(\mathbb{R}) \).

**Proof.** The theorem follows from (2.24) and Theorems 2.46 and 1.55.

At the end of this section we characterize the range of the Radon transformation when applied to functions \( f \in S^0_{n,\text{even}}(\mathbb{R}^n) \) of the form

\[
f(\omega) = f_m(r) Y_m(\omega) \quad \text{with} \quad f_m \in Q^m(S^0_{n,\text{even}}(\mathbb{R})), \quad Y_m \in \mathcal{Y}_m^m.
\]

In this characterization the Weyl-Gegenbauer transformation \( \mathcal{W}\mathcal{G}_{m,q/2-1} \) and its inverse \( \mathcal{W}\mathcal{G}_{m,q/2-1}^{-1} \) are involved. From (1.85), (1.86) we recall the definition

\[
\mathcal{W}\mathcal{G}_{m,q/2-1} = W_{-m} W_{m,(q-1)/2} Q^{-m}.
\]

\[
\mathcal{W}\mathcal{G}_{m,q/2-1}^{-1} = Q^m W_{m,(q-1)/2} W_m.
\]

Here the operators \( W_\lambda, \lambda \in \mathbb{R} \), are the Weyl operators defined in (1.34), while the operators \( W_m, \lambda \in \mathbb{R} \), are the Erdélyi-Kober operators defined in (1.23). Since the Erdélyi-Kober operator \( W_{m+(q-1)/2} \) maps \( S^0_{n,\text{even}}(\mathbb{R}) \) onto itself, by (2.24), we have

\[
\mathcal{W}\mathcal{G}_{m,q/2-1}^{-1}(Q^m(S^0_{n,\text{even}}(\mathbb{R}))) = P^m(S^0_{n,\text{even}}(\mathbb{R})).
\]

In this connection we refer to Theorem 2.17 where we have characterized the spaces \( Q^m(S^0_{n,\text{even}}(\mathbb{R}))) \) and \( P^m(S^0_{n,\text{even}}(\mathbb{R}))) \).

The operators \( \mathcal{W}\mathcal{G}_{m,q/2-1} \) and \( \mathcal{W}\mathcal{G}_{m,q/2-1}^{-1} \) are integral operators with the Gegenbauer polynomial \( C^m_q \), defined in (1.88), in their kernels; see (1.89), (1.90).

For the corresponding characterization of the range of the Radon transformation when applied to functions \( f \in S(\mathbb{R}^n) \) of the form

\[
f(\omega) = f_m(r) Y_m(\omega) \quad \text{with} \quad f_m \in Q^m(S_{\text{even}}(\mathbb{R})), \quad Y_m \in \mathcal{Y}_m^m,
\]

we refer to Theorem 1.56.

**Theorem 2.60**

(i) Let \( f_m \in Q^m(S^0_{n,\text{even}}(\mathbb{R}))) \), let \( Y_m \in \mathcal{Y}_m^m \) and define \( f : \mathbb{R}^n \to C \) by

\[
f(\omega) = f_m(r) Y_m(\omega), \quad r \geq 0, \quad \omega \in S^{r-1}.
\]

Then \( f \in S^0_{n,\text{even}}(\mathbb{R}^n) \) and its Radon transform is given by

\[
(Rf)(p, \omega) = (2\pi)^{n-1/2} \int_{0}^{\infty} (r^2 + p^2)^{(q-1)/2} C^m_q^{r-1}(p/r) f_m(r) r \, dr \, Y_m(\omega),
\]

\[
p \geq 0, \quad \omega \in S^{r-1}.
\]

The Weyl-Gegenbauer transform \( \mathcal{W}\mathcal{G}_{m,q/2-1} f_m \) belongs to \( P^m(S^0_{n,\text{even}}(\mathbb{R}))) \).

(ii) Let \( g_m \in P^m(S^0_{n,\text{even}}(\mathbb{R}))) \), let \( Y_m \in \mathcal{Y}_m^m \) and define \( g : Z_q \to C \) by
\[ g(p, \omega) = g_m(p) \, Y_m(\omega), \quad (p, \omega) \in \mathbb{Z}^q. \]

Then \( g \) is the Radon transform of the function \( R^{-1}g \) given by
\[
(R^{-1}g)(rw) = (2\pi)^{-(s-1)/2} \left( W G_{m_s/2-1}^{-1} \right) (r) \, Y_m(\omega)
\]
\[
= \frac{(-\sqrt{2\pi})^{1-s}}{r^{s-2}} \int \frac{C_{m/2-1}^{(s-1)/2}(p/r)}{r^{s-1}} \, dp \, Y_m(\omega),
\]
\[ r \geq 0, \quad \omega \in S^{s-1}. \]

The inverse Weyl-Gegenbauer transform \( W G_{m_s/2-1}^{-1} \) belongs to \( Q^m(S_{a,even}(\mathbb{R})). \)

**Proof.** The theorem follows from (2.27) and Theorems 2.45 and 1.56. \[\square\]
CHAPTER 3
Fractional Calculus and Gegenbauer transformations

In \( C^\infty(\mathbb{R}^+) \) we define the multiplication operators \( M_\alpha, \alpha \in \mathbb{R} \), and the composition operators \( T_p, p \in \mathbb{R}\{0\} \), by

\[
(M_\alpha f)(x) = x^\alpha f(x), \quad f \in C^\infty(\mathbb{R}^+), \quad x > 0,
\]

\[
(T_p f)(x) = f(x^p), \quad f \in C^\infty(\mathbb{R}^+), \quad x > 0.
\]

The operators \( M_\alpha, \alpha \in \mathbb{R} \), and \( T_p, p \in \mathbb{R}\{0\} \), constitute one-parameter groups on \( C^\infty(\mathbb{R}^+) \) with

\[
M_\alpha M_\beta = M_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R},
\]

\[
T_p T_q = T_{pq}, \quad p, q \in \mathbb{R}\{0\}.
\]

The differentiation operator \( D \) defined by \( Df = f' \) maps \( C^\infty(\mathbb{R}^+) \) into itself, but is not a bijection. We mention the intertwining relations

\[
T_p M_\alpha = M_{p^\alpha} T_p, \quad \alpha \in \mathbb{R}, \quad p \in \mathbb{R}\{0\}, \quad \tag{3.1}
\]

\[
DT_p = pM_{p^{-1}}T_pD, \quad p \in \mathbb{R}\{0\}. \quad \tag{3.2}
\]

3.1 The Weyl fractional operators

An appropriate space for establishing a fractional calculus based on Weyl operators is the space \( S_\ast(\mathbb{R}^+) \) which we define as follows.

Definition 3.1 The space \( S_\ast(\mathbb{R}^+) \) consists of all functions \( f \in C^\infty(\mathbb{R}^+) \) for which

\[
x^k f^{(l)} \in L^1([1, \infty)), \quad k, l \in \mathbb{N}_0.
\]

In fact \( S_\ast(\mathbb{R}^+) \) consists of the \( C^\infty(\mathbb{R}^+) \)-functions which, together with their derivatives, tend to zero more rapidly than any negative power of \( x \) as \( x \to \infty \). There are no restrictions on the behaviour of the functions at \( x = 0 \). For instance, the function

\[
f(x) = \exp(1/x - x), \quad x > 0,
\]

belongs to \( S_\ast(\mathbb{R}^+) \).

Along with the operators \( M_\alpha, \alpha \in \mathbb{R}, T_p, p > 0 \), and \( D \), which map \( S_\ast(\mathbb{R}^+) \) onto itself, we introduce the Weyl fractional integral operator \( W_\mu \) of order \( \mu > 0 \) on \( S_\ast(\mathbb{R}^+) \), defined by

\[
(W_\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_1^\infty f(t) (t-x)^{\mu-1} \, dt, \quad f \in S_\ast(\mathbb{R}^+), \quad x > 0, \quad \mu > 0. \quad \tag{3.3}
\]
It readily follows that $W_\mu$ maps $S_-(\mathbb{R}^+)$ into $C^\infty(\mathbb{R}^+)$ with

$$DW_\mu = W_\mu D, \quad \mu > 0.$$  \hfill (3.4)

**Lemma 3.2** For $\mu > 0$, the operator $W_\mu$ maps $S_-(\mathbb{R}^+)$ into itself.

**Proof.** Let $\mu > 0$ and let $f \in S_-(\mathbb{R}^+)$. The conclusion $W_\mu f \in S_-(\mathbb{R}^+)$ follows from the estimate

$$|\Gamma(\mu) z^k D^i W_\mu f (x)| = |\Gamma(\mu) z^k (W_\mu D^i f) (x)| = |z^k \int_0^\infty f^{(i)}(t + z) t^{\mu-1} dt|$$

$$\leq \int_0^\infty (t + z)^k \left| f^{(i)}(t + z) \right| \left( 1 + (t + z)^{2\mu} \right) t^{\mu-1} (1 + t^{2\mu}) dt$$

$$\leq \frac{\pi}{2\mu} \sup_{\xi > 1} |\xi^k (1 + \xi^{2\mu}) f^{(i)}(\xi)|, \quad k, i \in \mathbb{N}_0, \quad \mu > 1.$$

As usual in fractional calculus we want to extend the definition of the operator $W_\mu$ to non-positive values of $\mu$. This can be done in a natural way: By an $n$ times repeated integration by parts in the integral expression (3.3) for $W_\mu f$ we obtain

$$W_\mu = W_{n+\mu} (-D)^n, \quad n \in \mathbb{N}_0, \quad \mu > 0.$$  \hfill (3.5)

For fixed $n \in \mathbb{N}_0$ the right-hand side of the above equality makes sense for $\mu > -n$. Therefore, we can extend the definition of the operator $W_\mu$ to non-positive values of $\mu$ as follows:

$$W_\mu = I, \quad W_{\mu - n} = W_{[\mu - n]} = W_{\mu - [\mu]} (\mu > 0).$$

Here $[\mu]$ denotes the smallest integer which is greater than or equal to $\mu$.

Observe that $W_{-n} = (-D)^n$ for $n \in \mathbb{N}$. For this reason, the operator $W_{-\mu}, \mu > 0$, is called the Weyl fractional differential operator of order $\mu$. The operators $W_\mu, \mu \in \mathbb{R}$, constitute a group on $S_-(\mathbb{R}^+)$:

**Theorem 3.3** For $\mu, \nu \in \mathbb{R}$,

$$W_\mu W_\nu = W_{\mu + \nu}.$$  \hfill (3.6)

**Proof.** For $\mu = 0$ or $\nu = 0$ the identity is trivial. Throughout the remainder of this proof it is understood that $\mu, \nu > 0$. For $f \in S_-(\mathbb{R}^+)$ and $x > 0$ we derive

$$(W_\mu W_\nu f) (x) = \frac{1}{\Gamma(\mu)} \int_0^\infty (W_\nu f) (t) (t - x)^{\mu-1} dt$$

$$= \frac{1}{\Gamma(\mu) \Gamma(\nu)} \int_0^\infty \left( \int_t^\infty f(s) (s - t)^{\nu-1} ds \right) (t - x)^{\mu-1} dt$$
\[ \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int \int (s-t)^{\nu-1} (t-z)^{\mu-1} dt \, f(s) \, ds \, . \]

By substitution of \( \tau = (t-z)/(s-z) \) in the inner integral, an Euler-type integral appears and we obtain

\[ (W_\mu W_\nu f) (x) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int \int f(s) (s-z)^{\mu+n-1} ds \int_0^1 (1-\tau)^{\nu-1} \tau^{\mu-1} d\tau \]

\[ = \frac{1}{\Gamma(\mu+n)} \int f(s) (s-z)^{\mu+n-1} ds = (W_{\mu+n} f) (x) \, . \]

Through repeated integration by parts we find \( W_n(-D)^n = I, \, n \in \mathbb{N}_0 \). By use of this result and the relation (3.4) we derive

\[ W_{-\mu} W_\mu = W_{[\mu] - \mu} (-D)^{[\mu]} W_\mu = W_{[\mu] - \mu} W_\mu (-D)^{[\mu]} = W_{[\mu]} (-D)^{[\mu]} = I \, , \]

and therefore,

\[ W_{-\mu} W_\nu = W_{[\mu] - \mu} (-D)^{[\mu]} W_{[\nu] - \nu} (-D)^{[\nu]} = W_{-\mu + \nu} W_{\mu + \nu} W_{[\mu] + [\nu]} ^{[\mu] + [\nu]} \]

\[ = W_{-\mu + \nu} W_{[\mu] + [\nu]} ^{[\mu] + [\nu]} = W_{-\mu + \nu} \, . \]

Partitioning \( \nu \) or \(-\mu\) into two appropriate parts we then infer that

\[ W_{-\mu} W_\nu = \begin{cases} W_{-\mu} W_{\mu + (\nu - \mu)} = W_{-\mu} W_{\mu} W_{\nu - \mu} = W_{\nu - \mu} & \text{if } \mu \leq \nu \, , \\ W_{-\mu - (\mu - \nu)} = W_{-\mu - (\mu - \nu)} W_{\mu} W_{\nu} = W_{\mu - \nu} & \text{if } \mu > \nu \, , \end{cases} \]

and finally,

\[ W_\mu W_{-\nu} = W_\mu W_{[\nu] + (-D)^{[\nu]} W_\nu = W_{-\nu} W_\mu = W_{\nu - \mu} \, . \]

Since \( DM_1 - M_1 D = I \), it can be proved by induction (with respect to \( m \)) that

\[ D^m M_{m+n} D^n = M_m D^{m+n} M_n \, , \quad m, n \in \mathbb{N}_0 \, . \]

Here we prove the following generalization analogous to the second index law of Love [36, Sections 6,7].

**Theorem 3.4** For \( \mu, \nu \in \mathbb{R} \),

\[ W_\mu W_{\mu + \nu} W_\nu = M_{-\nu} W_\mu W_{\mu + \nu} \, . \]

**Proof.** For \( \mu = 0 \) or \( \nu = 0 \) the identity is trivial. Throughout the remainder of this proof it is understood that \( \mu, \nu > 0 \). For \( f \in S_-(\mathbb{R}^n) \) and \( x > 0 \) we derive

\[ (W_\mu W_{-\mu - \nu} W_\nu f) (x) \]

\[ = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int \int (s-t)^{\nu-1} (t-z)^{\mu-1} dt \, f(s) \, ds \, . \]
\[ \frac{1}{\Gamma(\mu) \Gamma(\nu)} \int_{\frac{1}{z}}^{\infty} (r^{-\mu} - s^{-\nu}) \int_{r}^{\infty} f(t) (t - r)^{\nu-1} dr \, (r - z)^{\mu-1} dr \]

\[ = \frac{1}{\Gamma(\mu) \Gamma(\nu)} \int_{\frac{1}{z}}^{\infty} f(t) \left( \int_{\frac{1}{z}}^{t} (r - x)^{\mu-1} (t - r)^{\nu-1} r^{-\mu} dr \right) dt . \]

By substitution of \( r = (1/r - 1/t) / (1/z - 1/t) \) in the inner integral, an Euler-type integral appears and we obtain

\[ (W_{\mu} M_{\nu} W_{\nu}, f)(x) = \frac{x^{-\mu}}{\Gamma(\mu) \Gamma(\nu)} \int_{\frac{1}{z}}^{\infty} t^{\mu} f(t) \int_{0}^{t} (1 - t)^{\nu-1} t^{-1} dt \]

\[ = \frac{x^{-\mu}}{\Gamma(\mu + \nu)} \int_{\frac{1}{z}}^{\infty} t^{\mu} f(t) \int_{0}^{t} (t - x)^{\mu+\nu-1} dt = (M_{\mu} W_{\nu}, M_{\nu} f)(x) . \]

By rewriting the identity we also have

\[ W_{\nu}, M_{\mu}, W_{\mu+\nu} = M_{\nu}, W_{\mu}, M_{\mu}, \]

\[ W_{\mu+\nu}, M_{\mu}, W_{\nu} = M_{\mu}, W_{\nu}, M_{\mu-\nu} , \]

and by inversion we obtain

\[ W_{\nu}, M_{\mu+\nu}, W_{\nu} = M_{\mu}, W_{\mu+\nu}, M_{\nu} , \]

\[ W_{\mu-\nu}, M_{\mu}, W_{\nu} = M_{\mu}, W_{\nu}, M_{\mu-\nu} , \]

\[ W_{\nu}, M_{\mu}, W_{\nu-\nu} = M_{\mu}, W_{\nu}, M_{\nu} . \]

Thus all possible cases are covered. \( \square \)

By Leibniz's differentiation rule we have

\[ D^{n} M_{1} = M_{1} D^{n} + n D^{n-1} , \quad n \in \mathbb{N} . \]

Here we prove the following generalization.

**Theorem 3.5** For \( \mu \in \mathbb{R} , \)

\[ W_{\mu}, M_{1} = M_{1} W_{\mu} + \mu W_{\mu+1} . \]

**Proof.** For \( \mu = 0 \) the identity is trivial. Throughout the remainder of this proof it is understood that \( \mu > 0 \). For \( f \in \mathcal{S}_{\mu}(\mathbb{R}^{+}) \) and \( x > 0 \) we derive

\[ ((M_{1} W_{\mu} + \mu W_{\mu+1}) f)(x) \]

\[ = \frac{x}{\Gamma(\mu)} \int_{\frac{1}{x}}^{\infty} f(t) (t - x)^{\mu-1} dt + \frac{\mu}{\Gamma(\mu + 1)} \int_{\frac{1}{x}}^{\infty} f(t) (t - x)^{\mu} dt . \]
By use of this result and Leibniz's differentiation rule we obtain
\[ W_{-\mu} M_1 = W_{[-\mu]}(W_{[\mu] - \mu} M_1) = W_{[-\mu]}(M_1 W_{[\mu] - \mu} + ([\mu] - \mu) W_{[\mu] - \mu + 1}) \]
\[ = (M_1 W_{[\mu]} - [\mu] W_{[\mu] + 1}) W_{\mu} - \mu W_{\mu + 1} = M_1 W_{\mu} - \mu W_{\mu + 1} \]

Next we introduce the Erdélyi-Kober operator \( BW_{\mu}, \mu \in \mathbb{R} \), on \( S_{-}(\mathbb{R}^+) \), defined by
\[ BW_{\mu} = 2^{-\mu} T^2 W_{\mu} T_{1/2}, \quad \mu \in \mathbb{R}. \] (3.6)

For \( \mu > 0 \), this definition can be elaborated by means of (3.3), viz.
\[ (BW_{\mu} f)(x) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_{0}^{\infty} f(t) \left( t^2 - x^2 \right)^{\mu-1} t \; dt, \quad f \in S_{-}(\mathbb{R}^+), \quad x > 0, \quad \mu > 0. \] (3.7)

The extension to non-positive values of \( \mu \) readily follows by means of (3.5), (3.2) and (3.1):
\[ BW_{0} = I, \quad BW_{-\mu} = BW_{[\mu] - \mu}(-2T^2 DT_{1/2})^{[\mu]} = BW_{[\mu] - \mu}(-M_{-1} D)^{[\mu]}, \quad \mu > 0. \] (3.8)

Observe that \( BW_{-\mu} = (-M_{-1} D)^{\mu} \) for \( \mu \in \mathbb{N} \). For this reason, the operator \( BW_{-\mu}, \mu > 0 \), is called a fractional differential operator of order \( \mu \).

The preceding results for the operators \( BW_{\mu}, \mu \in \mathbb{R} \), can be translated into results for the operators \( BW_{\mu}, \mu \in \mathbb{R} \). Thus it follows from Lemma 3.2 that for \( \mu > 0 \), the operator \( BW_{\mu} \) maps \( S_{-}(\mathbb{R}^+) \) into itself. The translation of Theorems 3.3, 3.4 and 3.5 is comprised in the following theorem.

**Theorem 3.6** For \( \mu, \nu \in \mathbb{R} \),

(i) \( BW_{\mu} BW_{\nu} = BW_{\mu + \nu} \),

(ii) \( BW_{\mu} M_{-2\mu - 2\nu} BW_{\nu} = M_{-2\nu} BW_{\mu + \nu} M_{-2\mu} \),

(iii) \( BW_{\mu} M_{2} = M_{2} BW_{\mu} + 2\mu BW_{\mu + 1} \).

By induction it can be proved that
\[ D^{2\mu + 1} = (M_{-1} D)^{\mu} M_{2\mu + 1}(M_{-1} D)^{\mu + 1}, \quad \mu \in \mathbb{N}. \]

The fractional version of this identity, given in the next theorem, is the key formula in our derivation of inversion formulas for Weyl-Gegenbauer transformations in Section 3.6.

**Theorem 3.7** For \( \mu \in \mathbb{R} \),
\[ W_{2\mu + 1} = BW_{\mu + 1} M_{-2\mu - 1} BW_{\mu}. \]
Proof. First we consider the case \( \mu > -1 \). For \( f \in \mathcal{S}_-(\mathbb{R}^+) \) and \( x > 0 \) we derive

\[
(W_{\mu+1}M_{-2\mu-1}W_\mu f)(x) = (W_{\mu+1}M_{-2\mu-1}W_{\mu+1}(W_{\mu+1}f))(x)
\]

\[
= \frac{2^{-2\mu}}{\Gamma^2(\mu + 1)} \int_0^\infty (r^{2\mu + 1}) \int_0^\infty (r^{-2\mu - 1}) \int_0^\infty (W_{-1}f)(t) \left( t^2 - r^2 \right)^\mu dt \left( r^2 - x^2 \right)^\mu r \, dr
\]

\[
= \frac{2^{-2\mu}}{\Gamma^2(\mu + 1)} \int_0^\infty (W_{-1}f)(t) \left( \int_0^t \left( r^2 - x^2 \right)^\mu \left( t^2 - r^2 \right)^\mu r^{-2\mu} dr \right) dt
\]

The inner integral can be evaluated by means of two formulas quoted from [18, 2.1.3 (10), 2.8 (6)], viz.

\[
\int_0^1 t^{\mu-1}(1 - t)^{\nu - 1} (1 - zt)^{-a} dt = B(b, c - b) \_2F_1(a, b; c; z),
\]

\[
\text{Re } c > \text{Re } b > 0, \ |z| < 1,
\]

\[
\_2F_1(a - \frac{1}{2}, a; 2a; z) = (\frac{1}{2} + \frac{1}{2}(1 - z)^{1/2})^{-2a}.
\]

Indeed, by means of the substitution \( \xi = (r^2 - x^2)/t^2 \) and the quoted formulas, we have

\[
\int_0^t (r^2 - x^2)^\mu \left( t^2 - r^2 \right)^\nu r^{-2\nu} dr
\]

\[
= \frac{1}{2}((t^2 - x^2)/x)^{2\mu + 1} \int_0^1 \xi^{\mu}(1 - \xi)^{\mu} \left( 1 - (1 - t^2/x^2)\xi \right)^{-\mu - 1/2} d\xi
\]

\[
= \frac{1}{2}((t^2 - x^2)/x)^{2\mu + 1} B(\mu + 1, \mu + 1) \_2F_1(\mu + \frac{1}{2}, \mu + 1; 2\mu + 2; 1 - t^2/x^2)
\]

\[
= 2^{2\mu} B(\mu + 1, \mu + 1) (t - x)^{2\mu + 1}.
\]

By combining the previous results we obtain

\[
(W_{\mu+1}M_{-2\mu-1}W_\mu f)(x) = \frac{1}{\Gamma(2\mu + 2)} \int_0^\infty (W_{-1}f)(t) \left( t - x \right)^{2\mu + 1} dt
\]

\[
= (W_{2\mu+1}W_{-1}f)(x) = (W_{2\mu+1}f)(x).
\]

Thus we have proved the stated identity for \( \mu > -1 \).

Next suppose \( \mu < 0 \), then \(-\mu - 1 > -1\). In the previous result we replace \( \mu \) by \(-\mu - 1\), then

\[
W_{-2\mu-1} = W_{\mu}M_{2\mu+1}W_{\mu-1},
\]

and by inversion we obtain

\[
W_{2\mu+1} = W_{\mu+1}M_{-(2\mu+1)}W_\mu, \quad \mu < 0.
\]
This completes the proof of the theorem.

Finally we introduce a subspace of \( S_{-\alpha}(\mathbb{R}^+) \).

**Definition 3.8** Let \( b \in \mathbb{R}^+ \). The space \( S_{-\alpha}(\mathbb{R}^+) \) consists of all functions \( f \in C^\infty(\mathbb{R}^+) \) for which

\[
    f(x) = 0, \quad x > b.
\]

Obviously \( S_{-\alpha}(\mathbb{R}^+) \), \( b \in \mathbb{R}^+ \), is a proper subspace of \( S_{-\alpha}(\mathbb{R}^+) \). Further we define

\[
    S_{-\alpha}(\mathbb{R}^+) = S_{-\alpha}(\mathbb{R}^+).
\]

The following statements can be verified easily.

**Theorem 3.9** Let \( b, p \in \mathbb{R}^+ \) and let \( \alpha, \mu \in \mathbb{R} \). Then

(i) \( M_{p}, \ W_{\mu} \) and \( W_{\nu} \) map \( S_{-\alpha}(\mathbb{R}^+) \) bijectively onto \( S_{-\alpha}(\mathbb{R}^+) \);

(ii) \( T_{p} \) maps \( S_{-\alpha}(\mathbb{R}^+) \) bijectively onto \( S_{-\alpha/\nu}(\mathbb{R}^+) \).

From (3.3) and (3.7) we infer that for \( \mu > 0 \), the operators \( W_{\mu} \) and \( W_{\nu} \) on \( S_{-\alpha}(\mathbb{R}^+) \) are given by

\[
    (W_{\mu}f)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{b} f(t) \left(t - x\right)^{\mu-1} dt, \quad f \in S_{-\alpha}(\mathbb{R}^+), \quad 0 < x < b, \quad \mu > 0, \quad (3.9)
\]

\[
    (W_{\nu}f)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{b} f(t) \left(t^{2} - x^{2}\right)^{\nu-1} dt, \quad f \in S_{-\alpha}(\mathbb{R}^+), \quad 0 < x < b, \quad \mu > 0. \quad (3.10)
\]

### 3.2 The Riemann-Liouville fractional operators

An appropriate space for establishing a fractional calculus based on Riemann-Liouville operators is the space \( S_{-\alpha}(\mathbb{R}^+) \) which we define as follows.

**Definition 3.10** The space \( S_{-\alpha}(\mathbb{R}^+) \) consists of all functions \( f \in C^\infty(\mathbb{R}^+) \) for which

\[
    \lim_{\varepsilon \to 0} f^{(k)}(x) = 0, \quad k \in \mathbb{N}_0.
\]

In fact we have

\[
    S_{-\alpha}(\mathbb{R}^+) = T_{-1}(S_{-\alpha}(\mathbb{R}^+)). \quad (3.11)
\]

Along with the operators \( M_{\alpha}, \ p \in \mathbb{R}, \ T_{p}, \ p > 0, \) and \( D \), which map \( S_{-\alpha}(\mathbb{R}^+) \) onto itself, we introduce the Riemann-Liouville fractional integral operator \( I_{\alpha} \) of order \( \mu > 0 \) on \( S_{-\alpha}(\mathbb{R}^+) \), defined by
\[(I_{\mu})^n(x) = \frac{1}{\Gamma(\mu)} \int_0^x f(t) (x-t)^{\mu-1} dt, \quad f \in S_-(R^+), \quad x > 0, \quad \mu > 0.\] (3.12)

Next, we want to extend the definition of the operator \(I_\mu\) to non-positive values of \(\mu\). By an \(n\) times repeated integration by parts in the integral expression (3.12) for \(I_{\mu}^n\), we obtain
\[I_\mu = I_{\mu + n}D^n, \quad n \in \mathbb{N}_0, \quad \mu > 0.\]

Therefore, we can extend the definition of the operator \(I_\mu\) to non-positive values of \(\mu\) as follows:
\[I_0 = I, \quad I_{-\mu} = I_{[\mu]}D^{[\mu]}, \quad \mu > 0.\] (3.13)

Observe that \(I_{-n} = D^n\) for \(n \in \mathbb{N}\). For this reason, the operator \(I_{-\mu}, \mu > 0\), is called the Riemann-Liouville fractional differential operator of order \(\mu\).

We prove the following relation between the Riemann-Liouville operator \(I_\mu\) and the Weyl operator \(W_\mu\).

**Theorem 3.11** For \(\mu \in \mathbb{R}\),
\[I_\mu = M_{\mu-1}T_{-1}W_\mu T_{-1}M_{\mu+1}.\]

**Proof.** If \(\mu > 0\), the relation can be verified by a straightforward calculation based on the definitions (3.3) and (3.12) of \(W_\mu\) and \(I_\mu\). If \(\mu = 0\), the relation holds trivially. Next, it can be shown by induction that
\[M_{-n-1}T_{-1}D^n = (-D)^nT_{-1}M_{n+1}, \quad n \in \mathbb{N}.\]

By means of the previous results, Theorem 3.4 and (3.1), we derive for \(\mu > 0\),
\[I_{-\mu} = I_{[\mu]}D^{[\mu]} = M_{[\mu]}^{-1}T_{-1}W_{[\mu]-\mu}M_{\mu}(M_{-[\mu]-1}T_{-1}D^{[\mu]})\]
\[= M_{[\mu]}^{-1}T_{-1}(W_{[\mu]-\mu}M_{\mu}W_{-[\mu]})T_{-1}M_{-[\mu]+1}\]
\[= M_{[\mu]}^{-1}T_{-1}M_{-[\mu]}W_{-\mu}M_{-[\mu]}T_{-1}M_{-[\mu]+1}\]
\[= M_{-\mu-1}T_{-1}W_{-\mu}T_{-1}M_{-\mu+1}.\] \(\Box\)

From Theorem 3.11 it follows that the operators \(I_\mu, \mu \in \mathbb{R}\), map \(S_-(R^+)\) bijectively onto itself. Furthermore, by means of Theorem 3.4 we can prove that the operators \(I_\mu, \mu \in \mathbb{R}\), constitute a group on \(S_-(R^+)\). The following theorem contains the analogues of Theorems 3.3, 3.4 and 3.5.

**Theorem 3.12** For \(\mu, \nu \in \mathbb{R}\),
(i) \(I_{\mu}I_{\nu} = I_{\mu+\nu}\),
(ii) \(I_{\mu}M_{-\nu}I_{\nu} = M_{-\nu}I_{\mu+\nu}M_{-\mu}\),
(iii) \(I_{\mu}M_1 = M_1I_{\mu} - \mu I_{\mu+1}\).
Proof. (i) By means of Theorems 3.11 and 3.4, and (3.1), we derive
\[ I_{\nu} I_{\nu} = (M_{\nu-1} T_{-1} W_{\nu} T_{-1} M_{\nu+1}) (M_{\nu-1} T_{-1} W_{\nu} T_{-1} M_{\nu+1}) \]
\[ = M_{\nu-1} T_{-1} W_{\nu} M_{\nu-2} W_{\nu} T_{-1} M_{\nu+1} \]
\[ = M_{\nu-1} T_{-1} (M_{\nu+1} W_{\nu+1} M_{\nu}) T_{-1} M_{\nu+1} \]
\[ = M_{\nu+1} T_{-1} W_{\nu+1} T_{-1} M_{\nu+1} = I_{\nu+1} . \]

(ii) By means of Theorems 3.11 and 3.3 we derive
\[ I_{\nu} M_{\nu-\mu} I_{\nu} = (M_{\nu-1} T_{-1} W_{\nu} T_{-1} M_{\mu+1}) M_{\nu-\mu-1} (M_{\nu-1} T_{-1} W_{\nu} T_{-1} M_{\mu+1}) \]
\[ = M_{\nu-1} T_{-1} W_{\nu+1} T_{-1} M_{\mu+1} = M_{\nu-1} I_{\nu+1} . \]

(iii) By means of Theorems 3.11 and 3.5, and (3.1), we derive
\[ I_{\nu} M_{1} - M_{1} I_{\nu} = M_{\nu-1} T_{-1} W_{\nu} T_{-1} M_{\mu+2} - M_{\nu} T_{-1} W_{\nu} T_{-1} M_{\mu+1} \]
\[ = M_{\nu} T_{-1} (M_{\nu} W_{\mu} - W_{\nu} M_{\nu}) T_{-1} M_{\mu+2} \]
\[ = M_{\nu} T_{-1} (\mu W_{\mu+1}) T_{-1} M_{\mu+2} = -\mu I_{\nu+1} . \]

Next we introduce the Erdélyi-Kober operator \( I_{\mu} \), \( \mu \in \mathbb{R} \), on \( S_{-}(R^{+}) \), defined by
\[ I_{\mu} = 2^{-\mu} T_{\mu/2} I_{\mu} , \quad \mu \in \mathbb{R} . \] (3.14)

For \( \mu > 0 \), this definition can be elaborated by means of (3.12), viz.
\[ (I_{\mu} f) (x) = \frac{\Gamma(\mu)}{\Gamma(\mu-1)} \int_{0}^{x} f(t) (x^{2} - t^{2})^{\mu-1} t \, dt , \quad f \in S_{-}(R^{+}) , \quad x > 0 , \quad \mu > 0 . \] (3.15)

The extension to non-positive values of \( \mu \) readily follows by means of (3.13), (3.2) and (3.1):
\[ I_{0} = I , \quad I_{-\mu} = I_{[-\mu]} (2 T_{\mu} D T_{-\mu/2})^{[\mu]} = I_{[-\mu]} (M_{-\mu} D)^{[\mu]} , \quad \mu > 0 . \] (3.16)

Observe that \( I_{-n} = (M_{-1} D)^{n} \) for \( n \in \mathbb{N} \). For this reason, the operator \( I_{-\mu} \), \( \mu > 0 \), is called a fractional differential operator of order \( \mu \).

By translation of Theorem 3.11 we obtain the following relation between the Erdélyi-Kober operators \( I_{\mu} \) and \( W_{\mu} \):

Theorem 3.13 For \( \mu \in \mathbb{R} \),
\[ I_{\mu} = M_{2\mu-1} T_{-1} W_{\mu} T_{-1} M_{2\mu+2} . \]

Also the results in Theorem 3.12 for the operators \( I_{\mu} \), \( \mu \in \mathbb{R} \), can be translated into results for the operators \( I_{\mu} \), \( \mu \in \mathbb{R} \).
Theorem 3.14 For $\mu, \nu \in \mathbb{R}$,

(i) $I_\mu I_\nu = I_{\mu+\nu}$,

(ii) $I_\mu M_{-2\mu-2\nu} I_\nu = M_{-2\nu} I_{\mu+\nu} M_{-2\mu}$,

(iii) $I_\mu M_2 = M_2 I_{\mu} - 2\mu I_{\mu+1}$.

The next theorem provides the key formula in our derivation of inversion formulas for Riemann-Liouville-Gegenbauer transformations in Section 3.6; cf. Theorem 3.7.

Theorem 3.15 For $\mu \in \mathbb{R}$,

$$I_{\mu+1} = I_{\mu+1} M_{-2\mu-1} I_{\mu}.$$

Proof. By means of Theorems 3.11, 3.7 and 3.13, and (3.1), we derive

$$I_{\mu+1} = M_{2\mu} T_{-1} W_{2\mu+1} T_{-1} M_{2\mu+2}$$

$$= M_{2\mu} T_{-1} (W_{2\mu+1} M_{-2\mu-1} W_{2\mu}) T_{-1} M_{2\mu+2}$$

$$= (M_{2\mu} T_{-1} W_{2\mu+1} T_{-1} M_{2\mu+4}) (M_{-2\mu-1} W_{2\mu} T_{-1} M_{2\mu+2})$$

$$= I_{\mu+1} M_{-2\mu-1} I_{\mu}.$$ \(\square\)

Finally, we introduce a subspace of $S_{\alpha}(\mathbb{R}^+)$. 

Definition 3.16 Let $\alpha \in \mathbb{R}$. The space $S_{\alpha}(\mathbb{R}^+)$ consists of all functions $f \in C^\infty(\mathbb{R}^+)$ for which

$$f(x) = 0, \quad 0 < x < \alpha.$$

Obviously $S_{\alpha}(\mathbb{R}^+), \alpha \in \mathbb{R}^+, \text{ is a proper subspace of } S_{\alpha}(\mathbb{R}^+)$. Observe that

$$S_{\alpha}(\mathbb{R}^+) = T_{-1}(S_{-\alpha/n}(\mathbb{R}^+)),$$  \(\alpha \in \mathbb{R}^+.\) \(\text{(3.17)}\)

Further, we define

$$S_{\alpha}(\mathbb{R}^+) = S_{\alpha}(\mathbb{R}^+).$$ \(\text{(3.18)}\)

The following statements can be verified easily.

Theorem 3.17 Let $a, p \in \mathbb{R}$ and let $\alpha, \mu \in \mathbb{R}$. Then

(i) $M_0, I_\mu$ and $I_\mu$ map $S_{\alpha}(\mathbb{R}^+)$ bijectively onto $S_{\alpha}(\mathbb{R}^+)$.\(\text{; (ii) } T_p \text{ maps } S_{\alpha}(\mathbb{R}^+) \text{ bijectively onto } S_{\alpha/n}(\mathbb{R}^+).\)
From (3.12) and (3.15) we infer that for $\mu > 0$, the operators $I_\mu$ and $\mathbb{I}_\mu$ on $S_{a_\nu}(R^+)$ are given by

$$
(I_\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x f(t) (x - t)^{\mu-1} \, dt, \quad f \in S_{a_\nu}(R^+), \quad 0 < a < x, \quad \mu > 0, \quad (3.19)
$$

$$
(\mathbb{I}_\mu f)(x) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_a^x f(t) (x^2 - t^2)^{\mu-1} \, dt, \quad f \in S_{a_\nu}(R^+), \quad 0 < a < x, \quad \mu > 0.
$$

$$
(3.20)
$$

### 3.3 Rodrigues-type formulas for Gegenbauer functions

In Sections 3.4 and 3.5 we study four types of operator products, namely

$$
W_\mu V_\lambda, \quad W_\nu W_\mu, \quad I_\mu I_\lambda, \quad \mathbb{I}_\mu I_\lambda.
$$

These operator products can be reduced to integral operators with Gegenbauer functions in their kernels. To prove this, we establish a formula for Gegenbauer functions which is an extension of the Rodrigues’ formula for the Gegenbauer polynomials.

We start with the introduction of some special functions.

For $\mu, \nu \in R$ the Legendre function $P_\nu^\mu$ is defined by means of the hypergeometric function $\mathbf{2} F_1$, cf. [18, 3.2(16)],

$$
P_\nu^\mu(z) = \frac{2^{-\nu}}{\Gamma(1 - \mu)} \frac{(z + 1)^{\nu + \mu/2}}{(z - 1)^{\mu/2}} \mathbf{2} F_1\left(-\nu, -\nu - \mu; 1 - \mu; \frac{z - 1}{z + 1}\right), \quad z \in C.
$$

The function $P_\nu^\mu(z)$ is known to be analytic in the complex $z$-plane cut along the real axis from $-\infty$ to 1.

For $z = x, \ x > 1$, the above definition passes into

$$
P_\nu^\mu(x) = \frac{2^{-\nu}}{\Gamma(1 - \mu)} \frac{(x + 1)^{\nu + \mu/2}}{(x - 1)^{\mu/2}} \mathbf{2} F_1\left(-\nu, -\nu - \mu; 1 - \mu; \frac{x - 1}{x + 1}\right), \quad x > 1.
$$

(3.21)

The Legendre function on the cut $-1 < x < 1$ is denoted by $P_\nu^\mu(x)$. From the definition in [18, 3.4 (1)] we have for $\mu, \nu \in R$,

$$
P_\nu^\mu(x) = \frac{1}{2} \left[e^{i\pi\mu/2} P_\nu^\mu(x + i0) + e^{-i\pi\mu/2} P_\nu^\mu(x - i0)\right]
$$

$$
= \frac{2^{-\nu}}{\Gamma(1 - \mu)} \frac{(1 + x)^{\nu + \mu/2}}{(1 - x)^{\mu/2}} \mathbf{2} F_1\left(-\nu, -\nu - \mu; 1 - \mu; \frac{x - 1}{x + 1}\right), \quad -1 < x < 1.
$$

(3.22)

The Legendre functions satisfy the relations (cf. [18, 3.3.1 (1), 3.4 (7)])

$$
P_\nu^\mu = P_{\nu-1}^\mu, \quad P_\nu^\mu = P_{-\nu}^\mu.
$$

(3.23)

as can be verified by means of [18, 2.1.4 (23)], viz.

$$
\mathbf{2} F_1(a, b; c; x) = (1 - x)^{-a + b} \mathbf{2} F_1(c - a, c - b; c; x), \quad |z| < 1.
$$

For $\mu, \nu \in R$ the Gegenbauer function $C_\nu^\mu$ is defined in terms of the Legendre function by
\[ C_\nu(x) = (x^2 - 1)^{1/4 - \mu/2} \, \mathcal{P}^{1/2-\mu}_{\nu+1/2}(x), \quad x \in \mathbb{C}. \] (3.24)

This definition has been adopted from [18, 3.15.1 (3)] with all multiplicative constants omitted. Expressed in terms of the hypergeometric function \( _2F_1 \), the Gegenbauer function is given by

\[ C_\nu(x) = \frac{2^{-\nu-1/2}}{\Gamma(\mu + \frac{1}{2})} \, (z + 1)^{\nu} \, _2F_1 \left( -\nu - \mu + \frac{1}{2}, -\nu; \mu + \frac{1}{2}; \frac{z-1}{x+1} \right), \quad z \in \mathbb{C}, \]

which shows that \( C_\nu(x) \) is analytic in the complex \( x \)-plane cut along the real axis from \(-\infty\) to \(-1\). For later convenience we introduce two notations for the Gegenbauer function of real argument: \( C^\circ_\nu \) and \( G^\circ_\nu \) denote the restrictions of \( C_\nu \) to \((1, \infty)\) and \((0, 1)\), respectively, so that

\[ C^\circ_\nu(x) = C^\circ_\nu(x) = (x^2 - 1)^{1/4 - \mu/2} \, \mathcal{P}^{1/2-\mu}_{\nu+1/2}(x), \quad x > 1, \] (3.25)

\[ G^\circ_\nu(x) = G^\circ_\nu(x) = (1 - x^2)^{1/4 - \mu/2} \, \mathcal{P}^{1/2-\mu}_{\nu+1/2}(x), \quad 0 < x < 1. \] (3.26)

Hence, by (3.23), the Gegenbauer functions satisfy the relations

\[ C^\circ_\nu = C^\circ_{\nu-2n}, \quad G^\circ_\nu = G^\circ_{\nu-2n}. \] (3.27)

For \( \mu > -\frac{3}{2} \) with \( \mu \neq 0 \) and \( n \in \mathbb{N}_0 \), we define the Gegenbauer polynomial \( C^\circ_\mu \) by the Rodrigues' formula, cf. [18, 3.15.1 (10)],

\[ C^\circ_\mu(x) = \frac{(-2)^n \, \Gamma(\mu+n) \, \Gamma(2\mu+n)}{n! \Gamma(\mu) \, \Gamma(2\mu+2n)} \, (1-x^2)^{1/2-\mu} \, (d/dx)^n \left[ (1-x^2)^{\mu+1/2} \right]. \] (3.28)

Chebyshev polynomials and Legendre polynomials are special cases of Gegenbauer polynomials. For \( \mu \to 0 \) we obtain the Chebyshev polynomial of the first kind, \( T_n \), given by

\[ C^\circ_0(x) = \lim_{\mu \to 0} 1 \, C^\circ_\mu(x) = \frac{2}{n} \, T_n(x) = \frac{2}{n} \cos(n \arccos x), \quad n \in \mathbb{N}. \] (3.29)

For \( \mu = 1 \) we obtain the Chebyshev polynomial of the second kind, \( U_n \), given by

\[ C^\circ_1(x) = U_n(x) = \sin((n+1) \arccos x) / \sin(\arccos x), \quad n \in \mathbb{N}_0. \] (3.30)

For \( \mu = \frac{3}{2} \) we obtain the Legendre polynomial \( P_n \) defined by

\[ C^\circ_{3/2}(x) = P_n(x) = (2^n \, n!)^{-1} \, (d/dx)^n \, (x^2 - 1)^n, \quad n \in \mathbb{N}_0. \] (3.31)

By [18, 3.15.1 (4)] we have the following relations between Gegenbauer functions and Gegenbauer polynomials: For \( \mu > -\frac{3}{2} \) with \( \mu \neq 0 \) and \( n \in \mathbb{N}_0 \),

\[ C^\circ_\nu(x) = 2^{1/2-\mu} \, \frac{\Gamma(2\mu) \, n!}{\Gamma(n+2\mu) \, \Gamma(\mu+1/2)} \, C^\circ_n(x), \] (3.32)

\[ C^\circ_\nu(x) = \sqrt{2 \pi} \, T_n(x), \] (3.33)

\[ C^\circ_\nu(x) = \frac{\sqrt{2 \pi}}{n+1} \, U_n(x), \] (3.34)

\[ C^\circ_{3/2}(x) = P_n(x). \] (3.35)
Furthermore, we observe that for $\mu > -\frac{1}{2}$,

$$ C_{\nu}^{\mu}(z) = 2^{1-\nu}/\Gamma(\mu + 1/2). \tag{3.36} $$

The following integral relations for Gegenbauer functions generalize the Rodrigues’ formula for Gegenbauer polynomials.

**Theorem 3.18**

(i) For $\nu \in \mathbb{R}$, $\mu > 0$, $\lambda > -\frac{1}{2}$ and $x > 1$, we have

$$ \frac{1}{\Gamma(\mu)} \int_0^1 \left( t^2 - 1 \right)^{1/2 - 1/2} C_{\nu}^{\mu}(t) (x - t)^{\nu - 1} dt = (x^2 - 1)^{1/2} C_{\nu}^{\mu}(x). $$

(ii) For $\nu \in \mathbb{R}$, $\mu > 0$, $\lambda > -\frac{1}{2}$ and $0 < x < 1$, we have

$$ \frac{1}{\Gamma(\mu)} \int_0^1 \left( t^2 - 1 \right)^{1/2 - 1/2} C_{\nu}^{\mu}(t) (t - x)^{\nu - 1} dt = (1 - x^2)^{1/2} C_{\nu}^{\mu}(x). $$

**Proof.** (i) By substitution of $t = x - (x - 1)z$ and by using the expansions

$$ 2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad (1 - z)^{\mu} = \sum_{k=0}^{\infty} \frac{(-d)_k}{k!} z^k, $$

derive

$$ \frac{(x - 1)^{-\lambda - \mu + 1/2}}{(x + 1)^{\lambda + \mu - 1/2}} \int_0^1 \left( t^2 - 1 \right)^{1/2 - 1/2} C_{\nu}^{\mu}(t) (x - t)^{\nu - 1} dt $$

$$ = \frac{2^{-\nu + 1/2}}{\Gamma(\lambda + 1/2)} \int_0^1 s^{\nu - 1/2} (1 - s)^{1/2 - 1/2} \left( 1 - \frac{x - 1}{x + 1} \right)^{\lambda + \mu - 1/2} $$

$$ + \frac{2^{-\nu + 1/2}}{\Gamma(\lambda + 1/2)} \sum_{k=0}^{\infty} \frac{(-\lambda - \nu/2)_k (-\nu)_k (-\lambda - \nu + k + 1/2)}{(\lambda + 1/2)_k k! k!} \left( 1 - \frac{x - 1}{x + 1} \right)^{k+1} $$

$$ = \frac{2^{-\nu + 1/2} \Gamma(\mu)}{\Gamma(\lambda + \mu + 1/2)} \sum_{k=0}^{\infty} \frac{(-\lambda - \nu + k + 1/2)_k (-\nu)_k (\mu)_k}{(\lambda + \mu + 1/2)_k k! k!} \left( 1 - \frac{x - 1}{x + 1} \right)^{k+1}. $$

The latter double series can be expressed in terms of the hypergeometric function $2F_1$. Indeed, by means of the summation formula in Hansen [25, (7.4.15)], viz.

$$ \sum_{k=0}^{\infty} \frac{(-n)_k (b)_k}{(c)_k k!} \left( \frac{c-b}{c} \right)_k \cdot \frac{1}{(c)_k}, \quad$$
we obtain
\[
\sum_{k,l=0}^{\infty} \frac{(-\lambda - \nu + \frac{1}{2})_{k+l} (-\nu)_{k} (\mu)_{l}}{(\lambda + \mu + \frac{1}{2})_{k+l} k! l!} \left( \frac{x-1}{x+1} \right)^{k+l} = \sum_{n=0}^{\infty} \frac{(-\lambda - \nu + \frac{1}{2})_{n}}{(\lambda + \mu + \frac{1}{2})_{n} n!} \sum_{k=0}^{n} \frac{(-\nu)_{k} (\mu)_{n-k}}{(n-k)! k!} \left( \frac{x-1}{x+1} \right)^{n} \sum_{l=0}^{n} \frac{(-n)_{l} (-\nu)_{k}}{(1-\mu - n)_{l} l!} \left( \frac{x-1}{x+1} \right)^{l} = \binom{\nu}{\lambda} F_{1} \left( -\nu + \frac{1}{2}, \mu - \nu; \lambda + \mu + \frac{1}{2}; \frac{x-1}{x+1} \right).
\]

Combination of the previous results yields the wanted integral formula.

(ii) The proof of the second integral formula runs along the same lines.

To see that the integral relations in Theorem 3.18 generalize the Rodrigues’ formula for Gegenbauer polynomials, take \( \mu = \nu = n \in \mathbb{N} \) in Theorem 3.18 and differentiate \( n \) times. Then the integral relations pass into
\[
C_{\lambda}^{n}(x) = \frac{2^{-\lambda-n+1/2}}{\Gamma(\lambda + n + \frac{1}{2})} (x^{2} - 1)^{1/2-\lambda} \left( \frac{d}{dx} \right)^{n} \left[ (x^{2} - 1)^{1/2-\lambda} \right], \quad x > 1,
\]
\[
G_{\lambda}^{n}(x) = \frac{2^{-\lambda-n+1/2}}{\Gamma(\lambda + n + \frac{1}{2})} (1 - x^{2})^{1/2-\lambda} \left( -\frac{d}{dx} \right)^{n} \left[ (1 - x^{2})^{1/2-\lambda} \right], \quad 0 < x < 1,
\]
which are equivalent to (3.32) and (3.28).

Further Rodrigues-type formulas for Gegenbauer functions are presented in the following theorem.

**Theorem 3.19**

(i) For \( \nu \in \mathbb{R}, n \in \mathbb{N}_{0}, \lambda > n - \frac{1}{2} \) and \( x > 1 \), we have
\[
C_{\lambda+n}^{n}(x) = (x^{2} - 1)^{n-\lambda+1/2} \left( \frac{d}{dx} \right)^{n} \left[ (x^{2} - 1)^{1/2-\lambda} C_{\lambda}^{n}(x) \right].
\]

(ii) For \( \nu \in \mathbb{R}, n \in \mathbb{N}_{0}, \lambda > n - \frac{1}{2} \) and \( 0 < x < 1 \), we have
\[
G_{\lambda+n}^{n}(x) = (1 - x^{2})^{n-\lambda+1/2} \left( -\frac{d}{dx} \right)^{n} \left[ (1 - x^{2})^{1/2-\lambda} G_{\lambda}^{n}(x) \right].
\]

**Proof.** For \( n = 0 \), the formulas are obvious. For \( n \in \mathbb{N} \), the stated formulas follow by setting \( \mu = n, \lambda := \lambda - n, \nu := \nu + n \) in Theorem 3.18, and by differentiating the integral relations thus obtained, \( n \) times.

With the Rodrigues-type formulas for Gegenbauer functions at hand we are fully prepared to examine the four operator products mentioned in the beginning of this section.
3.4 The Weyl-Gegenbauer transformations

In this section we consider the Weyl operator products $W_\mu W_\lambda$ and $BW_\lambda W_\mu$. We define the Weyl-Gegenbauer transformations in terms of these operator products.

First we consider the product $W_\mu W_\lambda$.

As a preliminary we present the following relation.

**Lemma 3.20** For $\lambda, \mu \in \mathbb{R}$,

$$W_\mu W_\lambda = W_{\mu+\lambda-1} B W_{-\lambda+1} M_{2\lambda-1} .$$

**Proof.** By means of Theorem 3.7 with $\mu = -\lambda$ we derive

$$W_\mu W_\lambda = W_{\mu+\lambda-1} W_{-2\lambda+1} B W_\lambda = W_{\mu+\lambda-1} (B W_{-\lambda+1} M_{2\lambda-1} B W_{-\lambda}) B W_\lambda$$

$$= W_{\mu+\lambda-1} B W_{-\lambda+1} M_{2\lambda-1} .$$

$\square$

**Theorem 3.21** Let $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu > 0$. For $f \in S_\infty(\mathbb{R}^+)$ we have

$$(W_\mu W_\lambda f)(x) = \frac{2^{1-\lambda}}{\Gamma(\lambda) \Gamma(\mu)} \int_x^\infty \left( \int_x^t (t^2 - s^2)^{1-\lambda-1/2} \frac{s}{s-x} ds \right) (t-x)^{\mu-1} dt .$$

**Proof.** First suppose $\lambda > 0$ and $\mu > 0$. We start from the definitions (3.3) and (3.7) of $W_\mu$ and $W_\lambda$. By means of (3.36) with $\mu = \lambda - 1/2$ and Theorem 3.18 (ii) with $\nu = 0$ and $\lambda$ replaced by $\lambda - 1/2$, we derive for $f \in S_\infty(\mathbb{R}^+)$ and $x > 0$,

$$(W_\mu W_\lambda f)(x) = \frac{2^{1-\lambda}}{\Gamma(\lambda) \Gamma(\mu)} \int_x^\infty \left( \int_x^t (t^2 - r^2)^{1-\lambda-1/2} (r-x)^{\mu-1} dr \right) f(t) t^{1-\mu} dt$$

$$= \frac{2^{1-\lambda}}{\Gamma(\lambda) \Gamma(\mu)} \int_x^\infty \left( \int_x^t (t^2 - r^2)^{1-\lambda-1/2} (r-x)^{\mu-1} dr \right) f(t) t dt$$

$$= \frac{1}{\Gamma(\mu)} \int_x^\infty \left( 1 - s^2 \right)^{1-\lambda-1/2} \frac{s}{s-x} (s-x)^{\mu-1} ds \int_x^t (t^2 - s^2)^{1-\lambda} G_{\mu-1/2}(s) f(t) t^{2\lambda+\mu-1} dt$$

$$= \int_x^\infty (t^2 - x^2)^{1-\mu-1/2} G_{\mu-1/2}(s) f(t) t^{1-\mu} dt .$$

Next suppose $\lambda > 0$, $\mu \leq 0$ with $\lambda + \mu > 0$. Choose $n \in \mathbb{N}$ such that $\mu + n > 0$. Then by using the previous result and Theorem 3.19 (ii) with $\nu = -\mu - n$ and $\lambda$ replaced by $\lambda + \mu + n - 1/2$, we derive for $f \in S_\infty(\mathbb{R}^+)$ and $x > 0$,

$$(W_\mu W_\lambda f)(x) = (W_{\mu+n} W_\lambda W_\mu) f(x)$$

$$= (-d/dx)^n \int_x^\infty (t^2 - x^2)^{1+\mu+n-1/2} G_{-\mu-n}(s) f(t) t^{1-\mu-n} dt$$
Thus we have proved the wanted integral relation for $\lambda, \mu \in \mathbb{R}$, subject to $\lambda > 0$ and $\lambda + \mu > 0$. Finally, suppose $\lambda \leq 0$, $\mu > 0$ with $\lambda + \mu > 0$. Then by using Lemma 3.20, the previous result and (3.27) with $\nu = -\mu$ and $\mu$ replaced by $\lambda + \mu - 1/2$, we derive for $f \in S_{-}(\mathbb{R}^{+})$ and $x > 0$,

\[(W_{\nu,\lambda}W_{\mu}f)(x) = (W_{2\lambda+\mu-1,\lambda+1}W_{\lambda+1}M_{2\lambda-1}f)(x)\]

\[= \int_{x}^{\infty} (t^{2} - z^{2})^{\lambda+\mu-1/2} G_{-2\lambda+\mu+1}^{\lambda+1}(x/t) f(t) t^{1-\mu} dt\]

\[= \int_{x}^{\infty} (t^{2} - z^{2})^{\lambda+\mu-1/2} G_{-\lambda}^{\lambda+1}(x/t) f(t) t^{1-\mu} dt . \]

We now come to the definition of the Weyl-Gegenbauer transformation of the first kind, denoted by $W_{G,\nu,\lambda}$, namely

\[W_{G,\nu,\lambda} = W_{\nu}W_{-\lambda+1/2}M_{\mu} , \quad \nu, \lambda \in \mathbb{R} . \quad (3.37)\]

Observe that $W_{G,\nu,\lambda}$ maps $S_{-}(\mathbb{R}^{+})$ bijectively onto itself. By means of Theorem 3.21 with $\mu = -\nu$ and $\lambda$ replaced by $\nu + \lambda + 1/2$, we obtain the following integral representation with a Gegenbauer function in its kernel.

**Theorem 3.22** Let $\nu, \lambda \in \mathbb{R}$ with $\lambda > -1/2$. For $f \in S_{-}(\mathbb{R}^{+})$ we have

\[(W_{G,\nu,\lambda}f)(x) = \int_{x}^{\infty} (t^{2} - z^{2})^{\lambda-1/2} G_{\lambda+1}^{\nu}(x/t) f(t) t dt , \quad x > 0 .\]

Next we consider the product $BV_{\nu}W_{\mu}$.

As a preliminary we present the following relation; cf. Lemma 3.20.

**Lemma 3.23** For $\lambda, \mu \in \mathbb{R}$,

\[BV_{\nu}W_{\mu} = M_{2\lambda+1}BV_{-\lambda-1}W_{\mu+2\lambda+1} .\]

**Proof.** By means of Lemma 3.20 with $\mu$ replaced by $-\mu$ and $\lambda$ by $-\lambda$, we derive

\[BV_{\nu}W_{\mu} = (W_{-\nu}W_{-\lambda})^{-1} = (W_{-\nu-2\lambda-1}W_{\lambda+1}M_{-2\lambda-1})^{-1}\]

\[= M_{2\lambda+1}BV_{-\lambda-1}W_{\mu+2\lambda+1} . \]

\[\square\]
Theorem 3.24 Let $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu > 0$. Then for $f \in S_\infty(\mathbb{R}^+)$ and $x > 0$ we have

\[
(W_\lambda W_\mu f)(x) = x^{1-\mu} \int_0^\infty (t^2 - x^2)^{1+\lambda-1} C_{1-\mu-1/2}(t/x) f(t) \, dt.
\]

Proof. First suppose $\lambda > -1$ and $\mu > 1$. Because of $W_{-1} = -M_{-1} D = M_{-1} W_{-1}$, we may write

\[W_\lambda W_\mu = W_{\lambda + 1} M_{-1} W_{\mu - 1},\]

whereupon we employ the definitions (3.7) and (3.3) of $W_{\lambda + 1}$ and $W_{\mu - 1}$. By means of (3.36) with $\mu = \lambda + 1/2$ and Theorem 3.18 (i) with $\nu = 0$, $\lambda$ replaced by $\lambda + 1/2$ and $\mu$ by $\mu - 1$, we derive for $f \in S_\infty(\mathbb{R}^+)$ and $x > 0$,

\[
(W_\lambda W_\mu f)(x) = (W_{\lambda + 1} M_{-1} W_{\mu - 1} f)(x)
= \frac{2^{-\lambda}}{\Gamma(\mu + 1) \Gamma(\mu - 1)} \int_0^\infty \int_0^r f(t) (t-r)^{\mu-2} \, dt \, (r^2 - x^2)^{1/2} \, dr
= \frac{2^{-\lambda}}{\Gamma(\mu + 1) \Gamma(\mu - 1)} \int_0^\infty \int_0^{r^2} (r^2 - z^2)^{1/2} (t-r)^{\mu-2} \, dr \, f(t) \, dt
= \frac{x^{2+\mu-1}}{\Gamma(\mu - 1)} \int_0^\infty \int_0^{x^2} (z^2 - s^2)^{1/2} C_{1-\mu-1/2}(s (t/x - s)^{\mu-2}) \, ds \, f(t) \, dt
= x^{1-\mu} \int_0^\infty (t^2 - x^2)^{1+\mu-1} C_{1-\mu-1/2}(t/x) f(t) \, dt.
\]

Next suppose $\lambda > -1$, $\mu \leq 1$ with $\lambda + \mu > 0$. Choose $n \in \mathbb{N}$ such that $\mu + n > 1$. Then by using the previous result, through integration by parts, and by means of Theorem 3.19 (i) with $\nu = 1 - \mu - n$ and $\lambda$ replaced by $\lambda + n + 1/2$, we derive for $f \in S_\infty(\mathbb{R}^+)$ and $x > 0$,

\[
(W_\lambda W_\mu f)(x) = (W_{\lambda + n} W_{\mu - n} f)(x)
= (-1)^n x^{1-\mu-n} \int_0^\infty (t^2 - x^2)^{1+\mu+n-1} C_{1-\mu-1/2}(t/x) f^{(n)}(t) \, dt
= x^{2+\mu-1} \int_0^\infty (d/d(t/x))^n (t^2/x^2 - 1)^{1+\mu+n-1} C_{1-\mu-1/2}(t/x) f(t) \, dt
= x^{1-\mu} \int_0^\infty (t^2 - x^2)^{1+\mu-1} C_{1-\mu-1/2}(t/x) f(t) \, dt.
\]
Thus we have proved the wanted integral relation for \(\lambda, \mu \in \mathbb{R}\), subject to \(\lambda > -1\) and \(\lambda + \mu > 0\).

Finally, suppose \(\lambda \leq -1, \mu > 1\) with \(\lambda + \mu > 0\). Then by using Lemma 3.23, the previous result and (3.27) with \(\nu = 1 - \mu\) and \(\mu\) replaced by \(\lambda + \mu - 1/2\), we derive for \(f \in S_{-}(\mathbb{R}^{+})\) and \(x > 0\),

\[
(W_{\lambda}W_{\mu}f)(x) = (M_{2\lambda+1}W_{-\lambda-1}W_{\mu+\lambda+1}f)(x)
\]

\[
= x^{1-\mu} \int_{x}^{\infty} (t^{2} - x^{2})^{\lambda+\mu-1} C_{-\lambda-\mu}^{\lambda+\mu-1/2}(t/x) f(t) \, dt
\]

\[
= x^{1-\mu} \int_{x}^{\infty} (t^{2} - x^{2})^{\lambda+\mu-1} C_{-\lambda-\mu}^{\lambda+\mu-1/2}(t/x) f(t) \, dt .
\]

We now come to the definition of the Weyl-Gegenbauer transformation of the second kind, denoted by \(WC_{\nu,\lambda}\), namely

\[
WC_{\nu,\lambda} = M_{1-\nu} W_{\lambda-1/2} W_{1-\nu} , \quad \nu, \lambda \in \mathbb{R} .
\]

Observe that \(WC_{\nu,\lambda}\) maps \(S_{-}(\mathbb{R}^{+})\) bijectively onto itself. By means of Theorem 3.24 with \(\mu = 1 - \nu\) and \(\lambda\) replaced by \(\nu + \lambda - 1/2\), we obtain the following integral representation with a Gegenbauer function in its kernel.

**Theorem 3.25** Let \(\nu, \lambda \in \mathbb{R}\) with \(\lambda > -1/2\). For \(f \in S_{-}(\mathbb{R}^{+})\) we have

\[
(WC_{\nu,\lambda}f)(x) = x \int_{x}^{\infty} (t^{2} - x^{2})^{\lambda-1/2} C_{-\nu}^{\lambda}(t/x) f(t) \, dt , \quad x > 0 .
\]

**Remark.** The operators \(M_{\nu}, W_{\nu}\) and \(W_{\lambda}\) map \(S_{-\nu}(\mathbb{R}^{+})\) bijectively onto itself, see Theorem 3.9 (i). So both \(WG_{\nu,\lambda}\) and \(WC_{\nu,\lambda}\) map \(S_{-\nu}(\mathbb{R}^{+})\) bijectively onto itself as well. Furthermore, in the integral representations for \((WG_{\nu,\lambda}f)(x)\) and \((WC_{\nu,\lambda}f)(x)\) of Theorems 3.22 and 3.25, valid for \(\lambda > -\frac{1}{2}\), we may replace the upper limit \(\infty\) by \(b\) whenever \(f \in S_{-\nu}(\mathbb{R}^{+})\) and \(0 < x < b\).

### 3.5 The Riemann-Liouville-Gegenbauer transformations

In this section we consider the Riemann-Liouville operator products \(I_{\mu}D_{\lambda}\) and \(D_{\lambda}I_{\mu}\). We define the Riemann-Liouville-Gegenbauer transformations in terms of these operator products. The derivations are omitted because they are fully analogous to the derivations in Section 3.4.

First we consider the product \(I_{\mu}D_{\lambda}\).

As a preliminary we present the following relation; cf. Lemma 3.20.

**Lemma 3.26** For \(\lambda, \mu \in \mathbb{R}\),

\[
I_{\mu}D_{\lambda} = I_{\mu+2\lambda-1}D_{-\lambda+1}M_{2\lambda-1} .
\]
Theorem 3.27 Let $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu > 0$. For $f \in S_-(\mathbb{R}^+)$ we have

$$(I_{\mu, \lambda} f)(x) = \int_0^x (x^2 - t^2)^{\lambda+\mu-1} C_{\mu}^{\lambda+\mu-1/2}(x/t) f(t) t^{1-\nu} dt, \quad x > 0.$$ 

The Riemann-Liouville-Gegenbauer transformation of the first kind, denoted by $IC_{\nu, \lambda}$, is defined by

$$IC_{\nu, \lambda} = I_{-\nu} \mathcal{H}_{\nu+\lambda+1/2} M_{-\nu}, \quad \nu, \lambda \in \mathbb{R}.$$ 

(3.39)

Observe that $IC_{\nu, \lambda}$ maps $S_-(\mathbb{R}^+)$ bijectively onto itself. By means of Theorem 3.27 with $\mu = -\nu$ and $\lambda$ replaced by $\nu + \lambda + 1/2$, we obtain the following integral representation with a Gegenbauer function in its kernel.

Theorem 3.28 Let $\nu, \lambda \in \mathbb{R}$ with $\lambda > -1/2$. For $f \in S_-(\mathbb{R}^+)$ we have

$$(IC_{\nu, \lambda} f)(x) = \int_0^x (x^2 - t^2)^{\lambda+\nu-1/2} C_{\nu}^\lambda(x/t) f(t) t dt, \quad x > 0.$$ 

Next we consider the product $\mathcal{H}_{\lambda} I_{\nu}$.

As a preliminary we present the following relation; cf. Lemma 3.23.

Lemma 3.29 For $\lambda, \mu \in \mathbb{R}$,

$$\mathcal{H}_{\lambda} I_{\mu} = M_{2\lambda+1} \mathcal{H}_{-\lambda-1} I_{\mu+2\lambda+1}.$$ 

Theorem 3.30 Let $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu > 0$. For $f \in S_-(\mathbb{R}^+)$ we have

$$(\mathcal{H}_{\lambda} I_{\mu} f)(x) = x^{1-\mu} \int_0^x (x^2 - t^2)^{\lambda+\nu-1} G_{1-\mu}^{\lambda+\nu-1/2}(t/x) f(t) dt, \quad x > 0.$$ 

The Riemann-Liouville-Gegenbauer transformation of the second kind, denoted by $IG_{\nu, \lambda}$, is defined by

$$IG_{\nu, \lambda} = M_{-\nu} \mathcal{H}_{\nu+\lambda+1/2} I_{-\nu}, \quad \nu, \lambda \in \mathbb{R}.$$ 

(3.40)

Observe that $IG_{\nu, \lambda}$ maps $S_-(\mathbb{R}^+)$ bijectively onto itself. By means of Theorem 3.30 with $\mu = 1 - \nu$ and $\lambda$ replaced by $\nu - \lambda - 1/2$, we obtain the following integral representation with a Gegenbauer function in its kernel.

Theorem 3.31 Let $\nu, \lambda \in \mathbb{R}$ with $\lambda > -1/2$. For $f \in S_-(\mathbb{R}^+)$ we have

$$(IG_{\nu, \lambda} f)(x) = x \int_0^x (x^2 - t^2)^{\lambda+\nu-1/2} G_{\nu}^\lambda(t/x) f(t) dt, \quad x > 0.$$
Remark. The operators \( M_\alpha, I_\alpha \) and \( \mathcal{I}_\alpha \) map \( S_{\omega}(\mathbb{R}^+ \rightleftharpoons \mathbb{R}^+) \) bijectively onto itself, see Theorem 3.17 (i). So both \( \mathcal{I} C_{\nu, \alpha} \) and \( \mathcal{I} G_{\nu, \alpha} \) map \( S_{\omega}(\mathbb{R}^+) \) bijectively onto itself as well. Furthermore, in the integral representations for \( (\mathcal{I} C_{\nu, \alpha} f)(x) \) and \( (\mathcal{I} G_{\nu, \alpha} f)(x) \) of Theorems 3.28 and 3.31, valid for \( \lambda > -\frac{1}{2} \), we may replace the lower limit 0 by \( a \) whenever \( f \in S_{\omega}(\mathbb{R}^+) \) and \( 0 < a < x \).

3.6 The inverse Gegenbauer transformations

The Weyl-Gegenbauer transformations and the Riemann-Liouville-Gegenbauer transformations introduced in the previous sections, are henceforth called Gegenbauer transformations for short. In this section we establish several relations between these Gegenbauer transformations. Of special interest are the formulas for the inverse Gegenbauer transformations. These formulas will be used in the next section to solve certain integral equations.

The following lemma is an easy consequence of the definitions (3.37)-(3.40) of the Gegenbauer transformations.

Lemma 3.32 For \( \nu, \lambda, \rho \in \mathbb{R} \),

(i) \( W_\rho W G_{\nu, \lambda} M_\rho = W G_{\nu - \rho, \lambda + \rho} \),

(ii) \( M_\rho W C_{\nu, \lambda} W_\rho = W C_{\nu - \rho, \lambda + \rho} \),

(iii) \( I_\rho I C_{\nu, \lambda} M_\rho = I C_{\nu - \rho, \lambda + \rho} \),

(iv) \( M_\rho I G_{\nu, \lambda} I_\rho = I G_{\nu - \rho, \lambda + \rho} \).

Corresponding to (3.27) viz. \( C_\lambda^{\nu} = C_{\nu - 2\lambda}, G_\lambda^{\nu} = G_{\nu - 2\lambda} \), we have the following counterparts for Gegenbauer transformations.

Theorem 3.33 For \( \nu, \lambda \in \mathbb{R} \),

(i) \( W G_{\nu, \lambda} = W G_{\nu - 2\lambda, \lambda} \),

(ii) \( W C_{\nu, \lambda} = W C_{\nu - 2\lambda, \lambda} \),

(iii) \( I C_{\nu, \lambda} = I C_{\nu - 2\lambda, \lambda} \),

(iv) \( I G_{\nu, \lambda} = I G_{\nu - 2\lambda, \lambda} \).

Proof. We only prove the first identity. The proofs of the other identities run along the same lines. By means of (3.37) and Lemma 3.26 we deduce

\[
W G_{\nu, \lambda} = W_{\nu - 2\lambda} W_{\nu - \lambda + 1/2 - \nu} M_{\nu} = W_{\nu + 2 \lambda} W_{\nu - \lambda + 1/2 - \nu} M_{\nu + 2 \lambda} = W G_{\nu - 2\lambda, \lambda}.
\]

The next theorem provides relations between the Weyl-Gegenbauer transformations and the Riemann-Liouville-Gegenbauer transformations.
Theorem 3.34 For \( \nu, \lambda \in \mathbb{R} \),

(i) \( \mathcal{W}_{\nu, \lambda} = M_{2\lambda} T_{-1} \mathcal{I}_{\nu, \lambda} T_{-1} M_{2\lambda+2} \),

(ii) \( \mathcal{W}_{\nu, \lambda} = M_{2\lambda} T_{-1} \mathcal{I}_{\nu, \lambda} T_{-1} M_{2\lambda+2} \).

Proof. (i) In the following derivation we use successively (3.37), Theorems 3.11, 3.13, 3.14 (ii), Lemma 3.26, Theorem 3.15, Lemma 3.29 and (3.40):

\[
\mathcal{W}_{\nu, \lambda} = W_{-\nu} B_{\nu+\lambda+1/2} M_{-\nu} \\
= M_{-\nu-1} T_{-1} I_{-\nu} (M_{-\nu-2} I_{\nu+\lambda+1/2} M_{-\nu-1}) T_{-1} M_{2\lambda+2} \\
= M_{-\nu-1} T_{-1} (I_{-\nu} I_{\nu+2\lambda+1} M_{-\nu-2\lambda-1} I_{\nu+2\lambda+1} T_{-1} M_{2\lambda+2} \\
= M_{-\nu-1} T_{-1} I_{-\nu-1/2} (I_{\nu+2\lambda+1} M_{-\nu-2\lambda-1} I_{\nu+2\lambda+1} T_{-1} M_{2\lambda+2} \\
= M_{-\nu-1} T_{-1} (I_{-\nu-1/2} I_{\nu+2\lambda+1} T_{-1} M_{2\lambda+2} \\
= M_{2\lambda} T_{-1} (M_{-\nu} I_{\nu+\lambda+1/2} T_{-1} M_{2\lambda+2} \\
= M_{2\lambda} T_{-1} \mathcal{I}_{\nu, \lambda} T_{-1} M_{2\lambda+2} .
\]

The proof of (ii) runs along the same lines.

The next part of this section is devoted to the inverse Gegenbauer transformations. It is a straightforward consequence of the definitions (3.37)–(3.40) of the Gegenbauer transformations that the inverse of a Gegenbauer transformation is again a Gegenbauer transformation.

Theorem 3.35 For \( \nu, \lambda \in \mathbb{R} \),

(i) \( \mathcal{W}_{\nu, \lambda}^{-1} = \mathcal{W}_{\nu+2\lambda+1, -1} \lambda \);

(ii) \( \mathcal{W}_{\nu, \lambda}^{-1} = \mathcal{W}_{\nu+2\lambda+1, -1} \lambda \);

(iii) \( \mathcal{I}_{\nu, \lambda}^{-1} = \mathcal{I}_{\nu+2\lambda+1, -1} \lambda \);

(iv) \( \mathcal{I}_{\nu, \lambda}^{-1} = \mathcal{I}_{\nu+2\lambda+1, -1} \lambda \).

Proof. We only prove the first identity. The proofs of the other identities run along the same lines. By means of the definitions (3.37), (3.38) of the Weyl-Gegenbauer transformations we have

\[
\mathcal{W}_{\nu, \lambda}^{-1} = M_{\nu} B_{\nu-\lambda-1/2} W_{\nu} = \mathcal{W}_{\nu+2\lambda+1, -1} \lambda .
\]
The second equality in (i) follows from Theorem 3.33 (ii).

In theorems 3.22, 3.25, 3.28, 3.31 we have seen that the Gegenbauer transformations are integral operators if \( \lambda > -\frac{1}{2} \). Then it follows from Theorem 3.35 that the inverse Gegenbauer transformations are integral operators if \( \lambda < -\frac{1}{2} \). Next we show that the inverse Gegenbauer transformation can be expressed as the product of (fractional) differential operators and an integral operator.

**Theorem 3.36** For \( \nu, \lambda, \rho \in \mathbb{R} \),

(i) \[ \mathcal{W}G_{\nu,\lambda}^{-1} = (M_{\rho} \mathcal{W}_{-\nu,\rho} M_{\nu}) \mathcal{W}C_{\nu+2\lambda-2\rho+1,\nu-1-\lambda+\rho}, \]

(ii) \[ \mathcal{W}C_{\nu,\lambda}^{-1} = \mathcal{W}G_{\nu+2\lambda-2\rho+1,\nu-1-\lambda+\rho}^{-1} (M_{\nu-\rho} \mathcal{W}_{-\nu,\rho} M_{\nu-1}) \]

(iii) \[ \mathcal{I}C_{\nu,\lambda}^{-1} = (M_{\rho} I_{-\nu,\rho} M_{\nu}) \mathcal{I}G_{\nu+2\lambda-2\rho+1,\nu-1-\lambda+\rho}, \]

(iv) \[ \mathcal{I}G_{\nu,\lambda}^{-1} = \mathcal{I}C_{\nu+2\lambda-2\rho+1,\nu-1-\lambda+\rho}^{-1} (M_{\nu-\rho} I_{-\nu,\rho} M_{\nu-1}) \]

**Proof.** We only prove the first identity. The proofs of the other identities run along the same lines. From the definition (3.37) we have

\[ \mathcal{W}G_{\nu,\lambda} = W_{-\nu} B_{\nu+\lambda-1/2,\nu} M_{-\nu} = (W_{-\nu} B_{\nu+\lambda-\rho+1/2,\nu}) M_{\nu} B_{\nu} M_{-\nu} \]

\[ = \mathcal{W}G_{\nu,\lambda-\rho} (M_{\rho} B_{\nu} M_{-\nu}), \quad \rho \in \mathbb{R} \]

Now the wanted result is obtained by inversion, using Theorem 3.35 (i). \( \square \)

In the next section we consider integral equations with Gegenbauer functions in their kernels. Various classical results for the solution of these integral equations will be explained in terms of the inversion formulas of Theorem 3.36. Alternative expressions for the solution can be obtained by means of the following inversion formulas for Gegenbauer transformations.

**Theorem 3.37** For \( \nu, \lambda \in \mathbb{R} \),

(i) \[ \mathcal{W}G_{\nu,\lambda}^{-1} = M_{-2\lambda-1} MC_{\nu,\lambda} W_{-2\nu-1} = W_{-2\nu-1} WC_{\nu,\lambda} M_{-2\lambda-1}, \]

(ii) \[ \mathcal{W}C_{\nu,\lambda}^{-1} = M_{-2\lambda-1} WB_{\nu,\lambda} W_{-2\nu-1} = W_{-2\nu-1} \mathcal{W}G_{\nu,\lambda} M_{-2\lambda-1}, \]

(iii) \[ \mathcal{I}C_{\nu,\lambda}^{-1} = M_{-2\lambda-1} IG_{\nu,\lambda} I_{-2\nu-1} = I_{-2\nu-1} \mathcal{I}G_{\nu,\lambda} M_{-2\lambda-1}, \]

(iv) \[ \mathcal{I}G_{\nu,\lambda}^{-1} = M_{-2\lambda-1} IC_{\nu,\lambda} I_{-2\nu-1} = I_{-2\nu-1} \mathcal{I}C_{\nu,\lambda} M_{-2\lambda-1}. \]

**Proof.** First we prove (i) and (iv) simultaneously.

By means of (3.37), (3.38) and Lemma 3.23 we deduce

\[ \mathcal{W}G_{\nu,\lambda}^{-1} = M_{\nu} (W_{-\nu-\lambda-1/2} W_{\nu}) = M_{-2\nu-1} (M_{\nu} W_{\nu+\lambda-1/2} W_{1-\nu}) W_{-2\nu-1} \]

\[ = M_{-2\nu-1} MC_{\nu,\lambda} W_{-2\nu-1}. \] (3.41)
Similarly, by means of (3.39), (3.40) and Lemma 3.26 we deduce

\[ \mathcal{I}G_{\nu, \lambda}^{-1} = (I_{\nu-1} \mathcal{I}G_{\nu-1, \lambda} M_{\nu-1}) M_{\nu-1} = I_{\nu-1} (I_{\nu-1} \mathcal{I}G_{\nu-1, \lambda} M_{\nu-1}) M_{\nu-1} \]

\[ = I_{\nu-1} \mathcal{I}G_{\nu, \lambda} M_{\nu-1}. \]  

(3.42)

By using Theorem 3.34, (3.42) and Theorem 3.11 we deduce

\[ \mathcal{W} \mathcal{G}_{\nu, \lambda}^{-1} = M_{\nu-1} \mathcal{I}G_{\nu, \lambda}^{-1} M_{\nu-1} \]

\[ = M_{\nu-1} I_{\nu-1} \mathcal{I}G_{\nu, \lambda} M_{\nu-1} \]

(3.41)

\[ = (M_{\nu-1} I_{\nu-1} \mathcal{I}G_{\nu, \lambda} M_{\nu-1}) \mathcal{W} \mathcal{C}_{\nu, \lambda} M_{\nu-1} \]

\[ = W_{\nu-1} \mathcal{W} \mathcal{C}_{\nu, \lambda} M_{\nu-1}. \]

Similarly, by using Theorem 3.34, (3.41) and Theorem 3.11 we deduce

\[ \mathcal{I}G_{\nu, \lambda}^{-1} = M_{\nu-1} \mathcal{I}G_{\nu, \lambda}^{-1} M_{\nu-1} \]

\[ = M_{\nu-1} I_{\nu-1} \mathcal{I}G_{\nu, \lambda} M_{\nu-1} \]

\[ = (M_{\nu-1} I_{\nu-1} \mathcal{I}G_{\nu, \lambda} M_{\nu-1}) \mathcal{W} \mathcal{C}_{\nu, \lambda} M_{\nu-1} \]

\[ = W_{\nu-1} \mathcal{W} \mathcal{C}_{\nu, \lambda} M_{\nu-1}. \]

Thus we have proved (i) and (iv). By inversion we obtain (ii) and (iii).

We conclude this section with the presentation of Gegenbauer transform pairs. As we have seen in Theorems 3.35, 3.36 and 3.37, there are many formulas for the inverse Gegenbauer transformations. Here we make a particular choice. From Theorems 3.32, 3.25 and 3.37 (i), (ii) we obtain the following Weyl-Gegenbauer transform pairs.

**Theorem 3.38** Let \( \nu, \lambda \in \mathbb{R} \) with \( \lambda > -\frac{1}{2} \), and let \( b \in \mathbb{R}^*_+ \). For \( f \in S_{-\lambda}(\mathbb{R}^+) \) we have

\[ (\mathcal{W} \mathcal{G}_{\nu, \lambda} f)(x) = \int \limits_{\mathbb{R}^+} (t^2 - x^2)^{\lambda-1/2} G_{\nu}^\lambda(x/t) f(t) \, dt, \quad 0 < x < b, \]

\[ (\mathcal{W} \mathcal{G}_{\nu, \lambda}^{-1} f)(x) = x^{-2\lambda} \int \limits_{\mathbb{R}^+} (t^2 - x^2)^{\lambda-1/2} G_{\nu}^\lambda(t/x) (W_{-2\lambda-1} f)(t) \, dt, \quad 0 < x < b, \]

and

\[ (\mathcal{W} \mathcal{C}_{\nu, \lambda} f)(x) = x \int \limits_{\mathbb{R}^+} (t^2 - x^2)^{\lambda-1/2} C_{\nu}^\lambda(t/x) f(t) \, dt, \quad 0 < x < b, \]

\[ (\mathcal{W} \mathcal{C}_{\nu, \lambda}^{-1} f)(x) = x^{-2\lambda-1} \int \limits_{\mathbb{R}^+} (t^2 - x^2)^{\lambda-1/2} G_{\nu}^\lambda(x/t) (W_{-2\lambda-1} f)(t) \, dt, \quad 0 < x < b. \]
From Theorems 3.25, 3.31 and 3.37 (iii), (iv) we obtain the following Riemann-Liouville-Gegenbauer transform pairs.

**Theorem 3.39** Let \( \nu, \lambda \in \mathbb{R} \) with \( \lambda > -\frac{1}{2} \), and let \( a \in \mathbb{R}^+_0 \). For \( f \in S_{\omega,1}(\mathbb{R}^+) \) we have

\[
(\mathcal{IC}_{\nu,\lambda} f)(x) = \int_a^x (x^2 - t^2)^{\lambda-1/2} C_\nu^\lambda(t/x) t f(t) \, dt, \quad x > a, \\
(\mathcal{IG}_{\nu,\lambda} f)(x) = \int_a^x (x^2 - t^2)^{\lambda-1/2} G_\nu^\lambda(t/x) \left( \mathcal{L}_{2\lambda-1} f \right)(t) \, dt, \quad x > a,
\]

and

\[
(\mathcal{IC}_{\nu,\lambda} f)(x) = \int_a^x (x^2 - t^2)^{\lambda-1/2} G_\nu^\lambda(t/x) f(t) \, dt, \quad x > a, \\
(\mathcal{IG}_{\nu,\lambda} f)(x) = \int_a^x (x^2 - t^2)^{\lambda-1/2} C_\nu^\lambda(t/x) \left( \mathcal{L}_{2\lambda-1} f \right)(t) \, dt, \quad x > a.
\]

### 3.7 Integral equations involving Gegenbauer functions

Based on the theory developed in the preceding sections of this chapter, we present a systematic method for solving Mellin-type convolution equations where the convolutor contains a Gegenbauer function. To be precise, we consider the integral equations

\[
\int_a^b (t^2 - x^2)^{\lambda-1/2} G_\nu^\lambda(t/x) t^{\mu+1} g(t) \, dt = f(x), \quad 0 < x < b; \tag{3.43}
\]

\[
x^{\mu+1} \int_a^x (t^2 - x^2)^{\lambda-1/2} C_\nu^\lambda(t/x) g(t) \, dt = f(x), \quad 0 < x < b; \tag{3.44}
\]

\[
\int_a^x (t^2 - x^2)^{\lambda-1/2} C_\nu^\lambda(t/x) t^{\mu+1} g(t) \, dt = f(x), \quad x > a; \tag{3.45}
\]

\[
x^{\mu+1} \int_a^x (t^2 - x^2)^{\lambda-1/2} G_\nu^\lambda(t/x) g(t) \, dt = f(x), \quad x > a. \tag{3.46}
\]

where \( \nu, \mu \in \mathbb{R}, \lambda > -\frac{1}{2}, a \in \mathbb{R}^+_0, b \in \mathbb{R}^+_\infty \).

In this section we discuss the integral equation (3.44) extensively. The other integral equations can be dealt with in the same manner.

For convenience we assume \( f, g \in S_{\omega,1}(\mathbb{R}^+) \). By Theorem 3.25, the integral equation (3.44) can be written shortly as

\[
M_\nu WC_{\nu,\lambda} g = f.
\]
Hence, the solution $g$ is given by

$$g = W_{-1}^{-1} g_{-1} M_{-\mu} f .$$

Now the various expressions for $W_{-1}^{-1} g_{-1}$, established in Theorems 3.35, 3.36, 3.37, provide integral/differential formulas for the solution $g$. The formulas of Theorem 3.35 (ii), viz. $W_{-1}^{-1} g_{-1} = W_{-1}^{-1} g_{+1} = W_{-1}^{-1} g_{-1} = W_{-1}^{-1} g_{+1}$ do not provide integral expressions for $g$, because $-1 - \lambda < -1/2$. From Theorem 3.36 (ii) with $\rho \in \mathbb{R}$ such that $-1 - \lambda + \rho > -1/2$, and Theorem 3.22, we find

$$g(t) = (W_{-1}^{-1} g_{+1} M_{-\mu} f) (t)$$

$$= \int_0^b (y^2 - t^2)^{\lambda - 1/2} G_{+1}^{\lambda - 1} (t/y) y^{\rho - 1} W_{-\rho} (y^{\rho - 1} f(y)) \, dy , \quad 0 < t < b .$$

(3.47)

From Theorems 3.37 (ii) and 3.22 we obtain the following alternative expressions for $g$:

$$g(t) = (M_{-1}^{-1} W_{+1}^{-1} W_{+1} M_{-\mu} f) (t)$$

$$= t^{-2\lambda - 1} \int_0^b (y^2 - t^2)^{\lambda - 1/2} G_{+1}^{\lambda} (t/y) y W_{-\lambda - 1} (y^{\lambda} f(y)) \, dy , \quad 0 < t < b .$$

(3.48)

and

$$g(t) = (W_{-1}^{-1} W_{+1} M_{-\mu} f) (t)$$

$$= W_{-1}^{-1} \int_0^b (y^2 - t^2)^{\lambda - 1/2} G_{+1}^{\lambda} (t/y) y W_{-\lambda - 1} (y^{\lambda} f(y)) \, dy , \quad 0 < t < b .$$

(3.49)

The expressions for $g$ presented follow as an immediate consequence of the inversion formulas for Gegenbauer transforms established in Section 3.6. We emphasize that the evaluation of the inversion formulas into integral expressions for $g$ uses two basic ingredients, namely,

1. Rodrigues-type formulas for Gegenbauer functions (Theorems 3.18, 3.19);
2. intertwining relations between multiplication operators and fractional integral/differential operators (Theorems 3.7, 3.15).

For specific values of the parameters $\lambda, \nu$, the Gegenbauer function kernel in the integral equation (3.44) reduces to a Gegenbauer polynomial or to one of the special cases, a Legendre polynomial or a Chebyshev polynomial. Convolution integral equations with such polynomial kernels have been treated by Li [35], Buschman [4], [5], [6], [7], and Higgins [30]. In these papers the solution of the integral equation seems to have been found by clever guessing, supported in [5] by a formal application of the Mellin transformation. The proposed solution is then verified by substitution into the original integral equation. This verification requires the (lengthy) computation of a special convolution integral. It is the aim of this section to show that the solutions of Li, Buschman, and Higgins, can be
found systematically as special cases of our solution (3.47) with specific values assigned to the parameters $\lambda, \mu, \nu, \rho, b$. In addition, we present alternative expressions for the solution of the integral equations considered, obtained by specialisation of (3.48) and (3.49).

In the course of his work in aerodynamics, Li [35] was led to an integral equation with a Chebyshev polynomial of the first kind as its kernel. He showed that the integral equation

$$
\int_{\frac{1}{2}}^{1} (t^2 - x^2)^{-1/2} T_n(t/x) \, g(t) \, dt = f(x) \,, \quad 0 < x < 1 \,.
$$

has the solution

$$
g(t) = -\frac{2}{\pi} \int_{t}^{1} (y^2 - t^2)^{-1/2} T_{n-1}(t/y) \, y^{1-n} \frac{d}{dy} (y^n f(y)) \, dy \,, \quad 0 < t < 1 \,.
$$

provided $f$ satisfies certain conditions. Here, in the case $n = 0$, $T_{-1}$ is taken to be $T_1$, and the integral equation (3.50) reduces to the classical Abel integral equation.

Now assume $f, g \in S_{-1}([0,\infty))$ and observe that $T_n = \sqrt{\pi/2} C_n^0$, see (3.33). Then the Chebyshev transform pair (3.50), (3.51) is precisely the special case of (3.44), (3.47) with $\lambda = 0$, $\mu = -1$, $\nu = n$, $\rho = 1$, $b = 1$. By assigning the same values to $\lambda, \mu, \nu, b$ in (3.48), (3.49), we obtain the following alternative expressions for $g$:

$$
g(t) = -\frac{2}{\pi} \int_{t}^{1} (y^2 - t^2)^{-1/2} T_n(t/y) \, y \frac{d}{dy} (y f(y)) \, dy \,, \quad 0 < t < 1 \,.
$$

Inspired by Li's work, Buschman [4] considered an integral equation with a Legendre polynomial as its kernel. He showed that the integral equation

$$
\int_{\frac{1}{2}}^{1} P_n(t/x) \, g(t) \, dt = f(x) \,, \quad 0 < x < 1 \,.
$$

has the solution

$$
g(t) = \int_{t}^{1} P_{n-2}(t/y) \, y^{2-n} \left(\frac{1}{y} \frac{d}{dy}\right)^2 (y^n f(y)) \, dy \,, \quad 0 < t < 1 \,.
$$

provided $f$ satisfies certain conditions. Now assume $f, g \in S_{-1}([0,\infty))$ and observe that $P_n = C_n^{1/2}$, see (3.35). Then the Legendre transform pair (3.52), (3.53) is recognized as the special case of (3.44), (3.47) with $\lambda = \frac{1}{2}$, $\mu = -1$, $\nu = n$, $\rho = 2$, $b = 1$. By assigning the same values to $\lambda, \mu, \nu, b$ in (3.48), (3.49), we obtain the following alternative expressions for $g$:

$$
g(t) = t^{-\frac{3}{2}} \int_{1}^{t} P_n(t/y) \, y \left(\frac{d}{dy}\right)^2 (y f(y)) \, dy \,, \quad 0 < t < 1 \,.
$$

$$
g(t) = \left(\frac{d}{dt}\right)^2 \left\{ \int_{1}^{t} P_n(t/y) \, f(y) \, dy \right\} \,, \quad 0 < t < 1 \,.
$$
Next, Buschman [5] considered an integral equation involving the Gegenbauer polynomial \( C_{n}^{k} \) with \( k, n \in \mathbb{N} \), \( 0 \leq k < n \). For convenience we rewrite \( C_{n}^{k} \) in terms of our notation \( C_{n}^{\lambda} \), cf. (3.32). Then Buschman’s integral equation takes the form

\[
\int_{x}^{1} (t^2 - x^2)^{k/2-1/2} C_{n}^{k/2}(t/x) \, g(t) \, dt = f(x) \,, \quad 0 < x < 1 \, ,
\]

(3.54)

with the solution

\[
g(t) = \int_{x}^{1} (y^2 - t^2)^{k/2-1/2} C_{n-k+1}^{k/2}(t/y) \, y^{2-n} \left( -\frac{1}{y} \frac{d}{dy} \right)^{k+1} (y^n f(y)) \, dy \,, \quad 0 < t < 1 \, ,
\]

(3.55)

provided \( f \) satisfies certain conditions.

Now assume \( f, g \in \mathcal{S}_{-1}(\mathbb{R}^+) \). Then the Gegenbauer transform pair (3.54), (3.55) is precisely the special case of (3.44), (3.47) with \( \lambda = k/2, \mu = -1, \nu = n, \rho = k+1, b = 1 \). The corresponding specialisation of (3.48), (3.49) yields the following alternative expressions for \( g \):

\[
g(t) = t^{-k-1} \int_{x}^{1} (y^2 - t^2)^{k/2-1/2} C_{n}^{k/2}(t/y) \, y \left( -\frac{d}{dy} \right)^{k+1} (y^n f(y)) \, dy \,, \quad 0 < t < 1 \, ,
\]

\[
g(t) = \left( -\frac{d}{dt} \right)^{k+1} \left\{ \int_{x}^{1} (y^2 - t^2)^{k/2-1/2} C_{n}^{k/2}(t/y) y^{1-k} f(y) \, dy \right\} \,, \quad 0 < t < 1 \, .
\]

Buschman’s transform pair (3.54), (3.55) covers Li’s transform pair (3.50), (3.51) (take \( k = 0 \)) and Buschman’s transform pair (3.52), (3.53) (take \( k = 1 \)).

Higgins [30] studied an integral equation involving the Gegenbauer polynomial \( C_{n}^{\lambda} \) with \( m \in \mathbb{N} \), \( \lambda > -\frac{1}{2} \). Again for convenience we rewrite \( C_{n}^{\lambda} \) in terms of our notation \( C_{n}^{\lambda} \), cf. (3.32). Then Higgins’ integral equation takes the form

\[
\int_{x}^{1} (t^2 - x^2)^{\lambda-1/2} C_{m}^{\lambda}(t/x) \, g(t) \, dt = f(x) \,, \quad 0 < x < 1 \, ,
\]

(3.56)

with the solution

\[
g(t) = \int_{x}^{1} (y^2 - t^2)^{\lambda-1/2} C_{m}^{\lambda}(t/y) \, y^{-m} B_{-\lambda-1} \{ y^m f(y) \} \, dy \,, \quad 0 < t < 1 \, ,
\]

(3.57)

where \( n, m \in \mathbb{N} \), \( n < m, \mu = (m - n - 1)/2, \lambda > -\frac{1}{2} \), provided \( f \) satisfies certain conditions.

Now assume \( f, g \in \mathcal{S}_{-1}(\mathbb{R}^+) \). Then the Gegenbauer transform pair (3.56), (3.57) is recognized as the special case of (3.44), (3.47) with \( \lambda = \lambda, \mu = -1, \nu = n, \rho = \lambda +(m-n+1)/2, b = 1 \). The corresponding specialisation of (3.48), (3.49) yields the following alternative expressions for \( g \):
\[ g(t) = t^{-2\lambda-1} \int_0^1 \left( y^2 - t^2 \right)^{\lambda-1/2} C_n(t/y) y W_{-2\lambda-1} \{ y f(y) \} \, dy, \quad 0 < t < 1, \]

\[ g(t) = W_{-2\lambda-1} \left\{ \int_0^1 \left( y^2 - t^2 \right)^{\lambda-1/2} C_m(t/y) y^{\lambda-2\lambda} f(y) \, dy \right\}, \quad 0 < t < 1. \]

Higgins' transform pair (3.56), (3.57) covers Buschman's transform pair (3.54), (3.55) (take \( \lambda = k/2, n = m - k - 1 \), so that \( \mu = k/2 \)).

Finally, Buschman [6] considered an integral equation with the Legendre function \( P_{\nu}^{1-\mu} \) as its kernel. He showed that the integral equation

\[ \int_0^1 \left( t^2 - x^2 \right)^{(\mu-1)/2} \mathbb{P}_{\nu}^{1-\mu}(t/x) g(t) \, dt = f(x), \quad 0 < x < 1, \]  

has the solution

\[ g(t) = \int_0^1 \left( y^2 - t^2 \right)^{(\mu-1)/2} \mathbb{P}_{\nu}^{1-\mu}(t/y) y^{\mu-\nu} \left( -\frac{1}{y} \frac{d}{dy} \right)^n \{ y^n f(y) \} \, dy, \quad 0 < t < 1, \]

(3.58)

(3.59)

where \( n \in \mathbb{N}, n > \mu > 0 \), provided \( f \) satisfies certain conditions.

Now assume \( f, g \in S_{\mu}(\mathbb{R}^+ \times \mathbb{R}^+) \) and observe that \( P_{\nu}^{1-\mu}(z) = (z^2 - 1)^{\mu-1/2} C_{\nu-\mu+1/2} \), cf. (3.24). Then the Legendre transform pair (3.56), (3.57) is precisely the special case of (3.44), (3.47) with \( \lambda := \mu - \frac{1}{2}, \mu := -\mu, \nu := \nu - \mu + 1, \rho = n, b = 1. \) By assigning the same values to \( \lambda, \mu, \nu, b \) in (3.48), (3.49), we obtain the following alternative expressions for \( g \):

\[ g(t) = t^{-2\mu} \int_0^1 \left( y^2 - t^2 \right)^{(\mu-1)/2} \mathbb{P}_{\nu}^{1-\mu}(t/y) y^{\mu} W_{-2\mu} \{ y^{\mu} f(y) \} \, dy, \quad 0 < t < 1, \]

\[ g(t) = W_{-2\mu} \left\{ \int_0^1 \left( y^2 - t^2 \right)^{(\mu-1)/2} \mathbb{P}_{\nu}^{1-\mu}(t/y) f(y) \, dy \right\}, \quad 0 < t < 1. \]

From (3.47) it follows that the expression (3.59) for \( g \) remains valid for non-integral \( n > \mu \), provided that \((-y^{-1} d/dy)^n\) is replaced by \( W_{-n} \). Thus, Buschman's transform pair (3.58), (3.59) covers Higgins' transform pair (3.56), (3.57) (take \( \mu := \lambda + \frac{1}{2}, \nu := m + \lambda - \frac{1}{2}, \lambda := \lambda + (m - n + 1)/2, f(x) := x^{-(\lambda+1/2)} f(x) \)).

We briefly discuss two further papers on the solution of convolution integral equations. Erdélyi [20] considered an integral equation involving the Legendre function \( P_{\nu}^{1-\mu} \), which he solved by techniques based on Riemann-Liouville fractional calculus. Erdélyi's integral equation is equivalent to our equation (3.45) and can be written shortly as

\[ IC_{\nu-\lambda+1/2} M_{-1}g = f. \]

His solution can be identified with one of the many expressions for \( M_{-\lambda+1/2} IC_{\nu-\lambda+1/2} f \), obtainable from Theorems 3.35, 3.36, 3.37. Alternative representations for the solution \( g \), analogous to (3.48) and (3.49), can be derived as well.
Sneddon [47] treated the general Mellin-type convolution equation which he solved by Mellin transform techniques. He recovered the previous solutions of Li, Buschman, Higgins, and Erdélyi, as special cases. However, his procedure is not very transparent and requires a thorough knowledge of Mellin transforms.

For a detailed review of the literature on convolution integral equations with special function kernels, we refer to Srivastava and Buschman [48].

More recently, Deans [7], [8] utilized the Radon transformation to determine a Gegenbauer transform pair with the Gegenbauer polynomial $C^{(q)}_{m}^{v}/2$, $m, q \in \mathbb{N}$, $q \geq 2$, as its kernel. For convenience we rewrite $C^{(q)}_{m}^{v}/2$ in terms of our notation $C^{(v)}_{m}$, cf. (3.32). Then Deans’ Gegenbauer transform pair takes the form

\[ f(x) = \int_{0}^{\infty} (t^2 - x^2)^{\nu/2} C^{(v)}_{m}(y/t) t g(t) \, dt, \quad x > 0, \quad (3.60) \]

\[ g(t) = t^{\nu-1} \int_{0}^{\infty} \left( \frac{d}{dy} \right)^{v-1} (t^2 - x^2)^{\nu/2} C^{(v)}_{m}(y/t) f(y) \, dy, \quad t > 0, \quad (3.61) \]

where $m, q \in \mathbb{N}$, $q \geq 2$, provided $f$ satisfies certain conditions.

Now assume $f, g \in S_{-}(\mathbb{R}^+)$. Then the Gegenbauer transform pair (3.60), (3.61) is identical to the first transform pair in Theorem 3.38 with $\lambda = q/2 - 1$, $\nu = m$, $b = \infty$. From Theorem 3.37 (i) with $\lambda = q/2 - 1$, $\nu = m$, we obtain the following alternative expression for $g$:

\[ g(t) = \left( \frac{-d}{dt} \right)^{v-1} \left\{ \int_{0}^{\infty} (t^2 - x^2)^{\nu/2} C^{(v)}_{m}(y/t) y^{\nu-1} f(y) \, dy \right\}, \quad t > 0. \]

The Gegenbauer transform pair (3.60), (3.61) has been used in Section 1.7, formulas (1.89), (1.90), and in Section 2.6. In these sections we studied the Radon transformation $\mathcal{R}$ acting on functions $f \in S(\mathbb{R}^v)$ or $\in S^\theta(\mathbb{R}^v)$, of the form

\[ f(rw) = f_m(r) Y_m(\omega), \quad r \geq 0, \quad \omega \in S^{v-1}, \]

where $r^{-m} f \in S_{\text{even}}(\mathbb{R})$ or $\in S^\theta(\mathbb{R})$, and $Y_m \in Y^\theta_m$. It was shown that

\[ (\mathcal{R} f)(p, \omega) = (2\pi)^{(v-1)/2} (\mathcal{W}_m Y_{m/2-1} f_m)(p) Y_m(\omega) \]

\[ = (2\pi)^{(v-1)/2} \left( \int_{r}^{\infty} (t^2 - p^2)^{(v-3)/2} C^{(v-1)}_{m}(t/r) r f_m(r) \, dr \right) Y_m(\omega), \]

\[ p \geq 0, \quad \omega \in S^{v-1}. \]

Conversely, let the function $g : Z_{-} \to C$ be defined by

\[ g(p, \omega) = g_m(p) Y_m(\omega), \quad (p, \omega) \in Z_{-}, \]

where $g_m \in F^m(S_{\text{even}}(\mathbb{R}))$ or $\in F^m(S^\theta_{\text{even}}(\mathbb{R}))$ and $Y_m \in Y^\theta_m$. Then we have the inverse Radon transform

\[ (\mathcal{R}^{-1} g)(r\omega) = (2\pi)^{-(v-1)/2} (\mathcal{W}_m Y_{m/2-1} g_m)(r) Y_m(\omega) \]

\[ = \frac{(-\sqrt{2\pi})^{v-1}}{\gamma_{v-2}} \int_{r}^{\infty} \left( \frac{t^2 - r^2}{t^{v-2}} \right)^{(v-3)/2} C^{(v-1)}_{m}(t/r) g_m(t)(p) \, dp Y_m(\omega), \]

\[ r \geq 0, \quad \omega \in S^{v-1}. \]

These results have been stated in Theorems 1.56 and 2.60.
BIBLIOGRAPHY


GLOSSARY OF SYMBOLS

Special functions

\( C_n^\nu \) Gegenbauer function, 116
\( C_0^\nu \) restriction of \( C_n^\nu \) to \((1, \infty)\), 116
\( G_n^\nu \) restriction of \( C_n^\nu \) to \((0, 1)\), 116
\( C_n \) Gegenbauer polynomial, 116
\( _2F_1 \) Hypergeometric function, 117
\( H_n \) Hermite polynomial, 12
\( \psi_n \) Hermite function of one variable, 12
\( \Psi_n \) Hermite function of \(q\) variables, 34
\( J_\nu \) Bessel function of the first kind, 17
\( L_n^\nu \) generalized Laguerre polynomial, 15
\( X_n^\nu \) Laguerre function, 15
\( L_n^\nu \) Fourier-Laguerre function, 24
\( P_n \) Legendre polynomial, 116
\( \mathcal{P}^\nu_n \) Legendre function, 115
\( \mathcal{P}^\nu \) Legendre function on the cut, 115
\( T_n \) Chebyshev polynomial of the first kind, 116
\( U_n \) Chebyshev polynomial of the second kind, 116

Basic symbols

\( l_2 \) space of square summable sequences
\( l_\infty \) space of bounded sequences
\( L_2(X) \) space of square integrable functions on \(X\)
\( L_\infty(X) \) space of bounded functions on \(X\)
\( \mathbb{R}^+ \) set of positive real numbers
\( \mathbb{R}_0^+ \) \( \mathbb{R}^+ \cup \{0\} \), 61
\( \mathbb{R}_0^\infty \) \( \mathbb{R}^+ \cup \{\infty\} \), 61
\( \mathbb{R}_0^{\infty, 0} \) \( \mathbb{R}^+ \cup \{0, \infty\} \), 61
\( \mathbb{R}^q \) \( q \)-dimensional Euclidean space
\( x \cdot y \) Euclidean inner product in \( \mathbb{R}^q \), 34
\( |x| \) \( (x \cdot x)^{1/2} \), 34
\( x_k \) \( x_1, x_2, \ldots, x_q \), 85
\( \partial^\nu \) \( \partial_1^{\nu_1} \partial_2^{\nu_2} \cdots \partial_q^{\nu_q} \), 85
\( \partial_\nu \) \( \partial_1 \partial_2 \cdots \partial_q \), 85
\( \mathcal{N}_n^\nu \) set of \( q \)-tuples of nonnegative integers
\( |n| \) \( n_1 + n_2 + \ldots + n_q \), 34
\( S^{q-1} \) unit sphere in \( \mathbb{R}^q \), 34
\( \sigma^{q-1} \) measure on \( S^{q-1} \), 34
\( \Omega_q \) \( \sigma^{q-1}(S^{q-1}) \), total surface area of \( S^{q-1} \), 35
\( Z_q \) \( \mathbb{R} \times S^{q-1} \), unit cylinder in \( \mathbb{R}^{q+1} \), 44
\( X_\nu \) \( L_2(\mathbb{R}^q; x^\nu dx) \), 15
\( \mathcal{Y}_m^\nu \) space of spherical harmonics of degree \(m\) on \( \mathbb{R}^q \), 36
\( N(q, m) \) number of linearly independent elements in \( \mathcal{Y}_m^\nu \), 36
$K^S_m$  reproducing kernel of $Y^S_m$, 38
$Y_{m,j}$  spherical harmonic of degree $m$ on $\mathbb{R}^s$, 39
$\tilde{Y}_{m,j}$  harmonic extension of $Y_{m,j}$, 36
$u_{s,m,j}(r\omega)$  $L^{m+s/2-1}_r(r)\ Y_{m,j}(r\omega)$, 39
$\otimes$  tensor product, 44
$\lfloor \mu \rfloor$  smallest integer greater than or equal to $\mu$, 106
$\Box$  end of proof

Operators
$T^*$  adjoint of operator $T$
$D(T)$  domain of operator $T$
$D^{\infty}(T)$  $C^{\infty}$-domain of operator $T$, 10
$S_\alpha(T)$  Gevrey space brought about by operator $T$, 67
$T$  $q$-tuple of operators $(T_1, \ldots, T_q)$, 30
$T_1^* T_2^* \ldots T_q^*$  30
$[T]$  $(T_1^2 + T_2^2 + \ldots + T_q^2)^{1/2}$, 30
$D^{\infty}(T)$  $C^{\infty}$-domain of $T$, 30
$S_\alpha(T)$  Gevrey space brought about by $T$, 85
$Q, Q_j, Q^j$  multiplication operators, 10, 30, 31
$P, P_j, P^j$  differentiation operators, 10, 30, 31
$F, F_j, F^j$  Fourier transformations, 9, 10, 30, 31
$H_\nu$  Hankel transformation, 17
$H_{-1/2}$  Fourier-cosine transformation, 17
$H_\nu$  Hankel-Clifford transformation, 24
$\mathcal{H}$  Hilbert transformation, 52
$\nabla$  differential operator, 39
$M$  momentum operator, 39
$\Delta$  Laplacian, 31
$\Delta_{LB}$  Laplace-Beltrami operator, 41
$V$  connection operator, 46
$\mathcal{R}$  Radon transformation, 50
$E_n$  orthogonal projection of $L_2(S^{s-1})$ onto $\mathcal{Y}^n_s$, 49
$M_{x\mu}$  multiplication by the function $\text{sgn}$, 52
$M_{x\nu}$  multiplication operator, 105
$T_x$  composition operator, 105
$D$  differentiation operator, 105
$W_\mu$  Weyl operator, 105, 106
$W_\mu$  Erdélyi-Kober operator, 109
$I_\mu$  Riemann-Liouville operator, 111, 112
$I_{\mu}$  Erdélyi-Kober operator, 113
$W_G, \lambda$  Weyl-Gegenbauer transformation of the first kind, 120
$W_G, \lambda$  Weyl-Gegenbauer transformation of the second kind, 122
$\mathcal{IC}_{n, \lambda}$  Riemann-Liouville-Gegenbauer transformation of the first kind, 123
$T\mathcal{G}_{n, \lambda}$  Riemann-Liouville-Gegenbauer transformation of the second kind, 123
Function spaces

$C^\infty(X)$ space of infinitely differentiable functions on $X$

$C_c^\infty(X)$ subspace of $C^\infty(X)$ of functions with compact support

$S(\mathbb{R})$ Schwartz space on $\mathbb{R}$, 9

$S_{\text{even}}(\mathbb{R})$ subspace of even functions in $S(\mathbb{R})$, 14

$S_{\text{odd}}(\mathbb{R})$ subspace of odd functions in $S(\mathbb{R})$, 14

$S(\mathbb{R}^+)$ Schwartz space on $\mathbb{R}^+$, 23

$S_{\text{even}}(\mathbb{R}^+)$ subspace of $C^\infty(\mathbb{R}^+)$, 105

$S_{\text{odd}}(\mathbb{R}^+)$ subspace of $S_{\text{even}}(\mathbb{R}^+)$, 111

$S_{\text{even}}(\mathbb{R}^+)$ subspace of $C^\infty(\mathbb{R}^+)$, 111

$S_{\text{odd}}(\mathbb{R}^+)$ subspace of $S_{\text{even}}(\mathbb{R}^+)$, 114

$S_\ell(\mathbb{R})$ $S_{\text{even}}(\mathbb{R}^+) \cap S_{\text{even}}(\mathbb{R}^+)$, 26

$S(\mathbb{R})$ Schwartz space on $\mathbb{R}$, 29

$S(Z_q)$ Schwartz space on $Z_q$, 45

$S_{\text{even}}(Z_q)$ subspace of even functions in $S(Z_q)$, 46

$S_{\text{odd}}(Z_q)$ subspace of odd functions in $S(Z_q)$, 46

$S^0(\mathbb{R})$ Gelfand-Shilov space on $\mathbb{R}$, 61

$S_{\text{even}}^0(\mathbb{R})$ subspace of even functions in $S^0(\mathbb{R})$, 74

$S_{\text{odd}}^0(\mathbb{R})$ subspace of odd functions in $S^0(\mathbb{R})$, 74

$G^0(\mathbb{R}^+)$ Gelfand-Shilov space on $\mathbb{R}^+$, 78

$S^0_2(\mathbb{R}^+)$ Gelfand-Shilov space on $\mathbb{R}^+$, 84, 85
# INDEX

<table>
<thead>
<tr>
<th>Term</th>
<th>Page Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>C**-domain</td>
<td>10</td>
</tr>
<tr>
<td>Cauchy problem</td>
<td>55, 56</td>
</tr>
<tr>
<td>Chebyshev polynomial</td>
<td>116, 130</td>
</tr>
<tr>
<td>Composition operator</td>
<td>105</td>
</tr>
<tr>
<td>Differentiation operator</td>
<td>10, 30, 31, 105</td>
</tr>
<tr>
<td>Erdélyi-Kober operator</td>
<td>20, 57, 75, 101, 109, 113</td>
</tr>
<tr>
<td>Expansion in spherical harmonics</td>
<td>41, 93</td>
</tr>
<tr>
<td>Factorization problem</td>
<td>49, 97</td>
</tr>
<tr>
<td>Fourier transformation</td>
<td>9, 10, 30, 31</td>
</tr>
<tr>
<td>Fourier-cosine transformation</td>
<td>17</td>
</tr>
<tr>
<td>Fourier-Laguerre function</td>
<td>24, 25, 79</td>
</tr>
<tr>
<td>Fractional calculus</td>
<td>105</td>
</tr>
<tr>
<td>Functions of rapid decrease</td>
<td>9</td>
</tr>
<tr>
<td>Funk-Hecke theorem</td>
<td>37</td>
</tr>
<tr>
<td>Gegenbauer function</td>
<td>115</td>
</tr>
<tr>
<td>Gegenbauer polynomial</td>
<td>37, 116</td>
</tr>
<tr>
<td>Gegenbauer transformation</td>
<td>124</td>
</tr>
<tr>
<td>Gel'fand-Shilov space</td>
<td>61, 84</td>
</tr>
<tr>
<td>Gevrey space</td>
<td>67, 85</td>
</tr>
<tr>
<td>Hankel transformation</td>
<td>17, 74</td>
</tr>
<tr>
<td>Hankel-Clifford transformation</td>
<td>24, 78</td>
</tr>
<tr>
<td>Harmonic polynomial</td>
<td>36</td>
</tr>
<tr>
<td>Hecke-Bochner theorem</td>
<td>38</td>
</tr>
<tr>
<td>Hermite function</td>
<td>12, 34, 70, 92</td>
</tr>
<tr>
<td>Hermite polynomial</td>
<td>12</td>
</tr>
<tr>
<td>Hilbert transformation</td>
<td>52</td>
</tr>
<tr>
<td>Homogeneous polynomial</td>
<td>36</td>
</tr>
<tr>
<td>Huygens' principle</td>
<td>57</td>
</tr>
<tr>
<td>Hyperplane</td>
<td>50</td>
</tr>
<tr>
<td>Integral equation</td>
<td>128</td>
</tr>
<tr>
<td>Intertwining relation</td>
<td>105, 129</td>
</tr>
<tr>
<td>Inverse Gegenbauer transformation</td>
<td>124</td>
</tr>
<tr>
<td>Kashpirovskii's intersection result</td>
<td>55, 89</td>
</tr>
<tr>
<td>Laguerre function</td>
<td>15, 74</td>
</tr>
<tr>
<td>Laguerre polynomial (generalized)</td>
<td>15</td>
</tr>
<tr>
<td>Laplacian</td>
<td>31</td>
</tr>
<tr>
<td>Laplace-Beltrami operator</td>
<td>41, 46</td>
</tr>
<tr>
<td>Legendre function</td>
<td>115</td>
</tr>
<tr>
<td>Legendre function on the cut</td>
<td>115</td>
</tr>
<tr>
<td>Legendre polynomial</td>
<td>116, 130</td>
</tr>
<tr>
<td>Mellin-type convolution equation</td>
<td>128</td>
</tr>
<tr>
<td>Momentum operator</td>
<td>39, 46</td>
</tr>
<tr>
<td>Multi-index notation</td>
<td>29, 85</td>
</tr>
<tr>
<td>Multiplication operator</td>
<td>10, 30, 31, 105</td>
</tr>
<tr>
<td>Projection theorem</td>
<td>51</td>
</tr>
<tr>
<td>Radially symmetric function</td>
<td>41, 57, 93, 101</td>
</tr>
<tr>
<td>Radon inversion formula</td>
<td>52, 53, 54, 55, 58, 59</td>
</tr>
<tr>
<td>Radon transformation</td>
<td>50</td>
</tr>
<tr>
<td>Reproducing kernel</td>
<td>38</td>
</tr>
<tr>
<td>Riemann-Liouville operator</td>
<td>111</td>
</tr>
<tr>
<td>Riemann-Liouville-Gegenbauer transformation</td>
<td>122, 123</td>
</tr>
<tr>
<td>Rodrigues' formula</td>
<td>115, 116, 117, 118</td>
</tr>
<tr>
<td>Schwartz space</td>
<td>9, 29</td>
</tr>
<tr>
<td>Spherical harmonic</td>
<td>34, 36</td>
</tr>
<tr>
<td>Tensor product</td>
<td>44</td>
</tr>
<tr>
<td>Weyl operator</td>
<td>25, 79, 105</td>
</tr>
<tr>
<td>Weyl-Gegenbauer transformation</td>
<td>58, 59, 102, 119, 120, 122</td>
</tr>
<tr>
<td>Zonal spherical harmonic</td>
<td>37</td>
</tr>
</tbody>
</table>
SAMENVATTING

In dit proefschrift wordt de werking van een aantal klassieke integraaltransformaties op de functionruimtes $S^0_0(\mathbb{R}^q)$, $\alpha, \beta \in \mathbb{R}_{0,\infty}^+$, $q \in \mathbb{N}$, bestudeerd. De notatie $S^\infty_0(\mathbb{R}^q)$ staat voor Gel'fand-Shilovruimte met als bijzonder geval de Schwartzruimte $S^\infty_0(\mathbb{R}^q) = S(\mathbb{R}^q)$ der snel afnemende functies. Genoemde functionruimten kunnen gekarakteriseerd worden met behulp van de zelfgeadjungeerde operatoren $Q_j$ (vermenigvuldiging met $x_j$) en $P_j = i\partial/\partial x_j$, $j = 1, \ldots, q$, in $L_1(\mathbb{R}^q)$. Uit deze karakteriseringen volgen de bekende fraaie eigenschappen

$$S^\infty_0(\mathbb{R}^q) \cap S^0_0(\mathbb{R}^q) = S^\omega_0(\mathbb{R}^q) \quad \text{en} \quad \mathcal{F}(S^\omega_0(\mathbb{R}^q)) = S^\omega_0(\mathbb{R}^q),$$

waarin $\mathcal{F}$ de Fouriertransformatie op $L_1(\mathbb{R}^q)$ is. Tevens is de ruimte $S^\omega_0(\mathbb{R}^q)$ te karakteriseren met behulp van de Fouriertransformatie. In geval $q = 1$ is de ruimte $S^\omega_0(\mathbb{R})$, i.e. de ruimte der even functies in $S^\omega_0(\mathbb{R})$, te beschrijven in termen van de Hankeltransformatie $\mathcal{H}_\nu$:

$$\mathcal{H}_\nu(S^\omega_0(\mathbb{R})) = S^\omega_0(\mathbb{R}), \quad \alpha, \beta \in \mathbb{R}_{0,\infty}^+, \quad \nu \geq -\frac{1}{2}.$$ 

Met behulp van deze karakteriseringen en het bekende verband tussen de Fouriertransformatie en de Hankeltransformatie (stelling van Hecke-Bochner) wordt de volgende karakterisering van de radiaalsymmetrische functies in $S^\omega_0(\mathbb{R}^q)$ afgeleid: Een functie $f(x) = g(|x|)$ behoort tot $S^\omega_0(\mathbb{R}^q)$ dan en slechts dan als $g \in S^\omega_0(\mathbb{R})$. Door ontwikkeling naar sferische harmonische wordt de overeenkomstige karakterisering van de niet-radiaalsymmetrische functies in $S^\omega_0(\mathbb{R}^q)$ gevonden.

CURRICULUM VITAE

27 februari 1964  geboren te Sint-Oedenrode
4 juni 1982  eindexamen Gymnasium β,
              Mgr. Zwijsen College te Veghel
2 juli 1987  doctoraal examen wiskunde (met lof),
              Technische Universiteit Eindhoven
van 21 november 1986  deeltijdsleraar wiskunde,
tot 1 augustus 1991  Hogeschool Eindhoven
vanaf 1 september 1987  assistent in opleiding,
                         Technische Universiteit Eindhoven
STELLINGEN
behorende bij het proefschrift

INTEGRAL TRANSFORMATIONS
AND
SPACES OF TYPE $S$

doors C.A.M. van Berkel
1. Beschouw de recurrente betrekking

\[(n + 1) (2n + 1) c_{j,n+1} = c_{j-1,n} + 2n^2 c_{j,n} + n(2n + 1) c_{j,n-1} \]

\[-2n(n - 1) c_{j,n-2}, \quad 0 \leq j \leq n + 1, \quad n \in \mathbb{N}_0,
\]

met randwaarden \(c_{0,0} = 1\), \(c_{j,n} = 0\) als \(j < 0\) of \(j > n\).

Voor \(n \in \mathbb{N}_0\) geldt

\[\sum_{j=0}^{n} c_{j,n} = 1 \quad \text{en} \quad c_{j,n} \geq 0, \quad 0 \leq j \leq n .\]

2. Voor \(p \in \mathbb{R}\) en \(n \in \mathbb{N}_0\) geldt

\[(x^{1-p} d/dx)^{2n+1} = (x^{1-2p} d/dx)^n x^{(2n+1)} (x^{1-2p} d/dx)^{n+1} .\]

3. Voor \(t \in \mathbb{R}^+\) en \(n, m, q \in \mathbb{N}_0\) met \(q \geq 2\) geldt

\[t^n e^{-t} |L_n^m|^{-1/2 - 1}(t)\]

\[\leq \frac{x^{q/2} \Gamma(n + m + q/2) \Gamma(2n + m + q) \Gamma(m + 1)}{(q - 1) \Gamma(2m + q - 2) \Gamma(q/2) \Gamma(m + q - 2) \Gamma(2n + m + 1) \Gamma(n + 1)} .\]

Hierin is \(L_n^m\) het gegenormaliseerde Laguerre-polyoom,

\[L_n^m(x) = x^{-m} \frac{e^x}{n!} \frac{d^n}{dx^n} [e^{-x} x^{-m}] .\]

(Vergelijk Duran [1])

4. Zij \(\nu \in C\) met \(\text{Re} \ \nu > 0\) en zij \(K_\nu\) de verzameling van alle functies van de vorm

\[t \mapsto \varphi(1 - 2 \tan^2 t) \cosh^{-\nu} t ,\]

met \(\varphi\) een geheel analytische functie. Het beeld van \(K_\nu\) onder de Fouriertransformatie is de verzameling van alle functies van de vorm

\[x \mapsto \Gamma((\nu + ix)/2) \Gamma((\nu - ix)/2) \psi(x) ,\]

met \(\psi\) een even en geheel analytische functie van sub-exponentiële groei, i.e.

\[\forall \varepsilon > 0 \sup_{x \in \mathbb{C}} \exp(-\varepsilon |x|) |\psi(x)| < \infty .\]

(Van Berkel en De Graaf [2])
5. Zij \( L_n^m \) het gegeneraliseerde Laguerrepolyoon als in Stelling 3. Definieer voor \( n, m \in \mathbb{N}_0 \) de functie \( U_{n,m} \) op \( \mathbb{R}^2 \) door

\[
U_{n,m}(x) = e^{-\frac{1}{2}(x_1^2 + x_2^2)} L_n^m(x_1^2 + x_2^2) (x_1 + ix_2)^m, \quad x \in \mathbb{R}^2.
\]

Dan wordt de Radongetransformeerde \( \mathcal{R}U_{n,m} \) gegeven door

\[
(\mathcal{R}U_{n,m})(p, \omega) = (-1)^n \frac{\sqrt{2\pi}}{n!} e^{-\frac{1}{2}p^2} H_n(p) H_{n+m}(p) (\omega_1 + i\omega_2)^m, \quad p \in \mathbb{R}, \quad \omega \in \mathbb{R}^2 \text{ met } \omega_1^2 + \omega_2^2 = 1.
\]

Hierin is \( H_n \) het Hermitepolyoon,

\[
H_n(x) = (-1)^{n} e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} [e^{-\frac{1}{2}x^2}], \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}.
\]

6. Zij \( f \) een snel afnemende functie op \( \mathbb{R}^2 \) en veronderstel dat haar Radongetransformeerde \( \mathcal{R}f \) de eigenschap heeft dat \( (\mathcal{R}f)(p, \omega) = 0 \) voor \( |p| > a \). Dan is \( f(x) = 0 \) voor \( |x| > a \).

Deze stelling kan eenvoudig bewezen worden met behulp van de Weyl-Gegenbauertransformaties en ontwikkelingen naar spherische harmonieën.

(Voor andere bewijzen zie Helgason [3] of Ludwig [4])

7. Zij \( p_m \) een homogene harmonisch polynoom van de graad \( m \) op \( \mathbb{R}^4 \). Dan geldt voor \( l \in \mathbb{N}_0^4 \) en \( \xi \in S^{4-1} = \{ \omega \in \mathbb{R}^4 : \omega_1^2 + \ldots + \omega_4^2 = 1 \} \),

\[
|\mathcal{A}(p_m)(\xi)|^2 \leq \lambda_m^l \int_{S^{4-1}} |p_m|^2 d\sigma^{4-1}
\]

met

\[
\lambda_m^l = \frac{\pi^{1/2-\epsilon/2} 2^{l-\epsilon-2} \Gamma(m+1) \Gamma(m+q/2) \Gamma(m-l+q/2)}{\Gamma(q/2 - 1/2) \Gamma(m-l+1) \Gamma(m-l+q/2-1)}.
\]

Hierin is \( \mathcal{A} = \partial_1^{\epsilon} \partial_2^{q/2} / (\partial_3^{4/2} \ldots \partial_4^{q/2}) \) en \( \sigma^{4-1}(S^{4-1}) = 2\pi^{2/3} / \Gamma(q/2) \).

(Van Berkel en Van Eijndhoven [5])
8. Zij \( F \) de Fresnelintegraal gedefinieerd door

\[
F(x) = \int_{-\infty}^{\infty} \exp(it^2) \, dt.
\]

Voor \( x > 0 \) geldt

\[
\int_{-\infty}^{x} \frac{e^{-2it} F(\sqrt{2t})}{(x-t)^{1/2}} \, dt = \frac{ix}{2\sqrt{2}} - e^{it/4} \sqrt{\pi/2} \, e^{-2ix} F(\sqrt{2x}).
\]

Met behulp van dit resultaat is de integraal \( I(\lambda) \) in Servadio [6, Appendix B] uit te drukken in een Fresnelintegral, waarna de asymptotische ontwikkeling van \( I(\lambda) \) voor \( \lambda \to \infty \) eenvoudig volgt uit die van de Fresnelintegral.

9. Zij \( J \) de Besselfunctie van de eerste soort en van orde \( \nu \). Voor \( m, n \in \mathbb{N}_0 \) en \( 0 < x \leq \pi \) geldt

\[
\sum_{k=0}^{\infty} k^{-1} J_{m+\frac{1}{2}}(kz) J_{n+\frac{1}{2}}(kz) = \frac{\delta_{m,n}}{n + \frac{1}{2}}.
\]

Deze betrekking is te beschouwen als een discrete versie van de orthogonaleiteitstrekking

\[
\int_{-\infty}^{\infty} t^{-1} J_{m+\frac{1}{2}}(t) J_{n+\frac{1}{2}}(t) \, dt = \frac{\delta_{m,n}}{n + \frac{1}{2}}.
\]

Literatuur


