A Variational Approach to Magneto-elastic Buckling Problems

PROEFSCHRIFT

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Aan mijn ouders
(To my parents)
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0. Introduction to the thesis

During the past twenty years many authors published scientific articles on the subject of high field devices, which are becoming more and more important in modern technology. One can think of:
- magnetically levitated trains,
- fusion reactors,
- NMR-scanners for medical purposes,
- MHD-generators.

Subjects that have been studied are:
- Calculation of the magnetic fields generated by the magnets,
- deformation in the magnets,
- magneto-elastic waves in the magnets,
- influence of the magnetization on crack propagation in magnets,
- stability, or buckling, of ferromagnetic and superconducting constructions.

In this thesis, especially the last subject will receive most attention. The strong magnetic fields are usually generated by superconducting magnets. Usually we have some knowledge about the devices, such as the shape, the global current and so on; but it is very difficult to make a good estimate of the magnitude of the generated magnetic fields. Hence, also the electromagnetic forces (e.g. the Lorentz forces) on the structure, causing the deformation and possibly, the instability of the structure, are unknown. So, initially, it is necessary to construct a model from which the magnetic and mechanical fields in a ferromagnetic or superconducting solid of given shape, can be calculated.

In this thesis we establish such a model. For the sake of simplicity the devices are assumed to deform in an elastic way. The model contains a complete description of magneto-elastic interactions in solid continua. Starting from a well-known system of differential equations and boundary conditions (the so-called Maxwell-Minkowski model) an equivalent variational formulation is derived. This variational model is based upon an expression for the Lagrangian density, which in the static case is related to the energy density (see chapter 1). We refrain from an investigation of the post-buckling behaviour of the devices and we discuss only bifurcation theory. Let us explain what the latter remark exactly means: Suppose a device is subjected to high electromagnetic forces; under the influence of these forces the device will slightly deform into a so-called intermediate state, which is an equilibrium state. If the forces are not too high this intermediate state is the one and only equilibrium state. However, for increasing values of the forces, at a certain bifurcation point, suddenly another equilibrium state, the "buckled" state, is possible, and the intermediate state becomes unstable.

In classical mechanics usually a system of differential equations, boundary conditions and other conditions is set up and solved in the form of an eigenvalue problem (see chapter 1, section 2). A reasonable approximation of the eigenfunctions, say of order $e$, then leads to an approximation of the eigenvalue, which is related to the buckling field or buckling current, of order $e$. The device is supposed to be slender (see also the chapters 2,3 and 4) and the differential equations
involve the well-known beam, plate (or shell) or ring equations. However, in this classical approach, the magnetic field for the buckled state has always to be solved. Within our variational theory (see the following chapters) two advantages arise. Firstly, we do not always have to solve the magnetic problem and secondly, a reasonable approximation of the eigenfunctions, of order ε, leads to an approximation of the buckling field or buckling current of order ε², as we shall explain further on.

Let us explain the exact mathematical issue with the use of a very simple example. Consider a rod loaded by two mechanical forces P at the ends, (see Fig. 0.1). If the forces P become higher and higher only a small displacement, in the direction

![Diagram of a rod loaded by two mechanical forces](image)

Fig. 0.1.: A rod loaded by two mechanical forces.

perpendicular to the forces P, is enough to let the rod buckle in another, stable situation (situation 2 or 3), the so-called buckled state. So for the values $P \geq P_1$ (say) (or $P \geq P_2 > P_1$ etcetera) the intermediate state (situation 1) is unstable.

For explanatory purposes let us consider a bifurcation-problem represented by the algebraic equations

$$f(x;P) = 0,$$

(0.1)

where $x = x(P)$ is a solution for every $P > 0$. We are looking for the infimum $P_1$ of the values $P$ for which there exists more than one solution $x(P)$. This infimum $P_1$ is a so-called bifurcation-point of the system (0.1), which is called stable for $P < P_1$ and unstable for $P > P_1$.

For the theoretical solution of the bifurcation problem and the determination of the bifurcation point $P_1$ it is necessary that the linear mapping $D_x$, with respect to $x$, on $f$, is not invertible anymore, so we arrive at the condition

$$A(P) := D_x f(x(P);P), \quad \det A(P) = 0,$$

(0.2)

where $\det$ stands for the determinant. It is assumed that the linear mapping $A(P)$ is symmetric and positive semi-definite. So there exists a vector $w$ such that
\[ A(P)w=0, \ w\neq 0, \ P=P_1, P_2, \ldots. \quad (0.3) \]

It should be noted that, mathematically, the condition (0.2) is necessary but not necessarily sufficient to find the correct value of \( P_1 \). In practice the condition (0.2) is usually sufficient. So we assume (0.3) to be equivalent to the bifurcation problem.

The function (the \( \delta \)-symbol is defined by formula (1.2.12))

\[ J(w; P)=\frac{1}{2} w^T A(P) w, \ \delta_J=0 \iff A(P)w=0. \quad (0.4) \]

is usually some energy function(s). For \( w\neq 0 \) this function is positive for \( P=P_2 \) and becomes zero for \( P=P_1 \). We now suppose that we have a good approximation \( h\neq 0 \) of a part of \( w \) at our disposal. Let us say

\[ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad |w_1-h|=O(\varepsilon), \ 0<\varepsilon<<1. \quad (0.5) \]

Approximations \( y \) and \( \bar{P}_1 \) of \( w_2 \) and \( P_1 \) are now found by solving the problem

\[ J(h, y; \bar{P})=\frac{1}{2} [h^T, y^T] \begin{bmatrix} A_{11}(P) & A_{12}(P) \\ A_{21}(P) & A_{22}(P) \end{bmatrix} \begin{bmatrix} h \\ y \end{bmatrix}, \]

\[ \delta_J=0, \ J(h, y; \bar{P}_1)=0(J(w_1, w_2; P_1)=0). \quad (0.6) \]

Write

\[ J(h, y; P)=\frac{1}{2} [h^T A_{11}(P) h + h^T A_{12}(P) y + y^T A_{21}(P) h + y^T A_{22}(P) y], \]

\[ \delta_J=[\frac{1}{2} (h^T A_{12} + A_{21}^T)(y) + \frac{1}{2} (2 A_{12} + A_{21}^T)(h) y]. \quad (0.7) \]

and since \( A(P) \) is a symmetric mapping \((A_{11}=A_{11}^T, A_{22}=A_{22}^T, A_{12}=A_{21}^T)\), we deduce from (0.6), (0.7) the best vector-value for \( y \), to be:

\[ y=y(\bar{P}_1)=-A_{22}^{-1} A_{21}(\bar{P}_1) h, \quad \text{det} A_{22}(\bar{P}_1) \neq 0. \quad (0.8) \]

The approximation \( \bar{P}_1 \) of \( P_1 \) follows from

\[ J(h; \bar{P}_1)=-\frac{1}{2} h^T (A_{11}-A_{21} A_{22}^{-1} A_{21})(h) h = 0 \Rightarrow P=\bar{P}_1(h), \]

\[ J(w_1; \bar{P}_1)=0, \ |h-w_1|=O(\varepsilon), \ 0<\varepsilon<<1. \quad (0.9) \]

We now apply the implicit function theorem for the equation \( J(h; \bar{P}_1(h))=0 \) around \( h=w_1 \), \( \bar{P}_1=P_1 \) and we derive

\[ (D_{\bar{P}_1} \bar{P}_1)(w_1)=0 \quad \text{if} \quad w_1^T \partial (A_{11}-A_{21}^{-1} A_{22} A_{21})/\partial P(P_1) w_1 \neq 0, \quad (0.10) \]

and the immediate conclusion is
\[ | \vec{P}_1 - P_1 | = O (\varepsilon^2). \] (0.11)

Hence, a reasonable approximation \( h \) of \( w_1 \) leads us by the equation (0.9)\(^1\) to a very good approximation of \( P_1 \), which is the thing we were interested in.

The procedure described above can be generalized for magneto-elastic systems of differential equations, boundary conditions and other conditions (such as conditions at infinity or Ampère's law) due to magneto-elastic interactions. Usually we have some knowledge about the energy functional \( J \) and about the displacements; however, we have no further information about the disturbances or perturbations of the magnetic fields, which are found by linearizing the whole system with respect to the intermediate state which is close to the undeformed state.

In the first chapter of this thesis we start from an energy functional, which for the static case is equal to the Lagrangian; the energy integral is evaluated at an intermediate state with respect to the perturbations up to terms of second order. Variation of this energy integral with respect to the perturbations under suitable constraints leads us to the well-known Maxwell-Minkowski model.

The variational principle for magneto-elastic buckling problems is further elaborated for the two cases: ferromagnetic and superconducting structures. For the superconducting case the variational principle can be adjusted in such a way that the boundary condition, implying the connection between the perturbed magnetic field and the displacements, is no longer a constraint. The chapter has been published in [1].

The second chapter shows a complete description of the magneto-elastic buckling problems of i) a single ferromagnetic or superconducting beam, ii) a system of two ferromagnetic or superconducting beams. These problems are solved analytically by means of using Green's identities, fundamental solutions and complex analysis. The buckling value of a system of two beams is found to be one order smaller than the one for a single beam with respect to the slenderness parameter (see chapter 2, sections 3 and 4).

In the third chapter the magneto-elastic buckling problems of two superconducting concentric or coaxial tori (or rings) are reduced to the buckling problem of two superconducting beams. So, these problems can also be solved by applying complex function theory. The buckling value is apart from a constant factor equal to the buckling value of a system of two superconducting beams.

The fourth and last chapter shows a numerical treatment of the magneto-elastic buckling problem of a system of an arbitrary number of superconducting beams. The numerical solution is based upon two-dimensional potential theory with the use of logarithms as fundamental solutions. In this chapter also all the buckling modes (the eigenfunctions for the displacements) are obtained.

The practical problems, treated in the chapters 3, 4 have been published in [2], [3], [4] and show a first outline for engineers to treat the magneto-elastic buckling problem for a system of beams and rings with the use of numerical and analytical methods. The magneto-elastic buckling problem of curved beams will be investigated in a forthcoming article.
List of symbols

\( A \quad : \) system matrix (-)
\( \mathbf{A} \quad : \) magnetic vector potential (N/A)
\( \mathbf{a} \quad : \) perturbed magnetic vector potential (N/A)
\( \mathbf{B} \quad : \) magnetic induction (N/Am)
\( \mathbf{b} \quad : \) perturbed magnetic induction (N/Am)
\( \mathbf{C} \quad : \) complex boundaries
\( \mathbf{c} \quad : \) characteristic cross-section-constant (-)
\( \mathbf{c}^{*} \quad : \) elastic tensor (N/m²)
\( \mathbf{c}^{**} \quad : \) piezo-magnetic tensor (N/Am)
\( \mathbf{D} \quad : \) dielectric displacement (C/m²)
\( \mathbf{D} , \mathbf{B} \quad : \) regions and boundaries (two-dim.)
\( \mathbf{E} \quad : \) electric field strength or intensity (N/C)
\( \mathbf{E} \quad : \) Lagrangian deformation tensor (-)
\( \mathbf{E} \quad : \) Young's modulus (N/m²)
\( \mathbf{e} \quad : \) linear deformation tensor/permittivity tensor (-)
\( \mathbf{F} \quad : \) deformation gradient/complex dimensionless magnetic induction (-/-)
\( \mathbf{f} \quad : \) adjusted dimensionless perturbed magnetic potential (-)
\( \mathbf{G} , \mathbf{G} \quad : \) regions and boundaries (three-dim.)
\( \mathbf{H} \quad : \) magnetic field strength or intensity (A/m)
\( \mathbf{h} \quad : \) perturbed magnetic field strength or intensity (A/m)
\( \mathbf{i} \quad : \) differentiation with respect to \( x \) or \( \xi \)
\( \mathbf{i}_{0} \quad : \) prescribed current (A)
\( \mathbf{J} \quad : \) surface current density (A/m)
\( \mathbf{J} \quad : \) functional of second order in perturbations (N/m)
\( \mathbf{J}_{F} \quad : \) Jacobian determinant (-)
\( \mathbf{L} \quad : \) Lagrangian (Nm)
\( \mathbf{L} \quad : \) Lagrangian density (N/m³)
\( \mathbf{l} \quad : \) support length (m)
\( \mathbf{M} \quad : \) magnetization per unit of mass (Am²/kg)
\( \mathbf{m} \quad : \) perturbed magnetization per unit of mass (Am²/kg)
\( \mathbf{m} \quad : \) ratio of distance and radius (-)
\( \mathbf{N} , \mathbf{n} \quad : \) unit normals (-)
\( \mathbf{P} \quad : \) polarization per unit of mass (Cm/Kg)
\( \mathbf{Q} \quad : \) dimensionless force (-)
\( \mathbf{r} , \mathbf{R} \quad : \) radii (m)
\( \mathbf{S} \quad : \) complex regions
\( \mathbf{T} \quad : \) stress tensor (N/m²)
\( \mathbf{t} \quad : \) perturbed stress tensor (N/m²)
\( \mathbf{U} \quad : \) internal energy density per unit of mass (m³/h²)
\( \mathbf{u} \quad : \) displacements (m)
\( \mathbf{v} \quad : \) buckling amplitudes (-)
\( \mathbf{w} \quad : \) central line displacements (m)
\( \mathbf{X}, \mathbf{x} \) : position vectors (m)
\( x, y \) : two-dimensional coordinates (m)
\( z \) : axial coordinate/complex variable (m/-)
\( \alpha \) : dimensionless complex radius (-)
\( \beta \) : normalized ratio of distance and radius (+)
\( \delta \) : slenderness parameter (-)
\( \epsilon \) : small parameter/dielectric constant \((-\text{F/m})\)
\( \theta \) : polar angle/deflection angle (-/-)
\( \chi \) : eigenvalue (-)
\( \Lambda \) : invariant magnetization \((\text{Amp}^2/\text{Ag})\)
\( \lambda \) : Helmholtz parameter \((\text{m}^{-1})\)
\( \mu \) : magnetic permeability \((\text{N/Am}^2)\)
\( \nu \) : Poisson's ratio (-)
\( \xi \) : position vector (m)
\( \rho \) : mass density \((\text{kg/m}^3)\)
\( \sigma \) : electric conductivity \((\text{SC/m})\)
\( \tau \) : unit tangent vector (-)
\( \tau \) : torsion angle (-)
\( \chi \) : ferromagnetic susceptibility (-)
\( \phi \) : two-dim. dimensionless perturbed magnetic potential/polar angle (-/-)
\( \psi \) : perturbed magnetic potential \((\text{N/Am})\)
\( \omega \) : dimensionless real potential (-)

\((\text{N}=\text{kg/m}^2, \text{C}=\text{As})\)
CHAPTER 1

A variational principle for magneto-elastic buckling

1. Introduction

The last two decades have shown a great progress in the research on magneto-elastic stability problems. Based on the pioneering work of F.C. Moon, several other authors have solved problems in this area of research. For an excellent survey and a very extensive list of references, we refer to the monograph of Moon [5]. Important parts of two IUTAM-symposia in Paris 1983, [6], and in Tokyo 1986, [7], were devoted to related subjects. Usually, these problems are treated in a classical mechanical way, e.g. by means of establishing a beam or a plate equation, in which the loading terms are of magnetic origin (for a general survey of this method, cf. [8]). An alternative way was followed by Goudjo and Maugin [9], who employed the principle of virtual power for the construction of a stability theory for soft ferromagnetic plates.

In the present chapter we shall introduce a variational principle on the basis of which a magneto-elastic stability (or buckling) problem can be formulated in terms of an eigenvalue problem. Explicit formulations for this eigenvalue problem will be given for two cases, which are the most important ones from a practical point of view, i.e.

- soft ferromagnetic, and
- superconducting

media. For these two cases, we shall show how the variational principle directly yields an explicit expression for the buckling value. The advantage of this method is that, whenever it is possible to determine the solution for the intermediate and perturbed electromagnetic fields, it is just a matter of a simple substitution to obtain the buckling value. However, in complex constructions, as occur in for example fusion reactors and high-field magnetic devices, such exact solutions are not available, and then the variational principle serves as a sound basis for a construction of approximation fields yielding an optimal approximation for the buckling value.

We start this chapter by showing in general terms how a magneto-elastic buckling problem can be related to an eigenvalue problem, and how this eigenvalue problem can be formulated as a variational principle (see also [10]). For the formulation of this principle the first and second vari-ation of a so-called Lagrangian $L$ is needed. In section 3 an expression for $L$ is given and the first and second variation of $L$ are evaluated in terms of the perturbed fields (which are perturbations with respect to some intermediate, or pre-buckled, state). In section 4 we will show that this specific choice for $L$ corresponds to the so-called Maxwell-Minkowski model for magneto-elastic interactions (cf. [11]). Finally, in section 5 the general buckling criterion is formulated and the main lines for the procedure to obtain a buckling value are described.

In the sections 6 and 7 more detailed formulations are given for soft ferromagnetic structures and superconductors.
2. A general eigenvalue problem

Every equilibrium state of a system of bodies, that is influenced by an external magnetic field in vacuum, is governed by a set of equations and boundary conditions (cf. [9], (11)). Let us denote this set schematically by

\[ S_i \{ B(x), M(x), T(x), x; \beta_0 \} = 0, \quad 1 \leq i \leq N. \quad (1.2.1) \]

The symbols B, M, T, x and \( \beta_0 \) refer to the magnetic induction, the magnetization, the stress tensor, the position and the external magnetic field parameter (e.g. the field at infinity), respectively. The symbols \( S_i \) contain various differential operators, some of these operators act on the boundaries of the bodies.

In the theory of stability three equilibrium configurations of the bodies are to be distinguished, namely the natural or unloaded state, given by (denoting \( x = X \))

\[ S_{0i} \{ 0, 0; 0, X; 0 \} = 0, \quad 1 \leq i \leq N, \quad (1.2.2) \]

the intermediate state, satisfying (\( x = \xi \))

\[ S_{\xi i} \{ B^0(\xi), M^0(\xi); T^0(\xi), \xi; \beta_0 \} = 0, \quad 1 \leq i \leq N, \quad (1.2.3) \]

and the present or spatial state, that differs only slightly from the intermediate state, and is characterized by (1.2.1) or (\( x = \xi + u(\xi) \))

\[ S_i \{ B^0 + b(\xi), M^0 + m(\xi); T^0 + r(\xi), \xi + u(\xi); \beta_0 \} = 0, \quad 1 \leq i \leq N. \quad (1.2.4) \]

The perturbations \( b, m, r \) and \( u \) are supposed to be small. Subtraction of (1.2.3) from (1.2.4) and neglecting second order terms in the perturbations yield a problem that is homogeneous with respect to the perturbations. In the sequel this homogeneous problem is denoted by

\[ s_i \{ b(\xi), m(\xi); r(\xi), u(\xi); \beta_0 \} = 0, \quad 1 \leq i \leq N. \quad (1.2.5) \]

The symbols \( s_i \) refer to linear operators, which contain various differential operators, some of them acting on the intermediate boundaries. For each value of the field parameter \( \beta_0 \) there exists the solution

\[ s_i \{ 0, 0; 0; \beta_0 \} = 0, \quad 1 \leq i \leq N, \]

but we are only interested in those values of \( \beta_0 \) for which

\[ (b, m; r, u) \ast \{ 0, 0; 0, 0 \}. \quad (1.2.6) \]

is a solution of (1.2.5). The problem posed by (1.2.5) and (1.2.6) is an eigenvalue problem; the perturbations and the field \( \beta_0 \) play the role of the eigenvector and the eigenvalue, respectively. In the theory of stability the eigenvalues are called buckling values. Of course we are especially interested in the lowest buckling value. The eigenvalue problem is linear with respect to the perturbations, but depends on \( \beta_0 \) in a non-linear way.

In many cases neglecting the intermediate deformations is justified; this simplification makes it possible to identify the natural and the intermediate boundaries, thus the intermediate configurations are no longer unknown.

However, no simplification of this kind makes the dependence on \( \beta_0 \) of the eigenvalue problem
less complicated. Generally it is impossible to obtain the buckling value directly from (1.2.5), (1.2.6).

The basic idea of a variational principle for magneto-elastic buckling is as follows: Assume that some of the equations (or boundary conditions) (1.2.1), (1.2.3), say 1 ≤ i ≤ k are satisfied a priori,

\[
S_i [B(x), M(x); T(x), x; B_0] = 0, \quad 1 \leq i \leq k < N,
\]

and consider these equations as constraints for the variations of the functionals, the so called Lagrangians,

\[
L \left[ B, M; T; B_0 \right] = \int_{\Omega} L [B(x), M(x); T(x), x; B_0] dV,
\]

\[
L^0 \left[ B^0, M^0; T^0; B_0 \right] = \int_{\Omega} L^0 [B^0(x), M^0(x); T^0(x), x; B_0] dV.
\]

The integrands of the integrals in the right hand sides of (1.2.8), the so called Lagrangian densities, are connected with the sets of equations and boundary conditions (1.2.1) and (1.2.3) and need to be specified later on. Evaluation of \( N_1 - S^0_i \) and \( L - L^0 \) in terms of the perturbations results in (compare with (1.2.5))

\[
L_1 [b(\xi), m(\xi); t(\xi), u(\xi); B_0] = 0, \quad 1 \leq i \leq k,
\]

\[
L - L^0 = \delta L + J + O(\varepsilon^3),
\]

in which \( \varepsilon \) denotes the order of magnitude of the perturbations and \( \delta L \) and \( J \) are the first and a factor one half times the second variation of \( L \) with respect to the intermediate state. Note that \( \delta L \) contains only terms of order \( \varepsilon \), whereas \( J \) contains only terms of order \( \varepsilon^2 \). If \( B^0, M^0 \) and \( T^0 \) are chosen in such a way that

\[
\delta L = 0 \quad \forall \; [b(\xi), m(\xi); t(\xi), u(\xi); B_0] = 0, \quad 1 \leq i \leq k \quad \Leftrightarrow \quad (1.2.3) \quad (1.2.10)
\]

then it can be proved that

\[
\delta J = 0 \quad \forall \; \left[ b(\xi), m(\xi); t(\xi), u(\xi); B_0 \right] = 0, \quad 1 \leq i \leq k \quad \Leftrightarrow \quad (1.2.5) \quad (1.2.11)
\]

Here, \( \delta J \) is the first variation of \( J \) which is defined as

\[
\delta J (b, m, t, u | b_1, m_1, t_1, u_1, B_0) = \frac{1}{\varepsilon} \left( J (b+\varepsilon b_1, m+\varepsilon m_1, t+\varepsilon t_1, u+\varepsilon u_1, B_0) - J (b, m, t, u, B_0) \right).
\]

Hence, the eigenvalue problem (1.2.5), (1.2.6) is equivalent to
\[ \delta J = 0 \quad \text{if} \quad (b, m; r, u) = (0, 0, 0, 0). \quad (1.2.13) \]

Here, it is assumed that the intermediate fields are already known.

From the fact that \( J \) is a homogeneous and quadratic functional with regard to the perturbations one can deduce the important property

\[ \delta J = 0 \Rightarrow J = 0. \quad (1.2.14) \]

The equivalence of the eigenvalue problem to (1.2.13) and the property (1.2.14) imply that any reasonable approximation for the perturbations leads us to a good approximation for the buckling value \( B_0 \) (see also section 5).

3. Statement and evaluation of the Lagrangian

In this section we shall postulate an explicit expression for the Lagrangian in terms of the magnetic field in and outside the deformed body. As indicated in the preceding section these fields are considered as perturbations with respect to some intermediate state. By an evaluation of the Lagrangian with respect to the perturbations, an explicit representation for formula (1.2.9) will be obtained.

We only consider static situations in which one single, simply connected body is influenced by a uniform field \( B_0 \). The body is assumed to be magnetizable and non-conducting.

For this case, a specific expression for the Lagrangian density \( L \) is postulated (note that in a static version \(- L \) is equal to the energy density). This choice is justified by the fact that a variation of the Lagrangian, under the proper constraints, yields a set of equations and boundary conditions, known in literature as the Maxwell-Minkowski model (cf. [11]). Our choice of \( L \) is based upon the form of the electromagnetic energy density for the Maxwell-Minkowski model as given in [11], page 55, i.e. (for \( E \neq 0 \))

\[ \frac{1}{4 \mu_0} (H, H). \]

We note that other forms for the energy density are possible and admissible. For instance, the choice of (see [11], page 72)

\[ \frac{1}{2 \mu_0} (B, B) \]

would result in the so called Ampèrean-current model as is used often by e.g. F. Moon, [5]. Hutter and van de Ven showed in [11] that these models are completely equivalent.

The present configuration of the body, its boundary and the vacuum are denoted by \( G^* \), \( \partial G \) and \( G^\circ \), respectively. An upper index \( \circ \) stands for a value outside the body and an index \( ^* \) for a value inside the body. Then the Lagrangian density is chosen as

\[ L^* = -\frac{1}{2 \mu_0} (H^*, H^*) + \frac{1}{2 \mu_0} B^2, \quad L^\circ = -\frac{1}{2 \mu_0} (H^\circ, H^\circ) + \frac{1}{2 \mu_0} B^2 + \rho U, \quad (1.3.1) \]

accompanied by the constraints
\[
\begin{align*}
\mathbf{B}^* &= \text{curl} \mathbf{A}^* , \quad \mathbf{M}^* = 0 , \quad \mathbf{x} \in G^* ; \\
\mathbf{A}^* &= \mathbf{A}^- , \quad \mathbf{x} \in \partial G ; \\
\mathbf{B}^- &= \text{curl} \mathbf{A}^- , \quad T = \rho \frac{\partial U}{\partial \mathbf{p}} \mathbf{F}^T , \quad \rho J_F = \rho_0 , \quad \mathbf{x} \in G^- ; \\
\mathbf{B}^* &\to \mathbf{B}_0 , \quad |\mathbf{x}| \to \infty ;
\end{align*}
\]

where \( \rho \) and \( \rho_0 \) are the mass densities in the present and the natural state respectively and \( \mathbf{H}, \mathbf{F} \) and \( J_F \) are the magnetic field, the deformation gradient and the Jacobian defined by

\[
\mathbf{H}^* = \frac{1}{\mu_0} \mathbf{B}^* , \quad \mathbf{H}^- = \frac{1}{\mu_0} \mathbf{B}^- - \rho \mathbf{M}^- , \quad \mathbf{F} = \frac{2\mathbf{x}}{\partial \mathbf{x}} , \quad J_F = \det \mathbf{F} .
\]

Here \( \mu_0 \) is the magnetic permeability in vacuum. Furthermore, the function \( U = U(F, \mathbf{M}) \) is the internal energy density. Finally \( \mathbf{A} = \mathbf{A}(\mathbf{x}) \) is some vector potential introduced in order to assure that \( \mathbf{B} \) satisfies

\[
\begin{align*}
\text{div} \mathbf{B}^* &= 0 , \quad \mathbf{x} \in G^* ; \\
(\mathbf{B}^*, \mathbf{n}) &= (\mathbf{B}^-, \mathbf{n}) , \quad \mathbf{x} \in \partial G ,
\end{align*}
\]

where \( \mathbf{n} \) is the unit normal on \( \partial G \) (see also appendix B).

**NOTE:** Requirements of objectivity imply that \( U \) can only be a function of tensorial variables which are invariant with respect to observer transformations. This condition can be satisfied by taking

\[
U = U(E, \mathbf{A}) ,
\]

where \( E \) is the Lagrangian deformation tensor and \( \mathbf{A} \) is the invariant magnetization given by

\[
E = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) , \quad \mathbf{A} = \mathbf{F}^T \mathbf{M} .
\]

respectively. For the moment however, there is no need for this somewhat more complex formulation, but we shall return to this later on.

### 3.1 General Evaluation Procedure

For the derivation of expressions for \( \delta L \) and \( f = \delta^2 L/2 \) an expansion of \( L - L^0 \) in terms of \( \epsilon \) up to and including terms of order \( \epsilon^2 \) is needed. For this purpose, a more precise definition of the perturbations is required, as will be given below. The relation between the Euler coordinates \( \mathbf{x} \) and \( \mathbf{\xi} \) of the perturbed (or present) and intermediate state, respectively, and the displacement \( \mathbf{u} \) is

\[
\mathbf{x} = \mathbf{\xi} + \mathbf{u}(\mathbf{\xi}) .
\]

Inside the body we prefer a formulation in terms of the coordinate \( \mathbf{\xi} \) and, therefore, we define
\[ B^-(x) = \hat{B}^-(\xi) = B_{\xi^0}^-(\xi) + b^- (\xi), \quad \xi \in \mathbb{R}^3, \]  
with analogous definitions for \( h^- (\xi), m^- (\xi), a^- (\xi) \) and \( t (\xi) \).

In the vacuum material coordinates are meaningless and we are bound to formulate the vacuum
perturbations in terms of the local coordinate \( x \); therefore, we define
\[ B^+(x) = B^{0+}(x) + b^+(x), \quad x \in \mathbb{R}^3, \]
with analogous definitions for \( h^+(x), m^+(x) \) and \( a^+(x) \).

The use of different coordinates in \( \mathbb{R}^- \) and \( \mathbb{R}^+ \) will give rise to some extra terms in the linearized boundary conditions as we shall see later (e.g., Equations (1.3.15), see also [12], (3.13), (3.14)).

The order of magnitude of the perturbations, \( \epsilon \), is expressed by
\[ \epsilon = | \partial u / \partial \xi | \ll 1, \]
and it is assumed that
\[ | b | = O (\epsilon | B^0 |), \quad | h | = O (\epsilon | H^0 |), \quad \text{etc.} \]

In the sequel most of the relations will be written in the Cartesian tensor notation, and \( \delta_{ij} \) and \( \varepsilon_{ijk} \) will be the Kronecker delta and the alternating tensor, respectively. Moreover differentiation with respect to a coordinate is denoted by a (lower case) letter preceded by a comma; However, we have to distinguish between differentiation in \( \mathbb{R}^- \) and \( \mathbb{R}^+ \). This means that one has to read \( \partial \) as
\[ \partial = \partial / \partial \xi_i, \quad \xi \in \mathbb{R}^3 \cup \partial \mathbb{R}^3 \cup \mathbb{R}^3, \quad i = 1, 2, 3, \]
\[ \partial = \partial / \partial x_i, \quad x \in \mathbb{R}^3 \cup \partial \mathbb{R}^3 \cup \mathbb{R}^3, \quad i = 1, 2, 3. \]

In order to obtain the linearized constraints (1.2.9), a linearization of the equations and boundary
conditions (1.2.3) is required. Since this linearization is straightforward we give the results right
away
\[ b^+_i = c_{\mu k} a^+_{\mu j}, \quad m^+_i = 0, \quad \xi \in \mathbb{R}^3, \]
\[ b^-_i = c_{\mu k} (a^-_{\mu j} - \frac{\partial U}{\partial w^0} u_{\mu j}), \quad \rho = \rho^0 (1 - u_{\lambda k}^0), \]
\[ b^-_i = -\rho^0 u_{\lambda k}^0 + \frac{\partial^2 U}{\partial w^0 \partial w^0} u_{\lambda k}^0 \xi \in \partial \mathbb{R}^3; \]
\[ a^+_i = c_{\mu k} a^+_{\mu j}, \xi \in \partial \mathbb{R}^3; \]
\[ b^+_i = 0, \quad | x | \to \infty. \]

in which the material coefficients \( c_{\mu k}^0, \rho^0, \) and \( c_{\mu k}^0 \) are defined by
\[ c_{\mu k}^0 = \left[ \frac{\partial^2 U}{\partial F_{\mu} \partial F_{\rho}} \right]^0, \quad c_{\mu k}^0 = \left[ \frac{\partial^2 U}{\partial F_{\mu} \partial M_{\rho}} \right]^0. \]

We note that in the derivation of the linearized boundary condition (1.3.13) the following result
is used (we give a second order expression directly because this is needed in the sequel)

let \( \xi \in \partial \mathbb{R}^3 \) and \( \xi \in \partial \mathbb{R}^3 \) be material points of the boundary, then
\[ A^T(x) - A^T(x) = (A^0 + a_i^T(x)) - (A^0 + a_i^T)(x) = (A^0 + a_i^T)(x) + a_i^T(x) - (A^0 + a_i^T)(x) = \]
\[ = [A^0(x) - A^0(x)] + [a_i^T(x) - a_i^T(x)] + \frac{1}{2} A^0 \mathcal{M} a_i^T u_j(x) + O(\varepsilon^2), \quad \xi \in \partial G^0. \]

Besides the linear constraints (1.3.13) we need an expression for the second order functional \( J \), as introduced in (1.2.9), for an explicit formulation of our variational principle. The derivation of the expression for \( J \) requires the second order approximations of \( B^T \) and \( H^T \) as can be derived from (1.3.2) and (1.3.3), respectively. Using

\[
\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} = \left[ \delta_{ij} - \frac{1}{\delta x_i} \frac{\partial}{\partial x_i} \right], \quad \xi \in G^0,
\]

and

\[
\frac{\partial}{\partial x_i} = 1 - \frac{1}{\delta x_i} \left( \frac{\partial}{\partial x_i} + O(\varepsilon^2) \right), \quad \xi \in G^0,
\]

we obtain

\[
b_i^T = \delta_{ij} \mathcal{M} - \frac{1}{2} (u_i^T u_j + u_j^T u_i), \quad \xi \in G^0.
\]

By substitution of

\[
A^0 = A^0 - \mathcal{M} \mathcal{M}^0,
\]

which is equivalent to (1.3.2) and (1.3.3), an alternative expression for \( b^T \) is obtained, which is more convenient for our later elaborations. The result is

\[
b_i^T = \delta_{ij} \mathcal{M}^0 - \frac{1}{2} (u_i^T u_j + u_j^T u_i), \quad \xi \in G^0.
\]

Substitution of (1.3.20) into (1.3.18) yields

\[
\mu_0 = \left[ \delta_{ij} \mathcal{M}^0 - \frac{1}{2} (u_i^T u_j + u_j^T u_i) + O(\varepsilon^2), \quad \xi \in G^0.\right.
\]

The relations (1.3.18), (1.3.20) and (1.3.21) will now be used for the determination of \( J \). This will be done in two steps, and we start with the material part.
3.2 Evaluation of the Material Lagrangian $L$

By virtue of the mass balance there exists a relation between the material volume elements $dV$ and $dV^0$ in the perturbed and intermediate state, respectively, which enables us to transform $L^-$ into an integral with domain $O^\sigma$. With $L^-$ according to (1.3.12) we then find

$$L^- - L^0 = \int_\sigma^{\sigma^0} \frac{\rho^0}{\rho} \frac{1}{2} \mu_0 \left( (H, H) - (H^0, H^0) \right) +$$

$$- \frac{\rho^0}{\rho} - 1 \right) \frac{1}{6} \mu_0 (H^0, H^0) + \left( \frac{\rho^0}{\rho} - 1 \right) \frac{1}{2} \mu_0 (H^0, H^0).$$

(1.3.22)

Since the present part only concerns fields inside the body, there is no difficulty in a temporary omission of the upper index "$\sigma$". Using the reciprocal relation of (1.3.17) for the mass density and (1.3.21), we obtain

$$- \frac{\rho^0}{\rho} \left( \mu_0 \left( (H, H) - (H^0, H^0) \right) - \frac{\rho^0}{\rho} - 1 \right) \frac{1}{6} \mu_0 (H^0, H^0) =$$

$$= - \frac{\rho^0}{\rho} \left( 2 (H^0, H) + (h, h) \right) - \frac{\rho^0}{\rho} - 1 \right) \frac{1}{6} \mu_0 (H^0, H^0) =$$

$$= \{ - \varepsilon_{ij} (u_i - A_{ii} u_i) H_j^0 + \mu_0 \rho^0\varepsilon_{ij} H_j +$$

$$+ \left( \frac{\rho^0}{\rho} - 1 \right) \frac{1}{6} \mu_0 (H^0, H^0) \right) +$$

$$= \{ - \varepsilon_{ij} (u_i - A_{ii} u_i) H_j^0 + \mu_0 \rho^0\varepsilon_{ij} H_j +$$

$$+ \frac{\rho^0}{\rho} - 1 \right) \frac{1}{6} \mu_0 (H^0, H^0) \right) +$$

(1.3.23)

in which $T_{ij}^\sigma$ is the so-called Maxwell stress tensor defined by

$$T_{ij}^\sigma = H_i b_j - \frac{1}{3} \mu_0 H_k b_k b_j.$$

(1.3.24)

A Taylor expansion of $\rho^0 \left( U - U^0 \right)$ in terms of derivatives with respect to the intermediate state yields

$$\rho^0 \left( U - U^0 \right) = \rho^0 \left[ \frac{\partial U}{\partial \varphi} \right]_m \left( F_{0m} - F_{0m}^0 \right) +$$

$$+ \frac{1}{2} \rho^0 \left[ \frac{\partial^2 U}{\partial \varphi^2} \right]_m \left( F_{0m} - F_{0m}^0 \right) \left( F_{0m} - F_{0m}^0 \right) +$$

$$+ \left[ \frac{\partial U}{\partial \varphi} \right]_m \left( m_{0m} - m_{0m}^0 \right) + O (\varepsilon^2) : \varepsilon \in G^\sigma,$$

or (with $F_{0m} - F_{0m}^0 = u_i F_{i0}^0$)
\begin{equation}
\rho^0 (U - U^0) = [T^0_{ij} u_{ij} + \rho^0 \left( \frac{\partial U}{\partial M_j} \right)^0 m_j] + \frac{1}{2} \rho^0 \left( c^{00}_{ij} u_{ij} u_{ij} + 2 c^{00}_{ijk} u_{ij} m_j + c^{00}_{ijk} m_i m_j \right) + O(\epsilon^3), \quad \xi \in \mathcal{O}^0,
\end{equation}

where \( T^0_{ij} \) is the intermediate stress tensor (cf. (1.3.23)), \( c^{00}_{ij} \) and \( c^{00}_{ijk} \) are the material coefficients as defined by (1.3.14) and

\begin{equation}
\xi^0 = \left( \frac{\partial U}{\partial M_j} \right)^0.
\end{equation}

Substitution of (1.3.25), (1.3.23) and (1.3.17) into (1.3.22) results in a formulation of \( (L^- - L^0^-)\) in terms of the independent perturbations \( a^-\), \( m^-\) and \( u^-\). Decomposing this result into a term \( \delta L^-\) that only contains terms of order \( \epsilon \) and a term \( J^-\) of order \( \epsilon^3 \), we obtain

\begin{equation}
L^- - L^0^- = \delta L^- + J^- + O(\epsilon^3),
\end{equation}

where

\begin{equation}
\delta L^- = \frac{1}{c^0} \left[ -\epsilon_{ijk} (a_k + A_{ik} u_k) \right] \frac{\partial U}{\partial M_j} - (T^0_{ij} + T^{00}_{ij}) u_{ij} + \rho^0 m_i \left( \mu_0 H^0_i \right)^0 - \frac{1}{2 \rho_0} B^0_i u_{ik} \Gamma dV^0
\end{equation}

and

\begin{equation}
J^- = \frac{1}{c^0} \left[ -\frac{1}{2} \rho^0 \left( c^{00}_{ijk} u_{ij} u_{ij} + 2 c^{00}_{ijk} u_{ij} m_j + c^{00}_{ijk} m_i m_j \right) \right] - \frac{1}{c^0} \mu_0 H^0_i \left( \mu_0 H^0_i \right)^0 - \frac{1}{c^0} \mu_0 (u_{ij} u_{ij}) H^0_i \frac{\partial U}{\partial M_j} + \frac{1}{2 \rho_0} B^0_i u_{ik} \Gamma dV^0.
\end{equation}

By means of Gauss' divergence theorem an alternative formula for \( \delta L^-\) can be deduced, in which derivatives of the perturbations \( a^-\), \( m^-\) and \( u^-\) are absent, namely

\begin{equation}
\delta L^- = \frac{1}{c^0} \left[ -\epsilon_{ijk} H^0_{ij} a_i + \left[ (T^0_{ij} + T^{00}_{ij}) \right] u_{ij} \right] + \left( \mu_0 H^0_i \right)^0 \rho^0 m_j \Gamma dV^0 + \frac{1}{c^0} \left[ -\epsilon_{ijk} H^0_{ij} N^0 a_i + \left[ (T^0_{ij} + T^{00}_{ij}) \right] N^0 - A_{ik} e_{ij} \mu H^0_i N^0 \right] u_{ij} \Gamma dS^0,
\end{equation}

where \( N^0\) is the unit normal on \( \partial \mathcal{O}^0\).

As the second step in the procedure for the determination of \( J^-\), we proceed with the vacuum part.
3.3 Evaluation of the Vacuum Lagrangian $L^*$

In terms of the Lagrangian densities $L^*$ and $L^{0*}$, the variation $(L^* - L^{0*})$ is defined by

$$L^* - L^{0*} = \int_G L^* (x) d\nu - \int_{\partial G} L^{0*} (x) d\nu^0,$$  \hfill (1.3.29)

i.e. as integrals over infinite domains $G^*$ and $G^{0*}$, respectively. The behaviour of the magnetic induction $B^*$ at infinity according to (1.3.29), however, guarantees the existence of $L^*$ and $L^{0*}$ (that is to say the above integral expressions for $L^*$ and $L^{0*}$ are convergent).

Firstly, we shall transform the integral for $L^*$ into one over the intermediate configuration $G^{0*}$. To this end, we introduce two auxiliary vector functions $W(x)$ and $W^0(x)$ by

$$L^* (x) = \text{div} W(x), \quad x \in G^*,$$  \hfill (1.3.30)

$$L^{0*} (x) = \text{div} W^0(x), \quad x \in G^{0*}.$$  

The existence of such functions is ensured but they are not determined by $(1.3.30)$ at all if $W(x)$ satisfies (1.3.30) then $W(x) + \text{curl} V(x)$ also satisfies (1.3.30). But, as the auxiliary functions will not occur in the final formula for $(L^* - L^{0*})$, this indeterminacy is totally irrelevant.

Using (1.3.30) and Gauss' divergence theorem we derive straightforward

$$L^* - L^{0*} = \int_G \text{div} W(x) d\nu - \int_{\partial G} \text{div} W^0(x) d\nu^0 =$$  \hfill (1.3.31)

$$= - \int_{\partial G} \langle W(x+u), n \rangle dS + \int_{\partial G^*} \langle W^0(x), N^0 \rangle dS^0,$$

in which $dS$ and $dS^0$ denote the surface elements on $\partial G$ and $\partial G^*$, respectively. The connection between the directed surface elements is (cf. [13], Eq. (21), page 61)

$$n dS = \text{det} (\frac{\partial x}{\partial \xi}) (\frac{\partial x}{\partial \eta})^T N^0 dS^0 = \frac{\partial x}{\partial \xi} (I - \frac{\partial u}{\partial \xi})^T N^0 dS^0,$$  \hfill (1.3.32)

in which $I$ is the unity tensor.

This relation is used in (1.3.31) to transform the integral over $\partial G$ into an integral over $\partial G^*$, resulting in

$$L^* - L^{0*} = \int_{\partial G^*} \langle W^0(x), \frac{\partial x}{\partial \xi} (I - \frac{\partial u}{\partial \xi}) W(x+u), N^0 \rangle dS^0.$$  \hfill (1.3.33)

With

$$W(x) = W^0(x) + W(x), \quad |W| = O (\epsilon W^0),$$  \hfill (1.3.34)

the integrand of the integral in the right-hand side of (1.3.33) can be evaluated in terms of $\epsilon$ yielding

$$\langle W^0, [I + u_{ik} + \frac{1}{2} (u_{ik} u_{ij} - u_{ij} u_{ik})] \{ \delta_{ij} - u_{ij} + u_{ik} u_{ij} \} \rangle.$$
\[ (W_0^2 + W_1^2 u_2 + \frac{1}{2} W_2^2 u_4 u_4 + w_j + w_1 u_1) N^3 + O(x^4) = \]
\[ (w_1 w_j w_j - w_1 w_j w_j) - w_j + [w_1 w_j w_j - w_1 w_j w_j] + \]
\[ \frac{1}{2} [W_1^2 (u_2 u_2 - u_1 u_1) + (W_2^2 u_4 u_4 - u_1 u_1) + \]
\[ (W_1^2 u_2 - u_1 u_1) + W_2^2 (u_2 u_2 - u_1 u_1)] + \]
\[ \frac{1}{2} [(u_2 u_2 - u_1 u_1) + W_2^2 (u_2 u_2 - u_1 u_1)] + \]
\[ - \frac{1}{2} W_3^2 u_2 u_4 + O(x^3) \]

after some rearrangement of terms.

For the further procedure, we need the following Lemma, which is a special case of Stokes' theorem.

Lemma

If \( f(\zeta) \) and \( g(\zeta) \) possess continuous derivatives in a neighbourhood of \( \partial C^0 \) then

\[ \int_{\partial \Omega} (f_i g_j - g_i f_j) + f_i g_j - g_i f_j) N^3 \, ds = 0 \]

Noticing that the integrand can be written as \( \text{curl} (f \times g) \) (a tangential derivative) and taking Stokes' theorem for granted, we have a trivial proof of this Lemma.

Substitution of (1.3.35) into (1.3.33) and use of the Lemma leads us to

\[ L^* - L^{ost} = \int_{\partial \Omega^3} W_1 N_1^3 \, ds - \int_{\partial \Omega^3} W_1^2 u_1 \, ds + \]
\[ \int_{\partial \Omega^3} [W_2^2 u_2 u_4 u_4 + \frac{1}{2} W_2^2 (u_2 u_2 - u_1 u_1) + N_2^3 \, ds] \]

After a transformation of the first integral in (1.3.36) into a volume integral by means of Gauss' theorem, we eliminate \( W^0 \) and \( w \) from (1.3.36) with the aid of the definitions (1.3.30) and (1.3.34). Thus, we arrive at

\[ L^* - L^{ost} = \int_{\partial \Omega^3} (L^* - L^{ost}) \, ds - \int_{\partial \Omega^3} L^{ost} u_1 N_1^3 \, ds + \]
\[ \int_{\partial \Omega^3} [L^{ost} u_1 + \frac{1}{2} L^{ost} u_1 u_1 + \frac{1}{2} L^{ost} (u_2 u_2 - u_1 u_1) + N_2^3 \, ds] \]

With \( L^* \) according to (1.3.1) and with (1.3.18), we have

\[ L^{ost} = -\frac{1}{2} \mu_0 H_2^2 + \frac{1}{2} \mu_0 B \]

and

\[ L^* - L^{ost} = -\frac{1}{2} \mu_0 (H_2^2 + h_j^2) + \frac{1}{2} \mu_0 H_2^2 + H_j^2 = \frac{1}{2} \mu_0 H_2^2 + H_j^2 \]
\[ L^* - L^{AA} = - \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 - \frac{1}{2} \mu_0 \int_{C'} H_{ik}^2 H_{ik}^2 u_i N^0_i dS^0 + \]

Substitution of (1.3.38)\(^1\) and (1.3.35)\(^2\) into (1.3.37) yields

\[ L^* - L^{AA} = - \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 + \frac{1}{2} \mu_0 \int_{C'} H_{ik}^2 H_{ik}^2 u_i N^0_i dS^0 + \]

\[ + \int_{C'} [e_{ik} H_{ik}^2 a_i a^*_j u_i + \frac{1}{2} \mu_0 H_{ik}^2 H_{ik}^2 u_i u_j a^*_j - \frac{1}{2} \mu_0 H_{ik}^2 H_{ik}^2 (u_{ij} u_i - u_{ij} u_j)] N^0_i dS^0 + \]

\[ - \frac{1}{2\mu_0} \int_{C'} b_i b^*_j dV^0 - \frac{1}{2\mu_0} \int_{C'} [u_i + \frac{1}{2} (u_{ij} u_i - u_{ij} u_j)] N^0_i dS^0. \]  

(1.3.39)

Consider the first integral of (1.3.39), i.e.

\[ - \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 = \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 + \int_{C'} e_{ik} H_{ik}^2 a_i N^0_i dS^0 = \]

\[ = \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 + \int_{C'} e_{ik} H_{ik}^2 [a_j A_{jk}^0 - A_{jk}^0 a_j u_i - A_{jk}^0 u_i u_j - \frac{1}{2} A_{jk}^0 u_i u_a] N^0_i dS^0. \]

(1.3.40)

where in the second integral \(a^*\) is eliminated in favour of \(a^*\) by means of (1.3.15). With the use of the relation (compare (1.3.19))

\[ A_{jk}^0 = A_{jk}^0 + \mu_0 e_{ik} H_{ik}^2 \]

(1.3.41)

one can derive

\[ e_{ik} H_{ik}^2 A_{jk}^0 u_i = \mu_0 (H_{ik}^2 H_{ij}^2 u_i - \frac{1}{2} H_{ik}^2 H_{ij}^2 \delta_{ij} u_i) \]

(1.3.42)

Substituting (*) and (**) into (1.3.39) and assembling terms of order \(\varepsilon\) and those of order \(\varepsilon^2\), we ultimately arrive at the following formula for \((L^* - L^{AA})\) in terms of the independent variables \(a\) (or \(b\)) and \(u\) (here \(\delta L^*\) contains only terms of order \(\varepsilon\) and \(J^*\) only those of order \(\varepsilon^2\) in analogy with (1.3.27) and (1.3.28))

\[ L^* - L^{AA} = \delta L^* + J^* + O(\varepsilon^3), \]

(1.3.42.1)

where

\[ \delta L^* = \int_{C'} e_{ik} H_{ik}^2 a_i a^*_j dV^0 + \]

\[ + \int_{C'} [e_{ik} H_{ik}^2 a_j a^*_i + H_{ik}^2 \delta_{ij} u_j - A_{jk}^0 u_i H_{ik}^2 u_j - \frac{1}{2\mu_0} \delta_{ij} u_i] N^0_i dS^0 + \]

(1.3.42.2)

and
\[ J^* = -\frac{1}{2\mu_0} \int_{\partial\Omega} b_{\gamma} b_{\gamma}^* d\Omega^0 + \int_{\partial\Omega} \left( H_0^2 (\varepsilon_{ijk} u_i - \varepsilon_{ijk} u_i) a_{ij} + \frac{1}{2} \mu_0 H_0^2 H_0^* u_i u_j \right) + \\
- \frac{1}{2} \varepsilon_{ijk} H_0^2 A_{\alpha\beta}^0 u_k u_i + \frac{1}{4} \left( \mu_0 H_0^2 H_0^* - \frac{1}{\mu_0} b_0^* \right) \left( u_{ij} u_i - u_{ij} u_i \right) \right)^* N_0^0 \Delta S^0 \]  

(1.3.42.3)

The rigorous analytical elaborations presented here enable us to state the final explicit version of the evaluation (1.2.9)\(^2\). Adding the formulas (1.3.27), (1.3.28) and (1.3.42) we conclude that

\[ L - L^0 = \delta L + J (\varepsilon \gamma) 
\]

(1.3.43.1)

in which the first variation \(\delta L\) of \(L\) with respect to the intermediate state is given by

\[ \delta L = \delta L^* + \delta L^* = \\
= \int_{\partial\Omega} \left( -\varepsilon_{ijk} H_0^2 a_i + (\varepsilon_{ijk} H_0^2 - \frac{2U^0}{\pm M_1}) \right) \mu_0 m_i + \\
+ \left( (T_0^0 + T_0^0) \right) + A_{\alpha\beta}^0 \varepsilon_{ijk} H_0^2 \left( u_i u_j \right) d\Omega^0 
\]

(1.3.43.2)

whereas the second variation \(J = \frac{1}{2} \delta^2 L\) of \(L\) reads

\[ J = J^* + J^* = \\
= \int_{\partial\Omega} \left( -\frac{1}{2} p_{ij} \left( e_{ijk} u_k u_j + 2 e_{ijk} u_i m_j + \varepsilon_{ijk} u_i m_j \right) + \\
+ e_{ijk} (u_i u_j - A_{\alpha\beta}^0 u_{\alpha\beta}) H_0^2 u_{ij} - (H_0^2 H_0^* - \frac{1}{2} \mu_0 H_0^2 H_0^*) \frac{1}{2} \left( u_{ij} u_j - u_{ij} u_i \right) + \\
- \frac{1}{2} \mu_0 H_0^2 \delta_l \delta_{ik} \right) d\Omega^0 + 
\]

(1.3.43.3)

It should be noted that the terms in \(L^*\) and \(L^*\) containing \(\delta\) cancel each other. Hence, the term \(\frac{B_0^2}{2\mu_0}\) in \(L^*\) is totally irrelevant to the value of \(\delta L\) or \(J\) and was merely added to the Lagrangian density to make the integral \(L^*\) converge.
The formulas (1.3.43) form the basis for the next sections in which our variational principle is further developed.

4. Consequences of the variations of $L$ and $J$

Now that we have the disposal of explicit expressions for $\delta L$ and $\delta J$ we shall show that variation of $L$ and of $J$ results in sets of equations corresponding to the Maxwell-Minkowski model for magneto-elastic interactions. Thus, we will have proved that the variational principle described in section 2 (i.e. (1.2.10), (1.2.11)) with the Lagrangian according to section 3 is equivalent with this model.

For the further procedures it is convenient to make some rearrangements in the constraints and the variables. From now on we shall consider $\lambda^0$, $\rho^0$, $\rho^1$ and $a$, $\alpha$, $\beta$ as basic variables and $B^0$, $B^1$, $\gamma^0$ and $b$, $\alpha$ and $\gamma$ as auxiliary variables. We then consider (1.3.26), (1.3.23), (1.3.3) and (1.3.24) as definitions, rather than as constraints, for $B^0$, $\gamma^0$, $\rho^0$, $B^1$ and $\gamma^0$, and in the same sense we consider (1.3.13) and (1.3.18) (from the latter only the linearized version) as definitions for $b^0$, $\gamma$ and $B^1$.

Thus, the only relevant constraints are (see (1.3.7) and (1.3.13))

for the intermediate state

$$\alpha_0 \rightarrow \lambda^0 \quad , \quad \xi \in \partial G^0 \; ;$$

$$\varepsilon_{ij} \alpha_{ij} = 0 \quad , \quad |x| \rightarrow \infty \; .$$

for the perturbed state

$$\alpha_1 = \alpha_0 = \lambda_{ij} \quad , \quad \xi \in \partial G^0 \; ;$$

$$\varepsilon_{ij} \alpha_{ij} = 0 \quad , \quad |x| \rightarrow \infty \; .$$

We proceed with a more detailed discussion of the procedure described in section 2. The requirement $\delta L = 0$ yields (1.3.43) yields

$$\varepsilon_{0i} H_{ij} = 0 \quad , \quad \varepsilon_{0i} H_{ij} = \frac{\partial U}{\partial M_i} \; , \quad \varepsilon_{ij} \gamma^0 = 0 \quad , \quad \xi \in \partial G^0 \; ;$$

$$\varepsilon_{ij} (H_{ij} - H_{ij}^P) = 0 \quad , \quad \varepsilon_{ij} (H_{ij}^P - H_{ij}^P) = 0 \quad , \quad \xi \in \partial G^0 \; ;$$

$$\varepsilon_{ij} H_{ij}^P = 0 \quad , \quad x \in \partial G^P \; .$$

Note: In the derivation of the boundary condition (1.3.26) it is used that $(\varepsilon_{ij} H_{ij} \alpha_{ij})$ is continuous across $\partial G^0$; this is because $\alpha^0$ (and, hence, also its tangential derivative) and $(N \times B^0)$ are continuous across $\partial G^0$.

Supplementing (1.4.3) by the constraints (1.4.1) and the definitions (1.3.2) and (1.3.3) and (1.3.24), we obtain a set of equations and boundary conditions known as the Maxwell-Minkowski model, here referring to the intermediate state.
The calculation of δJ, the variation of the functional J with respect to the independent perturbations a, m and u is similar to the derivation of the expression for J, (1.3.43)². Therefore, we only state the result

$$\delta J = \int \left( \epsilon_{ij} \left( \epsilon_{ij} \delta u - \epsilon_{ij} \delta u + \epsilon_{ij} \delta m \right) \right) \partial a_{i} + \left( \epsilon_{ij} \delta \partial a_{i} - \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} + \left( \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} - \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} \partial \delta a_{j} + \frac{1}{2} \int \left( \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} \partial \delta a_{j} \delta a_{j} \partial \delta a_{j} + \left( \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} - \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} \partial \delta a_{j} \delta a_{j} \partial \delta a_{j}$$

Here, J² is the linearization of J² (since we do not need this later on, we refrain from giving an explicit expression for J²). From (1.4.4) we conclude that the requirement δJ = 0 yields the following system of equations and boundary conditions

$$e_{ij} \left( \epsilon_{ij} \delta u - \epsilon_{ij} \delta u + \epsilon_{ij} \delta m \right) \partial a_{i} = 0 \quad \text{at} \quad \partial \delta a_{j} \partial \delta a_{j} \delta a_{j} \partial \delta a_{j}$$

$$\left( \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} - \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} \partial \delta a_{j} \delta a_{j} \partial \delta a_{j} + \left( \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} - \epsilon_{ij} \epsilon_{ij} \delta \partial a_{i} \right) \partial a_{j} \partial \delta a_{j} \delta a_{j} \partial \delta a_{j}$$

Here, the last term in the denominator of the boundary condition (1.4.5)⁵, which arises from the variation with respect to a, the last term in the coefficient of δa_i in (1.4.4) vanishes.

Together with the constraints (1.4.2) and the definitions (1.3.13)⁵, and (1.3.18), the set (1.4.5) amounts to the linearized Maxwell-Minkowski model (cf. [11], section 5.3).

At this stage we have proved the validity of the theory presented in section 2, that is we have shown the equivalence between the variational principle (1.2.10)-(1.2.11), with λ according to (1.3.1), and the Maxwell-Minkowski model.
5. General buckling criterion

In magneto-elastic stability theory it has been the usual procedure to start from a linearized set of equations for the perturbations, such as e.g. (1.4.2), (1.4.5), and to look for a value of the basic field parameter \( B_0 \) for which this set has a non-trivial solution. Since an exact 3-dimensional solution for this set is mostly very difficult, one starts looking for adequate approximate solutions. This is usually done in the following way (confer e.g. [5], [12], [14]; see also [10]), which is of special application for slender bodies:

For a slender body the 3-dimensional displacement \( \mathbf{u} \) is approximated by a 1- or 2-dimensional characteristic displacement parameter \( w \) (e.g. a deflection of a central line or plane of the slender body); this \( w \) is chosen in such a way that it satisfies the global equilibrium equations (i.e. integrated versions of (1.4.5)) together with (1.4.5) \( \alpha \), \( \beta \); \( h^\alpha \) and \( m^\alpha \) are solved from the remaining equations, i.e. (1.4.5) \( \alpha \), \( \beta \) in which \( w \) is replaced by its approximation \( w \); finally, the buckling value is then found as the first eigenvalue for \( B_0 \) for which this solution is unequal to the zero-solution.

However, as the 'solution' obtained by the procedure described above is not an exact solution of (1.4.5), but only a reasonable approximation, the calculated value for \( B_0 \) is also an approximation.

Let us introduce a scalar \( \eta \) (0 \( \leq \eta \leq 1 \)) as a measure for the approximation error in the perturbations; then it is evident that, due to the linear character of the perturbed equations, the error in the eigenvalue for \( B_0 \) is also of the first order in \( \eta \).

In this respect, the use of our variational principle clearly has an advantage over the method described above. For, in our procedure the error in \( B_0 \) is of the second order in \( \eta \). This can be explained best by first describing the main lines of our method. These lines are successively

i) choose a class of trial functions \( \{ \alpha, \mathbf{m}, \mathbf{u}; B_0 \} \) satisfying the constraints (1.4.2);

ii) determine the best member out of this class by setting \( \delta \mathbf{u}/\delta B_0 = \delta \mathbf{m}/\delta B_0 = \delta F/\delta B_0 = 0 \);

iii) calculate the buckling value for \( B_0 \) from the equation \( F = 0 \) (see (1.2.14)).

Due to the stationary behaviour of the quadratic functional \( F \) the deviation between the exact buckling value and the approximated one calculated in iii) is of the order of the square of the deviation between the exact and the approximated perturbations.

The choice of a class of trial functions (point i)) is usually based on a choice of a displacement field. In practice, buckling theory always applies to slender bodies, such as beams or rods, plates and rings. For slender bodies the displacement in buckling can be characterized by one or two global displacement parameters. Examples of such global displacement parameters are the deflection of the central line of a beam or the normal displacement of the central plane of a thin plate. Here, we always shall approximate the 3-dimensional displacement field \( \mathbf{u} \) by the global displacement parameter pertinent to the type of slender body under consideration. Of course, this global displacement has to satisfy the support conditions of the body. As soon as this choice is made, the constraint (1.4.2) \( \alpha \) for a can be made explicit.

Clearly, it is assumed that the intermediate fields are known (note that these are also needed in the formulation for \( \delta \), (1.3.43) \( \alpha \)). In principle these fields can be determined from (1.4.1) and (1.4.3), but if this is too complicated we have also the disposal of a variational principle for the \( \delta \)-state (see (1.3.43) \( \alpha \)). Thus, approximated intermediate fields can be calculated from the
variation \( \delta l = 0 \), if necessary.

In many practical problems, however, the deformations in the \( \xi \)-state have only a negligible influence on the buckling value. In these cases, the intermediate state may be replaced by the so-called rigid-body state. As long as the shape of the body is not too complicated, the determination of these rigid-body fields is rather simple (at least in comparison with the calculation of the perturbations).

In the next two sections more explicit applications of our variational principle will be given for

a) soft ferromagnetic structures,

b) superconductors.

6. Soft ferromagnetic structures

A soft ferromagnetic medium is characterized by a linear relationship between the magnetization and the magnetic field. In this section we shall consider soft ferromagnetic media, which, moreover, are isotropic, homogeneous and linearly elastic. Keeping in mind the note at the beginning of section 3, which states that the internal energy density \( U \) must be a function of \( E \) and \( A \) (see (1.3.5)), we assume \( U \) of the form

\[
U = \frac{E}{2
\rho_0(1 + \nu)} \left( \frac{\nu}{1 - 2\nu} \text{tr}(E^2) + \nu \text{tr}(E^2) \right) + \frac{\text{Poisson}}{\lambda A} \left( A , A \right) .
\]  

(1.6.1)

where \( E \) is Young's modulus, \( \nu \) is Poisson's ratio and \( \chi \) represents the ferromagnetic susceptibility. The first term in (1.6.1) is the elastic energy and the second term the ferromagnetic energy; magnetostrictive energy is not included in this expression.

In the sequel we suppose that the ferromagnetic susceptibility \( \chi \) is so large, that \( \chi^{-1} \) is negligible with respect to unity. As a consequence, all terms containing a factor \( \chi^{-1} \) will be neglected, which in essence implies that the ferromagnetic term in (1.6.1) vanishes. The direct consequences of this are that

\[
H^0 = \nabla^\prime = 0
\]  

(1.6.2)

and that (see (1.3.14) and (1.3.26))

\[
e_{\xi}^{\tau^{\mu\nu}} = 0 , \quad e_{\xi}^{\tau^{0}} = 0 .
\]  

(1.6.3)

Under these restrictions the system for the intermediate state and the explicit expression for \( J \) reduce considerably. From the intermediate state variables only \( B^{0\tau} \) and, eventually, \( \tau^{0} \) are relevant for the rest of this section. Use of (1.6.1), (1.6.2) in (1.4.1), (1.4.3) and (1.3.2) yield

\[
B_{i}^{0} = e_{\xi} \mathcal{A}_{i}^{0} \quad ( \text{or } B_{i}^{0} = 0 , \quad \int_{\mathcal{O}^{0}} B_{i}^{0} N_{i}^{0} dS^{0} = 0 ) ,
\]

\[
e_{\xi} \mathcal{A}_{i}^{0} N_{i}^{0} = 0 , \quad \xi \in \mathcal{O}^{0} ;
\]

\[
\tau_{i,j}^{0,j} = 0 , \quad \tau_{i}^{0} = \frac{\rho_{0}}{1 + \nu} E_{i}^{0} \int_{\mathcal{O}^{0}} \left( \frac{\nu}{1 - 2\nu} \eta_{i}^{0} \eta_{i}^{2} + \eta_{i}^{0} \eta_{i}^{2} \right) \, d\mathcal{O}^{0} .
\]  

(1.6.4)
\[ e_{ijk} \sigma_i^0 \chi_j^0 = 0 \ , \ \tau_i^0 \chi_j^0 = \frac{1}{2 \mu_0} \sigma_i^0 \sigma_j^0 \chi_t^0 \ , \ \chi_i \in \partial \mathcal{G}^0 ; \]

\[ \sigma_i^0 \rightarrow \sigma_i \ , \ | \chi | \rightarrow \infty . \]

Substitution of (1.6.1)- (1.6.3) into (3.4.3) and elimination of \( \mathbf{B}^{\text{ext}} = \mathbf{B}^{\text{ext}}/\mu_0 \) and \( \mathbf{A}^{\text{ext}} \) in favour of \( \mathbf{B}^{\text{ext}} \) (by use of (1.6.4)) and of \( \chi_t^0 \) in favour of \( \mathbf{b}^* \) (by means of the relation \( \mathbf{b}^*_i = e_{ijk} \partial_i \chi_j^0 \)), results in the following simplified expression for \( J \),

\[ J = -\frac{1}{2} \int \left[ T^0 \partial_i \partial_i u_i \right] \frac{E_0}{\mu_0} \frac{E}{2(1+\nu)} \left( -\frac{2\nu}{1-2\nu} B_{ij}^0 B_{ij}^0 + B_{ij}^0 B_{ij}^0 + B_{ij}^0 B_{ij}^0 \right) u_i \partial_i u_i \right] dV^0 + \]

\[ + \frac{1}{4 \omega} \int \left[ \frac{1}{2} \mathbf{B}^0 \cdot \mathbf{B}^0 \right] (\mathbf{B}^0 \cdot \mathbf{B}^0) u_i \partial_i u_i \right] dV^0 + \]

\[ - \frac{1}{2 \omega} \int \mathbf{b}^* \cdot \mathbf{b}^* dV^0, \tag{1.6.5} \]

where \( \mathbf{b}_{ij} \) is the left Cauchy-Green tensor, i.e.

\[ \mathbf{b}_i^0 \equiv F_i^0 F_i^0 . \tag{1.6.6} \]

The intermediate fields are to be calculated from (1.6.4), the only relevant constraints for the perturbations \( \mathbf{b}^* \) and \( \mathbf{u} \) are

\[ \mathbf{b}^*_i = e_{ijk} \partial_i \chi_j^0 , \quad \chi \in \mathcal{G}^0 \,, \quad \left( \text{or } \mathbf{b}^*_i = 0 \right) \quad \int \mathbf{b}^*_i \cdot \mathbf{b}^*_i dV^0 = 0 ; \]

\[ \mathbf{b}^*_i \rightarrow 0 , \quad | \chi | \rightarrow \infty , \tag{1.6.7} \]

possibly supplemented by some kinematical boundary conditions for \( \mathbf{u} \) if the body is supported.

Assuming for a moment that the intermediate fields are known, we have to choose the perturbations \( \mathbf{b}^* \) and \( \mathbf{u} \) out of some admissible class (satisfying the constraints) and to determine the optimal \( \mathbf{b}^* \) and \( \mathbf{u} \) in this class by variation of \( J \). It is not surprising that, if we choose the perturbations out of the complete class of admissible fields, our variation principle will yield an optimal \( \mathbf{b}^* \)-field that is conservative, i.e. a field that satisfies

\[ e_{ijk} \partial_i \chi_j^0 = 0 \,, \quad \chi \in \mathcal{G}^0 . \tag{1.6.8} \]

For every conservative field \( \mathbf{b}^* \) there exists a continuous potential \( \varphi = \varphi (x) \), such that

\[ \mathbf{b}^*_i = \nabla \varphi , \quad \chi \in \mathcal{G}^0 . \tag{1.6.9} \]

Motivated by this result, we now choose the perturbation \( \mathbf{b}^* \) such that it can be expressed in a scalar field \( \varphi (x) \) in the way as in (1.6.9). In order that this is consistent with (1.6.7), \( \varphi \) has to satisfy the constraints

\[ \Delta \varphi = \nabla \varphi , = 0 , \quad \chi \in \mathcal{G}^0 ; \]

\[ \varphi \rightarrow 0 , \quad | \chi | \rightarrow \infty , \tag{1.6.10} \]
\[ \int \frac{\partial \psi}{\partial \mathbf{w}} \, dS^0 = 0. \]

Note that \( \psi \) is not determined by (1.6.10), because the value of \( \psi \) on the boundary has not yet been specified.

The use of (1.6.9) enables us to transform the integral over \( G^0 \) in (1.6.5) into a surface integral over \( \partial G^0 \) by means of Cauchy's theorem. With the use of (1.6.9) and (1.6.11) we, thus, can write (1.6.5) in the form

\[ f = -\frac{1}{4\pi} \int (\mathbf{T}_j u_i u_{ij} + \frac{\mathcal{E}}{\rho_0} \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{R} \cdot \mathbf{R}) \, dV^0 + \frac{1}{2\mu_0} \int (\psi + B_0^a u_k) \frac{\partial \psi}{\partial \mathbf{w}} \right] \frac{\partial B_0^a}{\partial \mathbf{w}} \, dS^0 + \frac{1}{2\mu_0} \int \frac{\partial \psi}{\partial \mathbf{w}} \, dS^0 + \frac{1}{2}\mathbf{B}_0^a \cdot \mathbf{B}_0^a \, \mathbf{u}_i \cdot \mathbf{u}_j \cdot \mathbf{N}^0 \right] \, dS^0. \]

Equation (1.6.11) is valid for \( \psi \) under the constraints (1.6.10) results in

\[ \frac{\partial \psi}{\partial \mathbf{w}} = \frac{1}{2\mu_0} \int (\psi + B_0^a u_k) \frac{\partial \psi}{\partial \mathbf{w}} \, dS^0 + \frac{1}{2\mu_0} \int \frac{\partial \psi}{\partial \mathbf{w}} \, dS^0, \]

where we have used Green's second identity in the form

\[ \int \frac{\partial \psi}{\partial \mathbf{w}} \, dS^0 = 0. \]

This result is rather important because in many problems, especially for slender bodies, our knowledge about the form of the displacements is more extensive than that about the perturbed magnetic field. This means that it is easier to make a reasonable choice for \( u \) than for \( b^a \).

In this concept, however, it is necessary to derive from (1.6.10) and (1.6.14) by given \( u \) an exact solution for \( \psi \). As long as the shape of the body is not too complicated this can be done (as we shall show in the following chapters), but otherwise a different way must be followed. In the latter case we choose a set of trial functions for \( \psi \) out of a class restricted by (1.6.10) and we determine the optimal \( \psi \) by \( \xi_j, \psi = 0 \).

Before we can state an ultimate expression for the buckling value, we have one more step to go. In practice, buckling problems always apply to slender bodies. The buckling problem for a slender body often admits the neglect of the intermediate deformations. In that case we may identify the intermediate state by the undeformed or natural state of the body. Hence \( \xi \rightarrow X \) and
\begin{equation}
G^{\text{ext}} = G^E_0, \quad \partial G^N = \partial G_0, \quad N^E = N, \quad B^0_j = \delta_{ij},
\end{equation}

by which (1.6.9) reduces to

\begin{equation}
J = \frac{1}{2} \int \left( \sum_{k} T_{ik} u_i u_j + \frac{E}{1 + \nu} \left( \nabla^2 \epsilon_{ij} + \epsilon_{ij} \epsilon_{kl} \right) \right) \, dV_0 +
\end{equation}

\begin{equation}
+ \frac{1}{2\mu_0} \int \left( \psi + B_k u_k \right) \frac{\partial \psi}{\partial N} + B_i u_i \frac{\partial \psi}{\partial N} \left( \psi + B_k u_k \right) +
\end{equation}

\begin{equation}
- B_k u_k B_j u_{j,i} N_i + \frac{1}{2} B_k B_j \left( u_{i,j} u_i - u_{i,j} u_i \right) N_j \right) \, dS_0,
\end{equation}

where

\begin{equation}
\epsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right).
\end{equation}

Here \( B := B^F \) and \( T := T^F \) are the rigid-body fields which satisfy

\begin{equation}
\nabla \cdot B = 0, \quad \text{curl} B = 0, \quad x \in G^E_0;
\end{equation}

\begin{equation}
B \times N = 0, \quad x \in \partial G_0;
\end{equation}

\begin{equation}
\int_{\partial G_0} (B \cdot N) \, dS = 0;
\end{equation}

\begin{equation}
B \to B_0, \quad |x| \to \infty
\end{equation}

and

\begin{equation}
T^0_j = 0, \quad x \in G^E_0; \quad T_j N_j = \frac{1}{2\mu_0} (B \cdot B) N_j, \quad x \in \partial G_0.
\end{equation}

From (1.6.18) and (1.6.19) it is evident that the following normalized field quantities

\begin{equation}
\tilde{B} := B / B_0, \quad \tilde{T} := \mu_0 T / B_0^2,
\end{equation}

are independent of \( B_0 \) and the same is true for (see (1.6.10), (1.6.14))

\begin{equation}
\tilde{\psi} := \psi / B_0.
\end{equation}

After having chosen the displacement field \( u \) and the determination of the associated \( \psi \)-field (either exactly or by variation), we proceed with the calculation of \( J \) according to (1.6.16). Then, finally, the buckling value is determined by putting \( J = 0 \), yielding (with the use of (1.6.20), (1.6.21) and omitting the hats)

\begin{equation}
\frac{\mu_0 E}{B_0^2} \left( \int \left( (\psi + B_k u_k) \frac{\partial \psi}{\partial N} + B_i u_i \frac{\partial \psi}{\partial N} \left( \psi + B_k u_k \right) +
\end{equation}
\[ -B_k u_k \rho \mathbf{J}_i \mathbf{u}_j \mathbf{N}_i + \frac{1}{2} B_k B_k \mathbf{u}_j \mathbf{u}_j \mathbf{N}_i \mathbf{dS}_0 + \]

\[ - \int \frac{1}{1 + \nu} \left\{ \left[ \frac{1}{1 - 2\nu} \mathbf{e}_{ij} \mathbf{e}_{ij} \mathbf{dV}_0 \right] + \left[ \mathbf{e}_{ij} \mathbf{e}_{ij} \mathbf{dV}_0 \right] \right\}. \]

(1.6.22)

In this result, the pre-stressed \( T \) still occur. In some cases, e.g. for straight beams, the pre-stresses can be neglected, but a general statement for this is not possible at this stage.

7. Superconductors

The theory of the preceding deals specifically with the case in which a magnetizable body is influenced by an external uniform magnetic field \( B_0 \). However, as we shall show in this section, our general variational principle can be equally well applied to superconductors with a prescribed total electric current \( I_0 \). In that case the buckling value is the value of \( I_0 \), corresponding with the lowest eigenvalue of the general eigenvalue problem of section 2; here \( B_0 \) is replaced by \( I_0 \). Since the analysis runs essentially along the same lines, we shall confine ourselves to pointing out the main differences and giving only the results.

We consider a superconducting body as a non-magnetizable body, for which the current density \( J \) (per unit of length) is concentrated on the surface of the body, and for which the magnetic field \( B \) inside the body vanishes. The current density \( J \) is related to the boundary value of the vacuum field \( B \) by

\[ \mu_0 J = \mathbf{B}^*, \quad x \in \partial G. \]  

(1.7.1)

For reasons of simplicity we only consider one single, simply connected superconductor in a static situation.

Bearing in mind that \( \mu_0 \mathbf{H}^* = \mathbf{B}^* \), \( \mathbf{B}^* = \mathbf{B}_{0}=0 \) we introduce, in analogy with (1.3.1) and (1.3.2), the Lagrangian densities and the constraints as

\[ L^+ = -\frac{1}{2\mu_0} (\mathbf{B}^*, \mathbf{B}^*) , \quad L^- = -\rho U \]  

(1.7.2)

accompanied by the constraints

\[ \mathbf{B}^* = \text{curl} \mathbf{A}^* , \quad x \in G^* ; \]

\[ \mathbf{B}^- = \mathbf{0} , \quad T = \rho \frac{dU}{dT} , \quad \rho J_F = \rho_0 , \quad x \in G^* ; \]

\[ (n, \text{curl} \mathbf{A}^*) = 0 , \quad (\text{or } \mathbf{A}^* = \mathbf{0}) , \quad x \in \partial G ; \]  

\[ \mathbf{B}^* \rightarrow \mu_0 I_0 \mathbf{e}(x) , \quad |x| \rightarrow \infty . \]  

(1.7.3)

where the vector potential \( \mathbf{A} = \mathbf{A}(x) \) assures that \( \mathbf{B}^* \) satisfies (compare with (1.3.4) and appendix B)

\[ \text{div} \mathbf{B}^* = 0 , \quad x \in G^* ; \]
(B, n) = 0, x ∈ ∂Ω ;

(1.7.4)

and c(λ) is an explicit field, independent of the total current I_Ω, that needs to be specified for the particular case in question. In all cases c(λ) tends to zero at infinity. For a straight, infinitely long conductor,

[Equation not visible]

(1.7.5)

The linearization of the constraints (1.7.3) is straightforward and the result is

\[ b_1^+ = c_{i j k} a_{k, j}, \quad x \in G^0 ; \]
\[ b_0^+ = 0, \quad u_j = -T_0^i u_{i, k} + T_0^k u_{i, k} + \rho \delta^i_{0} (1 - u_{i, k}) \delta_{i, k}, \quad \xi \in G^0 ; \]
\[ c^{+}_{i j} = -A^{l}_{i j} u_l, \quad \xi \in \partial G^0 ; \]
\[ b_1^- \rightarrow 0, \quad | x | \rightarrow \infty, \]

(1.7.6)

with the material coefficients \( c_{i j}^{0} \) as given in (1.3.14).

The derivations of \( \delta L \) and \( J \) are analogous to those in section 3. We merely have to apply to (1.4.3) the substitutions

[Equation not visible]

(1.7.7)

and to put equal to zero the fields

[Equation not visible]

[Equation not visible]

[Equation not visible]

[Equation not visible]

where we have also used that the tangential derivative of \( A^{\xi} (= \Theta \text{ on } \partial G^0) \) along \( \partial G^0 \) is zero, or

\[ c_{i j} A^{\xi}_{i j} N_{0} = 0, \quad \xi \in \partial G^0, \]

(1.7.9)

while \( J \) becomes

[Equation not visible]

(1.7.10)
\[ + B^0_k \cdot (\varepsilon_{ijk} u_i - \varepsilon_{ijk} u_i (A^0_k + u_m)) + \frac{1}{2} \varepsilon_{ijk} B^0_k \cdot (u_{ij} u_i - u_{ij} u_i) \] \[ \cdot N^0 \, ds^0. \]

in the derivation of which we have used
\[ N^0 \cdot \nu = (\varepsilon_{ijk} u_i - \varepsilon_{ijk} u_i) = 0, \quad \xi \in \partial G^0, \]
and (1.7.11)

The requirement \( \delta L = 0 \) under the constraints (1.7.3) yield the following set of equations and boundary conditions for the intermediate fields
\[ B^0_k = \varepsilon_{ijk} A^0_j, \quad \varepsilon_{ijk} B^0_k = 0, \quad x \in G^0; \]
\[ \tau_{ij}^0 = 0, \quad \tau_{ij}^0 = \rho \left( \frac{\partial U}{\partial F_{ij}} \right) \cdot F_{ij}^0, \quad \xi \in \partial G^0; \]
\[ A^0_k = 0 \quad (or \ B^0_k = 0), \quad \tau_{ij}^0 \cdot N^0 = -\frac{1}{2\mu_0} B^0_j \cdot B^0_j \cdot N^0, \quad \xi \in \partial G^0; \]  
(1.7.12)

For an isotropic, homogeneous, linearly elastic and non-magnetizable superconductor the internal energy density \( U \) is given by
\[ U = \frac{E}{2\mu_0 (1+\nu)} \left( \frac{1}{1-2\nu} (\text{tr} \, E^2) + \text{tr} \, (B^2) \right). \]  
(1.7.13)

As done in the preceding section, we shall confine ourselves here to conservative fields \( b^0 \), i.e. as in (1.6.9) we introduce a potential \( \psi = \psi (x) \), such that
\[ b^0 = \nabla \psi, \quad x \in G^0. \]
(1.7.14)

In order that this solution is consistent with the constraints (1.7.6), \( \psi \) has to satisfy
\[ \Delta \psi = 0, \quad x \in G^0; \]
\[ \frac{\partial \psi}{\partial N} = (B^0_k \cdot u_i - B^0_k \cdot u_i)N^0, \quad x \in \partial G^0; \]  
(1.7.15)

With (1.7.14) and (1.7.15) the integral over \( G^0 \) in (1.7.10) can be transformed into a surface integral, as follows,
\[ \int_{\sigma^0} b^0 \cdot \nu \, d\sigma = \frac{\partial \psi}{\partial N} \, d\sigma = \int_{\partial G^0} \psi (B^0_k \cdot u_i - B^0_k \cdot u_i)N^0 \, d\sigma^0. \]  
(1.7.16)

NOTE: We note that the potential \( \psi \) is completely determined by (1.7.15) (this in contrast to the potential \( \psi \) in section 6 which still was free on the boundary \( \partial G \)). Hence, if we confine ourselves to conservative \( b^0 \) (i.e. to (1.7.14)), the potential must be solved from (1.7.14), and there is no degree of freedom left or a determination by variation.

Confining ourselves to conservative \( b^0 \), neglecting the influence of intermediate deformations
and using the \( I_0 \)-independent variables
\[
\hat{B} := B^0 / \mu_0 I_0 \quad \hat{A} := A^0 / \mu_0 I_0 \quad \hat{T} := T^0 / \mu_0 I_0^3 \quad \hat{\psi} := \psi / \mu_0 I_0
\]
(1.7.17)
in the buckling equation \( J=0 \), we obtain analogously to the preceding section the following formula for the buckling value \( I_0 \) (omitting the hats)
\[
\frac{\bar{E}}{\mu_0 I_0} = \left( \psi \left( B_{ij} u_{ij} - \frac{\bar{T}}{\bar{\psi}} u_{ij} \right) + B_k B_{ij} u_{ij} - e_{ijm} B_m A_{ij} u_{ik} u_{kj} + \frac{2}{3} B_k \left( e_{ijm} u_{ij} + e_{ijk} u_{ik} \right) - \frac{1}{2} B_k B_k \left( u_{ij} - u_{kl} u_{ij} \right) \right) \mathcal{N}_0 dS_0 + \frac{1}{4+\nu} \left[ \frac{\nu}{1-2\nu} \bar{e}_{ij} - \bar{e}_{kl} dV_0 \right]^{1/2}
\]
(1.7.18)
where the tensor \( e_{ij} \) is the same as defined in (1.6.17).

The variational principle based upon (1.7.2)-(1.7.3) can be used to solve the buckling problem of the superconductor. But then, as already mentioned in the note following (1.7.16), there is no freedom left for variation of the magnetic potential, which is awkward in numerical applications. This difficulty can be smoothed over using a Legendre transformation, or transformation in the reciprocal form (see [15], Ch. IV, §9). The formal procedure for Legendre transformation is as follows. Firstly, we pass from the variable \( B \) to the variable \( M \) defined by (see [15], Ch. IV, (87))
\[
M = \frac{1}{\mu_0} \frac{1}{\partial B} \left[ \left( \frac{1}{2} \partial B \right) \cdot \left( \frac{1}{2} \partial B \right) \right]
\]
(1.7.19)
Of course, \( M \) is the magnetic field intensity. Secondly, we add a term \( (H, B) \) to the Lagrangian density \( L^* \) and, thirdly, we replace the constraints \( \text{div} B = 0 \) and \( (B, n) = 0 \) by the constraint \( \text{curl} H = 0 \). As in our case \( B \) and \( H \) only differ a fixed multiplicative constant \( \mu_0 \), we can hold on to \( B \) as our fundamental variable. The Legendre transformation then amounts to a change of sign in the outer Lagrangian density,
\[
L^* = \frac{1}{2\mu_0} (B, B) \quad L^- = -\rho \psi
\]
(1.7.20)
while the constraints become
\[
B = 0 \quad T = \rho \psi \frac{d}{dF} \frac{d}{dF} \quad \rho J_p = \rho_0 \quad x \epsilon G^-;
\]
\[
\text{curl} B = 0 \quad x \epsilon G^+; B \rightarrow 0 \quad |x| \rightarrow \infty.
\]
(1.7.21)
As an extra constraint we prescribe the total current \( J_0 \) by means of Ampère's law, i.e.
\[
\oint_C (B \cdot \hat{t}) ds = \mu_0 I_0
\]
(1.7.22)
where \( C \) is a contour entirely in the vacuum and \( t \) is the tangent vector at \( C \). The contour \( C \) must be chosen suitably for the specific problem at hand.
In the same way as done in the preceding, we can formulate on the basis of (1.7.20)-(1.7.22) a variational principle that can be used in the study of the buckling problem for a superconducting body. Note that the inner Lagrangian density $L^\ast$ in (1.7.20) is equal to the one in (1.7.2), which at its turn is equal to $L^\ast$ according to (1.3.1), provided that in the latter the internal field $H$ and the field at infinity $B_0$ are taken equal to zero. Hence, we can adopt the calculation of the difference $L^\ast-L^0$ of the inner Lagrangians in section 3. Putting equal to zero all internal magnetic field quantities and $B_0$ we obtain from (1.3.27) (with omission of the upperindices $^\ast$)

$$L^\ast-L^0 = \delta L^\ast + J^\ast + O(\epsilon^2) ,$$  \hspace{1cm} (1.7.23.1)

where

$$\delta L^\ast = \oint \frac{1}{2} T_{ij} u_i dV - \int \frac{1}{2\mu_0} \delta G \delta N_i u_i dS ,$$  \hspace{1cm} (1.7.22.2)

and

$$J^\ast = -\frac{1}{2} \oint \frac{\rho c_{ij} \epsilon_i u_k u_j dV}{\mu_0} .$$  \hspace{1cm} (1.7.23.3)

Since the outer Lagrangian densities in (1.7.2) and (1.7.20) only differ in their signs, we can use the calculation of $L^\ast-L^0$ in section 3. Taking $\beta=0$, $H=0$, $\mu_0$ and $c_{ij}=\theta_i$ in (1.3.39), and multiplying its right-hand side by -1, we obtain

$$L^\ast-L^0 = \frac{1}{2\mu_0} \oint b_j B_j dV - \frac{1}{2\mu_0} \oint b_k B_k u_i N_i dS$$

$$- \frac{1}{\mu_0} \oint \left[ \frac{1}{2} b_k B_k u_i + \frac{1}{2} b_{ij} B_k u_i u_j + \frac{1}{4} b_k B_k (u_i u_j - u_j u_i) \right] N_i dS$$

$$+ \frac{1}{2\mu_0} \oint b_k b_j dV .$$  \hspace{1cm} (1.7.24)

Since the constraints (1.7.21)-(1.7.22) have to be satisfied for both the intermediate and the present state, the constraints for the perturbations are

$$b = \theta , \quad t_j = -T_{ij} u_k + T_{is} u_j + \rho c_{ij} \epsilon_i u_k , \quad \xi \in G^- ;$$

$$\text{curl} b = 0 , \quad x \in G^+ : b \to 0 , \quad |x| \to \infty .$$  \hspace{1cm} (1.7.25)

and

$$\oint (b, \tau) dS = 0 .$$  \hspace{1cm} (1.7.26)

The constraints (1.7.25) and (1.7.26) guarantee the existence of a continuous potential $\psi(x)$, such that

$$b = \nabla \psi , \quad x \in G^+ .$$  \hspace{1cm} (1.7.27)

To dispose of irrelevant constants in $\psi$, we replace (1.7.25) by the constraint

$$\psi \to 0 , \quad |x| \to \infty .$$  \hspace{1cm} (1.7.28)

Addition of (1.7.23.1) and (1.7.24), after use of (1.7.27), yields
\[ L - L^0 = \delta L + J + O(\varepsilon^3) , \quad \text{(1.7.29.1)} \]

where
\[
\delta L = \int \nabla_i u_i \, dV - \int_{\partial D} \left( \nabla_j N_j + \frac{1}{2\mu_0} B_k B_k N_i u_i \right) dS + \frac{1}{\mu_0} \int_{\partial D} \nabla \cdot B_i dV . \quad \text{(1.7.29.2)}
\]

and
\[
J = -\frac{1}{2} \int_{\partial D} \rho \varepsilon_{ij} u_i u_j dV \]
\[
- \frac{1}{\mu_0} \int_{\partial D} \left( \nabla \cdot B_i B_i + \frac{1}{2} \nabla_j B_i B_k u_i u_j + \frac{1}{2} B_k B_k (u_i u_j - u_j u_i) \right) N_i dS
\]
\[- + \frac{1}{2\mu_0} \int_{\partial D} \nabla \cdot \nabla \psi dV . \quad \text{(1.7.29.3)}
\]

The Legendre transformation ensures us that our variational principle is equivalent to the preceding variational principle. This means that variation of \( L \) and \( J \) results in sets of equations fully describing the magneto-elastic buckling of the superconducting body. Of course, it is also possible to verify this directly. Using Gauss' divergence theorem in the last term of (1.7.29.2) we can show that the variation \( \delta L \) of \( L \) is equal to
\[
\delta L = \int \nabla_i u_i \, dV - \int_{\partial D} \left( \nabla_j N_j + \frac{1}{2\mu_0} B_k B_k N_i u_i + \frac{1}{\mu_0} \nabla \cdot B_i dS
\]
\[- - \frac{1}{\mu_0} \int_{\partial D} \nabla \cdot \nabla \psi dV . \quad \text{(1.7.30)}
\]

Variation of \( L \), i.e. the requirement \( \delta L = 0 \) for all \( u \) and \( \psi \), yields the remaining intermediate equations
\[
T_{ij,j} = 0 , \quad \xi \in G^- ;
\]
\[
B_i N_j = 0 , \quad \nabla_j N_j + \frac{1}{2\mu_0} B_k B_k N_i = 0 , \quad \xi \in \partial G^- ;
\]
\[
B_{i,j} = 0 , \quad \xi \in G^+ .
\]

With use of the same instruments as before, e.g. Gauss' divergence theorem and the lemma following (1.3.35), it is possible to show that variation of \( J \) yields the remaining perturbed equations (compare with the results of section 4)
\[
\xi_{ij} - T_{ij,k} u_{k,j} = 0 , \quad \xi \in G^- ;
\]
\[
(\xi_{ij} - T_{ij,k} u_{k,j}) N_j - (T_{i,k} - \xi_{i,k}) u_{i,j} N_j - T_{ij,k} u_{k,j} N_j = 0 ,
\]
\[
\psi_{,i} N_i + \theta_{ij} u_{i,j} N_i - B_j u_{i,j} N_i = 0 , \quad \xi \in \partial G^- ;
\]
\[
\psi_{,i} = 0 , \quad \xi \in G^+ .
\]

In order to obtain a suitable form of the buckling equation \( J = 0 \), we rewrite expression (1.7.29.3). We assume that the superconducting body is isotropic, homogeneous and linearly elastic. As
before, we neglect intermediate deformations, and thus identify the intermediate configuration
and the undeformed or natural configuration of the body. Then (cf. (1.6.16))

\[ p \epsilon_{ijkl} u_{ik} u_{lj} = T_{ij} u_{ij} u_{ij} + \frac{E}{1+v} \left[ \frac{1}{1-2v} \epsilon_{ijkl} \epsilon_{ijkl} \right]. \quad (1.7.33) \]

where \( E \) is Young’s modulus, \( v \) is Poisson’s ratio and (cf. (1.6.17))

\[ \epsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right). \quad (1.7.34) \]

Furthermore, we rewrite half of the second term in the right-hand side of (1.7.29.3) as follows
(with the aid of the lemma following (1.3.33) and (1.7.31)): \( ^{2}\)

\[ \begin{align*}
-\frac{1}{2\mu_0} \int_{G} \psi B_{k} u_{k} N_{i} dS
&= \frac{1}{2\mu_0} \left( \left( \psi B_{i} u_{j} \right)_{,j} - \psi B_{i,j} u_{j} - \psi B_{i} u_{j,j} \right) N_{i} dS \\
&= \frac{1}{2\mu_0} \int_{G} \left( B_{i,j} u_{j} - B_{i} u_{j,j} \right) \psi N_{i} dS. \\
(1.7.35)
\end{align*} \]

Substituting (1.7.33), (1.7.35) into (1.7.29.3) we obtain

\[ \begin{align*}
J &= -\frac{1}{2} \int_{G} \left[ T_{jk} \epsilon_{ij,k} \epsilon_{ij,k} + \frac{E}{1+v} \left[ \frac{1}{1-2v} \epsilon_{ijkl} \epsilon_{ijkl} \right] \right] dV \\
&\quad -\frac{1}{2\mu_0} \int_{G} \left( \psi \left( B_{i,j} u_{j} + \left( B_{i,j} u_{j} - B_{i} u_{j,j} \right) \psi + B_{i} B_{i} u_{j} u_{j} + \frac{1}{2} B_{i} B_{i} \left( u_{j,j} - u_{j,j} \right) \right) \right) \\
&\quad + \psi \psi_{,j} N_{i} dS - \frac{1}{2\mu_0} \int_{G} \psi \psi_{,k} dV. \\
(1.7.36)
\end{align*} \]

For the first term of the first integral we use Gauss’ divergence theorem together with (1.7.31): \( ^{3}\)

On account of curl \( B = 0 \) we have

\[ B_{i,j} = B_{i,j}, \quad (1.7.37) \]

which we apply to the third term in the second integral in (1.7.36). Rearranging terms, we finally
arrive at the identity

\[ \begin{align*}
J &= -\frac{1}{2} \int_{G} \left[ T_{jk} \epsilon_{ij,k} \epsilon_{ij,k} + \frac{E}{1+v} \left[ \frac{1}{1-2v} \epsilon_{ijkl} \epsilon_{ijkl} \right] \right] dV \\
&\quad -\frac{1}{2\mu_0} \int_{G} \left( \psi \left( B_{i,j} u_{j} + \frac{1}{2} B_{i} B_{i} \left( u_{j,j} - u_{j,j} \right) \right) \right) \\
&\quad -\frac{1}{2\mu_0} \int_{G} \left( \psi \left( B_{i,j} u_{j} + \left( B_{i,j} u_{j} - B_{i} u_{j,j} \right) \psi + \frac{1}{2} B_{i} B_{i} \left( u_{j,j} - u_{j,j} \right) \right) \right) \psi_{,j} N_{i} dS - \frac{1}{2\mu_0} \int_{G} \psi \psi_{,k} dV. \\
(1.7.38)
\end{align*} \]

In order to dispose of the integral over the infinite region \( G^{*} \), we impose (1.7.32) as an extra
constraint on \( \psi \), so that the constraints for \( \psi \) now are

\[ \Delta \psi = 0, x \in G^{*}; \psi \rightarrow 0, |x| \rightarrow \infty. \quad (1.7.39) \]

In contrast to (1.7.15), \( \psi \) is not completely determined by (1.7.39), so that there is still freedom
for variation. We introduce the normalized variables
\[ \frac{\delta}{\beta} = \frac{2\beta a}{\beta_0}, \quad \frac{\delta}{\beta} = \frac{(2\beta a)^2}{\beta_0\beta}, \quad \frac{\delta}{\beta} = \frac{2\beta a}{\beta_0} \psi, \]
(1.7.40)

where \( a \) is some length parameter, which has to be chosen suitably for the problem under consideration. Thus the buckling equation \( J = 0 \) yields (immediately omitting the hats)
\[
\frac{E}{\mu_0} \frac{(2\beta a^2)^2}{\beta_0} = \left\{ \left[ \frac{1}{2} \left( B_k B_k + B_k B_k (u_{ij} + u_{ij}) - B_k B_k u_{ij} u_{ij} \right) \right]
+ \frac{1}{2} B_k B_k (u_{ij} + u_{ij}) u_{ij} - (u_{ij} + B_k B_k u_{ij} - B_k B_k u_{ij}) \psi \right\} + \int_G \beta_{ij} u_{ij} u_{ij} dV
\]

\[
\left[ \frac{1}{1 + \nu} \left( \frac{\nu}{1 - 2\nu} \sigma_{xx} \sigma_{xx} + \sigma_{yy} \sigma_{yy} \right) \right]^{-1}
\]
(1.7.41)

We have already noted that this formula for the buckling value \( J_0 \) is especially useful when it is difficult to determine \( \psi \) exactly. If, however, we are able to calculate \( \psi \) exactly, as we shall do in the following chapters, then the last term in the first integral in (1.7.41) drops out, and formula (1.7.41) becomes equivalent to (1.7.18) (see appendix A). It is in this latter form that we shall use the above relation in chapter 3.

8. Discussion

In the preceding sections we have derived on the basis of a variational principle explicit expressions for the magneto-elastic buckling value for two special cases, namely for a soft ferromagnetic structure and for a superconductor. Although in our opinion these two cases are from a practical point of view also the most important cases, we note that still other applications are possible. For instance, if electrical fields do play a role, one has to supplement the Lagrangian density \( L \) by electric fields, yielding (\( E \) is the electric field strength and \( P \) the polarization, see appendix C)
\[
L = \frac{1}{2} \epsilon_0 \left( E, E \right) - \frac{1}{2} \mu_0 \left( H, H \right) + \rho \left( P, E \right) - \rho U.
\]
(1.8.1)

Moreover, with a few adjustments, the principle can also be applied to non-linearly magnetic or to magnetically saturated media. Further possible extensions are to systems of several bodies, to bodies with internal interfaces (singular surfaces) or to infinite, but periodically supported bodies, such as rods, beams or plates.

In this chapter particularly the basic theory, resulting in the two expressions: (1.6.22) for \( B_{0,w} \) and (1.7.16) or (1.7.41) for \( J_{0,w} \), is presented. Specific applications to concrete systems will be given in the following chapters. Essentially this amounts to solving the problem for \( \psi \). In these following chapters, the buckling values will be calculated for systems of two parallel rods, both for the case that the rods are soft ferromagnetic and placed in a uniform magnetic field, as well as for the case of two superconducting rods with prescribed total current, and for systems of two superconducting tori.
CHAPTER 2

A variational approach to magneto-elastic buckling problems for systems of ferromagnetic or superconducting beams

1. Introduction

In chapter 1 we derived an explicit relation for a magneto-elastic buckling value by way of a variational principle. This relation was accompanied by equations and boundary conditions for both the intermediate (i.e., pre-buckled or rigid-body) fields and for the perturbed (due to buckling) fields. These fields must be solved first, and then mere substitution of the results into the expression for the buckling value immediately yields an explicit value for the critical or buckling field. In chapter 1 detailed evaluations were given for (i) soft ferromagnetic bodies, and (ii) superconductors.

We start here with recapitulating the main results of chapter 1. Firstly, for a soft ferromagnetic body in vacuum placed in a uniform field of field strength $B_0$, one has for the critical value of $B_0$ the relation (cf. (1.6.22); for the definitions of the symbols we refer to chapter 1)

$$\frac{\mu_0}{B_0^2} = \left\{ \left[ \frac{(\psi + B_k u_k) \frac{\partial}{\partial N} + B_k \frac{\partial}{\partial N} (\psi + B_k u_k)}{\partial \psi} \right] \right. - B_k u_k B_j u_j N_j + \frac{1}{2} B_k B_k (u_{ij} u_i - u_{ij} u_j) N_i \right\} dS \tag{2.1.1}

- \left[ T_{ik} u_i \frac{\partial}{\partial N} \right] \left\{ \frac{1}{1 + \nu} \right\} \left\{ \frac{1}{1 + \nu} \right\} \left\{ \frac{1}{1 - 2 \nu} e_{ik} e_{ii} + e_{ik} e_{ij} \right\} dV \right]^{-1} .

In this expression $B$ and $\psi$ are the normalized magnetic induction in the vacuum $G^*$ and the normalized stress tensor in the rigid-body state, which have to satisfy (cf. (1.6.18)-(1.6.21))

$$\text{div} B = 0 , \quad \text{curl} B = 0 , \quad x \in G^* ; \quad B \times N = 0 , \quad x \in \partial G ; \tag{2.1.2}

$$\int_{\partial G} (B \cdot N) dS = 0 ; \quad B \rightarrow B_0 \quad \text{as} \quad x \rightarrow \infty ,$$

and

$$T_{ij} = 0 , \quad x \in G^* ; \quad T_0 N_j = \frac{1}{2} (B \cdot B) N_j , \quad x \in \partial G \ . \tag{2.1.3}$$

Note that $T$ is not completely determined by (2.1.3), but this will do for our purposes. Moreover, since we have identified the intermediate state with the rigid-body state, there is no need anymore to distinguish between Lagrange and Euler coordinates.

The field $\psi$, occurring in (2.1.1), is the normalized perturbed magnetic potential, due to the deflection $u$ in buckling. For $\psi$, we have derived in chapter 1 the relations (cf. (1.6.10), (1.6.14))
The displacement field \( u \) must be chosen as such that it constitutes a reasonable representation for the deflection in buckling for the (mostly slender) body under consideration. Clearly, this choice can only be made after the shape of the body (e.g. a plate or a beam) is known. In the next section this will be made explicit for the case of a slender beam. The linear deformations \( \varepsilon_u \) are related to \( u \) by

\[
\varepsilon_u = \frac{1}{2} (u_{i,j} + u_{j,i}) .
\]

Whenever we can succeed in solving (2.1.2)-(2.1.4) and make an acceptable choice for \( u \), we only have to substitute the results in (2.1.1) to obtain a numerical value for the buckling field magnitude (in this case \( \Phi \)). It is this procedure that we shall follow in this chapter.

Secondly, we proceed with the recapitulation of the analogous results for a superconducting structure with total electric current \( I_0 \). For the critical current we have derived (cf. (1.7.18))

\[
\frac{E}{\mu_0 I_0} = \left( \lambda \right) = \left( \begin{array}{c}
\frac{E}{\mu_0 I_0} = \left( \begin{array}{c}
\int \left[ \gamma \left( B_{i,j} u_{i,j} - B_{i,j} u_j \right) + B_k B_{k,l} u_l u_j - \varepsilon_{i,j} \frac{B_m}{B_m} A_{j,l} u_k u_l + B_k B_\perp u_j + u_j u_k \right] \right] N_i \, dS + 2 B_k \left( \varepsilon_{k,i} u_i - \varepsilon_{k,j} u_j \right) A_{j,i} u_k u_l \right) \times 1 + \left( \frac{1}{B_k B_\perp u_j} \left( \varepsilon_{i,j} u_i + \varepsilon_{j,i} u_j \right) \right) \times dV \right)
\end{array} \right),
\]

while the constraints here are (cf. (1.7.12), (1.7.15))

\[
\gamma \left( \right) = \left( \begin{array}{c}
\gamma \left( \right) = \left( \begin{array}{c}
\text{div } B = 0 \right) , \varepsilon_{i,j} B_{k,j} = 0 \left( \text{ or curl } B = 0 \right) , x \in G^+ ;
\end{array} \right),
\end{array} \right),
\]

\[
A = 0 \left( \text{ or } \left( B, N \right) = 0 \right) , x \in G^+ ;
\]

\[
B \to \Phi (x) , \quad |x| \to \infty ,
\]

and

\[
\gamma \left( \right) = \left( \begin{array}{c}
\gamma \left( \right) = \left( \begin{array}{c}
T_{(x)} = 0 , x \in G^- ; T_\perp N_i = -\frac{1}{2} \left( B_i B_i \right) N_i , x \in G^- .
\end{array} \right),
\end{array} \right),
\]

and for the perturbed potential \( \psi \),

\[
\Delta \psi = 0 , x \in G^+ ; \quad \frac{\partial \psi}{\partial N} = (\gamma_{i,j} u_{i,j} - \gamma_{i,j} u_j) N_i , x \in G^+ ;
\]

\[
\psi \to 0 , \quad |x| \to \infty .
\]

In the next section, the above results will be further elaborated for the special case of an infinitely long beam, which is periodically supported. In section 3, explicit buckling values are calculated for one beam of arbitrary cross-section. The third section also serves as a first acquaintance with the mathematical methods that will be used in section 4 to solve the buckling problem for a set of two parallel rods. Buckling values are calculated for both ferromagnetic and superconducting
rods. In the final section we present some special results and we compare our results with those following from a more simplified approach, based upon a generalization of the law of Biot and Savart.

2. The slender beam

Consider an infinitely long beam of arbitrary cross-section. The beam is periodically supported (simply supported or clamped), the distance between the supports being \( l \). Let \( R \) be a characteristic length for the cross-section. Then, the beam is called slender if \( R/l \ll 1 \). A coordinate system \((O, e_1, e_2, e_3)\) is chosen with the \( e_2 \)-axis along the central line of the beam, and the \( e_1 \)- and \( e_3 \)-axes in the plane of the cross-section \( D^* \) along the principle axes of inertia. It is assumed that in buckling the beam deflects in the \( e_1 \)-direction. We denote the deflection of the central line of the beam in the \( e_1 \)-direction by \( w(z) \). In accordance with Bernoulli's theory for the bending of slender beams, we then choose the displacement field in an arbitrary point \((x, y, z)\) of the beam as

\[
\begin{align*}
    u_1 &= w(z) + \frac{1}{2} \nu (x^2 - y^2) w''(z) , \\
    u_2 &= \nu xy w''(z) , \\
    u_3 &= -x w'(z) ,
\end{align*}
\]

(2.2.1)

where \( \nu \) is Poisson's ratio and \( :=d/dz \).

The results recapitulated in section 1, have been derived in chapter 1 under the restriction that the body is of finite dimension. In the above example, however, this is no longer true. We can avoid this discrepancy by assuming that the fields are periodic in the \( x \)- or \( e_3 \)-direction with period \( p \) (\( p \) is related to \( l \), but does not need to be equal to \( l \), and depends on the type of support). In this case it is allowed to replace in (2.1.1) and (2.1.6) the infinite region \( G^- \) with boundary \( \partial G \) by the finite parts for one period. Considering a final cross-section, separating two periods, we notice that the contributions due to points just before and just after this cross-section cancel each other. Hence, the unacted part of \( \partial G \) only consists of the lateral surface of the beam. Therefore, from now on one must read for \( G^- \) the finite domain of one period, say \( z \in (0, p) \), and for \( \partial G \) the lateral surface of \( G^- \).

We now are able to evaluate the integral in the denominator of the right-hand side of (2.1.1) and (2.1.6). Since this integral represents the elastic energy of the beam, it is not surprising to find that (2.2.1) implies that this term is equal to the classical energy for a slender beam in bending (apart from a factor \( E/2 \)), i.e.

\[
\frac{1}{1 + \nu} \int \left( \frac{1}{1 - 2\nu} \varepsilon_{tt} \varepsilon_{tt} + \varepsilon_{tt} \varepsilon_{tt} \right) dV = l_z \int_0^p w'^2(z) dz ,
\]

(2.2.2)

where

\[
l_z = \int_0^x x^2 dS ,
\]

(2.2.3)

the moment of inertia about the \( y \)-axis. Note that in the derivation of (2.2.2) it is used that

\[
[R^*w^{(0)}(z)] = O \left( \frac{R^*l^2}{l^2} \right) \|w(z)\| ,
\]

(2.2.4)

and that \( O \left( \frac{R^*l^2}{l^2} \right) \) terms are neglected with respect to unity.
We assume that the bias field \(B_0\) for the ferromagnetic beam is perpendicular to the \(e_3\)-axis, and that for the superconducting beam the unperturbed current runs in the \(e_3\)-direction. For both cases, the problem for the rigid-body field is then purely two-dimensional, i.e. \(B = B(x, y)\) and \((\mathbf{B}, e_3) = 0\). The problem for the perturbed potential \(\psi\) can be reduced to a two-dimensional problem by the separation of variables

\[
\begin{align*}
\psi(x, y, z) = & -\psi_0 + \phi(x, y)w(z), & (F) \\
\psi(x, y, z) = & \phi(x, y)w(z), & (S).
\end{align*}
\]

(2.2.5)

NOTE We try, whenever possible, to treat the ferromagnetic and the superconducting case simultaneously. However, when distinction is necessary, we label the ferromagnetic relations with a suffix \((F)\) and the superconducting ones with \((S)\).

The separation according to (2.2.5) is only then consistent with the constraint \(\Delta \psi = 0\) if \(w(z)\) satisfies the relation

\[
w''(z) + \lambda^2 w(z) = 0,
\]

(2.2.6)

where the real parameter \(\lambda\) is a separation constant, which is related to \(l\) through the support conditions (e.g. for a cantilever \(\lambda = \pi/2l\), and for a simply supported beam \(\lambda = \pi/l\)). The parameter \(\lambda\) is proportional to and always of the same order as \(l^{-2}\) and, hence, the parameter \(\delta\) defined by

\[
\delta = \lambda R \quad (= O(K/l) \ll 1) .
\]

(2.2.7)

is very small. Note that \(\delta\) is a measure for the slenderness of the beam. With (2.2.5) and (2.2.6), the constraint \(\Delta \psi = 0\), for \(x \in G^+\), transforms into the following constraint for \(\phi\)

\[
\Delta \phi(x, y) = \lambda^2 \phi(x, y), \quad (x, y) \in D^+ .
\]

(2.2.8)

where, now, \(\Delta\) is the two-dimensional Laplace operator and \(D^+\) is the domain of the vacuum in the \(e_1-e_2\)-plane.

3. Buckling values for a single ferromagnetic or superconducting beam

Consider a slender beam as described in the preceding section. For the ferromagnetic case the beam is supposed to be placed in a uniform magnetic field \(B_0\). The basic field \(B_0\) is directed in the \(e_3\)-direction, which is taken as the axis of lowest bending stiffness. The deflection in buckling is then indeed in the \(e_3\)-direction. This also holds true for the superconducting beam. At this stage we can eliminate a small inconvenience in our formulation. The normalized fields \(\hat{B}, \hat{T}\) and \(\hat{\psi}\) (see (1.7.17)) are not dimensionless. Therefore, we introduce new dimensionless normalized variables by \((l\ell = \text{a characteristic measure for the cross-section})\)

\[
\hat{B} = \frac{2\pi R}{\mu_0 I_0} B, \quad \hat{A} = \frac{2\pi R}{\mu_0 I_0} A, \quad \hat{\psi} = \frac{2\pi R}{\mu_0 I_0} \psi, \quad \hat{T} = \frac{2\pi R}{\mu_0 I_0} T, \quad (S) .
\]

(2.3.1)

This normalization also implies that the normalized pre-stresses \(\hat{T}\) are of the same order of magnitude with respect to the small parameter \(\delta\) as the magnetic components \(\hat{B}\). There are only two minor changes due to this modification. Firstly, in the left-hand side of (2.1.6) we must replace
and secondly, with the current \( I_0 \) in the \( e_3 \)-direction, the constraint at infinity \((2.1.7)^4\) can be made explicit, yielding (see also \((1.7.5)\)) (omitting the hats from now on)

\[
B = \frac{R}{|x|} (-\sin \theta e_1 + \cos \theta e_2) , \quad |x| \to \infty , \quad (S).
\]

where \(|x| = (x^2 + y^2)^{1/2}\) and \(\theta\) is the polar angle.

We proceed with an evaluation of the numerator of the right-hand sides of \((2.1.1)\) and \((2.1.6)\). The magnetic vector potential \(A\) for the rigid-body problem is of the form \(A = A(x, y) e_3\), yielding \(B = B(x, y)\) and \((B, e_3) = B = 0\). Moreover, we assume that the supports of the beams are such that the stresses in the \(e_3\)- or \(z\)-direction are zero, i.e.

\[
T_{zz} = T_{zz} = T_{za} = 0 .
\]

Together with the above results, we use the constraint relations for \(B\) and \(w\) of section 1, and the equations \((2.2.1)\), \((2.2.2)\), \((2.2.5)\) and \((2.2.6)\) of section 2. Finally, for the sake of simplicity we neglect in the superconducting case \((S)\) lateral contraction (i.e. \(v = 0\)). This results in the following asymptotic relations for the buckling values, deduced from \((2.1.1)\) and \((2.1.6)\),

\[
\frac{\mu_0 EL^2 \lambda^2}{B_0^2} = \int \frac{1}{h} \frac{\partial}{\partial x} \left( \frac{\partial (\phi + B_x)}{\partial x} \right) ds + O(\delta^2) , \quad \delta \to 0 , \quad (F),
\]

\[
\frac{4\pi^2 EL^2 \lambda^4}{\mu_0 B_0^2} = \int [ - (\phi + B_x) \frac{\partial B_x}{\partial x} = \frac{\partial}{\partial y} (B_x + B_y) N_x ] ds
\]

\[
+ O(\delta^2) , \quad \delta \to 0 , \quad (S).
\]

We note that, due to \((2.1.4)^4\), the (irrelevant) constant \(w_0\) does not contribute to \((2.3.5.1)\) (in fact, the condition for \(w\) at infinity implies \(w_0 = 0\)). Moreover, we make the convention that any term in the subsequent analysis of the form \(O(\delta^4 \log^2 \delta)\) will be referred to as an \(O(\delta^4)\)-term.

For a complete solution we still need \(B\) and \(w\). The intermediate field \(B\) can be solved from \((2.1.2)\) or \((2.1.7)\), whereas the perturbed potential \(\phi = \phi(x, y)\) has to satisfy

\[
\Delta \phi = \lambda^2 \phi , \quad (x, y) \in D^+ ; \quad \phi \to 0 , \quad x^2 + y^2 \to \infty ;
\]

\[
\phi + B_0 = 0 , \quad (P), \quad \frac{\partial}{\partial N} (\phi + B_x) = 0 , \quad (S) , \quad (x, y) \in \partial D.
\]

It will turn out (see \((2.3.12)\)) that the leading terms in the right-hand sides of \((2.3.5)\) are of \(O(1)\) with respect to \(\delta\), for \((P)\), and \(O(\delta^2)\), for \((S)\), which means that the higher order terms in \((2.3.5)\) are indeed negligible.
Let the region $D^-$, occupied by the cross-section of the beam in the $x$-$y$-plane, be finite, simply connected and sufficiently regular (so that all of the manipulations that follow are allowed). Furthermore, let $z$ be the normalized complex variable
\[ z = (x + iy)/R, \] (2.3.7)
and $S^-$, $S^+$ and $C$ the regions in the $z$-plane corresponding with $D^-$, $D^+$ and $\partial D$, respectively. Then, there exists exactly one conformal mapping
\[ z = h(u) \] (2.3.8)
from the region $\{ u \mid |u| > 1 \}$ in the complex $u$-plane onto $S^+$, such that
\[ h(-1) = -A, \quad h(1) = B, \quad h(\infty) = \infty, \]
where $-A$ and $B$ are the intersections of the boundary $C$ of $S^+$ with the negative and positive real axis in the $z$-plane, respectively (see Fig. 2.1). For this conformal mapping, the number $\varepsilon$ defined by
\[ \varepsilon = \lim_{u \to \infty} |h(u)/u| = \lim_{u \to \infty} |h'(u)|, \] (2.3.9)
is finite and positive. Moreover, it is assumed that the cross-section is sufficiently regular providing that
\[ \varepsilon = O(1) \quad \text{and} \quad 1/\varepsilon = O(1), \quad \delta \to 0, \] (2.3.10)
implying that
\[ \lambda c = \delta \varepsilon = O(R/1) \ll 1. \] (2.3.11)
We can now give the final results for the buckling values. We shall present these first, together with some interpretations and specific results for special cross-sections, and we shall postpone the proof until subsection 3.1 at the end of this section.
The final formulae for the buckling values, which follow from (2.3.9) are
\[
\frac{2\pi}{\mu_0 EI_L} = \frac{1}{\Gamma(\delta)} \left(1 + O(\delta^2)\right), \quad (F), \quad (2.3.12.1)
\]
\[
\frac{1}{4\pi^2 L f_L} = \frac{3}{\Gamma(\delta_c) - \frac{1}{2}} \left(1 + O(\delta^2)\right), \quad (S), \quad (2.3.12.2)
\]
for \(\delta \to 0\), where
\[
\Gamma(\delta_c) = -\gamma - \log \left(\frac{1}{\delta_c}\right), \quad \gamma = 0.577, \quad (Euler's \ constant) \quad (2.3.13)
\]

Before proving these results (in subsection 3.1), we make some remarks. Firstly, the general form of the results (2.3.12) holds irrespective of the shape of the cross-section. In fact, the shape of the cross-section only enter these formulae through the number \(c\). Hence, realizing that \(f_L\) is proportional to \(R^4\), we see that, apart from a logarithmic factor, the buckling values \(B_L^0\) and \(l_0\) are proportional to \((R/h)^2\) and to \(R/h\), respectively, for every finite cross-section.

Secondly, for a circular cross-section one has \(c = 1\), and the then obtained results correspond completely with those known in literature (cf. [5], [14] for (F), and [16] for (S)). For a beam of elliptic cross-section \((a \times b; a \leq b\) one gets
\[
R_c = \frac{1}{2}(a + b), \quad I_f = \frac{a^3}{4} \pi a^3 b \quad (2.5.14)
\]

Restricting ourselves to case (F) for a cantilever \((\lambda = \pi/20\), we find from (2.3.12.1) for the buckling field
\[
\frac{B_L^0}{\mu_0 E} = \frac{\pi a}{2l} \left[\frac{\pi a}{2l} \Gamma \left(\frac{\pi a}{2l}\right)\right] \quad (2.5.15)
\]

This result is in correspondence with [14], eq. (6.13), with \(\mu^{-1} \to 0\).

Finally we consider a ferromagnetic cantilever of rectangular cross-section \((a \times b; a \leq b\) for a rectangle it can be proved that \(c\) becomes (analogously to [17], p. 178)
\[
R_c = \frac{a}{2(E(p^2) - (1 - p^2)K(p^2))} \quad (2.5.16)
\]

where \(p \in (0, \frac{1}{\sqrt{2}})\) is the root of the relation
\[
a - \frac{E(p^2) - (1 - p^2)K(p^2)}{b - E(1 - p^2) - p^2 K(1 - p^2)} = 0, \quad (0 < \frac{a}{b} \leq 1), \quad (2.5.17)
\]

and \(K\) and \(E\) are complete elliptic integrals of the first and second kind, respectively. Moreover
\[
I_f = \frac{4a^3b}{3} \quad (2.5.18)
\]

Then, (2.3.12.1) yields (with \(\lambda = \pi/20\))
\[
\frac{B_L^0}{\mu_0 E} = \frac{2b}{3 \pi} \left[\frac{\pi a}{2l} \Gamma \left(\frac{\pi R_c}{2l}\right)\right] \quad (2.5.19)
\]

In a previous paper, [18], the author stated that it was to be expected that the buckling values for a narrow rectangular cross-section may be approximated by the corresponding values for an elliptic cross-section. To check this statement, we shall compare the result (2.5.19) with (2.5.15) for
an ellipse \((a_1 \times b_1)\), such that

\[
a_1 = \sigma a , \quad b_1 = \sigma b , \quad \sigma = \frac{2}{(3 \pi)^{1/4}}.
\]  

(2.3.20)

In this case the rectangle and the ellipse have identical thickness to width ratio's and moments of inertia \(I_p\). Defining \(q\) as the quotient of the buckling values we then find from (2.3.15) and (2.3.19)

\[
q = \left(\frac{a_1}{b_1}\right)_{\text{rectangle}} = \left(\frac{\Gamma(a R/c^2)}{\Gamma(a (a_1 + b_1)/4\ell)}\right)^{1/2}.
\]  

(2.3.21)

Some \(q\)-values, for \(b/c = 0.1\) and varying \(a/b\) are listed in Table 2.1. These values justify the expectation stated above.

<table>
<thead>
<tr>
<th>(a/b)</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
<th>0.5</th>
<th>0.4</th>
<th>0.3</th>
<th>0.2</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R/c)</td>
<td>1.180</td>
<td>1.121</td>
<td>1.061</td>
<td>1.000</td>
<td>0.938</td>
<td>0.875</td>
<td>0.810</td>
<td>0.743</td>
<td>0.672</td>
<td>0.595</td>
</tr>
<tr>
<td>(q)</td>
<td>0.982</td>
<td>0.983</td>
<td>0.984</td>
<td>0.985</td>
<td>0.987</td>
<td>0.990</td>
<td>0.994</td>
<td>1.000</td>
<td>1.008</td>
<td>1.022</td>
</tr>
</tbody>
</table>

Table 2.1. Ratio of buckling values for rectangular and elliptic cross-sections.

We now proceed with the proof of the general results (2.3.12).

3.1. Proof of (2.3.12)

All manipulations in this section will be performed in the complex \(z\)-plane (with \(z\) according to (2.3.7)). We shall not give detailed references in all steps of our calculations, but for a general reference with respect to the methods we use here we refer to [19]. Introducing the complex line element \(dz\) by

\[
R \, dz = i (N_x + i N_y) \, ds = i N \, ds ,
\]

(2.3.22)

where \((N_x, N_y)\) denotes the unit outward normal on \(C\), and the complex derivative as

\[
\frac{\partial}{\partial z} = \frac{1}{2} R \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) .
\]

(2.3.23)

we immediately derive the useful relation

\[
2dz \frac{\partial}{\partial z} = ds \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \forall z \in C .
\]

(2.3.24)

With the function \(F\) defined as

\[
F = B_x - i B_y , \quad z \in S^* \cup C ,
\]

(2.3.25)

we rewrite the constraints (2.1.2) and (2.1.7), (2.3.3) for the intermediate state as

\[
F \text{ analytical} , \quad z \in S^* .
\]
\[ i F \, dz = R \, (F) ; \quad F \, dz = R \, (S), \quad z \in C, \quad \text{(2.3.26)} \]

\[ F = 1 + O(z^2), \quad (F) ; \quad F = -iz^{-1} + O(z^3), \quad (S), \quad z \to \infty. \]

For the perturbed potential \( \phi \) we have at our disposal the constraints (2.3.6). Consider \( \phi \) as \( \phi(z, \overline{z}) \), then the Helmholtz equation (2.3.6)\(^1\) can be written as (\( \delta = \Delta R \))

\[ \Delta_R \phi = 4 \frac{\partial^2 \phi(z, \overline{z})}{\partial z \partial \overline{z}} = \delta^2 \phi(z, \overline{z}). \quad \text{(2.3.27)} \]

With the introduction of the real valued function

\[ f = f(z, \overline{z}) = \phi + B_0, \quad z \in S^* \cup C, \quad \text{(2.3.28)} \]

the constraints (2.3.6)\(^2-3\) simplify to

\[ f = 0, \quad (F), \quad \frac{\partial f}{\partial N} = 0, \quad (S), \quad z \in C. \quad \text{(2.3.29)} \]

After substitution of (2.3.25) and (2.3.28) into (2.3.5) and with the use of (2.3.24) and the above constraints, the buckling formulae (2.3.5) can be transformed into

\[ \frac{\mu_0 E_1 \lambda^4}{2 B_0} = \text{Im} \int_C F \frac{\partial f}{\partial z} \, dz, \quad (F), \quad \text{(2.3.30.1)} \]

\[ \frac{4 \pi^2 E_1 \lambda^2 \lambda^2}{\mu_0 \delta} = \text{Im} \left[ \left( -f \frac{\partial f}{\partial z} + \frac{1}{4} \delta^2 (z + \overline{z}) f^2 \right) \, dz \right], \quad (S). \quad \text{(2.3.30.2)} \]

In the above equations \( \partial f/\partial z \) and \( f \) occur. Therefore, we first derive integral equations for \( \partial f/\partial z \) (F) and \( f \) on C.

The fundamental solutions of the Laplace and the Helmholtz equation are

\[ G(z, \overline{z}, z_0, \overline{z}_0) = \frac{1}{2\pi} \ln |z - z_0| = \frac{1}{4\pi} \left[ \ln(z - z_0) + \ln(\overline{z} - \overline{z}_0) \right], \quad \text{(2.3.31)} \]

\[ H(z, \overline{z}, z_0, \overline{z}_0) = \frac{1}{2\pi} K_0(\delta |z - z_0|), \quad \text{(2.3.32)} \]

respectively, where \( K_0 \) is the modified Bessel function of the second kind of order zero. These solutions satisfy

\[ \Delta_R G = -\delta_0(z - z_0); \quad \Delta_R H = -\delta_0(z - z_0), \quad \text{(2.3.33)} \]

where \( \delta_0(z) \) is Dirac's delta function. Green's second identity, together with (2.3.26) and (2.3.27), implies for \( (x_0, y_0) \in D^* \),

\[ \phi(x_0, y_0) = \int_{D^*} \left( \phi \frac{\partial H}{\partial N} - H \frac{\partial \phi}{\partial N} \right) \, ds, \quad \text{(2.3.34)} \]

and
\[ B_x(x_0, y_0) = B_x(\infty) + \int \left( \frac{\partial G}{\partial N} - \frac{\partial B_x}{\partial N} \right) ds \tag{2.3.34} \]

\[ = B_x(\infty) + \int \left( \frac{\partial G}{\partial N} - \frac{\partial B_x}{\partial N} \right) ds + \int \left( \frac{\partial (H - G)}{\partial N} - B_x \frac{\partial (H - G)}{\partial N} \right) ds , \]

where in the latter step it is used that \( \partial B_x/\partial N = -\partial B_y/\partial s \) on \( \partial \Omega \) and one partial integration is performed.

With use of (2.3.24) it can be shown that

\[ \phi \frac{\partial H}{\partial N} ds = \text{Re} \left[ \frac{2}{i} \phi \frac{\partial H}{\partial z} dz \right] . \tag{2.3.35} \]

Analogously, and with the use of (2.3.25), the last integral of (2.3.34) can be transformed into a complex integral. Then (2.3.33) and (2.3.34) add up to

\[ f(x_0, y_0) = \text{Re} \left\{ F(\infty) + \frac{2}{i} \int \frac{\partial H}{\partial z} dz - \frac{2}{i} \int \frac{\partial H}{\partial t} dz \right\} - \frac{2}{i} \left[ \int \frac{\partial (H - G)}{\partial z} dz \right] , \quad x_0 \in S^+ . \tag{2.3.36} \]

We note that both \( \partial H/\partial z \) and \( \partial G/\partial z \) are of the form

\[ \frac{1}{4\pi(z - z_0)} + \text{regular term} , \]

for \( z \to z_0 \). Therefore, for \( z_0 \to \infty \) only the first integral on the right-hand side of (2.3.36) becomes singular. Using Flamet's formula (cf. [19]), or see (2.3.57) we then obtain from (2.3.36) by letting \( z_0 \to \infty \) (since \( z_0 \) and \( \bar{z}_0 \) are coupled on \( \partial \Omega \), we denote \( f(z_0, \bar{z}_0) \) by \( f(z_0) \) for \( z_0 \) on \( C \)

\[ f(z_0) = 2 \text{Re} \left\{ F(\infty) + \frac{2}{i} \int \frac{\partial H}{\partial z} dz - \frac{2}{i} \int \frac{\partial H}{\partial t} dz \right\} - \frac{2}{i} \left[ \int \frac{\partial (H - G)}{\partial z} dz \right] , \quad z_0 \in C . \tag{2.3.37} \]

where \( \int \) stands for Cauchy's principal value (cf. [19]). At this point we have to consider for a moment the ferromagnetic and the superconducting case separately. From the relations (2.3.29) (with the first one written as \( \partial f/\partial t = 0 \)) it follows that

\[ \frac{\partial f}{\partial t} dz \in \mathbb{R} , (F) ; \frac{\partial f}{\partial z} dz \in \mathbb{R} , (S) , \quad z \in C . \tag{2.3.38} \]

Using these relations and the last rule of (2.3.26) in (2.3.37) we deduce successively

\[ \frac{2}{i} \left[ \int \frac{\partial f}{\partial z} dz \right] = 1 - \text{Re} \left[ \frac{2}{i} \left[ \int \frac{\partial f}{\partial z} dz \right] dz \right] , (F) . \tag{2.3.39.1} \]

and
both for \( z_0 \in C \).

In order to get a better uniformity between the (F) and (S) case, we introduce the auxiliary function \( \Lambda(s) \in \mathbb{R} \) by

\[
\Lambda(z) := \frac{1}{i} \int_0^L \frac{\partial f}{\partial t} (z(t)) \frac{dz(t)}{dt} \, ds - \frac{\gamma r}{L} \phi(z) \, , \quad 0 \leq s \leq L \, . \tag{2.3.40}
\]

where \( s \) is the arc length parameter along \( C \) which stands in a one-to-one relationship with \( z \) on \( C \), i.e., \( z = z(s) \) on \( C \). Moreover, \( L \) is the total arc length of \( C \) and

\[
\gamma r := \frac{1}{i} \int \frac{\partial f}{\partial t} \, dz \in \mathbb{R} \, . \tag{2.3.41}
\]

From (2.3.40) it follows that

\[
\frac{1}{i} \frac{\partial f}{\partial t} \, dz = \frac{\partial \Lambda(z)}{\partial s} + \frac{\gamma r}{L} \, ds \, , \quad z \in C \, , \quad 0 \leq s \leq L \, . \tag{2.3.42}
\]

After the substitution of (2.3.42) into (2.3.39.1) and one partial integration, (2.3.39.1) transforms into

\[
-2 \frac{1}{i} \frac{\partial H}{\partial t} \, ds = -4 \Re \left( \frac{\partial H}{\partial t} \right) \, ds = 1 - 2 \frac{\gamma r}{L} \int H \, ds
\]

\[\quad - \Re \left( \frac{1}{i} \frac{\partial (H-G)}{\partial t} \right) \, , \quad (F) \, , \quad z_0 \in C \, . \tag{2.3.43}
\]

The integral equations (2.3.39.2) and (2.3.43) are too complicated to solve them exactly. However, recalling that \( \delta \) is very small, we can write (for \( \Gamma(\delta) \) see (2.3.15))

\[
2 \pi (H-G) = \Gamma(\delta) + O(\delta^2) \, , \tag{2.3.44}
\]

\[
4 \pi \frac{\partial (H-G)}{\partial t} = \frac{1}{2} \delta^2 (\Gamma(\delta) + \frac{1}{2} \log |z-z_0| + \Gamma(\delta) + O(\delta^2)) \, ,
\]

for \( \delta \to 0 \), uniform in \( z \), \( z_0 \in C \). Introducing the first order approximations for \( i \Lambda(z) \) and \( f(z) \) by \( (g_F \in \mathbb{R} \) and \( g_S \in \mathbb{R}) \)

\[
i \Lambda(z) = g_F(z) (1 + O(\delta^2)) \, , \quad (F) \, ; \quad f(z) = \delta^2 g_S(z) (1 + O(\delta^2)) \, , \quad (S) \, , \tag{2.3.45}
\]

respectively, and neglecting terms of \( O(\delta^2) \) with respect to unity, we can approximate (2.3.43) and (2.3.39.2) by

\[
\Re \left( \frac{1}{2 \pi} \frac{\partial \phi(z)}{\partial t} \right) \, dz = \frac{1}{2} + \frac{\gamma r}{2 \pi} \left( \log |z-z_0| + \frac{\gamma r}{2 \pi} \Gamma(\delta) \right) \, , \quad (F) \, , \tag{2.3.46.1}
\]

and
\[ \frac{1}{2} \left[ \frac{1}{2} \frac{d}{dz} \left( \frac{ds}{dz} \right) \right] + \text{Re} \left( \frac{1}{2 \pi i} \oint_{\Gamma} \frac{ds}{dz} \ dz \right) \]

\[ = \text{Re} \left( \frac{1}{4 \pi i} \oint_{\Gamma} \left[ F(z) \left\{ \log |z - z_0| - \Gamma(\delta) \right\} - \frac{1}{4} (\overline{z} - \overline{z}_0) \ dz \right] \right), \tag{2.346.2} \]

for \( z_0 \in C \). Use of the above definitions and approximations in (2.3.30) results in the following set of buckling relations

\[ \frac{\mu_0 E I_s \lambda^2}{2 B^2} \mu \lambda^2 = \text{Im} \left( \int \frac{F(z) dz}{\lambda^2} \left\{ \log |z - z_0| - \Gamma(\delta) \right\} \ dz \right), \tag{2.347.1} \]

and

\[ \frac{4 \pi^2 E I_s \lambda^2}{\mu \lambda^2} = \text{Im} \left( \int \frac{\overline{F(z)} dz}{\lambda^2} \left\{ \log |z - z_0| - \Gamma(\delta) \right\} \ dz \right). \tag{2.347.2} \]

Hence for the calculation of the buckling values we do not need to know \( F \) and \( \overline{F} \) completely, but we only need the values of the first integrals on the right-hand sides of (2.3.47). Again the calculation of these integrals runs for (F) and (S) mainly along the same lines. Firstly, we define two real valued functions \( R_F(z_0, \overline{z}_0) \) and \( R_S(z_0, \overline{z}_0) \), for \( z_0 \in S^- \cup C \), which for \( z_0 \in C \) are equal to the right-hand sides of (2.3.46). i.e.

\[ R_F(z_0, \overline{z}_0) = \frac{1}{2} + \frac{\lambda}{2 \pi} \left\{ \log |z - z_0| - \frac{\lambda}{2 \pi} \Gamma(\delta) \right\}, \tag{2.348.1} \]

and

\[ R_S(z_0, \overline{z}_0) = \text{Re} \left( \frac{1}{4 \pi i} \oint_{\Gamma} \frac{ds}{dz} \left\{ \log |z - z_0| - \Gamma(\delta) \right\} \ dz \right). \tag{2.348.2} \]

For later use we calculate the first derivatives with respect to \( z_0 \) of these functions for \( z_0 \in S^- \).

They read (use (2.3.26) and)

\[ \frac{\partial R_F}{\partial z_0} = \frac{\lambda}{4 \pi} \left\{ \log |z - z_0| - \frac{\lambda}{2 \pi} \Gamma(\delta) \right\}, \tag{2.349.1} \]

and

\[ \frac{\partial R_S}{\partial z_0} = \frac{1}{16 \pi} \int_{\Gamma} \frac{ds}{dz} \left\{ \log |z - z_0| - \frac{\overline{z} - \overline{z}_0}{z - z_0} \right\} \ dz \right), \tag{2.349.2} \]

The real and continuous function \( R(z_0, \overline{z}_0) \) possesses continuous derivatives in \( S^- \) and furthermore, it can be proved that

\[ \frac{\partial^2 R}{\partial z_0 \partial \overline{z}_0} = 0, \quad z_0 \in S^- \tag{2.3.50} \]

For \( R_F \) the proof of (2.3.50) is trivial (see (2.3.49.1)), whereas, for \( R_S \), (2.3.50) follows from (2.3.49.2) with the use of the property that for \( z_0 \in S^- \).
$$
\frac{1}{2\pi i} \int \frac{F(z)}{z-z_0} \, dz = F(\infty) = 0 . \quad (2.3.51)
$$

We note that, due to (2.3.51) and because $F(z)$ is $R$ on $C$, the integral

$$
\frac{1}{2\pi i} \int \frac{F(z) \log |z-z_0|}{z-z_0} \, dz , \quad z_0 \in S^* .
$$

occurring in the right-hand side of (2.3.50) is a real constant which will be denoted by $w_2$, i.e. (take $z_0=0$)

$$
w_2 = \frac{1}{2\pi i} \int \frac{F(z) \log |z|}{z} \, dz . \quad (2.3.52)
$$

The relation (2.3.50) together with the properties of $R(z_0, z_0)$ mentioned above imply the existence of analytical functions $\Psi_F(z_0)$ and $\Psi_S(z_0)$ for $z_0 \in S^- \cup C$, such that

$$
R(z_0, z_0) = \Re \Psi(z_0) , \quad z_0 \in S^- \cup C . \quad (2.3.53)
$$

Differentiating (2.3.53) with respect to $z_0$, we obtain a relation, which will be used further on,

$$
\frac{d\Psi(z_0)}{dz_0} = 2 \frac{\partial R(z_0, z_0)}{\partial z_0} , \quad z_0 \in S^- . \quad (2.3.54)
$$

As a second step, we introduce the Cauchy integral

$$
\Phi(z_0) = \frac{1}{2\pi i} \int \frac{F(z)}{z-z_0} \, dz , \quad z_0 \in C \setminus C . \quad (2.3.55)
$$

Then

i) $\Phi(z)$ analytical , $z \in C \setminus C$ ;

ii) $\Phi(z) = O(z^{-1})$ , $z \to \infty$ ;

iii) $\Phi(z_0) = \frac{1}{2} \log(z_0) + \frac{1}{2\pi i} \int \frac{F(z)}{z-z_0} \, dz , \quad z_0 \in C . \quad (2.3.56)

The relations iii) are the well-known Flemelj formulae, already mentioned before. Combining these relations with the integral equations (2.3.46) and using that $ig_0$, $g_0$ and the right-hand sides of (2.3.46), i.e. $R_0$ and $R_0$, are all real we straightforwardly find that

$$
\Re \Phi(z_0) = R(z_0) , \quad z_0 \in C . \quad (2.3.57)
$$

Furthermore, for case (F) it follows that

$$
\Re \Phi(z_0) = \Re \Phi^*(z_0) , \quad z_0 \in C . \quad (2.3.58)
$$

Finally, subtraction of the Flemelj-formulae amounts to

$$
\frac{dg}{dz} - \frac{d}{dz} (\Phi - \Phi^*) , \quad \text{along } C . \quad (2.3.59)
$$

A comparison of (2.3.58) with (2.3.53) yields
\[ \Re(\Phi^+(z_0) - \Psi(z_0)) = 0, \quad z_0 \in C. \quad (2.3.61) \]

Use of a well-known result from the theory of complex functions, saying that if the real part of an analytical function is zero at a boundary \( C \), this function can at most be an imaginary constant in the interior region \( S^{-} \) of \( C \), now implies that
\[ \Phi^+(z_0) = \Psi(z_0) + i \sigma, \quad z_0 \in S^{-} \cup C, \quad (2.3.62) \]
where \( \sigma \) is an irrelevant real constant.

With the preceding results we can derive
\[ \int_C \frac{d\Phi}{dz} \, dz = \int_C \frac{d\Psi}{dz} \, dz = \int_C F \, d\Phi \, dz, \quad (2.3.63) \]
since
\[ \int_C F \, d\Phi^+ \, dz \, dz = 0, \quad (2.3.64) \]
because \( F \, d\Phi / dz \) tends to zero as (at least) \( O(z^{-2}) \) at infinity. Explicit expressions for \( d\Psi / dz \) can be deduced from (2.3.54) and (2.3.49). From (2.3.54) and (2.3.49.1) we obtain
\[ \frac{d\Psi}{dz_0} = -\frac{E}{2\pi L} \left[ \frac{d\Phi}{dz} \right] \bigg|_{z = z_0} \quad (2.3.65.1) \]

with the use of the Plemelj formulae. Analogously, we obtain from (2.3.54) and (2.3.49.2) together with (2.3.51) and (2.3.52)
\[ \frac{d\Psi}{dz_0} = -\frac{1}{8\pi i} \left[ \frac{\bar{F}(z)}{z - z_0} \right] + \frac{1}{4} \frac{1}{2} \left( \Gamma(z) - K_0 \right) \quad (2.3.65.2) \]

We now have to substitute (2.3.65) into (2.3.63) and, subsequently, the result into (2.3.47). After some elementary calculations we finally arrive at the following buckling relations
\[ \frac{\mu_0 E_0 \lambda^2}{2 B_0^2} = \kappa \left( 1 + \mathcal{O}(\delta^2) \right), \quad (F), \quad (2.3.66.1) \]
\[ \frac{4 \pi E_0 \lambda^2}{\mu_0 B_0^2} = \left( \Gamma(z) - K_0 - \frac{1}{2} \right) \left( 1 + \mathcal{O}(\delta^2) \right), \quad (S), \quad (2.3.66.2) \]

At this point we still have to determine the constants \( \kappa \) and \( \kappa_0 \). It is only in this last step that the conformal mapping (2.3.8) (and, hence, the specific shape of the cross-section) enters into our analysis. For the calculation of \( \kappa \) we do not use its definition (2.3.41), but a result that is a consequence of (2.3.58)-(2.3.59). Considering \( \Phi^+(z) = \Phi^+(h(u)) = \Phi^+(u), \quad |u| \geq 1, \) as a function of \( u \), we see that
\[
\frac{1}{2\pi i} \int_{|z| = 1} \text{Re} \Phi \frac{du}{u} = \text{Re} \left\{ \frac{1}{2\pi i} \int_{|z| = 1} \frac{\Phi}{u} \frac{du}{u} \right\} = \text{Re} \left\{ \Phi (\infty) \right\} = 0 \quad . \tag{2.3.67}
\]

On the other hand (2.3.59), (2.3.58) and (2.3.48.1) imply
\[
\frac{1}{2\pi i} \int_{|z| = 1} \text{Re} \Phi^* \frac{du}{u} = \frac{1}{2\pi i} \int_{|z| = 1} \text{Re} \frac{du}{u}
\]
\[
= \frac{1}{2} \left( 1 - \frac{\kappa_5}{\pi} \Gamma (\delta) \right) \frac{1}{2\pi i} \int_{|z| = 1} \frac{du}{u} + \frac{\kappa_5}{2\pi i} \int_{|z| = 1} \frac{du}{u} \int \frac{du}{u} \log |z - z_0| \, dz_0 \quad . \tag{2.3.68}
\]
\[
= \frac{1}{2} \left( 1 - \frac{\kappa_5}{\pi} \Gamma (\delta) \right) + \frac{\kappa_5}{2\pi i} \int (u_0, \bar{u}_0) \, dz_0 \quad ,
\]

where
\[
l (u_0, \bar{u}_0) = \frac{1}{2\pi i} \int_{|u_0| = 1} \log |h (u) - h (u_0)| \frac{du}{u} \quad (\in R) \quad . \tag{2.3.69}
\]

Extending the domain of \( l \) to \( |u_0| \geq 1 \), we see that \( l \) satisfies
\[
\begin{align*}
& l \frac{du_0}{du_0} = \frac{1}{2} \frac{h (u_0)}{2\pi i} \int_{|u_0| = 1} \frac{du}{u} \left( h (u) - h (u_0) \right) = \frac{1}{2\pi i} \quad , \quad |u_0| > 1 \quad . \tag{2.3.70}
\end{align*}
\]
\[
\begin{align*}
& l = \log |c u_0| + O (u_0^2) \quad , \quad |u_0| \to + \infty ;
\end{align*}
\]

in accordance with (2.3.9). Therefore, the real integral \( l \) is equal to
\[
l (u_0, \bar{u}_0) = \log |c u_0| = \log c + \log |u_0| \quad , \quad |u_0| \geq 1 \quad . \tag{2.3.72}
\]

Substitution of (2.3.72) into (2.3.68) with simultaneous use of (2.3.67) leads us to
\[
1 - \frac{\kappa_5}{\pi} \Gamma (\delta) + \frac{\kappa_5}{\pi} \log c = 0 \quad , \tag{2.3.73}
\]

or, with the definition (2.3.13),
\[
\frac{\kappa_5}{\pi} = \frac{s}{\Gamma (\delta)} - \log c = \frac{s}{\Gamma (\delta) c} \quad . \tag{2.3.74}
\]

With \( z = h (u) \) the expression (2.3.52) for \( \kappa_5 \) becomes
\[
\kappa_5 = \frac{1}{2\pi} \int F (z) \log |h (u)| \, dz
\]
\[
= \frac{1}{2\pi} \int F (z) \log c + \log |h (u)| \, dz
\]
\[
= \frac{\log c}{2\pi} \int F (z) \, dz + \text{Re} \left\{ \frac{1}{2\pi} \int F (z) \log \left( \frac{h (u)}{c} \right) \, dz \right\} \quad .
\]

because of the \((\delta)\)-property that \( F (z) \) is analytic in \( \mathcal{R} \). Here, \( \hat{c} \) is a complex constant such that \( h (u)/\hat{c} u \to 1 \), for \( u \to 0 \) (hence, \( \hat{c} = w \)). Since \( h (u)/\hat{c} u \) is an analytical function, unequal to zero, in \( \mathcal{S}^+ \), the function \( \log (h (u)/\hat{c} u) \) is also analytical in \( \mathcal{S}^+ \) and tends to zero for \( u \to \infty \). Moreover \( F (z) \to i z^{-1} + O (z^{-2}) \), for \( z \to \infty \) and, hence the second integral in the last right-hand
side of (2.3.75) is zero, while the first one becomes equal to 2π. Hence, (2.3.75) amounts to

\[ \kappa F = \log \sigma . \]  

(2.3.76)

Substitution of (2.3.74) and (2.3.76) into the buckling relations (2.3.66) ultimately results in

\[ \frac{\mu_0 E f \lambda^4}{2 b^3} = \frac{\pi}{\Gamma(8 e)} \left( 1 + O \left( \delta^2 \right) \right) , \quad (F). \]  

(2.3.77.1)

\[ \frac{4 \pi E f \lambda^2}{\mu_0 b^3} = \left( \Gamma(8 e) - \frac{1}{2} \right)(1 + O \left( \delta^2 \right)) , \quad (S). \]  

(2.3.77.2)

This completes the proof of (2.3.12).
4. A set of two parallel beams

In this section we consider systems of two identical, parallel, infinitely long beams (as described in section 2). The beams can be either soft ferromagnetic (F) or superconducting (S). In order to keep our analysis manageable, we restrict ourselves to cross-sections which show double symmetry. The distance between the centres of the cross-sections is 2a. A coordinate system \((Oe_1, e_2, e_3)\) is chosen with the origin \(O\) midway between the centres of the cross-sections, the \(e_3\)-axis parallel to the central lines of the beams and the \(e_1\)-axis through the centres of the cross-sections. The \(e_1\)-axis coincides with one of the symmetry axes of the cross-sections. In the \(e_1-e_2\)-plane the cross-sections are denoted by \(D_1\) and \(D_2\), with boundaries \(\partial D_1\) and \(\partial D_2\), respectively, and the vacuum region is denoted by \(D^*\). The centre of \(D_1\) lies on the positive \(e_1\)-axis (coordinates \((a, 0)\)). Our general approach applies to arbitrary, however doubly symmetric, cross-sections, but explicit numerical results will only be given for circular cross-sections. 

The basic field \(B_0\) (case (F)) is taken in the \(e_1\)-direction (in section 5 we shall discuss the somewhat more complicated case that \(B_0\) makes an arbitrary angle \(\theta_0\) with the \(e_1\)-axis). In case (S), we assume that the total currents in the two superconducting beams are equal in magnitude \((I_0)\), but these currents can be either in equal (case (S_a)) or in opposite directions (case (S_b)) (the current in the first beam, cross-section \(D_1\), is always in the positive \(e_1\)-direction). Then, in all cases these the buckling displacement is along the \(e_1\)-axis. Moreover, we assume that the buckling displacements of the two beams are equal but opposite. In fact, if the beams buckle in the same direction, the complete set behaves as one beam and it turns out that then the buckling load is much higher (i.e., at least \(O(S^{-1})\); see the preceding section) than the one we shall find in this section. Thus, we suppose that the displacement field (2.2.1) (for one beam) here generalizes into

\[
\begin{align*}
    u_1(x, y, z) &= -u_1(-x, y, z) = w(x) + \frac{1}{2} \nu ((x-a)^2 - y^2) w''(z), \\
    u_2(x, y, z) &= u_2(-x, y, z) = \nu ((x-a) y w''(z), \\
    u_3(x, y, z) &= u_3(-x, y, z) = -(x-a) w''(z), \quad (x, y) \in D_1^* 
\end{align*}
\]  

(2.4.1)

yielding the same expression for the elastic energy of one beam as found in (2.2.2). Obviously, the elastic energies of the two beams are equal.

Again, all manipulations in this section will be performed in the complex \(s\)-plane, with \(s\) according to (2.3.7) and with all notations as introduced in section 3 (e.g., \(\partial D_1 \to C_1\)). There is no difficulty in verifying that, in analogy with section 3, the relations (2.3.20) remain valid here. We only have to replace the explicit condition at infinity for case (S) by the more vague condition

\[
    F \to 0, \quad z \to \infty, \quad (S). \tag{2.4.2}
\]

This vagueness is due to the fact that the behaviour of \(F\) for \(z \to \infty\) is different in the cases (S_a) and (S_b). In case (S_a) one has \(F = O(z^{-1})\), \(z \to \infty\), whereas \(F = O(z^{-2})\), \(z \to \infty\), in case (S_b). Of great use in the following calculations are the symmetry relations (which are due to the double symmetry of the cross-sections)

\[
    F(-z) = F(z), \quad (F), \tag{2.4.3}
\]
\[ F(-z) = -F(z), \quad (S_4); \quad F(-z) = F(z), \quad (S_0). \]

Finally, we need the following results for the integrals of \( F(z) \) along \( C_1 \), which can easily be verified.

\[ \int_{C_1} F(z) \, dz = 0, \quad (F); \quad \int_{C_1} F(z) \, dz = 2\pi, \quad (S). \quad (2.4.4) \]

Likewise, the relations for the perturbed potential \( \varphi \) remain practically the same as in section 3. We only have to realize that (for \( w > 0 \)) the displacement of \( D_1 \) is in the positive \( e_1 \)-direction, but that of \( D_2 \) is in the negative \( e_1 \)-direction. Therefore, instead of \( f = \varphi + B_2 \), we must introduce

\[ f = f(z, \bar{z}) = \begin{cases} \varphi + B_2, & \text{Re} z > 0, \\ \varphi - B_2, & \text{Re} z < 0. \end{cases} \quad (2.4.5) \]

However, we do not have to worry about this somewhat peculiar relationship, if we make use of the trivial symmetry relations

\[ f(-z, -\bar{z}) = f(z, \bar{z}), \quad (F); \]
\[ f(-z, -\bar{z}) = f(z, \bar{z}), \quad (S_4); \quad f(-z, -\bar{z}) = -f(z, \bar{z}), \quad (S_0). \quad (2.4.6) \]

Thus, the relations (2.3.6) for \( \varphi \) remain valid (\( \partial \theta \to \partial \theta \)) and the same holds true for the relations (2.3.29) for \( f(z) \), which now refer to \( C_1 \). Finally, the formulae (2.3.30) for the buckling value may be applied here too, where the integration takes place over \( C_1 \). However, in the \( (S) \)-formulae the \( O(\delta^2) \)-term in the integral on the right-hand side of (2.3.30.2) may now be neglected, as the leading term in this case turns out to be of \( O(1) \). Since the influence of the pre-stresses is enclosed in this \( O(\delta^2) \)-term, this means that the pre-stresses may be neglected now (note that this was not the case for the single superconducting beam).

Considering (2.3.30) we conclude that we are only interested in the functions \( F(z) \) and \( f(z) \), for \( z \in C_1 \). In the same way as in the preceding section we can derive the following integral equation for \( f(z) \), \( z \in C_1 \) (compare with the derivation of (2.3.37) from (2.3.33)-(2.3.36), and realize that now \( C = C_1 \cup C_2 \) and \( f = \varphi - B_2 \) on \( C_2 \))

\[ f(z_0) = 2 \operatorname{Re} (\langle F \rangle) \int_{C_1} F \frac{\partial H}{\partial \bar{z}} \, dz - \frac{2}{i} \int_{C_1} H \frac{\partial F}{\partial \bar{z}} \, dz \]
\[ + \frac{2}{i} \int_{C_1} (H - G) \frac{\partial F}{\partial \bar{z}} \, dz - \frac{2}{i} \int_{C_1} \left[ \frac{\partial H}{\partial \bar{z}} \right] \, dz, \quad z_0 \in C_1. \quad (2.4.7) \]

The last term in the right-hand side of (2.4.7) is descended from

\[ 2 \int_{C_1} \left( B_2 \frac{\partial H}{\partial \bar{N}} - H \frac{\partial B_2}{\partial \bar{N}} \right) \, ds = \operatorname{Re} \left( \frac{2}{i} \int_{C_1} F \frac{\partial H}{\partial \bar{z}} \, dz \right), \quad z_0 \in C_1. \quad (2.4.8) \]

In the derivation of which a.o. the relation \( \partial B_2 / \partial \bar{N} = -\partial B_2 / \partial \bar{z} \) is used.

In exact analogy with the preceding section we introduce, in the \( (F) \)-case, the auxiliary function \( \Lambda(z) \) by (2.3.40) and we denote the first order approximation of \( i \Lambda \) by \( g_0(z) \) (see (2.3.45.1)).
Moreover, we introduce the first order approximation of \( f(z) \) in the (S)-case according to
\[
f(z) = g_\varepsilon(z)(1 + O(\bar{\delta}^2))
\]  
(2.4.9)
(note that this is in contrast with (2.3.45)). Using the approximations (2.3.44), the boundary conditions (2.3.29), and neglecting all terms of order \( O(\bar{\delta}^2) \), we derive the following two integral equations for \( g_\varepsilon(x) \) and \( g_\varepsilon(z) \)
\[
\text{Re}\left\{ \frac{1}{2\pi i} \int_{C_1} \frac{g_\varepsilon(z)}{z - z_0} \, dz \right\} = R_\varepsilon(z_0), \quad z_0 \in C_1.
\]  
(2.4.10.1)
and
\[
\frac{1}{2} \frac{d}{dz} g_\varepsilon(z_0) + \text{Re}\left\{ \frac{1}{2\pi i} \int_{C_1} \frac{g_\varepsilon(z)}{z - z_0} \, dz \right\} = R_\delta(z_0), \quad z_0 \in C_1.
\]  
(2.4.10.2)
where
\[
R_\varepsilon(z_0) = \frac{\varepsilon \psi}{2\pi} \int_{C_1} \frac{\log |z - z_0| \, ds}{z - z_0} \, \log |z - z_0| \, ds - \frac{\Gamma(\delta)}{2\pi} (\psi \psi + \psi \psi) + \frac{1}{2} - \text{Re}\left\{ \frac{1}{2\pi i} \int_{C_1} \frac{F(z)}{z - z_0} \, dz \right\}, \quad z_0 \in C_1.
\]  
(2.4.11)
and
\[
R_\delta(z_0) = \text{Re}\left\{ \frac{1}{\pi i} \int_{C_1} \frac{F(z)}{z - z_0} \, dz \right\}, \quad z_0 \in C_1.
\]  
(2.4.12)
Furthermore
\[
\psi \psi = \frac{1}{i} \int_{C_1} \frac{\partial f}{\partial z} \, dz, \quad \psi \psi = \frac{1}{i} \int_{C_1} \frac{\partial f}{\partial z} \, dz; \quad L = \int_{C_1} \, ds \quad (F).
\]  
(2.4.13)
From the symmetry relations (2.4.6) it is evident that
\[
\psi \psi = -\psi \psi.
\]  
(2.4.14)
Hence, in the expression (2.4.11) the third term vanishes and the first two terms can be taken together to yield
\[
R_\varepsilon(z_0) = \text{Re}\left\{ \frac{\varepsilon \psi}{2\pi} \int_{C_1} \frac{\log |z - z_0| \, ds}{z + z_0} \, \log |z - z_0| \, ds \right\} + \frac{1}{2}
\]  
(2.4.15)
Note that the integral equations (2.4.10) still contain integrals over \( C_2 \). However, these integrals can be transformed in integrals over \( C_1 \) (but this is postponed for the moment).
The integral equations (2.4.10) are similar to the equations (2.3.46) of section 3, and just as in
that section these equations will be solved by Hilbert-methods. The analysis is exactly the same as the one presented in section 3 between the eqs. (2.3.53)-(2.3.53) and, therefore, we immediately give the results, which read

\[
\frac{\mu_0 E F e^2}{2 R^2} \lambda = \int \frac{\frac{\partial E}{\partial z}}{E} \, dz = \int \frac{\frac{\partial E}{\partial z}}{E} \left( \frac{\partial E}{\partial z} + \frac{i \lambda}{L} \right) \, ds \tag{2.4.16.1}
\]

and

\[
\frac{4\pi^2 E F e^2}{\mu_0} \lambda = -\int \frac{\partial E}{\partial z} \, dz = \int \frac{\partial E}{\partial z} \left( \frac{\partial E}{\partial z} - \frac{\partial H}{\partial z} \right) \, ds \tag{2.4.16.2}
\]

Note that here (2.3.64) does not apply, because the region exterior to \( C_1 \) is not simply connected but contains as a hole the region \( S_2 \), corresponding to the cross-section of the second beam. The functions \( \Psi^E(z) \) and \( \Phi^E(z) \), occurring in (2.4.16), must be calculated from (compare with (2.3.57)-(2.3.58))

\[
\text{Re} \, \Phi^E(z) = \text{Re} \, \Phi^E(z) = R^E(z) = \text{Re} \, \Psi^E(z), \quad z_0 \in C_1, \quad (F);
\]

\[
\text{Im} \, \Phi^E(z) = \text{Im} \, \Phi^E(z) = R^E(z) = \text{Re} \, \Psi^E(z), \quad z_0 \in C_1, \quad (S).
\]

Up to here the results apply to arbitrary, but doubly symmetric cross-sections. For the explicit calculations of the right-hand sides of (2.4.16), however, we from now on restrict ourselves to circular cross-sections (radius \( R, I = \pi R^2/4 \)). The analysis is based on a conformal mapping from the exterior region \( S^* \) onto a ring. For other than circular cross-sections the use of a conformal mapping is in principle also possible, but in that case our considerations are much more complex. For two circular cross-sections the conformal mapping reads (in the \( z \)-plane all distances are normalized with respect to \( R \))

\[
z = \beta \frac{1 + i \alpha}{1 - i \alpha}, \quad \beta = \sqrt{\alpha^2 - 1}, \quad m = \frac{\alpha}{\beta} > 1. \tag{2.4.18}
\]

Under this mapping, the exterior region \( S^* \) transforms into a ring bounded by concentric circles of radii \( \alpha \) and \( \alpha^{-1} \), where

\[
\alpha = m - \beta, \quad \alpha^{-1} = m + \beta, \quad \alpha \in (0, 1), \tag{2.4.19}
\]

(see Fig.2.2.)
Fig. 2.2. The conformal mapping (2.4.18).

The cross-sections $S_1$ and $S_2$ are mapped onto the interior and exterior regions of the ring, respectively, and the boundaries $C_1$ and $C_2$ onto the circles $|z| = a$ and $|z| = a^{-1}$, respectively. The point $z = \infty$ corresponds to $u = 1$. Finally for $z \in C_1$ (i.e., $|u| = a$ or $u = a e^{i\phi}$) one has
\[
\frac{dz}{du} = \frac{2b}{(1-u)^2} \quad \text{and} \quad \frac{du}{dz} = \frac{2b/a}{(1-u)^2}
\]
(2.4.20)

From here on the paths for the two cases (F) and (S) diverge, and, therefore, we have to consider these cases separately. We start with

The ferromagnetic case (F)

For the calculation of $F(z)$, it is convenient to introduce the function
\[
G(u) = \frac{u(\tilde{F}(u) - 1)}{(1-u)^2}, \quad \tilde{F}(u) = F(z(u)), \quad a \leq |u| \leq a^{-1}.
\]
(2.4.21)

From (2.3.26) it follows that
\[
\tilde{F}(u) = 1 + O((1-u)^2), \quad u \to 1,
\]
(2.4.22)

and, hence, $G(u)$ is regular for $u=1$. From (2.3.26) and (2.4.4), we conclude that

i) $G(u)$ is analytical, $\quad a < |u| < a^{-1}$,

ii) $G(u) = \frac{u}{1-u^2}$, $\quad |u| = a$,

iii) $G(u) = G(u^{-1})$, $\quad a < |u| < a^{-1}$,

iv) $\int_{|u| = a} G(u) \frac{1}{u} du = 0$.

Developing $G(u)$ in a Laurent series and employing the properties of $G(u)$ listed in (2.4.23) we conclude that $G(u)$ must be of the form
\[ G(u) = \sum_{n=1}^{\infty} g_n (u^n + u^{-n}) ; \quad g_n = \frac{n \alpha^{2n}}{1 - \alpha^{2n}}, \quad n \geq 1. \]  

(2.4.24)

This yields for \( F \)

\[ \hat{F}(u) = F_0 + \sum_{n=1}^{\infty} F_n (u^n + u^{-n}), \]  

(2.4.25)

with

\[ F_0 = 1 + 2g_1, \quad F_n = g_{n+1} - 2g_n + g_{n-1}, \quad n \geq 1, \quad g_0 = 0. \]  

(2.4.26)

For the solution of (2.4.17) we need explicit expressions for the integrals occurring in the right-hand side of (2.4.15). Using the symmetry of \( F \) (i.e., \( (2.4.3) \)) and (2.4.20) we deduce with (2.4.25) that, for \( z_0 \in \mathbb{D}_1 \cup \mathbb{C}_1 \),

\[ \frac{1}{2\pi i} \oint_{\mathbb{C}_1} \frac{F(z)}{z - z_0} \, dz = -\frac{1}{2\pi i} \oint_{\mathbb{C}_1} \frac{F(z)}{z + z_0} \, dz = -\frac{1}{2\pi i} \oint_{\mathbb{C}_1} \hat{F}(u) \left( \frac{1}{1-u} + \frac{1}{u-u_0} \right) \, du \]

\[ = \sum_{n=1}^{\infty} F_n (a^n_0 - 1) = g_1 + \sum_{n=1}^{\infty} F_n a^n_0, \quad |u_0| \leq \alpha. \]  

(2.4.27)

The first integral in the right-hand side of (2.4.15) is calculated by transforming the path of integration \( C \) into the circle \( |u| = \alpha \), developing \( \log(1 - u) \), \( (v = u a^n_0 \) or \( v = u a^n_0 \)) in a power series in \( v \) and applying Cauchy's residue theorem. In this way we obtain \( (L = 2\alpha) \)

\[ \text{Re} \left( \frac{k^0}{2\pi L} \int_{\mathbb{C}_1} \log \left( \frac{z - z_0}{z + z_0} \right) \, dz \right) \]

\[ = \frac{k^0}{2\pi} \left( \text{Re} \left( \frac{1}{2\pi i} \int_{|z| = \alpha} \left[ \log(-u_0) + \log(1 - u_0 a^n) - \log(1 - u_0) \right] \right) \right) \]

\[ \frac{1}{u - a^n} - \frac{1}{u - 1} \, du \right) \]

\[ = \frac{k^0}{2\pi} \left( \log \alpha - \text{Re} \sum_{n=1}^{\infty} \left( -\alpha^n \frac{1}{a^n} \right) a^n_0 \right) \cdot |u_0| = \alpha. \]  

(2.4.28)

The right-hand side \( R_F (z_0) \) of (2.4.17) is now explicitly known. The symmetry relation for \( f(z, \bar{z}) \) implies (\( \Phi_F (u) = \Phi_F (u^{-1}) \))

\[ \Phi_F (-z) = \Phi_F (z), \quad \Phi_F (u^{-1}) = -\Phi_F (u). \]  

(2.4.29)

Furthermore, the function \( \Psi_F \) must be analytical in the inner region \( |u| < \alpha \). Consequently, the Laurent series for \( \Phi_F \) and \( \Psi_F \) are of the form

\[ \Phi_F (u) = \sum_{n=1}^{\infty} \phi_n (u^n - u^{-n}), \quad \alpha \leq |u| \leq \alpha^{-1}, \]  

(2.4.30)

and
\[ \psi_F(u) = \sum_{n=0}^{\infty} \psi_n u^n \quad , \quad |u| \leq \alpha . \]  
(2.4.30.2)

Substituting (2.4.27) and (2.4.28) and the series (2.4.30) into (2.4.17.1), we see that these integral equations are satisfied if
\[ \frac{\psi_j^{(j)}}{2\pi} = \frac{1 + \alpha_1}{\log \alpha^2} - \frac{1}{\log \alpha^2} \left( \frac{1 + \alpha^2}{1 - \alpha^2} \right) , \]
\[ \psi_n = \frac{\psi_j^{(j)}}{2\pi} \frac{1 - x^2}{n} , \quad n \geq 1 , \]
(2.4.31)
\[ \psi_0 = 0 , \quad \psi_n = -\frac{x^2}{1 - x^2} \psi_n , \quad n \geq 1 . \]

It is now a matter of simple arithmetic to derive that for \( \varepsilon \in C_j \) or \( |u| = \alpha \),
\[ \frac{d \psi_j}{dz} = \frac{d \psi_n}{dz} = \frac{d}{du} (\Phi_j - \psi_j) du \]
\[ = -\sum_{n=1}^{\infty} \left( \frac{\psi_j^{(j)}}{2\pi} \frac{n F_n}{1 - \alpha^2} (u^2 + \alpha^2 u^{-2}) \right) \frac{du}{u} . \]  
(2.4.32)

The final step consists of the substitution of (2.4.32) into the buckling formula (2.4.16.1) and the calculation of the thus obtained integrals. This leads to the following explicit result for the buckling field \( F = \pi R^2 / 4 \)
\[ \frac{m_0 \pi}{16 \delta_0} = \text{Im} \left[ \frac{x_0^{(j)}}{2\pi} \frac{1}{2m} \int_{C_j} \frac{F(z)}{z-m} dz \right] \]
\[ = \sum_{n=1}^{\infty} \left( \frac{\psi_j^{(j)}}{2\pi} \frac{n F_n}{1 - \alpha^2} \right) \frac{1}{2m} \int_{C_j} \frac{F(u)}{|u|^2} \left( u^2 + \alpha^2 u^{-2} \right) \frac{du}{u} . \]

In the above calculation we have used that
\[ \frac{1}{2\pi} \int_{C_j} \frac{F(z)}{z-m} dz = \frac{1}{2\pi} \int_{C_j} \frac{F(z)}{z-m} dz - \frac{1}{2\pi} \int_{C_j} \frac{F(z)}{z-m} dz . \]  
(2.4.34)

where the first integral in (2.4.34) is equal to \( F(\infty) = 1 \) and the second is given by (2.4.27) for \( u_0 = \alpha^2 \). Moreover, in the final step we have used that
\[ \sum_{n=1}^{\infty} \int_{C_j} F_n = -g_1 , \quad 1 + 2 g_1 = \frac{1 + \alpha^2}{1 - \alpha^2} . \]  
(2.4.35)

The coefficients \( F_n \) follow, for given \( \alpha \), from (2.4.26) and (2.4.24). According to its definition
(2.4.18)-(2.4.19),
\[ \alpha = m - \beta = \frac{a}{R} - \sqrt{\frac{a^2}{R^2} - 1}. \] (2.4.36)

the number \( \alpha \) is directly related to the ratio \( a/R \) and, hence, our final result (2.4.33) represents an explicit expression for the buckling field as a function of the ratio \( a/R \). Numerical results will be presented in the final section of this chapter.

The superconducting case (S)

In this case we introduce \( G(u) \) as
\[ G(u) = -i \beta u - \frac{\tilde{F}(u)}{(1-u)^2}, \quad \alpha \leq |u| \leq \alpha^{-1}. \] (2.4.37)

and this function satisfies (note that \( \tilde{F}(1) = 0 \), see (2.3.26))

i) \( G(u) \) analytical,
\[ \alpha \leq |u| \leq \alpha^{-1}, \quad u \neq 1. \] (2.4.38)

ii) \( G(u) \in \mathbb{R} \),
\[ |u| = \alpha. \]

iii) \( G(u) = O((1-u)^{-1}) \),
\[ u \rightarrow 1. \]

iv) \[ \frac{1}{2m} \int_{|u|=\alpha} \frac{G(u)}{u} \, du = \frac{1}{2}, \]

while the symmetry condition here reads (cf. (2.4.3))
\[ G(u) = -G(u^{-1}), \quad (S_u); \quad G(u) = G(u^{-1}), \quad (S_u). \] (2.4.39)

The properties (2.4.38) together with the symmetry condition (2.4.39) yield
\[ G(u) = \frac{1}{1-u} - \sum_{n=0}^{\infty} \delta_n u^n : \quad \delta_n = -\delta_{-n} = \frac{\alpha^{2n}}{1 + \alpha^{2n}}, \quad n \geq 1, \quad \delta_0 = \frac{1}{2}, \quad (S_u), \quad (S_u). \] (2.4.40.1)

and
\[ G(u) = \frac{1}{2}, \quad F = i ((z+\beta)^{-1} - (z-\beta)^{-1}). \] (2.4.40.2)

We can now determine \( F \) from (2.4.37) and next the integral in the right-hand side of (2.4.12). As before, \( \Psi_d \) is taken equal to this integral and, thus,
\[ \Psi_d(z_0) = -\frac{1}{m} \int_{z_{-1,0}}^{z_{1,0}} \frac{F(z)}{z} \, dz = \frac{1}{m} \int_{z_{-1,0}}^{z_{1,0}} \frac{F(z)}{z} \, dz \] (2.4.41.1)
\[ \Psi_d(z_0) = \frac{2i}{\beta} \left[ -\frac{1}{2} + \sum_{n=1}^{\infty} F_n u_0^n \right], \quad F_n = \delta_{n+1} - 2 \delta_n + \delta_{n-1}, \quad |u_0| \leq \alpha, \quad (S_u). \]

\[ \Psi_d(z_0) = \frac{2i}{\beta} \left[ -\frac{1}{2} (1 - u_0) \right], \quad |u_0| \leq \alpha, \quad (S_u). \] (2.4.41.2)

Accounting for the symmetry conditions for \( \Phi_d \), we write the function \( \Phi_d^+(u), \alpha \leq |u| \leq \alpha^{-1}, \) as
Fig. 2.3. The relative buckling value and the deflection angle as function of $\theta_0$ for given $m$.

As a second example, we shall apply the result (2.4.44) to the case of two infinitely long superconducting rods of circular cross-section, simply supported over periods of length $i$. Then, $8=\pi R/l$, and (2.4.44) yields

$$\sqrt{\frac{\mu_0}{E}} I_0 = \frac{\pi R}{\sqrt{Q_5}} \left[ \frac{\pi R}{l} \right]^2,$$

(2.5.17)

where $Q_5 = Q_s(m)$ stands for the right-hand side of (2.4.44). This relation formally resembles (2.5.1), and, hence, the behaviour of $I_0$ under varying $R/l$ or $m$ is the same as that of $\theta_0$. Values of $Q_5$ as function of $m$ are given in Table 2.4. We note that for larger values of $m$ the factor $1/\sqrt{Q_5}$ approaches 0.

Table 2.4. Values of $Q_5$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_5$</td>
<td>0.311</td>
<td>0.220</td>
<td>0.168</td>
<td>0.0935</td>
<td>0.0568</td>
<td>0.0266</td>
<td>0.0153</td>
<td>0.00985</td>
</tr>
<tr>
<td>$1/\sqrt{Q_5}$</td>
<td>1.79</td>
<td>2.13</td>
<td>2.44</td>
<td>3.27</td>
<td>4.20</td>
<td>6.13</td>
<td>8.09</td>
<td>10.08</td>
</tr>
</tbody>
</table>

It is of some technical interest to compare this result with the result of a less accurate but more simple solution which is based upon a generalization of the law of Biot and Savart. The basic relation for this method is given by Moon in [3], eq. (2.6.4). Let $L_1$ and $L_2$ be two curves in $R_3$ carrying the same electric current $I_0$. Moreover, let $P_1$ and $P_2$ be two points on $L_1$ and $L_2$ with
\begin{equation}
\hat{\Phi}_f(u) = n \sum_{n=1}^{\infty} a_n \phi_n(u^n + u^{-n}), \quad (S_x),
\end{equation}

\begin{equation}
\hat{\Phi}_f(u) = i \sum_{n=1}^{\infty} b_n \phi_n(u^n - u^{-n}), \quad (S_y).
\end{equation}

The coefficients \( \phi_n, \ n \geq 1 \) follow from the relation \( \text{Im} \Phi_f = \text{Im} \Psi_f \), for \( |u| = \alpha \) (the constant \( \phi_0 \) is irrelevant, but can be chosen such that \( \Phi_f(z \to \infty) = \Phi_f(1) = 0 \)). This yields

\begin{equation}
\phi_n = \frac{2}{\beta} \frac{\alpha^n}{1 + \alpha^n} F_n, \quad n \geq 1, \quad (S_x),
\end{equation}

and

\begin{equation}
\alpha_1 = \frac{\alpha}{2\beta}, \quad \phi_0 = 0, \quad n \geq 2, \quad (S_x).
\end{equation}

We now are able to evaluate the right-hand side of (2.4.16.2). Substituting the preceding results and calculating the integrals in the usual way, we finally arrive at (use the Laurent series for \( F \) and \( f_+ = xR^4/4 \))

\begin{equation}
\frac{x^2 E R^2}{\mu_0 I_b^2} = \frac{4}{\beta^2} \sum_{n=1}^{\infty} n F_n \left[ \frac{1 - \alpha^{2n}}{1 + \alpha^{2n}} \right]
\end{equation}

\begin{equation}
= \frac{4}{\beta^2} \sum_{n=1}^{\infty} \frac{n \alpha^{2n} (1 - \alpha^{2n})^2}{(1 + \alpha^{2n})^2 (1 + \alpha^{2n})^2} > 0, \quad (S_x),
\end{equation}

and

\begin{equation}
\frac{x^2 E R^2}{\mu_0 I_b^2} = -\frac{1}{2 \beta^2} (\alpha + \alpha^{-1}) \frac{m}{\beta} < 0, \quad (S_y).
\end{equation}

Hence, we conclude that in case the currents run in the same direction \( (S_x) \) the system buckles in a symmetric mode (i.e. opposite displacements), where the critical current is given by (2.4.44) as function of the ratio \( \alpha/R \). On the other hand, as the right-hand side of (2.4.45) is negative, there is no symmetric buckling in the \( (S_y) \)-case (opposite currents). This does not imply that the \( (S_x) \)-system is always stable, but the critical current is much higher (at least \( O(\delta^{-1}) \), compare with the case of one beam) than the one for the \( (S_y) \)-system.

5. Conclusions and Discussion

In this section we look at the results of section 4 into more detail for some specific cases. Let us first apply the result (2.4.33) to the case of two cantilevered rods of circular cross-section, radius \( R \), length \( L \). In this case is \( \delta = aR/2L \) and then, (2.4.33) yields
\[ \frac{B_0}{\sqrt{\mu_0}} = \frac{1}{\sqrt{Q_R}} \left( \frac{\pi R}{4} \right)^2 \]  

(2.5.1)

where \( Q_R = Q_{R_0}(m) \) stands for the right-hand side of (2.4.33). This result shows that for fixed \( m = a/R \) the buckling load is proportional to \( R^2/a^2 \) (just as in the case of one single beam). The dependence of \( B_0 \) on the distance between the rods is expressed by the factor \( Q_R \). In Table 2.2 some values for \( Q_R \) as function of \( m \) are given.

<table>
<thead>
<tr>
<th>( m )</th>
<th>1.04</th>
<th>1.25</th>
<th>1.43</th>
<th>1.67</th>
<th>2.00</th>
<th>2.50</th>
<th>3.30</th>
<th>5.00</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_R )</td>
<td>31.7</td>
<td>2.52</td>
<td>1.28</td>
<td>0.788</td>
<td>0.537</td>
<td>0.371</td>
<td>0.277</td>
<td>0.228</td>
<td>0.169</td>
</tr>
</tbody>
</table>

Table 2.2. Values for \( Q_R \).

The corresponding \( B_0/\sqrt{\mu_0} \)-values as function of \( m \) and for fixed \( (RI/c) \)-values (i.e. \( R/c = 0.01 \)) are given in Table 2.3. The data in this table indicate an increase in the buckling value with an increase in the distance between the rods.

Our numerical results are in good correspondence with those of [20] in case \( \mu_0 = 5 \times 10^4 \). In [20] the same problem as mentioned here is treated in a completely different way and for more general values of \( \mu_0 \) (i.e. here \( \mu_0 \) is assumed to be so large that even \( \mu_0 (\lambda R)^2 \gg 1 \), whereas in [20] it is only assumed that \( \mu_0 \gg 1 \) (e.g. \( \mu_0 > 100 \)), but \( (\lambda R)^2 \) may remain finite.

A second aspect deserving attention is the influence of the direction of the basic field \( B_0 \) with respect to the plane through the two rods. Thus far, we have taken the direction of \( B_0 \) parallel to this plane. Let us now consider the more general case that \( B_0 \) makes an angle \( \theta_0 \), \( \theta_0 \in [0, \pi/2] \), with the positive \( e_1 \)-axis. We investigate the influence of the value of \( \theta_0 \) on the buckling value and we determine the direction of the buckling deflection, which, as we shall show, is not always equal to the direction of \( B_0 \).

We assume a symmetrical buckling mode and we denote the angle between the deflection of the first beam and the positive \( e_1 \)-axis by \( \theta_1 \). This means that the displacements of the central lines of the first rod are given by

\[ u_1(a, 0) = w(x) \cos \theta_1 \quad ; \quad u_2(a, 0) = w(x) \sin \theta_1 \]  

(2.5.2)

whereas those of the second rod are equal but opposite. For circular cross-sections one has \( I_1 = I_2 = \pi R^4/4 \) and, thus, the elastic energy remains as given by (2.2.2).

Our basic formula (2.1.1) for the buckling field was derived in chapter 1 by putting a functional \( J \) equal to zero (cf. (1.6.16)-(1.6.22)). Starting from this formula we here derived a.o. (2.3.5.1) and (2.4.16.1). With the displacement field according to (2.5.2), the functional \( J \) depends on \( \theta_1 \). Analogously to the derivation of (2.3.5.1) we now obtain

\[ J(\theta_1) = \frac{\pi I_0 \sqrt{\B}}{4 B_0^2} \]  

(2.5.3)

\[ - \int \left( B_x \cos \theta_1 + B_y \sin \theta_1 \right) \frac{\partial}{\partial N} \left( y + B_x \cos \theta_1 + B_y \sin \theta_1 \right) ds . \]

According to chapter 1, the correct value \( \theta_1 \) of \( \theta_1 \) can be determined by variation of \( J \) with...
respect to $\theta$, i.e.
\[
\frac{d}{d\theta} (\hat{\theta}_1) = 0 \quad \text{and} \quad \frac{d^2}{d\theta^2} (\hat{\theta}_1) > 0.
\]
(2.5.4)

The lowest buckling value is then obtained from
\[
J (\hat{\theta}_1) = 0.
\]
(2.5.5)

The further analysis of this problem runs exactly along the same lines as the one for $\theta_0 = 0$ presented in section 4. Therefore, we refrain from giving the details of the calculations here. The only extra complication is due to a more general condition at infinity for the analytical function $F(z)$ introduced in (2.3.25). Instead of (2.3.26), we must use here
\[
F(z) = e^{-i\theta} + O (z^{-2}) , \quad |z| \to \infty.
\]
(2.5.6)

As in (2.4.21)-(2.4.26) we can solve $F(z)$, yielding (compare with (2.4.24)-(2.4.26), and note that the $F_\alpha$s are no longer real)
\[
F(z) = F_0 + \sum_{n=1}^{\infty} F_n (u^n + u^{-n}) ,
\]
(2.5.7)

with
\[
F_0 = e^{-i\theta} + 2 \xi_1 , \quad F_n = g_{n+1} - 2 g_n + g_{n-1} , \quad n \geq 1 , \quad g_0 = 0 ,
\]
(2.5.8)
\[
g_n = n \alpha^n \left( \frac{\cos \theta_0}{1 - \alpha^{2n}} + \frac{\sin \theta_0}{1 + \alpha^{2n}} \right) , \quad n \geq 1 .
\]

The evaluation of the right-hand side of (2.5.3) is a generalization of the derivation of (2.4.16.1) (in which $J$ is already put equal to zero). The result reads ($\xi$ corresponds to $g_0$ but is no longer an imaginary function; see (2.3.42) and (2.3.45))
\[
J (\theta_1) = \frac{\pi \mu_0 B^2}{4 B_1^2} - \text{Im} \left( \frac{F_0 + e^{2i\theta}}{C_1} \right) \left( \frac{dx}{ds} + \frac{i \xi(p)}{2\pi} \right) ds .
\]
(2.5.9)

The function $g$ satisfies (compare with (2.4.32))
\[
\frac{dx}{ds} = - \sum_{n=1}^{\infty} \left( \frac{\xi(p)}{2\pi} - \frac{n F_n}{1 - \alpha^{2n}} (u^n + u^{-n}) \right) \frac{du}{u}
\]
(2.5.10)
\[
+ \sum_{n=1}^{\infty} \frac{n F_n}{1 + \alpha^{2n}} (u^n - \alpha^{2n} u^{-n}) \frac{du}{u} ,
\]
while $\xi(p)$ is given by (compare with (2.4.31))
\[
\frac{\xi(p)}{2\pi} = \frac{e^{-i\theta} + 2 \xi_1}{\log \alpha^2} .
\]
(2.5.11)

Substituting (2.5.7), (2.5.8), (2.5.10) and (2.5.11) into (2.5.9) we obtain...
\[ \frac{1}{4\pi} f(\theta_1) = \frac{H_0 E_0^4}{16 B_0^2} \left( c_0 + c_1 \cos 2\theta_1 + c_2 \sin 2\theta_1 \right), \quad (2.5.12) \]

where the coefficients \( c_0, c_1, \) and \( c_2 \), which are independent of \( \theta_1 \), are given by
\[
\begin{align*}
    c_0 &= \frac{1}{2 \log \alpha} \left| F_0 \right|^2 + \sum_{n=1}^{\infty} \frac{2 \alpha_n \alpha_{n+1}}{1 + \alpha_n^2} \left| P_n \right|^2 > 0, \\
    c_1 &= \frac{1}{2 \log \alpha} \text{Re} \left( P_0^2 \right) + \sum_{n=1}^{\infty} \frac{1 + \alpha_n^2}{1 - \alpha_n^2} \text{Re} \left( P_n^2 \right), \\
    c_2 &= \frac{1}{2 \log \alpha} \text{Im} \left( P_0^2 \right) - \sum_{n=1}^{\infty} \frac{1 + \alpha_n^2}{1 - \alpha_n^2} \text{Im} \left( P_n^2 \right). 
\end{align*} \tag{2.5.13}
\]

Application of (2.5.4) to (2.5.12) yields
\[
\tan 2\theta_1 = \frac{c_2}{c_1} \quad \text{and} \quad \frac{\cos 2\theta_1}{c_1} > 0, \tag{2.5.14}
\]

which after substitution into (2.5.5) finally results in
\[
\frac{H_0 E_0^4}{16 B_0^2} = c_0 + (c_1^2 + c_2^2)^{1/2}. \tag{2.5.15}
\]

In Table 2.3 we have listed some critical \( B_0 \) values for various values of \( m \), for \( \theta_0 = 0, \pi/4, \pi/2 \) and for \( R/l = 0.01 \). In this table, \( \tilde{B} \) represents the buckling value for two rods relative to the value for one rod, which is given by (2.3.15) in case \( a = b = R \). Hence
\[
\tilde{B} = \frac{B_0^2}{B_0^1} = \frac{1}{(2 \Gamma(8)[c_0 + (c_1^2 + c_2^2)^{1/2}])^{1/2}}. \tag{2.5.16}
\]

<table>
<thead>
<tr>
<th>( m )</th>
<th>1.04</th>
<th>1.25</th>
<th>1.43</th>
<th>1.67</th>
<th>2.00</th>
<th>2.50</th>
<th>3.30</th>
<th>5.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_0 = 0 )</td>
<td>0.061</td>
<td>0.216</td>
<td>0.302</td>
<td>0.385</td>
<td>0.467</td>
<td>0.547</td>
<td>0.625</td>
<td>0.717</td>
</tr>
<tr>
<td>( \theta_0 = \frac{\pi}{4} )</td>
<td>0.065</td>
<td>0.305</td>
<td>0.415</td>
<td>0.477</td>
<td>0.558</td>
<td>0.609</td>
<td>0.665</td>
<td>0.733</td>
</tr>
<tr>
<td>( \theta_0 = \frac{\pi}{2} )</td>
<td>0.509</td>
<td>0.541</td>
<td>0.564</td>
<td>0.589</td>
<td>0.617</td>
<td>0.650</td>
<td>0.690</td>
<td>0.747</td>
</tr>
</tbody>
</table>

Table 2.3. Relative buckling values for two ferromagnetic rods \((R/l = 10^{-2})\)

From the above table we see that the critical \( B_0 \) value depends on the angle of incidence \( \theta_0 \) of \( B_0 \). This is illustrated in the first graph in Fig. 2.3, which shows a tendency for \( B_0 \) to increase when \( \theta_0 \) increases from 0 to \( \pi/2 \). In the second graph of Fig. 2.3 the difference between \( \theta_1 \) and \( \theta_0 \) is plotted against \( \theta_0 \); it turns out that \( \theta_1 = \theta_0 \) if \( \theta_0 = 0 \) or \( \theta_0 = \pi/2 \). Hence, the deflection and the basic field \( B_0 \) are in the same direction when \( B_0 \) is either parallel or normal to the plane of the rods. In both cases one has \( c_2 = 0 \), while in the first case \( c_1 > 0 \) and in the second case \( c_1 < 0 \). Furthermore, Fig. 2.3 shows that the difference between \( \theta_0 \) and \( \theta_1 \) is maximal for \( \theta_0 \) in the neighbourhood of \( \pi/4 \) and that this difference decreases with increasing \( m \).
position vectors \( r_1 (x_1) \) and \( r_2 (x_2) \), respectively. Here, \( x_1 \) and \( x_2 \) are the arc length parameters along \( L_1 \) and \( L_2 \), respectively. The force per unit of length in \( P \) acting on \( L_1 \) is now calculated as the Lorentz-force due to the current through \( L_1 \) times the magnetic field created by \( L_2 \). The latter follows from a generalization of the law of Biot and Savart (cf. (5), (2.6.3)). According to (5), (2.6.4) this force is then given by

\[
F(x_1) = \frac{\mu_0 I^2}{4\pi} \int_{L_2} \frac{(t_1 \times (t_1 \times R))}{R^3} \, dr_2 ,
\]

(2.5.18)

where \( t_1 \) and \( t_2 \) are unit tangent vectors along \( L_1 \) and \( L_2 \), respectively and \( R \) is the position vector from \( P \) to \( P_1 \), i.e.

\[
t_1 = \frac{dr_1}{ds_1} , \quad t_2 = \frac{dr_2}{ds_2} , \quad R = r_1 - r_2 .
\]

(2.5.19)

The above formula for \( F \) is in so far an approximation in that, firstly, the three dimensional current carrying bodies are considered as one dimensional curves (thus, for instance, the specific shape of the cross-section and the distribution of the current over this cross-section are disregarded) and, secondly, the force due to the self field of \( L_1 \) is neglected. Nevertheless, it will turn out that this approach will give good agreement with our results as long as the two current filaments are not too nearby.

We shall now apply the above formula to our problem of two straight, parallel, infinitely long current carriers with equal currents \( I_0 \). In the undeformed state, the filaments are a distance \( 2a \) apart and directed in the \( e_1 \)-direction. We define \( z := x_1 \) and \( \zeta := x_2 \). The deflections of the filaments are directed in the \( e_1 \)-direction and denoted by \( u_1 (z) \) and \( u_2 (\zeta) \), respectively. Hence,

\[
r_1 = \{ a + u_1 (z) \} e_1 + z e_3 , \quad r_2 = \{ -a + u_2 (\zeta) \} e_1 + \zeta e_3 ,
\]

\[
R = \{ 2a + u_1 (z) - u_2 (\zeta) \} e_1 + (z - \zeta) e_3 ,
\]

(2.5.20)

\[
t_1 = u_1' (z) e_1 + e_3 , \quad t_2 = u_2' (\zeta) e_1 + e_3 .
\]

These formulae enable us to evaluate (2.5.18). In doing so, we must realize that the displacements are small and, hence, that a linearization with respect to these displacements is allowed. In this way we approximate \( R^3 \) by

\[
R^3 = R_0^2 \left[ 1 + \frac{6z^2}{R_0^2} (u_1 - u_2) \right] ,
\]

(2.5.21)

where

\[
R_0 = R_0 (z, \zeta) = \sqrt{4a^2 + (z - \zeta)^2} \geq 2a .
\]

(2.5.22)

In the same way we linearize the numerator of the integrand in (2.5.18), thus finding an expression for \( F \) of the form

\[
F(z) = F^{(0)} (z) + f(z) ,
\]

(2.5.23)

where \( F^{(0)} \) is independent of and \( f \) linear in the displacements. Hence, \( F^{(0)} \) is the force in the pre-buckled state (causing the so called predeflections), which does not play any role in the determination of the buckling value for \( I_0 \). Therefore, we define \( q(z) \) as the force per unit of length in
the $e_1$-direction acting on the deflected beam by

$$q(z) = (f(z), e_1) ,$$

and this force density is related to the deflection by the well-known beam equation

$$E I_y u''(z) = q(z) .$$

The procedure described above yields the following expression for $q(z)$

$$q(z) = -\frac{\mu_0 I_0}{4\pi} \int \left[ \frac{u_1(z) - u_2(\zeta)}{R_0^2} - \frac{12 a^2 (u_1(z) - u_2(\zeta))}{R_0^2} \right] d\zeta ,$$

(2.5.26)

For the further evaluation of (2.5.26) we assume symmetrical buckling, i.e. $u_1(z) = u_2(z) = u(z)$. After two partial integrations, in which it is used that $u(\zeta)$ is a periodic function in $\zeta$, (2.5.26) becomes

$$q(z) = -\frac{\mu_0 I_0}{4\pi a^2} \int u(z) + a^2 \int \frac{u(z) - u(z)}{R_0^2} d\zeta .$$

(2.5.27)

Finally, we realize that the second term in the right-hand side of (2.5.27) is $O(a^2/l^2)$ with respect to the first term. Since we have restricted ourselves to the cases $a \ll l$, we may neglect this term. Thus (2.5.25) takes the form

$$\frac{d^2 u}{dx^2} - \frac{\mu_0 I_0}{4\pi a^2 E I_y} u = 0 .$$

(2.5.28)

The boundary conditions for $u(z)$ are (simply supported)

$$u(0) = u(l) = u''(0) = u''(l) = 0 .$$

(2.5.29)

The first buckling mode satisfying these four boundary conditions is

$$u(z) = A \sin \pi z / l ,$$

(2.5.30)

which after substitution into (2.5.28) with $I_y = \pi R^4/4$ leads to the following result for the buckling value

$$\sqrt{\frac{\mu_0 I_0}{E I_y}} = \pi a \left( \frac{\pi R}{l} \right)^{1/2} .$$

(2.5.31)

Table 2.4 shows that $1/\sqrt{\zeta} = m = a/R$ for $m$ large (relative difference is less than 5% for $m \geq 4$) and then (2.5.17) becomes equal to (2.5.31). Hence, we conclude that for $a/R \gg 4$ the Biot-Savart approach presented here gives a good approximation for the buckling value. However, when the filament comes nearer to each other the correspondence becomes worse. In the limit $m \to 1$ the formula (2.5.31) gives a buckling value that is about 80% lower than the one according to (2.5.17).
We conclude with the remark that the results of section 4 for two parallel, superconducting rods will be used in the following chapter, in which the buckling problem for two parallel toroidal superconductors is investigated. The fields for two rods, as found in the present chapter, constitute a useful first approximation for the fields for two tori in case these tori are slender.
CHAPTER 3

A variational approach to magneto-elastic buckling problems
for systems of superconducting tori

1. Introduction

In the preceding chapters, the problem of magnetoelastic buckling was studied on the basis
of a variational approach. In chapter 1 a variational principle, yielding explicit relations for magne-
toelastic buckling values, was formulated and in chapter 2 applications to systems of ferromag-
netic or superconducting beams were presented. As one of the results of chapter 2, it was proved
that a configuration of two equal parallel superconducting rods could become unstable, and the
pertinent buckling value for the current was calculated. The mechanical stability of supercon-
ducting structures has been subject of an increasing amount of research; an excellent survey of
this field is given by F. Moon in his monograph [5]. As one of the many subjects in [5], the sta-
bility of toroidal superconductors in a transverse or a toroidal magnetic field was discussed. The sta-
bility of one superconducting torus in its own field was investigated by Chattopadhyay, [21], and
by Van de Ven and Couwenberg, [22], both leading to the conclusion that the natural
configuration of the torus was stable.

In this chapter we use the variational principle (1.7.41). This principle is believed to be
more suitable for numerical purposes, because it contains constraints on the fundamental vari-
ables which are much weaker than the constraints in (1.7.16). As in chapter 1 and 2 and [22] we
assume that the electric current is confined to the surface of the superconducting body. In order to
arrive at analytical expressions for buckling values, we set up two integral equations, one for the
surface current density $J$ and one for a variable $\psi$, which is related to the perturbation potential $\psi$.

We shall examine two specific buckling problems for superconducting systems. The first
concerns in-plane buckling of a pair of concentric tori, and the second out-of-plane buckling of a
coaxial pair of equal tori. All tori have equal circular cross-sections. In both cases a small param-
eter $\epsilon$ is introduced representing the slenderness of the system. In the integral equations for $J$ and
$\psi$ the integration over the tangential coordinate $\phi$ is carried out exactly, and then the integral
equations are linearized with respect to $\epsilon$. It appears that in both cases $J$ and $\psi$, and therefore also
the magnetic field in initial and perturbed state, are in zeroth order in $\epsilon$ the same as in the case of
two parallel rods (see chapter 2). When the currents in the two tori are equally directed (which is
the technically relevant case) only these zeroth order fields together with the elastic energy of
the buckling mode, play a role in the computation of the buckling value. Therefore, the buckling
values of two concentric and two coaxial tori differ only a numerical factor from the buckling
value of an equivalent pair of parallel rods. The numerical factor depends on the elastic energy
only. When the currents in the two tori are directed opposite to each other, higher order develop-
ments of $J$ and $\psi$ are needed, and the analysis becomes laborious. For these cases, we confine our-
selves to stating that if the tori will buckle at all, the buckling value is much higher than in the
case of equally directed currents. In conclusion, we present some numerical results and we
compare these results with those obtained from a mathematically less complicated, but also less rigorous, method. This method, which was also applied in chapter 2, is based upon a generalization of the law of Biot and Savart (cf. [5], (2.6.4)). In general, the correspondence between the results of the two methods is good.

2. A variational principle

Consider a superconducting body, on the surface of which a current flows with density \( J \) per unit of length. The deformed configuration of the body is denoted by \( G^+ \), its boundary by \( \partial G \) and the vacuum outside the body by \( G^- \). We start here recapitulating the variation principle (1.7.19)-(1.7.41). For the critical value of the current \( I_0 \) we have the disposal of the relation (1.7.41)

\[
\frac{E(2\pi\kappa)^2}{\mu_0 J} = \left( \int_{\partial G} \left[ -\frac{1}{2} B_k B_j \left( u_{ik} + u_{kj} \right) - \left( \psi_i + B_j \mu_{ij} - B_j u_{ij} \right) \right] dS + \int_{\partial G} T_{ij} u_{ij} dV \right)^{-1}
\]

where \( B \) and \( T \) are the normalized magnetic induction in the vacuum \( G^+ \) and the normalized stress tensor in the rigid-body state, which have to satisfy (cf. (1.7.21), (1.7.22), (1.7.31), (1.7.40))

\[
\text{div} B = 0, \quad \text{curl} B = 0, \quad x \in G^+ ;
\]

\[
(B, N) = 0, \quad x \in \partial G ; B \to 0, \quad |x| \to \infty ;
\]

\[
\int_{\partial G} (B, \gamma) d\gamma = 2\pi a ;
\]

and

\[
T_{ij} = 0, \quad x \in G^- ; T_{ij} N_j = -\frac{1}{2} (B, B) N_i , \quad x \in \partial G .
\]

The pre-stresses \( T_{ij} \) are not completely determined by (3.2.3), but for our purposes this will do. The contour \( C \) (\( \gamma \) is the tangential vector at \( C \)) has to be chosen suitably for the specific problem at hand. The perturbed potential \( \psi \) satisfies the constraints

\[
\Delta \psi = 0, \quad x \in G^+ ; \psi \to 0, \quad |x| \to \infty .
\]

By variation with respect to \( \psi \) the connection between the potential \( \psi \) and the displacement field \( u \) is found to be

\[
\frac{\partial \psi}{\partial N} = (B, \mu_{ij} - B_j u_{ij}) N_i , \quad x \in \partial G .
\]

Finally the linear deformations \( e_{ij} \) are given by
\[ 
\varepsilon_j = \frac{1}{2} (u_{ij} + u_{ji}). 
\] (3.2.6)

In the next sections we will chose reasonable representations for the displacement field \( u \) and solve the problem (3.2.4)-(3.2.5) for the perturbed potential \( \psi \) exactly. Note that the last term in the surface integral in (3.2.1) then drops out. In the following sections we will introduce displacement fields with several unknowns which are computed by variation of the right-hand side of (3.2.1).

3. Integral equations

Since \((B,N) = 0\) on \( \partial G \), the derivative of \( \psi + (B,u) \) occurring in the first term in the right-hand side of (3.2.1) reduces to a purely tangential derivative. Hence, knowledge of the values of \( B \) and \( \psi + (B,u) \) on \( \partial G \) suffices for the calculation of the right-hand side of (3.2.1) (with the last term in the first integral equal to zero). These values can be calculated from two integral equations, derived below, namely one for the surface current density \( J \), which is related to the magnetic induction \( B \) on \( \partial G \) by (cf. (1.7.1))

\[ 
\varepsilon_0 J = N \times B, \quad \text{or} \quad B = \mu_0 J \times N, \quad x \in \partial G, 
\] (3.3.1)

and one for the variable \( \psi \), defined on \( \partial G \) by

\[ 
\tilde{\psi} = \psi + (B,u), \quad x \in \partial G, 
\] (3.3.2)

which we call the modified perturbation potential. From [23], Sec. 4.12, Eq. (3), we see that

\[ 
B(x_0) = \mu_0 \int \frac{J(x) \times V_4 G(x,x_0)}{\partial G} \, ds_4, \quad x_0 \in \mathbb{G}^*, 
\] (3.3.3)

where (see also appendix B)

\[ 
G(x,x_0) = \frac{1}{4\pi |x-x_0|}, \quad x \in \partial G, 
\] (3.3.4)

is the fundamental solution of Laplace's equation. Using the theory of single layer and double layer potentials (see e.g. [24], Chap. 11), we can determine the limiting behaviour of the integral in (3.3.3) when \( x_0 \) tends to a point on \( \partial G \). Taking the limit \( x_0 \to \partial G \) in (3.3.3) and using (3.3.1)², we arrive at the first integral equation

\[ 
J(x_0) \times N(x_0) = 2 \int \frac{J(x) \times V_4 G(x,x_0)}{\partial G} \, ds_4, \quad x_0 \in \partial G. 
\] (3.3.5)

On account of the equation \( \Delta \psi = 0 \) in \( \mathbb{G}^* \) and the condition \( \psi \to 0 \) for \( |x| \to \infty \), Green's second identity implies (compare with (2.3.33)-(2.3.34))

\[ 
\psi(x_0) = \int \frac{\partial G}{\partial N_4}(x,x_0) - \frac{\partial G}{\partial N_4}(x_0G(x,x_0)) \, ds_4, \quad x_0 \in \mathbb{G}^*. 
\] (2.3.6)

Analogously to (3.5) we find by letting \( x_0 \) tend towards a point on \( \partial G \) the integral representation for \( \psi \),
\[ \psi(x_0) = 2 \int_{\partial G} \left[ \psi(x_{\gamma} \frac{\partial G}{\partial N_x})(x, x_0) \right. - \left. \frac{\partial \psi}{\partial N_x}(x, x_0) \right] dS_x, \quad x_0 \in \partial G. \]  

(3.3.7)

Using subsequently (3.2.16)\(^3\), (3.2.15)\(^3,4\) and (3.3.1) we can derive (use the Lemma following (1.3.35), Stokes' theorem)

\[
- \int_{\partial G} \frac{\partial \psi}{\partial N_x}(x, x_0)G(x, x_0) dS_x = \int_{\partial G} \left[ B_1 \mu_2 - B_2 \mu_1 \right] G N_x dS_x
\]

\[
= \int_{\partial G} \left[ u_0 \mu_1 G \mu_1 dS_x \right]
\]

\[
= \int_{\partial G} \left[ -u_0 (u(x) \times J(x), \nabla_x G(x, x_0)) + (B(x), u(x)) \frac{\partial G}{\partial N_x}(x, x_0) \right] dS_x. 
\]

(3.3.8)

Substitution of (3.3.8) into (3.3.7) yields

\[
\psi(x_0) = 2 \int_{\partial G} \left[ \psi(x) + (B(x), u(x)) \frac{\partial G}{\partial N_x}(x, x_0) \right. 
\]

\[
- u_0 (u(x) \times J(x), \nabla_x G(x, x_0)) \left] dS_x, \quad x_0 \in \partial G. 
\]

(3.3.9)

Taking the inner product with \(u_0 u(x_0)\) on both sides of (3.3.5), using (3.3.1), and adding the resulting equation to (3.3.9), we finally arrive at the second integral equation

\[
- \psi(x_0) = 2 \int_{\partial G} \left[ \psi(x) \frac{\partial G}{\partial N_x}(x, x_0) \right. 
\]

\[
+ u_0 ((u(x_0) - u(x)) \times J(x), \nabla_x G(x, x_0)) \left] dS_x, \quad x_0 \in \partial G. 
\]

(3.3.10)

In the next two sections we shall use (3.3.5) and (3.3.10) to determine the exact \(J\) and \(\psi\) for sets of two concentric and two coaxial tori, respectively.

4. Two concentric superconducting tori

Consider two concentric superconducting tori, which both have a circular cross-section with radius \(a\). The central line of the outer torus has radius \(b+c\), and the central line of the inner torus has radius \(b-c\), where \(c > a\). A coordinate system \((O, e_x, e_y, e_z)\) is chosen with \(O\) in the joint center of the two central lines, \(e_x\), and \(e_y\) in the equatorial plane (i.e., the plane through the central lines) and \(e_z\) perpendicular to the equatorial plane. The corresponding cylindrical coordinates are denoted by \((r, \theta, z)\). A cross-section of the pair of tori is shown in Fig. 3.1.
The interiors of the outer and inner torus are denoted by $G^*_1$ and $G^*_2$ and their boundaries by $\partial G^*_1$ and $\partial G^*_2$, respectively. The intersections of the outer and inner torus with the half-plane $\phi=0$ are denoted by $D^*_1$ and $D^*_2$, with boundaries $\partial D^*_1$ and $\partial D^*_2$, respectively. We define $D^- := D^*_1 \cup D^*_2$ and $\partial D := \partial D^*_1 \cup \partial D^*_2$. We suppose that

$$\epsilon := \frac{a}{b} \ll 1,$$

and furthermore that

$$m := \frac{e}{d} = O(1), \quad (m \gg 1).$$

(3.4.1)

(3.4.2)

In view of (3.4.1)-(3.4.2), the system of the two tori is called slender, and $\epsilon$ is called the slenderness parameter.

In the intermediate state a current flows on the surfaces of the two tori with surface current density

$$J = J(r, \phi) e_\phi.$$  

(3.4.3)

The total current on the outer torus has the prescribed value $I_0$. The total current on the inner torus is taken equal to $-I_0$ or $I_0$, the currents on the two tori are called equally directed or oppositely directed, respectively. Because of rotational symmetry in the intermediate state, the Cauchy stresses $\sigma_{\theta\theta}$ and $\sigma_{rr}$, and the magnetic field component $B_\theta$ vanish and all intermediate fields are independent of $\phi$.

We only consider in-plane buckling. The deflection of the central line of the outer ($l=1$) and inner torus ($l=2$) can then be written as $w_1(\phi) e_\phi + w_2(\phi) e_\theta$. In analogy with Bernoulli's theory for the bending of slender inextensible beams, the displacement fields of the tori (considered as slender rings) may be written as (neglecting $O(e^3)$-terms)
\[ u_r = w_1 + \frac{\nu}{b_1^2} \left( \frac{r-b_1}{b_1} - \left( w'_1 - w'_\perp \right) \right) \]
\[ u_\perp = v_1 - \frac{r-b_1}{b_1} \left( w'_1 - w'_\perp \right) \]
\[ u_\parallel = \frac{\nu}{b_1^2} \left( \frac{r-b_1}{b_1} - \left( w''_1 - w''_\perp \right) \right) \quad \text{in } G_i, \ i=1,2. \]

where
\[ b_1 = b + c, \quad b_2 = b - c, \]
while the inextensibility of the rings is expressed by the condition
\[ v'_1(\phi) + w_1(\phi) = 0, \quad i=1,2. \]

The perturbation potential \( \psi(r,\phi,z) \) is separated according to
\[ \psi(r,\phi,z) = \zeta(r,z)\omega(\phi). \]

Using \( \Delta \psi = 0 \) and the periodicity of \( \omega(\phi), \) i.e. \( \omega(\phi + 2\pi) = \omega(\phi), \) we find that \( \omega(\phi) \) must be of the form
\[ \omega(\phi) = \Omega \cos(n\phi + \alpha). \]

where \( n \) is a natural number. We take \( \alpha = 0; \) this is always possible by redefining the coordinate \( \phi. \)

Substitution of (3.4.4) and (3.4.8) into the boundary condition (3.2.5) for \( \psi \) at \( N \), reveals that this boundary condition can only be satisfied for every \( \phi \in [0,2\pi] \) if \( w_1(\phi) \) and \( w_2(\phi) \) are proportional to \( \cos n\phi \). In the sequel we take \( n = 2 \), which corresponds to the first bending mode. Thus,
\[ w_i(\phi) = W_i \cos 2\phi, \quad i=1,2. \]

Because of \( B_\alpha = 0 \), the separation (3.4.7) induces a separation of \( \psi = \psi_0 + (B,u) \), which we write as follows
\[ \tilde{\psi}(r,\phi,z) = f(\phi,z) W \cos 2\phi, \quad W := \sqrt{W_1^2 + W_2^2}. \]

The ratio \( W_2/W_1 \) is yet unknown. Since (3.2.1) is based upon a variational principle, we can determine this ratio by variation of the right-hand side of (3.2.1).

We identify the parameter \( a \) in (3.2.1) with the radius \( a \) of the cross-sections of the tori. Furthermore, we introduce the dimensionless variables (which will be used continually in the sequel without explicit use of the hats)
\[ \tilde{\bar{J}} = \frac{2\pi a_1}{I_0}, \quad \tilde{\bar{J}}_{(r,z)} = \frac{2\pi a_1}{I_0} J_{(r,z)}, \quad \tilde{\bar{J}}_{(r,z)} = \frac{2\pi a_1}{\mu_0 a} f_{(r,z)} \]

We proceed with the determination of zeroth order approximations with respect to \( \epsilon \) for \( J \) and \( f \). It turns out that these approximations for \( J \) and \( f \) are identical to the corresponding functions for the case of two slender parallel beams, as calculated in chapter 2.

*) **Note:** In the sequel of this chapter we do not apply the summation convention with respect to the indices \( i \) and \( j. \)
4.1. The zeroth order approximation of $f(r,z)$

The current density $J$ can be determined from (3.3.5). Since $J(x_0)$ is independent of $a_0$, we may confine ourselves to $a_0=0$. Putting

$$x = r\mathbf{e}_r + z\mathbf{e}_z, \quad x_0 = r_0\mathbf{e}_r + z_0\mathbf{e}_z,$$

where, for $a_0=0$,

$$\mathbf{e}_r = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_z, \quad \mathbf{e}_z = \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_z, \quad \mathbf{e}_z = \mathbf{e}_z,$$

we obtain from (3.3.4)

$$\nabla_s G(x,x_0) = \frac{x_0-x}{4\pi |x_0-x|^3},$$

$$= \frac{1}{4\pi R^3} (r_0 \cos \theta - r \mathbf{e}_r - r_0 \sin \theta \mathbf{e}_z + (r_0 - r) \mathbf{e}_z),$$

(3.4.14)

with

$$R := |x_0 - x| = \sqrt{(r_0 - r)^2 + (z_0 - z)^2}.$$

(3.4.15)

For the normal vector $N(x_0)$ in (3.5) we substitute

$$N(x_0) = N_x \mathbf{e}_x + N_z \mathbf{e}_z,$$

(3.4.16)

while for the surface element $dS_x$ we have

$$dS_x = r \, d\theta \, ds,$$

(3.4.17)

in which $ds$ is the line element on the boundary 2D of the cross-section $D^*$. Taking the inner product with $e_n \cdot N(x_0)$ on both sides of (3.3.5), and using (3.4.3) and (3.4.12)-(3.4.17), we derive

$$J(r_0,z_0) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{N_x (r_0 \cos \theta - r \cos \theta) + N_z (r_0 \sin \theta - r \sin \theta)}{R^3} J(r,z) r \, ds \, d\theta.$$  

(3.4.18)

The integration over $\phi$ can be carried out exactly. For an arbitrary integrable function $g(\phi)$

$$\int_0^{2\pi} g(\phi) \, d\phi = \frac{k^2}{2} \int_0^{2\pi} g(\phi) \, d\phi,$$

(3.4.19)

where

$$\Delta := \sqrt{1 - k^2 \sin^2 \theta}, \quad k := \sqrt{\frac{4r_0}{(r_0 + r)^2 + (z_0 - z)^2}}.$$  

By means of (3.4.19), the integrals over $\phi$ occurring in (3.4.18) can be reduced to complete elliptic integrals of the first and second kind (cf. [25], pp. 904-905, for the relevant definitions). Formulas 37 and 42 from [25], 2.584, reveal that
\[
\text{The integral equations are:}
\begin{align*}
\int_0^1 \frac{1}{\Delta^2} d\theta &= \frac{1}{k^2} E(k), \\
\int_0^1 \cos^2 \theta \frac{1}{\Delta^2} d\theta &= \frac{1}{k^2} (K(k) - E(k)).
\end{align*}
\]

in which $K$ and $E$ are the complete elliptic integrals of the first and second kind, respectively, and

\[ K' := \sqrt{1 - k^2}. \]  

is the complementary modulus. Further, we define

\[ a_1(k) := \int_0^1 \frac{2 \cos^2 \theta}{\Delta^2} d\theta = -\frac{2}{k^2} (K(k) - E(k)). \]

Substituting (3.4.19-24) into (3.4.18), rearranging terms and using the identity

\[ k^2 = \frac{(r_0 - r)^2 + (s_0 - s)^2}{(r_0 + r)^2 + (s_0 + s)^2} = \frac{1}{4a_0} \left( (r_0 - r)^2 + (s_0 - s)^2 \right), \]

we find the integral equation (omitting the arguments of $J, E$ and $K$)

\[
J = \frac{1}{2\pi} \int_{\partial D} \frac{k^2 r}{n^2} \left[ \frac{4a_0}{k^2} \left( N_s (r_0 - r) N_s (s_0 - s) \right) - E \right] \frac{N_s (r_0 - r) N_s (s_0 - s)}{(r_0 - r)^2 + (s_0 - s)^2} \, ds.
\]

Before linearizing the above integral with respect to $\epsilon$, we introduce the notations

\[
\begin{align*}
 r &= b + \epsilon \xi, & s &= \epsilon \eta, & r_0 &= b + \epsilon \xi_0, & s_0 &= \epsilon \eta_0, \\
 N_s = N_s, & N_s = N_s, & N_s = N_s, & N_s = N_s, & N_s = N_s, & N_s = N_s, \\
 ds &= \epsilon \, dx, & \end{align*}
\]

\[
\begin{align*}
 h(\xi, \eta, \xi_0, \eta_0) &= \frac{\sqrt{(r_0 - r)^2 + (s_0 - s)^2}}, \\
 I(\xi, \eta, \xi_0, \eta_0) &= \frac{N_s (r_0 - r) N_s (s_0 - s)}{(r_0 - r)^2 + (s_0 - s)^2}.
\end{align*}
\]

The contour defined by the points $(\xi, \eta)$ for which $(r, s) \in \partial D$, is called $C_1 = C_1 \cup C_2$, where $C_1$ and $C_2$ are the circles

\[ C_1: (\xi - m)^2 + \eta^2 = 1, \quad C_2: (\xi + m)^2 + \eta^2 = 1. \]

In view of (3.4.2) both $\xi$ and $\eta$ are of order unity with respect to the small parameter $\epsilon$ when $(\xi, \eta) \in C$. Then it is easy to verify that (3.4.25) implies

\[
k^2 = \frac{1}{4} \epsilon^2 h^2 \frac{N_s (r_0 - r) N_s (s_0 - s)}{1 + O(\epsilon)},
\]

so we can use the developments of $K$ and $E$ for small $k'$ (see [25], 8.113, form. 3, and 8.114, form. 3).
\[ K(k) = \ln \frac{1}{k} + O(k^2 \ln k), \quad E(k) = 1 + O(k^2 \ln k), \quad k' = 0. \] (3.4.34)

From (3.4.33)-(3.4.34) we derive
\[ K(k) = \ln \frac{\xi}{k} - \ln h + O(e), \quad E(k) = 1 + O(e^2 \ln e), \] (3.4.33)
\[ k = 1 + O(e^2), \quad \alpha_k(k) = -2 \ln \frac{\xi}{k} + 2 \ln h - 2 + O(e). \] (3.4.36)

The zeroth order approximation with respect to \( e \) of \( J \) is denoted by \( J^{(0)} \), so
\[ J = J^{(0)}(1 + O(e)). \] (3.4.37)

Substitution of (3.4.27)-(3.4.31) into (3.4.26) and linearization with respect to \( e \) with the aid of (3.4.33)-(3.4.37), yields the simplified integral equation
\[ J^{(0)} = \frac{1}{\pi \xi} \int_{C} \frac{J^{(0)(\omega)}}{\omega} d\omega. \] (3.4.38)

We introduce the complex variables
\[ z = \frac{i}{\xi} + i \eta, \quad \omega = \frac{i}{\xi} + i \eta, \] (3.4.39)
\[ N = N_{k} + i N_{\eta}, \quad N_{0} = -N_{k} + i N_{\eta}, \quad S = i N_{\eta}, \quad S_{0} = -i N_{\eta}, \] (3.4.40)
so that according to (3.4.30)-(3.4.31)
\[ h = |z - \bar{z}|, \quad J = - \frac{N_{0}(\bar{z} - \bar{z})}{|z - \bar{z}|^2} = - \frac{N_{0}}{2z - \bar{z}}, \] (3.4.41)
where \( \bar{z} \) and \( \bar{z} \) denote the complex conjugates of \( z \) and \( \omega \). The contours \( C_{1} \) and \( C_{2} \) are given by
\[ C_{1}: |z + \bar{z}| = 1, \quad C_{2}: |z + \bar{z}| = 1, \] (3.4.42)
while for \( \omega \in C = C_{1} \cup C_{2} \)
\[ dw = S d\eta = i N_{\eta} d\eta, \quad d\omega = -i N_{\eta} dz. \] (3.4.43)

The exterior of \( C \) is denoted by \( S^{+} \) and the interior of \( C \) by \( S^{-} \). Substitution of (3.4.41) and (3.4.43) into (3.4.38) transforms this integral equation into
\[ J^{(0)}(z_{0}) = \text{Re} \left\{ \frac{h_{0}}{\pi} \int_{C} \frac{J^{(0)}(\omega)}{\omega} \frac{\eta}{\bar{\eta}} d\omega \right\}, \quad z_{0} \in C. \] (3.4.44)

where \( \text{Re} \) stands for Cauchy's principal value. To solve this integral equation, we introduce the Cauchy integral
\[ F(z_{0}) = \frac{1}{2\pi i} \int_{C} \frac{J^{(0)}(\omega)}{\omega - z_{0}} d\omega, \quad z_{0} \in C \cap C. \] (3.4.45)

The function \( F(z) \) has the following properties (see (2.3.55)-(2.3.57))
\( F(z) \) analytical, \( z \in S^- \cup \mathbb{S}^+ \),
\[
F(z) = O(z^{-1}), \quad z \to \infty. \tag{3.4.46}
\]
\[
F'(z_0) - F^+(z_0) = \frac{1}{2\pi i} \int_{z_0}^{\infty} \frac{F_0}{z-\zeta} d\zeta, \quad z_0 \in \mathbb{C}, \tag{3.4.48}
\]
\[
F'(z_0) + F^+(z_0) = \frac{1}{\pi i} \int_{z_0}^{\infty} \frac{F_0}{\zeta-z} d\zeta, \quad z_0 \in \mathbb{C}. \tag{3.4.49}
\]

where \( F^- \) and \( F^+ \) are defined by
\[
F^\pm(z_0) = \lim_{z \to z_0, \pm \mathbb{S}^+} F(z), \quad z_0 \in \mathbb{C}. \tag{3.4.50}
\]

Because of (3.4.48) and (3.4.49) we have
\[
\text{Re}(F^- N) = 0, \quad \text{on } C, \tag{3.4.51}
\]
\[
\text{Im}(F^- N - iF^+ N) = 0, \quad \text{on } C. \tag{3.4.52}
\]

The relations (3.4.46) (for \( z \in S^+ \)) and (3.4.51) constitute an interior Riemann-Hilbert problem (see [19], Chap. 5, §39) for \( S^- \), with trivial solution
\[
F(z) = 0, \quad z \in S^- . \tag{3.4.53}
\]

The relations (3.4.46) (for \( z \in S^+ \), (3.4.47) and (3.4.52) in which we use (3.4.53) and (3.4.43)) constitute the following exterior Riemann-Hilbert problem:

i) \( F(z) \) analytical, \( z \in S^+ \),

ii) \( \text{Im}(F^+ dz) = 0, \quad \text{on } C, \tag{3.4.54}
\]

As extra constraint there still remains (3.2.7'), saying that the total current over \( \partial D_1 \) must equal \( I_0 \), which under the neglect of \( O(z) \)-terms yields (recall that here \( \gamma \) is the dimensionless current density according to (3.4.11))
\[
I_0 = \frac{I_0}{2\pi} \int_{C_1} J d\lambda = \frac{I_0}{2\pi} \int C_1 F^+(z) dz, \tag{3.4.55}
\]
in which we have used (3.4.43), (3.4.48) and (3.4.53). Hence,
\[
\int_{C_1} F^+(z) dz = 2\pi, \tag{3.4.56}
\]

and in the same way it follows that
\[
\int_{C_1} F^+(z) dz = \begin{cases} 2\pi, & (S_1), \\ -2\pi, & (S_0), \end{cases} \tag{3.4.57}
\]

where \((S_0)\) indicates the case of equally directed currents and \((S_1)\) the case of opposite currents. Using the theory of (19), Chap. 5, §40 and §42, it can be shown that \( F(z) \) is completely determined by the relations (3.4.56) and (3.4.56)-(3.4.57). We note that (3.4.54) and (3.4.56) are identical to the relations (2.3.26) (S) and (2.4.6) (S) and, furthermore, that (3.4.56)-(3.4.57) are in accordance with the symmetry relations (2.4.3) \((S_1, S_0)\), reading
\[ F(-x) = -F(x), \quad (S_n), \quad F(-z) = F(z), \quad (S_n). \quad (3.4.58) \]

Hence, the present \( F(x) \) and the one in chapter 2, referring to the \((S)\)-case, are necessarily the same. This fact essentially means that the zeroth order approximations of the surface current density for two slender ions and for two slender beams are identical.

### 4.2. The zeroth order approximation of \( f(r,z) \)

We proceed with the determination of the function \( f(r,z) \) from (3.4.10), under the neglect of \( O(x^2) \)-terms. For this we may again confine ourselves to \( \phi = 0 \). In that case we deduce with (3.4.5), (3.4.4), (3.4.9) and (3.4.15)

\[
(u(x_0) - u(x)) \times j(x) = (W \cos \phi - W \cos 2\phi) j(x) \eta_4,
\]

\[ x_0 \pm \partial G_i, \quad \eta_4 \pm \partial G_j, \quad i,j = 1,2. \quad (3.4.59) \]

Substitution of (3.4.10), (3.4.14)-(3.4.17) and (3.4.59) into (3.4.10) for \( \phi = 0 \) yields the integral equation for \( f \) on \( \partial D_i \)

\[
W f (r_0, z_0) = \sum \frac{1}{2} \int \int \left\{ W f (r,z) \cos \theta \ \frac{N (r_0 \cos \theta - r) + N (z_0 \cos \theta - z)}{R^3} \right. \]

\[ + (W \cos \phi - W \cos 2\phi) j(x) \eta_4 \cos \frac{r - r_0}{R} d\phi \ dx \ \ (r_0, z_0) \in \partial D_i. \quad (3.4.60) \]

Again we carry out the integration over \( \phi \) exactly. To this end we first calculate with the aid of (3.4.19) and (3.4.21) the integrals

\[
\int \frac{2 \pi \cos^2 \theta}{R^3} d\phi = \frac{k^2}{2 (\sigma r)^2} \left[ \frac{1}{R} E(k) + a_2(k) \right], \quad (3.4.61) \]

and

\[
\int \frac{2 \pi \cos \phi \cos \phi}{R^3} d\phi = \frac{k^2}{2 (\sigma r)^2} \left[ \frac{1}{R} K(k) + a_1(k) \right], \quad (3.4.62) \]

where

\[
a_2(k) = \int_0^{\pi} \frac{2 \cos^4 \theta - 8 \cos^2 \theta}{\Delta^2} d\theta, \quad (3.4.63) \]

and

\[
a_3(k) = \int_0^{\pi} \frac{-16 \cos^6 \theta + 24 \cos^4 \theta - 10 \cos^2 \theta}{\Delta^3} d\theta. \quad (3.4.64) \]

The functions \( a_2(k) \) and \( a_3(k) \) can be expressed in terms of \( E(k) \) and \( K(k) \), but we only need the asymptotic behaviour of \( a_2(k) \) and \( a_3(k) \) for \( k \to 1 \). The asymptotic behaviour of the integrals of \( 1/\Delta^3 \) and \( \cos^2 \theta / \Delta^3 \) can be deduced directly from (3.4.21)-(3.4.22) and (3.4.34), and the asymptotic behaviour of the integrals of \( \cos^6 \theta / \Delta^3 \) and \( \cos^4 \theta / \Delta^3 \) can be computed elementarily, noting that \( \cos^6 \theta / \Delta^3 \) and \( \cos^4 \theta / \Delta^3 \) are bounded for \( 0 \leq \theta \leq \pi / 2 \), \( 0 \leq k \leq 1 \). Therefore, we only give the
results here, reading
\begin{align}
\alpha_2(k) &= -8\ln k + \frac{4}{k} + 16O(k^2 \ln k), \quad (3.4.65) \\
\alpha_3(k) &= -10\ln k + \frac{72}{k} + O(k^2 \ln k). \quad (3.4.66)
\end{align}

Substitution of (3.4.21), (3.4.24) and (3.4.61)-(3.4.62) into (3.4.60) yields
\begin{align}
W_f &= \sum_{j=0}^{2} \int_{\partial C} \left[ \frac{k^2}{2} \left( \frac{4r \alpha}{k^2} \right) + \frac{N_c \theta_c}{r} \right] \left( \frac{W_f - W_j}{k^2} + \frac{4}{k} + 16O(k^2 \ln k) \right) ds, \quad \text{on } \partial C, \quad i=1,2.
\end{align}

In addition to (3.4.27)-(3.4.31) we introduce
\begin{align}
l(h, \eta_0, \eta_0) &= \frac{N_c \theta_c (h_0 - \eta_0)}{k^2 + (\eta_0 - \eta)^2},
\end{align}
and we denote the zeroth order approximation of the function \(f(r,z)\) by \(f^{(0)}(r,z)\), so
\begin{align}
f(r,z) &= f^{(0)}(r,z) (1 + O(\varepsilon)). \quad (3.4.69)
\end{align}

Developing the integrand of (3.4.67) for small \(\varepsilon\), with the aid of (3.4.35)-(3.4.37) and (3.4.65)-(3.4.66), we find the following integral equation for \(f^{(0)}\)
\begin{align}
f^{(0)} &= \frac{1}{j_{\eta_0}} \sum_{j=0}^{2} \left[ \frac{1}{k^2} \left( f^{(0)} \right) \right] \left( \frac{W_f - W_j}{k^2} + \frac{4}{k} + 16O(k^2 \ln k) \right) ds, \quad \text{on } C_i, \quad i=1,2.
\end{align}

If \(W_f = W_2\), i.e. if the two tori have the same buckling patterns, the \(O(\varepsilon^2)\)-term between \(\left[ \right] \) vanishes for all \(i\) and \(j\), and then \(f^{(0)}\) will be \(O(\varepsilon^2)\) smaller than in the case \(W_f \approx W_1 \neq W_2\). In the next subsection it turns out that the order of magnitude of the lowest buckling value is directly related to the order of \(f^{(0)}\) and, therefore, we are primarily interested in the lowest order terms of \(f^{(0)}\). This brings us to assume
\begin{align}
q := \frac{W_1 - W_2}{W} = 0.
\end{align}

Under this restriction we may neglect the second and third term between \(\left[ \right]\) in (3.4.70), which simplifies this integral equation considerably. With the complex notations (3.4.39)-(3.4.45) and with \(f^{(0)} \alpha dh \approx f^{(0)} \int dh \approx dz\) (see (3.4.55)) this reduced version of (3.4.70) can be written in the form of the coupled pair of integral equations.
\[
\begin{align*}
\frac{1}{2} f^{(0)} + \text{Re} \left\{ \frac{1}{2i} \int_{\mathbb{C}} \frac{1}{z - z_0} F^{(0)} dz \right\} &= \frac{q}{2} \text{Re} \left\{ \frac{1}{2i} \int_{\mathbb{C}} \frac{1}{z - z_0} F^* dz \right\}, \quad z_0 \in C_1, \\
\frac{1}{2} f^{(0)} + \text{Re} \left\{ \frac{1}{2i} \int_{\mathbb{C}} \frac{1}{z - z_0} F^{(0)} dz \right\} &= -\frac{q}{2} \text{Re} \left\{ \frac{1}{2i} \int_{\mathbb{C}} \frac{1}{z - z_0} F^* dz \right\}, \quad z_0 \in C_2. 
\end{align*}
\] (3.4.72)

Equation (3.4.72) is, apart from a factor \(-q/2\) in its right-hand side, identical to the relation (2.4.10.2) for \(g_k\). Moreover, we note that (3.4.72) is in accordance with the symmetry relations \((2.4.6) (S_0, S_1)\), yielding
\[
\bar{g}_k(-\zeta) = g_k(\zeta), \quad (S_0): \quad g_k(-\zeta) = -g_k(\zeta), \quad (S_1). \quad (3.4.73)
\]

Hence, we conclude that
\[
f^{(0)} = -\frac{q}{2} g_1. \quad (3.4.74)
\]

The factor \(q/2\) in (3.4.73) is due to the fact that we did not a priori put \(W_2 = W_1\), as was done in chapter 2. The minus-sign is due to the fact that the directions of the current through \(\partial D_1\), i.e. \(e_i\) in chapter 2 and \(e_i\) here, are opposite \((e_i = -e_i)\).

### 4.3. Calculation of the buckling value \(l_0\)

We start with the calculation of the denominator in the right-hand side of (3.3.1), which is, apart from a factor \(E/2\), the elastic energy of the pair of tori. From (3.4.4) we can calculate the components in cylindrical coordinates of the deformations \(e_\phi\) and we find that
\[
e_{ss} = -\frac{r-b_1}{rb_1} (w'' + \lambda') (1 + O(\zeta)), \\
e_{st} = \frac{1}{rb_i} \left( \frac{1}{1+\zeta} \right) e_{\phi}(1 + O(\zeta)), \\
e_{ss} = 0, \quad \text{in} \ G_i, \quad i=1,2, \quad (3.4.75)
\]

while \(e_{ss}\) and \(e_{st}\) are \(O(\zeta)\) with respect to \(e_{\phi}\). With this result the elastic energy becomes
\[
\frac{1}{1+\zeta} \int \int \left[ \frac{1}{1-\zeta} e_{\phi} + e_{ss} + e_{tt} \right] dV = \int \int \frac{2k}{r^2 b_i^2} (w'' + \lambda')^2 + (1 + O(\zeta)) dV, \quad (3.4.76)
\]

where we have used (3.4.6), the relations \(r=b(1+O(\zeta)), b_i=b(1+O(\zeta))\), and the definition
\[
l_i = \frac{r}{b_i} \int (r-b_i)^2 dS = \frac{1}{2} r a^4. \quad (3.4.77)
\]

We note that (3.4.76) represents the classical expression for the elastic energy (apart from a factor \(E/2\)) for in-plane bending of a slender inextensible ring. With \(w_i\) as given by (3.4.9) we moreover have
\[
\int_0^{2\pi} \left( u_\varphi u_\varphi \right)^2 \, d\varphi = 9 \pi W^2.
\]  
(3.4.78)

For the evaluation of the first term in the first integrand in (3.2.1) we use (3.2.1)-(3.2.2) (in dimensionless form, so without the factor \(\mu_0\)), (3.4.3), (3.4.4), (3.4.9) and (3.4.10) and we neglect \(O(\varepsilon^2)\)-terms, resulting in

\[
-(\psi B_\mu u_k)_j B_\mu u_n N_m = -\mathcal{W} \cos 2\varphi (f_j N_j - f_k N_k) \mathcal{W} \cos 2\varphi N_r, \\
= \int \frac{df_{ij}}{d\varphi} N_r \mathcal{W}, \quad \text{on } \partial \Omega_j, \quad j = 1, 2.
\]  
(3.4.79)

Integration over \(\partial \Omega_j\) of the right-hand side of (3.4.79) yields (subsequently with use of (3.4.17), (3.4.29), (3.4.63), (3.4.29), and the relation \(J^{(0)} N_q = -\text{Im} F^+, \) following from (3.4.42) and (3.4.53))

\[
- \int \left( (\psi B_\mu u_k)_j B_\mu u_n N_m \right) \, dS \mathcal{W} \int \frac{df_{ij}}{d\varphi} N_r \, dS = -\pi \mathcal{W} \int \frac{df_{ij}}{d\varphi} N_r \, dS \\
= \pi \mathcal{W} \int \frac{df_{ij}}{d\varphi} N_r \, dS (1 + O(\varepsilon)) = -\pi \mathcal{W} \int \text{Im} \left[ F^+, \frac{df_{ij}}{d\varphi} \right] \, dS, \quad j = 1, 2.
\]  
(3.4.80)

after the omission of \(O(\varepsilon)\)-terms. For \(j = 1\), the integral in the right-hand side of (3.4.80) is computed in chapter 2; the corresponding result can be obtained from (2.4.16) (with \(J^m = R^m/4\) and \(\lambda = \beta/4\)) and (2.4.44)-(2.4.45). Bearing in mind that \(J^{(0)} = q G_{\varphi}, \) we thus obtain

\[
\text{Im} \left[ F^+, \frac{df_{ij}}{d\varphi} \right] \, dS = \left\{ \begin{array}{ll}
\frac{1}{2} \mathcal{Q}, & \left. (S) \right|_{\alpha}, \\
\frac{m}{4\pi} (\alpha - \alpha^2), & \left. (S) \right|_{\alpha},
\end{array} \right.
\]  
(3.4.81)

where

\[
\mathcal{Q} := \frac{4}{\beta^3} \sum_{n=1}^{\infty} \frac{\pi \alpha^4 (1 - \alpha^2)^5}{(1 + \alpha^2 - \alpha^2 \gamma^2)^5 (1 + \alpha^2 + \gamma^2)^5}, \\
\beta := \sqrt{m-1}, \quad \alpha := m - \beta, \quad m = \frac{c}{a}.
\]  
(3.4.82)

The integral for \(j = 2\) in the right-hand side of (3.4.80) is the opposite of the one for \(j = 1\), as follows from the symmetry relations (3.4.58) and (3.4.73)-(3.4.74). Adding the results for \(j = 1\) and \(j = 2\) and using (3.4.71) for \(W_1 - W_2\) we finally obtain for the first term in the right-hand side of (3.2.1)

\[
- \int \frac{df_{ij}}{d\varphi} N_r \, dS \mathcal{W} \int \frac{df_{ij}}{d\varphi} N_r \, dS = \left\{ \begin{array}{ll}
\frac{1}{2} \pi \mathcal{G}, & \left. (S) \right|_{\alpha}, \\
\frac{m}{4\pi} (\alpha - \alpha^2) W^2, & \left. (S) \right|_{\alpha},
\end{array} \right.
\]  
(3.4.84)

The calculation of the remaining terms in the right-hand side of (3.2.1) turns out to be redundant because, as we shall show, these terms are either identically zero or \(O(\varepsilon^2)\) with respect
to the first term and, hence, they are negligible. The third and fourth term vanish identically, the latter because we have determined \( y \) exactly. The second term in the integrand is equal to

\[
-\frac{1}{2} \sum_{e} \sum_{k} (n_{1}(u_{k}, m_{j}) - n_{1}(u_{m}, \bar{r}_{k})) \bar{r}_{e}
\]

\[
= \frac{1}{2} \int_{\gamma} \frac{r_{0j}}{\eta} \left( r_{0j} - \nu_{j} \right) n_{1} d\gamma_{j}, \quad \text{on } \partial \Omega_{j}, \ j = 1, 2.
\]

(3.4.85)

and, hence, the integral over \( \partial \Omega \) of this term is \( O(\epsilon^{2}) \) smaller than the first term, given by the right-hand side of (3.4.84). Finally, with \( T_{w} = T_{w0} = 0 \) and with the use of (3.4.4) it can be shown that

\[
\int_{\gamma_{j}} T_{w0} n_{1} d\gamma = \frac{1}{2b} \int_{\gamma_{j}} T_{w} d\gamma_{j} = \frac{1}{2b} \int_{\gamma_{j}} \eta_{j} d\gamma_{j} (1 + O(\epsilon^{2})), \quad j = 1, 2.
\]

(3.4.86)

Since the normalized stress \( T_{w0} \) is of order unity, the right-hand side of (3.4.86) has the order of magnitude \( a^{2} W^{2} b \); and, hence, is also \( O(\epsilon^{2}) \) with respect to the first term. Thus, it is shown that for small \( \epsilon \) the numerator in the right-hand side of (3.2.1) is indeed dominated by its first term.

Substitution of (3.4.77), (3.4.76) and (3.4.85) into (3.2.1) now yields

\[
\frac{4\pi^{2} E_{a} a^{2}}{\omega \beta^{2} b^{4}} \left[ \frac{1}{18} \pi \eta^{2} Q_{a}, \quad (S_{a}) \right.
\]

\[
\left. \frac{1}{36} \pi \beta^{4} \gamma^{2} \sin^{2} \lambda \right] (S_{a}).
\]

(3.4.87)

In the case of equally directed currents the lowest buckling value is found for the highest value of \( q^{2} \). According to (3.4.71) the maximal value of \( q^{2} \) is 2, and occurs for \( W_{2} = W_{1} \), implying that the buckling displacements of the two tori are equal but opposite to each other, in analogy with the results of chapter 2. This finally results in the following buckling value for \( I_{0} \)

\[
I_{0} = 6 \left[ \frac{n E_{a}^{2}}{b^{2} \left( \omega Q_{a} \right)} \right]^{1/6} = \frac{3}{b^{2} \left( \omega Q_{a} \right)}^{4}, \quad (S_{a}).
\]

(3.4.88)

In the case of opposite currents the tori do not buckle at \( q = 0 \). If \( q = 0 \), we have to review our analysis, starting from (3.4.70), in which now the first term between \( [ \) drops out. The resulting \( f^{(0)} \) is now \( O(\epsilon^{2}) \) smaller and in the computation of the right-hand side of (3.2.1), the integrals we have neglected before, play a role too. One would then expect a leading term in the right-hand side of (3.2.1) which is \( O(\epsilon^{2}) \) smaller than in the preceding analysis, but after performing the necessary laborious calculations it appears that this term vanishes too. By means of symmetry relations analogous to (3.4.58) and (3.4.73) it is possible to show that the leading term must be \( O(\epsilon^{2}) \) smaller than in the preceding analysis, which means that if the tori buckle at all, the buckling value is \( O(\epsilon^{2}) \) higher than in the case of equally directed currents.
5. Two coaxial superconducting tori

Consider two equal coaxial superconducting tori, which both have a circular cross-section with radius a. The central lines of both tori have radius b and the distance between the parallel equatorial planes is 2c. A coordinate system \((O, \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)\) is chosen with \(O\) on the joint axis of the tori midway between the equatorial planes, with \(\mathbf{e}_r\) parallel to the equatorial planes and with \(\mathbf{e}_z\) along the joint axis. The corresponding cylindrical coordinates are \((r, \phi, z)\). A cross-section of the pair of tori is shown in Fig. 3.2.

![Fig. 3.2. Cross-section of a pair of coaxial tori](image)

Variables pertaining to the upper torus are labelled with an index 1, and variables pertaining to the lower torus with an index 2. Relations (3.4.1)-(3.4.3) remain valid here. The total current on the upper torus is \(I_0\). The total current on the lower torus is either \(I_0\) or \(-I_0\), corresponding to equally directed or oppositely directed currents, respectively.

We assume out-of-plane buckling. The deflection of the central line of either torus is of the form \(w_i(\phi)e_z\) and, moreover, the cross-section rotates about the central line by an angle \(\varepsilon_i(\phi)\). For a slender ring (i.e. up to \(O(\epsilon^2)\) terms) the displacement field can then be expressed in \(w_i\) and \(\varepsilon_i\) as

\[
    w_i = (z-c_i)\varepsilon_i + \frac{(z-c_i)(r-b)}{b^2} \left( w_i^\prime - b \varepsilon_i \right),
\]

\[
    \varepsilon_i = \frac{b^2}{c_i} w_i^\prime, \quad \text{in } G_i, \quad i=1,2.
\]

where

\[
    c_1 = c, \quad c_2 = -c.
\]

Analogously to (3.4.9)-(3.4.10) we find (corresponding to the lowest periodical buckling mode)

\[
    w_i(\phi) = W_i \cos 2\phi, \quad \varepsilon_i(\phi) = T_i \cos 2\phi, \quad i=1,2.
\]
\[ f(r,\phi) = (r,\phi) \cos 2\phi, \quad W = \sqrt{W_1^2 + W_2^2}. \] (3.54)

At the end of this section we establish relationships between the unknowns \( W_1, W_2, T_1 \) and \( T_2 \), again by variation of the right-hand side of (3.2.1).

As in the preceding section, we construct the integral equations for \( f \) and \( \phi \) and linearize them with respect to \( \phi \). By means of a simple transformation (a rotation by \( \frac{1}{2} \pi \)) we relate the linearized integral equations to those of the preceding section. The procedure leading to the buckling value is then analogous to the one of the preceding section, except for the calculation of the elastic energy.

5.1. The zeroeth order approximation of \( f(r,\phi) \)

In subsection 4.1 we did not use the specific form of the contour \( C \) until we established the relationship with chapter 2, in the last paragraph of 4.1 (especially in the symmetry relations (3.4.58)). Therefore, the results of subsection 4.1 can immediately be used here. The only difference lies in the form of the contour \( C \), which in section 4 is defined by (3.4.42), whereas \( C \) is here given by \( C_1 \cup C_2, \) where \( C_1 \) and \( C_2 \) are the circles

\[ C_1: |x+im| = 1, \quad C_2: |x-im| = 1. \] (3.5.5)

However, by the simple conformal mapping \( z \rightarrow \zeta \)

\[ \zeta = iz, \quad z = i\zeta, \] (3.5.6)

the circles \( C_1 \) and \( C_2 \) are mapped onto \( \tilde{C}_1 \) and \( \tilde{C}_2 \) respectively, where

\[ \tilde{C}_1: |\zeta+m| = 1, \quad \tilde{C}_2: |\zeta-m| = 1, \] (3.5.7)

which are identical to \( C_1 \) and \( C_2 \) according to (3.4.42). The exterior of \( \tilde{C}_1 \cup \tilde{C}_2 \) in the complex \( \zeta \)-plane is denoted by \( \tilde{S}^{-} \) and the interior by \( \tilde{S}^{+} \). Furthermore, we define

\[ \tilde{F}(\zeta) = iF(i\zeta). \] (3.5.8)

With this definition \( \tilde{F} \) satisfies the relations (3.4.54) and (3.4.56)-(3.4.57), with \( z, F, C, S \) replaced by \( \zeta, \tilde{F}, \tilde{C}, \tilde{S} \). Hence, the function \( \tilde{F} \) is identical to the function \( F \) used in subsection 4.1 and thus also to the \( F \) known from chapter 2.

5.2. The zeroeth order approximation of \( f(r,\phi) \)

As in subsection 4.2 we take \( \varrho_0 = 0 \), calculate \( u(\varrho_0) - u(\varrho_0 + \phi(x)) \) and substitute the result together with (3.4.14)-(3.4.17) into (3.3.10), leading to the integral equation (compare with (3.4.60))

\[
Wf(r_0,\varphi_0) = \frac{1}{2\pi} \int_{\varphi_0}^{\varphi_0 + \varphi} \left( \int_{\varphi_0}^{\varphi} \frac{N_1(r_0\cos\varphi - \tau) + N_2(\varphi_0 - \varphi)}{R^2} \right) \left( \delta W_1 - (r_0 - \varphi)T_1 \right) \cos 2\phi \left( r_0 \cos \varphi - \tau \right) d\varphi.
\]
\begin{equation}
\frac{1}{2} f^0(\Theta) + \text{Re} \left[ \frac{1}{2 \pi i} \int_{-\infty}^{\infty} f^0(\Theta) \frac{dz}{z - z_0} \right] = \frac{Q}{2} \text{Re} \left[ \frac{1}{2 \pi i} \int_{-\infty}^{\infty} F^+ \frac{dz}{z - z_0} \right], \quad z_0 \in C_1.
\end{equation}
\begin{equation}
\frac{1}{2} f^0(\Theta) + \text{Re} \left[ \frac{1}{2 \pi i} \int_{-\infty}^{\infty} f^0(\Theta) \frac{dz}{z - z_0} \right] = -\frac{Q}{2} \text{Re} \left[ \frac{1}{2 \pi i} \int_{-\infty}^{\infty} F^+ \frac{dz}{z - z_0} \right], \quad z_0 \in C_2.
\end{equation}

Introducing \( \zeta \) by (3.5.6), \( \bar{F}(\zeta) \) by (3.5.8) and \( f^0(\Theta) \) by
\begin{equation}
f^0(\Theta) = F^0(i\Theta),
\end{equation}
we can confirm that \( F^0 \) satisfies the relations (3.4.70), with \( a, F, f^0, C \) replaced by \( \zeta, \bar{F}, f^0, \bar{C} \). Hence, the function \( f^0 \) is identical to the function \( f^0 \) from subsection 4.2, and thus also related to the function \( g_\epsilon \), calculated in chapter 2.

5.3. Calculation of the buckling value \( I_0 \)

The displacement field (3.5.1) yields the following, well-known, expression for the elastic energy for a slender ring in out-of-plane bending
\begin{equation}
\frac{1}{1 + v} \int \left[ \frac{v}{2(1 - 2v)} \epsilon_{xy} \epsilon_{xy} + \epsilon_{xz} \epsilon_{xz} \right] dV
= \frac{I_\perp}{b^3} \int_0^{2\pi} \left( 4W^+ \right) \frac{d\phi}{2(1 + v)b^3} + \frac{I_p}{2(1 + v)b^3} \int_0^{2\pi} \left( 2W^+ \right) \frac{d\phi}{b}.
\end{equation}

where in the last step (3.5.3) is used, and where
\begin{equation}
I_\perp := \int_{D_1} \left( s - c_i \right)^2 dS = \frac{1}{4} \pi a^4,
\end{equation}
\begin{equation}
I_p := \int_{D_1} \left( (r - b)^2 + (s - c_i)^2 \right) dS = \frac{1}{2} \pi a^4.
\end{equation}

For the first term in the first integrand in (3.2.1) we find analogously to (3.4.79)
\begin{equation}
-(W + B_1 u_1)_y B_1 u_1 N_m = \int \frac{d\phi}{d\theta} \cos^2 \left((s - c_i) - r - b - 1/2\right) N_m.
\end{equation}
\[ a_j^{(0)} \frac{dE_j^{(0)}}{dx} W \cos^2 2b W_j N_j (1+\Omega(\varepsilon)) \text{ on } \partial G_j, \ j=1,2, \quad (3.5.14) \]

as we assume that \( T_1 = \Omega \) (W/b) as is suggested by the specific form of the result (3.5.12), and as will be confirmed furtheron (see (3.5.18)). Integration over \( \partial G \) of the right-hand side of (3.5.14) yields (analogously to (3.4.80), but now with the use of \( a_j^{(0)} N_j = -\text{Im}(iF) \))

\[ - \int_{\partial G} \left( W + B_j + \Omega \right) \frac{dE_j^{(0)}}{d\lambda} \text{d}S = -\pi W J_1 \int_{\partial G} \frac{dE_j^{(0)}}{d\lambda} \text{d}\lambda = -\pi W J_1 \int_{\partial G} \text{Im} \left( \left\{ \left. \frac{dE_j^{(0)}}{d\lambda} \right|_{\lambda=0} \right\} \right), \quad (3.5.15) \]

with \( \tilde{C}_j, \tilde{F} \) and \( J_1^{(0)} \) according to (3.5.7), (3.5.8) and (3.5.11). The conclusions at the end of subsections 5.1 and 5.2 imply that the right-hand side of (3.5.15) is exactly equal to the right-hand side of (3.4.80) and, consequently, the result (3.4.84) for the integral over the first term in the right-hand side of (3.2.1) holds here, too. As in subsection 4.3 it can be shown that all other terms in the numerator of (3.2.1) may be neglected.

Using \( J_2 = \Omega \) in (3.5.12) and then substituting (3.5.12) and (3.4.84) into (3.2.1), we obtain

\[ \frac{4\pi^2 E_j a^4}{\mu_0 b^4} = \begin{cases} \pi \frac{a^2}{2} (Q_0)^{-1}, & (S_1), \\ -\pi \frac{a^2}{4b^2} Q_0 (2 + \Omega)^{-1}, & (S_2), \end{cases} \quad (3.5.16) \]

where

\[ p = W^2 \sum_{i=1}^{2} \left( 6W_i T_i + 4W_i T_i b \right) + \frac{1}{1+(2W_1 + 2W_2)^2}. \]

In the case of equally directed currents, minimization of \( p \) with respect to \( T_1 \) and \( T_2 \) yields

\[ T_i b = \frac{\frac{2}{3} \sqrt{2 + \Omega}}{4 + \sqrt{2 + \Omega}}, \quad i = 1,2, \text{ and } p = \frac{32}{3 + \sqrt{2 + \Omega}}. \quad (3.5.18) \]

Since \( q^2 \leq 2 \) (see (3.4.71)), the right-hand side of (3.5.16) (\( S_1 \)) attains its maximum value for \( q^2 = 2 \), corresponding to \( W_2 = -W_1 \). Hence, the buckling displacements are again opposite to each other. The lowest buckling value is thus found to be

\[ I_0 = \frac{12}{\sqrt{3 + \Omega}} \frac{a^2}{b^2} \left[ \frac{E_j}{\mu_0 Q} \right]^{1/2} = \frac{6a^2}{3 + \sqrt{2 + \Omega}} \left[ \frac{E_j}{\mu_0 Q} \right]^{1/2}, \quad (S_1). \quad (3.5.19) \]

In the case of opposite currents, the tori do not buckle for \( q \neq 0 \). If \( q = 0 \), then we have to review the calculation of \( J_0^{(0)} \). The resulting buckling value will again be much higher than for equally directed currents. Since the calculations involved are massive, and the results of little practical use, we refrain from diluting upon this calculation.
6. Conclusions and discussion

In the preceding two sections we have calculated the buckling current for sets of two concentric and two coaxial tori. In both cases the electric currents through the tori are equal, both in magnitude and in direction. The results, which are given by the formulas (3.4.88) and (3.5.19), are recapitulated below:

\[ I_0 = \frac{3\mu_0 a^2}{b^2} \left( \frac{E}{1 - \nu} \right)^{1/4}, \]  
(3.6.1)

for a pair of concentric tori, and

\[ I_0 = \frac{6\mu_0 a^3}{\sqrt{15} \nu + 2} \left( \frac{E}{1 - \nu} \right)^{1/6}, \]  
(3.6.2)

for a pair of coaxial tori. In chapter 2, a table showing values of \( Q \) as function of \( m = c/a \) is given (cf. Table 2.4). The above results are visualized in Figure 3.3. Here we have used the following numerical values

\[ E = 8 \times 10^6 \text{ N/m}^2; \quad \mu_0 = 4 \times 10^{-7} \text{ H/m}; \quad \nu = 0.3; \]
\[ b = 0.5 \text{ m}; \quad a = 5 \times 10^{-3} \text{ m}. \]

Fig.3.3. Buckling current as function of \( m \) (a: concentric pair; b: coaxial pair)

In chapter 2 the buckling current for a pair of parallel straight beams was calculated. The final result, according to (2.5.17), reads

\[ I_0 = \frac{\pi R^3}{8} \left( \frac{E}{1 - \nu} \right)^{1/4}. \]  
(3.6.3)

One can compare the result (3.6.3) with (3.6.1) and (3.6.2) by realizing that \( R \) and \( l \) must be related to \( a \) and \( b \). By taking \( a = R \) and \( nb = 2l \) (leading to equal periods for the buckling modes for the beam and the torus) we find that the buckling values according to (3.6.1) and (3.6.2) are a factor 3/4 and 3/2 of \( \pi R^2 \), respectively, times the buckling value (3.6.3). Hence, we notice that the
buckling values for pairs of parallel beams, concentric tori or coaxial tori all differ only a numerical factor from each other. Moreover, these numerical factors are completely determined by the elastic energies of the respective systems (see (3.4.76), (3.5.12) and (2.2.2)). This is due to the fact that the term which in fact is determinant for the buckling value, i.e. the numerator of the right-hand side of (3.2.1), for slender pairs of beams is dominated by its first term. This term takes the same value for all of the three systems mentioned above, at least in a zeroth order approximation with respect to \( e \) (see the comments in the final paragraphs of the subsections 4.1, 4.2, 5.1 and 5.2). Therefore, it is expected that the buckling value for any "slender" pair of parallel curved beams is equal to that of an equivalent pair of straight beams times the ratio of the elastic energies. The concept "slenderness" has to be defined properly in each problem in hand.

In chapter 2, a more simple, yet less accurate method for the solution of our buckling problem was presented. The method is based upon a generalization of the law of Biot and Savart, as described by Moon, [5], sect. 2.6. It was shown in chapter 2 how this method yields approximate buckling values \( l_0 \) for a set of two parallel rods, which are very close to the exact values as long as the two rods are not too near. These results were derived from the basic relation, (2.5.18), for the force on one current-carrying curve \( L_1 \) due to the current in a second curve \( L_2 \).

Let us now apply this relation to the buckling problems described in the sections 4 and 5 of the present chapter. We start with the system of section 4 as illustrated in Fig.3.1. The tori are considered as one-dimensional circles (rings) \( L_1 \) and \( L_2 \), which can be described by the sets of cylindrical coordinates \( (r_1, \phi_1, z_1) \) and \( (r_2, \phi_2, z_2) \), with bases \( \{e_r, e_\phi, e_z\} \) and \( \{e_r, e_\phi, e_z\} \), respectively. In the undeformed state of \( L_1 \) and \( L_2 \) one has \( r_1 = r_2 = 0 \) and \( \phi_1 = 0, 2\pi \), \( i = 1, 2 \). Restricting ourselves to in-plane bending, with the displacements of the central lines of the tori according to (3.4.4), we find for the position vectors \( r_1 \) and \( r_2 \) of two points \( P_1 \in L_1 \) and \( P_2 \in L_2 \), respectively, the relations

\[
\begin{align*}
  r_1 &= (r_1 + w_1(\phi_1))e_r + v_1(\phi_1)e_\phi, \\
  r_2 &= (r_2 + w_2(\phi_2))e_r + v_2(\phi_2)e_\phi.
\end{align*}
\]

(3.6.4)

The unit tangent vectors \( t_1 \) and \( t_2 \) along \( L_1 \) and \( L_2 \), respectively, and the position vector \( R \) from \( P_2 \) to \( P_1 \) are given by (2.5.19), and they become here (the inextensibility condition (3.4.6) is already taken into account)

\[
\begin{align*}
  t_1 &= e_r + \frac{1}{r_1} (w_1(\phi_1) - v_1(\phi_1))e_\phi, \\
  t_2 &= e_r + \frac{1}{r_2} (w_2(\phi_2) - v_2(\phi_2))e_\phi, \\
  R &= (r_1 + w_1(\phi_1))e_r + (r_2 + w_2(\phi_2))e_r + v_1(\phi_1)e_\phi - v_2(\phi_2)e_\phi.
\end{align*}
\]

(3.6.5)

These relations must be substituted into the force-relation (2.5.18), and then the result must be linearized with respect to the small displacements \( u_i \) and \( v_i \). This ultimately results in an expression for the force on the ring \( L_1 \) of the form (2.5.23), of which only the linear contribution \( \mathbf{f} \) is relevant. The calculation of this term is somewhat cumbersome but straightforward, and therefore we omit the underlying calculations. We only have to mention that in these calculations it has been assumed that \( c/d \ll 1 \) (being the criterion for the slenderness of the pair of rings), and that
we have neglected all terms that are $o(1)$ for $c/b \to 0$. This finally results in the following expression for the force per unit length acting in $P_1$ on $L_1$,

$$f(\phi) = f(\phi_0)e_n = \frac{\nu_o b^3}{8\pi c^2} [w_1(\phi_0) - w_2(\phi_0)] e_n.$$  \hspace{1cm} (3.6.6)

This purely radial load serves as the load parameter in the ring equation, which for an inextensible ring in in-plane bending reads (cf. [5], sect. 6.7, or [22], 7.2)

$$w_{1,\phi\phi}(\phi_0) + w_1(\phi_0) = \frac{b^4}{4E} f(\phi_0)$$

$$= \frac{\mu_o b^2}{8\pi} \frac{b^2}{c} \frac{\beta^2}{c} \left[ \frac{1}{2} (w_1(\phi_0) - w_2(\phi_0)) \right].$$  \hspace{1cm} (3.6.7)

An analogous ring equation holds for $w_2(\phi_0)$ on $L_2$. The lowest buckling value (for a periodical buckling mode) corresponds to

$$w_1(\phi_0) = -w_2(\phi_0) = W \cos 2\phi,$$

(in accordance with $\beta^2 = 2$ and (3.4.9)) and this yields

$$I_0 = \frac{3\pi c^2 (\frac{E}{b^4})}{\mu_o}.$$  \hspace{1cm} (3.6.8)

This result is in agreement with (3.4.88) if in the latter $1/\beta^2$ is replaced by $c/a$. As already shown in chapter 2, at the end of section 5, this is approximately true for $c/a$ not too close to unity (e.g. for $c/a \approx 4$ the relative difference is less than 1%). The worst discrepancy occurs for $c/a \to 1$, in that case relation (3.6.9) gives a buckling value that is about 45% lower than the one according to (3.4.88), or, equivalently, (3.4.88) is 80% higher than (3.6.9).

The above method can also be applied to the buckling problem of section 5. For this system (see Sec.3.2) and for out-of-plane buckling (see Sec.3.1) one has

$$r_1 = b_1 e_n + (c + w_1(\phi_0)) e_r,$$

$$r_2 = b_2 e_n + (c + w_2(\phi_0)) e_r.$$  \hspace{1cm} (3.6.9)

In exactly the same way as in the preceding problem an expression for the linearized perturbed force can be derived. In this case the force is in the $e_r$-direction and is equal to (under the neglect of $O(c/b)$-terms)

$$f_2(\phi_0) = \frac{\nu_o b^3}{8\pi c} [w_1(\phi_0) - w_2(\phi_0)].$$  \hspace{1cm} (3.6.10)

The ring equations for out-of-plane bending and torsion can be found in [5], (6.7.18). With the substitutions
\[ A = EI, \quad C = GL_p \left[ \frac{EI}{1 + v} \right]. \]

\[ u = w_1(\phi_t), \quad \phi = -\gamma(\phi_t), \quad \dot{\gamma} = \frac{d}{dx} = \frac{1}{b} \frac{d}{d\phi_t}. \]

These relations become

\[ \frac{EI}{b} \left[ w_1''(\phi_t) + b \gamma_0''(\phi_t) \right] + \frac{GL_p}{b} \left[ w_1(\phi_t) + b \gamma_0(\phi_t) \right] + f_3(\phi_t) = 0, \]  

\[ \frac{EI}{b^2} \left[ w_1'(\phi_t) - b \gamma_0'(\phi_t) \right] - \frac{GL_p}{b^2} \left[ w_1' (\phi_t) + b \gamma_0'(\phi_t) \right] = 0. \]

Using

\[ GL_p = \frac{EI}{1 + v}, \]

and the relations (3.5.3) for \( w_1 \) and \( \gamma_1 \), we obtain from the second relation of (3.6.13) (in accordance with (3.5.18))

\[ T_1 = -\frac{4(2 + 4v)}{3 + v} \cdot \frac{W_1}{b}. \]

With this result the first relation of (3.6.13) yields

\[ \frac{3EI}{(2 + 4v)} - \frac{w_1'(\phi) \cos 2\phi}{f_3(\phi)} = \frac{\mu_0 / \delta}{4\pi c^2} \cdot \frac{1}{2} (W_1 - W_2) \cos 2\phi. \]

An analogous relation holds for \( W_2 \), and it is then easily seen that the lowest buckling value occurs for \( W_2 = -W_1 \) and is equal to (with \( I = a^4 \))

\[ f_0 = \frac{6a^2 c^2}{\sqrt{3} \delta^2 b + 1 / 6}. \]

Again, this result is in agreement with (3.5.19) if \( 1/\sqrt{Q} \approx c / a. \)
A variational approach to the magneto-elastic buckling problem
of an arbitrary number of superconducting beams

1. Introduction

In this chapter the variational character of the method for the calculation of the magneto-elastic buckling value for superconducting structural systems is shown to full advantage. This variational method was derived in Chapter 1 and applied in the chapters 2 and 3 to pairs of superconducting beams and rings, respectively. Instead of using the explicit relations for the buckling values, as (2.1.6) and (3.2.1), we here start anew with the formulation of a functional $J = J(u; I_0)$ (taken from Chapter 1). In this, $u$ is the displacement field (in buckling) and $I_0$ is the electric current of the superconducting (slender) structure. This functional $J$ is given by (1.7.10). Moreover, we consider the relations (1.7.12), (2.1.7), (2.1.8) and (1.7.15), (2.1.9) as constraints. Since this chapter concerns systems of superconducting beams, we will use the normalized variables as introduced in (2.3.1). We then can derive from (1.7.10) (along the same lines as (1.7.18) is derived) the following expression for $J$ (for the definition of the symbols we refer to Chapter 1 and Chapter 2)

$$J(u; I_0) = -\frac{4\pi^2 ER^2}{\mu_0 l_0} \left[ \frac{1}{1+\nu} \left\{ \frac{1}{1-2\nu} \epsilon_{kk} \epsilon_{uu} + \epsilon_{uu} \epsilon_{kk} \right\} \right] dV$$
$$+ \int \left[ (B_j u_{ij} - B_{ij} u_j) + B_j B_{ij} u_i u_j - \epsilon_{ijm} B_m A_{ij} u_k u_i 
+ 2 B_j (\epsilon_{ijm} u_i - \epsilon_{ijm} u_i) (A_{jm} A_{km})_{ij} + \frac{1}{2} B_k B_j (u_{ij} u_j - u_{ij} u_j) \right] N_j dS$$
$$- \int T_{jk} u_{ik} u_{ij} dV \ . \quad (4.1.1)$$

where the intermediate (or rigid-body) fields $B$ and $A$ must be determined from (cf. (2.1.7), (2.1.8))

$$B_i = \epsilon_{ij} A_{ij} \ , \quad (4.1.2.1)$$
$$\epsilon_{ijm} B_{ij} = 0 \ , \quad (A_{ijm} - A_{ijm} = 0) \ , \quad x \in G^+ ;$$
$$B_i N_j = 0 \ , \quad (A = 0) \ , \quad x \in \partial G \ ; \quad (4.1.2.2)$$
$$B \to (x) \ , \quad x \to \infty \ ; \quad (4.1.2.3)$$

and the pre-stresses $T_{ij}$ have to satisfy

$$T_{ij} = 0 \ , \quad x \in G^- \ ; \quad T_{ij} N_j = -\frac{1}{2} (B \cdot B) N_j \ , \quad x \in \partial G \ ; \quad (4.1.3)$$

whereas the perturbed magnetic potential $\psi$ is related to the displacement field $u$ according to
\[ \Delta \psi = 0, \quad x \in \Omega^*; \quad \frac{\partial \psi}{\partial N} = (B_i u_i - B_j u_j) N_i, \quad x \in \partial \Omega; \]

\[ \psi \to 0, \quad |x| \to \infty. \quad (4.1.4) \]

The constraints for \( \psi \) are so severe that, by given \( u, \psi \) is completely determined by (4.1.4) (that is why we use the notation \( J = J(u, \psi; l_0) \) instead of \( J = J(u, \psi; l_0) \)).

We now can propose the following variational approach to the magneto-elastic buckling problem of a superconducting structural system:

the displacement field \( u \) is derived from the variation of \( J \) with respect to \( u \) and, then, the buckling value for the current \( l_0 \) is obtained by putting \( J \) equal to zero (see (1.2.14)); hence, this means that we have to solve

\[ \delta_u J = 0, \quad \text{and} \quad J = 0. \quad (4.1.5) \]

The function \( c(x) \) in the relation (4.1.2.3) is characteristic for the problem under consideration, but (after the normalization) independent of the current \( l_0 \) (see e.g. (2.3.3)). This relation is made more specific in section 2, eq. (4.2.3). Therefore, the current \( l_0 \) only enters into the functional \( J \) through the factor \( 4\pi R^2 / l_0 \) in the first term of \( J \) (see (4.1.1)), and so the buckling current can indeed be calculated by \( (4.1.5)^2 \). The approach to calculate \( u \) from (4.1.5) is different from that in the chapter 2 and 3, where an a priori choice for \( u \) was made (however, based on rather trivial physical arguments).

In the next section we shall apply the method described above to a system of an arbitrary number \( N \) of slender superconducting beams, placed parallel to each other in one plane. We shall chose the displacements of the respective beams out of a class of displacement fields representing the bending of a slender beam. The best member of this class is found by application of (4.1.5)\(^2\).

In this way an eigenvalue problem for the amplitudes of the buckling displacements of the beams is found. This eigenvalue problem is governed by a symmetric matrix \( A \). The highest eigenvalue of \( A \) corresponds to the lowest buckling value of \( l_0 \). For the calculation of the matrix \( A \) the fields \( B \) and \( \psi \) are needed. The main part of this chapter is concerned with the calculation of these fields. For \( N \geq 2 \) it seems no longer possible to find an analytical solution for \( B \) and \( \psi \) (as in chapter 2) and, therefore, we have to set up a numerical procedure for this calculation. This procedure is presented in section 4. In section 5 the numerical results are given. In the final section some specific results are presented and a comparison with the so-called Biot-Savart-method (cf. chapters 2 and 3) is made.

2. A set of \( N \) parallel beams

In chapter 2, section 4, we gave a detailed description of a system of two infinitely long parallel slender beams. For the choice of the coordinate axes \( e_1, e_2, \) and \( e_3 \), we refer to Fig.4.1. We restrict ourselves to beams with circular cross-sections, radius \( R \) (this is not necessary at this point, since the following analysis analogously holds for cross-sections which show double symmetry, cf. chapter 2, section 4). The centers of the cross-sections all lie on the \( e_3 \)-axis at distances \( 2a \) from each other. The infinitely long beams are periodically supported over length \( l \). We number the \( N \) beams with \( \kappa \), (\( 1 \leq \kappa \leq N \)). The central line of the first beam coincides with the \( e_3 \)-axis. The regions occupied by the cross-sections in the \( e_1-e_2 \)-plane are denoted by \( D^\kappa \).
\begin{align*}
(1 \leq n \leq N), & \text{ with boundaries } \partial D_n, \text{ and the 2-dimensional vacuum space outside the beams is } D^*. \\
\text{The position of the center of } D_n \text{ is } x_n = 2(n-1)a e_1.
\end{align*}

![Diagram of N parallel beams]

**Fig.4.1. A set of N parallel beams**

In the sequel it is supposed that the total currents, running along the surfaces of the superconducting beams, are all equal both in magnitude ($I_0$) and in direction. In the undeformed state of the system the currents are in the positive $e_3$ direction. Analogous to (2.4.1), the displacement field $w^{(n)}(x), x \in D_n^*$, of the $n$-th cross-section is expressed in terms of explicit functions of the in-plane variables $x$ and $y$ and the displacement $w_n(z)$ of the central line, according to

\begin{align*}
&w^{(n)}(x, y, z) = w_n(z) + \frac{1}{2} \nu ((x-x_n)^2 - y^2) w'_n(z), \\
&w^{(n)}(x, y, z) = \nu (x-x_n) y w'_n(z), \\
&w^{(n)}(x, y, z) = -(x-x_n) y w'_n(z), \quad (x, y) \in D_n^*; \\
&(x_n = 2(n-1)a; \quad n = 1 \leq n \leq N).
\end{align*}  \tag{4.2.1}

As in (2.2.5), the problem (4.1.4) for the perturbed magnetic potential $\psi$ is reduced to a 2-dimensional problem by the separation of variables

\begin{align*}
\psi(x, y, z) &= \Phi(x, y)w(z),
\end{align*}  \tag{4.2.2}

(the relationship between $w_n(z)$ and $w(z)$ will be derived furtheron, see (4.2.7)). The intermediate (or rigid-body) field $B$ (subjected to the constraints (4.1.2)) is already purely 2-dimensional, i.e. $B = B(x, y)$ and $(\partial B) = 0$. The condition at infinity, (4.1.2.3), is replaced by the set of conditions (compare with (3.2.3)) (it is the unit tangential vector along $\partial D_n$)

\begin{align*}
B &\to 0, \quad x^2 + y^2 \to \infty,
\end{align*}
\[ \int_{\partial D_0} (B \cdot \nu) \, ds = \frac{2\pi R}{\phi} \, , \quad 1 \leq n \leq N \, , \quad (4.2.3) \]

where the last condition (i.e. Ampere's law in the normalized variables) expresses the relation between the (normalized) rigid-body field \( B \) on the boundary \( \partial D_0 \) and the total current on the beam.

The constraints (4.1.2) for the rigid-body field \( B = B_0(x, y) \theta_1 + B_b(x, y) \theta_2 \), can now be rewritten out explicitly, yielding:

\[
\begin{align*}
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} &= 0 \, , \\
\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} &= (x, y) \in D^+ \\
\int_{\partial D_0} \left( -B_0 \theta_1 + B_b \theta_2 \right) ds &= 2\pi R \, , \quad 1 \leq n \leq N \\
(B_x, B_y) &\rightarrow 0 \, , \quad x^2 + y^2 \rightarrow \infty \\
\end{align*}
\]

(4.2.4)

As concerns (4.1.3) we only note that, in accordance with the boundary condition (4.1.3)\( ^2 \) the normalized pre-stresses \( T_{ij} \) are of the order of \( B^2 = (B, B) \).

The constraints (4.1.4) for \( \psi \) can be evaluated by substitution of (4.2.1) and (4.2.2) into them. In doing so we neglect terms of order \( R^2 \nu^2 R \); This means in practice, that we maintain in (4.2.1) only the zeroth order term, i.e.

\[
w_0 = w_0(x, y) \, , \quad w_0^{(1)} = w_0^{(2)} = 0 
\]

(4.2.5)

The boundary condition (4.1.4)\( ^2 \) thus becomes

\[
\frac{\partial \psi}{\partial N} = \frac{\partial \Phi(x, y)}{\partial N} \psi(x) + \frac{\partial B(x, y)}{\partial N} \psi(x) \nu(x) \, (x, y) \in \partial D_0 
\]

(4.2.6)

Since this relation must be satisfied for arbitrary \( \psi \), it is necessary that

\[
w_0(x) = \nu_0 \psi(x) \, , \quad \nu_0 \in \mathbb{R} \, , \quad 1 \leq n \leq N 
\]

(4.2.7)

We call the numbers \( \nu_0 \) the amplitudes of the buckling displacements, and we note that the \( \nu_0 \)'s are independent of each other. Furthermore, the separation (4.2.2) is only then consistent with the Laplace eq. (4.1.4)\( ^2 \) if there exists a parameter \( \lambda \in \mathbb{R}^+ \) such that

\[
\Delta \Phi(x, y) - \lambda^2 \psi(x, y) = 0 \quad \text{and} \quad \nu_0 \psi(x) = 0
\]

(4.2.8)

The parameter \( \lambda \) is related to \( l \) through the support conditions of the beams (which are supposed to be the same for all beams). For simply supported beams \( \lambda \) equals \( 2l \).

In this way the following constraint relations for \( \psi(x, y) \) are obtained from (4.1.4)

\[
\Delta \psi = \lambda^2 \psi \, , \quad (x, y) \in D^+ \\
\frac{\partial \psi}{\partial N} = -\nu_0 \frac{\partial B_0}{\partial N} \, , \quad (x, y) \in \partial D_0 \, , \quad 1 \leq n \leq N 
\]

(4.2.9)
\( \phi \to 0 \), \( x^2+y^2 \to \infty \).

The amplitudes \( v_n \) of the central line displacements and the buckling value for \( l_0 \) are still unknown and are to be solved from the variation c.q. the zero value of the functional \( J \), i.e.

\[
\frac{\partial J}{\partial v_n} = 0 \quad (1 \leq n \leq N) \quad \text{:} \quad J = 0.
\]  

(4.2.10)

We start now with the evaluation of the expression for \( J \) according to (4.1.1) for the displacement field (4.2.1). Firstly we note that in the formula (4.1.1) for the functional \( J \) the regions \( G^+, G^- \) and the boundary \( \partial G \) are to be restricted to the truncations \( D^+ \times (0, p) \), \( D^- \times (0, p) \) and \( \partial D \times (0, p) \), respectively, where \( D^+ \) and \( \partial D \) are the unions of the regions \( D_m^+ \) and the boundaries \( \partial D_m \), respectively. This is based upon the assumption that the fields are periodic in the \( z \) or \( x \)-direction with period \( p \) (see chapter 2, section 2, for more details).

The right-hand side of (4.1.1) contains three integrals. The first one, representing the elastic energy, yields in the usual way the classical energy for a slender beam in bending (see (2.2.3)). Since we neglect terms of \( O \left( R^2 \right) \) (or \( O \left( \lambda^2 R^2 \right) \), as \( \lambda \) is proportional to \( r^{-\frac{1}{2}} \) we may use in the elaboration of the second integral the reduced form (4.2.5) for the displacement field. Moreover we use (4.2.2), (4.2.4), (4.2.7) and (4.2.8)², and we introduce the set of functions \( \Phi_m \), \( (1 \leq m \leq N) \), by

\[
\Phi(x, y) = \sum_{n=1}^{N} \Phi_n(x, y).
\]

(4.2.11)

Then (4.2.9) implies that each \( \Phi_n \) is independent of the amplitudes \( v_1, v_2, \ldots, v_N \) and has to satisfy

\[
\Delta \Phi_n = \lambda^2 \Phi_n \quad (x, y) \in D^+,
\]

\[
\frac{\partial \Phi_n}{\partial N} = -\frac{\partial D}{\partial N} \quad (x, y) \in \partial D_n,
\]

\[
\frac{\partial \Phi_n}{\partial x} = 0 \quad (x, y) \in \partial D \setminus \partial D_n;
\]

\[
\Phi_n \to 0 \quad x^2+y^2 \to \infty.
\]

(4.2.12)

for each \( m \in [1, N] \).

Finally we note that (as \( T_0 \) is of the order \( B^2 \)) the third integral gives a contribution that is \( O \left( R^2 \right) \) and, hence, negligible (just as was found in the chapters 2 and 3). All this yields, apart from a factor

\[
\int_0^l \omega^2(x) \, dx
\]

(which, if desired, could be normalized to unity) the ultimate expression for the functional \( J \), i.e.

\[
J = J(v; l_0) = \langle A \cdot v, v \rangle - \epsilon(v, v),
\]

(4.2.13)

which is exact up to \( O \left( \lambda^2 R^2 \right) \langle v, v \rangle \). Here \( v \) is a \( N \)-vector, representing the buckling displacements, which possesses the following column representation with regard to the orthonormal, positively oriented base \( (E_1, \ldots, E_N) \) of \( R_N \).
\[ v = [v_1, v_2, \ldots, v_N]^T \quad (4.2.14) \]

\[ \kappa = \frac{4 \pi^2 E \pi R^2}{\mu_0 \beta^2}, \quad J = \int_0^l \frac{x^2}{\delta^2} \, dS = \frac{1}{4} \pi R^4 \quad (4.2.15) \]

and \( A \) is a linear transformation from \( E_N \rightarrow E_N \), with the following matrix with regard to the base \( \{E_1, \ldots, E_N\} \), (in \( A_{mn} \) summation is not applied)

\[ A_{mm} = - \int \frac{\partial b_m}{\partial x} \, ds \quad 1 \leq m, n \leq N, \quad m \neq n ; \]

\[ A_{mn} = - \int \left( \phi_n + B_m \right) \frac{\partial b_n}{\partial N} \, ds \quad 1 \leq n \leq N. \quad (4.2.16) \]

On account of the Helmholtz problem \((4.2.12)\) and Green's second identity we derive from the matrix representation formulas \((4.2.16)\) the property

\[ A_{mm} = \int \phi_m \frac{\partial b_m}{\partial N} \, ds = \int \phi_m \frac{\partial b_m}{\partial N} \, ds = A_{mm}, \quad n \neq m. \quad (4.2.17) \]

Hence, the linear transformation \( A \) is symmetric and elaboration of \((4.2.10)\) then yields

\[ A \, v = \kappa \, v, \quad \kappa \neq 0, \quad \kappa > 0; \quad v = \frac{(A, v)}{(v, v)}. \quad (4.2.18) \]

The set \((4.2.18)\) implies that the lowest buckling value for the current \( l_0 \) corresponds to the highest positive eigenvalue \( \kappa \) of the matrix \( A \). This matrix still depends on the parameter \( \lambda \), by means of the functions \( \phi_n \) (cf. \((4.2.12)\)). In the next section we shall prove that for slender beams the influence of the ratio \( R/l \) on the eigenvalue for \( \kappa \) is negligible.

3. Complex Formulation

In this section we shall use a great deal of the complex manipulations, which were already applied to the buckling problems for one single beam and for a set of two parallel beams in chapter 2. Therefore, we shall recapitulate only those notions and methods, which are indispensable to the understanding of the complete procedure. We introduce a small parameter \( \delta \) \((0 < \delta \ll 1)\), the normalized complex coordinate \( z \) and the complex function \( F \) in the same way as in \((2.2.7), (2.3.7), (2.3.25)\), i.e.

\[ \delta = \lambda R, \quad z = (x + iy) R, \quad F = F_z - i F_y, \quad z \in S^* \cup C. \quad (4.3.1) \]

where \( S^* \) and \( C \) stand for the region and curves in the complex \( z \)-plane corresponding to \( D^* \) and \( \partial D \), respectively. Moreover, we denote the \( \varepsilon \)-transformations of \( D_z, \partial D \), and \( v_n \) by \( S_{\varepsilon}, C_{\varepsilon}, \) and \( z_{\varepsilon}, \) respectively.

Analogous to \((2.3.26), (2.4.2), (2.4.4)\) the relations for the rigid-body state (see \((4.2.4)\)) can be transformed into (for the definition of the complex line element \( dz \) see \((2.3.22)\))
\[ F \text{ analytical }, z \in S^+ , \]
\[ F dz \in \mathbb{R}, z \in C , \]
\[ F \to 0, |z| \to \infty , \]
\[ \frac{F dz}{C} = 2\pi, 1 \leq n \leq N . \quad (4.3.2) \]

The introduction of the real-valued functions (compare with (2.3.38), (2.4.5) and note the difference between the definition of \( f_m \) used here and the one according to (2.4.5))

\[ f_m (z, z) = \begin{cases} 
\phi_m, & 1 \leq |z-z_m| \leq u/R, n \neq m , \\
(\phi_m + B_j), & 1 \leq |z-z_m| \leq a/R . 
\end{cases} \quad (4.3.3) \]

(1 \leq m, n \leq N) enables us to write (4.2.16) as (for the definition of the complex derivative \( \frac{dz}{dz} \), see (2.3.24))

\[ A_{nm} = -2 \int_{z_m}^{z_n} \text{Im} \frac{\partial \phi_m}{\partial z} dz = - \text{Im} \int_{z_m}^{z_n} \frac{df_m}{dz} dz . \quad (4.3.4) \]

and (4.2.12) as

\[ \frac{\partial f_m}{\partial \bar{z}} = 0, z \in C, 1 \leq m, n \leq N . \quad (4.3.5) \]

What we are looking for are the numerical values of the coefficients \( A_{nm} \) according to (4.3.4) and, hence, it is evident that our special interest is in the boundary values of the functions \( f_m \). For the calculation of those values an integral equation is constructed. Since the construction runs along the lines of the methods presented in (2.3.31)-(2.3.46) and (2.4.7)-(2.4.15) we do not enter into further details here, but only state the main results. Also, we use the convention that any \( O (\delta^2 \log \delta) \)-term is referred to as an \( O (\delta^2) \)-term.

The functions \( f_m \) are asymptotically approximated by the \( \delta \)-dependent functions \( g_m \) according to

\[ f_m (z) = g_m (z) (1 + O (\delta^2)), z \in C, 1 \leq m \leq N . \quad (4.3.6) \]

where \( g_m \) satisfies (compare with (2.4.10.2))

\[ \frac{1}{2} \int_{z_0}^{z_0} \text{Re} \left( \frac{1}{2 \pi i} \frac{g_m (z)}{z-z_0} \right) dz = R (z_0) . \quad (4.3.7) \]

with

\[ R (z_0) = \text{Re} \left( \frac{1}{2 \pi i} \int_{z_m}^{z_n} \frac{f_m (z)}{z-z_0} dz \right), \quad z_0 \in C \cap C_m . \quad (4.3.8.1) \]

and
\[ R(z_0) = \text{Re} \left( \frac{1}{2\pi i} \int_{C_0} \frac{F(z)}{z-z_0} \, dz \right), \quad z_0 \in C_m. \]  

(4.3.8.2)

Cauchy’s theorem for analytical functions states that

\[ \frac{1}{2\pi i} \oint_{\gamma} \frac{F(z)}{z-z_0} \, dz = \begin{cases} 
0, & z_0 \in S^- \setminus \gamma, \\
-F(z_0), & z_0 \in S^+.
\end{cases} \]

(4.3.9)

Introduction of the N analytical functions (so-called Cauchy-integrals)

\[ \Phi_m(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\Phi_m(z)}{z-z_0} \, dz - \frac{1}{2\pi i} \oint_{\gamma} \frac{F(z)}{z-z_0} \, dz, \quad z_0 \in C_m, \]

(4.3.10)

and use of (4.3.9) in (4.3.7)-(4.3.8) leads us to the following set of Riemann-Hilbert problems

\[ \text{Re} \Phi_m(z_0) = 0, \quad z_0 \in C_m. \]

(4.3.11.1)

and

\[ \text{Im} \left( \Phi_m(z_0) - \Phi_m^*(z_0) \right) = \begin{cases} 
-\text{Im} F(z_0), & z_0 \in C_m, \\
0, & z_0 \in C \setminus C_m.
\end{cases} \]

(4.3.11.2)

Furthermore, the functions \( \tilde{g}_m \) are related to the Cauchy-integrals \( \Phi_m \),

\[ \tilde{g}_m(z_0) = \begin{cases} 
\Phi_m^*(z_0) - \Phi_m^*(z_0), & z_0 \in C_m, \\
\Phi_m^*(z_0) - \Phi_m^*(z_0), & z_0 \in C_m.
\end{cases} \]

(4.3.12)

Since \( \Phi_m \) is analytical in \( S^- \) it follows from (4.3.11.1) that \( \Phi_m^* \) equals an imaginary constant, i.e.

\[ \Phi_m(z) = i \Phi_m^* \right, \quad z \in S^-, \quad \Phi_m^* \in \mathbb{R}. \]

(4.3.13)

Substituting (4.3.6), (4.3.12) and (4.3.13) into the expression for \( A_{m} \) according to (4.3.4) yields, under the neglect of \( O(\delta^3) \) terms,

\[ A_{m} = -\text{Im} \int_{\gamma} \tilde{g}_m \frac{dF}{dz} \, dz = \text{Im} \int_{\gamma} \frac{d\tilde{g}_m}{dz} \, dz = \]

(4.3.14)

Using the short-hand notation
we arrive at the ultimate mathematical formulation for the determination of the buckling current \( I_0 \):

**Calculate the matrix \( A \) from**

\[
A_{mn} = \left( \mathrm{Im} \left\{ \frac{1}{C_0} \int F_m(z) dz \right\} \right)_{1 \leq m, n \leq N},
\]

(4.3.16)

where the functions \( F(z) \) and \( F_n(z) \) satisfy

\[
F(z), F_n(z) \text{ analytical, } z \in S^+.
\]

\[
F(z), F_n(z) \rightarrow 0, \quad |z| \rightarrow \infty.
\]

\[
\begin{cases}
F(z)dz = 2\pi, & F_n(z)dz = 0, \\
\int_{C_0} F_n(z)dz = 0, & z \in C.
\end{cases}
\]

(4.3.17)

\[
\text{Im} \left( \frac{dF}{dz} \right) = 0, \quad z \in C.
\]

and, then, the amplitude-vector \( \mathbf{v} \) and the buckling current \( I_0 \) are obtained from the eigenvalue problem

\[
A \mathbf{v} = \lambda \mathbf{v}, \quad \mathbf{v} \neq 0, \quad \lambda > 0,
\]

(4.3.18)

and the relation

\[
I_0 = 2\pi \delta \sqrt{\frac{E\sigma_j\rho}{\mu_0 k R^2}}.
\]

(4.3.19)

On account of the fact that, within our approximation, the matrix \( A \) is independent of the parameter \( \delta \), it is evident that the buckling current \( I_0 \) is proportional to \( \delta^2 \). Moreover, we note that (4.3.17) directly implies that

\[
\sum_{n=1}^{N} F_n(z) = \frac{dF}{dz}, \quad z \in S^+ \cup C.
\]

(4.3.20)

and as a consequence, the column sums of the matrix \( A \) are equal to zero. Use of this property in (4.3.18) shows us that

\[
\sum_{n=1}^{N} A_{mn} = 0, \quad \sum_{n=1}^{N} v_n = 0.
\]

(4.3.21)

In other words, the amplitudes of the central line displacements always cancel each other.
Formally, we still have to prove that the approximation (4.3.16) of the original matrix \( A \) (see (4.2.16)) is again symmetric. The formulation of the problem (4.3.17) guarantees the existence of analytical functions \( \chi_m(v) (v \in S^* \cup C) \) such that (in \( c_m \) summation is not applied)

\[
F_m = \frac{d\chi_m}{dz} , \quad z \in S^* \cup C ;
\]

\[
\text{Im} \chi_m = c_m , \quad z \in C_m , \quad m \neq n , \quad 1 \leq n \leq N ;
\]

\[
\text{Im} \chi_m = \text{Im} F + c_m , \quad z \in C_m ;
\]

\[
\chi_m \to 0 \quad \text{as} \quad |z| \to \infty ;
\]

for each \( m, 1 \leq m \leq N \) (\( c_m \) are constants, \( 1 \leq m,n \leq N \)).

Using the fact that integration of a tangential derivative along a closed curve amounts to zero and applying the boundary conditions (4.3.22)\(^3\) and Green's second identity we derive for \( m \neq n, 1 \leq m,n \leq N \)

\[
A_{mn} = -\lim_{C_m} \int_{C_m} F_m dz = -\lim_{C_m} \int_{C_m} \frac{dF}{dz} \frac{d\chi_m}{dz} \frac{d}{dz} \frac{d}{dz} =
\]

\[
= \int_{C_m} \text{Re} \chi_m \frac{d}{dz} (\text{Im} \chi_m) dz = \int_{C_m} \text{Re} \chi_m \frac{d}{dz} (\text{Re} \chi_m) dz =
\]

\[
= \int_{C_m} \frac{d}{dz} (\text{Re} \chi_m) dz A_{mm} .
\]

Therefore, the matrix \( A \) (see (4.3.16)) is indeed symmetric.

It should be noted that

\[
A \left[ 1, 1, \ldots, 1 \right]^T = 0 ,
\]

so the matrix \( A \) is singular.

With the use of the symmetry (4.3.25) and the fact that the column- and row-sums of \( A \) are zero (see (4.3.23)), the following formula can be derived:

\[
(A v, v) = \sum_{n=1}^{N} \sum_{m=1}^{N} A_{nm} (v_n - v_m)^2 .
\]

The numerical method described in the next sections shows that all \( A_{nm} \) for \( 1 \leq m,n \leq N, m \neq n \) are negative, so the matrix \( A \) is positive semidefinite, although this is difficult to prove analytically.

From (4.3.16), (4.3.17), (4.3.21), (4.3.23) immediately follows

\[
F(z_0 - z) = -F(z) , \quad dF/dz (z_0 - z) = dF/dz (z) , \quad F_m (z_0 - z) = F_{m+1} - A_{nm} (z_0 - z) ,
\]

\[
A_{mn} = A_{nm} = A_{n+1,m+1} + \sum_{m}^{N} A_{mn} = 0 ,
\]

which means that the number of unknowns \( A_{nm} \) reduces to

\[
N(N-1) = \frac{1}{2} (N^2 - N) \mod 2).
\]
4. Numerical calculation of the matrix \( A \)

In chapter 2 the region \( S^+ \) was transformed into a ring-shaped region by conformal mapping for the case of two circular rods; the resulting problem was solved by complex analysis. For the case \( N > 2 \) such an analytical treatment is impossible and therefore we search for a numerical solution procedure for the eigenvalue problem (4.3.18). This, more specifically, amounts to numerically calculating the elements \( A_{mn} \) of the matrix \( A \), according to (4.3.16).

The first step is to reformulate the problem (4.3.16)-(4.3.20) in real terms, by introduction of the real functions \( \varphi = \varphi(x, y) \) and \( \varphi_m = \varphi_m(x, y) \) through

\[
F = -\frac{\partial \varphi}{\partial y} - i \frac{\partial \varphi}{\partial x}, \quad F_m = -\frac{\partial \varphi_m}{\partial y} - i \frac{\partial \varphi_m}{\partial x},
\]

(4.4.1)

for \( 1 \leq m \leq N \) and \( x = (x, y) \in S^+ \cup C \).

With \( ds = i N \, ds = (N_x - N_y) \, ds \), and \( \partial \varphi \partial s = 0 \) (see (4.4.33)) the problem (4.3.17) then transforms into

Find the positive eigenvalues \( \kappa \) of the matrix \( A \) (see (4.3.18)) with elements

\[
A_{mn} = \int \frac{\partial \varphi_m}{\partial \kappa} \frac{\partial \varphi}{\partial \kappa} \, ds, \quad 1 \leq m, n \leq N;
\]

(4.4.2)

where \( \varphi \) and \( \varphi_m \) satisfy

\[
\Delta \varphi = 0, \quad \Delta \varphi_m = 0, \quad x \in S^+;
\]

(4.4.3.1)

\[
\nabla \varphi \to 0, \quad \nabla \varphi_m \to 0, \quad |x| \to \infty;
\]

(4.4.3.2)

\[
\frac{\partial \varphi}{\partial \nu} = 0, \quad x \in C; \quad \frac{\partial \varphi_m}{\partial \nu} = \delta_{mn} \frac{\partial \varphi_m}{\partial \nu}, \quad x \in C_n;
\]

(4.4.3.3)

\[
\int_C \frac{\partial \varphi}{\partial \kappa} \, ds = 2\pi \varphi, \quad \int_C \frac{\partial \varphi_m}{\partial \kappa} \, ds = 0.
\]

(4.4.3.4)

for \( 1 \leq m, n \leq N \).

With (4.4.3.1) and (4.4.3.4) the conditions at infinity (4.4.3.2) can be made more explicit, yielding

\[
\varphi = N \log |x| + O(1), \quad \varphi_m = O(1), \quad |x| \to \infty.
\]

(4.4.4)

If desired for, the \( O(1) \)-terms (constants) in (4.4.4) can be made zero, i.e. replaced by \( o(1) \)-terms, because the potentials \( \varphi \) and \( \varphi_m \) are only relevant up to a constant term.

Moreover, the boundary conditions (4.4.3.3) can be integrated along each separate boundary \( C_n \), giving

\[
\varphi = \alpha, \quad \varphi_m = \delta_{mn} N_x \frac{\partial \varphi_m}{\partial N} + \beta_m, \quad x \in C_n;
\]

(4.4.5)

where \( \alpha \) and \( \beta_m \) are constants, which will be determined from (4.4.4.4) later on.

In the second step the functions \( \varphi \) and \( \varphi_m \) are split up in a set of harmonic functions (in \( S^+ \cup C \),
which are bounded at infinity and known on the boundary $C$, according to $\chi_k = 2(\kappa - 1) \alpha \theta^\kappa_1$, the center of the $k$-th cross-section)

$$
\omega = \sum_{k=1}^{N} \log |x - x_k| + \psi = \sum_{k=1}^{N} \alpha_k u_k ,
$$

$$
\psi_m = \psi_m + \sum_{k=1}^{N} \beta_{mk} u_k .
$$

(4.4.6)

The first term of (4.4.6) is chosen in such a way that the first condition of (4.4.3.4) is satisfied. The functions $\psi$ and $\psi_m$ have to satisfy the boundary conditions (4.4.5) with $\alpha_m = \beta_{mk} = 0$, whereas the remaining part of these boundary conditions are fulfilled by the parts with $u_k$. All the unknown functions (i.e., $\psi$, $\psi_m$ and $u_k$) can be found from an exterior Dirichlet problem, which form reads in general ($V = V(x, y)$)

$$
\Delta V = 0, \quad x \in S^*$ ,
$$

$$
V = O(1), \quad |x| \to \infty ,
$$

$$
V = f, \quad x \in C ;
$$

(4.4.7)

here $f$ is a given function of $x$ on the boundary $C$ of the exterior region $S^*$. In (4.4.7) we have to substitute for $V, \psi, \psi_m$ and $u_k$ successively. The associated boundary functions $f$ are given by:

$$
\text{for } V = \psi, \quad \rightarrow f(x) = \sum_{k=1}^{N} \log |x - x_k| ,
$$

(4.4.8.1)

$$
\text{for } V = \psi_m, \quad \rightarrow f(x) =
\begin{cases} 
0, & x \in C \cup C_m , \\
N \frac{\partial \psi_m}{\partial N}, & x \in C_m ,
\end{cases}
$$

(4.4.8.2)

$$
\text{for } V = u_k, \quad \rightarrow f(x) =
\begin{cases} 
0, & x \in C \cup C_k , \\
1, & x \in C_k .
\end{cases}
$$

(4.4.8.3)

The coefficients $\alpha_k$ and $\beta_{mk}$ are still to be determined from (4.4.3.4). This results in the following relations (for $1 \leq m, n \leq N$)

$$
\sum_{k=1}^{N} \alpha_k \int_{C_k} \frac{\partial u_k}{\partial N} ds = - \int_{C_k} \frac{\partial \psi_m}{\partial N} ds ,
$$

(4.4.9)

$$
\sum_{k=1}^{N} \beta_{mk} \int_{C_k} \frac{\partial u_k}{\partial N} ds = - \int_{C_k} \frac{\partial \psi_m}{\partial N} ds .
$$

It should be noted that the $N$ relations of the set (4.4.9)\(^1\) and the $N \times N$ relations of (4.4.9)\(^2\) are linearly dependent, because (for each $m, k \in \{1, N\}$)
derived from the fact that \( \psi, \psi_\omega \text{ and } u_\omega \) are harmonic in \( S^+ \) and bounded at infinity. Therefore in the sets of (4.4.9) one relation has to be dropped, which can be replaced by the following relations at infinity

\[
\sum_{k=1}^{N} \alpha_k u_k + \psi = 0, \quad |x| \to \infty,
\]

(4.4.11)

\[
\sum_{k=1}^{N} \beta_k \psi_k + \psi_\omega = 0, \quad (1 \leq m \leq N), \quad |x| \to \infty.
\]

For the derivation of these relations it is necessary to replace in (4.4.4) the \( O(1) \)-symbols by \( o(1) \)-symbols.

Using (4.4.3.3) the expression (4.4.2) for \( A_{\omega \omega} \) can be rewritten as

\[
A_{\omega \omega} = \int \left[ N_\omega \frac{\partial \psi_\omega}{\partial N} - \delta_{\omega \omega} N_\omega \frac{\partial}{\partial x} \left( N_\omega \frac{\partial \psi_\omega}{\partial N} \right) \right] \frac{\partial \psi_\omega}{\partial N} \, ds.
\]

(4.4.12)

For the calculation of these integrals we first have to solve the basic problems (4.4.7)-(4.4.8). However, from (4.4.11) we see that we are in fact interested only in the values of the normal derivatives along the boundaries, i.e. \( \partial V / \partial N \) for \( x \in C_n, \ 1 \leq n \leq N \).

The further procedure could be based on the use of layer potentials (cf. [24], [26]). However, introduction of a simple layer potential for the function \( V \) leads us to a situation in which it is difficult to determine the limit of \( V \) at infinity; moreover, the problem now involves a Fredholm integral equation of the first kind (weakly singular), i.e. an ill-posed problem for the density of the potential. On the other hand, introducing a double layer potential we arrive at a Fredholm integral equation of the second kind, which is singular in general.

To avoid these complications, we separate from \( V \) particular logarithmic solutions of the Laplace equation. The remaining part of \( V \) can then be expressed in double layer potentials, the densities of which satisfy ordinary integral equations. This separation is of the following form

\[
V(x) = V_1(x) + V_2(x), \quad x \in S^+ \cup C,
\]

(4.4.13)

where firstly

\[
V_1(x) = -\frac{1}{2\kappa} \int \frac{\mu(y)}{N_{xy}} \log |x-y| \, ds_y, \quad x \in S^+ \cup S^-.
\]

(4.4.14)

with \( \mu(x) \) satisfying

\[
\frac{1}{2\pi} \mu(x) = \frac{1}{2\pi} \int \mu(y) \frac{\partial}{\partial N_y} \log |x-y| \, ds_y = f(x), \quad x \in C_n,
\]

(4.4.15)

or

\[
L_n^+(\mu(x)) = f(x), \quad x \in C_n, \ 1 \leq n \leq N,
\]

(4.4.16)

for short. Secondly
\[ V_2(x) = c_0 + \sum_{i=0}^{N} c_i \left( \frac{1}{2\pi} \log |x-x_i| - V_i(x) \right), \]  
(4.4.17)

for \( x \in S^+ \cup (S^- \setminus \{x_1, x_2, \ldots, x_N\}) \), where

\[ \sum_{i=0}^{N} c_i = 0, \]  
(4.4.18)

\[ V_i(x) = -\frac{1}{2\pi} \int_{C} \mu'(y) \frac{\partial}{\partial \nu_y} \log |x-y| \, ds_y, \quad x \in S^+ \cup S^-. \]  
(4.4.19)

while \( \mu(x) \) has to satisfy

\[ L^* \mu(x) = -\frac{1}{2\pi} \log |x-x_i|, \quad x \in C_a. \]  
(4.4.20)

Evidently

\[ \Delta V_1 = 0, \quad x \in S \cup S^-, \]  
(4.4.21)

\[ \Delta V_2 = 0, \quad x \in S^+, \quad \Delta \nu_2 = c_0 \delta_0(x-x_i), \quad x \in S^- \]  
(4.4.22)

(\( \delta_0 \) is Dirac’s delta function) and

\[ V_1 \to 0, \quad V_2 \to c_0 = O(1), \quad |x| \to \infty, \]  
(4.4.23)

where the latter is a consequence of (4.4.18). From (4.4.23) together with (4.4.13) it follows that

\[ c_0 = V_\infty = \lim_{|x| \to \infty} V(x). \]  
(4.4.24)

As we shall show further on, the numbers \( c_0, c_1, \ldots, c_N \) can be chosen in such a way that \( V = f \) on \( C \). Note that the integral equations (4.4.15) (or (4.4.16)) and (4.4.20) possess regular kernels indeed. Moreover, the normal derivatives of the double layer potentials \( V_1 \) and \( V_2 \) are continuous across the boundaries \( C_a \) (see [27], p. 170), so (since \( \Delta V_1 = 0 \) and \( \Delta V_2 = 0, x \in S^a \))

\[ \int_{C_a} \frac{\partial V_1}{\partial N} \, ds = 0, \quad \int_{C_a} \frac{\partial V_2}{\partial N} \, ds = 0, \quad 1 \leq i \leq N, \]  
(4.4.25)

and then (from (4.4.17))

\[ c_a = \int_{C_a} \frac{\partial V_2}{\partial N} \, ds = \int_{C_a} \frac{\partial V_2}{\partial N} \, ds, \quad 1 \leq i \leq N. \]  
(4.4.26)

Taking for \( V_1(x) \) the exterior limit for \( x \to C_a \), in (4.4.24) to be denoted by \( V_1(x) \), we arrive at (cf. [24], p. 382; \( f \) stands for the principal value)

\[ V_1(x) = \frac{1}{2\pi} \mu(x) - \frac{1}{2\pi} \int_{C} \mu'(y) \frac{\partial}{\partial \nu_y} \log |x-y| \, ds_y \]  
(4.4.27)

where the last step follows immediately from (4.4.15). Writing for \( y \) and for \( x \in C_a \),

\[ = f(x) = \frac{1}{2\pi} \mu(x) \frac{\partial}{\partial \nu_y} \log |x-y| \, ds_y, \quad x \in C_a, \]  
(4.4.27)
\[ y = (x_r + r \cos \phi) e_1 + r \sin \phi e_2, \]

and

\[ x = (x_r + \cos \theta) e_1 + \sin \theta e_2, \]

respectively. We find for \( y \in C_n \) (\( N_y = \cos \theta e_1 + \sin \theta e_2 \))

\[
\frac{\partial}{\partial N_y} \log |x - y| = \left[ \frac{\partial}{\partial r} \log |x - y| \right]_{r=1} = \left[ \frac{(-x+y, N_y)}{|x - y|^2} \right]_{r=1} = \frac{1 - \cos(\theta - \phi)}{2(1 - \cos(\theta - \phi))} = \frac{1}{2}.
\]  

(4.4.28)

With (4.4.28) the integral on the right-hand side of (4.4.27) can be evaluated to (for \( x \in C_n \))

\[
\frac{1}{2\pi} \int_{C_n} \mu(y) \frac{\partial}{\partial N_y} \log |x - y| \, ds_y = \frac{1}{2} \bar{\mu}_n.
\]

(4.4.29)

where \( \bar{\mu}_n \) stands for the mean value of \( \mu \) on \( C_n \), i.e.

\[
\bar{\mu}_n = \frac{1}{2\pi} \int_{C_n} \mu \, ds.
\]

(4.4.30)

As a consequence of (4.4.29), (4.4.27) reduces to

\[
V_1(x) = f(x) = \frac{1}{2} \bar{\mu}_n, \quad x \in C_n.
\]

(4.4.31)

In a similar way one deduces

\[
V_2(x) = c_0 + \frac{1}{2} \sum_{m=1}^N c_m \bar{\mu}_m, \quad x \in C_n.
\]

(4.4.32)

where \( \bar{\mu}_m \) is the mean value of the density \( \mu^m \) on \( C_n \). The boundary condition

\[
V(x) = V_1(x) + V_2(x) = f(x).
\]

now yields

\[
\frac{1}{2} \sum_{m=1}^N c_m \bar{\mu}_m + c_0 = \frac{1}{2} \bar{\mu}_n, \quad 1 \leq n \leq N.
\]

(4.4.33)

This set, together with the relation (4.4.18), which is the necessary condition for the boundedness of \( V_2(x) \) at infinity, constitutes the basic set for the calculation of \( c_0, c_1, \ldots, c_N \) (after \( \mu \) and \( \mu' \) are known). We can write this total set in a more concise notation by introducing the \( N \)-column vectors \( e \) and \( \mu' \) and the \( (N \times N) \)-matrix \( B \) by

\[
e_m = \frac{1}{2} \bar{\mu}_m, \quad e_n = 1, \quad \beta_m = \frac{1}{2} \bar{\mu}_m^m.
\]

(4.4.34)

for \( 1 \leq m, n \leq N \). Then, the above mentioned set can be written as
\[
\begin{bmatrix}
B & e \\
e^T & 0
\end{bmatrix}
\begin{bmatrix}
e \\
c_0
\end{bmatrix}
=egin{bmatrix}
a
\end{bmatrix}.
\]

(4.4.35)

In this system of linear equations the vector \( (e^T, c_0)^T \) represents the unknown variables. The vector \( e \) is a fixed one, whereas the matrix \( B \) and the vector \( a \) are known once the ordinary integral equations (4.4.16) and (4.4.20) are solved (recall that this must be done for all \( f \)'s out of the three distinct sets presented in (4.4.8)). Note also that we do not need to calculate the functions \( V_1(x) \) or \( V_2(x) \) and \( V_3(x) \) for the solution of (4.4.35); these are only auxiliary functions.

Before proceeding to calculate \( A_{\text{aux}} \), we have to prove that the solutions of the integral equations (4.4.16) and (4.4.20), and the solution of the linear set (4.4.35) are unique. To this end we present the following two lemmas:

Lemma

The solution of the integral equation (see (4.4.15)-(4.4.16)) for unknown \( \sigma(x) \) and given \( \tau(x) \)

\[
L^+ \{ \sigma(x) \} = \tau(x), \quad x \in C_0, \quad 1 \leq n \leq N,
\]

is unique.

Proof

For the proof we only have to show that if \( \tau(x) = 0 \) then the integral equation only has the trivial solution \( \sigma(x) = 0 \). For this purpose we consider the double layer potential

\[
U(x) = -\frac{1}{2\pi} \int \sigma(y) \frac{\partial}{\partial N_y} \log |x - y| \, ds_y, \quad x \in S^+ \cup S^-.
\]

The integral equation (with \( \tau(x) = 0 \)) and the limiting values of \( U \) for \( x \to C \) yield

\[
U^+(x) = -\frac{1}{2} \sigma_0, \quad U^-(x) = -\sigma(x) - \frac{1}{2} \sigma_0, \quad x \in C_0, \quad 1 \leq n \leq N.
\]

Moreover we have

\[
A U = 0, \quad x \in S^+ \cup S^-;
\]

\[
U(x) = O(|x|^{-1}), \quad |x| \to \infty;
\]

\[
\begin{bmatrix}
\frac{\partial U}{\partial N}^+
\end{bmatrix} = \begin{bmatrix}
\frac{\partial U}{\partial N}^-
\end{bmatrix}, \quad x \in C.
\]

So,

\[
\int S | \nabla U |^2 \, dS = -\int U^+ \begin{bmatrix}
\frac{\partial U}{\partial N}
\end{bmatrix}^+ \, dS = \frac{1}{2} \sum_{a=1}^{2N} \int_{C_a} \begin{bmatrix}
\frac{\partial U}{\partial N}
\end{bmatrix}^- \, ds
\]

is finite for all \( x \in S^+ \cup S^- \).
\[ \frac{1}{2} \sum_{i=1}^{N} \frac{\partial U_i}{\partial S_i} dS = 0, \]

and \( U \to 0, \quad |x| \to \infty \), hence, \( U(x) = 0, \quad x \in S^+ \) and thus \( \partial \alpha_i = 0, \quad 1 \leq i \leq N \). Therefore, \( (\partial U/\partial S)^+ = (\partial U/\partial S)^- = 0, \quad x \in C \), and \( \Delta U = 0, \quad x \in S^- \). We conclude \( U^-(x) = \text{constant} \). This yields

\[ U^- (x) = -\partial \alpha_i^2 - \sigma(x) = -\sigma(x) = \text{constant}, \]

but then \( \sigma(x) = \alpha_i = 0, \quad x \in C, \quad 1 \leq i \leq N \).

Next we prove

**Lemma**

The solution \((e^t, c_0)^T\) of the system (4.36) of linear equations is unique.

**Proof**

The potential \( V_2 \) satisfies:

\[ \Delta V_2 = 0, \quad x \in S^+; \]

\[ \Delta V_2 = c_0 \delta_{ij}(x - x_i), \quad x \in S_i, \quad 1 \leq i \leq N; \]

\[ |\nabla V_2| = O(|x|^{-1}), \quad |x| \to \infty; \]

\[ V_2 = c_0 + \sum_{i=1}^{N} B_{ii} c_i, \quad x \in C, \quad 1 \leq k \leq N; \]

\[ \frac{\partial V_2^+}{\partial N} = \frac{\partial V_2^-}{\partial N}, \quad x \in C. \]

Therefore

\[ 0 \leq \frac{1}{\varepsilon} |\nabla V_2|^2 dS = -\varepsilon \sum_{i=1}^{N} \left( c_0 + \sum_{i=1}^{N} B_{ii} c_i \right) \left( \frac{\partial V_2^-}{\partial N} \right) dS \]

\[ = -\sum_{i=1}^{N} \left( c_0 + \sum_{i=1}^{N} B_{ii} c_i \right) c_i = (c_0 e^t e + c^T B e), \]

which reveals \( c_0 e^t e + c^T B e > 0 \) if \( V_2 \neq 0 \) (i.e. \( c \neq 0 \)).

If \( V_2 = c_0 \) i.e. \( c = 0 \). It is sufficient to prove that the set of equations for \( a = 0 \) only has the trivial zero-solution. This is evident since for \( a = 0 \) the equations show

\[ B e + c_0 e = 0, \quad c^T e = 0. \]

This yields \( c_0 e^T e + c^T B e = 0 \) and thus \( c = 0 \) and \( c_0 = 0 \).
In fact we are only interested in the values of the normal derivative of \( V \) at the boundaries \( C_m \). For this purpose we consider the function

\[
V_3(x) = V(x) - \frac{1}{2\pi} \sum_{m=1}^{N} c_m \log |x-x_m|.
\]  

(4.4.36)

hence,

\[
V_3(x) = V_1(x) + c_0 - \sum_{m=1}^{N} c_m V_m.
\]  

(4.4.37)

From the foregoing analysis it then follows that \( V_3(x) \) is harmonic in \( S^+ \), bounded at infinity and such that (from (4.4.25))

\[
\int C \frac{\partial V_3}{\partial n} \, ds = 0, \quad 1 \leq k \leq N.
\]  

(4.4.38)

These features guarantee the existence of a harmonic function \( W(x) \), \( x \in S^+ \cup C \), the conjugate function of \( V_3 \), such that

\[
\Delta W = 0, \quad x \in S^+,
\]

\[
W = O(1), \quad |x| \to \infty,
\]

\[
\frac{\partial W}{\partial n} = \frac{\partial V_3}{\partial n} - \frac{1}{2\pi} \sum_{m=1}^{N} c_m \log |x-x_m|, \quad x \in C.
\]

(4.4.39)

since \( V = f \), for \( x \in C \).

The above problem for \( W \) is, apart from an irrelevant constant, uniquely solved by writing \( W \) as a simple layer potential, the density of which satisfies an ordinary integral equation with regular kernal. Thus (cf. [24])

\[
W(x) = -\frac{1}{2\pi} \int \frac{v(y) \log |x-y|}{c(C \cup C_y)} \, dy,
\]  

(4.4.40)

with \( v \) following from

\[
\frac{1}{2\pi} \frac{v(x)}{c(C \cup C_y)} \int \frac{v(y) \frac{\partial}{\partial n} \log |x-y|}{c(C \cup C_y)} \, ds_y = \frac{\partial V_3}{\partial n}, \quad x \in C_y.
\]  

(4.4.41)

with \( \partial V_3/\partial n \) as given by (4.39) \( (W(x) \) is continuous and \( v = 0 \)). Since \( W \) is the conjugate of \( V_3 \), the normal derivative of \( V_3 \) on \( C \) equals the tangential derivative of \( W \) along \( C \), so

\[
\frac{\partial V}{\partial n} = \frac{\partial W}{\partial t} - \frac{1}{2\pi} \sum_{m=1}^{N} c_m \log |x-x_m|.
\]  

(4.4.42)

According to (4.4.40)

\[
\frac{\partial W}{\partial t} = \frac{1}{2\pi} \int \frac{v(y) (x-y, s_y)}{|x-y|^2} \, ds_y
\]
\[
\begin{align*}
\psi = -\frac{1}{2\pi} \int_{C_\infty} \frac{\nu(y) (x-y, s_\nu)}{|x-y|^2} \, dy - \frac{1}{2\pi} \int_{C_\infty} \frac{[\nu(y) - \nu(x)] (x-y, s_\nu)}{|x-y|^2} \, dy \\
-\nu(x) \int_{C_\infty} \frac{(x-y, s_\nu)}{|x-y|^2} \, ds_\nu, \quad x \in C_\infty, \ 1 \leq \eta \leq N. \tag{4.4.43}
\end{align*}
\]

Analogous to (4.4.28) it can be shown that
\[
\frac{(x-y, s_\nu)}{|x-y|^2} = \frac{\sin(\theta - \phi)}{2(1 - \cos(\theta - \phi))},
\tag{4.4.44}
\]

which is an odd function of \( \phi \) around \( \theta = \pi \), and, hence, the last integral in the right-hand side of (4.4.43) is equal to zero. Thus we obtain from (4.4.42)-(4.4.43)
\[
\frac{\partial V}{\partial N} = -\frac{1}{2\pi} \sum_{n=1}^{N} \epsilon_n \log |x - x_n| - \frac{1}{2\pi} \int_{C_\infty} \frac{\nu(y) (x-y, s_\nu)}{|x-y|^2} \, ds_\nu
\]
\[-\int_{C_\infty} \frac{\nu(y) - \nu(x)}{|x-y|^2} \, ds_\nu, \quad x \in C_\infty. \tag{4.4.45}
\]

When \( \nu(x) \) is known, i.e. solved from (4.4.41), \( \partial V/\partial N \) can be calculated from (4.4.45).

Before proceeding with the explicit numerical calculations that will be presented in the next section, we recapitulate here the main steps in calculating \( A_{\infty} \). This procedure consists of three parts, namely

1. \( V = \psi \); 2. \( V = u_k \); 3. \( V = \psi_m \).

**Part 1: \( V = \psi \).**

i) Calculate \( \mu(x) \) from (4.4.16) with \( f(x) \) according to (4.4.8.1).

ii) Calculate \( \mu'(x) \) from (4.4.20) (note that this relation and, hence, also \( \mu' \), is identical for each \( V \)).

iii) Determine \( a \) and \( B \) from their definitions (i.e. (4.4.30), (4.4.34)) and solve (4.4.35) for \( (c^2, c_0)^T \); this also yields \( \psi(\infty) = \psi_0 \) (see (4.4.24)).

iv) Calculate \( \nu(x) \) from (4.4.41) together with (4.4.39).

v) Find \( \partial \psi/\partial N \) from (4.4.45).

**Part 2: \( V = u_k \); \( k \leq N \).**

i-v) Analogous to Part 1, only with \( f(x) \) from (4.4.8.3), whereas in iii) \( u_k(\infty) \) and \( \partial \psi/\partial N \), respectively, are obtained.

vi) Calculate \( \omega_k \) from (4.4.9) and (4.4.11).

vii) Find \( \partial \omega/\partial N \) from (4.4.6).

**Part 3: \( V = \psi_m \); \( m \leq N \).**

i) Use the result from Part 2 vii) to obtain \( f(x) \) from (4.4.8.2), and calculate \( \mu(x) \) from (4.4.16).
i) Take \( \mu'(x) \) from Part 1 ii)

ii) Solve \( (C', c_0)^T \) analogous to Part 1 iii), which also gives \( \psi_{n}(\infty) = c_0 \).

iii) Calculate \( \psi_x(x) \) from (4.4.41) and (4.4.39).

iv) Find \( \partial \psi_{n} / \partial N \) from (4.4.45).

v) Calculate \( \delta_{\psi_{n}} \) from (4.4.9)

vi) Calculate \( \delta_{\psi_{n}} \) from (4.4.11).

vii) Find \( \partial \delta_{\psi_{n}} / \partial N \) from (4.4.6).

The final step is then:

Use the results of Part 2 vii) and Part 3 vii) for the calculation of \( A_{\text{max}} \) \((1 \leq m, n \leq N)\) from (4.4.12).
5. Numerical evaluation and results

In the preceding section we described a procedure for the solutions of the exterior Dirichlet problem in two dimensions, especially directed towards the calculation of the normal derivatives of the magnetic potentials on the boundaries. In this procedure the Dirichlet problem was reformulated in terms of integral equations. In our numerical program all occurring integral equations are approximated by systems of linear algebraic equations by means of discretization. For the approximations of the integrals and of the tangential derivative of $V_3$ we use trapezoidal rules and central differences, respectively. The integrand of the last term on the right-hand side of (4.4.5) in case $y \to x$ equals $\partial v / \partial x$, and, again, the latter is approximated by a central difference. The discretization is accomplished by dividing the circles $C_1, \ldots, C_M$ in $M$ segments, each with angle $h = 2\pi / M$. The $x$- and $y$-coordinates of the associated nodal points are consecutively numbered as

$$x_{k-1} + M, j = 2[(k-1)a/R + \cos(j-1)\theta] e_1 + [\sin(j-1)\theta] e_2,$$

for $x \in C_k$, and

$$y_{k-1} + M, j = 2[(j-1)a/R + \cos(j-1)\theta] e_1 + [\sin(j-1)\theta] e_2,$$

for $y \in C_j$, with

$$k, l \in [1, N], j \in [1, M], \text{ and } h = \frac{2\pi}{M}.$$

In our numerical program we follow the calculation scheme recapitulated at the end of section 4, but we compute the matrix elements $A_{nm}$ for $m < n < N + 1 - m$ only; the remaining ones follow from the identities

$$\sum_{n=1}^{N} A_{nm} = 0 \quad \text{and} \quad A_{nN} = A_{NN} = A_{N-n, N+1-n}.$$(4.5.2)

(see (4.3.26)). Standard runtimes, such as the partial pivoting process, are used for the solution of the obtained linear systems and for the calculation of the eigenvalues and eigenvectors of an $N \times N$-matrix. As a check for the accuracy of our numerical procedure we compare our results for $N = 2$ with those obtained earlier in chapter 2. Our results for $\kappa / \pi$ correspond to the values of $Q$ in Table 2.4 (see page 57). The results for $Q$, obtained for $M = 40$, and for $Q_0$ are listed in Table 4.1. We conclude that a very close agreement between $\kappa / \pi$ and $Q_0$ exists.

<table>
<thead>
<tr>
<th>$a/R$</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa / \pi$</td>
<td>0.2205</td>
<td>0.1678</td>
<td>0.09328</td>
<td>0.05661</td>
<td>0.02653</td>
<td>0.01270</td>
<td>0.009810</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>0.220</td>
<td>0.168</td>
<td>0.0935</td>
<td>0.0568</td>
<td>0.0266</td>
<td>0.0153</td>
<td>0.00985</td>
</tr>
</tbody>
</table>

Table 4.1: Values of $\kappa / \pi$ for $N = 2$ and $M = 40$ and of $Q_0$ (from Table 2.4) for various values of $a/R$.

For $N = 2$, the first buckling mode (corresponding to the lowest buckling value or largest
eigenvalue \( \kappa \) is found to be

\[
\nu = \frac{1}{2} \sqrt{\kappa}, \quad -\frac{1}{2} \sqrt{\kappa}^T
\]

(4.5.3)

again in accordance with the results of chapter 2.

Of course, also the eigenvalue \( \kappa = 0 \) appears, with buckling mode

\[
\nu = \frac{1}{2} \sqrt{\kappa}, \quad -\frac{1}{2} \sqrt{\kappa}^T
\]

(4.5.4)

for \( A \) is singular. However, this eigenvalue has no practical relevance, because it yields an infinitely high buckling current. The same phenomenon arises for \( N > 2 \). Therefore, in the sequel the eigenvalue \( \kappa = 0 \) is left out of consideration.

In the following Tables 4.2, 4.3 and 4.4 one finds the numerical results for the eigenvalue \( \kappa \) (related to the buckling current according to (4.3.19)) and the eigenvector (or buckling modes) for \( N = 3, 4 \) and 5, respectively; here we have used \( M = 40 \) and \( a/R = 3 \).

<table>
<thead>
<tr>
<th>( \kappa/\pi )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \nu_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1393</td>
<td>-0.408</td>
<td>0.816</td>
<td>-0.408</td>
</tr>
<tr>
<td>0.0724</td>
<td>0.707</td>
<td>0</td>
<td>-0.707</td>
</tr>
</tbody>
</table>

Table 4.2 The eigenvalues and buckling modes for \( N = 3 \) and \( a/R = 3 \), computed for \( M = 40 \).

<table>
<thead>
<tr>
<th>( \kappa/\pi )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \nu_3 )</th>
<th>( \nu_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1640</td>
<td>-0.238</td>
<td>0.666</td>
<td>-0.666</td>
<td>0.238</td>
</tr>
<tr>
<td>0.1183</td>
<td>0.500</td>
<td>-0.500</td>
<td>-0.500</td>
<td>0.500</td>
</tr>
<tr>
<td>0.0592</td>
<td>-0.666</td>
<td>-0.238</td>
<td>0.238</td>
<td>0.666</td>
</tr>
</tbody>
</table>

Table 4.3 The eigenvalues and buckling modes for \( N = 4 \) and \( a/R = 3 \), computed for \( M = 40 \).

<table>
<thead>
<tr>
<th>( \kappa/\pi )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \nu_3 )</th>
<th>( \nu_4 )</th>
<th>( \nu_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1790</td>
<td>-0.144</td>
<td>0.490</td>
<td>-0.692</td>
<td>0.490</td>
<td>0.144</td>
</tr>
<tr>
<td>0.1459</td>
<td>0.335</td>
<td>-0.623</td>
<td>0</td>
<td>0.623</td>
<td>-0.335</td>
</tr>
<tr>
<td>0.1028</td>
<td>-0.528</td>
<td>0.245</td>
<td>0.566</td>
<td>0.245</td>
<td>-0.528</td>
</tr>
<tr>
<td>0.0501</td>
<td>-0.623</td>
<td>-0.335</td>
<td>0</td>
<td>0.335</td>
<td>0.623</td>
</tr>
</tbody>
</table>

Table 4.4 The eigenvalues and buckling modes for \( N = 5 \) and \( a/R = 3 \), computed for \( M = 40 \).
The values for the buckling current $I_0$, associated with the computed highest values of $\kappa$, can be obtained from (4.3.19). With

$$I_0 = \frac{\kappa}{4} R^4,$$  
(4.5.5)

for circular cross-sections, and with

$$\delta = \lambda R = \frac{\pi R}{l},$$  
(4.5.6)

for simply supported rods, (4.3.19) yields

$$I_0 = \frac{1}{\sqrt{k_0 / \kappa}} \frac{\pi^2 R^4}{l^2} \sqrt{\frac{E}{\mu_0}}.$$  
(4.5.7)

Using this formula we have compared the results for 3, 4 and 5 rods with the buckling current for a set of 2 rods. The results are listed in Table 4.5.

<table>
<thead>
<tr>
<th>$N$</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(I_0)^2/4(\kappa)$</td>
<td>0.818</td>
<td>0.754</td>
<td>0.722</td>
</tr>
</tbody>
</table>

Table 4.5

6. Discussion

As an alternative way in the chapters 2 and 3 a more technical approach to the solution of buckling problems for (superconducting) structural systems was discussed. The method is based upon a generalization of the law of Biot and Savart (cf. [5], sect. 2.6). In chapter 2 this method was applied to the problem of two parallel rods. In a straightforward derivation, completely analogous to that of chapter 2, this method can be generalized to systems of more than 2 rods. For instance, for three rods the following equations are obtained

$$E \int_0^L v_1^2(x) = k_1 (v_1 - v_2) + \frac{1}{4} k_1 (v_1 - v_3),$$

$$E \int_0^L v_2^2(x) = k_1 (2v_2 - v_1 - v_3),$$

$$E \int_0^L v_3^2(x) = k_1 (v_3 - v_2) + \frac{1}{4} k_1 (v_3 - v_1),$$

(4.6.1)

with

$$k_1 = \frac{\mu_0 I_0^2}{8 \pi a^2}.$$  
(4.6.2)

Under the boundary conditions

$$v_i(0) = v_i(L) = v_i(L), \quad i = 1, 2, 3,$$  
(4.6.3)

the lowest eigenvalue of (4.6.1) is
\[ k_1 = \frac{x^4 E l_f}{3 l^4}, \quad (4.6.4) \]

associated with the buckling mode

\[ v_1(x) = v_2(x) = -\frac{1}{2} v_3(x), \quad v_2(x) = A \sin \left( \frac{\pi x}{l} \right) \quad (4.6.5) \]

This buckling mode is identical to the first one of Table 4.2.

From (4.6.4) with (4.6.2) the following formula for the buckling current is found (with \( I_0 = \pi R^4 / 4 \))

\[ I_0 = \sqrt{\frac{2}{3}} \frac{x^3 a R^2}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.6) \]

Let us compare this result with (4.5.7). For \( a / R = 3 \) we obtain from (4.5.7)

\[ I_0 = 2.679 \frac{x^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.7) \]

and from (4.6.5)

\[ I_0 = 2.449 \frac{x^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.8) \]

We see that the buckling value found by the Biot-Savart method is about 8% lower than the value from the variational method. The same difference was also found in chapter 2 for the set of two rods.

For the system of 5 rods, the Biot-Savart method yields the buckling mode

\[ v_2 = v_4 = -0.72 v_3, \quad v_1 = v_5 = 0.22 v_3 \quad (4.6.9) \]

which differs only slightly from the first buckling mode from Table 4.4, where

\[ v_2 = v_4 = -0.708 v_3, \quad v_1 = v_5 = 0.208 v_3 \quad (4.6.10) \]

For the buckling current we obtained

\[ I_0 = 0.723 \frac{x^3 a R^2}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.11) \]

yielding, for \( a / R = 3 \),

\[ I_0 = 2.168 \frac{x^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.12) \]

On the other hand, (4.5.7) gives for \( a / R = 3 \)

\[ I_0 = 2.364 \frac{x^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}} \quad (4.6.13) \]

and again a difference of about 8% is observed. Hence, we conclude that this relative difference is
independent of the number $N$.

Finally, we also calculated by the Biot-Savart method the buckling current for an infinite set of parallel rods. The result was that the buckling modes were related to each other by

$$\nu_{j+1} = -\nu_j, \quad j = 1, 2, \cdots$$  \hspace{1cm} (4.6.14)

while the buckling current was found to be

$$I_0 = \frac{\pi^2 \alpha R^2}{2 l^2} \sqrt{\frac{E}{\mu_0}}.$$  \hspace{1cm} (4.6.15)

It is striking to note that this value for the infinite set is exactly a factor $(\pi/2)$ lower than the value for the set of two rods, which according to (2.5.31) is equal to

$$I_0 = \frac{\pi \alpha R^2}{l^2} \sqrt{\frac{E}{\mu_0}}.$$  \hspace{1cm} (4.6.16)

We proceed with the analogous version of Table 4.5, but now with the results from the Biot-Savart method.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$3$</th>
<th>$5$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_{0N}/U_{02}$</td>
<td>0.816</td>
<td>0.723</td>
<td>0.637</td>
</tr>
</tbody>
</table>

**Table 4.6** Ratios of the buckling currents for $N$ rods and for 2 rods, calculated by means of the Biot-Savart method.

We note that the above ratios are independent of the value of $a/R$. Moreover, the differences in the ratios according to Table 4.5 and to Table 4.6 (for $N = 3$ or 5) are negligible. Hence, we may write (the subindices $V$ and $BS$ denote values according to the variational method and the Biot-Savart method, respectively)

$$\left[ \frac{I_{0N}}{I_{02}} \right]_V = \left[ \frac{I_{0N}}{I_{02}} \right]_{BS} = q_N(N),$$  \hspace{1cm} (4.6.17)

where $q_N$ depends only on $N$ and not on $a/R$. With the use of (2.5.17), this relation implies that

$$U_{0N} = \frac{q_N}{\sqrt{\alpha}} \frac{\pi^2 R^2}{l^2} \sqrt{\frac{E}{\mu_0}}.$$  \hspace{1cm} (4.6.18)

If we assume this relation of general validity (i.e., for all values of $a/R$ and $N$) we can extrapolate the results of Table 4.5 for $N = 3$ and $N = 5$ to other values of $a/R$. To this end we use the $U_{0N}^{-1}$-values as given in Table 2.4, for several values of $a/R$. Furthermore, we can also find a corresponding value for the infinite system. In this way we find for the coefficient $I_0$ defined by
\[ I_0 = i_0 \frac{a^2 R^3}{R^2} \sqrt{\frac{E}{\mu_0}}. \]  

(4.6.19)

the relation

\[ I_0 = \frac{a}{\sqrt{Qt}} = i_0 (\frac{a}{R}, N). \]  

(4.6.20)

Values for this normalized buckling current are listed in Table 4.7.

<table>
<thead>
<tr>
<th>( \frac{a}{R} )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3.429</td>
</tr>
<tr>
<td>6</td>
<td>5.005</td>
</tr>
<tr>
<td>8</td>
<td>6.606</td>
</tr>
<tr>
<td>10</td>
<td>8.230</td>
</tr>
</tbody>
</table>

Table 4.7 Values of the normalized buckling current \( i_0 \) found by extrapolation from the Biot-Savart results.

In conclusion, we state that we have found here a simple algorithm to extrapolate from the Biot-Savart results the more exact but also much harder to obtain, buckling values as they should be found by the variational method. Due to the striking correspondence between systems of rods and (parallel) rings, as found in chapter 3, it may be expected that this result can be generalized to systems of \( N \) \( (N \geq 2) \) rings. This will enable us to apply a combined method (based partially upon a variational approach and partially on Biot-Savart like calculations) to more complex systems such as, for instance, helical or spiral shaped conductors (cf.[28]).
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APPENDIX A

On the equivalence of (1.7.18) and (1.7.41)

Immediately follows from subtraction of (1.7.18) and (1.7.41) (expressed in the normalized variables (1.7.17) or (1.7.40) and with omission of the subindex $0$) the condition

$$0 = \int_{\partial \Omega} [(-\psi B_{ij} u_j + (\psi B_{ij} u_i)_j)] N_i dS + \int_{\partial \Omega} \varepsilon_{ilm} B_m A_{ikl} u_k u_i N_i dS +$$

$$+ \int_{\partial \Omega} [2 \varepsilon_{ilm} B_m A_{ikl} u_k - 2 \varepsilon_{ilm} B_m A_{ikl} u_i + B_k B_i (u_{ij} u_i - u_{ij} u_j)] N_i dS +$$

$$- \frac{1}{2} \int_{\partial \Omega} B_k B_i u_k u_i N_i dS - \int_{\partial \Omega} (T_{ik} u_k) N_i dV,$$  \hspace{1cm} (A.1)

where we have used that $\partial\psi/\partial N = \text{curl}(u \times B) \cdot N$ (see (1.7.15) or (1.7.32)), $\text{div} B = 0$, $B = \text{curl} A$ and $T_{ij} = 0$ (see (1.7.12)$^{1,2}$). The first integral in (A.1) is zero on account of $(B \cdot N) = 0$ (see (1.7.13)$^2$) and Stokes' theorem (see the Lemma on page 11, which will be used several times). The last two integrals in (A.1) add up to zero because of $\text{div} A = -(B \cdot N)/2$ (see (1.7.12)$^2$) and Gauss' divergence theorem. So we arrive at (after substitution of $\varepsilon_{ilm} B_m A_{ikl} = A_{ij} - A_{ji}$ in the integrands of the second and third integral in (A.1)) the following condition

$$0 = \int_{\partial \Omega} [(A_{ij} - A_{ji}) A_{ikl} (u_{ik} u_l - u_{kl} u_i)] N_i dS +$$

$$+ (A_{ij} - A_{ji}) (2 A_{ikl} u_k u_l + A_{kik} u_k u_i) N_i dS .$$  \hspace{1cm} (A.2)

Note that $A = 0$ (see (1.7.12)$^2$), so with the aid of Stokes' theorem we derive for the first term in (A.2)

$$\int_{\partial \Omega} (A_{ij} - A_{ji}) A_{ikl} (u_{ik} u_l - u_{kl} u_i) N_i dS =$$

$$\int_{\partial \Omega} [(A_{ij} - A_{ji}) A_{ikl} (u_{ik} u_l - u_{kl} u_i)] N_i dS =$$

$$\int_{\partial \Omega} [(A_{ij} - A_{ji}) A_{ikl} (u_{ik} u_l - u_{kl} u_i)] N_i dS =$$

$$\int_{\partial \Omega} [(A_{ij} - A_{ji}) A_{ikl} (u_{ik} u_l - u_{kl} u_i)] N_i dS .$$  \hspace{1cm} (A.3)

Therefore (A.1) can be elaborated in the form

$$0 = \int_{\partial \Omega} [(A_{ij} - A_{ji}) (A_{ikl} u_k u_l)] N_i dS - 2 (A_{ij} - A_{ji}) A_{ikl} u_k u_l N_i dS .$$  \hspace{1cm} (A.4)

With the use of again Stokes' theorem and the aid of (1.7.12)$^{1,2}$ $(A_{ij} - A_{ji}, i, j) = 0$ the first term becomes

$$\int_{\partial \Omega} (A_{ij} - A_{ji}) (A_{ikl} u_k u_l) N_i dS =$$  \hspace{1cm} (A.5)
\[-\int_{\partial \Omega} (A_{ij} A_{kl}) u_k u_l u_j u_i \, dS + \int_{\partial \Omega} (A_{ij} - A_{kl}) (A_{kl} u_k u_j) N_i \, dS.\]

The condition (A.1) now becomes with (A.5) substituted into the equivalent condition (A.4):

\[0 = \int_{\partial \Omega} ((A_{ij} - A_{kl}) (A_{kl} u_k u_j) u_i N_j) \, dS = \int_{\partial \Omega} (A_{ij} - A_{kl}) A_{kl} u_k u_j N_i \, dS \]  

\[= \int_{\partial \Omega} (A_{ij} - A_{kl}) A_{kl} u_k u_j N_i \, dS \]  

In Appendix A, the second term in (A.6) can be rewritten with Stokes' theorem in the following way (A.7)

\[= \int_{\partial \Omega} (A_{ij} - A_{kl}) A_{kl} u_k u_j N_i \, dS = \int_{\partial \Omega} (A_{ij} - A_{kl}) A_{kl} u_k u_j N_i \, dS \]

Substitution of (A.7) into (A.6) leads to the condition (equivalent to (A.1))

\[0 = \int_{\partial \Omega} (A_{ij} - A_{kl}) A_{kl} (u_k u_j - u_k u_i) N_i \, dS.\]

Since $A_{kl} v_k = 0$ if $v_k N_k = 0$ and $(u_k u_j - u_k u_i) N_k = 0$ the condition (A.8) ($\implies$ (A.1)) is fulfilled, so (1.7.18) and (1.7.41) are equivalent if $\partial \Psi \partial N = (\nabla \times B \cdot N)$ on $\partial \Omega$. 

APPENDIX B

On the vectorpotential $A$

Consider the class of functions
\[ A(x_0) = \hat{A}(x_0) + (\nabla \Omega)(x_0) + \frac{1}{2} \nabla \times x_0 \cdot \hat{A}(x_0) = \int_{\partial G} J(x) G(x; x_0) dS , \]  
(B.1)

where $G(x; x_0)$ and $\Omega(x)$ are some scalar functions satisfying
\[ G(x; x_0) \to 0 \quad |x_0| \to \infty ; \quad (\nabla \Omega)^* = (\nabla \Omega)^* , \quad x \in \partial G , \]  
(B.2)

then, if $B$ is defined by $B = \text{curl} A$,
\[ A^* = A - (\Phi (B^*, n) = (B^*, n)) \cdot x \in \partial G ; \]  
(B.3)
\[ \text{div} B^* = 0 , \quad x \in \Omega^* ; \quad B \to B_0 \quad |x| \to \infty ; \]

So the class defined by (B.1) is a class of vectorpotentials for $B$ in (1.3.2). If we confine ourselves to
\[ J = n \times \hat{B} , \quad \langle \text{curl} \hat{B}, n \rangle = 0 , \quad x \in \partial G ; \quad G = 1/4\pi |x-x_0| , \]  
(B.4)

then we have
\[ \Delta_0 G(x; x_0) = -\delta_0(x-x_0) ; \]
\[ \Delta \hat{A} = 0 , \quad x \in \Omega^* ; \]  
(B.5)

and
\[ \text{div} \hat{A} = \int_{\partial G} \epsilon_{ijk} \hat{B}_{ik} G_{jk} dS = \]  
(B.6)
\[ = \int_{\partial G} \epsilon_{ijk} \hat{B}_{ik} G_{jk} dS + \int_{\partial G} \epsilon_{ijk} \hat{B}_{ik} G_{jk} dS \]

Therefore we arrive at
\[ \text{div} \hat{A} = 0 , \quad x \in \Omega^* \cup \Omega^* \cup \partial G , \]  
(B.7)

and moreover, (use (B.5), (B.7))
\[ \text{curl} B = \text{curl} \text{curl} A = \Delta \hat{A} - \nabla (\text{div} \hat{A}) + \text{curl} \nabla \Omega + \text{curl} B_0 = \]  
(B.8)
\[ = 0 - 0 + 0 + 0 = 0 , \quad x \in \Omega^* ; \]

So now $A$ is a vectorpotential for $B$ such that
\[ \text{div} B^* = 0 \quad \text{curl} B^* = 0 , \quad x \in \Omega^* ; \]
\[ (B^*, n) = (B^*, n) , \quad x \in \partial G ; \quad B \to B_0 , \quad |x| \to \infty . \]  
(B.9)

For a soft ferromagnetic structure with high ferromagnetic susceptibility we have the extra condition
\[ n \times B^* = 0, \quad x \in \partial G, \quad (F). \] (B.10)

or

\[ n \times \left( \text{curl} \int_{G} (n \times \vec{B}) \, dS \right)^* = -n \times B_0; \]

\[ (n, \text{curl} \vec{B}) = 0, \quad x \in \partial G, \quad (F). \] (B.11)

For a superconducting structure we have

\[ (B^*, n) = 0, \quad x \in \partial G; \quad B_0 = 0, \quad \int_{\Omega} (B^*, \tau) \, dS = \mu_0 I_0, \quad (S). \] (B.12)

or

\[ A^* = 0, \quad x \in \partial G; \quad B_0 = 0, \quad \int_{\Omega} (\text{curl} A^*, \tau) \, dS = \mu_0 I_0, \quad (S). \] (B.13)

or \((\Omega = \Omega^*)\)

\[ \int_{\partial G} (n \times \vec{B}) \, dS + \nabla \Omega = 0; \]

\[ (n, \text{curl} \vec{B}) = 0, \quad x \in \partial G; \]

\[ \int_{\Omega} \left( \text{curl} \int_{G} (n \times \vec{B}) \, dS \right)^*, \tau) \, dS = \mu_0 I_0, \quad (S). \] (B.14)

In the second part of this appendix we want to prove that (3.2.2) and (3.3.5) are equivalent. Consider the superconducting problem (3.2.2) (in the normalized variables)

\[ \text{div} B = 0, \quad \text{curl} B = 0 \quad \text{or} \quad B_{ij} = B_{ij}, \quad x \in G^+; \]

\[ (B, n) = 0 \quad \text{or} \quad B_{ij} \, (N_j) = 0, \quad x \in \partial G; \quad B \rightarrow 0, \quad |x| \rightarrow \infty; \] (B.15)

\[ \int_{\Omega} (B, \tau) \, dS = 2\pi a; \quad B = 0, \quad x \in G^-. \]

The equations (B.15) imply \( \Delta B_i = B_{i,ij} = 0, \) so we have for \( x_0 \in G^+ \) (see also (B.15)34)

\[ B_i(x_0) = \int_{G} (G \Delta B_i - B_{i} \Delta G) \, dV = \int_{\partial G} (B_i G_{i,j} N_j - G B_{i,j} N_j) \, dS_x = \]

\[ = \int_{\partial G} (B_i G_{i,j} N_j - (G \delta_{ij} B_j) N_j) \, dS_x = \int_{\partial G} (B_i G_{i,j} N_j - (G \delta_{ij} B_j) N_j) \, dS_x = \]

\[ = \int_{\partial G} B_i \frac{\partial G}{\partial N} \left( \nabla_x G \cdot B \right) N_j) \, dS_x. \]

or

\[ B(x_0) = \int_{\Omega} (N \times B) \times \nabla_x G \, dS_x. \]

With
\[ J = N \times B, \quad B = J \times N, \]
and \( x_0 \to \partial G \) follows the problem (3.3.5)
\[
(J \times N)(x_0) = 2 \int_{\partial G} J \times V_x G \ dS_x, \quad x_0 \in \partial G. \tag{B.16}
\]
\[
(J, N) = 0, \quad x \neq \partial G; \quad \int_{\partial G} (J \times N, \tau) ds = 2na.
\]

On the other hand, if \( J \) satisfies (B.16) then define \( B \) by
\[
B = \int_{\partial G} J \times V_x G \ dS_x = \text{curl} \int_{\partial G} J G \ dS_x \tag{B.17}
\]
Then indeed \( \text{div} \ B = 0 \) and (B.16) reads
\[
B^* = J \times N; \quad B^* = 0, \quad x \neq \partial G.
\]
Furthermore follows from (B.17)
\[
\Delta B^* = 0, \quad x \in G^+; \tag{B.18}
\]
so
\[
B^* = 0, \quad x \in G^-; \tag{B.19}
\]
Moreover, we calculate
\[
(\text{curl } B) = (\text{curl } \text{curl} \int_{\partial G} J G \ dS_x) = (\Delta_{x_0} \int_{\partial G} J G \ dS_x - \nabla \text{div}_{x_0} \int_{\partial G} J G \ dS_x) =
\]
\[
= \int_{\partial G} \epsilon_{ji} N_j B^*_i G_{ij} \ dS_x = \int_{\partial G} \epsilon_{ji} B^*_i L_j G_j N_j \ dS_x - \int_{\partial G} (\epsilon_{ji} B^*_i G_j) N_j \ dS_x =
\]
\[
= \left( \int_{\partial G} (\text{curl } B^*, N) \nabla_x G \ dS_x \right),
\]
or
\[
\text{curl } B = - \int_{\partial G} (\text{curl } B^*, N) \nabla_x G \ dS_x. \tag{B.20}
\]
So we arrive at \( (x_0 \in G^-, x_0 \to \partial G) \) and \( x_0 \in G^+, x_0 \to \partial G \) and use (B.19)
\[
0 = - \int_{\partial G} (\text{curl } B^*, N) \nabla_x G \ dS_x + \frac{1}{2} (\text{curl } B^+, N) N(x_0), x_0 \in \partial G, \tag{B.21}
\]
\[
\text{curl } B^*(x_0) = - \int_{\partial G} (\text{curl } B^*, N) \nabla_x G \ dS_x - \frac{1}{2} (\text{curl } B^+, N) N(x_0), x_0 \in \partial G. \tag{B.22}
\]
From (B.21) and (B.22) we deduce directly
\[
(\text{curl } B^+, N) = 0, \quad x \neq \partial G, \tag{B.23}
\]
and according to (B.20) and (B.23)
\text{curl} \, B^2 = 0, \quad x \in G^3 \cup \partial G.

\text{(B.24)}

Therefore the problems (3.2.2) (or (B.15)) and (3.3.5) (or (B.16)) are completely equivalent.
APPENDIX C

On the addition of an electric term to $L$

Define the dielectric displacement $D$ and the magnetic field intensity $H$ (see (1.3.3)) by

\[ D = \varepsilon_0 \varepsilon \mathbf{E} + \varepsilon \mathbf{P}, \quad H = \frac{1}{\mu_0} \mathbf{B} - \mu_0 \mathbf{M}, \]  

(C.1)

where $P$ and $M$ are the polarization and the magnetization per unit of mass, respectively. Consider a body, polarizable, magnetizable and non-conducting, influenced by an external magnetic field $B_0$, then the possible Lagrangian densities are (for the Maxwell Minkowski model, see chapter 1, section 2)

\[ L_1 = -\frac{1}{2} \varepsilon_0 (\mathbf{E}, \mathbf{E}) - \frac{1}{2} \mu_0 (\mathbf{H}, \mathbf{H}) - \rho U, \]  

(C.3.1)

with constraints

\[ D = \text{curl} \mathbf{A}^D, \quad B = \text{curl} \mathbf{A}^B, \quad [A^D] = [A^B] = 0, \]  

(C.2.2)

and always

\[ \rho = \rho_0 \text{det} F, \quad T = \rho \frac{\partial U}{\partial \mathbf{F}^T}, \quad F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}; \]  

(C.2.3)

\[ (U = 0, \mathbf{F} = 0 \text{ outside the body}) \]

or

\[ L_2 = L_1 + (\mathbf{B}, \mathbf{H}). \]  

(C.3.1)

with constraints (besides (C.2.3))

\[ D = \text{curl} \mathbf{A}^D, \quad \mathbf{H} = \mathbf{\nabla} \Phi^H, \quad [A^D] = 0, \quad [\Phi^H] = 0; \]  

(C.3.2)

or

\[ L_3 = L_1 + (\mathbf{D}, \mathbf{E}), \]  

(C.4.1)

with constraints

\[ B = \text{curl} \mathbf{A}^B, \quad E = \mathbf{\nabla} \Phi^E, \quad [A^B] = 0, \quad [\Phi^E] = 0; \]  

(C.4.2)

(see chapter 1, section 8)

or

\[ L_4 = L_1 + (\mathbf{D}, \mathbf{E}) + (\mathbf{B}, \mathbf{H}). \]  

(C.5.1)

with constraints

\[ E = \mathbf{\nabla} \Phi^E, \quad H = \mathbf{\nabla} \Phi^H, \quad [\Phi^E] = 0, \quad [\Phi^H] = 0. \]  

(C.5.2)

The addition of the terms to $L_1$ is called Legendre transformation (see also chapter 1, section 7.
for the case of superconductors.
If the body is (non-linear) magnetizable, non-polarizable, and linear conducting (electric conductivity \( \sigma \)) then
\[
L_\Sigma = \frac{1}{2} \epsilon_0 \left( \mathbf{E}, \mathbf{E} \right) - \frac{1}{2} \mu_0 \left( \mathbf{H}, \mathbf{H} \right) + \sigma \left( \mathbf{E}, \mathbf{A} \right) - \rho \mathbf{U} .
\]  
(C.6.1) 
with constraints 
\[
\mathbf{E} = \nabla \Phi , \quad \mathbf{B} = \text{curl} \mathbf{A} , \quad \text{div} \mathbf{A} = 0 ;
\]
\[
[\Phi] = 0 , \quad [\mathbf{A}] = 0 ;
\]  
(C.6.2) 
Essential is now the "gauge" condition \( \text{div} \mathbf{A} = 0 \).
Summary

A variational principle that can serve as the basis for a magneto-elastic stability (or buckling) problem is constructed. The formulation starts from a specific choice for a magneto-elastic Lagrangian (associated with the so-called Maxwell-Minkowski model for magneto-elastic interactions). For the evaluation of the principle the first and second variations of the Lagrangian are calculated both inside and outside the solid magneto-elastic body. Thus, a general buckling criterion, consisting of an expression for the critical field value, together with a set of constraints for the field variables, is constructed. The first variation yields the total set of differential equations and boundary conditions for the intermediate state which is the one and only equilibrium state for fields lower than the buckling field. The second variation yields in the same way the differential equations and boundary conditions for the perturbed state (or buckled state). In this thesis we refrain from post-buckling analysis, so only bifurcation theory is discussed. The zerosness of the second variation (which is homogeneous and quadratic with respect to the perturbations) yields the general buckling equation. More detailed formulations are given for, successively, soft ferromagnetic and superconducting structures. The principle is applied to i) one single ferromagnetic or superconducting beam, ii) a system of two parallel ferromagnetic or superconducting beams, iii) a system of two concentric or coaxial superconducting tori (or rings), iv) a system of an arbitrary number of parallel superconducting beams. The problems i), ii) and iii) are solved analytically (for slender cases) with the use of Green's identities, integral equations, fundamental solutions and complex analysis. The fourth problem is solved with the use of a numerical method involving standard procedures (such as the partial pivoting process for the solution of a set of algebraic equations) for the solution of an eigenvalue problem. The advantage of the variational method, compared to the classical approach (see the introduction of this thesis), is in fact that only a reasonable approximation of the eigenfunctions leads us already to a very good approximation of the buckling value.
Samenvatting

Er wordt een variatie-principe geconstreerd, dat als basis dient voor een magneto-elastisch stabilitaats- (of knik-) probleem. De formulering begint met een specifieke keuze voor een magneto-elastische Lagrangeaan (welke samenhangt met het Maxwell-Minkowski model voor magneto-elastische interacties). Voor de uitwerking van het principe worden de eerste en tweede variatie van de Lagrangeaan berekend, zowel binnen als buiten het magneto-elastische lichaam, gemaakt van een vast materiaal. Daarna wordt een algemeen knik-criterium geconstreerd, bestaande uit een expressie voor de kritische vlakwaarde, samen met een set nevenvoorwaarden voor de variabelen. De eerste variatie geeft een totaal stelsel van differentiaalvergelijkingen en randvoorwaarden voor de intermediaire toestand, welke de enige evenwichtstoestand is voor velden lager dan de knikwaarde. De tweede variatie geeft op dezelfde manier de differentiaalvergelijkingen en randvoorwaarden voor de gestoorde toestand (of uitgeknipte toestand). In dit proefschrift behandelen we geen post-buckling analyse en we bekijken alleen de bifurcatietheorie. De nullwaarde van de tweede variatie (welke homogen kwadraatisch is met betrekking tot de stortingen) levert het algemene knik-criterium. Meer gedetailleerde formuleringen worden beschreven voor achtereenvolgens soft-ferromagnetische en supergeleidende structuren. Het principe wordt toegepast op de volgende problemen: i) een enkele ferromagnetische of supergeleidende baan, ii) een systeem van twee parallelle ferromagnetische of supergeleidende balken, iii) een systeem van twee concentrische of coaxiale tori (of ringen), iv) een systeem van een willekeurig aantal parallelle supergeleidende balken. De problemen i), ii) en iii) worden analytisch opgelost (voor slanke gevallen) met behulp van de identiteiten van Green, integraalvergelijkingen, fundamentele oplossingen en complexe analyse. Het vierde probleem wordt opgelost met behulp van een numerieke methode, die standaard procedures bevat (zoals het pivot-verkoorproces voor de oplossing van een stelsel algebraïsche vergelijkingen) voor de oplossing van een eigenwaarde-probleem. Het voordelen van de variationele methode, vergelijken met de klassieke benadering (zie de inleiding van dit proefschrift), ligt in het feit dat een redelijke benadering van de eigenfuncties leidt tot een reeds erg goede benadering van de knikwaarde.
Curriculum vitae

I

Voor een niet-lineair magnetiseerbaar, niet-polariserbaar, deformeerbaar lichaam (c.q. een stelsel van lichamen) wordt de energiedichtheid $E$ gegeven door

$$E = \frac{1}{2} \mu_0 (E, E) + \frac{1}{2} \mu_0 (H, H) + (D, E) + \rho U,$$

waarin

$$E = \nabla \Phi, \quad H = \frac{1}{\mu_0} B - \rho M, \quad B = \mu_0 A, \quad D = \varepsilon E + \rho P,$$

$$U = U (F^T M, F^T P; \frac{1}{2} (F^T F - I)), \quad F = I + \frac{\partial U}{\partial X}, \quad \rho = \rho_0 / \det F.$$ 

Onder de randcondities $\Phi = 0, [A] = 0$ en de oplegvoorwaarden uitgedrukt in $U$ is de integraal van $E$ stationair voor evenwichtstoestanden.

Literatuur: Dit proefschrift, Hdst. 1 en App. C.

II

Voor een niet-lineair magnetiseerbaar, niet-polariserbaar, lineair geleidend (elektrisch geleidingsvermogen $\sigma$) lichaam wordt de energiedichtheid $E$ gegeven door

$$E = \frac{1}{2} \mu_0 (E, E) + \frac{1}{2} \mu_0 (H, H) - \sigma (E, A) + \rho U,$$

met dezelfde definities, verschillende als in I, behalve $P = 0, (\partial U / \partial X) = 0$. Onder de randcondities (zie I) en de nu essentiële "gaugé" conditie

$$\text{div} A = \varepsilon$$

is de integraal van $E$ stationair voor evenwichtstoestanden.

Literatuur: Dit proefschrift, Hdst. 1 en App. C.

III

Bij de klassieke, zuiver elastische kinetiek spreekt de voorspanning een essentiële rol. Op enkele uitzonderingen na is dit niet het geval bij electromagneto-elastische kinetiekvoorspanningen.
IV

De in dit proefschrift analytisch bepaalde knikwaarde voor een systeem van twee parallelle ferromagneteiske balken met oerzichtige doorsnede leent overeen met de in onderstaande referentie nummeriek bepaalde waarde.


V

De in dit proefschrift behulp van complexe funktieanalyse bepaalde knikwaarde voor een ferromagnetische balk met elliptische doorsnede leent overeen met de met behulp van Mathieufuncties berekende waarde in onderstaande referentie.


VI

De invloed van magnetostatische effecten op de magneto-elasticke knikwaarde voor slanke lichamen en/of systemen is in de praktijk verwaarloosbaar.

VII

Het verwaarlozen van randeffecten bij elliptische problemen is toegestaan (het principe van de Saint-Venant); dit is echter niet het geval bij hyperbolische problemen. Bij 3-dimensionale elliptische problemen leidt een separatie tot een 2-dimensionaal hyperbolisch probleem. In ondergenoemd artikel maken de auteurs in dit opzicht een essentiële fout doordat zij een eindige maar smalle rechtboogige doorsnede vervangen door een oneindig brede strip. Het gevolg van deze fout is het verkrijgen van een verkeerde exponent in de relatie voor de knikwaarde van het verwendige veld.


VIII

De criticale waarde t.a.v. magneto-elasticke knik van de slankheidsparameter van een slank magnetisch verzaagd lichaam kan worden gevonden door het analoge resultaat voor een soft-ferromagnetisch lichaam te extrapolver naar het verzaagingspunt.

IX

De mode-indieken (genormaliseerde propagatieconstanten) voor elektromagnetische harmonische golven met een 2-dimensionaal VEL-opstel door oplossing door geleidende en/of richtkoppelingen kunnen worden bepaald uit een homogene integralvergelijking over een begrensd gebied. Een eenvoudigere methode is een variationele aanpak waarbij betreffende idee voor de bepaling van eigenwaarden voorhanden is als in dit proefschrift.

Literatuur: P.H. van Lieshout, Golflvoorplanting door optische richtkoppelingen, Stagereisling (1985), TUE.
N.H.G. Baken, Computational modelling of integrated optical waveguides, Proefschrift, PTT-Research/TUD (1990), (verschijnen in oktober).

X

De quasi-statische benadering voor een probleem van een plotseling belaste visco- elastisch lichaam, leidt bij bepaalde visco-elastische media in begind fase van de deformatie tot fysisch onacceptabele resultaten.

XI

De lefde van een analytisch gaat door de complexe functietheorie.