TRANSMISSION, REFLECTION AND RADIATION
AT JUNCTION PLANES
OF DIFFERENT OPEN WAVEGUIDES

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ABSTRACT

The devices used for optical point-to-point communication typically consist of a series-connection of sections of different types of cylindrical open waveguides. At a junction of two different sections, one has a discontinuity of the electromagnetic properties, which results in reflection, transmission and radiation of electromagnetic waves at the junction plane. The main theme of the present thesis is the quantitative analysis of these phenomena.

To start the analysis, both the propagation of electromagnetic waves along a uniform (infinite) waveguide section and the interaction of waves at the junction plane need to be described in mathematical terms. This description is based on Maxwell's equations for the electromagnetic field, the frequency-domain reciprocity theorem, and the electromagnetic Green's states. It is shown that the fields in a uniform (infinite) open waveguide section can be represented by a modal expansion involving surface-wave modes and radiation modes. Two methods for the computation of surface-wave modal fields are discussed and illustrated by numerical results for planar open waveguides.

Next, integral representations are derived for the fields in a finite open waveguide section in terms of the transverse fields in the boundary planes, and for the fields in a semi-infinite section in terms of the transverse field in the terminal plane and the transverse incident field propagating towards the terminal plane. By means of these representations, systems of integral equations are established for the fields in the junction plane(s) of two (three) series-connected open waveguide sections.
One of these systems of integral equations has been selected and solved numerically, for various combinations of two series-connected planar open waveguide sections and for a semi-infinite waveguide terminating in free space, whereby the incident field is a TE-surface-wave mode. More specifically, the system of integral equations is subjected to a spatial Fourier Transformation, whereupon the resulting Fourier transformed system is numerically solved by the method of moments. The solution obtained for the Fourier transform of the junction-plane field, is used to calculate the transverse field in the junction plane and the reflection of the incident surface-wave mode at the junction plane. In addition, the transmission at the junction plane is computed for the series-connection of two waveguide sections, whereas for the terminating waveguide the forward radiation from the terminal plane is determined.
1. INTRODUCTION

In communication engineering, optical systems for signal transmission are becoming of ever increasing importance. As any communication system, they contain devices for signal generation, signal transmission, signal detection and signal processing. In the present thesis we investigate in more detail the transmission of optical, i.e., electromagnetic, signals along waveguiding structures. In the early years, the transverse dimensions of these structures were of the order of some tens of wavelengths of the electromagnetic radiation employed, and, hence, they could be analysed with the aid of optical ray theory. However, the tendency is that the sizes of the cross-sections will go down to the order of the wavelength; therefore, an analysis based on the full electromagnetic equations becomes necessary. An introductory overview of waveguide theory is provided in some standard textbooks on the subject; we mention Kapany (1967), Marcuse (1974), Unger (1977), and Snyder and Love (1983).

As far as the waveguiding structures are concerned, we concentrate on the cylindrical, open, waveguides that are used in optical point-to-point communication systems. Ideally, a single straight waveguide would suffice, but in practice, a series-construction of different types of waveguides is technically inevitable. As a consequence, both the analysis of wave propagation along a straight section, and the interaction of waves at junctions of two such sections are of importance. The junction of two different sections amounts to a discontinuity in waveguiding properties. At such a discontinuity, reflection, transmission and radiation of electromagnetic waves take place. The quantitative analysis of this kind of phenomena is the main theme of this thesis.
Now, for the calculation of electromagnetic fields, several methods are available. The most direct one would be to solve, in practice numerically, Maxwell's equations, taking into account the appropriate boundary conditions and causality conditions (radiation conditions). In open-waveguide configurations, this method would require a numerical solution of Maxwell's electromagnetic differential equations in the entire $\mathbb{R}^3$, since the fields in general extend considerably outside the directly waveguiding region. Due to insurmountable difficulties with regard to the storage requirements in the computer, this method is outside the range of practical application. Hence, other methods have to be called for.

First of all, we can take advantage, in an analytical manner, of the translational invariance of the waveguide in the axial direction. For a straight open waveguide section, the electromagnetic field can be decomposed into its axial-spectral constituents by subjecting it to an axial Fourier Transformation. This method leads to the well-known modal description of the fields in a waveguide. For open waveguides, two types of modes are distinguished, viz. the surface-wave modes (for optical transmission the desired ones) and the radiation modes (usually of an unwanted nature). In order to include the description of the excitation of the modal field constituents by localised sources, we carry out the analysis by applying the axial Fourier Transformation to the electromagnetic field equations in which source terms have been included. Then, upon analytically continuing the axial Fourier transforms into the complex $k_x$-plane ($k_x$ being the parameter of the axial Fourier Transformation), the propagation coefficients of the surface-wave modes show up as poles, and the propagation coefficients of the radiation modes fill up on branch cuts in the complex $k_x$-plane, the latter being related to causal wave propagation in the outermost medium. The former propagation coefficients are often referred to as the discrete modal spectrum, the latter as the continuous modal spectrum. For the computation of the propagation coefficients and the corresponding transverse field
distributions, several methods are available. In the present thesis, the integral-equation method and the transfer-matrix formalism are discussed.

For the computation of the field in the junction plane(s) of two (or more) open waveguide sections, several methods have been presented in the literature. Firstly, we mention the application of (semi-)analytical methods (Wiener-Hopf technique) to the junction of two different semi-infinite structures (Angulo and Chang, 1989; Itilipboon and Hamid, 1981; Aoki et al., 1982; Uchida and Aoki, 1984).

A second method, which has been applied by many authors, is the full modal analysis, which comprises the matching, in a junction plane, of both the surface-wave modal fields and the radiation modal fields of the two waveguides at either side of the junction plane. In an early paper by Angulo (1987), this method is used to derive an integral equation for the electric field in the terminal plane of a terminating slab waveguide. From it, Angulo derived variational expressions that yield upper and lower bounds for the terminal admittance, and expressions for the forwardly radiated power flow density. Rulf (1977) employed this method in the matching problem for two semi-infinite slab waveguides. He reduced the problem to a system of singular integral equations for the forward and backward scattering coefficients of the surface-wave modes and the radiation modes. For small discontinuities in the waveguides' properties or axial alignments, he obtained an approximate solution for these equations by means of a perturbation analysis. Mostly, the continuous spectrum is discretised by employing an expansion into a sequence of functions, the integrals of products of which can readily be calculated (Clarricoats and Sharpe, 1972; Mahmoud and Beal, 1975; Brooke and Kharadly, 1976; Rozzi, 1978; Morishita et al., 1979; Rozzi and In 't Veld, 1980). Then, systems of linear algebraic equations are obtained, which can be solved by standard methods. A somewhat different method for solving the equations obtained by mode matching was employed by Gelin et al. (1981) and by
Takenaka et al. (1983); these authors determined the modal field coefficients by means of an iterative procedure. For small step discontinuities, Marcuse (1970) simplified the equations for the scattering coefficients of the surface-wave modes; by ignoring the backward scattered radiation modes, he obtained closed-form expressions for the reflection and transmission coefficients of the surface-wave modes; next, by ignoring the reflected surface-wave mode in the calculation of the scattering coefficients of the forward and backward scattered radiation modes, he obtained closed-form expressions for the scattering coefficients of the radiation modes. The same method was applied by Ittipiboon and Hamid (1979).

The third method for the computation of the fields in the junction planes of different open waveguide sections employs surface-source type integral representations for the fields in each of the joining waveguide sections. The latter fields are considered to be excited by surface-source distributions at the junction planes. These source distributions, which are simply related to the tangential electromagnetic fields in the junction planes, enter into the integral representations mentioned, together with appropriate Green's functions. By using, in each of the waveguide sections, these integral representations for the fields right at the junction planes, and by imposing the condition that the tangential fields should be continuous across the junction planes, a system of integral equations for the fields in the junction planes is obtained. The kernel functions in these integral equations are the Green's tensor elements of the joining waveguide sections. This method was employed by Nobuyoshi et al. (1983) and by Nishimura et al. (1983). These authors used approximate expressions for the Green's tensor elements occurring in the integral equations, in the sense that they either ignored the effect of the reflections at the transverse boundaries of the waveguide (Nobuyoshi et al.), or partly ignored this effect and partly took it into account by expressions based on geometrical optics or on image-method approximations (Nishimura et al.). These procedures restrict the application of their
methods to weakly guiding structures.

In the present thesis, the surface-source type integral formalism that involves Green's tensors, is developed in a rigorous manner. Exact expressions are used for the Green's tensor elements occurring in the integral equations. With it, a general method is provided for the computation of the reflection, transmission and radiation in a series-connection of an arbitrary number of waveguide sections (which can be used to model other, more general, discontinuities in a waveguide). To calculate the as yet unknown field distributions in the junction planes, the integral equations are subjected to a transverse Fourier Transformation. In this way, the behaviour of the fields in the junction plane, that may be both oscillatory and slowly decreasing away from the guiding structure due to the presence of continuous spectrum (radiation) field components, can be accounted for. Another advantage of this Fourier-transform computational method is, that the spatial singularities in the Green's tensors (cf. Lee et al., 1980) are more easily handled in the transform domain. The Fourier-transformed integral equations thus obtained are solved numerically. From the solutions, the scattering coefficients for the surface-wave modes are obtained, and the forward radiation of a terminating planar open waveguide is determined. Subsequent application of a Fast Fourier Transformation yields the fields in the junction planes. With this method, a number of configurations has been analysed. A brief outline of the contents of the subsequent chapters concludes this introduction.

In Chapter 2, the equations for the electromagnetic field, the frequency-domain reciprocity theorem, and the electromagnetic Green's states for a general structure are discussed.

Chapter 3 deals with the representation of the fields in straight open waveguide sections in terms of surface-wave modes and radiation modes (discrete and
continuous spectrum). Two methods for the computation of surface-wave modal fields (that will be taken as excitations for the discontinuities in the waveguide) are discussed and results are presented for several types of planar open waveguides.

In Chapter 4, integral representations for the fields in a straight open waveguide section are derived. Depending on the conditions that are imposed on the Green's tensors, representations are obtained in terms of either the transverse electric field at the boundary planes, or the transverse magnetic field at the boundary planes, or both.

In Chapter 5, the integral representations of Chapter 4 are used to derive integral equations for the transverse fields (electric, magnetic, or both) in the junction plane(s) of two and three series-connected open waveguide sections.

In Chapter 6, the theory developed in Chapter 5 is applied to the junction of two planar (two-dimensional) open waveguide sections. The transverse Fourier Transformation is applied to the relevant integral equations. Numerical results are presented for a number of configurations; a TE surface-wave modal field is taken as the incident field. A comparison is made with the results obtained by Rozzi (1978). Finally, the computing times involved are discussed.

Various auxiliary calculations and derivations are given in Appendices A–F.
2. BASIC RELATIONS OF ELECTROMAGNETIC FIELD THEORY

2.1. BASIC EQUATIONS FOR THE ELECTROMAGNETIC FIELD QUANTITIES
IN AN INHOMOGENEOUS MEDIUM

In this section we briefly discuss the equations that govern the frequency-domain electromagnetic field quantities in a medium with linear, time-invariant electromagnetic properties. The latter vary continuously with position, except at sufficiently smooth surfaces, across which the electromagnetic properties may exhibit a finite jump. Position in space is denoted by the position vector \( \mathbf{r} \) with respect to a fixed reference frame. The frequency component with angular frequency \( \omega \) has a time dependence \( \exp(j\omega t) \), where \( j \) denotes the imaginary unit and \( t \) is the time coordinate; the time factor \( \exp(j\omega t) \) is suppressed throughout. In a domain in space where the electromagnetic properties vary continuously with position, the electromagnetic field quantities are continuously differentiable and satisfy Maxwell's equations

\[
\nabla \times \mathbf{H}(t) - j\omega \mathbf{D}(t) = \mathbf{J}(t),
\]

\[
\nabla \times \mathbf{E}(t) + j\omega \mathbf{B}(t) = -\mu \mathbf{J}(t).
\]

The quantities occurring in these equations are listed in Table I. SI-units are used throughout the presentation. For a bounded domain, the electromagnetic field must satisfy prescribed boundary conditions at the boundary of the domain; for an unbounded domain, the field must satisfy the radiation condition at infinity (Felsen and Marcuvitz, 1973, p.87). The medium under consideration is assumed to be locally
Table 1. Quantities, symbols and SI-units.

<table>
<thead>
<tr>
<th>quantity</th>
<th>time domain</th>
<th>frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>electric field intensity</td>
<td>V/m</td>
<td>E</td>
</tr>
<tr>
<td>magnetic field intensity</td>
<td>A/m</td>
<td>H</td>
</tr>
<tr>
<td>electric flux density</td>
<td>C/m²</td>
<td>D</td>
</tr>
<tr>
<td>magnetic flux density</td>
<td>T</td>
<td>D</td>
</tr>
<tr>
<td>volume density of electric current</td>
<td>A/m²</td>
<td>J V</td>
</tr>
<tr>
<td>volume density of magnetic current</td>
<td>V/m²</td>
<td>K µ</td>
</tr>
<tr>
<td>surface density of electric current</td>
<td>A/m</td>
<td>J s</td>
</tr>
<tr>
<td>surface density of magnetic current</td>
<td>V/m</td>
<td>K µ</td>
</tr>
<tr>
<td>frequency-domain permittivity</td>
<td>F/m</td>
<td>ε</td>
</tr>
<tr>
<td>frequency-domain permeability</td>
<td>H/m</td>
<td>µ</td>
</tr>
</tbody>
</table>

* in vacuo $\varepsilon = \varepsilon_0 = 1/\mu_0\varepsilon_0^2$ with $c_0 = 2.99792458 \times 10^8$ m/s

** in vacuo $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

Fig. 2.1. Surface of discontinuity for the electromagnetic properties.
reacting, isotropic, and, as stated before, time-invariant. Under these circumstances its constitutive equations are

\[ \mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}), \quad (2.3) \]

\[ \mathbf{B}(\mathbf{r}) = \mu(\mathbf{r}) \mathbf{H}(\mathbf{r}). \quad (2.4) \]

In general, \( \varepsilon \) and \( \mu \) are complex-valued, with \( \text{Re}(\varepsilon) > 0 \) and \( \text{Re}(\mu) > 0 \). For a passive medium, \( \text{Im}(\varepsilon) \leq 0 \) and \( \text{Im}(\mu) \leq 0 \). A medium is called lossy (dissipative) when \( \text{Im}(\varepsilon) < 0 \) and/or \( \text{Im}(\mu) < 0 \); it is called lossless when \( \text{Im}(\varepsilon) = 0 \) and \( \text{Im}(\mu) = 0 \).

Across a surface of discontinuity \( \Sigma \) for the electromagnetic properties the electromagnetic field quantities must satisfy the boundary conditions

\[ (\mathbf{n} \cdot \mathbf{E})_1 = (\mathbf{n} \cdot \mathbf{E})_2, \quad (2.5) \]

\[ (\mathbf{n} \cdot \mathbf{H})_1 = (\mathbf{n} \cdot \mathbf{H})_2, \quad (2.6) \]

that express the continuity of the tangential components of \( \mathbf{E} \) and \( \mathbf{H} \). \( \mathbf{n} \) denotes the unit vector normal to the surface of discontinuity \( \Sigma \) (Fig. 2.1). On the surface of an electrically perfectly conducting object the condition

\[ \mathbf{n} \times \mathbf{E} = 0 \quad (2.7) \]

must hold, while on the surface of a magnetically impenetrable object

\[ \mathbf{n} \times \mathbf{H} = 0 \quad (2.8) \]
must be satisfied.

2.2. THE FREQUENCY-DOMAIN RECIPROCITY THEOREM

One of the most fundamental theorems in electromagnetic field theory is the Lorentz reciprocity theorem (Van Bladel, 1964). This theorem interrelates two different electromagnetic states that can occur in one and the same bounded domain \( \mathcal{V} \) and have the same angular frequency \( \omega \) (Fig. 2.2). Each of the two states satisfies the equations (2.1)–(2.6), applying to the relevant state.

Let us mark the quantities of state A by the superscript A and the quantities of state B by the superscript B. Then, with the aid of (2.1)–(2.6), it can be shown that

\[
\oint_{\partial \mathcal{V}} \left[ \mathbf{p} \cdot (\mathbf{E}^A \times \mathbf{H}^B - \mathbf{E}^B \times \mathbf{H}^A) \right] d\mathbf{A} = \int_{\mathcal{V}} \left[ -\mathbf{H}^B \cdot \mathbf{K}^A + \mathbf{E}^A \cdot \mathbf{z}^B - \mathbf{H}^A \cdot \mathbf{K}^B + \mathbf{E}^B \cdot \mathbf{z}^A \right] d\mathbf{V}. \quad (2.9)
\]

where \( \mathbf{n} \) is the unit vector normal to \( \partial \mathcal{V} \), the boundary surface of \( \mathcal{V} \) pointing away from \( \mathcal{V} \). Here it is understood that \( \varepsilon^A = \varepsilon^B \) and \( \mu^A = \mu^B \) for all \( \mathbf{r} \in \mathcal{V} \) (Fig. 2.3).

![Fig. 2.2. Bounded domain \( \mathcal{V} \) in space with closed boundary surface \( \partial \mathcal{V} \); \( \mathbf{n} \) is the unit vector normal to \( \partial \mathcal{V} \) pointing away from \( \mathcal{V} \) and \( \mathcal{V}^c \) is the complement of \( \mathcal{V} \cup \partial \mathcal{V} \) in \( \mathbb{R}^3 \).](image)
Fig. 2.3. Identical bounded domains in space with the same permittivity and permeability and two different field distributions with the same angular frequency.

Across surfaces of discontinuity for the electromagnetic properties the fields are assumed to satisfy the conditions (2.5) and (2.6), while on the boundary surfaces of impenetrable objects (2.7) or (2.8) must hold.

2.3. THE ELECTROMAGNETIC GREEN'S STATES

From the reciprocity relation (2.9) we want to derive source-type integral representations for the electromagnetic field quantities. To that end, we consider the fields generated by (vectorial) unit point sources with volume current densities proportional to the three-dimensional unit pulse \( \delta(t-t') \). The corresponding states are denoted as the electric Green's state \( \{ E^E, H^E, J^E, K^E \} \) if

\[
J^E(t) = J^E(t') = \delta^E(t-t'),
\]

\[
K^E(t) = K^E(t') = 0.
\]

(2.10)
and as the magnetic Green's state \( \{ \mathbf{E}^{\text{GM}}, \mathbf{H}^{\text{GM}}, J^{\text{GM}}, K^{\text{GM}} \} \) if

\[
J^{\text{GM}}(t) = J^{\text{GM}}(t') = 0, \tag{2.12}
\]

\[
K^{\text{GM}}(t) = K^{\text{GM}}(t') = \mathbf{M} \mathbf{I} - \omega^2. \tag{2.13}
\]

In an unbounded domain these Green's states are required to represent waves travelling away from the source point \( t' \) towards infinity, i.e., they must satisfy the radiation condition. With the use of (2.1)–(2.4) we arrive at the following systems of equations for the Green's states:

\[
\nabla \times \mathbf{H}^{\text{GE}}(\mathbf{r}, t) - j\omega t \mathbf{E}^{\text{GE}}(\mathbf{r}, t) = J^{\text{GE}}(\mathbf{r}, t), \tag{2.14}
\]

\[
\nabla \times \mathbf{E}^{\text{GE}}(\mathbf{r}, t) + j\omega t \mathbf{H}^{\text{GE}}(\mathbf{r}, t) = 0, \tag{2.15}
\]

and

\[
\nabla \times \mathbf{H}^{\text{GM}}(\mathbf{r}, t) - j\omega t \mathbf{E}^{\text{GM}}(\mathbf{r}, t) = 0, \tag{2.16}
\]

\[
\nabla \times \mathbf{E}^{\text{GM}}(\mathbf{r}, t) + j\omega t \mathbf{H}^{\text{GM}}(\mathbf{r}, t) = -K^{\text{GM}}(\mathbf{r}, t). \tag{2.17}
\]

These equations are to be supplemented by the appropriate boundary conditions at surfaces of discontinuity for the electromagnetic properties. In view of the linearity of the governing equations, \( \{ \mathbf{E}^{\text{GE}}, \mathbf{H}^{\text{GE}} \} \) and \( \{ \mathbf{E}^{\text{GM}}, \mathbf{H}^{\text{GM}} \} \) may be written as

\[
\mathbf{E}^{\text{GE}}(\mathbf{r}, t) = \mathbf{F}^{\text{GE}}(\mathbf{r}, t'), \tag{2.18}
\]

\[
\mathbf{H}^{\text{GE}}(\mathbf{r}, t) = -\mathbf{G}^{\text{GE}}(\mathbf{r}, t'), \tag{2.19}
\]
and

$$E_{GM}(t) = -\mathbf{a}_M \cdot \mathbf{q}^{ME}(t', t),$$  \hspace{1cm} (2.20)$$

$$H_{GM}(t) = \mathbf{a}_M \cdot \mathbf{q}^{MM}(t', t).$$  \hspace{1cm} (2.21)$$
in which $\mathbf{q}$ are the so-called Green's tensors of rank two. The dependence of the
position $t'$ of the point source is explicitly indicated in the notation for $\mathbf{q}$. Up to now,
the Green's states are not unique. They can be made so by imposing appropriate
boundary conditions (in case of a bounded domain) or the radiation condition (for an
infinite domain). The equations for the elements of the Green's tensors follow upon
substitution of (2.18) and (2.19) into (2.14) and (2.15), substitution of (2.20) and
(2.21) into (2.16) and (2.17), and by taking for $\mathbf{a}^E$ and $\mathbf{a}^M$ the successive unit vectors
of the coordinate system employed.

Let $\mathcal{Y}$ be a bounded domain with boundary surface $\partial \mathcal{Y}$ and let $\mathcal{Y}^\prime$ denote the
domain exterior to $\partial \mathcal{Y}$. Consider an electromagnetic state $\{E, H, \frac{1}{c} \mathcal{K} \mathcal{Y}, K \mathcal{Y}\}$ which
satisfies the equations (2.1)–(2.8). In the Lorentz reciprocity relation we take for state
$A: \{E^A, H^A, \frac{1}{c} \mathcal{K} \mathcal{Y}^A, K^A \mathcal{Y}\} = \{E, H, \frac{1}{c} \mathcal{K} \mathcal{Y}, K \mathcal{Y}\}$, and for state $B$ the electric Green's
state: $\{E^B, H^B, \frac{1}{c} \mathcal{K} \mathcal{Y}^B, K^B \mathcal{Y}\} = \{E^G, H^G, \frac{1}{c} \mathcal{K} \mathcal{Y}^G, K^G \mathcal{Y}\}$. Upon using (2.10), (2.11),
(2.18) and (2.19), we then arrive at

$$\int_{\partial \mathcal{Y}} [\mathbf{q}^{EM}(t', t) \cdot \mathbf{K} \mathcal{K} (t)] + \mathbf{q}^{EE}(t', t) \cdot \mathbf{J} \mathcal{K} (t)] dA(t)$$

$$+ \int_{\mathcal{Y}} [\mathbf{q}^{EM}(t', t) \cdot \mathbf{K} \mathcal{K} (t) + \mathbf{q}^{EE}(t', t) \cdot \mathbf{J} \mathcal{K} (t)] dV(t)$$

$$= (1, \frac{1}{c}, 0) E(t') \text{ when } t' \in \{ \mathcal{Y}, \partial \mathcal{Y}, \mathcal{Y}^\prime \}. \hspace{1cm} (2.22)$$
Likewise, when we take for state B the magnetic Green’s state: \( \{ E^B, H^B, \omega^B, \sigma^B, K^B \} \)

\[ \{ E^{GM}, H^{GM}, \omega^{GM}, \sigma^{GM}, K^{GM} \} \]

and use (2.12), (2.13), (2.20) and (2.21), we arrive at

\[
\int_{\partial \mathcal{V}} \left[ G^{MM}(t', t) \cdot K_{\omega}(t) + G^{ME}(t', t) \cdot J_{\omega}(t) \right] dA(t)
\]

\[ + \int_{\mathcal{V}} \left[ G^{MM}(t', t) \cdot K_{\omega}(t) + G^{ME}(t', t) \cdot J_{\omega}(t) \right] dV(t)
\]

\[ = \{1, \frac{1}{\mu}, 0\} H(t') \quad \text{when} \ t' \in \{ \mathcal{V}, \partial \mathcal{V}, \mathcal{V}^c \}. \quad (2.23)
\]

In (2.22) and (2.23) the surface current densities \( J_{\omega} \) and \( K_{\omega} \) are given by

\[
J_{\omega}(t) = -e \cdot H(t) \quad \text{with} \ t \in \partial \mathcal{V} \quad (2.24)
\]

\[
K_{\omega}(t) = e \cdot E(t) \quad \text{with} \ t \in \partial \mathcal{V} \quad (2.25)
\]

The factor 1/2 occurring in (2.22) and (2.23) applies to smooth boundaries, i.e., the
surface \( \partial \mathcal{V} \) is assumed to have a tangent plane.

In the preceding analysis we have assumed that \( \mathcal{V} \) is a bounded domain with
boundary surface \( \partial \mathcal{V} \). We can extend the validity of the expressions to cases in which
\( \mathcal{V} \) is an unbounded domain having (parts of) its boundary at infinity, provided that
the fields involved satisfy the radiation condition. Then, the contribution of the parts
at infinity to the surface integrals in (2.9) and (2.22), (2.23) vanishes. For the
unbounded domain exterior to a bounded closed surface only the contribution of the
latter surface remains (Fig. 2.4).

To prove reciprocity relations for the Green’s tensors, with respect to their
Fig. 2.4 Domain $\mathcal{V}$ with boundary $\partial \mathcal{V} = \partial \mathcal{V}_1 \cup \partial \mathcal{V}_2$, with $\partial \mathcal{V}_2 = \omega$. On the application of the Lorentz reciprocity theorem, the contribution of $\partial \mathcal{V}_2$ vanishes, and only the contribution of $\partial \mathcal{V}_1$ remains.

dependence on the two space arguments, we take $\mathcal{V} = \mathbb{R}^3$ in (2.22) and (2.23). We then obtain

$$\int_{\mathcal{V}} [G^{EM}(t', z') \cdot \mathbf{k} \mathbf{p}(t) + G^{EE}(t', z') \cdot \mathbf{j}(t)] dV(t) = \mathbf{E}(t'), \tag{2.26}$$

$$\int_{\mathcal{V}} [G^{MM}(t', z') \cdot \mathbf{k} \mathbf{p}(t) + G^{ME}(t', z') \cdot \mathbf{j}(t)] dV(t) = \mathbf{H}(t'). \tag{2.27}$$

By substituting for the field $\{\mathbf{E}, \mathbf{H}\}(t')$ in (2.26) and (2.27), the electric Green's field due to a unit point source at $t'$, i.e., by setting $\mathbf{j}(t) = \delta^{n}(t-t')$, $\mathbf{k} \mathbf{p}(t) = 0$ and $\{\mathbf{E}, \mathbf{H}\}(t') = \{\mathbf{E}^{GE}, \mathbf{H}^{GE}\}(t')$ and using (2.18)–(2.19), we arrive at
\[ G_{EE}(r', r'') \cdot \mathbf{E} = E_{GE}(r') = \mathbf{E} \cdot G_{EM}(r', r''), \]  

(2.28)

\[ G_{ME}(r', r'') \cdot \mathbf{E} = H_{GE}(r') = -\mathbf{E} \cdot G_{EM}(r', r''). \]  

(2.29)

Similarly, by substituting for the field \( \{ \mathbf{E}, \mathbf{H} \}(r') \) in (2.26) and (2.27), the magnetic Green's field due to a unit point source at \( r'' \), i.e., by setting \( \mathbf{j}(r) = 0 \), \( K \mathbf{\epsilon}(r) = \mathbf{a}^M \delta(r-r'') \) and \( \{ \mathbf{E}, \mathbf{H} \}(r') = \{ \mathbf{0}, \mathbf{H}^M \}(r') \) and using (2.20)–(2.21), we obtain

\[ G_{EM}(r', r'') \cdot \mathbf{a}^M = E_{GM}(r') = -\mathbf{a}^M \cdot G_{ME}(r', r''), \]  

(2.30)

\[ G_{MM}(r', r'') \cdot \mathbf{a}^M = H_{GM}(r') = \mathbf{a}^M \cdot G_{MM}(r', r''). \]  

(2.31)

From (2.28)–(2.31) we arrive at the reciprocity relations for the Green's tensors (Felsen and Marcuvitz, 1973, p 92)

\[ E_{EE}(r, r') = [G_{EE}(r', r')]^T, \]  

(2.32)

\[ M_{MM}(r, r') = [G_{MM}(r', r')]^T, \]  

(2.33)

\[ E_{EM}(r, r') = [G_{EM}(r', r')]^T, \]  

(2.34)

where the superscript \( T \) denotes transposition.

In subsequent chapters we shall use the integral representations (2.22) and (2.23) for the electromagnetic field intensities at \( r' \in \partial \mathcal{V} \) to describe the transmission and reflection properties of sections of straight open waveguides.
3. FIELD REPRESENTATIONS IN OPEN WAVEGUIDE SECTIONS

3.1. THE STRAIGHT OPEN WAVEGUIDE SECTION

In this chapter, the electromagnetic fields in a straight open waveguide section will be investigated. In Fig. 3.1, the pertaining configuration is shown. The axial coordinate is $z$. The terminal planes of the waveguide section are the transverse planes $z=z_1$ and $z=z_2$, with $z_1 < z_2$. The $z$-interval $z_1 < z < z_2$ is denoted by $\mathcal{S}$; the boundary of $\mathcal{S}$, i.e., $\{z=z_1\} \cup \{z=z_2\}$, is denoted by $\partial \mathcal{S}$; $\{-\infty < z < z_1\} \cup \{z_2 < z < \infty\}$ is denoted by $\mathcal{S}^-$. The configuration is translation invariant in the $z$-direction. This implies that the permittivity and the permeability of the medium are functions of the transverse position $r_T$, only, i.e., $\varepsilon = \varepsilon(r_T)$, $\mu = \mu(r_T)$, where

$$I = I_T + z I_z$$  \hspace{1cm} (3.1)

Fig. 3.1. Uniform section of an open waveguide.
Outside the bounded cross-sectional domain \( \mathcal{D} \) (see Fig. 3.1), whose boundary contour is \( \partial \mathcal{D} \), \( \epsilon \) and \( \mu \) are constants, to be denoted by \( \epsilon_1 \) and \( \mu_1 \). The domain outside \( \partial \mathcal{D} \) is denoted by \( \mathcal{D}' \). In \( \mathcal{D} \), the values of \( \epsilon \) and/or \( \mu \) differ from their values in \( \mathcal{D}' \).

Dependent on the specification of \( \epsilon \) and \( \mu \) as functions of \( z_T \in \mathcal{D} \) several types of waveguides are distinguished: step-index, where \( \epsilon \) and \( \mu \) are constants in \( \mathcal{D} \); multi-step-index, where \( \epsilon \) and \( \mu \) are piecewise constant functions of \( z_T \) in \( \mathcal{D} = \bigcup_{n=2}^{N} \mathcal{D}_n \); and graded-index, where \( \epsilon \) and \( \mu \) are continuous functions of \( z_T \) in \( \mathcal{D} \) (Table II). Some special waveguide shapes often encountered in practice are: the rotationally symmetric fibre and the planar waveguide for integrated optics (Fig. 3.2).

Table II. Permittivity and permeability distribution in a straight waveguide section.

<table>
<thead>
<tr>
<th>type of waveguide</th>
<th>permittivity and permeability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z_T \in \mathcal{D} )</td>
</tr>
<tr>
<td>step-index</td>
<td>( \epsilon(z_T) = \epsilon_2 )</td>
</tr>
<tr>
<td></td>
<td>( \mu(z_T) = \mu_2 )</td>
</tr>
<tr>
<td>multi-step-index</td>
<td>( \epsilon(z_T) = \epsilon_n ) ( z_T \in \mathcal{D}_n ) ( n=2, \ldots, N )</td>
</tr>
<tr>
<td></td>
<td>( \mu(z_T) = \mu_n ) ( z_T \in \mathcal{D}_n ) ( n=2, \ldots, N )</td>
</tr>
<tr>
<td>graded-index</td>
<td>( \epsilon(z_T) ) and ( \mu(z_T) ) continuous functions of ( z_T )</td>
</tr>
<tr>
<td></td>
<td>( \mu(z_T) = \mu_1 )</td>
</tr>
</tbody>
</table>
2.2. MODAL EXPANSION OF THE FIELDS IN AN OPEN WAVEGUIDE SECTION

In this section the modal expansion of the fields in open waveguides is discussed. This type of expansion is often used in describing the transmission properties of waveguide sections. In the following, irrelevant dependences on coordinates will be suppressed in the notation.

In order to investigate the transmission properties of the waveguide section, in which the field distributions in the end planes serve as excitations, we subject the field equations in a section to a finite Fourier Transformation with respect to the axial coordinate. To this end we introduce

Fig. 3.2. Planar waveguide (a), and rotationally symmetric fibre (b), and permittivity/permeability profiles: step-index (I), multi-step-index (II) and graded-index (III).
\[ \hat{F}(r_T,k_z) = \int_{s_1}^{s_2} \exp(ik_z z) \hat{F}(r_T,z) \, dz \text{ with } k_z \in \mathbb{R}. \] (3.2)

Inversely, we have

\[ (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ik_z z) \hat{F}(r_T,k_z) \, dk_z = \{1,0,0\} \hat{F}(r_T,z) \text{ when } z \in \{ \mathbb{R}, \mathbb{S}, \mathbb{S}' \}. \] (3.3)

The electromagnetic field equations (2.1) and (2.2) then transform into

\[ \vec{v} \times \hat{H}(r_T,k_z) - j\omega \hat{E}(r_T,k_z) = \int \gamma(r_T,k_z) + \int_{-\infty}^{\infty} \alpha(r_T,z_2) \exp(jk_z z_2) \exp(jk_z z_1) \, dz_2, \] (3.4)

\[ \vec{v} \times \hat{E}(r_T,k_z) + j\omega \hat{H}(r_T,k_z) = -K \gamma(r_T,k_z) - \int_{-\infty}^{\infty} \omega \alpha(r_T,z_2) \exp(jk_z z_2) \exp(jk_z z_1) \, dz_2, \] (3.5)

in which \( \int \alpha \) and \( K \omega \) are given in (2.24) and (2.25) with \( \mathbb{N} = -\mathbb{S}_z \) at \( z = s_1 \) and \( \mathbb{N} = \mathbb{S}_z \) at \( z = s_2 \); and

\[ \vec{v} = \gamma_T - jk_z \mathbb{S}_z. \] (3.6)

Since \( \epsilon \) and \( \mu \) in the waveguide are independent of \( z \), (3.3) and (2.4) transform into

\[ \hat{D}(r_T,k_z) = \epsilon(r_T) \hat{E}(r_T,k_z), \] (3.7)
\( \hat{H}(z_1, k_y) = \mu(z_1) \hat{H}(z_1, k_y). \) \hfill (3.8)

The surface source terms in (3.4) and (3.8) can be regarded as the axial Fourier transforms over the interval \(-a < s < a\) of the transverse end-plane current sheets with volume distributions of the electric type \( J_\varphi(T_{z_1}) \delta(z-z_1) \), \( J_\varphi(T_{z_2}) \delta(z-z_2) \), and volume distributions of the magnetic type \( K_\varphi(T_{z_1}) \delta(z-z_1) \), \( K_\varphi(T_{z_2}) \delta(z-z_2) \). In the usual transmission case they serve as excitations, while the volume source distributions \( J_\varphi \) and \( K_\varphi \) in the interior of the section vanish. Consequently, our case is fully covered once the fields excited by a single transverse electric current source distribution \( J_\varphi(s) \) and the fields excited by a single transverse magnetic current source distribution \( K_\varphi(s) \) have been determined.

For a transverse electric current source \( J_\varphi(s) \), the Fourier transforms of the fields over the interval \(-a < s < a\) satisfy the equations

\[
\vec{v} \times \vec{H}^E - j\omega \vec{E}^E = J_\varphi, \tag{3.9}
\]

\[
\vec{v} \times \vec{E}^E + j\omega \vec{H}^E = 0, \tag{3.10}
\]

in which the superscript \( E \) indicates the type of excitation. By separating these equations into transverse and axial parts, the symmetry properties of the field components with respect to \( k_y \) are readily established. Since \( J_\varphi \) is independent of \( k_y \), the transverse component of the left-hand side of (3.9) must be even in \( k_y \), and we arrive at:

\[
(E^E_T, H^E_T)(k_y) = (E^E_T, H^E_T)(-k_y), \quad (E^E_S, H^E_T)(k_y) = -(E^E_S, H^E_T)(-k_y). \tag{3.11}
\]

In order to reveal the modal structure of the fields, the functions \( \vec{E}^E \) and \( \vec{H}^E \) are
analytically continued into the complex $k_z$-plane. This analytic continuation is assumed to have the property: $|\{E^E, H^E\}(k_z)| \rightarrow 0$ as $|k_z| \rightarrow \infty$ (by virtue of the Riemann-Lebesgue lemma, this assumption is met for real values of $k_z$). From experience with configurations for which the transformed quantities can be evaluated analytically, we expect $E^E$ and $H^E$ to have the following singularities in the complex $k_z$-plane: a finite number of simple poles $\{k_n^E\}$, $n = 1, \ldots, N^E$, (under certain circumstances, there may be no poles) and a branch point $k_b = k_1 = \omega_1 = \rho_0^{1/2}$ (see Appendix A) in the fourth quadrant of the $k_z$-plane; and, symmetrically, a finite number of simple poles $\{-k_n^E\}$ and a branch point $k_2 = -k_1$ in the second quadrant of the $k_z$-plane (Fig. 3.3). In general, $k_1$ is complex-valued (lossy medium). The lossless case is considered as a limiting case of the lossy one. The branch points $k_z = \pm k_1$ are due to the occurrence of the square root $(k_1^2 - k_2^2)^{1/2}$ which is specified as that branch for which $\text{Im}(k_1^2 - k_2^2)^{1/2} \leq 0$ (Appendix A). Accordingly, we have the branch cuts $\mathcal{C}^+$ and $\mathcal{C}^-$ (on which $\text{Im}(k_1^2 - k_2^2)^{1/2} = 0$) as shown in Fig. 3.3.

By use of Cauchy's integral formula for the functions $\mathbf{E}^E$ and $\mathbf{H}^E$ and the contour shown in Fig. 3.3 (in the interior of which $\mathbf{E}^E$ and $\mathbf{H}^E$ are analytic functions of $k_z$) and by taking into account the symmetry properties (3.11) of the fields, we obtain

\begin{equation}
(E^E, H^E) = \sum_{n=1}^{N^E} \left\{ (E^E_{n, \mathbf{T}}, H^E_{n, \mathbf{T}, \kappa}) \frac{2j k_n^E}{k_z^2 - k_n^E} \right\} d\kappa + \int_{\mathcal{C}^+} \left\{ (E^E_{\kappa, \mathbf{T}}, H^E_{\kappa, \mathbf{T}, \kappa}) \frac{2j k_z}{k_z^2 - \kappa^2} \right\} d\kappa, \tag{3.12}
\end{equation}

\begin{equation}
(E^E, H^E) = \sum_{n=1}^{N^E} \left\{ (E^E_{n, \mathbf{H}}, H^E_{n, \mathbf{H}, \kappa}) \frac{2j k_n^E}{k_z^2 - k_n^E} \right\} d\kappa + \int_{\mathcal{C}^-} \left\{ (E^E_{\kappa, \mathbf{H}}, H^E_{\kappa, \mathbf{H}, \kappa}) \frac{2j k_z}{k_z^2 - \kappa^2} \right\} d\kappa, \tag{3.13}
\end{equation}

in which $j(E^E_{n, \mathbf{T}}, H^E_{n, \mathbf{T}})$ are the residues of $(E^E, H^E)$ at the pole $k_n^E$. The integration along $\mathcal{C}^\pm$ is taken from the branch point $\kappa = k_1$ towards infinity, and $-2\pi i(E^E_{\kappa, \mathbf{H}}, H^E_{\kappa, \mathbf{H}})$ denotes the "jump" in $(E^E, H^E)$ across the branch cut $\mathcal{C}^\pm$; this jump is defined as the
Fig. 3.3. Complex $k_z$-plane with branch points $k_z = n k_1$, branch cuts $\mathcal{A}^+$ and $\mathcal{A}^-$ \((\text{Im}(k_1^2 - k_z^2)^{1/2} = 0)\) and surface-wave poles \(\{\kappa_n^E\}\) and \(\{-\kappa_n^E\}\), which are either \(\{\kappa_n^E\}\) and \(\{-\kappa_n^E\}\), or \(\{\kappa_n^M\}\) and \(\{-\kappa_n^M\}\). Also shown is the contour for the application of Cauchy's integral formula.

The difference of the values of \(\{E^E, H^E\}\) at the branch cut on the side where \(\text{Re}(k_1^2 - k_z^2)^{1/2} > 0\) (indicated by a plus sign in Fig. 3.3), and the values of \(\{E^E, H^E\}\) on the side where \(\text{Re}(k_1^2 - k_z^2)^{1/2} < 0\) (indicated by a minus sign in Fig. 3.3).

By inverse Fourier Transformation of (3.12) and (3.13), evaluated by closing the path of integration in the lower half of the $k_z$-plane, the electromagnetic field \(\{E, H\}\) is obtained as

\[
\{E^E, H^E\} = \sum_{n=1}^{N^E} \{E_{n}^{E}, H_{n}^{E}\} \exp(-j x_n^E z) + \int_{\mathcal{B}^-} \{E_{n}^{E}, H_{n}^{E}\} \exp(-j x z) \, dx \quad (z > 0).
\]

When \(z < 0\), \(E^E\) and \(H^E\) can be obtained by using their symmetry properties.
\[(E_E^E, H_M^E)(z) = (E_E^E, H_M^E)(-z), \quad (E_E^E, H_M^E)(z) = -(E_E^E, H_M^E)(-z), \quad (3.15)\]

which follow from the symmetry properties (3.11) of \(\{E_E^E, H_M^E\}\). In (3.14), the summation over the poles can be interpreted as the contribution of the surface-wave modes to the fields in the waveguide (discrete part of the spectrum); the surface-wave poles \(\kappa_n^E\) also appear as propagation coefficients of the surface-wave modes. The integration along \(\mathcal{E}\) represents the contribution of the radiation modes (continuous part of the spectrum).

In the same way, we can analyse the excitation by a single transverse magnetic current source distribution with volume density \(K_T\delta(x)\). The Fourier transforms of the fields generated then satisfy the equations

\[\hat{x} \times \hat{H}_M^M - j \omega \hat{E}_M^M = 0, \quad (3.16)\]
\[\hat{x} \times \hat{E}_M^M + j \omega \mu \hat{H}_M^M = -K_T, \quad (3.17)\]

in which the superscript \(M\) refers to magnetic current source excitation. Since \(K_T\) is independent of \(k_x\), we now obtain the symmetry relations

\[(E_M^E, H_M^M)(k_2) = -(E_M^E, H_M^M)(-k_2), \quad (3.18)\]

As before, \(E_M^E\) and \(H_M^M\) are analytically continued into the complex \(k_z\)-plane. We expect \(E_M^E\) and \(H_M^M\) to have a finite number of simple poles \(\{\kappa_n^M\}, n = 1, \ldots, N^M\), (that may be different from the poles \(\{\kappa_n^E\}\) in the fourth and second quadrants, and again the branch points \(k_z = \pm k_1\).

Taking into account (3.18), the representation analogous to (3.12) and (3.13) is now

\[\]


\[
\begin{align*}
\{E^M \cdot H^M\}_n &= \sum_{n=1}^{N^M} \left( E^M_{n,T} \cdot H^M_{n,T} \right) \frac{2jk_z}{k_z^2 - \kappa_n^2} + \int_{\mathbb{R}^+} \left( E^M_{\kappa,T} \cdot H^M_{\kappa,T} \right) \frac{2jk_z}{k_z^2 - \kappa^2} \, d\kappa, \quad (3.19) \\
\{E^M \cdot H^M\} &= \sum_{n=1}^{N^M} \left( E^M_{n,z} \cdot H^M_{n,T} \right) \frac{2jk_z}{k_z^2 - \kappa_n^2} + \int_{\mathbb{R}^+} \left( E^M_{\kappa,z} \cdot H^M_{\kappa,T} \right) \frac{2jk_z}{k_z^2 - \kappa^2} \, d\kappa. \quad (3.20)
\end{align*}
\]

From these expressions, the fields in \( z > 0 \) are obtained as

\[
\{E^M \cdot H^M\} = \sum_{n=1}^{N^M} \left( E^M_{n,z} \cdot H^M_{n,T} \right) \exp(-jk_z n) + \int_{\mathbb{R}^+} \left( E^M_{\kappa,z} \cdot H^M_{\kappa,T} \right) \exp(-jk_z \kappa) \, d\kappa \quad (z > 0). \quad (3.21)
\]

When \( z < 0 \), \( E^M \) and \( H^M \) can be found by using their symmetry properties.

\[
\{E^M \cdot H^M\}(z) = -\{E^M \cdot H^M\}(-z), \quad \{E^M \cdot H^M\}(z) = -\{E^M \cdot H^M\}(-z). \quad (3.22)
\]

Again, the summation over the poles represents the contribution of the surface-wave modes with propagation coefficients \( \kappa_n \), and the integration along \( \mathbb{R}^+ \) can be interpreted as the contribution of the radiation modes.

From (3.14) and (3.21) it is apparent, that the fields due to an arbitrary excitation at \( z = 0 \) can be represented by

\[
\begin{align*}
\{E,H\} &= \sum_{n=1}^{N} \{E_n,H_n\} \exp(-jk_n z) + \int_{\mathbb{R}^+} \{E_{\kappa},H_{\kappa}\} \exp(-jk_{\kappa} z) \, d\kappa \quad \text{when} \ z > 0, \quad (3.23)
\end{align*}
\]

and a similar representation for the fields when \( z < 0 \). In (3.23) the field contributions due to transverse electric and transverse magnetic current source distributions have been taken together. In Appendix B it is shown that the modal field constituents for \( z > 0 \), \( \{E_n,H_n\} \exp(-jk_n z), \ n = 1, \ldots, N \), and \( \{E_{\kappa},H_{\kappa}\} \exp(-jk_{\kappa} z), \ k \in \mathbb{R}, \) form a
complete orthogonal set of functions. Next we introduce the normalised field constituents for \( z > 0 \), denoted by \( \{ \mathbf{e}_\alpha, \mathbf{h}_\alpha \} \) and \( \{ \mathbf{e}_\alpha^*, \mathbf{h}_\alpha^* \} \), which satisfy the Lorentz normalisation conditions

\[
\int_{\mathcal{S}_T} (\mathbf{e}_\alpha \times \mathbf{h}_\alpha^*) \cdot \mathbf{i}_z \, dA(z_T) = \frac{i}{2} \delta_{\alpha,\alpha^*},
\]

\[
\int_{\mathcal{S}_T} (\mathbf{e}_\alpha^* \times \mathbf{h}_\alpha) \cdot \mathbf{i}_z \, dA(z_T) = -\frac{1}{2} k_{T,1} \delta(k_{T,1}^*, k_{T,1}-k_{T,1}^*),
\]

(3.24)

(3.25)

where \( \mathcal{S}_T \) denotes the total transverse cross-sectional domain of the waveguide and its surroundings, and \( k_{T,1} = \left( k_1^2 - n^2 \right)^{1/2} \), \( k_{T,1}^* = \left( k_1^2 - n^* \right)^{1/2} \), (note that \( k_{T,1} \) and \( k_{T,1}^* \) are real and positive). For \( z < 0 \), the normalised field constituents follow by applying the symmetry properties (3.15) and (3.22).

It can be shown that the Green's tensors of the waveguide are expressible in terms of the Lorentz–normalised modal field constituents as

\[
\mathcal{G}_{EE}(\mathbf{r}_1' + z \mathbf{i}_z, \mathbf{r}_2 + z \mathbf{i}_z) = -\sum_{n=1}^{N} \mathbf{e}_n(\mathbf{r}_1') \cdot \mathbf{e}_n(\mathbf{r}_2) \exp(-j k_n(z'-z))
\]

\[
- \int_{\mathcal{S}^+} \mathbf{e}_n(\mathbf{r}_1') \cdot \mathbf{e}_n(\mathbf{r}_2) \exp(-j k_n(z'-z)) \, ds,
\]

(3.26)

\[
\mathcal{G}_{ME}(\mathbf{r}_1' + z \mathbf{i}_z, \mathbf{r}_2 + z \mathbf{i}_z) = -\sum_{n=1}^{N} \mathbf{h}_n(\mathbf{r}_1') \cdot \mathbf{e}_n(\mathbf{r}_2) \exp(-j k_n(z'-z))
\]

\[
- \int_{\mathcal{S}^+} \mathbf{h}_n(\mathbf{r}_1') \cdot \mathbf{e}_n(\mathbf{r}_2) \exp(-j k_n(z'-z)) \, ds,
\]

(3.27)
\[ \omega \sum \left( \nabla \times + \nabla \cdot \right) \mathbf{J} = - \sum_{n=1}^{N} \left( c_n(z_T) \cdot b_n(z_T) \exp(-ij_n(z'-z)) \right) 
\]

\[ - \int_{z'}^{z} c_n(z_T) \cdot c_n(z_T) \exp(-ij_n(z'-z)) ds, \quad (3.28) \]

\[ \omega \sum \left( \nabla \times + \nabla \cdot \right) \mathbf{J} = - \sum_{n=1}^{N} \left( b_n(z_T) \cdot b_n(z_T) \exp(-ij_n(z'-z)) \right) 
\]

\[ - \int_{z'}^{z} b_n(z_T) \cdot b_n(z_T) \exp(-ij_n(z'-z)) ds, \quad (3.29) \]

when \( z' > z \) (Blok and De Hoop, 1983; the difference in sign between their expressions and (3.28)–(3.29) is due to a difference in normalisation). When \( z' < z \), the expressions for the Green's tensors can be obtained by carrying out the appropriate changes according to symmetry (cf. (3.16) and (3.22)).

In the next section we shall discuss some methods for calculating the solutions of the source–free field equations that correspond to the surface–wave modes.

3.3. METHODS FOR THE CALCULATION OF SURFACE–WAVE MODES IN OPEN WAVEGUIDES

Several methods exist for the computation of the propagation coefficients and the field distributions of the surface–wave modes in open waveguiding structures. We mention: the direct numerical solution of the source–free electromagnetic field equations (Mur, 1978); the numerical solution of the system of source–type integral equations resulting from the source–free electromagnetic field equations (De Ruijter, 1980); the transfer–matrix formalism (for special geometries) (Suematsu and Furuya, 1972;
Claricoats et al., 1966); and methods of an approximate nature, such as the weak-
guidance approximation (Snyder and Young, 1978). In this section, two methods will
be treated in more detail, viz. the integral-equation method and the transfer-matrix
formalism.

3.3.1. The integral-equation method

The field of a surface-wave mode \((e_n, h_n)\exp(-j\kappa_n z)\) with propagation coefficient \(\kappa_n\)
satisfies the source-free electromagnetic field equations

\[
\nabla \times \mathbf{h}_n - j\omega c \mathbf{e}_n = 0, \quad (3.30)
\]

\[
\nabla \times \mathbf{e}_n + j\omega \mu_h \mathbf{h}_n = 0, \quad (3.31)
\]

in which

\[
\mathbf{v}_n = \mathbf{v}_T - j\kappa_n \mathbf{e}_n, \quad (3.32)
\]

and must be quadratically integrable over the total cross-sectional domain of the
waveguide and its surroundings. In fact, \(\kappa_n\) is an eigenvalue of equations (3.30) and
(3.31). The deviations of the permittivity and permeability in the waveguide from
their values \(\varepsilon_1\) and \(\mu_1\) in the surrounding medium are now conceived as
z-independent disturbances. In accordance with this point of view, equations (3.30)
and (3.31) are rewritten as

\[
\nabla \times \mathbf{h}_n - j\omega \varepsilon_1 \mathbf{e}_n = \mathbf{j}_n, \quad (3.33)
\]

\[
\nabla \times \mathbf{e}_n + j\omega \mu_1 \mathbf{h}_n = -\mathbf{k}_n, \quad (3.34)
\]
where

\[ j_n = j_0(\epsilon_n \kappa_n) g_n, \quad (3.35) \]

\[ \kappa_n = j_0(\mu_n \kappa_n) g_n. \quad (3.36) \]

Equations (3.33) and (3.34) have the appearance of electromagnetic field equations in a homogeneous medium with constitutive coefficients \( \epsilon_n \) and \( \mu_n \), and with volumetric source terms \( j_n \) and \( k_n \). In terms of these volume sources the solutions of these equations can be written as (De Hoop, 1977)

\[ \xi_n(\tau_T) = (j_0 \omega_n)^{-1} \sum_{n=1}^{N} g_n(\epsilon_n \kappa_n) j_n(\tau_T) g_n. \quad (3.37) \]

\[ h_n(\tau_T) = (j_0 \omega_n)^{-1} \sum_{n=1}^{N} g_n(\mu_n \kappa_n) k_n(\tau_T) g_n. \quad (3.38) \]

with

\[ p_n(\tau_T) = \int_{\mathcal{G}} g(\tau_T, \tau_T, \kappa_n) j_n(\tau_T) dA(\tau_T), \quad (3.39) \]

\[ q_n(\tau_T) = \int_{\mathcal{G}} g(\tau_T, \tau_T, \kappa_n) k_n(\tau_T) dA(\tau_T), \quad (3.40) \]

in which \( g \) is the two-dimensional free-space Green's function

\[ g(\tau_T, \tau_T, \kappa_n) = -(1/4)H^{(2)}_0(k_n |\tau_T - \tau_T|), \quad (3.41) \]

with
\[ k_T = (k_1^2 - k_n^2)^{1/2}, \quad \text{Im}(k_T) \leq 0, \quad k_1 = \omega \sqrt{\varepsilon_1 \mu_1}^{1/2}. \] (3.42)

For \( z_T \in \mathbb{Z} \) equations (3.37) and (3.38) constitute a system of homogeneous integral equations. Upon solving these equations (which, in general, has to be done numerically), we obtain the propagation coefficients \( \{a_n\} \) as eigenvalues, and the corresponding modal field distributions as eigenfunctions.

3.3.2. The transfer–matrix formalism

For configurations in which the geometry, permittivity and permeability are functions of a single coordinate only (e.g., the planar waveguide and the circularly cylindrical waveguide), the problem of determining the surface–wave modes can be reduced to a problem of solving ordinary differential equations and a corresponding transfer–matrix formalism can be developed. This formalism will be applied to a source–free configuration.

The configurations for which the transfer–matrix formalism can be used are shown in Fig. 3.4. The coordinate on which the waveguide properties depend, is denoted by \( u \); for the planar waveguide, \( u \) stands for the \( x \)-coordinate \((-a < u < a)\), and for the circularly cylindrical waveguide, \( u \) stands for the distance \( \rho \) to the axis \((0 \leq u < a)\).

The waveguide is divided into one or more layers, bounded by surfaces \( u = \) constant, in which the permittivity and permeability are continuous functions of \( u \). Across the interface of two successive layers, these quantities may exhibit a finite jump. Now, the four electromagnetic field components perpendicular to the direction of \( u \) are continuous upon crossing these interfaces. They are combined into a column matrix, the field matrix \( \xi \).

Let \( u = u_p \ (p = 1,2,...,N-1) \) denote the location of the interfaces, then in the interior...
Fig. 3.4. Configurations to which the transfer-matrix formalism can be applied:
(a) planar waveguide with piecewise continuous permittivity $\varepsilon(x)$ and
permeability $\mu(x)$; (b) circularly cylindrical waveguide with piecewise continuous
permittivity $\varepsilon(\rho)$ and permeability $\mu(\rho)$.

of the layer $u_{p-1} < u < u_p$, the field matrices at two positions $u$ and $u'$ are
interrelated by the transfer matrix $T_p$ (Walter, 1976), viz.

\[ f(u) = T_p(u, u', \kappa_{u}), f(u'), u_{p-1} \leq u, u' \leq u_p \]  (3.43)

in which $\kappa_{u}$ is the propagation coefficient of the surface-wave mode to be determined.
The columns of $T_p$ are the special fundamental solutions of the system of first-order
differential equations for the elements of \( f \) in the layer, that are uniquely defined by

\[
T_p^q(u, v, \nu_u) = I, \quad (3.44)
\]

in which \( I \) denotes the unit matrix. Since \( f \) only contains field components that are continuous upon crossing the interfaces between successive layers, the field at an arbitrary position \( u \) in the configuration can be expressed in terms of the field at another arbitrary position \( u' \). Let \( u_{q-1} \leq u \leq u_q \) and \( u_{p-1} \leq u' \leq u_p \), then we have when \( q > p \) (Fig. 3.5)

\[
f(u) = T_q^p(u, u_{q-1}) \cdot T_{q-1}^{q-1}(u_{q-1}, u_{q-2}) \cdots T_{p+1}^{p+1}(u_{p+1}, v_p) \cdot T_p^p(u_p, u') \cdot f(u'). \quad (3.45)
\]

A similar expression can be obtained in the case \( q < p \).

The present relation between the field matrices at different positions is used in the "interior" layers of the waveguide, i.e., \( u_1 < u < u_{N-1} \). In each of the "outer" domains, i.e., \(-\infty < u < u_1 \) and \( u_{N-1} < u < \infty \) for the planar waveguide, and \( 0 < u < u_1 \) and \( u_{N-1} < u < \infty \) for the circularly cylindrical waveguide, it is required that the fields must remain bounded as \( u \rightarrow \infty \) (planar waveguide) or as \( u \rightarrow 0 \) and \( u \rightarrow \infty \) (circularly cylindrical waveguide). As an example consider the outer domain \(-\infty < u < u_1 \) of the planar waveguide. In this domain the governing differential equations have four linearly independent solutions for the field matrix \( f \) consisting of the transverse field components \( \{E_y, E_z, H_x, H_y\} \). Two of these solutions can be chosen to be bounded as \( u \rightarrow \infty \), while the remaining two solutions grow exponentially as \( u \rightarrow -\infty \). Obviously the latter two solutions must be excluded, which leads to two linear relations to be imposed on the components of the field matrix \( f \). By means of these relations two components of \( f(u_1) \) can be eliminated. Similarly, by retaining only the bounded solutions in the outer domain \( u_{N-1} < u < \infty \) of the planar waveguide, two
components of the field matrix \( f(u_{N-1}) \) can be eliminated. The same procedure also applies to the solutions in the outer domains of the circularly cylindrical waveguide. Thus we conclude that after elimination of two components as indicated, both \( f(u_1) \) and \( f(u_{N-1}) \) contain two unknown field components only.

To determine the propagation coefficients and the field distributions of the surface-wave modes we now proceed as follows. By means of the transfer matrices, the field matrix at an arbitrarily chosen level \( u_0 \) is expressed in terms of the field matrix at \( u = u_1 \) by

\[
f(u_0) = \mathcal{T}(u_0, u_1) \cdot f(u_1),
\]

where \( \mathcal{T} \) is a product of transfer matrices of the layers between the levels \( u_1 \) and \( u_0 \), as in (3.45). At the same level \( u_0 \), the field matrix can also be expressed in terms of
the field matrix at \( u = u_{N-1} \) by

\[
\mathbf{T}(u_0, u_{N-1}) \cdot \mathbf{f}(u_{N-1}) = \mathbf{T}(u_0, u_1) \cdot \mathbf{f}(u_1).
\] (3.47)

Since the field matrices at \( u = u_0 \) must be identical, (3.46) and (3.47) lead to

\[
\mathbf{T}(u_1, u_{N-1})^{-1} \cdot \mathbf{T}(u_0, u_{N-1}) \cdot \mathbf{f}(u_{N-1}) = \mathbf{T}(u_0, u_1) \cdot \mathbf{f}(u_1).
\] (3.48)

This is a homogeneous system of four linear algebraic equations for the two unknown field components of \( \mathbf{f}(u_1) \) and the two unknown field components of \( \mathbf{f}(u_{N-1}) \). This system has a non-zero solution only for particular values of \( \kappa_n \), which are called eigenvalues. Having solved the resulting eigenvalue equation for \( \kappa_n \), we can obtain the unknown field distribution up to a complex multiplicative constant, which is determined by imposing the normalisation condition. The field matrices at \( u = u_1 \) and \( u = u_{N-1} \) are then known; the field matrix at an arbitrary position results by reusing the transfer-matrix formalism.

From (3.48) it is easily seen that the values of \( \kappa_n \) and of the field matrices do not depend on the choice of \( u_0 \), since

\[
[\mathbf{T}(u_0, u_1)]^{-1} = \mathbf{T}(u_1, u_0), \text{ so } [\mathbf{T}(u_0, u_1)]^{-1} \cdot \mathbf{T}(u_0, u_{N-1}) = \mathbf{T}(u_1, u_{N-1}),
\]

and the latter matrix, which is the transfer matrix from level \( u_{N-1} \) to level \( u_1 \), is independent of \( u_0 \). In practice, the level \( u_0 \) is chosen on computational grounds. Ideally, this level should correspond to the maximum of the transverse field distribution of the mode under consideration.

The transfer-matrix formalism is particularly suitable for waveguides that consist of layers for which closed-form expressions for the fundamental solutions are available. Examples are:
layers with a constant permittivity and permeability profile, for which the fundamental solutions involve trigonometric and exponential functions in the case of a planar waveguide (Suematsu and Furuya, 1972), and Bessel functions in the case of a circularly cylindrical waveguide (Clarricoats et al., 1968);

- layers with a linear refractive index profile for which the fundamental solutions in the case of a planar waveguide are expressible in terms of Airy functions (Brekhovskikh, 1980, pp. 181 — 188);

- layers with an Epstein-type refractive index profile for which the fundamental solutions for a planar waveguide are expressible in terms of hypergeometric functions or Heun's functions, depending on the type of polarisation (Blokh, 1907; Brekhovskikh, 1980, pp. 164 — 180; Van Duin, 1981).

Some authors have used a step-function approximation to an (arbitrary) graded-index profile (Clarricoats and Chan, 1970; Suematsu and Furuya, 1972) and have used the transfer-matrix formalism to perform computations of the propagation coefficients and the field distributions of the surface-wave modes of a graded-index waveguide. When the thickness of the layers used in the discretisation of the actual profile is sufficiently small as compared to the transverse wavelength of the surface-wave mode under consideration and to the variation of the profile, this approach will yield good approximate results for the propagation coefficients of the graded-index waveguide. For a specific example, the influence of the number of layers on the value of the propagation coefficient obtained for a particular surface-wave mode in a circularly cylindrical waveguide has been investigated by Clarricoats and Chan (1970).

In the next section, we shall apply the two methods discussed here to the wave propagation in a planar open waveguide, and we shall present some numerical results obtained by the two methods.
3.4. THE COMPUTATION OF SURFACE-WAVE MODES IN A PLANAR OPEN WAVEGUIDE

In this section the methods of computation discussed in the previous section are applied to the computation of the surface-wave modes in a planar open waveguide.

The configuration at hand is shown in Fig. 3.6. The geometry, permittivity and permeability only depend on the x-coordinate. The waveguide's thickness is $d = 2a$. When $-a < x < a$, $\varepsilon$ and $\mu$ are functions of $x$; outside the waveguide, $\varepsilon = \varepsilon_1$ and $\mu = \mu_1$ are constants. In this configuration we investigate the fields that are y-independent; then $\partial_y = 0$ and $\vec{E}_m = \vec{i}_x \theta_x - j \vec{n} \vec{h}_y$. From (3.30) and (3.31) it is easily seen that the field equations separate into two independent systems of equations, viz. one system for $\text{TE}$-fields with \( \{ \varepsilon_y, h_x, h_y \} \neq 0 \) and \( \{ h_y, e_x, e_y \} = 0 \), and one system for $\text{TM}$-fields with \( \{ h_y, e_x, e_y \} \neq 0 \) and \( \{ e_y, h_x, h_y \} = 0 \). In view of the duality of the electric and magnetic field quantities, the equations for the $\text{TM}$-field quantities follow from the $\text{TE}$-field equations by making the appropriate substitutions.

![Diagram](image)

Fig. 3.6. Straight planar waveguide and coordinate system. The slab thickness is $d = 2a$. 
3.4.1. The integral-equation method

In view of the $y$-independence of the configuration and the fields, the results (3.37) and (3.38) for TE-fields simplify to

$$e_{n,y} = -j\omega p_{n,y} + j\kappa q_{n,x} + jq_{n,y}$$  \hspace{1cm} (3.49)

$$h_{n,x} = (j\omega \kappa)^{-1}(-j\omega \kappa q_{n,x} - jq_{n,y}) - j\omega p_{n,x} + j\kappa p_{n,y}$$  \hspace{1cm} (3.50)

$$h_{n,y} = (j\omega \kappa)^{-1}(-j\omega \kappa q_{n,y} - jq_{n,x}) - j\omega p_{n,y} + j\kappa p_{n,x}$$  \hspace{1cm} (3.51)

in which $p_{n,y}$ and $q_{n,x,y}$ are now given by

$$p_{n,y}(x) = \int_0^d g(x,x',\kappa) j_{n,y}(x')dx'$$  \hspace{1cm} (3.52)

$$q_{n,x,y}(x) = \int_0^d g(x,x',\kappa) k_{n,x,y}(x')dx'$$  \hspace{1cm} (3.53)

Here $j_n$ and $k_n$ are given by (3.35) and (3.36), respectively; $d$ denotes the $x$-interval occupied by the slab; and the one-dimensional free-space Green's function is now

$$g(x,x',\kappa) = (2jk_x)^{-1}\exp(-jk_x|x-x'|)$$  \hspace{1cm} (3.54)

with $k_x$ given in (3.42). After inserting (3.52) and (3.53) into (3.49)–(3.51), the orders of integration and differentiation can be interchanged. The operator $\partial_x$ acts on the Green's function $g$ only. This differentiation can be performed analytically. From (3.54) it follows that $\partial_x g(x,x',\kappa)$ is discontinuous at $x = x'$, and that $\partial^2_x g(x,x',\kappa)$ has a singularity $-\kappa(x-x')$. 
In most cases, the permeability of the waveguide is constant and equal to the permeability of its surroundings, so that, according to (3.36) and (3.40), \( g_n = 0 \). Then (3.49), together with (3.52) and (3.35), provides a homogeneous integral equation for \( c_y \) in the slab, while (3.50) and (3.51) together with (3.52) and (3.35) are integral representations for \( h_x \) and \( h_z \), respectively, in terms of \( c_y \) in the slab.

For TM-fields, on the other hand, the duals of (3.49)–(3.33) with \( g_n = 0 \) lead to a system of homogeneous integral equations for \( c_x \) and \( c_z \) in the slab. The latter system follows from the duals of (3.50) and (3.51), together with the dual of (3.33) and (3.35). The dual of (3.49), together with the duals of (3.53) and (3.35), then provides an integral representation for \( h_y \) in terms of \( c_x \) and \( c_z \) in the slab.

By discretising the expressions (3.49)–(3.33) for TE-modes, (i.e., surface-wave modes having a TE field), or their duals for TM-modes, we arrive at a system of linear algebraic equations that is amenable to numerical solution. The discretisation procedure leads to a homogeneous system of the form

\[
\Delta \mathbf{f} = \mathbf{0},
\]

(3.55)

in which \( \mathbf{f} \) is a column matrix that is related to the field values used in the discretisation scheme, and \( \Delta \) is a square matrix, the elements of which are determined by the discretised versions of (3.49)–(3.33). The propagation coefficients \( \kappa_n \) are then computed from \( \det(\Delta) = 0 \). Next, the field distribution of the corresponding surface-wave mode is obtained by substituting the value of \( \kappa_n \) into (3.55) and solving this system, subject to a convenient normalisation.

The discretisation procedure to be used here is the method of moments (Kantorowitsch and Krylow, 1956; Harrington, 1968). In this method, the field
quantities are expanded with respect to the expansion functions \( \{ \psi_j(x); j=1,\ldots,J \} \). Suppressing the subscript \( n \) referring to the mode number, we write

\[
\{a_{j,k}\} = \sum_{j=1}^{J} \{a_{j,k}\} \psi_j(x). \tag{3.56}
\]

Upon inserting (3.56) into (3.49)–(3.53) and (3.35)–(3.36), the left- and right-hand sides of the resulting expressions are multiplied by the weighting functions \( \{ \varphi_k(x); k=1,\ldots,K \} \), and integrated over the slab domain. Then by eliminating the coefficients \( j \) and \( k \), a system of \( 3J \) equations is obtained for the \( 3J \) unknown field coefficients \( e_{j,k} h_{j,k} \). In this system the left-hand sides contain the integrals of products of weighting and expansion functions \( \int_a^b \varphi_k(x) \psi_j(x) dx \), while the right-hand sides contain the integrals

\[
\int_a^b \varphi_k(x) \partial_x g(x,x',\kappa) \psi_j(x') dx' dx, \quad \int_a^b \varphi_k(x) \partial_{xx} g(x,x',\kappa) \psi_j(x') dx' dx, \quad \text{and} \quad \int_a^b \varphi_k(x) (\epsilon - \kappa) \psi_j(x) dx, \quad \int_a^b \varphi_k(x) (\mu - \rho_1) \psi_j(x) dx,
\]

whereby the latter two integrals are evaluated numerically. In case the weighting and expansion functions are differentiable, the integrals involving \( \partial_x g \) and \( \partial_{xx} g \) can be transformed by an integration by parts. We thus obtain

\[
\int_a^b \varphi_k(x) \partial_x g(x,x',\kappa) \psi_j(x') dx' dx = - \int_a^b \partial_x \varphi_k(x) g(x,x',\kappa) \psi_j(x') dx' dx
\]

\[
+ \varphi_k(x) \int_a^b g(x,x',\kappa) \psi_j(x') dx' \bigg|_{x=a}^{x=a}, \tag{3.57}
\]

in which \( x = -a \) and \( x = a \) are the boundary planes of the slab, and, since \( \partial_x g(x,x',\kappa) = -\partial_{xx} g(x,x',\kappa) \),
\[
\frac{\partial}{\partial x} \varphi_k(x') g(x,x',\kappa) \psi_l(x') \, dx' \, dx = - \frac{\partial}{\partial x} \varphi_k(x) g(x,x',\kappa) \partial_{x'} \psi_l(x') \, dx' \, dx \\
+ \left[ \varphi_k(x) g(x,x',\kappa) \psi_l(x') \right]_{x'=-a}^{x'=a} + \left[ \varphi_k(x) \partial_{x'} \psi_l(x') \right]_{x'=-a}^{x'=a} \\
- \left[ \partial_{x'} \varphi_k(x) g(x,x',\kappa) \psi_l(x') \right]_{x'=-a}^{x'=a} 
\]

(3.58)

For special choices of the expansion and weighting functions, the above integrals may be evaluated analytically.

The simplest choice for the expansion functions is

\[\psi_j(x) = \text{Rect}_j(x) = \begin{cases} 
1 & \text{when } x \in d_j, \\
0 & \text{when } x \notin d_j,
\end{cases} \]

(3.59)

while for the weighting functions we take

\[\psi_k(x) = \delta(x-x_k). \]

(3.60)

Here \(d_j, j = 1, 2, \ldots, J\), are the subintervals into which \([-a,a]\) is divided, and \(x_k\) is an interior point of the subinterval \(d_k\) (Fig. 3.7). In our case, the subintervals have equal lengths and \(x_k\) is taken as the centre point of \(d_k\). Note that with this choice of expansion and weighting functions, (3.57) and (3.58) are not applicable. With this choice, the method of solution for the integral equation is called the point–matching method or the method of collocation. The integrals \(\int_{d_j} g(x,x',\kappa) \, dx'\), \(\int_{d_j} \partial_{x'} g(x,x',\kappa) \, dx'\), and \(\int_{d_j} \partial_{x'}^2 g(x,x',\kappa) \, dx'\) occurring when using the point–matching method are calculated analytically.
The zeros of $\text{det}(\mathbf{A})$ are computed by using Muller's method (Muller, 1956; Frank, 1958) for the iterative determination of a complex zero. The number and location of the zeros is frequency dependent. We have computed the zeros of $\text{det}(\mathbf{A})$ that correspond to some specific surface-wave modes. It appears that there is a tendency for the diagonal elements of the matrix $\mathbf{A}$ in (3.55) for the case of TE-modes, to be more dominant than the diagonal elements of $\mathbf{A}$ for TM-modes, especially for higher values of the contrast $\epsilon(x)/\epsilon_1 - 1$ and for values of $\kappa_n$ relatively close to $k_1 = \omega(\epsilon_1\mu_1)^{1/2}$. In these cases, the TE-system of equations is better conditioned than the TM-system, and hence the results for TE-modes will be more accurate than those for TM-modes.

In the subsequent tables and figures numerical results are presented for various waveguide configurations with symmetric profiles of the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$ and the relative permeability $\mu/\mu_0$. The resulting symmetry of the fields has been used in the computations in order to reduce the integration interval in (3.52) and (3.53) to one half of the slab (De Ruster, 1980).

First we have obtained results for the propagation coefficients $\kappa_0$ of a step-index planar waveguide. In this case there exists an analytical expression for the eigenvalue equation to be satisfied by the propagation coefficients and the field distributions (Unger, 1977, pp. 93 – 100). To illustrate the accuracy of the present implementation
of the integral-equation method, we have listed in Table III the values of the normalised propagation coefficients $\kappa_n/k_0$, with $k_0=\epsilon_0\mu_0^{1/2}$, for a planar waveguide with $\epsilon_{r,2}=1.01$, $\epsilon_{r,1}=1$, $\mu_{r,2} = \mu_{r,1} = 1$, for some modes. They have been obtained by the integral-equation method with point-matching using 8 and 16 equally spaced matching points in one half of the slab ($a = d/2$).

For a configuration with lossless media, the propagation coefficients $\kappa_n$ are real. The lossless configuration can be considered as the limiting case of a corresponding lossy configuration for vanishingly small losses. The modes are numbered in ascending order of their cut-off frequencies; the cut-off frequency of a specific mode is the frequency below which the mode is non-existent.

In Fig. 3.8, the electric field distribution of the $\text{TE}_3^-$ mode at $k_0a=1.101\times10^2$ ($\kappa_{3}^E/k_0=1.0013914$) is shown. The solid curve is obtained from the analytical

Table III. Some values of $\kappa_n/k_0$ for the step-index, $\epsilon_r=1.01$ symmetrical slab waveguide, obtained by the integral-equation method and point-matching with 8 and 16 matching points; values from the analytical expression for comparison.

<table>
<thead>
<tr>
<th>$k_0a$</th>
<th>mode</th>
<th>$\kappa_n/k_0$</th>
<th>mode</th>
<th>$\kappa_n/k_0$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>int. eq. J=0</td>
<td>int. eq. J=16</td>
<td>analytical</td>
<td>int. eq. J=0</td>
</tr>
<tr>
<td>2.142</td>
<td>$\text{TE}_0$</td>
<td>1.0004370</td>
<td>1.0004369</td>
<td>1.0004369</td>
</tr>
<tr>
<td>1.101\times10^2</td>
<td>$\text{TE}_0$</td>
<td>1.0004397</td>
<td>1.0004394</td>
<td>1.0004394</td>
</tr>
<tr>
<td>1.825\times10^1</td>
<td>$\text{TE}_1$</td>
<td>1.0003135</td>
<td>1.0003146</td>
<td>1.0003146</td>
</tr>
<tr>
<td>1.101\times10^2</td>
<td>$\text{TE}_2$</td>
<td>1.0003827</td>
<td>1.0003832</td>
<td>1.0003832</td>
</tr>
<tr>
<td>8.108\times10^1</td>
<td>$\text{TE}_3$</td>
<td>1.0003012</td>
<td>1.0003017</td>
<td>1.0003017</td>
</tr>
<tr>
<td>1.101\times10^2</td>
<td>$\text{TE}_4$</td>
<td>1.0001815</td>
<td>1.0001824</td>
<td>1.0001824</td>
</tr>
</tbody>
</table>
eigenvalue equation and the pertaining analytical expressions for the field distribution; the field values at the matching points obtained by the integral-equation method with point-matching are indicated by * (8 matching points) and o (16 matching points).

In Table IV, results are presented for a much larger contrast between the waveguide and its surroundings, viz. \( \epsilon_2 = 2.25, \epsilon_1 = 1, \mu_2 = \mu_1 = 1 \). In general, the results obtained from the integral-equation method using point-matching are in good agreement with the results obtained from the analytical eigenvalue equation. Furthermore, the results pertaining to TE-modes turn out to be more accurate than those pertaining to TM-modes. This better accuracy becomes more pronounced for larger values of the contrast between the waveguide and its surroundings. This is probably due to the TE-system of equations being better conditioned than the TM-system, as observed previously.

![Fig. 3.8. Electric field distribution \( e_y(x) \) of the TE\(_5\) modes at \( k_0a = 1.101 \times 10^2 \) (\( a_5 = 1.0019914 \)) obtained from the analytical eigenvalue equation (solid curve) and with the integral-equation method (*: 8 matching points, o: 16 matching points). The electric field distribution is normalised such that its maximum value is unity.](image-url)
The same computational scheme has been applied to a planar waveguide with a quadratic permittivity profile, for which \( \epsilon_{r,2}(x) = \epsilon_{r,\text{max}}(1-2\Delta x^2/a^2) \) when \(-a < x < a\). The value of \( \epsilon_{r,\text{max}} \) is 1.01, while \( \epsilon_{r,1} = 1, \mu_{r,2} = \mu_{r,1} = 1 \). Two values of \( \Delta \) have been taken, viz. \( \Delta = 2.475 \times 10^{-3} \) and \( \Delta = 4.950 \times 10^{-3} \) (Fig. 3.9). Some

Table IV. Some values of \( \kappa_i^2/k_0^2 \) for the step-index, \( \epsilon_r = 2.25 \) symmetrical slab waveguide, obtained by the integral-equation method and point-matching with 8 and 16 matching points; values from the analytical expression for comparison. The question mark indicates that the pertaining zero was not found numerically.

<table>
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<th>( \kappa_i^2/k_0^2 )</th>
<th>mode</th>
<th>( \kappa_i^2/k_0^2 )</th>
</tr>
</thead>
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Fig. 3.9. Permittivity profiles of:

- a step-index waveguide, \( \epsilon_{r,2}=1.01 \);
- a quadratic-index waveguide, \( \epsilon_{r,\text{max}}=1.01, \Delta=2.475 \times 10^{-3} \);
- a quadratic-index waveguide, \( \epsilon_{r,\text{max}}=1.01, \Delta=4.950 \times 10^{-3} \).
values of the normalised propagation coefficients, obtained with 8 and 16 matching points in one half of the slab, are listed in Tables V and VI. For the various planar waveguides we consider the differences of the values of the propagation coefficients obtained with 8 and 16 matching points. Then it appears that for the step-index waveguide and for the quadratic-index waveguide these differences are of the same order of magnitude. Hence, the differences between the computed and exact values of the propagation coefficients of the quadratic-index waveguide are likely to be also of the same order of magnitude as those for the step-index planar waveguide. In Fig. 3.10 the propagation coefficients of the TE_{0} and TE_{1}-modes for the three permittivity profiles of Fig. 3.9 are plotted as functions of k_{1}a. In Fig. 3.11, the electric field component of the TE_{1}-mode in the waveguide is shown for these profiles at two different values of k_{1}a.

Table V. Some values of \( \kappa_{n}/k_{0} \) for a graded-index symmetrical slab waveguide having a quadratic permittivity profile with \( \epsilon_{r,\text{max}}=1.01, \Delta=2.475\times10^{-3} \), obtained by the integral-equation method using point-matching with \( J=8 \) and \( J=16 \) matching points.

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Table VI. Some values of $\kappa_n/k_0$ for a graded-index symmetrical slab waveguide having a quadratic permittivity profile with $\epsilon_{\text{r, max}} = 1.01$, $\Delta = 4.950 \times 10^{-3}$, obtained by the integral-equation method using point matching with $J=8$ and $J=16$ matching points.

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<td></td>
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Fig. 3.10. Propagation coefficients $\kappa_n/k_0$ of the TE$_0$- and TE$_1$-modes of a slab waveguide with step-index profile, $\epsilon_{\text{r, max}} = 1.01$ (---); quadratic permittivity profile, $\epsilon_{\text{r, max}} = 1.01$, $\Delta = 2.475 \times 10^{-3}$ (----); quadratic permittivity profile, $\epsilon_{\text{r, max}} = 1.01$, $\Delta = 4.950 \times 10^{-3}$ (-----).
Fig. 3.11. Lorentz–normalised electric field component \( e_\perp(x) \) of the TE\(_3\)–mode in
the slab waveguide (a) at \( k_0a = 2.620\times10^9 \) (near cut–off) and (b) at \( k_0a = 1.101\times10^9 \) (far from cut–off),
for the step–index profile with \( \varepsilon_{r,2} = 1.01 \) (———);
for a quadratic permittivity profile, \( \varepsilon_{r,\text{max}} = 1.01, \Delta = 2.475\times10^{-3} \) (………);
for a quadratic permittivity profile, \( \varepsilon_{r,\text{max}} = 1.01, \Delta = 4.930\times10^{-3} \) (———).

The integral–equation method with point–matching has also been applied to a
strongly lossy step–index planar waveguide, for which \( \varepsilon_{r,2} = 2.25–2.235, \varepsilon_{r,1} = 1, \)
\( \mu_{\gamma,2} = \mu_{\gamma,1} = 1. \) Values for the propagation coefficients of the TE\(_0\), TE\(_2\), and
TE\(_4\)–modes are listed in Table VII. In Fig. 3.12 the electric field distribution of the
TE\(_4\)–mode at \( k_0a = 9.848 \) as obtained with the integral–equation method by
point–matching with 8 and 16 matching points, is compared with the electric field
distribution that has been obtained from the solution of the analytical eigenvalue
equation. For lossy structures, too, a good agreement is observed between the values
obtained from the analytical expressions and those computed by the method of
moments, both for the propagation coefficient and for the field distribution.

We now briefly discuss the behaviour of the propagation coefficient and of the field
distribution of a surface–wave mode as a function of frequency in general. At a very
low frequency, only one TE–surface–wave mode (the TE\(_0\)–mode) and one TM–
surface–wave mode (the TM\(_0\)–mode) are present in a planar waveguide; the
frequency is below the cut–off frequency of the other surface–wave modes. At the
Table VII. Values of the propagation coefficients $\kappa_n/k_0$ of the TE$_{0+}$, TE$_{1-}$ and TE$_{2-}$-modes in a lossy step-index planar waveguide with $K_{r.t} = 2.25 - 2.25j$, $\epsilon_r = 1$, $\mu_r = 1$, obtained by the integral-equation method using point-matching with $J = 16$ matching points. Some values resulting from the analytical eigenvalue equation are given for comparison. Dashes indicate that the corresponding root of the eigenvalue equation is absent, i.e., the corresponding mode is below cut-off.

<table>
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Fig. 3.12. Electric field distribution $\varepsilon_y(x)$ of the $\text{TE}_4$-mode at $k_0\lambda=9.848$ in a lossy step-index planar waveguide with $\varepsilon_{1,2}=2.25-2.25j$, $\varepsilon_{1,1}=1$, $\mu_{1,2}=\mu_{1,1}=1$, real part (---) and imaginary part (---) of the exact field distribution determined from the analytical eigenvalue equation $(\kappa_x^2/k_0^2=1.49295-7.14011\times10^{-1}j)$; 
- indicates the values from the integral-equation method with 8 matching points $(\kappa_x^2/k_0=1.47577-7.35179\times10^{-1}j)$;
- indicates the values from the integral-equation method with 16 matching points $(\kappa_x^2/k_0=1.48873-7.37499\times10^{-1}j)$. Normalisation is such that $\varepsilon_y(0)=1$.

The cut-off frequency of a surface-wave mode, the corresponding surface-wave pole is located on the branch cut $\text{Im}(k_{T,1}) = \text{Im}(k_1^2-k_x^2)^{1/2} = 0$, with $k_1 = \alpha(\varepsilon_x\mu_y)^{1/2}$. With increasing frequency the pole subsequently enters the Riemann sheet on which $\text{Im}(k_{T,1}) < 0$. Consequently, the modal field has an exponential decay at infinity in the transverse plane. At very high frequencies the modal fields have the tendency to concentrate in those parts of the waveguide where the refractive index $(\varepsilon_x\mu_y)^{1/2}$ is maximal. Then the value of $\alpha_n$ approaches $k_0(\varepsilon_x\mu_y)^{1/2}$. The root loci of the $\text{TE}_0^-$, $\text{TE}_1^-$ and $\text{TE}_2^-$ surface-wave poles for a lossy step-index planar waveguide, as shown
in Fig. 3.13, exhibit the outlined behaviour. It can be proved that these root loci lie in a restricted part of the complex \( k_z \)-plane only (De Ruiter, 1981).

The average computing times for finding the propagation coefficient and the field distribution of a surface-wave mode with the integral-equation method combined with the point-matching technique were 10 s for 8 matching points and 35 s for 16 matching points in one half of the slab. The computer programme was written in PL-I and run on an IBM 370/158 computer.

In solving the integral equations by the method of moments, we thus far used the simplest types of weighting and expansion functions, viz. delta functions and rectangle functions, respectively. In order to investigate the effect of a different choice for the

![Diagram](image_url)

Fig. 3.18. Root loci of the surface-wave poles corresponding to the \( \text{TE}_0^- \), \( \text{TE}_1^- \), and \( \text{TE}_2^- \) modes in a lossy step-index planar waveguide with \( \varepsilon_{r,2} = 2.25 - 2.25 \text{i} \), embedded in vacuum. The arrows along the curves indicate the direction of change at increasing frequency.
weighting and expansion functions, we now take \( \psi_j(x) = \varphi_j(x) = T_j(x) \), in which the triangle function \( T_j(x) \) is defined by (Fig. 3.14)

\[
T_j(x) = \begin{cases} 
(x-x_{j-1})/\Delta & \text{when } x \in d_{j-1} \\
(x_{j+1}-x)/\Delta & \text{when } x \in d_j, \quad j=2,\ldots,J-1, \\
0 & \text{when } x \notin d_{j-1} \cup d_j.
\end{cases}
\]

\[
T_1(x) = \begin{cases} 
(x-x_1)/\Delta & \text{when } x \in d_1 \\
0 & \text{when } x \notin d_1.
\end{cases}
\]

\[
T_J(x) = \begin{cases} 
(x-x_{J-1})/\Delta & \text{when } x \in d_{J-1} \\
0 & \text{when } x \notin d_{J-1}.
\end{cases}
\]

Here, the interval \([-a,a]\) has been divided into an even number of subintervals \( d_j = [x_j, x_{j+1}] \), \( j=1,2,\ldots,J-1 \), of equal lengths \( \Delta \). With this choice for the expansion and

![Fig. 3.14. The triangle functions \( T_j(x) \), used as expansion and weighting functions.](image)
weighting functions, the integrations by parts in (3.57) and (3.58) can be carried out, and the resulting integrals can be evaluated analytically.

In the implementation of the method we have not used a possible symmetry of the permittivity profile to reduce the integrations to one half of the cross-section, as has been done in the computations carried out with the point-matching technique.

We have computed the propagation coefficients and the field distributions of some TE-surface-wave modes. It is understood that \( \mu = \mu_2 \), hence \( q_2 = 0 \) in (3.49)-(3.51). Then (3.49), together with (3.52) and (3.55), provides a homogeneous integral equation for the field component \( \epsilon_y \) in the slab. Next, by discretisation of (3.49) the field coefficients \( \epsilon_{j,y} \) are found by solving a system of \( J \) homogeneous algebraic equations. Subsequently, the values of \( h_{j,x} \) and \( h_{j,a} \) are determined by means of the relations that are obtained from the discretised versions of (3.55) and (3.56)-(3.58).

This procedure has been used in the computations with triangle expansion and weighting functions. Some results obtained with this procedure are listed in Tables VIII - XII. For comparison, the results obtained with the point-matching technique (PM) are listed as well. For the point-matching results the integer \( J \) indicates the number of expansion functions employed if the integration had been carried out over the entire cross-section without using the symmetry of the permittivity profile. Thus \( J = 2M - 1 \), where \( M \) is the number of expansion functions and matching points in one half of the slab.

In Table VIII, some results are listed for the propagation coefficients of the \( \text{TE}_{0} \), \( \text{TE}_{1} \), and \( \text{TE}_{5} \)-modes in a step-index planar waveguide with \( \epsilon_{t,2} = 1.01, \epsilon_{t,1} = 1, \mu_{t,2} = \mu_{t,1} = 1 \). In Table IX, some results are listed for the propagation coefficients of the \( \text{TE}_{0} \), \( \text{TE}_{1} \), and \( \text{TE}_{5} \)-modes in a step-index planar waveguide with a higher contrast, viz. \( \epsilon_{t,2} = 2.25, \epsilon_{t,1} = 1, \mu_{t,2} = \mu_{t,1} = 1 \). In Tables X and XI, some results
are listed for the two quadratic-index planar waveguides with permittivity profiles as shown in Fig. 3.9. Finally, in Table XII a comparison is made of the results obtained with triangle expansion and weighting functions and with the point-matching technique, for the propagation coefficient of the TE<sub>14</sub>-mode in a lossy step-index planar waveguide in which \( \varepsilon_{r,2} = 2.25 - 2.25j \). In Fig. 3.15, the analytically determined exact field distribution \( e_y \) of the TE<sub>14</sub>-mode is plotted, together with the field values \( e_y \) at the positions \( x_j \) obtained from the integral-equation method with triangle expansion and weighting functions (cf. Fig. 3.12).

From the results in Tables VIII - XII it appears that in general the moment method using triangle expansion and weighting functions is superior to the point-matching technique, in the sense that fewer expansion functions are needed to achieve a certain

Table VIII. Values of the propagation coefficients \( \kappa / k_0 \) in a step-index planar waveguide with \( \varepsilon_{r,2} = 1.01 \), \( \varepsilon_{r,1} = 1 \), \( \mu_{r,2} = \mu_{r,1} = 1 \), obtained with triangle expansion and weighting functions, as compared with results from the point-matching technique (PM) and with results from the analytical expression.

The number of expansion functions used is J. Question marks indicate that the pertaining zero was not found numerically.

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</tr>
<tr>
<td>J=61</td>
<td>TE&lt;sub&gt;2&lt;/sub&gt;</td>
<td>1.0003008</td>
</tr>
<tr>
<td>J=9</td>
<td>TE&lt;sub&gt;3&lt;/sub&gt;</td>
<td>1.0002982</td>
</tr>
<tr>
<td>J=17</td>
<td>TE&lt;sub&gt;3&lt;/sub&gt;</td>
<td>1.0002982</td>
</tr>
<tr>
<td>J=33</td>
<td>TE&lt;sub&gt;3&lt;/sub&gt;</td>
<td>1.0002982</td>
</tr>
<tr>
<td>J=61</td>
<td>TE&lt;sub&gt;3&lt;/sub&gt;</td>
<td>1.0002982</td>
</tr>
<tr>
<td>J=129</td>
<td>TE&lt;sub&gt;3&lt;/sub&gt;</td>
<td>1.0002982</td>
</tr>
</tbody>
</table>

-53-
Table IX. Values of the propagation coefficients \( \kappa_n/k_0 \) in a stop–index planar waveguide with \( \epsilon_{r,2} = 2.26, \ \epsilon_{r,1} = 1, \mu_{r,2} = \mu_{r,1} = 1 \), obtained with triangle expansion and weighting functions, as compared with results from the point–matching technique (PM) and with results from the analytical expression. The number of expansion functions used is J. Question marks indicate that the pertaining zero was not found numerically.

<table>
<thead>
<tr>
<th>( k_0 )</th>
<th>mode ( J )</th>
<th>J=1</th>
<th>J=5</th>
<th>J=9</th>
<th>PM J=15</th>
<th>PM J=21</th>
<th>analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.811×10^{-2}</td>
<td>( \text{TE}_{0} )</td>
<td>1.03310</td>
<td>1.03374</td>
<td>1.03325</td>
<td>1.03328</td>
<td>1.03323</td>
<td>1.03323</td>
</tr>
<tr>
<td>9.848</td>
<td>( \text{TE}_{2} )</td>
<td>?</td>
<td>1.42078</td>
<td>1.42075</td>
<td>1.42075</td>
<td>1.42077</td>
<td>1.42035</td>
</tr>
<tr>
<td>1.886</td>
<td>( \text{TE}_{1} )</td>
<td>1.03193</td>
<td>1.03186</td>
<td>1.03182</td>
<td>1.03185</td>
<td>1.03181</td>
<td>1.03184</td>
</tr>
<tr>
<td>9.848</td>
<td>( \text{TE}_{1} )</td>
<td>1.46737</td>
<td>1.46736</td>
<td>1.46734</td>
<td>1.46733</td>
<td>1.46709</td>
<td>1.46727</td>
</tr>
<tr>
<td>8.061</td>
<td>( \text{TE}_{5} )</td>
<td>1.56652</td>
<td>1.08336</td>
<td>1.08477</td>
<td>1.08492</td>
<td>1.08492</td>
<td>1.08492</td>
</tr>
<tr>
<td>9.848</td>
<td>( \text{TE}_{5} )</td>
<td>1.16658</td>
<td>1.22221</td>
<td>1.28452</td>
<td>1.22043</td>
<td>1.21975</td>
<td>1.22412</td>
</tr>
</tbody>
</table>

Table X. Values of the propagation coefficients \( \kappa_n/k_0 \) of the \( \text{TE}_{1} \)-mode in a quadratic–index planar waveguide with \( \epsilon_{r,1,\text{max}} = 1.01, \Delta = 2.475×10^{-3}, \epsilon_{r,1} = 1, \mu_{r,2} = \mu_{r,1} = 1 \), obtained with triangle expansion and weighting functions, as compared with results from the point–matching technique (PM). The number of expansion functions used is J.

<table>
<thead>
<tr>
<th>( k_0 )</th>
<th>( \kappa_n/k_0 )</th>
<th>TE ( J )</th>
<th>J=17</th>
<th>J=33</th>
<th>PM J=15</th>
<th>PM J=21</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.524×10^{-1}</td>
<td>1.0005936</td>
<td>1.0007605</td>
<td>1.0007849</td>
<td>1.0007974</td>
<td>1.0007850</td>
<td></td>
</tr>
<tr>
<td>1.249×10^{-2}</td>
<td>1.00042106</td>
<td>1.00042106</td>
<td>1.00042106</td>
<td>1.00042106</td>
<td>1.00042106</td>
<td></td>
</tr>
</tbody>
</table>
Table XI. Values of the propagation coefficients $\kappa_n / k_0$ of the $TE_1$-mode in a quadratic-index planar waveguide with $\epsilon_{r,\text{max}} = 1.01$, $\Delta = 4.950 \times 10^{-3}$, $\epsilon_{r,1} = 1$, $\mu_{r,2} = \mu_{r,1} = 1$, obtained with triangle expansion and weighting functions, as compared with results from the point-matching technique (PM). The number of expansion functions used is $J$.

<table>
<thead>
<tr>
<th>$k_0 a$</th>
<th>$\kappa_n / k_0$</th>
<th>$TE_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$J=9$</td>
<td>$J=17$</td>
</tr>
<tr>
<td>$2.820 \times 10^1$</td>
<td>1.0001274</td>
<td>1.0001418</td>
</tr>
<tr>
<td>$1.101 \times 10^2$</td>
<td>1.0038154</td>
<td>1.0038302</td>
</tr>
</tbody>
</table>

Table XII. Values of the propagation coefficient $\kappa_n / k_0$ of the $TE_4$-mode at $k_0 a = 9.848$ in a lossy step-index planar waveguide with $\epsilon_{r,2} = 2.25-2.25j$, $\epsilon_{r,1} = 1$, $\mu_{r,2} = \mu_{r,1} = 1$, obtained with triangle expansion and weighting functions, as compared with results from the point-matching technique (PM) and with results from the analytical expression. The number of expansion functions used is $J$.

<table>
<thead>
<tr>
<th>$k_0 a$</th>
<th>$\kappa_n / k_0$</th>
<th>$TE_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$J=7$</td>
<td>$J=13$</td>
</tr>
<tr>
<td>$9.848$</td>
<td>1.45171</td>
<td>1.49044</td>
</tr>
<tr>
<td>$-0.74263j$</td>
<td>$-0.73936j$</td>
<td>$-0.74096j$</td>
</tr>
</tbody>
</table>
Fig. 3.15. Electric field distribution $e_y(x)$ of the $TE_{21}$-mode at $k_0a=9.848$ in a lossy step-index planar waveguide with \( \epsilon_{t,2} = 2.25 - 2.25i \), \( \epsilon_{t,1} = 1 \), \( \mu_{t,2} = \mu_{t,1} = 1 \); real part (---) and imaginary part (— — ) of the exact field distribution determined from the analytical eigenvalue equation; the values from the integral-equation method with triangle expansion and weighting functions are indicated by o, ⋄, ⊗, corresponding to the use of 7, 13, 25 expansion functions, respectively. Normalisation is such that $e_y(0) = 1$.

accuracy of the value of the propagation coefficient. Comparison of Figs. 3.12 and 3.15 shows that the same holds for the corresponding field distribution.

The CPU-times for determining the propagation coefficient and the field distribution of a surface-wave mode by the method of moments with $J$ triangle expansion and weighting functions ranged from 0.7 s for $J = 3$ to 36 s for $J = 29$. The computer programme was written in ALGOL 60 and was run on a Burroughs B7700 computer. It is difficult to compare these computing times with those of the point-matching technique, since the computer programmes were written in different programming languages and were run on different computers.
3.4.2. The transfer matrix formalism

In this subsection we apply the transfer matrix formalism discussed in Subsection 3.3.2 to the computation of surface-wave modes in a multi-step-index planar waveguide. The configuration is shown in Fig. 3.16. The waveguide consists of \( N \) homogeneous layers \( \{ d_p ; p=2,...,N\} \) in between \( N-1 \) planes \( \{ x=x_p ; p=1,...,N-1 \} \), embedded in two homogeneous media present in the semi-infinite domains \( d_1 : -\infty < x < x_1 \), and \( d_N : x_{N-1} < x < \infty \). The (constant) permittivity and permeability in the layer \( d_p \) are \( \varepsilon_p \) and \( \mu_p \), respectively. In the case considered here, the media occupying the semi-infinite domains \( d_1 \) and \( d_N \) are identical, with permittivity \( \varepsilon_1 = \varepsilon_N \) and permeability \( \mu_1 = \mu_N \).

In this configuration we investigate the fields that are \( y \)-independent. As has been pointed out at the beginning of Section 3.4, the field equations then separate into two independent systems of equations, viz. one system for TE-fields with components \( E_z \) and \( B_y \) tangential to the layer boundaries, and one system for TM-fields with components \( E_y \) and \( B_z \) tangential to the layer boundaries. Consequently, the field matrix \( f \) and the transfer matrix \( T_p \) can each be decomposed into two smaller

\[
\begin{array}{cccccccccc}
\varepsilon_1 & & & & & & & & & \\
& \mu_1 & & & & & & & & \\
\vdots & & \varepsilon_p & \mu_p & \varepsilon_{p+1} & \mu_{p+1} & \varepsilon_{p+2} & \mu_{p+2} & \varepsilon_p & \mu_p \varepsilon_{p+1} & \mu_{p+1} \varepsilon_{p+2} & \mu_{p+2} & \varepsilon_1 & \mu_1
\end{array}
\]

Fig. 3.16. Configuration of the multi-step-index planar waveguide.
matrices: $E_E$ and $M_M$ for TE-fields, and $M_M$ and $E_E$ for TM-fields. We shall restrict ourselves to the TE-case; the TM-case follows by making the appropriate changes.

For TE-fields, the field components $e_y$ and $h_z$ are continuous upon crossing the interfaces $x = x_p$. Hence we have the field matrix

$$\begin{bmatrix} E_E(x) \\ M_M(x) \end{bmatrix} = \begin{bmatrix} e_y(x) \\ h_z(x) \end{bmatrix}$$

(3.62)

where the subscript $p$ refers to the layer in which $x$ is located. From (3.30) and (3.31) we derive the system of equations that has to be satisfied by $e_y$ and $h_z$ inside the homogeneous, source-free layer $d_p$ as

$$k_{T,p}^2 e_{y,p} - j\omega \mu_h h_{z,p} = 0,$$

(3.63)

$$\delta_x e_{y,p} + j\omega \varepsilon e_{y,p} = 0,$$

(3.64)

in which

$$k_{T,p} = \left( k_p^2 - \sigma^2 \right)^{1/2}, \quad k_p = \omega^2 \varepsilon_p \mu_p, \quad \text{Im}(k_{T,p}) \leq 0,$$

(3.65)

and in which the subscript $n$ referring to the $n$-th surface-wave mode has been suppressed ($x$ now denotes the propagation coefficient of some surface-wave mode).

The transfer matrix $T_E(x,x')$ is easily constructed as

$$T_E(x,x') = \begin{bmatrix} \cos(k_{T,p}(x-x')) & -(j/Y_{x,p})\sin(k_{T,p}(x-x')) \\ -(j/Y_{x,p})\sin(k_{T,p}(x-x')) & \cos(k_{T,p}(x-x')) \end{bmatrix},$$

(3.66)
with

\[ Y_{x,p}^E = k_{x,p}^E / (\omega_{x,p}), \tag{3.67} \]

where (3.44) has been taken into account.

The solutions of (3.63) and (3.64) in the lower and upper half-spaces can be written as

\[
\begin{bmatrix}
  e_{y,1}(x) \\
  h_{z,1}(x)
\end{bmatrix} = \begin{bmatrix}
  1 \\
  -Y_{x,1}^E
\end{bmatrix} e_{y,1}(x_1) \exp(ik_{x,1}(x-x_1)) \text{ when } x \in d_1, \tag{3.68}
\]

and

\[
\begin{bmatrix}
  e_{y,N}(x) \\
  h_{z,N}(x)
\end{bmatrix} = \begin{bmatrix}
  1 \\
  Y_{x,N}^E
\end{bmatrix} e_{y,N}(x_{N-1}) \exp(-ik_{x,N}(x-x_{N-1})) \text{ when } x \in d_N, \tag{3.69}
\]

respectively. By means of (3.68) and the transfer matrices of the intermediate layers, the fields at the (conveniently chosen) position \( x_0 \in d_p \) are expressed in terms of \( e_{y,1}(x_1) \) as

\[
\begin{bmatrix}
  e_{y,p}(x_0) \\
  h_{z,p}(x_0)
\end{bmatrix} = T_{p}^{E}(x_0 \cdots x_{p-1}) \cdot T_{p-1}^{E}(x_{p-1} \cdots x_{p-2}) \cdots T_{2}^{E}(x_2 \cdots x_1) \cdot \begin{bmatrix}
  1 \\
  Y_{x,1}^E
\end{bmatrix} e_{y,1}(x_1). \tag{3.70}
\]

Similarly, by means of (3.69) and the relevant transfer matrices, the fields at \( x_0 \) are expressed in terms of \( e_{y,N}(x_{N-1}) \) as
Equations (3.70) and (3.71) are exploited to arrive at a computational scheme for the determination of the propagation coefficients. Although somewhat different, this scheme is equivalent to the transfer–matrix formalism as described in Subsection 3.3.2.

From (3.70) a relation between $e_{y,p}(x_0)$ and $h_{x,p}(x_0)$ is obtained that is written as

$$h_{x,p}(x_0) = \frac{e_{y,p}(x_0)}{y_{y,p}(x_0)}.$$  \hspace{1cm} (3.72)

while the relation between $e_{y,p}(x_0)$ and $h_{x,p}(x_0)$ obtained from (3.71) is written as

$$h_{x,p}(x_0) = y_{x,p}^{\text{E}}(x_0) e_{y,p}(x_0).$$  \hspace{1cm} (3.73)

The quantities $y_{x,p}^{\text{E}}(x_0)$ and $y_{y,p}^{\text{E}}(x_0)$ are the TE-field input admittances at either side of the level $x_0$ towards the lower and upper boundary planes of the planar waveguide, respectively. The requirement that (3.72) and (3.73) must be identical then leads to the condition

$$y_{x,p}^{\text{E}}(x_0) + y_{y,p}^{\text{E}}(x_0) = 0.$$

(transverse resonance condition, Felsen and Marcuvitz, 1973, pp. 215 – 217). The condition (3.74) is to be considered as an equation in $\kappa$; its solutions are the propagation coefficients of the TE–surface–wave modes of the planar waveguide. The normalised field distribution of a particular surface–wave mode is determined by
means of the transfer-matrix formalism applied in the upward and downward directions, using the field components $\epsilon_{x,p}(x_0)$ and $h_{x,p}(x_0)$ as starting values. These field components are connected by (3.72) or (3.73), and the field distribution is uniquely determined by imposing an appropriate normalisation condition.

From the expressions (3.70) and (3.71) together with (3.66) and (3.67), it is easily seen that the admittances $Y^{E}_{x,p}(x_0)$ and $Y^{E}_{x,p}(x_0)$, considered as functions of $\kappa$, will have branch cuts associated with the outer media only (compare the discussion in Appendix A). Indeed, the transfer matrices contain only single-valued functions of $\kappa$, viz. $\cos(k_{T,p}(x-x'))$, $Y^{E}_{x,p}(x-x')$ and $\sin(k_{T,p}(x-x'))/Y^{E}_{x,p}$. Hence, branch cuts are introduced due to the occurrence of $Y^{E}_{x,1}$ and $Y^{E}_{x,N}$, which contain an odd power of $k_{T,1}$ and $k_{T,N}$, respectively. Since we have chosen $\epsilon_1 = \epsilon_N$ and $\mu_1 = \mu_N$, only a single pair of branch cuts in the complex $\kappa$-plane results.

Some graphical results obtained by the transfer-matrix formalism and due to Huberts (1980), are shown in Figs. 3.17, 3.18 and 3.20, 3.21. In Fig. 3.17 the propagation coefficients of the $TE_0^+$, $TE_1^-$ and $TE_4^-$-modes in a step-index planar waveguide

![Graph](image)

Fig. 3.17. Propagation coefficients $\kappa_n/k_0$ of the $TE_0^+$, $TE_1^-$ and $TE_4^-$-modes as functions of $k_{0,a}$ for a step-index planar waveguide with permittivity $\epsilon_{r,2} = 2.28$, in a surrounding medium with permittivity $\epsilon_{r,1} = 2.25; \mu_{r,2} = \mu_{r,1} = 1$. 
with $\varepsilon_{r,2} = 2.28$, $\varepsilon_{r,1} = 2.25$, $\mu_{r,2} = \mu_{r,1} = 1$, and thickness $d = 2a$ are plotted as functions of $k_0a$. The electric field distribution of the $TE_4$-mode at $k_0a = 45.7$ is plotted in Fig. 3.18.

In Figs. 3.20 and 3.21 we present some results for the multi-step-index waveguide shown in Fig. 3.19 (core/cladding type). The surrounding medium is vacuum ($\varepsilon_{r,1} = \varepsilon_{r,5} = 1$, $\mu_{r,1} = \mu_{r,5} = 1$); the "core" with thickness $2a$ has a relative permittivity $\varepsilon_{r,3} = 2.28$ and a relative permeability $\mu_{r,3} = 1$, while the "cladding" consists of the layers $-3a < x < -a$ and $a < x < 3a$, where $\varepsilon_{r,2} - \varepsilon_{r,4} = 2.25$ and $\mu_{r,2} = \mu_{r,4} = 1$.

![Electric field distribution](image.png)

Fig. 3.18. Electric field distribution $E_y(x)$ of the $TE_4$-mode at $k_0a = 45.7$ in a step-index planar waveguide with permittivity $\varepsilon_{r,2} = 2.28$, in a surrounding medium with permittivity $\varepsilon_{r,1} = 2.25$; $\mu_{r,2} = \mu_{r,1} = 1$. The maximum value of $E_y$ is normalised to unity.

![Symmetrical multi-step-index planar waveguide](image.png)

Fig. 3.19. Symmetrical multi-step-index planar waveguide, for which $\varepsilon_{r,1} = \varepsilon_{r,5} = 1$, $\varepsilon_{r,2} = \varepsilon_{r,4} = 2.25$, $\varepsilon_{r,3} = 2.28$, $\mu_{r,1} = \mu_{r,2} = \mu_{r,3} = \mu_{r,4} = \mu_{r,5} = 1$. 
Fig. 3.20. Propagation coefficients \( \kappa_0/k_0 \) of the TE\(_0\)-, TE\(_4\)- and TE\(_9\)-modes as functions of \( k_0a \) for the multi-step-index planar waveguide of Fig. 3.19.

The propagation coefficients of the TE\(_0\)-, TE\(_4\)- and TE\(_9\)-modes are shown in Fig. 3.20. The electric field distribution of the TE\(_9\)-mode at \( k_0a = 62, 72, 82 \) is plotted in Fig. 3.21.

In core/cladding waveguides, two types of surface-wave modes are distinguished, viz. core modes and cladding modes. For lossless waveguides, the distinction between these two types is usually based on the behaviour of the field in the cross-section: for core modes, the field in the cladding shows an exponential decay with increasing distance from the core (Fig. 3.21c), i.e., \( k_{T,clad} \) is imaginary; for cladding modes, the field in the cladding shows an oscillatory behaviour (Fig. 3.21a,b), i.e., \( k_{T,clad} \) is real.

For lossy waveguides however, a distinction of this kind cannot be made, since \( k_{T,clad} \) is then, in general, complex-valued. For that reason, a better criterion for the distinction between core and cladding modes can be based on the ratio of the time-averaged power flows in core and cladding. This power flow is the axial component of the time-averaged Poynting vector \( S \), defined by

\[
S = \frac{1}{2} \text{Re}(E \times H^*).
\] (3.75)
Fig. 3.21. Electric field distribution $e_y(x)$ of the TE$_2$-mode in the multi-step-index planar waveguide of Fig. 3.19 (a) at $k_0a = 62$ (cladding mode), (b) at $k_0a = 72$ (cladding mode), and (c) at $k_0a = 82$ (core mode). Note the difference in the ratio of field intensities in core and cladding at $k_0a = 62$ and $k_0a = 72$. The maximum value of $e_y$ is normalised to unity.

Thus the ratio of the time-averaged power flows is given by

$$W = \left(\frac{1}{d_{core}} \int S_{d} dx\right) / \left(\frac{1}{d_{clad}} \int S_{d} dx\right).$$

(3.76)

Then a surface-wave mode is called a core mode (cladding mode) if $W \geq 1$ ($W < 1$).

From Fig. 3.21 it is observed that for a particular mode, the ratio $W$ may vary considerably with $k_0a$. Note that in this terminology, a surface-wave mode which
exhibits an exponential decay in the cladding, may nevertheless be a cladding mode, e.g. when $k_{T,\text{clad}}$ is imaginary but small in magnitude.

The programme for these computations (which can also deal with lossy structures) was written in ALGOL 60 and was run on a Burroughs B7700 computer. The surface-wave propagation coefficients were computed from (3.74) with the aid of a Cauchy integral method (Singaravelu et al., 1976). Computing times were 12 s CPU-time for finding the first five TE-surface-wave propagation coefficients of the step-index waveguide, and 40 s CPU-time for finding the propagation coefficients of the first ten TE-surface-wave modes of the core/cladding waveguide.

The results determined with the transfer-matrix formalism are more accurate, both in the propagation coefficients and in the field distributions, than those obtained by the integral-equation method combined with the method of moments. The reason for this is, that in the former method the computation of the surface-wave propagation coefficients is based on the exact equation (3.74), while the corresponding field distributions are determined by a procedure that is also basically exact. Hence, in the computations there only arise round-off errors due to the finite accuracy of the representation of numbers in the computer, and of the implementation of the mathematical functions occurring in (3.74). On the other hand, in the integral-equation method the propagation coefficients are computed from the equation $\det(\Delta) = 0$, based on (3.55), which is an approximation (obtained by discretisation of the integral equations by using a finite number of expansion and weighting functions) to the actual equation for the propagation coefficients. The computation of the field distributions is also based on the (approximate) equation (3.55). In view of these different starting points, the accuracy of the transfer-matrix formalism will, in general, compare favourably to that of the integral-equation method. Especially for higher-order surface-wave modes, computing times with the transfer-matrix
formalism are much less than those with the integral-equation method, since in the latter method a large number of expansion functions must be taken into account to approximate the fields with sufficient accuracy. However, this advantage gets lost when the refractive index profile of the waveguide is such that no analytical expressions for the fields in the layers are available (the transfer matrices are then to be determined by numerically solving the field equations in the layers). When an arbitrary profile is approximated by a profile of the multi-step-index type, the number of layers necessary for a good approximation may become prohibitively large, especially for the higher-order surface-wave modes, in view of the fact that their transverse wavelength is small.

In this chapter, we have treated the modal structure in general straight open waveguides (Sections 3.1 and 3.2). We have discussed two methods of computation for the surface-wave modes (Section 3.3), and we have applied these methods to the computation of surface-wave modes in a straight planar open waveguide (Section 3.4). In the next chapter, we shall consider finite sections of open waveguides, and we shall derive integral representations for the fields in the sections in terms of the tangential field components in the boundary planes.
4. INTEGRAL REPRESENTATIONS FOR THE FIELDS IN A STRAIGHT OPEN WAVEGUIDE SECTION IN TERMS OF THE TANGENTIAL FIELDS IN THE BOUNDARY PLANES

4.1. INTEGRAL REPRESENTATIONS AND THE COUPLING PROBLEM

In Section 3.2 it has been pointed out that the electromagnetic field in a source-free finite waveguide section can be expressed in terms of the tangential electric and magnetic field distributions in the end planes. These distributions can be conceived as electric and magnetic current sources with surface densities \( J^e \) and \( \mathbf{K}^m \). In order to describe the transmission of electromagnetic fields via a number of series connected open waveguide sections, integral representations will be derived for the fields in each section, expressed in terms of the surface sources \( J^e \) and \( \mathbf{K}^m \) in the junction planes. By imposing the condition that the tangential field components be continuous across the junction planes, a system of integral equations for the tangential fields in the junction planes follows.

4.2. INTEGRAL REPRESENTATIONS CONTAINING \( J^e \) AND \( \mathbf{K}^m \)

We consider a finite section of an infinite open waveguide with arbitrary cross-section, permittivity and permeability, which is schematically shown in Fig. 4.1. The interior of the section is free from volume sources. We apply the reciprocity relations (2.22) and (2.23) to the fields in this configuration. In these relations \( \mathcal{Y} \) is taken as the domain \( s_1 < s < s_2, \ \partial \mathcal{V}_1 \) and \( \partial \mathcal{V}_2 \) as the boundary planes \( s = s_1 \) and \( s = s_2 \).
Fig. 4.1. Configuration of a finite open waveguide section, coordinate system and unit vectors $\mathbf{n}$ normal to the boundary planes $\partial \gamma_1$ and $\partial \gamma_2$.

respectively, and $\gamma'$ is the union of the domains $x < x_1$ and $x > x_2$. The "closed surface" $\partial \gamma'$ occurring in (2.22) and (2.23) then consists of $\partial \gamma_1$, $\partial \gamma_2$ and a cylindrical surface $\omega_\Delta$ at an infinitely large distance around the waveguide, joining $\partial \gamma_1$ and $\partial \gamma_2$.

The contribution of $\omega_\Delta$ to the surface integrals in (2.22) and (2.23) vanishes by virtue of the radiation condition. Since $\mathbf{J}_\omega = 0$ and $\mathbf{K}_\omega' = \Omega$, (2.22) and (2.23) yield

$$
(1 \frac{\omega}{c^2}) \mathbf{E}(\tau) = \iiint_{\partial \gamma_1 \cup \partial \gamma_2} \left[ \mathbf{S}^{\text{EE}}(\tau, \tau') \cdot \mathbf{J}_\omega(\tau') + \mathbf{S}^{\text{EE}}(\tau', \tau') \cdot \mathbf{J}_\omega(\tau') \right] \mathbf{dA}(\tau')
$$

when $\tau \in \{ \gamma, \gamma_1, \gamma_2, \gamma', \gamma_1', \gamma_2' \}$, (4.1)

$$
(1 \frac{\omega}{c^2}) \mathbf{H}(\tau) = \iiint_{\partial \gamma_1 \cup \partial \gamma_2} \left[ \mathbf{S}^{\text{MM}}(\tau, \tau') \cdot \mathbf{J}_\omega(\tau') + \mathbf{S}^{\text{ME}}(\tau, \tau') \cdot \mathbf{J}_\omega(\tau') \right] \mathbf{dA}(\tau')
$$

when $\tau \in \{ \gamma, \gamma_1, \gamma_2, \gamma', \gamma_1', \gamma_2' \}$, (4.2)

in which $\mathbf{J}_\omega$ and $\mathbf{K}_\omega$ are given by (2.24) and (2.25) and in which the roles of $\tau$ and $\tau'$ have been interchanged in accordance with notational practice.
When \( r \in \chi \) the expressions (4.1) and (4.2) are integral representations for the fields in the waveguide. They consist of a contribution originating from the surface sources at \( \partial \chi_1 \) (these generate waves travelling towards \( \partial \chi_2 \)) and a contribution originating from the surface sources at \( \partial \chi_2 \) (these generate waves travelling towards \( \partial \chi_1 \)). In case the reflection and transmission properties of only a single boundary plane, \( \partial \chi \) are investigated, the wave travelling towards \( \partial \chi \) may be viewed upon as an incident field \( \{E^1, H^1\} \) (Fig. 4.2). In that case, (4.1) and (4.2) are rewritten as

\[
\left\{1, \frac{1}{2}, 0\right\} E(t) = E^1(t) + \int_{\partial \chi} \left[ G^{EM}(t, r') \cdot \mathbf{K} \cdot \mathbf{E}(r') + G^{FE}(t, r') \cdot \mathbf{J} \cdot \mathbf{E}(r') \right] dA(r')
\]

when \( \text{Re} \{ \chi, \phi, \chi', \phi' \} \),

\[
\left\{1, \frac{1}{2}, 0\right\} H(t) = H^1(t) + \int_{\partial \chi} \left[ G^{MM}(t, r') \cdot \mathbf{K} \cdot \mathbf{H}(r') + G^{MF}(t, r') \cdot \mathbf{J} \cdot \mathbf{H}(r') \right] dA(r')
\]

when \( \text{Re} \{ \chi, \phi, \chi', \phi' \} \),

in which \( \{E^1, H^1\} \) is the incident field in the infinite waveguide without boundary

![Fig. 4.2](image-url)

Fig. 4.2. Configuration of an open waveguide section with a single boundary plane \( \partial \chi \) and incident field \( \{E^1, H^1\} \).
plane. In case the wave travelling towards \( \partial \mathcal{V} \) is absent, \( \mathcal{E}^i = \mathcal{H}^i = 0 \) in (4.3) and (4.4).

The Green's tensors occurring in (4.1)–(4.4) are the Green's tensors of the infinite open waveguide, defined by (2.10)–(2.21) and satisfying the radiation condition at infinity. In particular, no boundary conditions are imposed on these tensors at the boundary planes. However, by using Green's tensors that satisfy appropriate boundary conditions, it is possible to eliminate either \( \mathcal{J}_{G} \) or \( \mathcal{K}_{G} \) from the integral representations. This will be shown in subsequent sections.

4.3. INTEGRAL REPRESENTATIONS CONTAINING EITHER \( \mathcal{J}_{G} \) OR \( \mathcal{K}_{G} \)

By imposing appropriate boundary conditions on the Green's tensors occurring in (4.1), (4.2) and (4.3), (4.4), at \( \partial \mathcal{V}_1 \) and \( \partial \mathcal{V}_2 \) or at \( \partial \mathcal{V} \), respectively, we can obtain representations that contain \( \mathcal{J}_{G} \) or \( \mathcal{K}_{G} \) only.

4.3.1. Representations containing \( \mathcal{J}_{G} \)

First, we impose on the Green's field \( \mathcal{H}^{GE}(\mathbf{r}) \), related to \( \mathcal{G}^{EM} \) by (2.19), the boundary condition

\[
\mathbf{n} \times \mathcal{H}^{GE}(\mathbf{r}) = 0 \quad \text{when} \quad \mathbf{r} \in \partial \mathcal{V}_1 \cup \partial \mathcal{V}_2 \tag{4.5}
\]

Now, by applying the reciprocity theorem (2.9) with state B: \( \{ \mathcal{E}^B, \mathcal{B}^B, \mathcal{J}^B, \mathcal{K}^B \} \)

\[
= \{ \mathcal{E}^{GE}, \mathcal{G}^{GE}, \mathcal{J}^{GE}, \mathcal{K}^{GE} \}
\]

subject to the condition (4.5), we obtain

\[
\mathcal{E}(\mathbf{r}) = \int_{\partial \mathcal{V}_1 \cup \partial \mathcal{V}_2} \mathcal{G}^{PE}(\mathbf{r}, \mathbf{r}') \cdot \mathcal{J}_{G}(\mathbf{r}') dA(\mathbf{r}') \quad \text{when} \quad \mathbf{r} \in \partial \mathcal{V}_1 \cup \partial \mathcal{V}_2 \tag{4.6}
\]
Likewise, by applying the reciprocity theorem (2.9) with state B: \( \{ E^B, H^B, J^B, K^B \} \)
= \( \{ E^{GM}, H^{GM}, J^{GM}, K^{GM} \} \) subject to the condition

\[ \mathbf{n} \cdot H^{GM}(r) = 0 \quad \text{when} \ r \in \partial \mathcal{X}_1 \cup \partial \mathcal{X}_2, \quad (4.7) \]

where \( H^{GM} \) is related to \( G^{MM} \) by (2.21), we arrive at

\[ H(t) = \int_{\partial \mathcal{X}_1 \cup \partial \mathcal{X}_2} \frac{G^{Me}(r, r') \cdot J_{\mathcal{M}}(r') \, dA(r')} {r}, \quad \text{when} \ r \in \mathcal{X} \setminus \partial \mathcal{X}_1 \cup \partial \mathcal{X}_2. \quad (4.8) \]

The superscripts \( e \) in \( G^{Ec}, G^{Me} \) have been introduced to distinguish these Green's tensors, which satisfy the boundary conditions (4.5), (4.7) in the boundary planes and the radiation condition in the transverse direction, from the Green's tensors \( G^{EE}, G^{ME} \) which satisfy the radiation condition in all directions.

Note that for \( r \in \partial \mathcal{X}_1 \cup \partial \mathcal{X}_2 \), the relation (4.8) for the transverse field \( H_{\mathcal{M}}(r) \) reduces to an identity as implied by the self-reproducing property of the Green's tensor with boundary condition (4.7) (compare Kellogg, 1953). The Green's tensors \( G^{Ec} \) and \( G^{Me} \) can explicitly be obtained by the method of images.

4.3.2 Representations containing \( K_{\mathcal{M}} \)

In order to eliminate \( J_{\mathcal{M}} \) from (4.1) we impose the following boundary condition on the Green's field \( E^{GE}(r) \) (related to \( G^{EE} \) by (2.18)):

\[ \mathbf{n} \times \mathbf{E}^{GE}(t) = 0 \quad \text{when} \ r \in \partial \mathcal{X}_1 \cup \partial \mathcal{X}_2. \quad (4.9) \]

By applying the reciprocity theorem (2.9) with state B: \( \{ E^B, H^B, J^B, K^B \} \)
= \( \{ E^{GE}, H^{GE}, J^{GE}, K^{GE} \} \) subject to the condition (4.9), we obtain
\[ E(z) = \int_{\partial \mathcal{K}_1 \cup \partial \mathcal{K}_2} G^{E_m}(z, z') \cdot K_{\mathcal{M}}(z') dA(z') \quad \text{when} \quad z \in \mathcal{K}_1 \cup \mathcal{K}_2. \quad (4.10) \]

For \( z \in \partial \mathcal{K}_1 \cup \partial \mathcal{K}_2 \), the relation (4.10) for the transverse field \( E_T(z) \) reduces to an identity. By applying the reciprocity theorem (2.9) with state \( B : \{ E^B, H^B, \mathcal{M}^B, K^B \} \)

\[ = \{ E^{G_{GM}B}, H^{G_{GM}B}, \mathcal{M}^{G_{GM}B}, K^{G_{GM}B} \} \quad \text{subject to the condition} \]

\[ \mathbf{n} \cdot E^{G_{GM}} = 0 \quad \text{when} \quad z \in \partial \mathcal{K}_1 \cup \partial \mathcal{K}_2. \quad (4.11) \]

where \( E^{G_{GM}} \) is related to \( G^{M_E} \) by (2.20), we arrive at

\[ H(z) = \int_{\partial \mathcal{K}_1 \cup \partial \mathcal{K}_2} G^{M_m}(z, z') \cdot K_{\mathcal{M}}(z') dA(z') \quad \text{when} \quad z \in \mathcal{K}_1 \cup \mathcal{K}_2. \quad (4.12) \]

The superscripts \( m \) in \( G^{E_m}, G^{M_m} \) have been introduced to distinguish these Green's tensors, which satisfy the boundary conditions (4.9), (4.11) in the boundary planes and the radiation condition in the transverse direction, from the Green's tensors \( G^{E_m}, G^{M_m} \) which satisfy the radiation condition in all directions.

The Green's tensors \( G^{E_m} \) and \( G^{M_m} \) can explicitly be obtained by the method of images.

4.4. THE METHOD OF IMAGES

The Green's tensors that are subject to certain boundary conditions, can be constructed from the Green's tensors of the infinite open waveguide by the method of images. Here, the symmetry properties of the latter Green's tensors, derived in Appendix C, and the fact that they depend on \( z \) and \( z' \) through \( z - z' \) only (both are
consequences of the translational invariance of the waveguide in the axial direction) are exploited. We shall give the details for a single boundary plane \( \partial \mathcal{X} \) whereby the method of images is applied directly to the field representations.

4.4.1. **Representations containing \( J_{\sigma} \)**

From (2.24) and (2.25) and the fact that \( \mathbf{n} = \mathbf{i}_z \) in the boundary plane \( \partial \mathcal{X} \) we infer that \( J_{\sigma} \) and \( K_{\epsilon} \) have transverse components only. Therefore, only the Green's tensor elements \( G_{\alpha T} \) \( (\alpha = T, x) \) enter into the representations (4.3) and (4.4). We introduce the subtensors \( \mathcal{G}_{TT}, \mathcal{G}_{Tx}, \mathcal{G}_{xT} \) and \( G_{xx} \) of the tensor \( \mathcal{G} \) as

\[
\mathcal{G} = \begin{bmatrix}
G_{TT} & G_{Tx} \\
G_{xT} & G_{xx}
\end{bmatrix},
\]

(4.13)

where

\[
G_{TT} = \begin{bmatrix}
G_{xx} & G_{xy} \\
G_{yx} & G_{yy}
\end{bmatrix},
\]

(4.14)

\[
G_{Tx} = \begin{bmatrix}
G_{xx} \\
G_{yx}
\end{bmatrix},
\]

(4.15)

\[
G_{xT} = \begin{bmatrix}
G_{xx} & G_{xy}
\end{bmatrix},
\]

(4.16)

and \( G_{xx} \) is the \( xx \)-element of the Green's tensor. Then the expressions (4.3) and (4.4) can be decomposed into representations for the transverse and axial field components, namely,
\[ (1, 0, 0) E_T(\tau) = E_T^I(\tau) + \int_{\partial \mathcal{V}} \left[ \mathcal{G}_{TT}^{EM}(\tau, \tau') \cdot \mathbf{K} \mathcal{A}(\tau') + \mathcal{G}_{TT}^{EE}(\tau, \tau') \cdot \mathbf{J} \mathcal{A}(\tau') \right] dA(\tau') \]

when \( \mathcal{R}(\mathcal{V}, \mathcal{V}') \), \hfill (4.17)

\[ (1, 0, 0) E_x(\tau) = E_x^I(\tau) + \int_{\partial \mathcal{V}} \left[ \mathcal{G}_{xT}^{EM}(\tau, \tau') \cdot \mathbf{K} \mathcal{A}(\tau') + \mathcal{G}_{xT}^{EE}(\tau, \tau') \cdot \mathbf{J} \mathcal{A}(\tau') \right] dA(\tau') \]

when \( \mathcal{R}(\mathcal{V}, \mathcal{V}') \), \hfill (4.18)

\[ (1, 0, 0) H_T(\tau) = H_T^I(\tau) + \int_{\partial \mathcal{V}} \left[ \mathcal{G}_{TT}^{MM}(\tau, \tau') \cdot \mathbf{K} \mathcal{A}(\tau') + \mathcal{G}_{TT}^{ME}(\tau, \tau') \cdot \mathbf{J} \mathcal{A}(\tau') \right] dA(\tau') \]

when \( \mathcal{R}(\mathcal{V}, \mathcal{V}') \), \hfill (4.19)

\[ (1, 0, 0) H_y(\tau) = H_y^I(\tau) + \int_{\partial \mathcal{V}} \left[ \mathcal{G}_{xT}^{MM}(\tau, \tau') \cdot \mathbf{K} \mathcal{A}(\tau') + \mathcal{G}_{xT}^{ME}(\tau, \tau') \cdot \mathbf{J} \mathcal{A}(\tau') \right] dA(\tau') \]

when \( \mathcal{R}(\mathcal{V}, \mathcal{V}') \). \hfill (4.20)

We now consider the configuration of Fig. 4.3. The boundary plane \( \partial \mathcal{V} \) is taken to be located at \( z = 0 \) (which can be done without loss of generality), \( \mathcal{V} \) is the domain \( z < 0 \)

![Fig. 4.3. Application of the method of images to a waveguide section with a single boundary plane \( \partial \mathcal{V} \)].
and \( \mathcal{Y} \) is the domain \( z > 0 \). Suppressing the dependences on \( x \) and \( y \), which are irrelevant for the moment, (4.17)–(4.20) are written as

\[
\begin{align*}
\{1, \frac{1}{2}, 0\} E_T(z) &= E_T^i(z) + \int_{z^1=0} \left[ G_{EM}^{MT}(z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{EE}^{MT}(z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}, \\
\{1, \frac{1}{2}, 0\} E_z(z) &= E_z^i(z) + \int_{z^1=0} \left[ G_{EM}^{zT}(z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{EE}^{zT}(z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}, \\
\{1, \frac{1}{2}, 0\} H_T(z) &= H_T^i(z) + \int_{z^1=0} \left[ G_{MM}^{MT}(z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{ME}^{MT}(z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}, \\
\{1, \frac{1}{2}, 0\} H_z(z) &= H_z^i(z) + \int_{z^1=0} \left[ G_{MM}^{zT}(z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{ME}^{zT}(z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}.
\end{align*}
\]

We now introduce the image point \( z_1 \) of \( z \) with respect to the plane \( \partial \mathcal{K} \); hence, if \( z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \} \), one has \( z_1 \in \{ \mathcal{Y} \setminus \partial \mathcal{K} \} \).

From (4.17)–(4.20) with \( z \) replaced by \( z_1 \), we then obtain the relations

\[
\begin{align*}
\{0, \frac{1}{2}, 1\} E_T(-z) &= E_T^i(-z) + \int_{z^1=0} \left[ G_{EM}^{MT}(-z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{EE}^{MT}(-z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}, \\
\{0, \frac{1}{2}, 1\} E_z(-z) &= E_z^i(-z) + \int_{z^1=0} \left[ G_{EM}^{zT}(-z,0) \cdot \mathbf{K} \cdot \sigma(0) + G_{EE}^{zT}(-z,0) \cdot \mathbf{J} \cdot \sigma(0) \right] dA(0) \\
&\quad \text{when } z \in \{ \partial \mathcal{K} \setminus \mathcal{Y} \}.
\end{align*}
\]
\[ \{0, \frac{1}{2}, \frac{3}{2}\} \mathcal{H}_T(z) = \mathcal{H}_T(z) + \sum_{z' = 0}^\infty \left[ \mathcal{G}^{MM}_{TT}(z', 0) \cdot \mathcal{K}_\sigma(z) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{ME}_{TT}(z', 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \]

when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \).

\[ \{0, \frac{1}{2}, \frac{3}{2}\} \mathcal{H}_z(z) = \mathcal{H}_z(z) + \sum_{z' = 0}^\infty \left[ \mathcal{G}^{MM}_{zT}(z, 0) \cdot \mathcal{K}_\sigma(z) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{ME}_{zT}(z, 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \]

when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \).

In Appendix C it is shown that \( \mathcal{G}^{EE}_{TT}, \mathcal{G}^{MM}_{TT}, \mathcal{G}^{EM}_{TT}, \mathcal{G}^{ME}_{TT} \), \( \mathcal{G}^{EM}_{zz}, \mathcal{G}^{EE}_{zz}, \mathcal{G}^{ME}_{zz}, \mathcal{G}^{MM}_{zz} \), and \( \mathcal{G}^{ME}_{zT} \) are even functions of \( z - z' \), while \( \mathcal{G}^{EM}_{zT}, \mathcal{G}^{EE}_{zT}, \mathcal{G}^{ME}_{zT}, \mathcal{G}^{MM}_{zT} \), and \( \mathcal{G}^{MM}_{zT} \) are odd functions of \( z - z' \). Taking into account these symmetry properties, we obtain by adding \((4.25)\) to \((4.21)\) and \((4.28)\) to \((4.24)\), and by subtracting \((4.26)\) from \((4.22)\) and \((4.27)\) from \((4.23)\),

\[ \mathcal{E}_{T}(z) = \mathcal{E}_{T}(z) + \mathcal{E}_{T}(z) + 2 \sum_{z' = 0}^\infty \left[ \mathcal{G}^{EE}_{TT}(z, 0) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{EE}_{zT}(z, 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \] when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \) \( (4.29) \)

\[ \mathcal{E}_{z}(z) = \mathcal{E}_{z}(z) - \mathcal{E}_{z}(z) + 2 \sum_{z' = 0}^\infty \left[ \mathcal{G}^{EE}_{zz}(z, 0) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{EE}_{zT}(z, 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \] when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \) \( (4.30) \)

\[ \mathcal{H}_{T}(z) = \mathcal{H}_{T}(z) - \mathcal{H}_{T}(z) + 2 \sum_{z' = 0}^\infty \left[ \mathcal{G}^{ME}_{TT}(z, 0) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{ME}_{zT}(z, 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \] when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \) \( (4.31) \)

\[ \mathcal{H}_{z}(z) = \mathcal{H}_{z}(z) + \mathcal{H}_{z}(z) + 2 \sum_{z' = 0}^\infty \left[ \mathcal{G}^{ME}_{zz}(z, 0) \cdot \mathcal{J}_\sigma(0) + \mathcal{G}^{ME}_{zT}(z, 0) \cdot \mathcal{J}_\sigma(0) \right] dA(z) \] when \( \sigma \in \{ \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{Y} \} \) \( (4.32) \)

4.4.2. Representations containing \( \mathcal{K}_\sigma \)

The representations containing \( \mathcal{K}_\sigma \) only can easily be obtained from \((4.21)\)–\((4.28)\) by adding \((4.26)\) to \((4.22)\) and \((4.27)\) to \((4.23)\), and by subtracting \((4.25)\) from \((4.21)\) and \((4.28)\) from \((4.24)\), and taking into account the symmetry properties of the Green's tensors. We obtain
\[ E_T(z) = E_T^1(z) - E_T^1(-z) + 2 \int_{z'=0}^{\infty} G_{TT}^{EM}(z,0) \cdot K_{\phi}(0) dA(0) \text{ when } z \in \mathcal{K} \quad (4.33) \]

\[ E_z(z) = E_z^1(z) + E_z^1(-z) + 2 \int_{z'=0}^{\infty} G_{zT}^{EM}(z,0) \cdot K_{\phi}(0) dA(0) \text{ when } z \in \mathcal{K} \quad (4.34) \]

\[ H_T(z) = H_T^1(z) - H_T^1(-z) + 2 \int_{z'=0}^{\infty} G_{TT}^{MM}(z,0) \cdot K_{\phi}(0) dA(0) \text{ when } z \in \mathcal{K} \quad (4.35) \]

\[ H_z(z) = H_z^1(z) - H_z^1(-z) + 2 \int_{z'=0}^{\infty} G_{zT}^{MM}(z,0) \cdot K_{\phi}(0) dA(0) \text{ when } z \in \mathcal{K} \quad (4.36) \]

The representations (4.29)–(4.36) contain the Green's tensors of the infinite open waveguide, the tangential magnetic or electric fields through \( J_{\phi} \) or \( K_{\phi} \), and the incident fields in \( z \) and its image point \( z' \). Note that when \( z \in \partial \mathcal{K} \) we have \( z = z' \).

In case two different boundary planes are involved, application of the method of images would give rise to an infinite number of image points. Consequently, in that case the method of images is of less practical value.

In this chapter, integral representations have been obtained for the field quantities in an open waveguide section, expressed in terms of the tangential field distributions in the boundary planes, or in terms of the field distribution in a single boundary plane and the field incident from the section towards the boundary plane. In the next chapter, these representations will be used to derive integral equations for the tangential field components in the junction planes of series-connected sections of different open waveguides.
5. INTEGRAL EQUATIONS FOR THE FIELDS IN THE JUNCTION PLANES OF
SERIES–CONNECTED STRAIGHT OPEN WAVEGUIDE SECTIONS

5.1. CONFIGURATION OF SERIES–CONNECTED WAVEGUIDE SECTIONS

In the previous chapter we have derived integral representations for the fields in a
straight open waveguide section, expressed in terms of the tangential fields in the
boundary planes of the section. In the present chapter we shall employ these
representations to establish integral equations for the tangential fields in the junction
plane(s) of a number of series–connected open waveguide sections.

We consider two cases: the coupling of two waveguide sections with a single junction
plane (Fig. 5.1a), and the coupling of \( n + 1 \) waveguide sections with \( n \) junction
planes, where \( n \geq 2 \) (Fig. 5.1b). The latter case will be illustrated for \( n = 2 \),
corresponding to three coupled sections with two junction planes (Fig. 5.1c). In the
case of a single junction plane, the integral equations involve the tangential fields in
this junction plane as unknowns. In the case of several junction planes, a system of
integral equations is established involving the tangential fields in each of the junction
planes as unknowns.
Fig. 5.1. Configuration of coupled waveguide sections (a) with a single junction plane; (b) with n junction planes; (c) with two junction planes.
5.2. INTEGRAL EQUATIONS FOR THE TANGENTIAL FIELDS IN THE JUNCTION PLANE OF TWO SERIES-CONNECTED STRAIGHT OPEN WAVEGUIDE SECTIONS

The configuration of two series-connected waveguide sections, A and B, is shown in Fig. 5.2. The junction plane \( \partial \mathcal{V} \) is located at \( x = 0 \). The unit vector \( \mathbf{n} \) normal to \( \partial \mathcal{V} \) is chosen to be \( i_y \). Accordingly, we introduce as the surface current densities in \( \partial \mathcal{V} \):

\[
J_0(t) = -i_x \times \mathbf{H}(t) = -i_x \times \mathbf{H}_T(t) \quad \text{with} \quad \mathbf{z} \in \partial \mathcal{V} \quad (5.1)
\]

\[
K_0(t) = i_x \times \mathbf{E}(t) = i_x \times \mathbf{E}_T(t) \quad \text{with} \quad \mathbf{z} \in \partial \mathcal{V} \quad (5.2)
\]

Radiation is incident in waveguide A; no radiation is assumed to be incident in waveguide B. Note that in view of (2.24) and (2.28), \( \{J_0K_0\} = \{J_{\sigma B}K_{\sigma B}\} = -\{J_{\sigma A}K_{\sigma A}\} \). Using the integral representations obtained in the previous chapter, integral equations involving either both \( \mathbf{E}_T \) and \( \mathbf{H}_T \), or \( \mathbf{H}_T \) only, or \( \mathbf{E}_T \) only, will be derived.

![Diagram]

Fig. 5.2. Configuration of two coupled waveguide sections with junction plane \( \partial \mathcal{V} \) at \( x = 0 \) and unit normal vector \( \mathbf{n} = i_y \).
For the fields in sections A and B, the representations (4.3) and (4.4) hold. In Appendix C it is shown that $G^{EM}_{TT}$ and $G^{ME}_{TT}$ are odd functions of $z - z'$. Therefore, $G^{EM}_{TT}(z, z') = 0$ and $G^{ME}_{TT}(z, z') = 0$ when $z' \in \partial \mathcal{K}$ Using (4.3) and (4.4) in waveguide section A with $z \in \partial \mathcal{K}$ we obtain

$$\frac{1}{2} E_1 (z) = \left[ \frac{\partial}{\partial z} \right] E_0 (z') dA(z') \text{ with } z \in \partial \mathcal{K} \quad (5.3)$$

$$\frac{1}{2} H_1 (z) = \left[ \frac{\partial}{\partial z} \right] H_0 (z') dA(z') \text{ with } z \in \partial \mathcal{K} \quad (5.4)$$

In waveguide section B, we use (4.3) and (4.4) with $\vec{E}_1 = 0$ and $\vec{H}^i = 0$, and $z \in \partial \mathcal{K}$. We then obtain

$$\frac{1}{2} E_1 (z) = - \int \left[ \frac{\partial}{\partial z} \right] E_0 (z') dA(z') \text{ with } z \in \partial \mathcal{K} \quad (5.5)$$

$$\frac{1}{2} H_1 (z) = - \int \left[ \frac{\partial}{\partial z} \right] H_0 (z') dA(z') \text{ with } z \in \partial \mathcal{K} \quad (5.6)$$

Here, $G_A$ and $G_B$ are the Green's tensors of the infinite open waveguide with the cross-section, permittivity and permeability of waveguide sections A and B, respectively. The fields $E_0 (z')$ in (5.3) and (5.5), and the fields $H_0 (z')$ in (5.4) and (5.6) can be identified because of the property that the tangential field components are continuous upon crossing a surface of discontinuity for the electromagnetic properties, i.e., $\lim_{z \to 0} \{ E_T H_T \} = \lim_{z \to 0} \{ E_T H_T \}$. Upon adding (5.3) to (5.5) and (5.4) to (5.6), we obtain

$$E_T (z) - E_T (z) = \int \left[ \left[ \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right] A(z') + \frac{\partial}{\partial z} G^{EM}_{TT} (z, z') \right] dA(z') \text{ with } z \in \partial \mathcal{K} \quad (8.7)$$
\[ H_{T}(\xi) - H_{T}^{1}(\xi) = \int_{\partial \mathcal{K}} \left( [G_{T,T,A}^{MM}(\xi, \xi') - G_{T,T,B}^{MM}(\xi, \xi')] \cdot K_{0}(\xi') dA(\xi') \right) \quad \text{with} \quad \xi \in \partial \mathcal{K} \quad (5.8) \]

Since \( K_{0}(\xi') \) and \( J_{0}(\xi') \) are simply related to the tangential fields \( E_{T} \) and \( H_{T} \) in \( \partial \mathcal{K} \), equations (5.7) and (5.8) constitute a system of linear inhomogeneous integral equations of the second kind for \( E_{T} \) and \( H_{T} \) in the junction plane.

Upon subtracting (5.5) from (5.3) we obtain a linear inhomogeneous integral equation of the first kind for \( H_{T} \) in the junction plane:

\[ E_{T}^{1}(\xi) = - \int_{\partial \mathcal{K}} \left( [G_{T,T,A}^{EE}(\xi, \xi') + G_{T,T,B}^{EE}(\xi, \xi')] \cdot J_{0}(\xi') dA(\xi') \right) \quad \text{with} \quad \xi \in \partial \mathcal{K} \quad (5.9) \]

Upon subtracting (5.6) from (5.4) we obtain a linear inhomogeneous integral equation of the first kind for \( E_{T} \) in the junction plane:

\[ H_{T}^{1}(\xi) = - \int_{\partial \mathcal{K}} \left( [G_{T,T,A}^{MM}(\xi, \xi') + G_{T,T,B}^{MM}(\xi, \xi')] \cdot K_{0}(\xi') dA(\xi') \right) \quad \text{with} \quad \xi \in \partial \mathcal{K} \quad (5.10) \]

The equations (5.7)–(5.10) can also be derived from (4.29) and (4.35), respectively, when applied to waveguide sections A and B.

Note that in the case of coupling of two identical sections, one has \( E_{T} = E_{T}^{1} \) and \( H_{T} = H_{T}^{1} \). This is obvious from (5.7) and (5.8), since then \( G_{A} = G_{B} \).
5.3. INTEGRAL EQUATIONS FOR THE TANGENTIAL FIELDS IN THE JUNCTION PLANES OF THREE SERIES-CONNECTED STRAIGHT OPEN WAVEGUIDE SECTIONS

The configuration of three series-connected open waveguide sections, A, B and C, is shown in Fig. 5.3. The junction planes $\partial \mathcal{V}_1$ and $\partial \mathcal{V}_2$ are located at $z = z_1$ and $z = z_2$, respectively. The unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ normal to $\partial \mathcal{V}_1$ and $\partial \mathcal{V}_2$ are chosen to be $\mathbf{i}_z$.

In accordance with this choice we introduce as the surface current densities in $\partial \mathcal{V}_1$:

$$
J_1(t) = -\mathbf{i}_z \times \mathbf{H}(t) = -\mathbf{i}_z \times \mathbf{H}_T(t) \quad \text{with } t \in \partial \mathcal{V}_1,
$$

(5.11)

$$
K_1(t) = \mathbf{i}_z \times \mathbf{E}(t) = \mathbf{i}_z \times \mathbf{E}_T(t) \quad \text{with } t \in \partial \mathcal{V}_1,
$$

(5.12)

and as the surface current densities in $\partial \mathcal{V}_2$:

$$
J_0(t) = -\mathbf{i}_z \times \mathbf{H}(t) = -\mathbf{i}_z \times \mathbf{H}_T(t) \quad \text{with } t \in \partial \mathcal{V}_2,
$$

(5.13)

Fig. 5.3. Configuration of three coupled waveguide sections with junction planes $\partial \mathcal{V}_1$ at $z = z_1$ and $\partial \mathcal{V}_2$ at $z = z_2$, and unit normal vectors $\mathbf{n}_1 = \mathbf{i}_z$ and $\mathbf{n}_2 = \mathbf{i}_z$, respectively.
\[ \mathbf{K}_2(t) = i_2 \times \mathbf{E}(t) = -\mathbf{E}_T(t) \quad \text{with} \quad \xi \in \partial \mathcal{K}_2 \]

(5.14)

Radiation is incident in waveguide A; no radiation is assumed to be incident in waveguide C. In view of (2.24) and (2.25), one has \[ \{J_1; \mathbf{K}_1\} = \{J_1; \mathbf{B}_A; \mathbf{B}_A\} \]
\[ = -\{J_1; \mathbf{B}_A; \mathbf{B}_A\}_{\xi = z_1} \quad \text{and} \quad \{J_2; \mathbf{K}_2\} = \{J_2; \mathbf{B}_C; \mathbf{B}_C\}_{\xi = z_2} = -\{J_2; \mathbf{B}_C; \mathbf{B}_C\}. \]

Using the integral representations derived in Chapter 4, integral equations involving either both \( \mathbf{E}_{T1,2} \) and \( \mathbf{H}_{T1,2} \) or \( \mathbf{E}_{T1,2} \) only, or \( \mathbf{H}_{T1,2} \) only, will be established in the subsequent subsections.

5.3.1. Integral equations containing \( \mathbf{E}_T \) and \( \mathbf{H}_T \)

For the fields in sections A and C, the representations (4.3) and (4.4) hold, with \( \mathbf{E}^i = 0 \) and \( \mathbf{H}^i = 0 \) in waveguide section C. In waveguide section A, we use (4.3) and (4.4) with \( \xi \in \partial \mathcal{K}_1 \). By employing the odd symmetry (in \( z-z' \)) of \( \mathcal{O}_{EM}^{TT} \) and \( \mathcal{O}_{ME}^{TT} \), we then obtain for the transverse components of \( \mathbf{E} \) and \( \mathbf{H} \) in \( \partial \mathcal{K}_1 \):

\[ \frac{1}{2} E_T(t) = E_T^i(t) + \int_{\partial \mathcal{K}_1} \mathcal{O}_{EM}^{TT,A}(\xi, \xi') \cdot J_1(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \mathcal{K}_1. \]

(5.15)

\[ \frac{1}{2} H_T(t) = H_T^i(t) + \int_{\partial \mathcal{K}_1} \mathcal{O}_{ME}^{TT,A}(\xi, \xi') \cdot K_1(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \mathcal{K}_1. \]

(5.16)

In waveguide section B, we use (4.1) and (4.2). First, we let \( \xi \in \partial \mathcal{K}_1 \), and obtain
\[
\frac{1}{2} E_T(t) = - \int_{\partial K_1} \left( \frac{\partial E}{\partial t} \cdot B(t, r') \right) \cdot J_2(r') \, dA(r') \\
+ \int_{\partial K_2} \left( \frac{\partial E}{\partial t} \cdot B(t, r') \right) \cdot J_1(r') \, dA(r') \\
\text{with } r \in \partial K_1. \tag{5.17}
\]

\[
\frac{1}{2} H_T(t) = - \int_{\partial K_1} \left( \frac{\partial H}{\partial t} \cdot B(t, r') \right) \cdot J_1(r') \, dA(r') \\
+ \int_{\partial K_2} \left( \frac{\partial H}{\partial t} \cdot B(t, r') \right) \cdot J_2(r') \, dA(r') \\
\text{with } r \in \partial K_1. \tag{5.18}
\]

Next, when \( r \in \partial K_2 \), we obtain

\[
\frac{1}{2} E_T(t) = - \int_{\partial K_1} \left( \frac{\partial E}{\partial t} \cdot B(t, r') \right) \cdot J_1(r') \, dA(r') \\
+ \int_{\partial K_2} \left( \frac{\partial E}{\partial t} \cdot B(t, r') \right) \cdot J_2(r') \, dA(r') \quad \text{with } r \in \partial K_2. \tag{5.19}
\]

\[
\frac{1}{2} H_T(t) = - \int_{\partial K_1} \left( \frac{\partial H}{\partial t} \cdot B(t, r') \right) \cdot J_2(r') \, dA(r') \\
+ \int_{\partial K_2} \left( \frac{\partial H}{\partial t} \cdot B(t, r') \right) \cdot J_1(r') \, dA(r') \quad \text{with } r \in \partial K_2. \tag{5.20}
\]

In waveguide section C, we again use (4.3) and (4.4). Let \( r \in \partial K_2 \), then we obtain
\[\frac{1}{2} E_T(t) = - \int_{\partial Y_2} g_{\text{EE}}^{\text{TT},C}(x,t') \cdot J_2(t') dA(t') \quad \text{with } t \in \partial Y_2. \quad (5.21)\]

\[\frac{1}{2} H_T(t) = - \int_{\partial Y_2} g_{\text{MM}}^{\text{TT},C}(x,t') \cdot K_2(t') dA(t') \quad \text{with } t \in \partial Y_2. \quad (5.22)\]

Upon adding (5.13) to (5.17), (5.16) to (5.18), (5.19) to (5.21) and (5.20) to (5.22), we obtain a system of four integral equations of the second kind for the tangential field components $E_T$ and $H_T$ in the junction planes $\partial Y_1$ and $\partial Y_2$, namely

\[E_T(t) - E_T^1(t) = \int_{\partial Y_1} g_{\text{EE}}^{\text{TT},AB}(x,t') \cdot J_1(t') dA(t') + \int_{\partial Y_2} g_{\text{EM}}^{\text{TT},B}(x,t') \cdot K_2(t') dA(t') + \int_{\partial Y_2} g_{\text{EE}}^{\text{TT},B}(x,t') \cdot J_2(t') dA(t') \quad \text{with } t \in \partial Y_1, \quad (5.23)\]

\[H_T(t) - H_T^1(t) = \int_{\partial Y_1} g_{\text{MM}}^{\text{TT},AB}(x,t') \cdot K_1(t') dA(t') + \int_{\partial Y_2} g_{\text{EM}}^{\text{TT},B}(x,t') \cdot J_2(t') dA(t') + \int_{\partial Y_2} g_{\text{EE}}^{\text{TT},B}(x,t') \cdot J_2(t') dA(t') \quad \text{with } t \in \partial Y_1, \quad (5.24)\]

\[E_T(t) = - \int_{\partial Y_1} g_{\text{EM}}^{\text{TT},B}(x,t') \cdot K_1(t') dA(t') + \int_{\partial Y_2} g_{\text{EE}}^{\text{TT},BC}(x,t') \cdot J_2(t') dA(t') \quad \text{with } t \in \partial Y_2. \quad (5.25)\]
\[ H_T(t) = - \int \frac{[G^{\text{MM}}_{TT,T,B}(\vec{r},\vec{r}'); \cdot \mathbf{K}_1(\vec{r}')] + G^{\text{ME}}_{TT,T,B}(\vec{r},\vec{r}'); \cdot \mathbf{J}_1(\vec{r}')]dA(\vec{r}')}{\partial \mathbf{Y}_1} \]

\[ + \int \frac{G^{\text{MM}_{\text{BC}}}_{TT,T,B}(\vec{r},\vec{r}'); \cdot \mathbf{K}_2(\vec{r}')]dA(\vec{r}')}{\partial \mathbf{Y}_2} \quad \text{with } \tau \in \partial \mathbf{Y}_2 \]  

(5.20)

In these equations the difference Green's tensors \( G^{\text{PQ}_{\text{RS}}} \) are defined by

\[ G^{\text{PQ}_{\text{RS}}} = G^\text{PQ}_R - G^\text{PQ}_S. \]  

(5.27)

A system of integral equations of the first kind is obtained upon subtracting (5.15) from (5.17), (5.16) from (5.18), (5.21) from (5.19) and (5.22) from (5.20), viz.

\[ E^i_T(t) = - \int \frac{G^{\text{EE}_{\text{TT}},A_B}(\vec{r},\vec{r}'); \cdot \mathbf{J}_1(\vec{r}')]dA(\vec{r}')}{\partial \mathbf{Y}_1} \]

\[ + \int \frac{G^{\text{EM}_{\text{TT},T,B}(\vec{r},\vec{r}'); \cdot \mathbf{K}_2(\vec{r}')} + G^{\text{EE}_{\text{TT},T,B}(\vec{r},\vec{r}'); \cdot \mathbf{J}_2(\vec{r}')]dA(\vec{r}')}{\partial \mathbf{Y}_2} \quad \text{with } \tau \in \partial \mathbf{Y}_1. \]  

(5.28)

\[ H^i_T(t) = - \int \frac{G^{\text{MM}_{\text{TT},T,B}(\vec{r},\vec{r}'); \cdot \mathbf{K}_1(\vec{r}')}dA(\vec{r}')}{\partial \mathbf{Y}_1} \]

\[ + \int \frac{G^{\text{MM}_{\text{TT},T,B}(\vec{r},\vec{r}'); \cdot \mathbf{K}_2(\vec{r}')} + G^{\text{ME}_{\text{TT},T,B}(\vec{r},\vec{r}'); \cdot \mathbf{J}_2(\vec{r}')]dA(\vec{r}')}{\partial \mathbf{Y}_2} \quad \text{with } \tau \in \partial \mathbf{Y}_1. \]  

(5.29)
\[ 0 = - \int \frac{[\mathcal{G}^E_{TT,B}(\xi,\eta) - K_1(\xi)] + \mathcal{G}^E_{TT,B}(\xi,\eta) \cdot J_1(\eta)] dA(\xi) }{\partial \chi_1} \]
\[ + \int \frac{[\mathcal{G}^{E,E,+}_{TT,B}(\xi,\eta) = J_2(\eta)] dA(\xi)}{\partial \chi_2} \quad \text{with } \eta \neq \chi_2, \]  
\text{(5.30)}

\[ 0 = - \int \frac{[\mathcal{G}^M_{TT,B}(\xi,\eta) - K_1(\xi)] + \mathcal{G}^M_{TT,B}(\xi,\eta) \cdot J_1(\eta)] dA(\xi) }{\partial \chi_1} \]
\[ + \int \frac{[\mathcal{G}^{M,E,+}_{TT,B}(\xi,\eta) = J_2(\eta)] dA(\xi)}{\partial \chi_2} \quad \text{with } \eta \neq \chi_2, \]  
\text{(5.31)}

in which the sum Green's tensors \( \mathcal{G}^{PQ,+}_{RS} \) are defined by

\[ \mathcal{G}^{PQ,+}_{RS} = \mathcal{G}^{PQ}_{R} + \mathcal{G}^{PQ}_{S}. \]  
\text{(5.32)}

By taking an appropriate combination of four equations from the systems (5.23)–(5.26) and (5.28)–(5.31) (i.e., by replacing one, two, or three of the equations (5.23+p) \((p=0,1,2,3)\) by equations (5.28+p)), a system of integral equations of a mixed kind is obtained for \( \mathcal{E}_T \) and \( \mathcal{H}_T \) in the junction planes \( \partial \chi_1 \) and \( \partial \chi_2 \).

5.3.2. Integral equations containing \( \mathcal{H}_T \)

By using the integral representations from Subsections 4.3.1 and 4.4.1 that only contain \( \mathcal{H}_T \), we can establish a system of integral equations involving only \( \mathcal{E}_T \) in the junction planes \( \partial \chi_1 \) and \( \partial \chi_2 \) as unknowns.

By taking \( \eta \in \partial \chi_1 \) in (4.29), we have
\[ E_T(t) = 2E_T^1(t) + 2 \int_{\partial \Omega_1} G_{E}^{EE}(\xi', \xi) \cdot J_1(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \Omega_1. \quad (5.33) \]

Taking \( \xi \in \partial \Omega_1 \) in (4.6), we obtain for the transverse electric field

\[ E_T(t) = -\int_{\partial \Omega_1} G_{E}^{EE}(\xi', \xi) \cdot J_1(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \Omega_1, \quad (5.34) \]

and taking \( \xi \in \partial \Omega_2 \), we have

\[ E_T(t) = -\int_{\partial \Omega_1} G_{E}^{EE}(\xi', \xi) \cdot J_1(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \Omega_2, \quad (5.35) \]

In waveguide section C we use again (4.29) with now \( \Omega^* \equiv \emptyset \) and obtain, upon taking \( \xi \in \partial \Omega_2 \),

\[ E_T(t) = -2 \int_{\partial \Omega_2} G_{E}^{EE}(\xi', \xi) \cdot J_2(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \Omega_2. \quad (5.36) \]

Upon subtracting (5.33) from (5.34) and (5.36) from (5.35), we obtain a system of two integral equations of the first kind for \( E_T \) in the junction planes \( \partial \Omega_1 \) and \( \partial \Omega_2 \), namely,

\[ E_T^1(t) = -\int_{\partial \Omega_1} G_{E}^{EE}(\xi', \xi) \cdot J_1(\xi') dA(\xi') \]

\[ + \int_{\partial \Omega_2} G_{E}^{EE}(\xi', \xi) \cdot J_2(\xi') dA(\xi') \quad \text{with} \quad \xi \in \partial \Omega_1. \quad (5.37) \]
\[ \phi = -\frac{1}{2} \int_{\partial \mathcal{H}_1} \mathbf{G}_{\text{TT},B}(\mathbf{y}, y') \cdot \mathbf{J}_1(y') dA(y') \]
\[ + \int_{\partial \mathcal{H}_2} \left[ \mathbf{G}_{\text{TT},C}(\mathbf{y}, y') \cdot \mathbf{J}_2(y') dA(y') \right] \quad \text{with } y \in \partial \mathcal{H}_2. \quad (5.38) \]

5.3.3. Integral equations containing \( \mathbf{E}_T \)

By using the integral representations from Subsections 4.3.2 and 4.4.2 that only contain \( \mathbf{E}_T \), we can establish a system of integral equations involving only \( \mathbf{E}_T \) in the junction planes \( \partial \mathcal{H}_1 \) and \( \partial \mathcal{H}_2 \) as unknowns.

By taking \( y \in \partial \mathcal{H}_1 \) in (4.35), we have
\[ \mathbf{H}_T(t) = 2\mathbf{H}_T(t) + 2 \int_{\partial \mathcal{H}_1} \mathbf{G}_{\text{TT},A}(\mathbf{y}, y') \cdot \mathbf{K}_1(y') dA(y') \quad \text{with } y \in \partial \mathcal{H}_1. \quad (5.39) \]

Taking \( y \in \partial \mathcal{H}_2 \) in (4.12), we obtain for the transverse magnetic field
\[ \mathbf{H}_T(t) = -\int_{\partial \mathcal{H}_1} \mathbf{G}_{\text{TT},B}(\mathbf{y}, y') \cdot \mathbf{K}_1(y') dA(y') + \int_{\partial \mathcal{H}_2} \mathbf{G}_{\text{TT},B}(\mathbf{y}, y') \cdot \mathbf{K}_2(y') dA(y') \]
\[ \quad \text{with } y \in \partial \mathcal{H}_1, \quad (5.40) \]

and taking \( y \in \partial \mathcal{H}_2 \), we have
\[ \mathbf{H}_T(t) = -\int_{\partial \mathcal{H}_1} \mathbf{G}_{\text{TT},B}(\mathbf{y}, y') \cdot \mathbf{K}_1(y') dA(y') + \int_{\partial \mathcal{H}_2} \mathbf{G}_{\text{TT},B}(\mathbf{y}, y') \cdot \mathbf{K}_2(y') dA(y') \]
\[ \quad \text{with } y \in \partial \mathcal{H}_2. \quad (5.41) \]
In waveguide section C we use again (4.35) with now $H^{4} = 0$ and obtain, upon taking $1 \in \partial \mathscr{K}_{2}$,

$$H_{\mathcal{T}}(t) = -2 \int_{\partial \mathscr{K}_{2}} G_{TT, C}(r, r') \cdot K_{2}(r') dA(r') \quad \text{with } r \in \partial \mathscr{K}_{2}. \quad (5.42)$$

Upon subtracting (5.39) from (5.40) and (5.42) from (5.41), we obtain a system of two integral equations of the first kind for $E_{\mathcal{T}}$ in the junction planes $\partial \mathscr{K}_{1}$ and $\partial \mathscr{K}_{2}$, namely,

$$H_{\mathcal{T}}(t) = -\int_{\partial \mathscr{K}_{1}} [G_{TT, A}^{MM}(r, r') + G_{TT, B}^{MM}(r, r')] \cdot K_{1}(r') dA(r')$$

$$+ \frac{1}{2} \int_{\partial \mathscr{K}_{2}} G_{TT, B}^{MM}(r, r') \cdot K_{2}(r') dA(r') \quad \text{with } r \in \partial \mathscr{K}_{1}, \quad (5.43)$$

$$0 = -\frac{1}{2} \int_{\partial \mathscr{K}_{1}} G_{TT, B}^{MM}(r, r') \cdot K_{1}(r') dA(r')$$

$$+ \int_{\partial \mathscr{K}_{2}} [G_{TT, A}^{MM}(r, r') + G_{TT, B}^{MM}(r, r')] \cdot K_{2}(r') dA(r') \quad \text{with } r \in \partial \mathscr{K}_{2}. \quad (5.44)$$

The integral equations derived in this chapter serve as the basis for the computation of the reflection, transmission, and radiation from the junction planes. Their numerical handling is illustrated in Chapter 6, where a number of examples is considered.
5. REFLECTION, TRANSMISSION AND RADIATION AT THE JUNCTION OF
TWO PLANAR OPEN WAVEGUIDES

6.1. DESCRIPTION OF THE CONFIGURATION

In Section 5.2 we have derived integral equations for the tangential electromagnetic
fields in the junction plane of two open waveguide sections with arbitrarily shaped
cross-sections. The present chapter deals with the numerical solution of these integral
equations in the case of planar open waveguides, these being the simplest form of open
waveguide structures. Yet this case exhibits all characteristic features as far as the
reflection, transmission and radiation properties of junctions of open waveguides are
concerned.

The configuration studied in this chapter is shown in Fig. 6.1. Two planar open
waveguide sections, A and B, are joined at the junction plane \( \partial \mathcal{V} \) located at \( z = 0 \).
The unit vector \( \mathbf{\hat{n}} \) normal to \( \partial \mathcal{V} \) is chosen to be \( \mathbf{i}_z \). The geometry, permittivity and
permeability of the configuration only depend on the \( x \)-coordinate. We investigate
the fields that are \( y \)-independent; then \( \partial_y = 0 \). As has been pointed out in Section 3.4,
these fields can be separated into TE- and TM-fields. In the next section it will be
shown that the integral equations for the fields in the junction plane separate
correspondingly. As a consequence, no coupling between TE- and TM-fields in the
junction plane occurs. The integral equations for the TE- and TM-fields in the
junction plane will be derived from the integral equations pertaining to the junction of
two waveguide sections with arbitrary cross-sections. The two-dimensional
Fig. 6.1. Configuration of two planar open waveguide sections with junction plane \( \theta \) at \( z = 0 \) and unit normal vector \( \mathbf{n} = \mathbf{e}_z \).

Integrations occurring in the latter equations reduce, in the present case, to one-dimensional integrations along the cross-section of the junction plane with the plane \( y = 0 \).

6.2. INTEGRAL EQUATIONS FOR THE FIELDS IN THE JUNCTION PLANE OF TWO PLANAR OPEN WAVEGUIDE SECTIONS

Consider a planar waveguide in which the electromagnetic fields are \( y \)-independent. The Green's tensors, as introduced in (2.10)–(2.21), now simplify because the excitations of the Green's states are to be taken \( y \)-independent as well. Thus, instead
of (2.10)-(2.13), the electric current excitation is taken as

\[ \int_{\gamma'} E_{xy} = \int_{\gamma'} \delta(x-x') \delta(z-z'), \]

(6.1)

\[ k \Gamma^{GE} = 0, \]

(6.2)

while the magnetic current excitation is taken as

\[ \int_{\gamma'} H_{xy} = 0, \]

(6.3)

\[ k \Gamma^{GM} = \int_{\gamma'} \delta(x-x') \delta(z-z'). \]

(6.4)

Setting \( \partial_{\gamma'} = 0 \) in (2.14)-(2.17), taking for \( \bar{E}^E \) and \( \bar{E}^M \) the successive unit vectors \( i_{x'}, i_{y'} \) and \( i_{z'} \), and using (2.18)-(2.21), we observe that several elements of the Green's tensors vanish, namely those that give rise to a coupling between TE- and TM-fields. Furthermore, we refer to the symmetry properties established in Appendix C, by which the tensor elements are either even or odd functions of \( z-z' \). Thus it is found that the Green's tensors have the following structure:

\[ G^{EE}(x,x',y,y'), G^{MM}(x,x',z,z'), G^{ME}(x,x',z,z'), G^{EM}(x,x',y,y') \]

\[ \begin{bmatrix} x & y & z \\ x & y & z \\ x & y & z \end{bmatrix} \begin{bmatrix} \text{even} & 0 & \text{odd} \\ 0 & \text{even} & 0 \\ \text{odd} & 0 & \text{even} \end{bmatrix}, \]

(5.5)

\[ \begin{bmatrix} x & y & z \\ x & y & z \end{bmatrix} \begin{bmatrix} 0 & \text{odd} & 0 \\ \text{odd} & 0 & \text{even} \end{bmatrix}. \]

(6.6)

The tensor elements that are odd functions of \( z-z' \) vanish when \( z-z' = 0 \). These properties of the Green's tensors are now used in the integral equations derived in
Section 5.2. Then the integral equations uncouple into a system of equations for TE-fields, which are excited by the incident field components \( \{ E_{\text{y}}^i, H_{\text{y}}^i \} \), and a system of equations for TM-fields, which are excited by the incident field components \( \{ E_{\text{y}}^i, H_{\text{y}}^i \} \). In the two subsections to follow we shall derive the integral equations for the tangential TE-fields and the tangential TM-fields in the junction plane of two planar waveguides by starting from the integral equations of Section 5.2.

6.2.1. Integral equations for TE-fields

For TE-fields, the field components \( \{ E_{\text{y}}, E_{\text{x}}, H_{\text{y}} \} \) are identically zero. Consequently, we have

\[
E = E_{\text{y}}^i y', \quad \text{(8.7)}
\]

\[
H = H_{\text{x}}^i x + H_{\text{y}}^i y', \quad \text{(8.8)}
\]

while the surface current densities in \( \partial \mathcal{V} \) are given by

\[
J_0 = -H_{\text{y}}^i y, \quad \text{when } z = 0, \quad \text{(6.9)}
\]

\[
K_0 = -E_{\text{y}}^i y, \quad \text{when } z = 0. \quad \text{(6.10)}
\]

The two-dimensional integrations over the junction plane \( \partial \mathcal{V} \) reduce to one-dimensional integrations over the range \(-a < x' < a\). Substituting (6.7) - (6.10) into (5.7) and (5.8), and taking into account the structure (6.5) of the Green’s tensors \( \mathcal{E}, \sum_{\text{MM}} \mathcal{M} \), we arrive at the system of integral equations.
\( E_y(x) - E_y^1(x) = - \int_{-\infty}^{\infty} \left[ G_{yy,A}(x,x') - G_{yy,B}(x,x') \right] H_x(x') \, dx', \quad (6.11) \)

\( H_x(x) - H_x^1(x) = - \int_{-\infty}^{\infty} \left[ G_{xx,A}(x,x') - G_{xx,B}(x,x') \right] E_y(x') \, dx', \quad (6.12) \)

in which the arguments \( z = 0, \, z' = 0 \), have been suppressed in the notation. Here, the subscript \( A \) refers to the waveguide section in \( z < 0 \), and the subscript \( B \) refers to the waveguide section in \( z > 0 \). Likewise, the integral equations (5.9) and (5.10) reduce to

\[ E_y^1(x) = \int_{-\infty}^{\infty} \left[ G_{yy,A}(x,x') + G_{yy,B}(x,x') \right] H_x(x') \, dx', \quad (6.13) \]

\[ H_x^1(x) = \int_{-\infty}^{\infty} \left[ G_{xx,A}(x,x') + G_{xx,B}(x,x') \right] E_y(x') \, dx'. \quad (6.14) \]

6.2.2. **Integral equations for TM-fields**

For TM-fields, the field components \( \{ E_y, H_x, H_z \} \) vanish identically. Consequently, we have

\[ E = E_{z=0}, \quad (6.15) \]

\[ H = H_{y=0}. \quad (6.16) \]

while the surface current densities in \( \partial \mathcal{V} \) are given by
\[ J_0 = H_{y}j_{x} \text{ when } z = 0, \] (6.17)

\[ K_{0} = E_{x}i_{y} \text{ when } z = 0. \] (6.18)

As before, the two-dimensional integrations over the junction plane \( \partial \mathcal{V} \) reduce to one-dimensional integrations over the range \(-a < x' < a\). Substituting (6.15)–(6.18) into (5.7) and (5.8), and taking into account the vanishing of the tensor elements as indicated in (6.5), we obtain the system of integral equations

\[ E_{x}'(x) - E_{x}(x) = \int_{-a}^{a} [G_{EE}^{XX}(x,x') - G_{EE}^{XX}(x,x')] H_{y}(x') \, dx', \] (6.19)

\[ H_{y}(x) - H_{y}'(x) = \int_{-a}^{a} [G_{MM}^{YY}(x,x') - G_{MM}^{YY}(x,x')] E_{x}(x') \, dx'. \] (6.20)

Likewise, the integral equations (5.9) and (5.10) reduce to

\[ E_{x}'(x) = -\int_{-a}^{a} [G_{EE}^{XX}(x,x') + G_{EE}^{XX}(x,x')] H_{y}(x') \, dx', \] (6.21)

\[ H_{y}'(x) = -\int_{-a}^{a} [G_{MM}^{YY}(x,x') + G_{MM}^{YY}(x,x')] E_{x}(x') \, dx'. \] (5.22)

6.3. TRANSVERSE FOURIER TRANSFORMATION OF THE INTEGRAL EQUATIONS

In a uniform waveguide section the fields can be represented by modal expansions
that consist of a contribution of the surface-wave modes (discrete part of the spectrum) which show an exponential decay as $|x| \rightarrow \infty$, and a contribution of the radiation modes (continuous part of the spectrum) which show an oscillatory behaviour and an algebraic decay as $|x| \rightarrow \infty$; cf. Section 3.2. In the junction plane, the spatial behaviour of the fields is not known. However, since the entire structure is open, it is expected that the fields in the junction plane have a considerable spatial extent as well as an oscillatory behaviour. To solve the integral equations, the unknown fields in the junction plane have to be expanded in some set of basis functions. In this respect, we shall use $\exp(-jk_x x)$ with $k_x \in \mathbb{R}$, which is a convenient choice in view of the expected behaviour of the fields in the junction plane. Another feature of this set of basis functions is that many integrations can be carried out analytically. Also, those integrations that cannot be handled analytically, can be carried out computationally by means of Fast Fourier Transform techniques (Brigham, 1974; Kong, 1981), be it that the range of integration must be truncated to a bounded one. The expansion in terms of basis functions of the type $\exp(-jk_x x)$ is equivalent to a transverse Fourier Transformation with respect to the variables $x$, $x'$.

Thus for functions $f(x)$, $h(x')$ we introduce the Fourier transforms

$$
\tilde{f}(k_x) = \int_{-a}^{a} \exp(jk_x x) f(x) \, dx, \tag{6.23}
$$

$$
\tilde{h}(k_x') = \int_{-a}^{a} \exp(jk_x' x') h(x') \, dx'. \tag{6.24}
$$

Then inversely, the functions $f(x)$, $h(x')$ are represented by the expressions

$$
f(x) = (2a)^{-1} \int_{-a}^{a} \exp(-jk_x x) \tilde{f}(k_x) \, dk_x, \tag{6.25}
$$
\[ h(x') = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-jk'_x x') \hat{h}(k'_x) \, dk'_x. \]  \hspace{1cm} (6.26)

Integral expressions of the form

\[ f(x) = \int_{-\infty}^{\infty} g(x,x') \hat{h}(x') \, dx', \]

then transform into

\[ \hat{f}(k'_x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{g}(k'_x,k'_x) \hat{h}(k'_x) \, dk'_x, \]

with

\[ \hat{g}(k'_x,k'_x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-jk'_x x - jk'_x x') g(x,x') \, dx \, dx'. \]  \hspace{1cm} (6.27)

In the two subsections to follow we shall present the Fourier-transformed integral equations for \( TE \)- and \( TM \)-fields as functions of the transform variables \( k_x \) and \( k'_x \).

### 6.3.1. Fourier-transformed integral equations for \( TE \)-fields

By transverse Fourier transformation of (6.11) and (6.12) we obtain the system of integral equations

\[ \hat{E}_y(k'_x) - \hat{E}_y^1(k'_x) = -\int_{-\infty}^{\infty} \left[ \hat{G}^{EE,1}_{yy}(k'_x,k'_x) - \hat{G}^{EE}_{yy}(k'_x,k'_x) \right] \hat{H}_x(k'_x) \, dk'_x. \]  \hspace{1cm} (6.28)
\[ \hat{H}_{x}(k_{x}) - \hat{H}_{x}(k_{x}) = - \int_{-\infty}^{\infty} \left[ \hat{G}_{\text{MM}}^{\text{MM}}(k_{x}, k_{x}') \hat{E}_{y}(k_{x}') \right] dk_{x}' \]  \hspace{1cm} (6.29)

Likewise, the integral equations (6.13) and (6.14) transform into

\[ \hat{E}_{y}(k_{x}) = \int_{-\infty}^{\infty} \left[ \hat{G}_{\text{EE}}^{\text{EE}}(k_{x}, k_{x}') + \hat{G}_{\text{EE}}^{\text{MM}}(k_{x}, k_{x}') \right] \hat{H}_{x}(k_{x}') dk_{x}' \]  \hspace{1cm} (6.30)

\[ \hat{H}_{x}(k_{x}) = \int_{-\infty}^{\infty} \left[ \hat{G}_{\text{MM}}^{\text{MM}}(k_{x}, k_{x}') + \hat{G}_{\text{MM}}^{\text{MM}}(k_{x}, k_{x}') \right] \hat{E}_{y}(k_{x}') dk_{x}' \]  \hspace{1cm} (6.31)

It is recalled that the arguments \( z = 0, z' = 0 \), have been suppressed in the notation. The elements of the tensor \( \hat{G}(k_{x}, k_{x}') \) occurring in (6.28)–(6.31) are determined by a numerical integration of the representation

\[ \hat{G}(k_{x}, k_{x}') \big|_{z=z'=0} = \frac{(2\pi)^{-1}}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k_{x}, k_{x}', k_{x}) dk_{x} \]  \hspace{1cm} (6.32)

Here, \( \hat{G} \) denotes the combined axial and transverse Fourier transform of the tensor \( \hat{G} \), defined by

\[ \hat{G}(k_{x}, k_{x}', k_{x}) = \int \int \int \exp(\mathbf{k}_{x}', \mathbf{x} - \mathbf{k}_{x}' \mathbf{x}' + \mathbf{j}_{k_{x}} \mathbf{x}) \hat{G}(x, x', 0) dx \, dk_{x}' \, dx. \]  \hspace{1cm} (6.33)

The required elements of the tensor \( \hat{G} \) are determined in Appendix D. The factor \( (2\pi)^{-1} \) multiplying the integral (6.32) results from the transformation of (6.11)–(6.14) into (6.28)–(6.31) and has been incorporated into \( \hat{G} \).
6.3.2. Fourier-transformed integral equations for TM-fields

By transverse Fourier transformation of (6.19) and (6.20) we obtain the system of integral equations

\[ \hat{E}_x^1(k_x) - \hat{E}_x^0(k_x) = \int \frac{d}{dk_x'} [G_{xx,A}(k_x,k_x') - G_{xx,B}(k_x,k_x')] \hat{H}_y(k_x') \, dk_x', \]  
(6.34)

\[ \hat{H}_y(k_x) - \hat{H}_y^0(k_x) = \int \frac{d}{dk_x'} [G_{y,A}(k_x,k_x') - G_{y,B}(k_x,k_x')] \hat{E}_x(k_x') \, dk_x', \]  
(6.35)

Likewise, the integral equations (6.21) and (6.22) transform into

\[ \hat{E}_x^1(k_x) = -\int \frac{d}{dk_x'} [G_{xx,A}(k_x,k_x') + G_{xx,B}(k_x,k_x')] \hat{H}_y(k_x') \, dk_x', \]  
(6.36)

\[ \hat{H}_y(k_x) = -\int \frac{d}{dk_x'} [G_{yy,A}(k_x,k_x') + G_{yy,B}(k_x,k_x')] \hat{E}_x(k_x') \, dk_x'. \]  
(6.37)

The required elements of the tensor \( \hat{G}(k_x,k_x') \) can be computed in the same manner as indicated in the previous subsection.

6.4. NUMERICAL METHODS EMPLOYED

Numerical results have been obtained for the reflection and transmission at the junction of two planar step-index waveguides. The configuration is shown in Fig. 6.2. The permittivity and the permeability of the outer media of the two waveguide
sections have the same values: \( \varepsilon_2 \) and \( \mu_2 \), respectively. The permittivity, permeability and width of the slab of waveguide section A are denoted by \( \varepsilon_A, \mu_A \) and \( d_A \), respectively; the permittivity, permeability and width of the slab of waveguide section B are denoted by \( \varepsilon_B, \mu_B \) and \( d_B \), respectively. The centre planes of the sections A and B are a distance \( t \cdot d_A \) apart; this distance is called the offset of the waveguides. We also investigate the radiation into free space from a terminating planar step-index waveguide. In the computations, only the case of TE-fields is considered. As excitation we have taken one of the lower-order TE-surface-wave modes of the relevant waveguide. The general method of calculating the transverse Fourier transforms of the field distributions of these modes is discussed in Appendix D, Subsection D.5.

The field distributions in the junction plane can be obtained by solving numerically either (6.30), or (6.31), or the system of equations (6.28), (6.29). Equations (6.30) and (6.31) are linear inhomogeneous integral equations of the first kind. In deriving either of them, one field component has undergone a preferential treatment. This is

![Diagram of two planar open step-index waveguide sections with junction plane \( \partial \Sigma \) at \( z = 0 \) and unit normal vector \( n = \hat{z} \).]

Fig. 6.2. Configuration of two planar open step-index waveguide sections with junction plane \( \partial \Sigma \) at \( z = 0 \) and unit normal vector \( n = \hat{z} \).
important from the point of view of solving the integral equations numerically: the requirement of continuity across the junction plane is automatically and completely satisfied for one tangential field component ($\dot{E}_y$ in (6.30) and $\dot{H}_x$ in (6.31)); after solving the integral equation, the continuity requirement for the other field component will be met only up to the accuracy of the numerical approximation involved. For either of these equations, only a single tensor element, $G_{y y}^{\text{EE}}$ or $G_{x x}^{\text{MM}}$, has to be calculated for both waveguides; on the other hand, only one field component, $\dot{H}_x$ or $\dot{E}_y$, respectively, is obtained. This suffices for the calculation of many quantities of interest, such as the transmission, reflection and radiation at the junction plane. The other field component is to be obtained from either (6.28) or (6.29): once $\dot{H}_x$ is known, (6.28) provides an expression for $\dot{E}_y$ (which contains the tensor elements $G_{y y}^{\text{EE}}$ of both waveguides), and once $\dot{E}_y$ is known, (6.29) provides an expression for $\dot{H}_x$ (which contains the tensor elements $G_{x x}^{\text{MM}}$ of both waveguides).

The equations (6.28)-(6.29) form a system of two linear inhomogeneous integral equations of the second kind. In deriving this system, both field components have been treated in an equal manner. For this system of equations, both tensor elements $G_{y y}^{\text{EE}}$ and $G_{x x}^{\text{MM}}$ have to be calculated for both waveguides.

It is not easy to anticipate which of the equations will yield the most accurate results in a numerical solution procedure. Numerical solution of the integral equation (6.30) or (6.31) gives rise to a smaller system of algebraic equations to be solved and less tensor elements to be computed. However, it is uncertain whether the asymmetry due to the preferential treatment of one of the tangential field components will affect the convergence or the stability of the procedure in an unfavourable manner. Moreover, there is a slight indication that integral equations of the second kind are to be preferred over those of the first kind as far as the stability of the approximating system of algebraic equations is concerned. In view of these considerations, we have...
chosen the system of integral equations (6.28)–(6.29) as the starting point of our numerical solution procedure.

6.4.1. Numerical solution of the integral equations and methods of computation

As has been described in Subsection 6.3.1, the tensor elements $\tilde{G}^{EE}$ and $\tilde{G}^{MM}$ occurring in (6.28) and (6.29) are determined from their axial Fourier transforms $\hat{G}^{EE}$ and $\hat{G}^{MM}$, derived in Appendix D, by a numerical real-axis integration with respect to $k_z$. In Appendix D, Subsection D.4, it is shown that the relevant elements of the tensor $\hat{G}$, considered as functions of the complex variable $k_z$, have simple poles at $k_z = \pm \kappa_n$, $n = 1, \ldots, N$, i.e., at the surface-wave poles; branch points $k_z = \pm \kappa_1$, where $\kappa_1 = \omega (\epsilon_1 \mu_1)^{1/2}$, and associated branch cuts described by $\text{Im}(k_1^2 - k_2^2)^{1/2} = 0$; and simple poles at $k_z = \pm \xi_1$, and $k_z = \pm \xi_2$, where $\xi_1 = (k_1^2 - k_2^2)^{1/2}$, $\xi_2 = (k_2^2 - k_1^2)^{1/2}$, with $\text{Im} (\xi_1) < 0$, and $\text{Im} (\xi_2) < 0$. Furthermore, the elements of the tensor $\hat{G}$ have removable singularities, due to compensating zeros of the numerator and the denominator, at $k_z = \pm (k_2^2 - k_1^2)^{1/2}$ and $k_z = \pm (k_1^2 - k_2^2)^{1/2}$, where $k_2 = \omega (\epsilon_2 \mu_2)^{1/2}$ with $\epsilon_2$ and $\mu_2$ denoting the permittivity and permeability of either waveguide slab.

In order to avoid difficulties in the numerical integration in the vicinity of the poles and the removable singularities, we have taken all media of the configuration slightly lossy. To that end, the relative permittivities of the slab medium and of the outer medium have been assigned a small imaginary part to the amount of $-10^{-3}$. Since the real parts of these permittivities are of the order of one, the introduction of an imaginary part of this magnitude is expected to affect the numerical results up to the relative order $10^{-3}$. For the surface-wave modes, this can be inferred from the influence of an imaginary part of the permittivity on the propagation coefficient (cf. Section 3.4). For the radiation modes, it is observed that the branch cuts in the
complex \( k_x \)-plane for the lossy waveguide are only a small distance apart from those for the corresponding lossless waveguide; differences between the radiation modal fields of the lossy and the lossless waveguide are expected to be accordingly small.

After having carried out the integration with respect to \( k_x \), the values of the tensor elements \( G_{yy}^{EE} \) and \( G_{xx}^{MM} \) are known. From the occurrence of the factor \( \left( k_x^2 - k_1^2 \right)^{-1} \) and (because of the symmetry in \( k_x \) and \( k_x' \)) of the factor \( \left( k_x^2 - k_1^2 \right)^{-1} \) in \( G_{yy}^{EE} \) and \( G_{xx}^{MM} \) (see Appendix D, Subsection D.4), it follows that \( G_{yy}^{EE} \) and \( G_{xx}^{MM} \) contain the factors \( \left( k_x^2 - k_1^2 \right)^{-1/2} \) and \( \left( k_x^2 - k_1^2 \right)^{-1/2} \), stemming from the residues at the poles \( k_x = +k_1 \) and \( k_x = -k_1 \). As a consequence, \( G_{yy}^{EE} \) and \( G_{xx}^{MM} \) are functions of \( k_x \) and \( k_x' \) that are strongly peaked at \( k_x = +\text{Re}(k_1) \) and/or \( k_x' = +\text{Re}(k_1) \). This behaviour is confirmed by the computations: \( G_{yy}^{EE} \) is strongly peaked at \( k_x = +\text{Re}(k_1) \) for any \( k_x' \), and at \( k_x' = +\text{Re}(k_1) \) for any \( k_x \), whereas \( G_{xx}^{MM} \) is strongly peaked only when both \( k_x = +\text{Re}(k_1) \) and \( k_x' = +\text{Re}(k_1) \).

We now turn to the numerical solution of the system of integral equations (6.28)-(6.29). As in Chapter 3 we shall use the method of moments (Kantorowitsch and Krylow, 1938) with properly chosen expansion and weighting functions. For the expansion functions we take triangle functions with apices at the points \( K_j \) (Fig. 6.3), and defined by

\[
\text{Tr}_j(k_x') = \begin{cases} 
\frac{(k_x' - K_{j-1})}{\Delta_j} & \text{when } k_x' \in d_{j-1}, \\
\frac{(K_{j+1} - k_x')}{\Delta_j} & \text{when } k_x' \in d_j, \\
0 & \text{when } k_x' \notin d_{j-1} \cup d_j 
\end{cases} 
\quad (m = 2, \ldots, j-1), 
\]

\[
\text{Tr}_1(k_x') = \begin{cases} 
\frac{(K_2 - k_x')}{\Delta_1} & \text{when } k_x' \in d_{1}, \\
0 & \text{when } k_x' \notin d_{1} 
\end{cases} 
\]

\[
\text{Tr}_j(k_x') = \begin{cases} 
\frac{(k_x' - K_{j-1})}{\Delta_j} & \text{when } k_x' \in d_{j-1}, \\
0 & \text{when } k_x' \notin d_{j-1} 
\end{cases} 
\]

(6.38)
Fig. 6.3. The triangle functions $T_{rj}(k_x)$ with apices at the (non-equidistant) points $K_{j}$ used as expansion functions.

Here, \( J \) is the total number of expansion functions, and \( \Delta_j \) is the length of the subinterval \( d_j = [K_j, K_{j+1}] \). As weighting functions we take delta functions in the apices of the expansion functions, namely, \( \delta(k_x - K_j) \) (method of collocation). It is recalled that the integration kernels in (6.28) and (6.29) exhibit a rapid variation in the vicinity of \( k_x \); \( k'_x = \text{Re}(k_1) \). It is therefore expected that also the field functions \( \hat{E}_x(k_x) \) and \( \hat{H}_x(k_x) \) vary rapidly near \( k_x = \text{Re}(k_1) \). In order to allow for this expected behaviour without introducing more expansion functions than necessary in those ranges where the functions involved show less variation, we choose the distribution of the points \( K_j \) to be more dense near \( k_x = \text{Re}(k_1) \) and less dense away from these values. As a result, we have quite a few expansion functions concentrated in the vicinity of \( k_x = \text{Re}(k_1) \), while away from these regions we use expansion functions with a larger base. In the subsequent computations we shall employ three different subdivisions of the \( k_x, k'_x \)-intervals. The corresponding distributions of grid points \( K_j \)
are listed in Table XIII, columns (1), (2), and (3), and are also shown in Fig. 6.4. The variables $k_x, k_x'$, and the grid points $K_j$ have been normalised with respect to the free-space wave number $k_0 = \omega \sqrt{\varepsilon_0 \mu_0} 1/2$. In the waveguide configurations studied in the next section, the permittivity and the permeability of the outer media are taken as $\varepsilon_1 = \varepsilon_0 (1 - 10^{-3})$, $\mu_1 = \mu_0$, hence, $k_1 = k_0 (1 - 0.5 \times 10^{-3})$ and $\text{Re}(k_1/k_0) = 1$. In principle, the values of $k_x$ and $k_x'$ vary over the whole interval $-\omega < k_x, k_x' < \omega$. In practice, the integration has to be truncated to one over a finite interval $-k_{\text{max}} < k_x < k_{\text{max}}$, while satisfaction of the integral equations is also restricted to the interval $-k_{\text{max}} < k_x < k_{\text{max}}$. In our first set of computations we have taken $k_{\text{max}}/k_0 = 11, k_{\text{max}}/k_0 = 9, \text{and } k_{\text{max}}/k_0 = 8$. The numerical results obtained with these values of $k_{\text{max}}$ did not significantly differ. When $k_{\text{max}}/k_0 = 8$, computations have shown that the results determined from $\vec{E}_y$ are more accurate than the results determined from $\vec{H}_y$. On the other hand, when $k_{\text{max}}/k_0 = 6$, the results determined from $\vec{E}_y$ turn out to be the more accurate ones. The reason for the latter is that, with increasing $|k_x|$, $\vec{E}_y(k_x)$ decreases faster than $\vec{H}_y(k_x)$, as found from an inspection of the computed values of $\vec{E}_y$ and $\vec{H}_y$. The faster decrease of $\vec{E}_y$ may be explained from the behaviour of the integration kernels $\hat{G}_{yy}^{\text{EE}}$ and $\hat{G}_{xx}^{\text{MM}}$ in (6.28) and (6.29). Both kernels decrease rapidly with increasing $|k_x|$ in the range $\text{Re}(k_1) \leq |k_x| < \omega$; however, the decrease of $\hat{G}_{xx}^{\text{MM}}$ is less rapid than that of $\hat{G}_{yy}^{\text{EE}}$ due to an extra multiplicative factor of order $k_x^2$ (cf. (D.61), (D.62) and, for the free space tensors, (D.72) and (D.73)).

In the computations reported in the next section, we have taken $k_{\text{max}}/k_0 = 8$ and $k_{\text{max}}/k_0 = 6$. The following configurations are considered:

(a) the junction of two (mutually different, or identical) planar step-index waveguides with offset of their axes, whereby the case of zero offset is included for mutually different waveguides only (Fig. 6.5);

(b) the terminating planar step-index waveguide radiating into the homogeneous
Table XIII. Subdivisions of the interval $0 \leq |k_x|, |k_x'| \leq k_{\text{max}}$ used in the computations. The total interval $-k_{\text{max}} \leq k_x, k_x' \leq k_{\text{max}}$ is subdivided symmetrically around $k_x, k_x' = 0$.

<table>
<thead>
<tr>
<th>Intervals for</th>
<th>$K_x/k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>k_x</td>
</tr>
<tr>
<td>$[0, 0.5)$</td>
<td>$0, 0.125, 0.25, 0.375$</td>
</tr>
<tr>
<td>$[0.5, 1.5]$</td>
<td>$0.5, 0.7, 0.9, 0.9, 0.9, 0.9, 0.9$</td>
</tr>
<tr>
<td>$\text{Re}(k_x/k_0)$</td>
<td>$0.9975$</td>
</tr>
<tr>
<td>$\text{Re}(k_x'/k_0)$</td>
<td>$1.0025$</td>
</tr>
<tr>
<td>$1.01, 1.02, 1.03, 1.04, 1.05$</td>
<td>$1.01, 1.02, 1.03, 1.04, 1.05$</td>
</tr>
<tr>
<td>$(2.5, k_{\text{max}}/k_0]$</td>
<td>$2.5, 2.5, 2.5, \ldots, k_{\text{max}}/k_0$</td>
</tr>
</tbody>
</table>

Fig. 6.4. Subdivision of the interval $0 \leq k_x \leq k_{\text{max}}$:
- : location of points $K_x/k_0$ as in Table XIII(1);
- : location of points $K_x/k_0$ as in Table XIII(2);
- : location of points $K_x/k_0$ as in Table XIII(3).
half-space \( z > 0 \) with permittivity and permeability \( \varepsilon_1 = \varepsilon_0(1-10^{-3}) \), \( \mu_1 = \mu_0 \) (Fig. 6.6).

In case (b), the tensors \( \hat{\mathbb{G}}^{\text{EE}} \) and \( \hat{\mathbb{G}}^{\text{MM}} \) of the waveguide in \( z > 0 \) are to be replaced by their counterparts for the homogeneous half-space. This half-space can be considered as "free space" with small losses added. For reasons of conciseness, we shall henceforth designate this homogeneous half-space as free space. The calculation of the free-space Green's tensors is discussed in Appendix D, Subsection D.2, and explicit results for \( \hat{\mathbb{G}}^{\text{EE}}_{yy} \) and \( \hat{\mathbb{G}}^{\text{MM}}_{xx} \) are presented in (D.72)–(D.73).

For the configurations considered under (a) and (b), we have numerically solved the system of integral equations (6.28)–(6.29). From the results for \( \hat{E}_y(k_x) \) and \( \hat{H}_x(k_x) \) we have computed the field distributions \( E_y(x) \) and \( H_x(x) \) in the junction plane.

Next, we have determined the reflection coefficients, and, only for the configurations (a), the transmission coefficients by use of the expressions for these coefficients derived in Appendix E. By representing the reflected field in waveguide A by a modal expansion, the reflection coefficient \( R^m_n \) is defined as the ratio of the amplitudes of the reflected \( \text{TE}_m \)-modal field and the incident \( \text{TE}_n \)-modal field. Likewise, by representing the transmitted field in waveguide B by a modal expansion, the transmission coefficient \( T^m_n \) is defined as the ratio of the amplitudes of the transmitted \( \text{TE}_m \)-modal field and the incident \( \text{TE}_n \)-modal field. Here, the amplitude of a modal field is the ratio of that field and the Lorentz–normalised field of the same mode. In addition, for the configurations (b) we have computed the directive gain of the terminating planar waveguide and the forward radiated power, by use of the expressions for these quantities derived in Appendix F. The directive gain is the normalised far-field power radiation pattern in forward direction as a function of the directive angle.
Fig. 6.5. Junction of two planar step-index waveguides with centre planes having a mutual offset \( t \cdot d_A \), incident \( \text{TE}_m \)-surface-wave mode, reflection coefficients \( R_n^m \) and transmission coefficients \( T_n^m \).

Fig. 6.6. Terminating planar step-index waveguide, direction of observation \( i_r \) and angle of observation \( \theta \), incident \( \text{TE}_m \)-surface-wave mode and reflection coefficients \( R_n^m \).
6.4.2. Outline of the computational procedure

We conclude this section with an outline of the computational procedure that we have used.

Firstly, the propagation coefficient \( \kappa_n \) of the incident modal field is computed. A method for the computation of \( \kappa_n \) for a multi-step-index planar waveguide is given in Appendix D, Subsection D.5; for the three-media, layered configurations that we consider, we use a simple analytical eigenvalue equation (Unger, 1977, pp. 93–100) that relates the propagation coefficient to the slab thickness \( k_0 d \) and the slab permittivity \( \varepsilon \). Since the expressions for the modal field distributions in this configuration are simple as well, their Fourier transforms have been calculated analytically. Use of the relevant propagation coefficient in these Fourier transforms yields the expression for the incident field \( \{ \hat{R}_y^{(1)}(k_x), \hat{R}_x^{(1)}(k_x) \} \). These expressions are evaluated at the grid points \( \{ K_i \} \) (see Table XIII for the distribution of the grid points).

Secondly, the values of the Green's tensor elements \( \hat{G}_{xy}^{EE} \) and \( \hat{G}_{xx}^{MM} \) at the grid points \( k_x = K_i, k_x' = K_j, i, j = 1, \ldots, J \) are determined; this is done as follows. The expressions for \( \hat{G}_{xy}^{EE}(k_x, k_x', k_x) \) and \( \hat{G}_{xx}^{MM}(k_x, k_x', k_x) \) derived in Appendix D are simplified by adapting them to our three-media configuration. Next, the inverse Fourier Transformation with respect to \( z \) is carried out for \( z = z' = 0 \). This involves integrals of the type (cf. (6.32))

\[
\int_{-\infty}^{\infty} \hat{G}(k_x, k_x', k_x) \, dk_x.
\]

These are evaluated numerically by means of a standard integration routine using higher-order Gaussian integration formulae; the latter have been reported to be economical as far as evaluations of the integrand are concerned. The integration
interval for \( k_x \) has been restricted to \( |k_x/k_0| \leq 10 \), since extension of the integration interval to \( |k_x/k_0| \leq 20 \) did not significantly alter the values of the tensor elements \( G_{y}^{EE} \) and \( G_{x}^{MM} \) obtained.

The third step consists of the construction and the solution of the system of linear algebraic equations from which the coefficients in the expansions of the unknown field quantities are to be determined. Using the expansion functions (6.38), we write

\[
\{ \hat{E}_{y,j} \} \{ \hat{H}_{x,j} \} (k_x') = \sum_{j=1}^{J} \{ \hat{E}_{y,j} \} \{ \hat{H}_{x,j} \} \text{Tr}_{j}(k_x'),
\]

(6.39)

then \( \hat{E}_{y,j} = \hat{E}_{y}(K_j) \) and \( \hat{H}_{x,j} = \hat{H}_{x}(K_j) \). Similar expansions are used for the incident field. Inserting these expansions into the system of integral equations (6.28)–(6.29), multiplying their left- and right-hand sides by the weighting functions \( \delta(k_x'-K_j) \), \( i = 1, \ldots, J \), and integrating with respect to \( k_x' \), we obtain

\[
\hat{E}_{y,i} - \hat{E}_{y,j} = \sum_{j=1}^{J} \int_{k_{max}}^{k_{max}} \{ G_{y}^{EE} (K_j,k_x') - G_{y}^{EE} (K_i,k_x') \} \hat{H}_{x,j} \text{Tr}_{j}(k_x') \, dk_x',
\]

(6.40)

\[
i = 1, \ldots, J,
\]

\[
\hat{H}_{x,i} - \hat{H}_{x,j} = \sum_{j=1}^{J} \int_{k_{max}}^{k_{max}} \{ G_{x}^{MM} (K_j,k_x') - G_{x}^{MM} (K_i,k_x') \} \hat{E}_{y,j} \text{Tr}_{j}(k_x') \, dk_x',
\]

(6.41)

\[
i = 1, \ldots, J.
\]

The integrals in (6.40), (6.41) are approximately evaluated as

\[
\int_{k_{max}}^{k_{max}} G(K_i,k_x') \text{Tr}_{j}(k_x') \, dk_x' \approx \int_{k_{max}}^{k_{max}} G(K_i,K_j) \int_{k_{max}}^{k_{max}} \text{Tr}_{j}(k_x') \, dk_x'.
\]
Then we are led to the following inhomogeneous system of linear algebraic equations for \( \hat{E}_{y,i} \) and \( \hat{H}_{x,i} \):

\[
\begin{align*}
\hat{E}_{y,i} - \hat{E}_{y,i}^i &= - \sum_{j=1}^{J} \left[ G_{y,y,x}^{EE}(K_{i},K_{j}) - G_{y,y,x}^{EE}(K_{i},K_{j}) \right] O_j \hat{H}_{x,j}^i, \\
i &= 1, \ldots, J, \\
(6.42)
\end{align*}
\]

\[
\begin{align*}
\hat{H}_{x,i} - \hat{H}_{x,i}^i &= - \sum_{j=1}^{J} \left[ G_{x,x,x}^{MM}(K_{i},K_{j}) - G_{x,x,x}^{MM}(K_{i},K_{j}) \right] O_j \hat{E}_{y,j}^i, \\
i &= 1, \ldots, J, \\
(6.43)
\end{align*}
\]

in which

\[
O_j = \int_{k_{\text{max}}}^{k_{\text{max}}} \frac{\gamma_j}{\Delta_j} \, dk_k = \begin{cases} 
(\Delta_{j-1} + \Delta_j)/2 & \text{when } j = 2, \ldots, J-1, \\
\Delta_j/2 & \text{when } j = 1, \\
\Delta_{J-1}/2 & \text{when } j = J.
\end{cases}
(6.44)
\]

Thus we have obtained a system of 23 algebraic equations for the unknown (complex) field components \( \{\hat{E}_{y,i}, \hat{H}_{x,i}\} \). By separating the field coefficients \( \hat{E}_{y,i}, \hat{H}_{x,i} \) and the tensor coefficients \( G(K_{i},K_{j}) \) into real and imaginary parts, a system of 4J inhomogeneous, linear, real, algebraic equations results. The latter system is solved by means of the NAG-library numerical routine F04ATA (Numerical Algorithms Group, 1981, Vol. 3).

After having determined the expansion coefficients \( \{\hat{E}_{y,i}, \hat{H}_{x,i}\} \), we compute the field distributions \( \{E_y, H_x\}(x) \) in the junction plane. This is done by means of a Fast Fourier Transform technique (Brigham, 1974). Since this technique requires equidistant sampling points of the quantity to be transformed, the values of \( \hat{E}_y(k_x) \) and \( \hat{H}_x(k_x) \) are needed for arguments \( k_x \) in between the chosen grid points. An obvious way to compute these values is by linear interpolation, since this is suggested
by the expansion of \( \hat{\mathbf{E}}_y \) and \( \hat{\mathbf{H}}_x \) in (6.39). We have checked the accuracy by computing the (known) incident field \( \langle \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{E}}_y^{1}(x) \rangle \) in the junction plane from \( \{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(K) \} \) with the Fast Fourier Transform method and a subsequent linear interpolation in between the grid points. Next, we have compared these computed field values with those determined from the analytical expressions for \( \{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(x) \} \) in the junction plane. It was found that the linear interpolation gave rise to a clearly noticeable (\( \approx 5\% \)) sinusoidal error superimposed on the actual field values. The periodicity of this error was found to correspond with the periodicity of the Fourier transform of a triangle function with base \( k_x/k_0 = 1 \), i.e., the base of the expansion functions \( T_{k}(k_x) \) with apices in the interval \( 1.5 k_0 < |k_x| < k_{\text{max}} \) (cf. Table XIII). For this reason, a different interpolation method has been used, viz. a cubic spline interpolation using the NAG--library routines E01BAA and E02BBA (Numerical Algorithms Group, 1981, Vol. 1). The latter interpolation method proved to give sufficiently accurate results for the incident field when 512 sample values were used. However, since the field distribution \( \hat{\mathbf{E}}_y^{1}(k_x) \) varies rapidly near \( k_x = \pm \text{Re}(k_1) \) (due to a square root singularity at \( k_x = \pm k_1 \) in the complex plane), the interpolation by cubic splines cannot be used in the vicinity of \( k_x = \pm \text{Re}(k_1) \). For that reason we have applied a linear interpolation in between the values \( \{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(K) \} \) in the intervals \( (1-\alpha)\text{Re}(k_1) < |k_x| < (1+\alpha)\text{Re}(k_1) \), where \( \alpha \) has been chosen as 0.1 or, in most cases, as 0.05. When \( M \) samples of \( \{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(k_x) \} \) are used in the Fast Fourier Transform method, and \( \Delta k \) is the length of the \( k_x \)-interval between two successive sample points, the Fast Fourier Transform method yields the values of \( \{ \hat{\mathbf{E}}_y, \hat{\mathbf{H}}_x(x) \} \) at \( M \) discrete positions \( x = x_p = 2\pi(p - M/2)/(M \cdot \Delta k) \), \( p = 0, \ldots, M-1 \). Then the values of \( \hat{\mathbf{E}}_y(x) \) and \( \hat{\mathbf{H}}_x(x) \) at an arbitrary position in the junction plane are obtained from the values \( \hat{\mathbf{E}}_y(x_p) \) and \( \hat{\mathbf{H}}_x(x_p) \) at the positions \( x_p \) by using the formula

\[
\{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(x) \} = \sum_{p=0}^{M-1} \{ \hat{\mathbf{E}}_y^{1}, \hat{\mathbf{H}}_x^{1}(x_p) \} \frac{\sin \left( \frac{k_x \cdot \Delta k (x-x_p)}{2M \cdot \Delta k} \right)}{\frac{k_x \cdot \Delta k (x-x_p)}{2M \cdot \Delta k}}.
\]  

(6.45)
For the actual computation of $E_y(x)$ and $H_x(x)$ in the junction plane with the Fast Fourier Transform method, $M = 2048$ sample points have been used.

The expressions for the reflection and transmission coefficients are presented in Appendix E; see (E.6)–(E.9) and (E.12)–(E.14). The power flow $P^f$ in the junction plane associated with a reflected surface-wave mode is easily obtained from

$$\frac{P^r}{P^i} = |R|^2,$$  \hspace{1cm} (6.46)

in which $P^i$ is the power flow in the junction plane associated with the incident TE-mode, given by

$$P^i = -\frac{1}{2} \int_{-\infty}^{\infty} \text{Re}[E_y^i(x) \, H_x^{i*}(x)] \, dx,$$  \hspace{1cm} (5.47)

and $R$ is the relevant reflection coefficient. Henceforth, $P^i$ and $P^r$ are called incident power and reflected power, respectively. By means of Parseval's theorem (E.15), alternative integral expressions are derived which permit a calculation of the reflection and transmission coefficients and of the power $P^i$ directly from $E_{y'}$, $H_x$ and $E_{y'}$, $H_x^{i*}$. The integrations involved are carried out numerically by means of the integration routine D01GAA (Numerical Algorithms Group, 1981, Vol. 1) over the interval $-k_{\text{max}} \leq k_x \leq k_{\text{max}}$.

For the terminating planar waveguide (Fig. 6.6) we have computed $P^f/P^i$, i.e., the power $P^f$ radiated from the terminal plane into free space as a fraction of the incident power $P^i$, and the directive gain $D(\theta)$ of the terminal plane for $-\pi/2 \leq \theta \leq \pi/2$. Henceforth, $P^f$ will be called the forward radiated power. The computations are based on the following approximate expressions for $D(\theta)$ and $P^f/P^i$, derived in Appendix F.
(cf. (F.31) and (F.30)):

\[
D(\theta) = \left| \text{Re}(k_1) \text{Re}((\epsilon_1/\mu_1)^{1/2}) | \cos(\theta) \hat{E}_y(k_3) - (\mu_1/\epsilon_1)^{1/2} \hat{H}_y(k_3) |^2 / (8F^4) \right|
\]

\[
+ \text{Re}(k_1) \text{Re}((\epsilon_1/\mu_1)^{1/2}) | \cos^2(\theta) \hat{E}_y(k_3) |^2 / (2F^3),
\]

where \( k_0 = \text{Re}(k_1) \sin(\theta) \), and

\[
P_f/P_i = (2\pi)^{-1} \int_{-\pi/2}^{\pi/2} D(\theta) \, d\theta.
\]

The computation of \( D(\theta) \) for \( \theta = \theta_0 \), with \( \text{Re}(k_1) \sin(\theta) = K_1 \), simply amounts to a substitution of the values \( \hat{E}_y(K_1) \) and/or \( \hat{H}_y(K_1) \) in the appropriate formulae.

For an arbitrary angle \( \theta \neq \theta_0 \), \( D(\theta) \) is determined by a cubic spline interpolation based on the values \( D(\theta_0) \). The forward radiated power \( P_f \) is computed by means of the NAG-library integration routine E02BDA (Numerical Algorithms Group, 1991, Vol. 1), which employs again the cubic splines that underlie the calculation of \( D(\theta) \) by interpolation.

6.5. NUMERICAL RESULTS

In this section we report on the computations we have carried out on the transmission, reflection and radiation at the junction plane of two planar step-index waveguides.
The basic configuration is shown in Fig. 6.7. Two planar step-index waveguides are joined at the junction plane \( z = 0 \). They are surrounded by a homogeneous medium with permittivity \( \epsilon_1 = \epsilon_0(1 - 10^{-3}) \), and permeability \( \mu_1 = \mu_0 \). The permittivity and permeability of the waveguide slabs are given by \( \epsilon_A \), \( \epsilon_B \) and \( \mu_A = \mu_B = \mu_0 \), respectively. The waveguides may differ in width and in permittivity (and hence in the number of supported surface-wave modes); their centre planes (henceforth referred to as axes) may be mutually offset. Also, the waveguide in \( z > 0 \) may be absent, in which case the waveguide in \( z < 0 \) (terminating waveguide) radiates into a homogeneous half-space with constitutive parameters \( \epsilon_1 = \epsilon_0(1 - 10^{-3}) \), \( \mu_1 = \mu_0 \) (free space).

The incident field is one of the TE-surface-wave modal fields. We have computed the field distributions \( E_y(x) \) and \( H_z(x) \) in the junction plane, the reflection coefficients and the transmission coefficients. For the terminating waveguide, we have computed the directive gain and the forward radiated power (instead of the transmission coefficients).

![Diagram](image)

**Fig. 6.7.** Configuration of (a) the junction of two planar step-index waveguides; (b) the terminating planar step-index waveguide radiating into free space.
First, we have carried out computations on a configuration studied before by Rossi (1978) and we have compared his results with ours. Next, we have studied the dependence of the reflection, transmission and radiation on the parameters: permittivity values, difference in permittivities, offset of the axes, number of supported surface-wave modes, and frequency of operation. Detailed numerical results are presented in the following Subsections 6.5.1 – 6.5.7. In each subsection we start with a description of the waveguide configuration considered. The various parameters for the configuration are listed in a table with the following column headings:

- incident field: specification of the incident $TE_{n}$-mode and of the waveguide of incidence;
- $f$: frequency of operation, $f = \omega/(2\pi)$;
- $k_0$: free-space wavenumber, $k_0 = \omega/(\epsilon_0^0 c_0)$;
- $\lambda_0$: free-space wavelength, $\lambda_0 = 2\pi/k_0$;
- $d_A$, $d_B$: widths of the waveguide slabs A and B;
- $\epsilon_{r,A}$, $\epsilon_{r,B}$: relative permittivities of the waveguide slabs A and B;
- $\epsilon_{r,A} = \epsilon_A/\epsilon_0$, $\epsilon_{r,B} = \epsilon_B/\epsilon_0$;
- $\kappa^E/k_0$: propagation coefficient of the incident $TE_{n}$-mode.

In the plots of the field distributions, the positions of the waveguides present in the configuration, are indicated by dashed vertical lines.

In the final subsection, some data on computation times and storage requirements are collected.

6.5.1. On-axis junction of two waveguides with different widths and equal permittivities $\epsilon_0 = 5 \times 10^{-3}$.

In order to check our method and its numerical implementation, we consider the
on–axis junction of two step–index waveguides with different widths and equal permittivities (Fig. 6.8). The values of the parameters for this configuration are given in Table XIV. Waveguide A only supports the TE_0–mode, whereas waveguide B supports the TE_0– and TE_1–modes. As incident fields we take the TE_0–mode of either waveguide A (Table XIV(1)) or waveguide B (Table XIV(2)); the propagation coefficients of these modes are given in Table XIV. The same configuration, but with the corresponding real permittivity values \( \epsilon_{r,A} = \epsilon_{r,B} = 5 \) and \( \epsilon_{r,1} = 1 \), has been studied by Rossi (1978).

In Fig. 6.9 we present the field distributions \( E_y(x) \) and \( H_x(x) \) in the junction plane that are generated by a TE_0–mode incident in waveguide A (Table XIV(1)), and by a TE_0–mode incident in waveguide B (Table XIV(2)). In Fig. 6.9 we have also plotted the Lorentz–normalised TE_0–modal fields incident in waveguides A and B. The fields \( E_y(x) \) and \( H_x(x) \) have been numerically determined from their transverse Fourier transforms \( E_y(k_x) \) and \( H_x(k_x) \), by the method described in Subsection 6.4.2.

![Fig. 6.8. On–axis junction of two planar waveguides with different widths. Parameters are listed in Table XIV.](image)

| Table XIV. Parameters for the configuration of Fig. 6.8 (SI–Units). |
|-------------------|---|---|---|---|---|---|
| Incident Field | \( f \) | \( k_0 \) | \( \lambda_0 \) | \( d_A \) | \( d_B \) | \( \epsilon_{r,A} \) | \( \epsilon_{r,B} \) | \( \lambda_0/k_0 \) |
| (1) TE_0 | 1.5989 \times 10^{13} | 4.192 \times 10^2 | 1.5712 \times 10^{-6} | 10^{-6} | 5 \times 10^{-6} | 5 \times 10^{-2} | 1.2355-4.0433 \times 10^{-3}j |
| (2) TE_0 | 1.9398 \times 10^{13} | 4.192 \times 10^2 | 1.5712 \times 10^{-6} | 10^{-6} | 5 \times 10^{-6} | 5 \times 10^{-2} | 1.8946-2.8192 \times 10^{-3}j |
Fig. 5.9. Transverse field distributions $E_x(x)$ and $Z_0 H_x(x)$ ($Z_0 = \left(\rho_0/\epsilon_0\right)^{1/2}$) in the junction plane of two planar waveguides with parameters as listed in Table XIV:

- --- : junction plane field for a TE$_0$-mode incident in waveguide A;
- ---- : junction plane field for a TE$_0$-mode incident in waveguide B;
- ----- : TE$_0$-modal field incident in waveguide A;
- ····· : TE$_0$-modal field incident in waveguide B.

Our results agree quite well with those of Rozzi (1978), who has only plots of $|E_y(x)|$.

Further numerical results have been obtained for the reflection coefficient $R_0^0$ and the transmission coefficient $T_0^0$ in the cases (1) and (2) of incidence in waveguide A and B, respectively. For the sake of distinctness, these coefficients are denoted by $R_0^{AB}$ and $T_0^{AB}$.
Table XV. Reflection coefficients $R_{0}^{AB}$, $R_{0}^{BA}$ and transmission coefficients $T_{0}^{AB}$, $T_{0}^{BA}$ for the on-axis junction of two planar waveguides (Fig. 6.6) with parameters as listed in Table XIV, obtained by Rosai (1978) and by our method.

Our results marked by E, H and EH, were determined from $\hat{E}_y$ in the junction plane, from $\hat{H}_x$ in the junction plane, and from both $\hat{E}_y$ and $\hat{H}_x$ in the junction plane, respectively. The indications (1), (2), (3) refer to the subdivision of the truncated integration interval as presented in Table XIII, columns (1), (2), (3).

<table>
<thead>
<tr>
<th></th>
<th>$R_{0}^{AB}$</th>
<th>$T_{0}^{AE}$</th>
<th>$T_{0}^{RA}$</th>
<th>$R_{0}^{BA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rosai</td>
<td>$-0.3031 + 0.0171j$</td>
<td>$0.0416 - 0.0071j$</td>
<td>$0.3472 + 0.0186j$</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>$-0.4376 - 0.0004j$</td>
<td>$0.8669 + 0.0465j$</td>
<td>$0.3848 + 0.0123j$</td>
<td>$0.3723 - 0.0000j$</td>
</tr>
<tr>
<td>(1) H</td>
<td>$0.0172 + 0.0272j$</td>
<td>$0.3271 + 0.0160j$</td>
<td>$0.2944 - 0.0174j$</td>
<td>$0.2634 + 0.0298j$</td>
</tr>
<tr>
<td>H</td>
<td>$-0.3203 + 0.0010j$</td>
<td>$0.8986 - 0.0131j$</td>
<td>$0.8156 - 0.0028j$</td>
<td>$0.2814 + 0.0001j$</td>
</tr>
<tr>
<td>E</td>
<td>$-0.7137 + 0.0183j$</td>
<td>$0.3021 - 0.0171j$</td>
<td>$0.3603 + 0.0033j$</td>
<td>$0.2754 + 0.0027j$</td>
</tr>
<tr>
<td>(2) E</td>
<td>$-0.3018 + 0.0394j$</td>
<td>$0.9441 - 0.0000j$</td>
<td>$0.8387 - 0.0100j$</td>
<td>$0.2439 + 0.0500j$</td>
</tr>
<tr>
<td>H</td>
<td>$-0.1111 + 0.0183j$</td>
<td>$0.8969 - 0.0133j$</td>
<td>$0.9485 - 0.0028j$</td>
<td>$0.2834 + 0.0028j$</td>
</tr>
<tr>
<td>E</td>
<td>$-0.7121 + 0.0137j$</td>
<td>$0.3597 - 0.0182j$</td>
<td>$0.3627 + 0.0022j$</td>
<td>$0.2702 + 0.0076j$</td>
</tr>
<tr>
<td>(3) H</td>
<td>$-0.3034 + 0.0201j$</td>
<td>$0.9441 - 0.0006j$</td>
<td>$0.8431 - 0.0066j$</td>
<td>$0.2437 + 0.0015j$</td>
</tr>
<tr>
<td>H</td>
<td>$-0.3055 + 0.0130j$</td>
<td>$0.9441 - 0.0127j$</td>
<td>$0.9485 - 0.0085j$</td>
<td>$0.2564 + 0.0025j$</td>
</tr>
</tbody>
</table>

In case (1), and by $R_{0}^{BA}$, $T_{0}^{BA}$ in case (2). In Table XV our results for the reflection and transmission coefficients are compared with those of Rosai (1978). In our computations we have set $k_{max}/k_0 = 8$, and we have employed three different subdivisions of the truncated $k_x k_x'$-interval, as listed in Table XIII, columns (1), (2), (3). Notice that these subdivisions differ in the number of grid points in the interval $0.99 k_0 \leq |k_x|,|k_x'| \leq 1.01 k_0$, where especially $G_{y y}$ is sharply peaked. Our numerical solution of the integral equations (6.28)–(6.29) yields both the electric field $\hat{E}_y(k_x)$ and the magnetic field $\hat{H}_x(k_x)$ in the junction plane. After that, the reflection and transmission coefficients can be calculated from $\hat{E}_y$ only, or from $\hat{H}_x$ only, or
from both \( E_y \) and \( H_x \), as described in Appendix E. From Table XV we observe that the calculations based on \( H_x \) yield the most accurate values for the reflection and transmission coefficients as compared with Roesl's results. Furthermore, the accuracy of the calculation based on \( E_y \) benefits most from refining the subdivision of the interval \( 0.99 \leq |k_x|, |k_y| \leq 1.01 \) \( k_0 \). Because of the reciprocity property, the transmission coefficients should satisfy the relation \( T_0^{AB} = T_0^{BA} \). This relation is satisfied best by the transmission coefficients calculated from \( H_x \). Note that the differences between Roesl's results and ours may be partly due to the small non-vanishing imaginary part that we have assigned to the permittivities.

6.5.2. On-axis junction of two waveguides with different widths and equal permittivities \( \varepsilon_{q} = 2.25 \times 10^{-3} \).

Next, we compute the field distributions in the junction plane and the reflection and transmission coefficients for another on–axis junction of two step–index waveguides with different widths and equal permittivities (Fig. 6.10). The parameters for this

Fig. 6.10. On–axis junction of two planar waveguides with different widths. Parameters are listed in Table XVI.

| Table XVI. Parameters for the configuration of Fig. 6.10 (SI-units). |
|------------------|------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| Incident Field   | \( k_0 \)         | \( \lambda_0 \)     | \( \varepsilon_A \) | \( \varepsilon_B \) | \( \varepsilon_{p,A}^{\hbar}\varepsilon_{p,B} \) | \( \varepsilon_{p,C} \) |
| \( TE_{0,A} \)   | \( 6.675 \times 10^{-13} \) | \( 1.994 \times 10^{-6} \) | \( 3.188 \times 10^{-8} \) | \( 10^{-6} \) | \( 1.4 \times 10^{-8} \) | \( 2.95 \times 10^{-12} \) |
| \( TE_{0,B} \)   | \( 9.465 \times 10^{-13} \) | \( 1.994 \times 10^{-6} \) | \( 3.188 \times 10^{-8} \) | \( 10^{-6} \) | \( 1.4 \times 10^{-8} \) | \( 2.95 \times 10^{-12} \) |
configuration are listed in Table XVI. Each of the waveguides supports only the TE₀-mode. As incident fields we take the TE₀-mode of either waveguide A (Table XVI(1)) or waveguide B (Table XVI(2)). In the computations, the interval for \(k_x, k_y\) has been truncated to \([-k_{\text{max}}, k_{\text{max}}]\) with \(k_{\text{max}}/k_0 = 6\). The subdivision of this interval is taken in compliance with Table XIII, column (1). The field distributions \(E_y(x)\) and \(H_x(x)\) in the junction plane are shown in Fig. 6.11, together with the

Fig. 6.11. Transverse field distributions \(E_y(x)\) and \(Z_0H_x(x)\) (\(Z_0 = (\mu_0/\varepsilon_0)^{1/2}\)) in the junction plane of two planar waveguides with parameters as listed in Table XVI;

--- junction plane field for a TE₀-mode incident in waveguide A;
--- --- junction plane field for a TE₀-mode incident in waveguide B;
--- --- --- TE₀-modal field incident in waveguide A;
--- --- --- --- TE₀-modal field incident in waveguide B.
Lorentz–normalised $\text{TE}_0$–modal fields incident in waveguides A and B. It is seen that the total field hardly differs from the incident field.

Numerical results for the reflection and transmission coefficients are presented in Table XVII. As could be expected in view of the relatively small differences between waveguide A and waveguide B, most of the incident field is transmitted, and only a small fraction is reflected and radiated.

6.5.3. Offset junction of two identical two–modeed waveguides, and radiation from a terminating waveguide

Our next example deals with the reflection and transmission at the offset junction of two identical waveguides for various values of the offset $t \cdot d$ (Fig. 6.12a). We also consider the reflection and radiation into free space of one terminating waveguide (Fig. 6.12b). The parameters for the configuration are listed in Table XVIII. The waveguides under consideration support the $\text{TE}_0$– and $\text{TE}_1$–modes. As incident fields

Table XVII. Reflection coefficients $R_{0A}^{AB}$, $R_{0B}^{BA}$ and transmission coefficients $T_{0A}^{AB}$, $T_{0B}^{BA}$ for the on–axis junction of two planar waveguides (Fig. 6.10) with parameters as listed in Table XVI. Results marked by E, H and EH, were determined from $\vec{E}_y$ in the junction plane, from $\vec{H}_x$ in the junction plane, and from both $\vec{E}_y$ and $\vec{H}_x$ in the junction plane, respectively. The subdivision of the truncated $k_x \cdot k_z$–interval is that of Table XIII, column (1), with $k_{\max \, x} / k_0 = 6$.

<table>
<thead>
<tr>
<th></th>
<th>$R_{0A}^{AB}$</th>
<th>$T_{0A}^{AB}$</th>
<th>$R_{0B}^{BA}$</th>
<th>$T_{0B}^{BA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>$-2.05 \cdot 10^{-2} + 1.00 \cdot 10^{-3} j$</td>
<td>$1.00 + 1.53 \cdot 10^{-3} j$</td>
<td>$1.00 + 2.46 \cdot 10^{-3} j$</td>
<td>$2.70 \cdot 10^{-2} + 1.35 \cdot 10^{-3} j$</td>
</tr>
<tr>
<td>H</td>
<td>$-2.50 \cdot 10^{-2} + 1.06 \cdot 10^{-3} j$</td>
<td>$1.00 + 1.53 \cdot 10^{-3} j$</td>
<td>$1.00 + 2.52 \cdot 10^{-3} j$</td>
<td>$2.31 \cdot 10^{-2} + 1.72 \cdot 10^{-3} j$</td>
</tr>
<tr>
<td>EH</td>
<td>$-2.48 \cdot 10^{-2} + 1.29 \cdot 10^{-3} j$</td>
<td>$1.00 + 8.57 \cdot 10^{-3} j$</td>
<td>$1.00 + 2.74 \cdot 10^{-3} j$</td>
<td>$2.50 \cdot 10^{-2} + 1.77 \cdot 10^{-3} j$</td>
</tr>
</tbody>
</table>
Fig. 6.12. (a) Offset junction of two identical two–moded planar waveguides; (b) terminating planar waveguide radiating into free space. Parameters are listed in Table XVIII.

<table>
<thead>
<tr>
<th>Incident field</th>
<th>( t )</th>
<th>( \kappa_0 )</th>
<th>( \lambda_0 )</th>
<th>( d_A=d_B )</th>
<th>( t_A=t_B )</th>
<th>( \kappa^r/\kappa_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE(_{0,A})</td>
<td>( 1.061-10^{18} )</td>
<td>( 4-10^5 )</td>
<td>( 1.971-10^{-3} )</td>
<td>( 2-10^{-6} )</td>
<td>( 5-10^{-6} )</td>
<td>( 1.9845-2.2320-10^{-4} )</td>
</tr>
<tr>
<td>TE(_{1,A})</td>
<td>( 1.809-10^{13} )</td>
<td>( 4-10^5 )</td>
<td>( 1.971-10^{-3} )</td>
<td>( 5-10^{-4} )</td>
<td>( 3-10^{-4} )</td>
<td>( 1.1869-4.3213-10^{-4} )</td>
</tr>
</tbody>
</table>

we take the TE\(_{0}\)–mode (Table XVIII(1)) and the TE\(_{1}\)–mode (Table XVIII(2)), incident in waveguide A. In the computations we have set \( \kappa_{\text{max}}/\kappa_0 = 8 \), and the subdivision of the truncated \( k, k' \)–interval is that of Table XIII, column (3).

First, we discuss the numerical results in the case of excitation by a TE\(_{0}\)–mode. In Fig. 6.13 we have plotted the field distributions of the incident Lorentz–normalised TE\(_{0}\)–mode (which is identical to the junction plane field in case \( t = 0 \), i.e., the on–axis junction of two identical waveguides), the field distributions \( E_y(x) \) and \( H_x(x) \) in the junction plane for offset parameter values \( t = 0.75, 1.5, 5 \), and the field distributions in the terminal plane \( (t = w) \) of the terminating waveguide under consideration.

In Fig. 6.14, the reflection coefficients \( R_{0}^0 \) and \( R_{0}^1 \) (for reflected TE\(_{0}\)– and
Fig. 6.13. Transverse field distributions $E_y(x)$ and $Z_0 H_x(x)$ in the junction plane of two planar waveguides at offset $t \cdot d$, and with parameters as listed in Table XVIII(1):

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.75$;
(c) $t = 1.5$; (d) $t = 5$; (e) $t = \infty$ (terminating waveguide radiating into free space).
The transmission coefficients $T_0^0$ and $T_0^1$ (for transmitted TE$_0^-$ and TE$_1^-$ modes, respectively) are presented as functions of the offset parameter $t$. We have also indicated the value of the reflection coefficient of the terminating planar waveguide, which is equal to the limiting value $R_0^0(t)$ of the reflection coefficient $R_0^0$ as $t \to \infty$. In view of the symmetry of the configuration as $t \to \infty$, $R_0^1(t)$ should vanish, numerical values for $R_0^1(t)$ were of the order of $10^{-4}$ at most. The reflection coefficients for the terminating waveguide have been computed by taking for $\hat{G}_d$ in (6.28) and (6.29) the Green's tensors of a homogeneous medium with constitutive parameters $\epsilon_1 = \epsilon_0 (1 - 10^{-3})$, $\mu_1 = \mu_0$. For $t > 2$, $R_0^0$ is oscillating around the limiting value $R_0^0(\infty)$ with decreasing amplitude for increasing $t$.

Next, we present numerical results for the case of excitation by a TE$_1^-$ mode. In Fig. 6.15, the field distributions of the incident Lorentz-normalised TE$_1^-$ mode (which is

![Fig. 6.14: Reflection coefficients $R_0^0$, $R_0^1$, and transmission coefficients $T_0^0$, $T_0^1$ of the incident TE$_0^-$ mode at the junction plane of two planar waveguides with parameters as listed in Table XVIII(1), as functions of the offset parameter $t$: $R_0^0$, $T_0^0$, $R_0^1$, $T_0^1$. The limiting value $R_0^0(\infty)$ of $R_0^0$ as $t \to \infty$ is indicated.](image)
Fig. 6.15 Transverse field distributions $E_y(x)$ and $Z_0 H_y(x)$ in the junction plane of two planar waveguides at offset $t \cdot d$, and with parameters as listed in Table XVIII(2);

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.75$; (c) $t = 1.5$;
(d) $t = 5$; (e) $t = \infty$ (terminating waveguide radiating into free space).
identical to the junction plane field in case $t = 0$) have been plotted, together with
the field distributions $E_x(x)$ and $H_x(x)$ in the junction plane for offset parameter
values $t = 0.75, 1.5, 5$, and the field distributions in the terminal plane ($t = \infty$) of the
terminating waveguide under consideration.

In Fig. 6.16, the reflection coefficients $R_1^0, R_1^1$, and the transmission coefficients $T_1^0, T_1^1$ are presented as functions of the offset parameter $t$. Again, the limiting value $R_1^1(a)$ of $R_1^1$ as $t \to a$ is indicated. Now, for symmetry reasons, $R_1^0(a)$ should vanish (numerical values were of the order of $10^{-4}$).

In Fig. 6.17 we have plotted, as an example, the spectral field distributions $	ilde{E}_x(k_x)$ and $	ilde{H}_x(k_x)$ for $t = 0$ (identical to the spectral field distributions of the incident

Fig. 6.16. Reflection coefficients $R_1^1, R_1^0$ and transmission coefficients $T_1^1, T_1^0$ of the incident $TE_1$-mode at the junction plane of two planar waveguides with parameters as listed in Table XVIII(2), as functions of the offset parameter $t$:

$---$: $R_1^1$; $---$: $T_1^1$; $---$: $R_1^0$; $---$: $T_1^0$.

The limiting value $R_1^1(a)$ of $R_1^1$ as $t \to a$ is indicated.
Fig. 6.17. Spectral field distributions $\tilde{E}_y(k_x)$ and $Z_0\tilde{H}_x(k_x)$ in the junction plane of two planar waveguides at offset $t-d$, and with parameters as listed in Table XVIII(1);

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.7$;
(c) $t = 1.5$; (d) $t = 5$ (terminating waveguide radiating into free space);

--- : (a) and (c), scaling on the left-hand side of the plots;
--- : (b) and (d), scaling on the right-hand side of the plots.
Fig. 6.18. Spectral field distributions $\hat{E}_y(k_x)$ and $Z_0\hat{H}_y(k_x)$ in the junction plane of two planar waveguides at offset $t-d$, and with parameters as listed in Table XVIII(2);

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.75$;
(c) $t = 1.5$; (d) $t = \infty$ (terminating waveguide radiating into free space);

--- : (a) and (c), scaling on the left-hand side of the plots;
--- : (b) and (d), scaling on the right-hand side of the plots.
mode), $t = 0.75$, $t = 1.5$ and $t = \infty$ (corresponding to the terminating waveguide), in the case of TE$_0$-mode incidence. Note that $\hat{E}_y(k_x)$ has extremely sharp peaks at $k_x/k_0 = \pm \text{Re}(k_1/k_0) = \pm 1$, whereas $\hat{H}_x(k_x)$ behaves much smoother. In Fig. 6.18 we have plotted $\hat{E}_y(k_x)$ and $\hat{H}_x(k_x)$, for the same values of the offset parameter $t$, in the case of TE$_1$-mode incidence.

In Fig. 6.19, plots are shown for the directive gain $D(\theta)$, over the range

![Diagram showing directive gain](image)

**Fig. 6.19.** Directive gain $D(\theta)$ ($-\pi/2 \leq \theta \leq \pi/2$) of a terminating planar waveguide radiating into free space, with parameters as listed in Table XVIII;

- - - : results computed from $\hat{H}_x$;
- - - : results computed from $\hat{E}_y$;

(a) TE$_0$-mode incidence (Table XVIII(1)); (b) TE$_1$-mode incidence (Table XVIII(2)).
\[-\pi/2 \leq \theta \leq \pi/2\], of the terminating waveguide radiating into free space, for the two cases of TE_0^- and TE_1^- mode incidence. The results for \( D(\theta) \) have been computed from \( \hat{E}_y \) and from \( \hat{H}_x \), by use of the second and third expressions in (6.48).

In Table XIX we have listed the radiated power fraction \( P^f/P^i \) and the reflected power fractions \( P^I_0/P^i \), \( P^I_1/P^i \), again for the two cases of TE_0^- and TE_1^- mode incidence. Here, \( P^i \) is the incident power, \( P^f \) is the forward radiated power, while \( P^I_0 \), \( P^I_1 \) denote the reflected power associated with the reflected TE_0^- and TE_1^- modes, respectively. The power fractions have been calculated either from \( \hat{E}_y \) only, or from \( \hat{H}_x \) only, or from \( \hat{E}_y \) and \( \hat{H}_x \), as described in Appendices E and F.

Table XIX. Forward radiated power fraction \( P^f/P^i \) and reflected power fractions \( P^I_0/P^i \), \( P^I_1/P^i \), for the terminating waveguide of Fig. 6.12b, with parameters as listed in Table XVIII. Results marked by E, H, and EH, were determined from \( \hat{E}_y \) in the terminal plane, from \( \hat{H}_x \) in the terminal plane, and from both \( \hat{E}_y \) and \( \hat{H}_x \) in the terminal plane, respectively. Dashes indicate that the pertaining values of the reflected power fractions should vanish for symmetry reasons; numerically, they were of the order of \( 10^{-4} \) at most.

<table>
<thead>
<tr>
<th>Incident field</th>
<th>( P^f/P^i )</th>
<th>( P^I_0/P^i )</th>
<th>( P^I_1/P^i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{TE}_0 )</td>
<td>( 0.745 )</td>
<td>( 0.248 )</td>
<td>-</td>
</tr>
<tr>
<td>H</td>
<td>( 0.697 )</td>
<td>( 0.222 )</td>
<td>-</td>
</tr>
<tr>
<td>EH</td>
<td>( 0.722 )</td>
<td>( 0.231 )</td>
<td>-</td>
</tr>
<tr>
<td>( \text{TE}_1 )</td>
<td>( 0.731 )</td>
<td>-</td>
<td>( 0.218 )</td>
</tr>
<tr>
<td>H</td>
<td>( 0.558 )</td>
<td>-</td>
<td>( 0.188 )</td>
</tr>
<tr>
<td>EH</td>
<td>( 0.598 )</td>
<td>-</td>
<td>( 0.194 )</td>
</tr>
</tbody>
</table>
When setting up the power balance in the terminal plane, we have to include the backward radiated power $P^b$, which has not been calculated. Since $P^f + P^s + P^b = P^l$ and $P^b > 0$, we should always have $P^f/P^l + P^s/P^l < 1$; this inequality is satisfied by all results in Table XIX.

### 6.5.4. Offset junction of two identical three–moded waveguides, and radiation from a terminating waveguide

Another configuration that has been investigated is the offset junction of two identical waveguides that can support three TE–surface–wave modes, namely, the TE$_{0}^{-}$, TE$_{1}^{-}$ and TE$_{2}^{-}$–modes (Fig. 6.20). The parameters for the configuration are listed in Table XX. As incident fields in waveguide A we take either one of the TE$_{0}^{-}$, TE$_{1}^{-}$ or

![Diagram of waveguide configurations](image)

Fig. 6.20. (a) Offset junction of two identical three–moded planar waveguides; (b) terminating planar waveguide radiating into free space. Parameters are listed in Table XX.

### Table XX. Parameters for the configuration of Fig. 6.20 (SI–units).

<table>
<thead>
<tr>
<th>Incident field</th>
<th>$f$</th>
<th>$k_0$</th>
<th>$\lambda_0$</th>
<th>$d_{A} =-2a$</th>
<th>$s_{A} =-1_A$</th>
<th>$P^f/k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) TE$_{0}^{-}$,$A$</td>
<td>$1.000 \times 10^{13}$</td>
<td>$4 \times 10^3$</td>
<td>$1.571 \times 10^{-8}$</td>
<td>$10^{-5}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$2.1839 \times 2.5929 \times 10^{-6}$</td>
</tr>
<tr>
<td>(2) TE$_{1}^{-}$,$A$</td>
<td>$1.000 \times 10^{13}$</td>
<td>$4 \times 10^5$</td>
<td>$1.571 \times 10^{-8}$</td>
<td>$10^{-5}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.6536 \times 2.6982 \times 10^{-6}$</td>
</tr>
<tr>
<td>(3) TE$_{2}^{-}$,$A$</td>
<td>$1.000 \times 10^{13}$</td>
<td>$4 \times 10^5$</td>
<td>$1.571 \times 10^{-8}$</td>
<td>$10^{-5}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.2369 \times 2.7809 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
$TE_2$-modes. Their propagation coefficients are given in Table XX (1), (2) and (3), respectively. In the computations the truncation parameter $k_{\text{max}}$ is set at $k_{\text{max}}/k_0 = 8$, and the subdivision of the truncated $k_x,k_y$-interval is that of Table XIV, column (3).

In Fig. 6.21 we have plotted the field distributions of the incident Lorentz-normalised

![Graphs showing field distributions](image)

Fig. 6.21. Transverse field distributions $E_y(x)$ and $Z_0H_z(x)$ in the junction plane of two planar waveguides at offset $s-d$, and with parameters as listed in Table XX(1):

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.5$;

(c) $t = 1$; (d) $t = \alpha$ (terminating waveguide radiating into free space).
TE₀-mode (which is identical to the junction plane field in case \( t = 0 \)), the field distributions \( E_y(x) \) and \( H_x(x) \) in the junction plane for offset parameter values \( t = 0.5, 1 \), and the field distributions in the terminal plane (\( t = a \)) of the terminating waveguide under consideration. In Figs. 6.22 and 6.23, the same transverse field distributions are shown for the cases of TE₁-mode and TE₂-mode incidence, respectively.

![Graphs of field distributions](image)

Fig. 6.22. Transverse field distributions \( E_y(x) \) and \( Z_x H_x(x) \) in the junction plane of two planar waveguides at offset \( t \cdot d \), and with parameters as listed in Table XX(2);
(a) \( t = 0 \) (junction plane field = incident field); (b) \( t = 0.5 \);
(c) \( t = 1 \); (d) \( t = a \) (terminating waveguide radiating into free space).
Next, we have computed the reflection and transmission coefficients as functions of the offset parameter \( t \) over the range \( 0 \leq t \leq 1 \), and the reflection coefficients for a terminating waveguide radiating into free space. The results are presented in Fig. 6.24 for the three cases of TE\(_0^-\), TE\(_1^-\) and TE\(_2^-\) mode incidence.

![Graphs showing reflection and transmission coefficients](image)

Fig. 6.23. Transverse field distributions \( E_y(x) \) and \( Z_0 H_z(x) \) in the junction plane of two planar waveguides at offset \( t-d \), and with parameters as listed in Table XX(3):

(a) \( t = 0 \) (junction plane field = incident field); (b) \( t = 0.5 \);

(c) \( t = 1 \); (d) \( t = a \) (terminating waveguide radiating into free space).
Fig. 6.24. Reflection coefficients $R_n^m$ and transmission coefficients $T_n^m$ of the incident $TE_n$-mode at the junction plane of two planar waveguides with parameters as listed in Table XX, as functions of the offset parameter $t$;

- : $R_n^0$, $T_n^0$
- : $R_n^1$, $T_n^1$
- : $R_n^2$, $T_n^2$

(a) $TE_0$-mode incidence; (b) $TE_1$-mode incidence; (c) $TE_2$-mode incidence. The limiting value $R_n^m(\infty)$ of $R_n^m$ as $t \to \infty$ is indicated.
In Fig. 6.25, plots are shown for the directive gain D(θ), over the range $-\pi/2 \leq \theta \leq \pi/2$, of the terminating waveguide radiating into free space, due to $TE_0^-$, $TE_1^-$ and $TE_2^-$mode incidence. In Table XXI we have listed the forward radiated power fraction $P_f^i/P^i$ and the reflected power fractions $P_r^f/P^i$, $P_r^i/P^i$, $P_r^f/P^i$; the three cases of $TE_0^-$, $TE_1^-$ and $TE_2^-$mode incidence have been considered. Note that in the case of $TE_0^-$mode incidence, the sum of the power fractions $P_f^i/P^i$ and $P_r^f/P^i$ computed from the transverse electric field $E_y^r$, exceeds unity (while the backward radiated power has not even been taken into account). This is consistent with a previous observation in Subsection 6.5.1, where we compared our results for the reflection coefficients with those of Rosai (1978). It was found there that the

Table XXI. Forward radiated power fraction $P_f^i/P^i$ and reflected power fractions $P_r^i/P^i$, $P_r^i/P^i$, $P_r^r/P^i$, for the terminating waveguide of Fig. 6.20b, with parameters as listed in Table XX. Results marked by $E$, $H$, and $EH$, were determined from $E_y$ in the terminal plane, from $H_x$ in the terminal plane, and from both $E_y$ and $H_x$ in the terminal plane, respectively. Dashes indicate that the pertaining values of the reflected power fractions should vanish for symmetry reasons; numerically, they were of the order of $10^{-4}$ at most.

<table>
<thead>
<tr>
<th>Incident Field</th>
<th>$P_f^i/P^i$</th>
<th>$P_r^i/P^i$</th>
<th>$P_r^r/P^i$</th>
<th>$P_r^f/P^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_0^-$</td>
<td>0.749</td>
<td>0.233</td>
<td>-</td>
<td>0.034</td>
</tr>
<tr>
<td>$TE_1^-$</td>
<td>0.739</td>
<td>0.217</td>
<td>-</td>
<td>0.029</td>
</tr>
<tr>
<td>$EH$</td>
<td>0.743</td>
<td>0.217</td>
<td>-</td>
<td>0.029</td>
</tr>
<tr>
<td>$TE_2^-$</td>
<td>0.628</td>
<td>-</td>
<td>0.472</td>
<td>-</td>
</tr>
<tr>
<td>$EH$</td>
<td>0.629</td>
<td>-</td>
<td>0.472</td>
<td>-</td>
</tr>
<tr>
<td>$TE_0^-$</td>
<td>0.612</td>
<td>0.419</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$EH$</td>
<td>0.619</td>
<td>0.420</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$TE_2^-$</td>
<td>0.372</td>
<td>0.034</td>
<td>-</td>
<td>0.480</td>
</tr>
<tr>
<td>$EH$</td>
<td>0.369</td>
<td>0.032</td>
<td>-</td>
<td>0.454</td>
</tr>
</tbody>
</table>
Fig. 6.25. Directive gain $D(\theta)$ ($-\pi/2 \leq \theta \leq \pi/2$) of a terminating planar waveguide radiating into free space, with parameters as listed in Table XX; solid: results computed from $\tilde{H}_x$; dotted: results computed from $\tilde{E}_y$.

(a) $\text{TE}_0$-mode incidence (Table XX(1)); (b) $\text{TE}_1$-mode incidence (Table XX(2)); (c) $\text{TE}_2$-mode incidence (Table XX(3)).
reflection coefficients calculated from $\tilde{E}_y$ are in general too large, whereas the reflection coefficients calculated from $\tilde{H}_x$ are the more accurate ones.

6.5.5. On-axis junction of two waveguides with different widths and different permittivities

In this subsection we compute the field distributions in the junction plane and the reflection and transmission coefficients for the on-axis junction of two waveguides with different permittivities, for three widths of the lower-permittivity waveguide. The configurations considered are shown in Fig. 6.26. The parameters for the

Fig. 6.26. On-axis junction of two planar waveguides with different permittivities and different widths of the lower-permittivity waveguide. The parameters for the nine combinations (a1),...,(c3) are listed in Table XXII.
Table XXII. Parameters for the configurations of Fig. 6.26 (SI-units).

<table>
<thead>
<tr>
<th>incident field</th>
<th>$f$</th>
<th>$k_0$</th>
<th>$\lambda_0$</th>
<th>$d_A$</th>
<th>$d_B$</th>
<th>$\xi_A$</th>
<th>$\xi_B$</th>
<th>$\xi_0/k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $TE_{0,A}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(2) $TE_{0,A}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(3) $TE_{0,A}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(4) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(5) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(6) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(7) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(8) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>(9) $TE_{0,B}$</td>
<td>$1.9092 \times 10^{-13}$</td>
<td>$4 \times 10^6$</td>
<td>$1.571 \times 10^{-3}$</td>
<td>$6.778 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2.25 - 2.5 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$1.9248 - 2.3 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

configurations are listed in Table XXII. The waveguide configurations are numbered (1), (2), (3), and differ only in the width $d_A$ of waveguide A; the successive values of $d_A$ are henceforth denoted by width (1), (2), (3), respectively. Waveguide A only supports the $TE_0$-mode, whereas waveguide B supports the $TE_0$- and $TE_1$-modes. As incident fields we take (a) the $TE_0$-mode in waveguide A, (b) the $TE_0$-mode in waveguide B, and (c) the $TE_1$-mode in waveguide B. The nine combinations of waveguide configurations (1), (2) or (3), and incident fields (a), (b) or (c), are henceforth referred to as the cases (a1), (a2), . . . (c3). In the computations we have set $k_{max}/k_0 = 6$; the subdivision of the truncated $k_0/k_{max}$-interval is that of Table XIII, column (3).

In Fig. 6.27 we have plotted the field distributions of the Lorentz-normalised $TE_0$-mode incident in waveguide A, together with the resulting field distributions $E_y(x)$ and $H_z(x)$ in the junction plane, for the two cases (a1) and (a2) of Fig. 6.26 and Table XXII. In Fig. 6.28 we have plotted the field distributions of the Lorentz-normalised $TE_0$-mode incident in waveguide B, together with the resulting
Fig. 6.27. Transverse field distributions $E_y(x)$ and $Z_0H_x(x)$ in the junction plane of two planar waveguides with parameters as listed in Table XXII(a);
(a) $\text{TE}_0$-modal field incident in waveguide A for case (a1);
(b) junction plane field for case (a1);
(c) $\text{TE}_0$-modal field incident in waveguide A for case (a3);
(d) junction plane field for case (a3).

Field distributions $E_y(x)$ and $H_x(x)$ in the junction plane, for the two cases (b1) and (b3) of Fig. 6.26 and Table XXII. The field distributions in the junction plane corresponding to the cases (a2) and (b2), are not shown; they hardly differ from the field distributions corresponding to the cases (a3) and (b3). The field distributions of
Fig. 6.28. Transverse field distributions $E_y(x)$ and $Z_0 H_x(x)$ in the junction plane of two planar waveguides with parameters as listed in Table XXII(b);

(a) $TE_0$-modal field incident in waveguide B for cases (b1), (b3), (b5);

(b) junction plane field for case (b1); (c) junction plane field for case (b3).

the Lorentz-normalised $TE_1$-mode incident in waveguide B and the resulting field distributions in the junction plane for the three different widths (1), (2) and (3) of waveguide A (corresponding to the cases (c1), (c2), (c3)), are plotted in Fig. 6.29. As already mentioned, the junction plane fields in the cases (a2) and (a3) and in the cases (b2) and (b3) are not noticeably different. On the other hand, there is a distinct difference between the junction plane fields in the cases (c2) and (c3). A possible explanation for this difference in behaviour is postponed till the discussion of the numerical results for the reflection and transmission coefficients in these cases.
Fig. 6.29 Transverse field distributions $E_y(x)$ and $Z_0 H_x(x)$ in the junction plane of two planar waveguides with parameters as listed in Table XXII(c).

(a) $TE_1$-modal field incident in waveguide B for cases (c1), (c2), (c3);
(b) junction plane field for case (c1);
(c) junction plane field for case (c2);
(d) junction plane field for case (c3).

Numerical results for the reflection and transmission coefficients are presented in Table XXIII for the nine combinations of (a) a $TE_0$-mode incident in waveguide A, (b) a $TE_0$-mode incident in waveguide B, (c) a $TE_1$-mode incident in waveguide B, and waveguide A having width (1), (2), (3) (cf. Table XXII (1), (2), (3)). In view of the symmetry of the configurations, the reflection and transmission coefficients that connect modes of different parity (even or odd), must vanish. Numerically we found
Table XXIII. Reflection coefficients $R_0$, $R_1$ and transmission coefficient $T_0$ for the on-axis junction of two planar waveguides (Fig. 6.26) with parameters as listed in Table XXII. The indications (1), (2), (3) and (a), (b), (c) refer to the corresponding indications in Fig. 6.26 and Table XXII. Results marked by E, H, and EH, were determined from $E_y$ in the junction plane, from $H_x$ in the junction plane, and from both $E_y$ and $H_x$ in the junction plane, respectively. The coefficients $R_0$, $R_1$ and $T_0$, $T_1$ (which are not included in this table) should vanish for symmetry reasons; numerically, they were of the order of $10^{-4}$ at most.

<table>
<thead>
<tr>
<th></th>
<th>$R_0^0$</th>
<th>$R_0^1$</th>
<th>$R_1^0$</th>
<th>$R_1^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>-0.104 + 0.030j</td>
<td>0.066 + 0.060j</td>
<td>0.439 + 0.111j</td>
<td>0.564 + 0.043j</td>
</tr>
<tr>
<td>H</td>
<td>-0.997 + 0.032j</td>
<td>0.991 + 0.041j</td>
<td>0.413 + 0.103j</td>
<td>0.610 + 0.038j</td>
</tr>
<tr>
<td>EH</td>
<td>-0.101 + 0.031j</td>
<td>0.298 + 0.021j</td>
<td>0.430 + 0.106j</td>
<td>0.611 + 0.038j</td>
</tr>
<tr>
<td>E</td>
<td>-0.192 + 0.031j</td>
<td>0.919 + 0.028j</td>
<td>0.363 + 0.046j</td>
<td>0.586 + 0.029j</td>
</tr>
<tr>
<td>H</td>
<td>-0.183 + 0.030j</td>
<td>0.919 + 0.028j</td>
<td>0.365 + 0.044j</td>
<td>0.592 + 0.012j</td>
</tr>
<tr>
<td>EH</td>
<td>-0.188 + 0.028j</td>
<td>0.919 + 0.028j</td>
<td>0.368 + 0.036j</td>
<td>0.586 + 0.016j</td>
</tr>
<tr>
<td>E</td>
<td>-0.153 + 0.044j</td>
<td>0.917 + 0.034j</td>
<td>0.365 + 0.056j</td>
<td>0.597 + 0.021j</td>
</tr>
<tr>
<td>H</td>
<td>-0.162 + 0.044j</td>
<td>0.917 + 0.034j</td>
<td>0.368 + 0.038j</td>
<td>0.590 + 0.018j</td>
</tr>
<tr>
<td>EH</td>
<td>-0.169 + 0.037j</td>
<td>0.918 + 0.033j</td>
<td>0.369 + 0.047j</td>
<td>0.348 + 0.021j</td>
</tr>
</tbody>
</table>

The values of such coefficients to be of the order of $10^{-4}$ at most. By reciprocity, the transmission coefficients $T_0$ for the incident fields (a) and (b) should be identical. From Table XXIII it is seen that this identity is met best by the values of $T_0$ calculated from $E_y$ in the junction plane. Probably this is a consequence of the choice $k_{max}/k_0 = 6$, which will make the results calculated from $H_x$ less accurate, since $H_x$ decreases less rapidly than $E_y$ does for increasing $|k_x|$ (cf. Fig. 6.17).

The reflection and transmission coefficients in the cases (a2) and (a3) and in the cases
(b2) and (b3) differ relatively little, as do the field distributions. This can be attributed to the fact that the field distributions of the TE₀-modes in waveguide A of widths (2) and (3) are practically identical in shape. From column (c) of Table XXIII it is observed that for the TE₄-mode incident in waveguide B, the reflection coefficients R₄¹ for the cases (c2) and (c3) show a significant difference, which even exceeds the difference between the coefficients R₄¹ for the cases (c1) and (c2) (in which the difference in width of waveguide A is much larger). On the other hand, the differences between the reflection coefficients R₀⁰ for the cases (a2) and (a3) and for the cases (b2) and (b3), are much smaller than the differences between R₀⁰ for the cases (a1) and (a2) and for the cases (b1) and (b2), respectively. The smaller reflection (and hence larger radiation, since waveguide A does not support a TE₄-mode) for the case (c3) as compared to the cases (c1) and (c2), is due to the fact that for waveguide A of width (3) the TE₄-mode in it is just below cut-off. Indeed, the cut-off frequency of the TE₄-mode in waveguide A is given by \( f = 1.975 \times 10^{14} \) Hz for waveguide width (1), \( f = 2.703 \times 10^{13} \) Hz for waveguide width (2), and \( f = 1.931 \times 10^{13} \) Hz for waveguide width (3). Though not yet present, the "immanent presence" of the TE₄-mode in waveguide A for case (c3) manifests itself in a larger amount of radiation (Rossi, 1978). This may also account for the difference between the junction plane fields in cases (c2) and (c3) in Fig. 6.29: although the difference between widths (1) and (2) of waveguide A is much larger than the difference between widths (2) and (3), the difference between the junction plane fields in the former case is smaller than the difference in the latter case.

6.5.8. Offset junction of two waveguides with different widths and different permittivities

Next we report on our computations of the field distributions in the junction plane, and of the reflection and transmission coefficients, for the offset junction of two planar
The waveguide configurations are numbered (1), (2), (3), and the corresponding parameters are listed in Table XXIV. The present configurations are composed of the same waveguides as considered in Subsection 6.5.5 (see Fig. 6.26(a) and Table XXII(a)), whereby the waveguide axes now have an offset $t \cdot d_B$. Notice that the offset has been related to the width $d_B$ of waveguide $B$, since the width $d_A$ of waveguide $A$ is varying. Waveguide $A$ only supports the TE$_0$-mode, whereas waveguide $B$ supports the TE$_0$- and TE$_1$-modes. As incident field we take the TE$_0$-mode of waveguide $A$.

In the computations we have set $k_{\text{max}}/k_0 = 6$, and the subdivision of the truncated $k_x \cdot k_y'$-interval is that of Table XIII, column (3).

For several values of the offset $t \cdot d_B$, we have computed the field distributions in the junction plane and the reflection and transmission coefficients, for a TE$_0$-mode

![Image of waveguide configurations](image)

Fig. 6.30. Offset junction of two planar waveguides with different widths and different permittivities. Parameters are listed in Table XXIV.

<table>
<thead>
<tr>
<th>Incident field</th>
<th>$f$</th>
<th>$k_0$</th>
<th>$\lambda_0$</th>
<th>$d_A$</th>
<th>$d_B$</th>
<th>$e_{r,A}$</th>
<th>$e_{r,B}$</th>
<th>$k_B/k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) TE$_0,A$</td>
<td>1.098-10$^{13}$</td>
<td>6.0$^9$</td>
<td>1.171-10$^{-5}$</td>
<td>6.778-10$^{-6}$</td>
<td>5.10$^{-1}$</td>
<td>2.25-10$^{-5}$</td>
<td>5-10$^{-3}$</td>
<td>1.0138-4.931610$^{-10}$</td>
</tr>
<tr>
<td>(2) TE$_0,A$</td>
<td>1.009-10$^{13}$</td>
<td>5.0$^9$</td>
<td>1.571-10$^{-5}$</td>
<td>4.938-10$^{-6}$</td>
<td>5.10$^{-3}$</td>
<td>2.25-10$^{-5}$</td>
<td>5-10$^{-3}$</td>
<td>1.0764-4.9224-10$^{-10}$</td>
</tr>
<tr>
<td>(3) TE$_0,A$</td>
<td>1.006-10$^{13}$</td>
<td>5.0$^9$</td>
<td>1.571-10$^{-5}$</td>
<td>6.942-10$^{-6}$</td>
<td>5.10$^{-4}$</td>
<td>2.25-10$^{-5}$</td>
<td>5-10$^{-3}$</td>
<td>1.3255-4.726510$^{-10}$</td>
</tr>
</tbody>
</table>
incident in waveguide A. In Fig. 6.31 we have plotted the field distributions $E_y(x)$ and $H_z(x)$ in the junction plane for configuration (1) at various values of the offset parameter $t$. In Fig. 6.32 the same field distributions have been plotted for

![Graphs showing field distributions](image)

Fig. 6.31. Transverse field distributions $E_y(x)$ and $Z_0H_z(x)$ in the junction plane of two planar waveguides at offset $t\cdot d_B$, and with parameters as listed in Table XXIV(1);
(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.25$, (c) $t = 0.5$;
(d) $t = 0.75$, (e) $t = 1$. 
configuration (3). The field distributions in the junction plane for the configurations (2) and (3) were found to be practically identical; therefore, no plots are shown for configuration (2).

![Graphs showing field distributions](image)

Fig. 6.32. Transverse field distributions $E_y(x)$ and $Z_0H_x(x)$ in the junction plane of two planar waveguides at offset $t-d_B$, and with parameters as listed in Table XXIV(3):

(a) $t = 0$ (junction plane field = incident field); (b) $t = 0.25$; (c) $t = 0.5$; (d) $t = 0.75$; (e) $t = 1$. 
In Fig. 6.33, the reflection coefficient $R_0^0$ and the transmission coefficients $T_0^0, T_1^0$ are presented as functions of the offset parameter $t$. Again we observe that the reflection and transmission coefficients for the configurations (2) and (3), corresponding to widths $d_A = 4.959 \times 10^{-6}$ m and $d_A = 6.943 \times 10^{-6}$ m, hardly differ.

0.5.7. Offset junction of two identical single-mode waveguides, and radiation from a terminating waveguide (dependence on offset and frequency of operation)

Our final example deals with the reflection and transmission of the TE$_0$-mode at the offset junction of two identical waveguides for various values of the offset $t \cdot d$ (Fig. 6.34a). We also consider the reflection and radiation into free space from the terminal.

![Graphs showing reflection and transmission coefficients](image)

**Fig. 6.33.** Reflection coefficient $R_0^0$ and transmission coefficients $T_0^0, T_1^0$ of the incident TE$_0$-mode at the junction plane of two planar waveguides with parameters as listed in Table XXIV, as functions of the offset parameter $t$;

- - - - : $d_A = 6.778 \times 10^{-7}$ m (Table XXIV(1));
- - - : $d_A = 4.959 \times 10^{-6}$ m (Table XXIV(2));
- - - : $d_A = 6.943 \times 10^{-6}$ m (Table XXIV(3)).
plane of one such waveguide (Fig. 6.34b). The parameters for the configuration are listed in Table XXV. For various values of the offset parameter \( t \) and of the frequency of operation \( f \), we have computed the field distributions in the junction plane, the reflection and transmission coefficients, and the directive gain of the terminating waveguide. The frequencies of operation employed are such that the waveguide supports only the lowest TE-surface-wave mode, the TE\(_{0}\)-mode. The frequencies \( f \) are listed in Table XXV, together with the corresponding values of the free-space wavelength \( \lambda_0 \) and the free-space wavenumber \( k_0 \), and of the propagation coefficient \( \kappa^E \) of the incident TE\(_{0}\)-mode; the successive values of \( f \) are henceforth denoted by frequency (1), (2), (3). In the computations we have set \( k_{\text{max}}/k_0 = 6 \) for the frequencies (1) and (3), and \( k_{\text{max}}/k_0 = 8 \) for the frequency (2); the subdivision of the truncated \( k_x^E \)-interval is that of Table XIII, column (1).

![Diagram](image)

*Fig. 6.34. (a) Offset junction of two identical single-moded planar waveguides; (b) terminating single-moded planar waveguide radiating into free space. Parameters are listed in Table XXV.*

<table>
<thead>
<tr>
<th>Incident Field</th>
<th>( f )</th>
<th>( \lambda_0 )</th>
<th>( k_0 )</th>
<th>( \kappa^E )</th>
<th>( \kappa_A^E )</th>
<th>( \kappa_B^E )</th>
<th>( \kappa_B^E/\kappa_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) TE(_{0})(_A)</td>
<td>1.094\times10'^13</td>
<td>2.797\times10'^5</td>
<td>2.318\times10'^6</td>
<td>10'^8</td>
<td>2.05 \times 10'^3</td>
<td>1.038 \times 10'^4</td>
<td></td>
</tr>
<tr>
<td>(2) TE(_{0})(_A)</td>
<td>5.465\times10'^13</td>
<td>2.797\times10'^6</td>
<td>3.198\times10'^6</td>
<td>10'^8</td>
<td>2.91 \times 10'^3</td>
<td>1.274 \times 10'^4</td>
<td></td>
</tr>
<tr>
<td>(3) TE(_{0})(_A)</td>
<td>1.323\times10'^14</td>
<td>2.797\times10'^6</td>
<td>2.933\times10'^6</td>
<td>10'^8</td>
<td>2.81 \times 10'^3</td>
<td>1.392 \times 10'^4</td>
<td></td>
</tr>
</tbody>
</table>
In Fig. 6.35 we have plotted the field distributions $E_y(x)$ and $H_x(x)$ of the incident TE$_0$-mode with the frequencies (1), (2) or (3); these field distributions have been normalised such that $E_y(x) = 1$ where $|E_y(x)|$ is maximal. In Figs. 6.36–6.38 we have plotted the field distributions $E_y(x)$ and $H_x(x)$ in the junction plane for offset parameter values $t = 0.25$, $t = 0.5$ and $t = 1$, respectively, and frequencies of operation (1), (2) or (3). In Fig 6.39 we have plotted the field distributions $E_y(x)$ and $H_x(x)$ of the TE$_0$-mode in a planar waveguide with parameters as listed in Table XXV;

(a) $f = 1.294 \times 10^{12}$ Hz (Table XXV(1)); (b) $f = 9.465 \times 10^{13}$ Hz (Table XXV(2)); (c) $f = 1.325 \times 10^{14}$ Hz (Table XXV(3)).

The field distributions have been normalised such that $E_y(x) = 1$ where $|E_y(x)|$ is maximal.
Fig. 6.36. Transverse field distributions $E_y(x)$ and $Z_0 H_x(x)$ in the junction plane of two offset planar waveguides with parameters as listed in Table XXV and offset parameter $t = 0.25$;
(a) $f = 1.294 \times 10^{13}$ Hz (Table XXV(1)); (b) $f = 9.465 \times 10^{13}$ Hz (Table XXV(2));
(c) $f = 1.325 \times 10^{14}$ Hz (Table XXV(3)).

The incident TE$_{01}$-modal field has been normalised such that $E_y^1(x) = 1$ where $|E_y^1(x)|$ is maximal.

$H_x(x)$ in the terminal plane of the terminating waveguide radiating into free space.

In Table XXVI we present numerical results for the reflection coefficient $R_0^0$ and the transmission coefficient $T_0^0$, and, in the case of a terminating waveguide ($t = a$), for the forward radiated power fraction $P_f^f/P_1^1$; the results are listed as functions of the offset parameter $t$ and the frequency of operation $f$. 
Fig. 6.37. Transverse field distributions \( E_y(x) \) and \( Z_0 H_z(x) \) in the junction plane of two offset planar waveguides with parameters as listed in Table XXV and offset parameter \( t = 0.5 \);

(a) \( f = 1.294 \times 10^{15} \) Hz (Table XXV(1)); (b) \( f = 9.465 \times 10^{13} \) Hz (Table XXV(2));
(c) \( f = 1.323 \times 10^{14} \) Hz (Table XXV(3)).

The incident \( TE_0 \)-modal field has been normalised such that \( E_y^1(x) = 1 \) where \( |E_y^1(x)| \) is maximal.

In Fig. 6.40, plots are shown for the directive gain \( D(\theta) \), over the range \(-\pi/2 \leq \theta \leq \pi/2\), of the terminating waveguide radiating into free space, again for the three different frequencies mentioned. Note that for the lowest frequency the reflection at the junction is negligible at all offsets considered, and the incident field is almost completely transmitted into the second waveguide (see Table XXVI). This is
Fig. 6.38. Transverse field distributions $E_y(x)$ and $Z_0 H_x(x)$ in the junction plane of two offset planar waveguides with parameters as listed in Table XXV and offset parameter $t = 1$;

(a) $f = 1.294 \times 10^{13}$ Hz (Table XXV(1)); (b) $f = 9.465 \times 10^{13}$ Hz (Table XXV(2));
(c) $f = 1.325 \times 10^{14}$ Hz (Table XXV(3)).

The incident $TE_0$-modal field has been normalised such that $E_y^1(x) = 1$ where $|E_y^1(x)|$ is maximal.

to be expected, since at the lowest frequency the field distributions of the $TE_0$-mode are very slowly varying functions of the transverse coordinate $x$ (see Fig. 6.35a). Furthermore, the incident $TE_0$-mode and the transmitted field in the offset waveguide hardly differ, since the offset is small compared to the wavelength. For the terminating waveguide it is seen from Fig. 6.40 that the radiated power flow is
Fig. 6.39. Transverse field distributions $E_y(x)$ and $Z_0 H_z(x)$ in the terminal plane of a terminating planar waveguide radiating into free space, with parameters as listed in Table XXV;
(a) $f = 1.294 \times 10^{13}$ Hz (Table XXV(1));
(b) $f = 9.465 \times 10^{13}$ Hz (Table XXV(2));
(c) $f = 1.325 \times 10^{14}$ Hz (Table XXV(3)).

The incident $TE_0$-modal field has been normalised such that $E_y(x) = 1$ where $|E_y(x)|$ is maximal.

concentrated in a small angular region around the direction $\theta = 0$. This behaviour is most pronounced at the lowest frequency, when also the reflection at the terminal plane is very small. As an explanation we observe that the incident $TE_0$-mode has a propagation coefficient $\kappa_E$ for which $\kappa_E / k_0$ is close to unity, hence, it closely resembles a plane wave travelling in the forward direction.
Table XXVI. Reflection and transmission coefficients $R_0$ and $\tau_0$ for the offset junction of two planar waveguides (Fig. 6.34a), and reflection coefficient $R_0^P$ and forward radiated power fraction $P_f^p$ for a terminating planar waveguide (Fig. 6.34b), for three different frequencies of operation. The parameters for the configuration are listed in Table XXV. Results marked by E, H, and EH, were determined from $E_y$ in the junction plane, from $H_y$ in the junction plane, and from both $E_y$ and $H_x$ in the junction plane, respectively.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$R_0^0$</th>
<th>$\tau_0^0$</th>
<th>$R_0^E$</th>
<th>$\tau_0^E$</th>
<th>$R_0^H$</th>
<th>$\tau_0^H$</th>
<th>$R_0^{EH}$</th>
<th>$\tau_0^{EH}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.10^-4</td>
<td>2.10^-4j</td>
<td>1.000</td>
<td>0.024 + 0.013j</td>
<td>0.968 - 0.007j</td>
<td>0.020 - 0.004j</td>
<td>0.941 - 0.002j</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>1.50^-5</td>
<td>2.10^-5j</td>
<td>1.000</td>
<td>0.017 + 0.013j</td>
<td>0.968 - 0.007j</td>
<td>0.022 - 0.004j</td>
<td>0.941 - 0.002j</td>
<td></td>
</tr>
<tr>
<td>EH</td>
<td>1.50^-5</td>
<td>2.10^-5j</td>
<td>1.000</td>
<td>0.008 + 0.014j</td>
<td>0.968 - 0.007j</td>
<td>0.022 - 0.004j</td>
<td>0.941 - 0.002j</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>6.00^-5</td>
<td>2.10^-5j</td>
<td>1.000</td>
<td>0.077 + 0.048j</td>
<td>0.923 - 0.023j</td>
<td>0.061 + 0.016j</td>
<td>0.784 - 0.012j</td>
<td></td>
</tr>
<tr>
<td>EH</td>
<td>6.00^-5</td>
<td>2.10^-5j</td>
<td>1.000</td>
<td>0.068 + 0.046j</td>
<td>0.878 - 0.025j</td>
<td>0.062 + 0.014j</td>
<td>0.784 - 0.012j</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.00^-4</td>
<td>2.10^-4j</td>
<td>1.000</td>
<td>0.190 + 0.086j</td>
<td>0.604 - 0.034j</td>
<td>0.224 - 0.078j</td>
<td>0.401 - 0.031j</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>6.00^-4</td>
<td>2.10^-4j</td>
<td>1.000</td>
<td>0.170 + 0.070j</td>
<td>0.604 - 0.034j</td>
<td>0.212 - 0.069j</td>
<td>0.401 - 0.031j</td>
<td></td>
</tr>
<tr>
<td>EH</td>
<td>6.00^-4</td>
<td>2.10^-4j</td>
<td>1.000</td>
<td>0.178 + 0.071j</td>
<td>0.604 - 0.034j</td>
<td>0.217 - 0.069j</td>
<td>0.401 - 0.031j</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$R_0^P$</th>
<th>$P_f^p$</th>
<th>$R_0^P$</th>
<th>$P_f^p$</th>
<th>$R_0^P$</th>
<th>$P_f^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>0.610 + 0.001j</td>
<td>0.999</td>
<td>0.334 + 0.038j</td>
<td>0.835</td>
<td>0.257 + 0.047j</td>
</tr>
<tr>
<td>H</td>
<td>0.610 + 0.002j</td>
<td>0.998</td>
<td>0.334 + 0.038j</td>
<td>0.881</td>
<td>0.140 + 0.043j</td>
</tr>
<tr>
<td>EH</td>
<td>0.614 + 0.002j</td>
<td>0.999</td>
<td>0.336 + 0.038j</td>
<td>0.831</td>
<td>0.250 + 0.047j</td>
</tr>
</tbody>
</table>
Fig. 6.40. Directive gain $D(\theta)$ ($-\pi/2 \leq \theta \leq \pi/2$) of a terminating planar waveguide radiating into free space, with parameters as listed in Table XXV:

- : results computed from $\hat{H}_y$; - - - : results computed from $\hat{E}_y$;

(a) $f = 1.294 \times 10^{13}$ Hz (Table XXV(1)); (b) $f = 9.465 \times 10^{13}$ Hz (Table XXV(2));
(c) $f = 1.325 \times 10^{14}$ Hz (Table XXV(3)).
6.5.8. Computation times and storage requirements

The computer programmes by which the numerical results of this section have been obtained, were written in ALGOL 60 and were run on a Burroughs B7700 computer.

Most time-consuming are the computation of the Green's tensor elements $G_{EE}(K_i, K_j)$ and $G_{MM}(K_i, K_j)$ and the computation of the spectral field values $E_{y_i}(K_i)$ and $H_{x_i}(K_i)$ from the system of linear algebraic equations that results by the discretisation of the original integral equations. For various cases the computation times and the average data storage requirements are listed in Table XXVII. The computation times and the storage requirements mentioned are the ones used for:

(a) the computation of all necessary values of the tensor elements $G_{EE}$ and $G_{MM}$ of one waveguide (second computation step in Subsection 6.4.2);

(b) the construction and the solving of the system of linear algebraic equations for $E_{y_i}(K_i)$ and $H_{x_i}(K_i)$ using the previously computed values of $G_{EE}(K_i, K_j)$ and $G_{MM}(K_i, K_j)$ (third computation step in Subsection 6.4.2).

We observe that the computation times for the Green's tensor elements increase with increasing permittivity $\varepsilon_r$ and with an increasing number of supported surface-wave modes. The latter trend may be due to the fact that each surface-wave mode corresponds to a pole in the complex $k_x$-plane, of the Fourier-transformed Green's tensor $G(k_x, k_x', k_z)$ (Appendix D, Subsection D.4). These poles are located close to the real axis. It is recalled that the tensor elements $G_{EE}(k_x, k_x')$ and $G_{MM}(k_x, k_x')$ are determined from $G$ by a numerical integration with respect to $k_x$ along the real axis. Hence, the presence of an additional surface-wave mode means another passage close to a pole singularity. From a comparison of the computation times needed when using the successive subdivisions of the truncated $k_x, k_x'$-interval (Table XIII, columns (1), (2), (3)), we conclude that especially the computation of $G(k_x, K_j)$ with $|K_1|, |K_j|$ close to $\text{Re}(k_x)$, takes much more time than the computation of $G(K_j, K_j)$ for other
Table XXVII. Computation times and storage requirements involved in (a) the computation of the tensor elements $G^{PE}_{xy}(K_i,K_j)$ and $G^{MM}_{xx}(K_i,K_j)$, and (b) the solution of the system of linear algebraic equations that results from the integral equations. Cases (1), (2) and (3) refer to the subdivisions of the truncated $k_y,k_x$-interval as listed in Table XIII, columns (1), (2) and (3). The width and relative permittivity of the waveguide are $d$ and $\varepsilon_r$.

(a)

<table>
<thead>
<tr>
<th>$\varepsilon_r$</th>
<th>$k_0d$</th>
<th>number of $k_{max}/k_0$</th>
<th>CPU-time</th>
<th>CPU-time</th>
<th>CPU-time</th>
<th>average memory usage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TE-modes</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>for data, (3)</td>
<td></td>
</tr>
<tr>
<td>$5 \times 10^{-3}$</td>
<td>0.4</td>
<td>1</td>
<td>8</td>
<td>2h 13m</td>
<td>2h 55m</td>
<td>3h 18m</td>
</tr>
<tr>
<td>$5 \times 10^{-3}$</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>2h 16m</td>
<td>2h 44m</td>
<td>3h 23m</td>
</tr>
<tr>
<td>$5 \times 10^{-3}$</td>
<td>4</td>
<td>3</td>
<td>8</td>
<td>2h 16m</td>
<td>2h 15m</td>
<td>3h 24m</td>
</tr>
<tr>
<td>$7.25 \times 10^{-3}$</td>
<td>0.271</td>
<td>1</td>
<td>6</td>
<td>2h 19m</td>
<td>2h 18m</td>
<td>3h 06m</td>
</tr>
<tr>
<td>$7.25 \times 10^{-3}$</td>
<td>1.864</td>
<td>1</td>
<td>6</td>
<td>2h 31m</td>
<td>2h 23m</td>
<td>3h 08m</td>
</tr>
<tr>
<td>$7.25 \times 10^{-3}$</td>
<td>2.777</td>
<td>1</td>
<td>6</td>
<td>2h 41m</td>
<td>2h 28m</td>
<td>3h 10m</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>waveguide parameters</th>
<th>CPU-time (3)</th>
<th>average memory usage for data, (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{1,A} = s_{1,B}$ and $d_A = d_B$</td>
<td>2h 14m</td>
<td>1048 Kbytes</td>
</tr>
<tr>
<td>$t_{1,A} \neq s_{1,B}$ and/or $d_A \neq d_B$</td>
<td>2h 14m</td>
<td>1520 Kbytes</td>
</tr>
</tbody>
</table>

values of $K_i$, $K_j$. This has also been verified by comparing the computation times of $\hat{G}(K_i,K_j)$ for single pairs of values $(K_i,K_j)$. Possibly a less refined subdivision of the $k_y,k_x$-interval in the vicinity of $|k_y|,|k_x| = \text{Re}(k_z)$ could be used, by properly taking into account (e.g. analytically) the poles of $\hat{G}(k_x,k_y,k_z)$, the singularity factors $(k_1^2 - k_y^2)^{-1/2}$, $(k_1^2 - k_x^2)^{-1/2}$ in $\hat{G}(k_x,k_y)$ (Subsection 6.4.1), and the singularity factor $(k_1^2 - k_x^2)^{-1/2}$ in $\hat{G}(k_x)$ (Appendix F). Such a coarser subdivision would lead to a reduction in computation time, however, we have not pursued this point.
The numerical solution of the system of linear algebraic equations that results by the discretisation of the integral equations, involves the storage of matrices of size $4J \times 4J$, in which $J$ is typically 52. Again, the storage requirements and the computation times can be reduced by using a less refined subdivision of the $k_y, k_y'$-interval.

The computations of the incident modal fields, the field distributions in the junction plane, the reflection and transmission coefficients, and of the directive gain of a terminating waveguide radiating into free space, require comparatively little time, of the order of 10 s or less.

In this chapter we have discussed in detail the reflection, transmission and radiation at the junction of two straight planar open waveguide sections, and we have presented numerical results for the pertaining field quantities for a variety of junctions of two step-index waveguide sections.
APPENDICES

A. ON THE BRANCH CUTS OCCURRING IN THE SPECTRAL–DOMAIN FIELD EXPRESSIONS FOR OPEN WAVEGUIDES

The axial–spectral representation for the fields generated by a localised source in a straight open waveguide has branch points and branch cuts associated with the propagation of waves away to infinity in the outer medium (or media, in case of a planar waveguide). To show this, we observe that in any source–free homogeneous cylindrical subdomain with constant permittivity $\varepsilon$ and constant permeability $\mu$ the transverse field components can be expressed in terms of the axial field components as

\[ \tilde{E}_T = k_T^{-2} (j \omega \tilde{u}_z \times \nabla_T \tilde{H}_z - j k_T \nabla_T \tilde{E}_z) \], \hspace{1cm} (A.1)

\[ \tilde{H}_T = k_T^{-2} (-j \omega \tilde{u}_z \times \nabla_T \tilde{E}_z - j k_T \nabla_T \tilde{H}_z) \], \hspace{1cm} (A.2)

while $\tilde{E}_z$ and $\tilde{H}_z$ satisfy the two–dimensional Helmholtz equation

\[ [(\nabla_T \cdot \nabla_T) + k_T^2] \{ \tilde{E}_z, \tilde{H}_z \} = 0. \] \hspace{1cm} (A.3)

with

\[ k_T^2 = \omega^2 \varepsilon - k_T^2 \quad \text{Im}(k_T) \leq 0. \] \hspace{1cm} (A.4)

Here, $\tilde{E}$, $\tilde{H}$ stand for the axial Fourier transforms of the fields $E$, $H$ over the interval
\[ -\pi < \pi < \pi, \text{ similar to (3.2). The results (A.1)-(A.4) can be derived from equations (3.9), (3.10) with } J_T = 0. \]

Application of Green's third identity to the solutions of (A.3) in a two-dimensional domain \( \mathcal{D} \) located outside the boundary contour \( \partial \mathcal{D} \) of a (possibly inhomogeneous) bounded domain \( \mathcal{D} \) (in our case: the guiding part of the open waveguide) yields

\[
\int_{\partial \mathcal{D}} \mathbf{H}^{(1)}(z_T, z_T^+)(\mathbf{H}^{(1)}_+ E_2(z_T^+)) - \mathbf{E}_2(z_T^+)(\mathbf{H}^{(1)}_+ \mathbf{G}(z_T, z_T^+)) ds(z_T^+) = \{1, \mathbf{0}, \mathbf{0}\} \mathbf{E}_2(z_T)
\]

when \( z_T \in \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{D}' \} \), \hspace{1cm} (A.5)

\[
\int_{\partial \mathcal{D}} \mathbf{H}^{(1)}(z_T, z_T^+)(\mathbf{H}^{(1)}_+ \mathbf{H}_2(z_T^+)) - \mathbf{H}_2(z_T^+)(\mathbf{H}^{(1)}_+ \mathbf{G}(z_T, z_T^+)) ds(z_T^+) = \{1, \mathbf{0}, \mathbf{0}\} \mathbf{H}_2(z_T)
\]

when \( z_T \in \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{D}' \} \). \hspace{1cm} (A.6)

Here, \( \mathbf{n} \) is the unit vector normal to \( \partial \mathcal{D} \) pointing away from \( \mathcal{D}' \), and \( \mathbf{G} \) is the spectral-domain scalar Green's function

\[
\mathbf{G}(z_T, z_T^+) = -\frac{j}{4}k_T^2(\mathbf{G}(z_T, z_T^+) - \mathbf{G}(z_T^+, z_T))
\]

that satisfies the inhomogeneous Helmholtz equation

\[
[(\mathbf{D}^2 - k_T^2)\mathbf{G}(z_T, z_T^+) = \mathbf{E}(z_T^+ - z_T^+), \hspace{1cm} (A.7)
\]

and the radiation condition as \( |z_T - z_T^+| \to \pi. \)
A.1. The planar waveguide

The configuration is shown in Fig. A.1. The waveguide geometry, permittivity, permeability and the sources only depend on the \(x\)-coordinate. We investigate fields that are \(y\)-independent. In the domain \(b < x < c\), the medium may be inhomogeneous, and sources may be present. For \(-a < x < b\), the medium is homogeneous with \(\epsilon = \epsilon_1, \mu = \mu_1\); for \(c < x < a\), the medium is homogeneous with \(\epsilon = \epsilon_N, \mu = \mu_N\). In these two half-spaces, no sources are present.

For the present configuration we can determine the spectral-domain field distributions \((\hat{E}_x, \hat{E}_y)\) in the outer media by a simpler procedure than by using (A.5) and (A.6). When \(-a < x < b\), the pertaining field distributions satisfy the ordinary homogeneous second-order differential equation

\[
[E_x^2 + k_{T,1}^2(\hat{E}_y, \hat{H}_y)] = 0,
\]  

(A.9)

![Diagram of a planar waveguide and source domain enclosed by planes \(x = b\) and \(x = c\). In the domains \(-a < x < b\) and \(c < x < a\), the medium is source-free and homogeneous with \(\epsilon = \epsilon_1, \mu = \mu_1\) and \(\epsilon = \epsilon_N, \mu = \mu_N\), respectively.](image)
with the boundary values

\[ \{ E_n(x), H_n(x) \} = \{ E_n(b), H_n(b) \} \] at \( x = b \), \hspace{1cm} (A.10) \]

and

\[ \{ E_n(x), H_n(x) \} \to 0 \hspace{0.5cm} \text{as} \hspace{0.5cm} x \to -\infty. \hspace{1cm} (A.11) \]

As usual, \( k_{T,p}^2 = \omega^2 \epsilon_p \mu_p - k_z^2 \) and \( \text{Im}(k_{T,p}) \leq 0 \). The field distributions that satisfy (A.9)–(A.11) are

\[ \{ E_n(x), H_n(x) \} = \{ E_n(b), H_n(b) \} \exp(jk_{T,1}(x-b)) \hspace{0.5cm} \text{when} \hspace{0.5cm} -\infty < x < b. \hspace{1cm} (A.12) \]

Likewise, we obtain for the field distributions in \( c < x < \infty \),

\[ \{ E_n(x), H_n(x) \} = \{ E_n(c), H_n(c) \} \exp(-jk_{T,N}(x-c)) \hspace{0.5cm} \text{when} \hspace{0.5cm} c < x < \infty. \hspace{1cm} (A.13) \]

From (A.1), (A.2), (A.4), (A.12) and (A.13) we conclude that, in view of the presence of the quantities \( k_{T,1} \) and \( k_{T,N} \) in the exponentials, the axial–spectral representation for the fields in a planar waveguide has branch points \( k_z = \pm k_1 = \pm \omega(\epsilon_1 \mu_1)^{1/2}, \)

\( k_z = \pm k_N = \pm \omega(\epsilon_N \mu_N)^{1/2}, \) and branch cuts \( \text{Im}(k_1^2 - k_z^2)^{1/2} = 0, \text{Im}(k_N^2 - k_z^2)^{1/2} = 0, \) associated with the outer media.

A.2. The waveguide with bounded cross–section

For the waveguide with bounded cross–section we consider the configuration of Fig.

A.2. For the description of the configuration and the field quantities we employ circular–cylinder coordinates \((r, \phi, z)\). In the domain \( b < r < a, \) the medium is homo–
geneous with \( \epsilon = \epsilon_1 \) and \( \mu = \mu_1 \), and no sources are present. Inside the cylinder \( r = b \), i.e., for \( 0 \leq r < b \), the medium may be inhomogeneous, and sources may be present.

We now apply (A.5) and (A.6) to the fields in the domain
\( \varphi' = \{r_T^{-} \mid b < r' < a, 0 \leq \varphi' < 2\pi\} \). On \( \partial \varphi' \) we have \( \mathbf{n} = -i \varphi' \), and the operator \( \varphi_T \) is given by

\[
\varphi_T = i \varphi^{-1} \varphi' = i \varphi^{-1} \varphi' \tag{A.14}
\]

We then obtain

\[
\left[ \tilde{G}(r_T, r_T) \right]_{r = b, \varphi = b} \varphi_T \left( \tilde{E}_z \right)_{r = b} + \left[ \tilde{E}_z \right]_{r = b, \varphi = b} \tilde{G}(r_T, r_T) \varphi_T \left( \tilde{E}_z \right)_{r = b} + \tilde{G}(r_T, r_T) \right]_{r = b, \varphi = b} \tilde{E}_z \varphi_T \right)_{r = b} \right. \left. \varepsilon \varphi' \right\}
\]

with \( r_T \in \varphi' \), \( \varphi_T \). \( \tilde{E}_z \)(A.15)

Fig. A.2. Cylindrical waveguide with bounded cross-section and source-domain enclosed by the circular cylinder \( r = b \). In the domain \( b < r < a \), the medium is source-free and homogeneous with \( \epsilon = \epsilon_1 \) and \( \mu = \mu_1 \).
\[ 2\pi \int_{0}^{2\pi} \left( -\tilde{G}(r_2, r_2) |_{r_2 = b} \tilde{H}_n(r_2) |_{r_2 = b} + \tilde{H}_n(r_2) |_{r_2 = b} \tilde{G}(r_2, r_2) |_{r_2 = b} \right) \delta \varphi \, d\varphi = \tilde{H}_n(r_2) \]

with \( r_2 \in \mathbb{R}^+ \).

(A.6)

In circular–cylinder coordinates, \( |r_2 - r_1|^2 = (r^2 + r_1^2 - 2rr_1\cos(\varphi - \varphi'))^{1/2} \). Using Graf's addition theorem (Magaus, Oberhettinger and Soni, 1966), we can write

\[ H_0^{(2)}(k_T | r_2 - r_1 |) \]

as

\[ H_0^{(2)}(k_T (r^2 + r_1^2 - 2rr_1\cos(\varphi - \varphi'))^{1/2}) = \sum_{n=-\infty}^{\infty} H_n^{(2)}(k_T r) J_n(k_T r) \exp(jn(\varphi - \varphi')) \]

(A.17)

valid for \( r > r_1 \). Inserting (A.7) and (A.17) into (A.15) and (A.16) we arrive at

\[ \tilde{E}_g(r, \varphi) = -(j/4)b \sum_{n=-\infty}^{\infty} H_n^{(2)}(k_T r) \left[ \int_{0}^{2\pi} \left( -J_n(k_T b) \delta \varphi \tilde{E}_g(r_2, \varphi') |_{r_2 = b} \right) \right] \exp(jn(\varphi - \varphi')) \, d\varphi \]

when \( b < r < \infty \),

(A.18)

\[ \tilde{H}_e(r, \varphi) = -(j/4)b \sum_{n=-\infty}^{\infty} H_n^{(2)}(k_T r) \left[ \int_{0}^{2\pi} \left( -J_n(k_T b) \delta \varphi \tilde{H}_e(r_2, \varphi') |_{r_2 = b} \right) \right] \exp(jn(\varphi - \varphi')) \, d\varphi \]

when \( b < r < \infty \).

(A.19)

From (A.1), (A.2), (A.4) and (A.18), (A.19) we conclude that, in view of the presence of the quantity \( k_T = k_{T,1} \) in the argument of the Hankel functions, the axial–spectral representation for the fields in a waveguide with bounded cross-section has branch points \( k_2 = \pm k_1 = \pm a(k_1r_1)^{1/2} \) and branch cuts \( \text{Im}(k_1^2 - k_2^2)^{1/2} = 0 \), associated with the outer medium.
B. ORTHOGONALITY PROPERTIES OF THE MODAL FIELD DISTRIBUTIONS

The modal field constituents propagating in the direction of increasing \( z \), as introduced in (3.22), satisfy the source–free electromagnetic field equations (cf. (3.30), (3.31))

\[
y^k \times H_k - j\omega E_k = 0, \tag{B.1}
\]

\[
y^k \times E_k + j\omega\mu H_k = 0, \tag{B.2}
\]

in which \( y^k = \sqrt{\frac{\gamma}{c}} - jk_z \); here, \( k = \kappa_n \ (n = 1, \ldots, N) \) and \( \{ E_k, H_k \} = \{ E_m, H_m \} \) for surface-wave modes, and \( k = \kappa \in S^+ \) and \( \{ E_k, H_k \} = \{ E_R, H_R \} \) for radiation modes.

It is pointed out that \( \{ \kappa_n \} \) and \( S^+ \) are located in the fourth quadrant of the complex \( k_z \)-plane (Fig. B.1).

![Fig. B.1. Complex \( k_z \)-plane with branch points \( k_z = \pm \kappa_1 \), branch cuts \( S^+ \) and \( S^- \) (Im(\( k_z^2 - k_2^2, 1/2 = 0 \)), and surface-wave poles \( \{ \pm \kappa_n \} \).](image-url)
In this appendix it is shown that the field constituents \( \{E_k, H_k\} \) form an orthogonal set of functions. By use of the orthogonality properties, a convenient normalisation for these modal field constituents (the Lorentz normalisation) will be proposed.

Throughout the analysis two assumptions are made. The surface-wave poles \( \{k_n\}, n=1, \ldots, N \), are supposed not to lie on the branch cut \( \mathcal{A}^+ \) where \( \text{Im}(k_n^2 - k^2)^{1/2} = 0 \).

Furthermore, we assume the modal spectrum to be non-degenerate, i.e., the (non-zero) solution of the equations (B.1), (B.2) with \( k = k_n \) or \( k = k \in \mathcal{A}^+ \), is unique except for a multiplicative constant.

As a first preliminary, we introduce the special electromagnetic Green's state \( \{E^G, H^G\} \) that is generated by a transverse electric or magnetic current source in the plane \( z = 0 \); the pertaining volume current densities are given by \( J^G \), \( K^G \). From (2.10)–(2.17) it readily follows that the axial Fourier transforms of the Green's fields \( \{E^G, H^G\} \) satisfy the equations

\[
\begin{align*}
\hat{V} \times \hat{H}^G - j\omega \hat{E}^G &= a_T^G \delta(z - z_T), \\
\hat{V} \times \hat{E}^G + j\omega \hat{H}^G &= -a_T^G \delta(z - z_T),
\end{align*}
\]

(B.3)

(B.4)

in which \( \hat{V} = V_T - jk \cdot \hat{z} \). The notation \( \{E^G, H^G\} \) combines the two cases of an electric Green's state: \( \{E^G, H^G\} = \{E^{GE}, H^{GE}\} \) if \( a_T^E \neq 0, a_T^M = 0 \), and a magnetic Green's state: \( \{E^G, H^G\} = \{E^{GM}, H^{GM}\} \) if \( a_T^E = 0, a_T^M \neq 0 \). It has been shown in (3.14) and (3.21), that the electromagnetic fields due to an arbitrary transverse electric or magnetic current excitation can be represented by a modal expansion involving surface-wave modes and radiation modes. Hence, also the electromagnetic Green's fields \( \{E^{GE}, H^{GE}\} \) and \( \{E^{GM}, H^{GM}\} \) can be expanded as
\[
\{E_{GE}^n H_{GE}^n\} = \sum_{n=1}^{N^E} \{E_{GE}^n, H_{GE}^n\} \exp(-j\omega z_n) + \int_{\gamma^+} \{E_{GE}^\kappa, H_{GE}^\kappa\} \exp(-j\kappa z) d\kappa \\
(z>0), \quad (B.5)
\]

\[
\{E_{GM}^n H_{GM}^n\} = \sum_{n=1}^{N^M} \{E_{GM}^n, H_{GM}^n\} \exp(-j\omega z_n) + \int_{\gamma^+} \{E_{GM}^\kappa, H_{GM}^\kappa\} \exp(-j\kappa z) d\kappa \\
(z>0), \quad (B.6)
\]

\[
\text{i.e., in terms of their modal field constituents } \{E_{GE}^n, H_{GE}^n\}, \{E_{GE}^\kappa, H_{GE}^\kappa\}, \text{ and } \{E_{GM}^n, H_{GM}^n\}, \{E_{GM}^\kappa, H_{GM}^\kappa\}. \text{ In view of the non-degeneracy of the modal spectrum, these constituents are proportional to the corresponding modal field constituents in (3.14) and (3.21). We can express this proportionality by}
\]

\[
\{E_{GE}^n, H_{GE}^n\} = \alpha_n \{E_{GE}^n, H_{GE}^n\}, \quad \{E_{GE}^\kappa, H_{GE}^\kappa\} = \alpha_\kappa \{E_{GE}^\kappa, H_{GE}^\kappa\}, \quad (B.7)
\]

\[
\{E_{GM}^n, H_{GM}^n\} = \alpha_n \{E_{GM}^n, H_{GM}^n\}, \quad \{E_{GM}^\kappa, H_{GM}^\kappa\} = \alpha_\kappa \{E_{GM}^\kappa, H_{GM}^\kappa\}, \quad (B.8)
\]

\[
\text{for the surface-wave modes and the radiation modes. Here, the coefficients } \alpha_n^{E,M} \text{ and } \alpha_\kappa^{E,M} \text{ depend on } E_T^m, \text{ and are functions of } L_T^m; \text{ expressions for these coefficients will be derived at the end of this appendix. For later use we establish the following representations for the axial Fourier transforms of the transverse components of the}
\]

\[
\tilde{E}_T^E = \sum_{m=1}^{N^E} \alpha_m E_{m,T}^E \frac{2j\kappa m}{k^2 - \kappa^2} + \int_{\gamma^+} \alpha_\kappa^E E_{\kappa,T}^E \frac{2j\kappa}{k^2 - \kappa^2} d\kappa, \quad (B.9)
\]

\[
\tilde{E}_T^H = \sum_{m=1}^{N^E} \alpha_m E_{m,T}^H \frac{2j\kappa m}{k^2 - \kappa^2} + \int_{\gamma^+} \alpha_\kappa^H E_{\kappa,T}^H \frac{2j\kappa}{k^2 - \kappa^2} d\kappa, \quad (B.10)
\]
\[ \frac{\lambda^\Sigma_{GM}}{\Sigma m = 1} a_m^M \delta^M_{m,T} \frac{2jk_z}{\kappa_s^2 - \kappa_m^2} + \int_{\delta^\Sigma} \alpha^M_{\kappa,T} \frac{2jk_z}{\kappa_s^2 - \kappa^2} d\kappa \] (B.11)

\[ \frac{\lambda^\Sigma_{GM}}{\Sigma m = 1} a_m^M \delta^M_{m,T} \frac{2jk_z}{\kappa_s^2 - \kappa_m^2} + \int_{\delta^\Sigma} \alpha^M_{\kappa,T} \frac{2jk_z}{\kappa_s^2 - \kappa^2} d\kappa \] (B.12)

cf. (3.12) and (3.13), (3.19) and (3.20). The representations (B.9)–(B.12) are valid in the entire cut \( k_s \)-plane.

As a second preliminary, we shall derive a relation that is basic in the proof of the orthogonality properties of the modal field constituents. To that end, we scalarly multiply (B.3) by \( E_k^G \) and (B.1) by \( -E_k^G \), and add the resulting equations. We then find

\[ E_k \cdot (\nabla_k \cdot \tilde{E}^G) - \tilde{E}^G \cdot (\nabla_k \cdot \tilde{H}^G) - jk_z E_k \cdot (i_z \times \tilde{H}^G) + \mu \tilde{E}^G \cdot (i_z \times \tilde{H}^G) = \gamma_{r_2} E_k \delta (\tau_{r_2} - \tau_k). \] (B.13)

Upon scalar multiplication of (B.4) by \( H_k \) and of (B.2) by \( -\tilde{H}^G \), and by subsequent addition of the resulting equations, we obtain

\[ H_k \cdot (\nabla_k \cdot \tilde{E}^G) - \tilde{E}^G \cdot (\nabla_k \cdot \tilde{H}^G) - jk_z H_k \cdot (i_z \times \tilde{E}^G) + \mu \tilde{H}^G \cdot (i_z \times \tilde{E}^G) = -\gamma_{r_2} H_k \delta (\tau_{r_2} - \tau_k). \] (B.14)

Addition of (B.13) and (B.14) yields

\[ \nabla_{r_2} \cdot (\tilde{E}^G \cdot H_k - \tilde{H}^G \cdot E_k) - j(k_z + k) i_z (\tilde{E}^G \times H_k - \tilde{H}^G \times E_k) = (\gamma_{r_2} E_k - \gamma_{r_2} H_k) \delta (\tau_{r_2} - \tau_k). \] (B.15)
Integration of (8.15) over the entire cross-sectional plane \( \mathcal{R}_T \) and subsequent application of Gauss' theorem lead to

\[
j(k_x + i\nu) \int_{\mathcal{R}_T} i_z \cdot (E_{k,T} + H_{k,T} + G_{k,T}) dA(z_T) = \sum_{l} E_{j,l}^E \cdot H_{k,T}(z_T) - \sum_{l} M_{j,l}^M \cdot H_{k,T}(z_T), \quad (B.16)
\]

which is the desired basic relation. In arriving at (B.16) we have used the property that the integral over the boundary of \( \mathcal{R}_T \) (at infinity) resulting from the first term of (B.15) vanishes. This property is obvious because the Fourier transforms of the Green's fields decay exponentially as \( |z_T| \to -\infty \), while the surface-wave and radiation modal fields remain bounded as \( |z_T| \to \infty \). In (B.16) it is explicitly indicated that only the transverse field components enter into the analysis.

We now come to the actual derivation of the orthogonality properties of the modal field constituents. In (B.16) we first take \( E_{j,l}^E \neq 0, \ m^M = 0, \ k = \kappa_n \); then, \( \langle E_{j,l}^E, E_{l',l}^E \rangle = \langle G_{j,l}^G, G_{l',l}^G \rangle, \ (E_{j,l}^E, H_{l',l}^M) = \langle E_{j,l}^E, H_{l',l}^M \rangle \). By inserting the representations (B.9), (B.10) for \( E_{j,l}^E, G_{j,l}^G \), we have for any \( \delta^E_T, \kappa_n, \delta^M_T \) and \( k \),

\[
j(k_x + i\nu) \sum_{l=1}^{N} \frac{2i\alpha_m^E}{k_x^2 - \kappa_x^2} \int_{\mathcal{R}_T} i_z \cdot [E_{j,l}^E \cdot H_{l,n}^M - \kappa_n^2 E_{j,l}^E \cdot H_{l,n}^M] dA(z_T) = \sum_{l=1}^{N} E_{j,l}^E \cdot H_{k,T}(z_T) - \sum_{l=1}^{N} M_{j,l}^M \cdot H_{k,T}(z_T) \quad (B.17)
\]

Since the right-hand side is independent of \( k_x \), the left-hand side must be so too. The finite sum in the left-hand side of (B.17) apparently has simple poles at \( k_x = \kappa_n^E \).
m = 1, ..., N^E, with the exception of \( k_{\mathbf{z}} = -\kappa_n \) (if \( \kappa_n \notin \{\kappa_m : m = 1, ..., N^E\} \)), where the zero value of the denominator is compensated by the factor \( j(k_{\mathbf{z}} + \kappa_n) \). To annihilate the pole at \( k_{\mathbf{z}} = \kappa_m^E \), it is necessary that the accompanying integral over \( \mathcal{S}_T \) vanishes at \( k_{\mathbf{z}} = \kappa_m^E \). This results into the relation

\[
\int_{\mathcal{S}_T} i_{\mathbf{z}} \cdot (E_{m,T}^E \cdot \mathbf{H}_{m,T}^E - E_{m,T}^E \cdot \mathbf{H}_{n,T}^E) i dA_T = 0 \quad \text{for all } \kappa_m^E \text{ and } \kappa_n^E, \tag{B.18}
\]

the result for \( \kappa_n = \kappa_m^E \) being trivial. Similarly, to annihilate the pole at \( k_{\mathbf{z}} = -\kappa_m^E \) with \( \kappa_m^E \neq \kappa_n^E \), the relation

\[
\int_{\mathcal{S}_T} i_{\mathbf{z}} \cdot (E_{m,T}^E \cdot \mathbf{H}_{m,T}^E + E_{m,T}^E \cdot \mathbf{H}_{n,T}^E) i dA_T = 0 \quad \text{for all } \kappa_m^E \neq \kappa_n^E, \tag{B.19}
\]

should hold. Addition of (B.18) and (B.19) yields

\[
\int_{\mathcal{S}_T} i_{\mathbf{z}} \cdot (E_{m,T}^E \cdot \mathbf{H}_{m,T}^E) i dA_T = 0 \quad \text{for all } \kappa_m^E \neq \kappa_n^E, \tag{B.20}
\]

and subtraction of (B.18) from (B.19) yields

\[
\int_{\mathcal{S}_T} i_{\mathbf{z}} \cdot (E_{m,T}^E \cdot \mathbf{H}_{n,T}^E) i dA_T = 0 \quad \text{for all } \kappa_m^E \neq \kappa_n^E. \tag{B.21}
\]

Equations (B.20) and (B.21) are the orthogonality relations for an arbitrary surface-wave mode (index \( n \)) and a surface-wave mode due to excitation by a transverse electric current source (index \( m \)).

In (B.16) we next take \( \mathbf{z}_T^E \neq 0, \mathbf{z}_T = \mathbf{0}, \mathbf{k} = \kappa_n \in \mathcal{S}_T^+ \); then, \( (E_{T}^E, H_{T}^E) = (\mathbf{E}^G, \mathbf{G}^G) \),
\begin{align}
\{E_{k}, T^{-1}H_{k}, T\} &= \{E_{\kappa'}, T^{-1}H_{\kappa'}, T\}. \text{ By inserting the representations (B.9) and (B.10) into (B.16), we have for any } \Sigma_{T}^{E}, \Sigma_{T}^{H} \text{ and } k_{z},

j(k_{z} + \kappa')\{E_{m}, T^{-1}H_{m}, T\} &= \sum_{m=1}^{N_{E}} \int_{\Sigma_{T}^{E}} \frac{2j_{0} E_{m}}{2k_{z} - \kappa'} \left( E_{m} - E_{m}^{E} \right) dA(t_{T})

+ \int_{\Sigma_{T}^{H}} \frac{2j_{0} \kappa'}{2k_{z} - \kappa'} \left( E_{m} - E_{m}^{H} \right) dA(t_{T}) \left( \kappa' + E_{m}^{H} \right) dA(t_{T}) = \Sigma_{T}^{E} \cdot E_{\kappa'}, T(t_{T}). \tag{B.22}
\end{align}

As before, the right-hand side is independent of } k_{z}; \text{ hence, the left-hand side must be so too. Again, the finite sum in the left-hand side of (B.22) apparently has simple poles at } k_{z} = \kappa_{m}, m = 1, ..., N_{E}. \text{ In order to annihilate these poles, the accompanying integral over } \Sigma_{T}^{E} \text{ should vanish for } k_{z} = \kappa_{m}. \text{ This leads to the relations}

\begin{align}
\int_{\Sigma_{T}^{E}} i_{E} \left( E_{m} - E_{m}^{E} \right) dA(t_{T}) = 0 \text{ for all } \kappa_{m} \text{ and } \kappa' \in \Sigma_{T}^{E}, \tag{B.23}
\end{align}

and

\begin{align}
\int_{\Sigma_{T}^{H}} i_{E} \left( E_{m} - E_{m}^{H} \right) dA(t_{T}) = 0 \text{ for all } \kappa_{m} \text{ and } \kappa' \in \Sigma_{T}^{H}. \tag{B.24}
\end{align}

By addition and subtraction of (B.23) and (B.24) we obtain

\begin{align}
\int_{\Sigma_{T}^{E}} i_{E} \left( E_{m} - E_{m}^{E} \right) dA(t_{T}) = 0 \text{ for all } \kappa_{m} \text{ and } \kappa' \in \Sigma_{T}^{E}, \tag{B.25}
\end{align}
\[
\int_{\partial T} i_{x'}(E_{m,T}^{E} \cdot H_{x',T}) dA(t_T) = 0 \quad \text{for all } \kappa_m^E \text{ and } x' \in \partial T. \tag{B.26}
\]

Equations (B.25) and (B.26) are the orthogonality relations for a radiation mode and a surface-wave mode due to excitation by a transverse electric current source.

In (B.16) we next take \( E_T^E = 0, \ E_T^M \neq 0 \), then, \( \{E_T^G, H_T^G\} = \{E_T^G M, H_T^G M\} \), corresponding to the magnetic Green's state generated by a transverse magnetic current source. Furthermore we set \( k = \kappa_n^E \) and \( \{E_n, H_n\} = \{E_n^G M, H_n^G M\} \). Then by inserting the representations (B.11), (B.12) for \( E_T^G M, H_T^G M \) into (B.16), we have for any \( \kappa_m^E, \kappa_n^E \) and \( \kappa_x^E \),

\[
j(k_x^E + \kappa_n^E) \left\{ \sum_{m=1}^{N^M} \frac{2j\omega}{k_x^E - \kappa_m^E} \int_{\partial T} i_{x'} \left[ \kappa_m^E (E_{m,T}^E \cdot H_{x',T}^M) - k_x^E (E_{m,T}^E \cdot H_{n,T}^M) \right] dA(t_T) \right\}
\]

\[
+ \int_{\partial T} \frac{2j\omega}{k_x^E - \kappa_x^E} \left[ \kappa_x^E (E_{n,T}^E \cdot H_{x',T}^M) - k_x^E (E_{n,T}^E \cdot H_{n,T}^M) \right] dA(t_T) dk_x^E \right\}
\]

\[
= -E_T^G M \cdot H_n^G M (I_T). \tag{B.27}
\]

Since the right-hand side is independent of \( k_x^E \), the left-hand side must be so too. The finite sum in the left-hand side of (B.27) apparently has simple poles at \( k_x^E = \kappa_m^E, \ m = 1, \ldots, N^M \), with the exception of \( k_x^E = -\kappa_n^E \) (if \( \kappa_n^E \notin \{\kappa_m^E, m = 1, \ldots, N^M\} \)), where the zero value of the denominator is compensated by the factor \( j(k_x^E + \kappa_n^E). \) To annihilate the pole at \( k_x^E = \kappa_m^E \), it is necessary that the accompanying integral over \( \partial T \) vanishes at \( k_x^E = \kappa_m^E \). This results into the relation
\[ \int_{\mathcal{F}_T} i_\kappa^M (E_{n,T}^M \cdot \mathbb{H}_{n,T}^M - E_{m,T}^M \cdot \mathbb{H}_{n,T}^M) \, dA(\mathcal{F}_T) = 0 \quad \text{for all } \kappa_m^M \text{ and } \kappa_n^M, \tag{B.28} \]

the result for \( \kappa_n^M = \kappa_m^M \) being trivial. Similarly, to annihilate the pole at \( k^M = -\kappa_m^M \) with \( \kappa_m^M \neq \kappa_n^M \), the relation

\[ \int_{\mathcal{F}_T} i_{\kappa'}(E_{n,T}^M \cdot \mathbb{H}_{m,T}^M + E_{m,T}^M \cdot \mathbb{H}_{n,T}^M) \, dA(\mathcal{F}_T) = 0 \quad \text{for all } \kappa_m^M \neq \kappa_n^M \tag{B.29} \]

should hold. Addition of (B.28) and (B.29) yields

\[ \int_{\mathcal{F}_T} i_{\kappa}^M (E_{n,T}^M \cdot \mathbb{H}_{m,T}^M) \, dA(\mathcal{F}_T) = 0 \quad \text{for all } \kappa_m^M \neq \kappa_n^M. \tag{B.30} \]

and subtraction of (B.28) from (B.29) yields

\[ \int_{\mathcal{F}_T} i_{\kappa}^M (E_{m,T}^M \cdot \mathbb{H}_{n,T}^M) \, dA(\mathcal{F}_T) = 0 \quad \text{for all } \kappa_m^M \neq \kappa_n^M. \tag{B.31} \]

Equations (B.30) and (B.31) are the orthogonality relations for an arbitrary surface-wave mode (index \( n \)) and a surface-wave mode due to excitation by a transverse magnetic current source (index \( m \)).

In (B.16) we next take \( \frac{E_T}{T} = 0 \), \( \frac{E^n_T}{T} \neq 0 \), \( k = \kappa' \in \mathcal{F}_T \); then, \( \{E_T^G \cdot \mathbb{H}_T^G\} = \{E_T^M \cdot \mathbb{H}_T^M\} \), \( \{E_{k,T}^G \cdot \mathbb{H}_{k,T}^G\} = \{E_{\kappa',T}^G \cdot \mathbb{H}_{\kappa',T}^G\} \). By inserting the representations (B.11) and (B.12) into (B.16), we have for any \( h_m^M, t_m^M \) and \( k_n^M \),
\[ j(k_z + \kappa)( \sum_{m=1}^{N^M} \frac{2 \sigma_m^M}{k_z^2 - \kappa_m^2} ) \int_{\mathcal{A}_T} i_\varepsilon \cdot [ \kappa_m^M(E_{m,T}^T \cdot \mathbf{H}_{m,T}^M) - k_z(E_{m,T}^T \cdot \mathbf{H}_{m,T}^M)] dA(\varepsilon_T) \]

\[ + \int_{\mathcal{A}_T} \frac{2 \sigma_{\kappa}^M}{k_z^2 - \kappa^2} \int_{\mathcal{A}_T} i_\varepsilon \cdot [\kappa(E_{\kappa,T}^T \cdot \mathbf{H}_{\kappa,T}^M) - k_z(E_{\kappa,T}^T \cdot \mathbf{H}_{\kappa,T}^M)] dA(\varepsilon_T) d\kappa \]

\[ = -2M_{T} \cdot \mathbf{H}_{\kappa, T}(\varepsilon_T). \quad (B.32) \]

As before, the right-hand side is independent of \( k_z \); hence, the left-hand side must be so too. Again, the finite sum in the left-hand side of (B.32) apparently has simple poles at \( k_z = \kappa_m^M, \ m = 1, ..., N^M \). In order to annihilate these poles, the accompanying integral over \( \mathcal{A}_T \) should vanish for \( k_z = \kappa_m^M \). This leads to the relations

\[ \int_{\mathcal{A}_T} i_\varepsilon \cdot [E_{\kappa,T}^T \cdot \mathbf{H}_{m,T}^M - E_{m,T}^T \cdot \mathbf{H}_{\kappa,T}^M] dA(\varepsilon_T) = 0 \quad \text{for all } \kappa_m^M \text{ and } \kappa' \in \mathcal{A}_T^+, \quad (B.33) \]

and

\[ \int_{\mathcal{A}_T} i_\varepsilon \cdot [E_{\kappa,T}^T \cdot \mathbf{H}_{m,T}^M + E_{m,T}^T \cdot \mathbf{H}_{\kappa,T}^M] dA(\varepsilon_T) = 0 \quad \text{for all } \kappa_m^M \text{ and } \kappa' \in \mathcal{A}_T^+. \quad (B.34) \]

By addition and subtraction of (B.33) and (B.34) we obtain

\[ \int_{\mathcal{A}_T} i_\varepsilon \cdot [E_{\kappa,T}^T \cdot \mathbf{H}_{m,T}^M] dA(\varepsilon_T) = 0 \quad \text{for all } \kappa_m^M \text{ and } \kappa' \in \mathcal{A}_T^+, \quad (B.35) \]

and
\[ \int_{\mathcal{S}_T} i_{x'} \left( \mathbf{E}_{m,T}^M \times \mathbf{H}_{m,T} \right) dA(z_T) = 0 \quad \text{for all } \kappa_m^M \text{ and } \kappa' \in \mathcal{A}^+. \quad (B.36) \]

Equations (B.35) and (B.36) are the orthogonality relations for a radiation mode and a surface-wave mode due to excitation by a transverse magnetic current source.

By combining the results for transverse electric and transverse magnetic excitations, it is seen either from (B.20) and (B.30), or from (B.21) and (B.31), that any two different surface-wave modes satisfy

\[ \int_{\mathcal{S}_n} i_{y'} \left( \mathbf{E}_{n,T} \times \mathbf{H}_{m,T} \right) dA(z_T) = 0 \quad \text{for all } m,n. \quad (B.37) \]

Likewise, it is recognised from (B.25) and (B.35), and from (B.26) and (B.36), that for any surface-wave mode and any radiation mode one has the relations

\[ \int_{\mathcal{S}_T} i_{y'} \left( \mathbf{E}_{m,T} \times \mathbf{H}_{m,T} \right) dA(z_T) = 0 \quad \text{for all } m \text{ and } \kappa' \in \mathcal{A}^+. \quad (B.38) \]

\[ \int_{\mathcal{S}_T} i_{y'} \left( \mathbf{E}_{m,T} \times \mathbf{H}_{m',T} \right) dA(z_T) = 0 \quad \text{for all } m \text{ and } \kappa' \in \mathcal{A}^+. \quad (B.39) \]

Equation (B.37) is the orthogonality relation for two surface-wave modal field distributions, while equations (B.38) and (B.39) are the orthogonality relations for a surface-wave mode and a radiation mode.

We now use (B.37), (B.38) and (B.39) in (B.17). Then it is found that all terms in the left-hand side of (B.17) drop out, except the term of the finite sum for which \( \kappa_m = \kappa \). As a consequence, the relation (B.17) simplifies to
\[ 2a_{n}^{T} \int i_{z} \cdot (E_{n,T}^{E} \times H_{n,T}^{E}) dA(\tau_{T}) = -2a_{T} E_{n,T}^{E}(\xi_{T}). \] (B.40)

Similarly, when using (B.37), (B.38) and (B.39) in (B.27), it is recognised that all terms in the left-hand side of (B.27) drop out, except the term of the finite sum for which \( \kappa_{m}^{M} = \kappa_{n}^{a} \). As a consequence, the relation (B.27) simplifies to

\[ 2a_{n}^{T} \int i_{z} \cdot (E_{n,T}^{M} \times H_{n,T}^{M}) dA(\tau_{T}) = -a_{T} H_{n,T}^{M}(\xi_{T}). \] (B.41)

For computational purposes it is advantageous to normalise the transverse modal field distributions appropriately. Thus we introduce, both for the modes excited by a transverse electric current source and for the modes excited by a transverse magnetic current source, the normalised surface-wave modal field constituents \( \{ e_{n}^{a}, b_{n}^{a} \} \), which satisfy the Lorentz normalisation condition

\[ \int i_{z} \cdot (e_{n,T}^{a} \times b_{n,T}^{a}) dA(\tau_{T}) = \frac{1}{2}. \] (B.42)

Then equations (B.37) and (B.42) can be combined to

\[ \int i_{z} \cdot (e_{n,T}^{a} \times b_{m,T}^{a}) dA(\tau_{T}) = \frac{1}{2} \delta_{n,m}. \] (B.43)

Next we use (B.38) and (B.39) in (B.22), then the finite sum drops out and we are left with the relation
\[ j(k_x + \kappa) \int_{\mathcal{A}^+} \frac{2\nu E^E}{k_x^2 - \kappa^2} \int_{\mathcal{F}^T} i_x \left[ k_x (E_{\kappa^x, T}^E H_{\kappa^x, T}^E) - \kappa (E_{\kappa^x, T}^E H_{\kappa^x, T}^E) \right] dA(z_T) d\kappa \]

\[ = \left[ \mathcal{E}^E_{\kappa^x, T}(z_T) \right]. \quad (B.44) \]

When considered as a function of \( k_x \), the left-hand side of (B.44) is apparently discontinuous across \( \mathcal{A}^+ \), whereas the right-hand side is independent of \( k_x \). We now take the limits as \( k_x \) tends to \( \kappa \in \mathcal{F}^+ \), from either side of \( \mathcal{A}^+ \). Then the jump in the left-hand side of (B.44) across \( \mathcal{A}^+ \) is proportional to the integrand of the \( \kappa \)-integration, taken at \( \kappa = \kappa' \). Since this jump should vanish, we are led to the relation

\[ \int_{\mathcal{F}^T} i_x \left[ (E_{\kappa^x, T}^E H_{\kappa^x, T}^E) - \kappa E_{\kappa^x, T}^E H_{\kappa^x, T}^E \right] dA(z_T) = 0 \quad \text{for all } \kappa', \kappa'' \in \mathcal{F}^+. \quad (B.45) \]

the result for \( \kappa = \kappa'' \) being trivial. Likewise, we take the limits in (B.44) as \( k_x \to -\kappa'' \), where \( \kappa'' \in \mathcal{F}^+ \) and \( \kappa'' \neq \kappa' \). The discontinuity of the left-hand side of (B.44) is again proportional to the integrand of the \( \kappa \)-integration, taken at \( \kappa = \kappa'' \). Since this discontinuity should vanish, we are led to the relation

\[ \int_{\mathcal{F}^T} i_x \left[ (E_{\kappa^x, T}^E H_{\kappa^x, T}^E) + \kappa E_{\kappa^x, T}^E H_{\kappa^x, T}^E \right] dA(z_T) = 0 \quad \text{for all } \kappa' \neq \kappa'', \kappa', \kappa'' \in \mathcal{F}^+. \quad (B.46) \]

When \( \kappa'' = \kappa' \), the discontinuity of the outer integral in the left-hand side of (B.44) as \( k_x \to -\kappa'' \), is annihilated by the factor \( j(k_x + \kappa) \). Addition of (B.45) and (B.46) yields

\[ \int_{\mathcal{F}^T} i_x \left[ (E_{\kappa^x, T}^E H_{\kappa^x, T}^E) \right] dA(z_T) = 0 \quad \text{for all } \kappa' \neq \kappa'', \kappa', \kappa'' \in \mathcal{F}^+, \quad (B.47) \]
and subtraction of (B.45) from (B.46) yields

\[
\int_{\mathcal{B}_T} i_x \cdot \left( \mathbf{E}_{\kappa''}^{\mathcal{E}} \times \mathbf{H}_{\kappa, T}^{\mathcal{E}} \right) dA(T_T) = 0 \quad \text{for all } \kappa' \neq \kappa'', \kappa', \kappa'' \in \mathcal{E}^T. \tag{B.48}
\]

Equations (B.47) and (B.48) are the orthogonality relations for an arbitrary radiation mode (index \(\kappa'\)) and a radiation mode due to excitation by a transverse electric current source (index \(\kappa''\)).

When using (B.38) and (B.39) in (B.32), the finite sum drops out and we are left with the relation

\[
\langle \kappa_2, \kappa' \rangle \int \frac{2 j_0 \kappa}{\mathcal{E}^T} i_x \cdot \left( \mathbf{E}_{\kappa', T}^{\mathcal{M}} \times \mathbf{H}_{\kappa, T}^{\mathcal{M}} \right) dA(T_T) d\kappa
\]

\[
= -i_x \mathbf{E}_{\kappa', T}^{\mathcal{M}} \cdot \mathbf{H}_{\kappa, T}^{\mathcal{M}} (T). \tag{B.49}
\]

Again, when considered as a function of \(k_x\), the left-hand side of (B.49) is apparently discontinuous across \(\mathcal{E}^T\), whereas the right-hand side is independent of \(k_x\). We now take the limits as \(k_x\) tends to \(\kappa'' \in \mathcal{E}^T\), from either side of \(\mathcal{E}^T\). Then the jump in the left-hand side of (B.49) across \(\mathcal{E}^T\) is proportional to the integrand of the \(\kappa\)-integration, taken at \(\kappa = \kappa''\). Since this jump should vanish, we are led to the relation

\[
\int_{\mathcal{B}_T} i_x \cdot \left( \mathbf{E}_{\kappa', T}^{\mathcal{E}} \times \mathbf{H}_{\kappa, T}^{\mathcal{E}} \right) dA(T_T) = 0 \quad \text{for all } \kappa', \kappa'' \in \mathcal{E}^T, \tag{B.50}
\]

the result for \(\kappa' = \kappa''\) being trivial. Likewise, we take the limits in (B.49) as \(k_x \to -\kappa''\), where \(\kappa'' \in \mathcal{E}^T\) and \(\kappa'' \neq \kappa'\). The discontinuity of the left-hand side of (B.49) is again
proportional to the integrand of the $\kappa$-integration, taken at $\kappa = \kappa'''$. Since this discontinuity should vanish, we are led to the relation

$$
\int_{\mathcal{S}_T} \left( E_{\alpha''', T} \cdot \mathcal{H}_{\alpha''', T}^M + E_{\alpha''', T} \cdot \mathcal{H}_{\alpha''', T}^M \right) dA(\xi_T) = 0 \quad \text{for all } \kappa' \neq \kappa''', \kappa', \kappa'' \in \mathcal{S}^+ . \tag{B.51}
$$

When $\kappa''' = \kappa'$, the discontinuity of the outer integral in the left-hand side of (B.49) as $k_x \to -\kappa'''$ is annihilated by the factor $i[k_x + \kappa']$. Addition of (B.50) and (B.51) yields

$$
\int_{\mathcal{S}_T} \left( E_{\alpha', T} \cdot \mathcal{H}_{\alpha', T}^M \right) dA(\xi_T) = 0 \quad \text{for all } \kappa' \neq \kappa''', \kappa', \kappa'' \in \mathcal{S}^+ , \tag{B.52}
$$

and subtraction of (B.50) from (B.51) yields

$$
\int_{\mathcal{S}_T} \left( E_{\alpha'''', T} \cdot \mathcal{H}_{\alpha'''', T}^M \right) dA(\xi_T) = 0 \quad \text{for all } \kappa' \neq \kappa''', \kappa', \kappa'' \in \mathcal{S}^+ . \tag{B.53}
$$

Equations (B.52) and (B.53) are the orthogonality relations for an arbitrary radiation mode (index $\kappa'$) and a radiation mode due to excitation by a transverse magnetic current source (index $\kappa'''$).

Combining (B.47) and (B.52), or (B.48) and (B.53), we obtain the orthogonality relation for any two different radiation modes:

$$
\int_{\mathcal{S}_T} \left( E_{\alpha', T} \cdot \mathcal{H}_{\alpha''', T}^M \right) dA(\xi_T) = 0 \quad \text{for all } \kappa' \neq \kappa''', \kappa', \kappa'' \in \mathcal{S}^+ . \tag{B.54}
$$

Using (B.45) in (B.44), we obtain
\[ j(k_z + \kappa) \int_{\mathcal{S}^2} \frac{2i \alpha M}{k_z + \kappa} \int_{\mathcal{A}_T} i_E \cdot (E_{\kappa', T} \times B_{\kappa, T}) dA(t_T) \, dk = \frac{2}{k_{T,1}} E_{\kappa, T}(t_{T,1}). \]  

(B.55)

and using (B.50) in (B.49), we obtain

\[ j(k_z + \kappa) \int_{\mathcal{S}^2} \frac{2i \alpha M}{k_z + \kappa} \int_{\mathcal{A}_T} i_E \cdot (E_{\kappa', T} \times H_{\kappa, T}) dA(t_T) \, dk = \frac{2}{k_{T,1}} H_{\kappa', T}(t_{T,1}). \]  

(B.55)

From (B.54), (B.55) and (B.56), we infer that the integral

\[ I(\kappa, \kappa') = \int_{\mathcal{A}_T} i_E \cdot (E_{\kappa', T} \times \mathbf{H}_{\kappa, T}) dA(t_T) \]  

(B.57)

behaves like a delta function with its singularity at \( \kappa = \kappa' \). Since the delta function is only defined for real values of its argument, we change the variable of integration in (B.56) and (B.57). By introducing \( k_{T,1} = (k_z^2 - \kappa^2)^{1/2} \), \( k'_{T,1} = (k_z^2 - (\kappa')^2)^{1/2} \), \( \kappa, \kappa' \in \mathcal{S}^2 \), and by taking \( k_{T,1} \) as the (real) variable of integration in (B.55) and (B.56), we have

\[ \int_{\mathcal{S}^2} [...] \, dk = \int_{\mathcal{S}^2} [...] \frac{k_{T,1}}{k} \, dk_{T,1}. \]

Then it is immediately recognised that the integral \( I(\kappa, \kappa') \) is proportional to \( \delta(k_{T,1} - k'_{T,1}) \). It is customary to normalise the radiation modes by the Lorentz normalisation condition, viz.

\[ \int_{\mathcal{S}^2} i_E \cdot (E_{\kappa, T} \times h_{\kappa, T}) dA(t_T) = -\frac{1}{2} \frac{\kappa}{k_{T,1}} \delta(k_{T,1} - k'_{T,1}). \]  

(B.58)

where \( \{e_{\kappa}, h_{\kappa}\} \) denote the normalised radiation modal field constituents.
Finally, we determine the coefficients $\alpha^{EM}$ introduced in (B.7) and (B.8). To that end, we replace the modal expansion functions $\{E^E_n, H^E_n\}$ and $\{E^M_\kappa, H^M_\kappa\}$ in (B.7) and (B.8) by the Lorentz-normalised modal field constituents $\{e_n^E, h_n^E\}$ and $\{e_n^M, h_n^M\}$. Thus we re-define the coefficients $\alpha^{EM}$ by

\[
\begin{align*}
\{E^E_n, H^E_n\} &= \alpha_n^E \{e_n^E, h_n^E\}, & \{E^M_\kappa, H^M_\kappa\} &= \alpha_\kappa^E \{e_\kappa^E, h_\kappa^E\}, \\
\{E^M_n, H^M_n\} &= \alpha_n^M \{e_n^M, h_n^M\}, & \{E^M_\kappa, H^M_\kappa\} &= \alpha_\kappa^M \{e_\kappa^M, h_\kappa^M\}.
\end{align*}
\]

Then it is found from (B.40) and (B.42) that

\[
\alpha_n^E = -\frac{1}{T^2} \varepsilon_{n, T}(e^T_n),
\]

and from (B.41) and (B.42) that

\[
\alpha_n^M = -\frac{1}{T^2} \varepsilon_{n, T}(e^T_n).
\]

Likewise, (B.55) and (B.58) imply

\[
\alpha_\kappa^E = -\frac{1}{T^2} \varepsilon_{\kappa, T}(e^T_\kappa),
\]

and (B.56) and (B.58) imply

\[
\alpha_\kappa^M = -\frac{1}{T^2} \varepsilon_{\kappa, T}(e^T_\kappa).
\]

The coefficients $\alpha^{EM}$ from (B.61)–(B.64) are inserted into the representations (B.9)–(B.12) with $E^E_m, H^E_m, E^M_\kappa, H^M_\kappa$ replaced by the Lorentz-normalised fields $e_n^E, h_n^E, e_n^M, h_n^M$. By inverse Fourier Transformation of these representations we
obtain modal expansions of the form (3.28) for the transverse field components $E^G_T$, $H^G_T$, $E^M_T$, $H^M_T$. Next, by use of (2.18)–(2.21), one may determine modal expansions for the transverse parts of the Green's tensors $G^{EE}$, $G^{EM}$, $G^{ME}$, $G^{MM}$, the results are found to agree with (3.26)–(3.29).

In the foregoing analysis, it has been assumed that the surface–wave poles $\{\kappa_n\}$, $n = 1, \ldots, N$, do not lie on the branch cut $\mathcal{C}^\tau$. Now, it may happen that some $\kappa_n$ is located on the branch cut, namely at a certain frequency which is called the cut–off frequency of the pertaining surface–wave mode. This case can be handled as the limit of the case where $\kappa_n$ is located away from the branch cut.
C SYMMETRY PROPERTIES OF THE GREEN'S TENSOR ELEMENTS OF AN INFINITE OPEN WAVEGUIDE

In this appendix we establish the symmetry properties in $z - z'$ of the elements of the Green's tensors for an infinite open waveguide with constitutive parameters $\varepsilon$ and $\mu$, that are independent of the axial coordinate $z$. These Green's tensors are defined by (2.18)-(2.21) and satisfy the radiation condition at infinity; no further boundary conditions are imposed.

The relevant symmetry properties can be derived from equations (2.10)-(2.21). Since the properties of the medium are independent of $z$, the only $z$-dependence in the equations (2.14)-(2.17) for the Green's states arises from the delta functions in $J^G_T$ and $K^G_M$. Consequently, the Green's tensors depend on $z$ and $z'$ through $z - z'$ only.

We decompose the Green's tensor $G$ into the constituents $G_{TT}$, $G_{xT}$, $G_{Tx}$, $G_{zz}$ as given in (4.13)-(4.16). These constituents and their related Green's fields and excitations are listed in Table C I.

The symmetry in $z - z'$ of the fields excited by an arbitrary transverse electric current distribution $J_T(z')\delta(z-z')$ is derived from (3.15), while the symmetry in $z - z'$ of the fields excited by an arbitrary transverse magnetic current distribution $K_T(z')\delta(z-z')$ is derived from (3.22); note that the argument $z$ in (3.15) and (3.22) is to be replaced by $z - z'$. As a result, $G_{TT}(z',z)$ and $G_{Tx}(z',z)$ (which are related to transverse excitation), exhibit the symmetry indicated in Table C II. The symmetry of the Green's tensor constituents that are related to axial electric excitation is determined by considering the axial component of (2.14), together with (2.15). Since $J^G_{TE}(z)$ is an even function of $z - z'$, $E_{z}^G$ and $H_{z}^G$ due to axial excitation are even functions as
Table CI. Constituents of the Green's tensors and related Green's fields and excitations.

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Electric Green's field</th>
<th>Tensor</th>
<th>Magnetic Green's field</th>
<th>Excitation $g^{E,M}<em>N N</em>{T-T'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{G}^{EE}(c'c)$</td>
<td>$\mathbf{E}^{GE}(c)$</td>
<td>$\mathbf{G}^{EM}(c'c)$</td>
<td>$-\mathbf{E}^{GE}(c)$</td>
<td>transverse electric $\mathbf{E}_T$</td>
</tr>
<tr>
<td>$\mathbf{G}^{EE}(c'c)$</td>
<td>$\mathbf{E}^{GE}(c)$</td>
<td>$\mathbf{G}^{EM}(c'c)$</td>
<td>$-\mathbf{E}^{GE}(c)$</td>
<td>axial electric $\mathbf{E}_s$</td>
</tr>
<tr>
<td>$\mathbf{G}^{EE}(c'c)$</td>
<td>$\mathbf{E}^{GE}(c)$</td>
<td>$\mathbf{G}^{EM}(c'c)$</td>
<td>$-\mathbf{E}^{GE}(c)$</td>
<td>transverse electric $\mathbf{E}_T$</td>
</tr>
<tr>
<td>$\mathbf{G}^{EE}(c'c)$</td>
<td>$\mathbf{E}^{GE}(c)$</td>
<td>$\mathbf{G}^{EM}(c'c)$</td>
<td>$-\mathbf{E}^{GE}(c)$</td>
<td>axial electric $\mathbf{E}_s$</td>
</tr>
</tbody>
</table>

well, while $H^{GE}_z$ and $E^{GE}_T$ are odd functions of $z - z'$. Likewise, the symmetry of the Green's tensor constituents that are related to axial magnetic excitation is determined by considering the axial component of (2.17), together with (2.16). Since $H^{GM}_z(z)$ is an even function of $z - z'$, $H^{GM}_z$ and $E^{GM}_z$ due to axial excitation are even functions as well, while $E^{GM}_z$ and $H^{GM}_z$ are odd functions of $z - z'$.

With these results, the symmetry properties of all constituents of the four Green's tensors $G^{E,M}(t',c')$ are known (see Table CI). To establish the symmetry of the constituents of the tensors $G^{E,M}(t',c')$, we use the reciprocity relations (2.32)–(2.34). The resulting symmetry properties of the constituents of $G^{E,M}(t',c')$ are also listed in Table CI.
Table CII. Symmetry in \( z - z' \) of the Green's tensors \( \mathcal{G}(r,r') \) and \( \mathcal{G}(r,r')' \) of the infinite open waveguide.

<table>
<thead>
<tr>
<th>( \mathcal{G}(r,r') )</th>
<th>( \mathcal{G}(r,r')' )</th>
<th>Symmetry</th>
<th>( \mathcal{G}(r,r) )</th>
<th>( \mathcal{G}(r,r)' )</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{G}_{T T}^{E E} )</td>
<td>( \mathcal{G}_{T T}^{E E} )</td>
<td>even</td>
<td>( -\mathcal{G}_{T T}^{E M} )</td>
<td>( \mathcal{G}_{T T}^{M E} )</td>
<td>odd</td>
</tr>
<tr>
<td>( \mathcal{G}_{T z}^{E E} )</td>
<td>( \mathcal{G}_{z T}^{E E} )</td>
<td>odd</td>
<td>( -\mathcal{G}_{z T}^{E M} )</td>
<td>( \mathcal{G}_{z T}^{M E} )</td>
<td>even</td>
</tr>
<tr>
<td>( \mathcal{G}_{T z}^{E E} )</td>
<td>( \mathcal{G}_{z T}^{E E} )</td>
<td>odd</td>
<td>( -\mathcal{G}_{z T}^{E M} )</td>
<td>( \mathcal{G}_{z T}^{M E} )</td>
<td>even</td>
</tr>
<tr>
<td>( \mathcal{G}_{z z}^{E E} )</td>
<td>( \mathcal{G}_{z z}^{E E} )</td>
<td>even</td>
<td>( -\mathcal{G}_{z z}^{E M} )</td>
<td>( \mathcal{G}_{z z}^{M E} )</td>
<td>odd</td>
</tr>
<tr>
<td>( \mathcal{G}_{T T}^{M M} )</td>
<td>( \mathcal{G}_{T T}^{M M} )</td>
<td>odd</td>
<td>( \mathcal{G}_{T T}^{M M} )</td>
<td>( \mathcal{G}_{T T}^{M M} )</td>
<td>even</td>
</tr>
<tr>
<td>( \mathcal{G}_{z T}^{M M} )</td>
<td>( \mathcal{G}_{z T}^{M M} )</td>
<td>even</td>
<td>( \mathcal{G}_{z T}^{M M} )</td>
<td>( \mathcal{G}_{z T}^{M M} )</td>
<td>odd</td>
</tr>
<tr>
<td>( \mathcal{G}_{T z}^{M M} )</td>
<td>( \mathcal{G}_{T z}^{M M} )</td>
<td>even</td>
<td>( \mathcal{G}_{T z}^{M M} )</td>
<td>( \mathcal{G}_{T z}^{M M} )</td>
<td>odd</td>
</tr>
<tr>
<td>( \mathcal{G}_{z Z}^{M M} )</td>
<td>( \mathcal{G}_{z z}^{M M} )</td>
<td>odd</td>
<td>( \mathcal{G}_{z z}^{M M} )</td>
<td>( \mathcal{G}_{z z}^{M M} )</td>
<td>even</td>
</tr>
</tbody>
</table>
D. CALCULATION OF THE AXIAL AND TRANSVERSE FOURIER TRANSFORMS OF THE GREEN'S TENSOR ELEMENTS OF A MULTI-STEP-INDEX PLANAR WAVEGUIDE

The present appendix deals with the calculation of the Fourier transforms, with respect to both the axial and the transverse coordinates, of the Green's tensor elements $G_{yy}^{EE}$ and $G_{xx}^{MM}$ of a multi-step-index planar waveguide. These tensor elements are related to the Green's state field components $E_{y}^{GE}$ and $H_{x}^{GM}$, respectively (cf. (2.18) and (2.21)). The required Fourier transforms of $G_{yy}^{EE}$ and $G_{xx}^{MM}$ are obtained by a Fourier Transformation of the relevant differential equations, the coefficients of which are piecewise constant functions of $x$.

The configuration of the multi-step-index waveguide is shown in Fig. D.1. The waveguide consists of $N-2$ homogeneous layers $\{d_{p} ; p=2, \ldots, N-1\}$ in between $N-1$ planes $\{x_{p} ; p=1, \ldots, N-1\}$, embedded in two homogeneous media present in the semi-infinite domains $d_{1} : -\infty < x < x_{1}$, and $d_{N} : x_{N-1} < x < x_{N} = \infty$. The (constant) permittivity and permeability in the layer $d_{p}$ are $\epsilon_{p}$ and $\mu_{p}$, respectively.

![Fig. D.1. Configuration of the multi-step-index planar waveguide.](image-url)
D.1. Calculation of the Fourier transforms $\hat{G}_{y'y}(k_x',k_y',k_z)$ and $\hat{G}_{x'x}(k_x',k_y',k_z)$

The integral equations (6.28)–(6.29) contain as kernels the tensor elements $\hat{G}_{y'y}(k_x',k_y')$ and $\hat{G}_{x'x}(k_x',k_y')$. These elements are determined by a numerical integration of the representation (cf. (6.32))

$$\hat{G}(k_x',k_y',k_z)|_{z'=0} = (2\pi)^{-1} \int \frac{dk_x'k_y'k_z}{k_x'k_y'k_z} \hat{G}(k_x',k_y',k_z) \, dk_x'$$  \hspace{1cm} (D.1)

Here, $\hat{G}$ denotes the combined axial and transverse Fourier transform of the tensor $G$, defined by (cf. (6.33))

$$\hat{G}(k_x',k_y',k_z) = \int \int \int \exp(ik_x(x-x') + ik_y(y-y') + ik_z(z-z')) \hat{G}(x,y,z,0) \, dx\,dy\,dz.$$  \hspace{1cm} (D.2)

In this section we shall evaluate the tensor elements $\hat{G}_{y'y}(k_x',k_y',k_z)$ and $\hat{G}_{x'x}(k_x',k_y',k_z)$.

From (2.32) and (2.18) it is seen that

$$G_{y'y}(x,x',0) = G_{y'y}(x',0,x) = E_{y}(x,x),$$  \hspace{1cm} (D.3)

where $E_{y}$ is the $y$-component of the electric field due to an electric line source with volume density of electric current $\hat{J} = \hat{J}(x-x')\delta(s)$. Likewise, it is seen from (2.33) and (2.21) that

$$G_{x'x}(x,x',0) = G_{x'x}(x',0,x) = H_{x}(x,x),$$  \hspace{1cm} (D.4)
where $H_x^GM$ is the $x$-component of the magnetic field due to a magnetic line source with volume density of magnetic current $K^G = k_x^G \delta(x-x') \delta(z)$. The field components $E_y^{GE}$ and $H_x^GM$ are to be determined from the systems of field equations (2.14)–(2.16) and (2.16)–(2.17), respectively. Since the fields are $y$-independent, the field equations separate into two independent systems of equations, for TE-fields and for TM-fields. In the sequel we only retain the system for TE-fields with $\{E_y, H_x, E_z\} \neq 0$ and $\{E_x, E_z, H_y\} \equiv 0$.

We first apply a Fourier Transformation with respect to the axial coordinate $z$. For this, we employ the notation

$$\tilde{F}(x,x',z) = \int_{-\infty}^{\infty} \exp(-ik_zz) F(x,x',z) \, dz. \quad (D.5)$$

Then we have, inversely,

$$F(x,x',z) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ik_zz) \tilde{F}(x,x',k_z) \, dk_z. \quad (D.6)$$

In the case of the field component $E_y^{GE}$, the TE-field equations transform into

$$jk_x \tilde{E}_y(x) + j\omega \mu(x) \tilde{H}_x(x) = 0, \quad (D.7)$$

$$-jk_x \tilde{H}_x(x) - \partial_x \tilde{E}_y(x) - j\omega \varepsilon(x) \tilde{E}_y(x) = \tilde{J}_y^G = \delta(x-x'), \quad (D.8)$$

$$\partial_x \tilde{E}_y(x) + j\omega \varepsilon(x) \tilde{H}_x(x) = 0. \quad (D.9)$$
For simplicity, the superscript GE and the dependence of the transformed field components on $x'$ and $k_x$ have been suppressed in the notation. In the case of the field component $H_{x}^{GM}$, the TE-field equations transform into

\begin{align}
  jk_x \tilde{E}_y(x) + j \omega \rho(x) \tilde{H}_x(x) = - \tilde{K}_x^G = - \delta(x - x'), \\
  -jk_x \tilde{H}_x(x) - \frac{\partial}{\partial x} \tilde{E}_y(x) - j \omega \rho(x) \tilde{E}_y(x) = 0, \\
  \frac{\partial}{\partial x} \tilde{E}_y(x) + j \omega \rho(x) \tilde{H}_x(x) = 0.
\end{align}

Again, the superscript GM and the dependence of the transformed field components on $x'$ and $k_x$ have been suppressed in the notation.

Secondly, we apply a Fourier Transformation with respect to the source-point coordinate $x'$. For this, we employ the notation

\begin{equation}
  \mathcal{F}(x, k_x, k_y) = \int_{-\pi}^{\pi} \exp(-j k_x x') \tilde{F}(x, x', k_y) \, dx'.
\end{equation}

Then we have, inversely,

\begin{equation}
  \tilde{F}(x, x', k_y) = (2\pi)^{-1} \int_{-\pi}^{\pi} \exp(j k_x x') \mathcal{F}(x, k_x, k_y) \, dk_x.
\end{equation}

By transformation of (D.7)–(D.9) we obtain the equations

\begin{equation}
  jk_x \tilde{E}_y(x) + j \omega \rho(x) \tilde{H}_x(x) = 0,
\end{equation}
\[-j k_x H_y(x) - \partial_x H_z(x) - j \omega(x) E_y(x) = \int_{\pi}^{0} \hat{H}_x \exp(-j k_x x') \, dx' = \exp(-j k_x x), \quad (D.16)\]

\[\partial_x \bar{E}_y(x) + j \omega(x) \bar{H}_z(x) = 0, \quad (D.17)\]

and by transformation of (D.10)–(D.12) we obtain the equations

\[jk_x F_y(x) + j \omega(x) H_z(x) = - \int_{\pi}^{0} \hat{H}_x \exp(-j k_x x') \, dx' = -\exp(-j k_x x), \quad (D.18)\]

\[-j k_x H_y(x) - \partial_x H_z(x) - j \omega(x) E_y(x) = 0, \quad (D.19)\]

\[\partial_x \bar{E}_y(x) + j \omega(x) \bar{H}_z(x) = 0. \quad (D.20)\]

Thirdly, we apply a Fourier Transformation with respect to the transverse coordinate \(x\). To take advantage of the fact that \(\epsilon(x)\) and \(\mu(x)\) are piecewise constant functions of \(x\), we write the Fourier integral with respect to \(x\) as the sum of the contributions over the intervals \(d_x, p = 1, \ldots, N\), i.e.,

\[\hat{\Phi}(k_x, k_y, k_z) = \sum_{p=1}^{N} \hat{\Phi}_p(k_x, k_y, k_z), \quad (D.21)\]

in which

\[\hat{\Phi}_p(k_x, k_y, k_z) = \int_{x_{p-1}}^{x_p} \exp(j k_x x) \hat{F}(x, k_y, k_z) \, dx. \quad (D.22)\]

Then we have, inversely,
\[ F_p(x, k_x, k_z) = (2\pi)^{-1} \int_{-a}^{a} \exp(-jk_x x) \hat{B}_p(k_x, k_z, k_z) \, dk_z, \]  
(D.23)

and

\[ F(x, k_x, k_z) = \begin{cases} F_p(x, k_x, k_z) & \text{when } x_{p-1} < x < x_p, \\ F_p(x, k_x, k_z) + F_{p+1}(x, k_x, k_z) & \text{when } x = x_p. \end{cases} \]  
(D.24)

Application of the transformation (D.22) to (D.15)–(D.20) gives rise to boundary terms containing \( F_y(x_p), F_y(x_{p-1}), H_z(x_p), \) and \( H_z(x_{p-1}). \) This follows from

\[ \int_{x_a}^{x_b} \exp(jk_x x) \frac{\partial f(x)}{\partial x} \, dx = f(x_b)\exp(jk_x x_b) - f(x_a)\exp(jk_x x_a) - jk_x \int_{x_a}^{x_b} \exp(jk_x x) f(x) \, dx. \]  
(D.25)

In the case of the field component \( B_y \) corresponding to an electric current excitation, the relevant equations (D.15)–(D.17) transform into

\[ jk_x \hat{B}_{y,p} + j\omega_p \hat{H}_{x,p} = 0, \]  
(D.26)

\[ -jk_x \hat{H}_{x,p} + jk_x \hat{H}_{x,p} - j\omega_p \hat{B}_{y,p} - \hat{F}_{y,p}(k_x) + \hat{F}_{y,p+1}(k_x) = \hat{S}_p(k_x - k_x), \]  
(D.27)

\[ -jk_x \hat{B}_{y,p} + j\omega_p \hat{H}_{x,p} + \hat{R}_{x,p}(k_x) - \hat{R}_{x,p+1}(k_x) = 0, \]  
(D.28)

in which
\[ \mathbf{J}_{y,p}(k_x) = \mathbf{H}^{y,p}_z(\mathbf{x}_p) \exp(\mathbf{j}k_x x_p), \quad \mathbf{H}_{z,p}(k_x) = \mathbf{E}^{z,p}_y(\mathbf{x}_p) \exp(\mathbf{j}k_x x_p). \quad \tag{D.29} \]

and the function \( \mathcal{S}_p \) is given by

\[ \mathcal{S}_p(k) = \int_{x_{p-1}}^{x_p} \exp(\mathbf{j}kx) \, dx. \tag{D.30} \]

Notice that the boundary terms (D.29) can be regarded as being due to electric and magnetic surface current distributions in the plane \( x = x_p \). The term \( \mathcal{S}_p(k_x - k'_x) \) in (D.27) stems from the original electric current excitation with volume density \( \mathbf{j}_e = \mathbf{i}_e \delta(x-x') \delta(z) \).

Likewise, in the case of the field component \( \mathbf{H}^{G,\mathbf{M}}_z \) corresponding to a magnetic current excitation, the relevant field equations (D.18)–(D.20) transform into

\[ \mathbf{j}_x \mathbf{A}_{y,p} + j \omega \mu \mathbf{E}_{z,p} = -\mathcal{S}_p(k_x - k'_x), \tag{D.31} \]

\[ -j k_x \mathbf{A}_{z,p} + j k_z \mathbf{A}_{x,p} - j \omega \mu \mathbf{E}_{y,p} + \mathbf{J}_{y,1}(k_x) + \mathbf{J}_{y,p-1}(k_x) = 0, \tag{D.32} \]

\[ -j k_x \mathbf{E}_{y,p} + j \omega \mu \mathbf{A}_{z,p} + \mathbf{H}_{z,1}(k_x) - \mathbf{H}_{z,p-1}(k_x) = 0. \tag{D.33} \]

Here, the boundary terms \( \mathbf{J}_{y,p}(k_x), \mathbf{H}_{z,p}(k_x) \) and the function \( \mathcal{S}_p \) are again given by (D.29) and (D.30). The term \( -\mathcal{S}_p(k_x - k'_x) \) in (D.31) now stems from the original magnetic current excitation with volume density \( \mathbf{H}^{G}_z = j \omega \mathbf{E}(x-x') \delta(z) \).

The function \( \mathcal{S}_p(k) \), as given by (D.30), can easily be evaluated. For later use we present the following results:
for \( p = 2, \ldots, N-1 \),

\[
S_p(k) = \begin{cases} 
\int_{x_{p-1}}^{x_p} \exp(jkx) \, dx = -jk^{-1}(\exp(jkx_p) - \exp(jkx_{p-1})) & \text{when } k \neq 0, \\
-x_{p-1} & \text{when } k = 0 
\end{cases}
\]  
(D.34)

for \( p = 1 \),

\[
S_1(k) = \int_{-\infty}^{x_1} \exp(jkx) \, dx = -jk^{-1}\exp(jkx_1) \quad \text{when } \text{Im}(k) < 0; 
\]  
(D.35)

for \( p = N \),

\[
S_N(k) = \int_{x_{N-1}}^{a} \exp(jkx) \, dx = jk^{-1}\exp(jkx_{N-1}) \quad \text{when } \text{Im}(k) > 0. 
\]  
(D.36)

Equations (D.26)–(D.28) and (D.31)–(D.33) are now solved for the transformed field constituents \( \hat{\mathbf{h}}_{x,p} \) and \( \hat{\mathbf{h}}_{s,p} \). To eliminate \( \hat{\mathbf{h}}_{x,p} \), we infer from (D.26) and (D.31) that

\[
\hat{\mathbf{h}}^{\text{GE}}_{x,p} = -(j\omega\mu_p)^{-1}jk^2 S_{y,p}^{\text{GE}} 
\]  
(D.37)

in the case of electric current excitation, and

\[
\hat{\mathbf{h}}^{\text{GM}}_{x,p} = -(j\omega\mu_p)^{-1}jk^2 S_{y,p}^{\text{GM}} + \hat{S}_p(k-k') 
\]  
(D.38)

in the case of magnetic current excitation. To properly distinguish between the two cases, we have restored the superscripts GE and GM in (D.37) and (D.38). Next, by substitution of (D.37) into (D.27) and (D.28), the resulting equations can readily be
solved for $\Phi_{y,p}$ and $\Phi_{z,p}$ in the case of electric current excitation. Likewise, by substitution of (D.38) into (D.32) and (D.33), the resulting equations can be solved for $\Phi_{y,p}$ and $\Phi_{z,p}$ in the case of magnetic current excitation. Except for a different multiplicative factor $\alpha_p$, the solutions obtained turn out to be of the same form in the two cases, namely

\[
\Phi_{y,p} = (k_x^2 - k_{z,p}^2)^{-1}\left\{-\alpha_p \beta_{p} (k_x - k_z) - \left[jk_x \Phi_{z,p} + j\omega_p \Phi_{y,p}\right](k_x) \right. \\
\left. + \left[jk_x \Phi_{z,p-1} + j\omega_p \Phi_{y,p-1}\right](k_x)\right\}, \tag{D.39}
\]

\[
\Phi_{z,p} = (k_x^2 - k_{T,p}^2)^{-1}\left\{-\left(k_x^2/\omega_p\right)\beta_{p} \beta_{p} (k_x - k_z) - \left[jk_{T,p} \gamma_{x,p} \Phi_{z,p} + jk_x \Phi_{y,p}\right](k_x) \right. \\
\left. + \left[jk_{T,p} \gamma_{x,p} \Phi_{z,p-1} + jk_x \Phi_{y,p-1}\right](k_x)\right\}, \tag{D.40}
\]

in which (cf. (3.67))

\[
\gamma_{x,p} = k_{T,p}/(\omega_p), \quad k_{T,p} = (k_x^2 - k_z^2)^{1/2}, \quad \text{Im}(k_{T,p}) \leq 0, \quad k_x = \omega \epsilon_{p} \mu_p^{1/2}. \tag{D.41}
\]

The factor $\alpha_p$ in (D.39) and (D.40) is given by

\[
\alpha_p = j\omega_p \quad \text{for electric current excitation (superscript GE)}, \tag{D.42}
\]

\[
\alpha_p = -jk_x \quad \text{for magnetic current excitation (superscript GM)}. \tag{D.43}
\]

In (D.39) and (D.40) it is understood that

\[
\Phi_{y,0}(k_x) = \Phi_{z,0}(k_x) = 0, \quad \Phi_{y,N}(k_x) = \Phi_{z,N}(k_x) = 0. \tag{D.44}
\]
because of

\[ E_y(x_0) = \lim_{x \to -\infty} E_y(x) = 0, \quad H_z(x_0) = \lim_{x \to -\infty} H_z(x) = 0, \]

\[ E_y(x_N) = \lim_{x \to +\infty} E_y(x) = 0, \quad H_z(x_N) = \lim_{x \to +\infty} H_z(x) = 0. \]

Then (D.39) and (D.40) are valid for \( p = 1, \ldots, N \).

Through (D.39) and (D.40), the transformed field constituents \( \hat{E}_{y,p} \) and \( \hat{H}_{z,p} \) are expressed in terms of the known function \( \hat{S}_p \) (which stems from the original Green's type current excitation) and the boundary terms \( \hat{I}_{y,p}^{\prime}, \hat{R}_{z,p} \), which involve, by (D.29), the as yet unknown values of \( E_y, H_z \) at the interfaces \( x = x_p, p = 1, \ldots, N-1 \). The latter values are interrelated by transfer matrices in accordance with the transfer-matrix formalism discussed in Subsection 3.4.2. The interrelation by transfer matrices can also be deduced from (D.39) and (D.40). To that end we observe that the right-hand sides of (D.39) and (D.40) apparently have simple poles at \( k_x = \pm k_{T,p} \). Now for \( p = 2, \ldots, N-1 \), the integration (D.22) reduces to a finite Fourier transform; hence, \( \hat{E}_{y,p} \) and \( \hat{H}_{z,p} \) are regular functions of \( k_x \) in the entire complex \( k_x \)-plane. Consequently, the residues of \( \hat{E}_{y,p} \) and \( \hat{H}_{z,p} \) at \( k_x = \pm k_{T,p} \), \( p = 2, \ldots, N-1 \), must vanish. Thus we obtain for \( k_x = k_{T,p} \), both from (D.39) and (D.40),

\[
-\alpha_p \hat{S}_p (k_{T,p} - k_x) - \left[ k_{T,p} \hat{R}_{z,p} + j \omega \mu_p \hat{I}_{y,p} \right](k_{T,p})
\]

\[
+ \left[ k_{T,p} \hat{R}_{z,p-1} + j \omega \mu_p \hat{I}_{y,p-1} \right](k_{T,p}) = 0, \quad p = 2, \ldots, N-1, \quad (D.45)
\]

while for \( k_x = -k_{T,p} \) we obtain, both from (D.39) and (D.40),
\[-\alpha p \mathbf{\hat{E}}_\pmb{p}(-k_{T,p} - k_x') + [j k_{T,p} \mathbf{\hat{E}}_{z,p} - j \omega \mu_p \mathbf{\hat{E}}_{y,p}(-k_{T,p}) - j \omega \mu_p \mathbf{\hat{E}}_{y,p}(-k_{T,p})]
\]
\[= 0, \quad p = 2, \ldots, N-1. \quad (D.46)\]

The transformed field constituents \(\{\mathbf{\hat{E}}_{y,1}, \mathbf{\hat{H}}_{z,1}\}\) and \(\{\mathbf{\hat{E}}_{y,N}, \mathbf{\hat{H}}_{z,N}\}\) result from a Fourier integration over the intervals \(-\alpha < x \leq x_1\) and \(x_{N-1} \leq x < \alpha\), respectively; see (D.22). Hence, \(\mathbf{\hat{E}}_{y,1}\) and \(\mathbf{\hat{H}}_{z,1}\) are regular functions of \(k_x\) in the lower half-plane \(\text{Im}(k_x) < 0\), whereas \(\mathbf{\hat{E}}_{y,N}\) and \(\mathbf{\hat{H}}_{z,N}\) are regular functions of \(k_x\) in the upper half-plane \(\text{Im}(k_x) > 0\). We now take the media present in the semi-infinite domains \(d_1\) and \(d_N\), as slightly lossy. Then \(k_1\) and \(k_N\) become complex-valued with \(\text{Im}(k_1) < 0, \text{Im}(k_N) < 0\), and consequently we have \(\text{Im}(k_{T,1}) < 0, \text{Im}(k_{T,N}) < 0\). To annihilate the pole of \(\mathbf{\hat{E}}_{y,1}\) and \(\mathbf{\hat{H}}_{z,1}\) at \(k_x = k_{T,1}\), the corresponding residue must vanish. This condition leads to

\[-\alpha p \mathbf{\hat{E}}_\pmb{p}((k_{T,1} - k_x') - j \omega \mu_p \mathbf{\hat{E}}_{y,1}(k_{T,1}) = 0, \quad (D.47)\]

obtained both from (D.39) and (D.40) with \(p = 1\). To annihilate the pole of \(\mathbf{\hat{E}}_{y,N}\) and \(\mathbf{\hat{H}}_{z,N}\) at \(k_x = -k_{T,N}\), again the corresponding residue must vanish. This condition leads to

\[-\alpha p \mathbf{\hat{E}}_\pmb{p}((k_{T,N} - k_x') - j \omega \mu_p \mathbf{\hat{E}}_{y,N}(k_{T,N}) = 0, \quad (D.48)\]

obtained both from (D.39) and (D.40) with \(p = N\).

The set of equations (D.45)–(D.48) is reduced by expressing \(\mathbf{\hat{E}}_{y,p}\) and \(\mathbf{\hat{H}}_{z,p}\) in terms of \(\mathbf{\hat{E}}_{y}(x_p)\) and \(\mathbf{\hat{H}}_{z}(x_p)\) by use of (D.29). Furthermore, the functions \(\mathbf{\hat{E}}_{y,p}, \mathbf{\hat{E}}_{z,1}, \mathbf{\hat{E}}_{z,N}\) are evaluated by means of (D.34)–(D.36). Next, the resulting set of equations is solved for \(\mathbf{\hat{E}}_{y}(x_p), \mathbf{\hat{H}}_{z}(x_p)\), expressed in terms of \(\mathbf{\hat{E}}_{y}(x_{p-1}), \mathbf{\hat{H}}_{z}(x_{p-1})\). Omitting the details of
the derivation, we present the solution in the format of the transfer-matrix formalism, viz.

\[
\begin{bmatrix}
E_y(x_p) \\
H_z(x_p)
\end{bmatrix}
= T_p \begin{bmatrix}
E_y(x_{p-1}) \\
H_z(x_{p-1})
\end{bmatrix} - \frac{\alpha_p}{2\mu_p}\begin{bmatrix}
(I_p^+ + I_p^-) / \gamma_{x,p}^E \\\n-I_p^+ + I_p^-
\end{bmatrix}
\]
\[p = 2, ..., N-1,
\]
(D.49)

\[
\begin{bmatrix}
E_y(x_1) \\
H_z(x_1)
\end{bmatrix}
= \begin{bmatrix}
1 \\
-\gamma_{x,1}^E
\end{bmatrix} \begin{bmatrix}
E_y(x_1) \\
H_z(x_1)
\end{bmatrix} - \frac{\alpha_1}{2\mu_1} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
(D.50)

\[
\begin{bmatrix}
E_y(x_{N-1}) \\
H_z(x_{N-1})
\end{bmatrix}
= \begin{bmatrix}
1 \\
-\gamma_{x,N}^E
\end{bmatrix} \begin{bmatrix}
E_y(x_{N-1}) \\
H_z(x_{N-1})
\end{bmatrix} + \frac{\alpha_N}{2\mu_N} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
(D.51)

in which the transfer matrix \(T_p \begin{bmatrix}
E_y(x_{p-1}) \\
H_z(x_{p-1})
\end{bmatrix}\) is given by

\[
T_p \begin{bmatrix}
E_y(x_{p-1}) \\
H_z(x_{p-1})
\end{bmatrix}
= \begin{bmatrix}
\cos(\gamma_{x,p}^E) & -(1/\gamma_{x,p}^E)\sin(\gamma_{x,p}^E) \\
-j\gamma_{x,p}^E \sin(\gamma_{x,p}^E) & \cos(\gamma_{x,p}^E)
\end{bmatrix}
\]
(D.52)

in accordance with (3.66). Here, the following short notations have been employed:

\[
I_p^+ = -j(\gamma_{x,p}^E + k_x^+)^{-1}[\exp(-j\gamma_{x,p}^E) - \exp(-j\gamma_{x,p}^E)]\exp(j\gamma_{x,p}^E d_p), \quad p = 2, ..., N-1.
\]
(D.53)

\[
I_p^- = -j(\gamma_{x,p}^E + k_x^-)^{-1}[\exp(-j\gamma_{x,p}^E) - \exp(-j\gamma_{x,p}^E)]\exp(-j\gamma_{x,p}^E d_p), \quad p = 2, ..., N-1.
\]
(D.54)

\[
I_1 = -j(\gamma_{x,1}^E + k_x^+)^{-1}\exp(-j\gamma_{x,1}^E),
\]
(D.55)

\[
I_N = -j(\gamma_{x,N-1}^E + k_x^-)^{-1}\exp(-j\gamma_{x,N-1}^E),
\]
(D.56)
in which \( d_p = x_p - x_{p-1} \). The factor \( \alpha_p \) is given by (D.42) and (D.43) in the two cases of electric and magnetic current excitation.

Equations (D.49)–(D.51) constitute a system of \( 2(N-1) \) linear, inhomogeneous, algebraic equations for the \( 2(N-1) \) unknown field quantities \( \{ E_y, H_z \} \) at the layer boundaries \( x = x_p, \ p = 1, \cdots, N-1, \) of the waveguide. The equations can be solved directly by a numerical method. However, when \( N \) is large, it might be advantageous to first express \( \{ E_y(x_p), H_z(x_p) \}, \ p = 1, \cdots, N-2, \) in terms of \( E_y(x_1) \) and \( E_y(x_{N-1}) \), and then to solve for \( E_y(x_1) \) and \( E_y(x_{N-1}) \). Such a solution proceeds as follows. By inversion of (D.49) we establish the relation

\[
\begin{bmatrix}
E_y(x_{p-1}) \\
H_z(x_{p-1})
\end{bmatrix} = \alpha_p(x_{p-1} - x_p) \begin{bmatrix}
E_y(x_p) \\
H_z(x_p)
\end{bmatrix}
+ \frac{\alpha_p}{2 \sqrt{\omega_p}} \left[ \frac{\Gamma_p^+ \exp(-j k_{T_p} d_p) + \Gamma_p^- \exp(j k_{T_p} d_p)}{\gamma_p} \right], \quad p = 2, \cdots, N-1, \quad (D.57)
\]

in which the transfer matrix \( \alpha_p(x_{p-1} - x_p) \) is found from (D.52) with \( x_p \) and \( x_{p-1} \) interchanged. We now derive two representations for the field quantities \( E_y, H_z \) at \( x = x_p \). Starting from (D.50), we have, by repeated use of (D.49),

\[
\begin{bmatrix}
E_y(x_p) \\
H_z(x_p)
\end{bmatrix} = \sum_{\mu=1}^{p-1} \begin{bmatrix}
\frac{1}{\omega_p} \\
-1
\end{bmatrix} E_y(x_{\mu}) - \frac{\alpha_p}{\sum_{m=2}^{p-1} 2 \sqrt{\omega_p}} \sum_{m=2}^{\mu=p-1} \begin{bmatrix}
\Gamma_0^+ + \Gamma_1^+ \\
-\Gamma_0^{-1} + \Gamma_1^{-1}
\end{bmatrix} x_m
\]

\[
\begin{bmatrix}
E_y(x_p) \\
H_z(x_p)
\end{bmatrix} = \frac{\alpha_1}{\sqrt{\omega_1}} \sum_{p=1}^{\mu=p-1} \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

(D.58)
Likewise, by starting from (D.51) we have, by repeated use of (D.57),

\[
\begin{align*}
\begin{bmatrix}
E_y(x_p) \\
H_z(x_p)
\end{bmatrix}
&= \prod_{p=0}^{N-1} \begin{bmatrix} 1 \\ \gamma_{x_p, x_{p+1}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \gamma_{y, x_{p+1}} \end{bmatrix} \cdot E_y(x_{N-1}) \\
+ \frac{C_n}{\mu N} \prod_{p=0}^{N-1} \begin{bmatrix} 0 \\ I_{N_p} \end{bmatrix}
\end{align*}
\]

Here, the matrix \( \prod_{p=0}^{N-1} \) stands for the product of the transfer matrices involved, viz.

\[
\prod_{p=a}^{b} E = \begin{bmatrix} \gamma_{a+1, a} \cdot \gamma_{a, a-1} & \gamma_{a+2, a} \cdot \gamma_{a+1, a-2} & \cdots & \gamma_{b+1, a} \cdot \gamma_{b+1, a} \\
1 & \gamma_{a+2, a} \cdot \gamma_{a+1, a-2} & \cdots & \gamma_{b+1, a} \cdot \gamma_{b+1, a} \\
0 & \gamma_{a+2, a} \cdot \gamma_{a+1, a-2} & \cdots & \gamma_{b+1, a} \cdot \gamma_{b+1, a} \\
0 & \gamma_{a+2, a} \cdot \gamma_{a+1, a-2} & \cdots & \gamma_{b+1, a} \cdot \gamma_{b+1, a} 
\end{bmatrix}
\]

When \( a > b \),

\[
\prod_{p=a}^{b} E = \begin{bmatrix} \gamma_{a+1, a} & \gamma_{a+2, a} & \cdots & \gamma_{b+1, a} \\
1 & \gamma_{a+2, a} & \cdots & \gamma_{b+1, a} \\
0 & \gamma_{a+2, a} & \cdots & \gamma_{b+1, a} \\
0 & \gamma_{a+2, a} & \cdots & \gamma_{b+1, a} 
\end{bmatrix}
\]

When \( a = b \),

\[
\prod_{p=a}^{b} E = \begin{bmatrix} \gamma_{a+1, a} \\
1 & \gamma_{a+2, a} \\
0 & \gamma_{a+2, a} \\
0 & \gamma_{a+2, a} 
\end{bmatrix}
\]

By equating (D.58) and (D.59) for some \( p \in \{1,\ldots,N-1\} \), we are led to a system of two linear algebraic equations from which \( E_y(x_p) \) and \( E_y(x_{N-1}) \) can easily be solved. The result is inserted into (D.59) and (D.81), whereupon the field quantities \( \{E_y(x_p), H_z(x_p)\} \), \( p = 1,\ldots,N \), can be determined by repeated use of (D.49) and/or (D.57).

We now return to the calculation of the tensor elements \( \gamma_{y, y'}(k_x, k_y, k_z) \) and \( \gamma_{x, x'}(k_x, k_y, k_z) \). Consider first the case of electric current excitation and restore the superscript \( GE \). Then, by use of (D.29) and (D.42), the expression (D.39) for \( \gamma_{y, p} = \gamma_{y, p}^{GE} \) is rewritten as
\[ \hat{\mathbf{E}}_{y, p}^{GE} = -j\omega \mathbf{p} \left( k_x^2 - k_{-p}^2 \right)^{-1} \mathbf{S}_{p}^{(k_x - k_{-p})} \]

\[ + \left( k_x^2 - k_{-p}^2 \right)^{-1} \left\{ -j k_x \mathbf{E}_{y, p}^{\mathcal{C}}(x, p) + j \omega \mathbf{p} \mathbf{H}_{z, p}^{\mathcal{C}}(x, p) \right\} \exp(j k_x x_p) \]

\[ + [j k_x \mathbf{E}_{y, p}^{\mathcal{C}}(x_{p-1}) + j \omega \mathbf{p} \mathbf{H}_{z, p}^{\mathcal{C}}(x_{p-1})] \exp(j k_x (x_{p-1})) \].

(D.61)

To completely determine \( \hat{\mathbf{E}}_{y, p}^{GE} \), one should substitute the relevant values of \( \{ \mathbf{E}_{y, p}^{\mathcal{C}}(x, p), \mathbf{H}_{z, p}^{\mathcal{C}}(x, p) \} \) into (D.61). These values are obtained by solving the system of equations (D.49)-(D.51) with \( \alpha_p = j \omega \mathbf{p} \) in accordance with (D.42).

Consider next the case of magnetic current excitation and restore the superscript GM. By use of (D.29) and (D.43), the expression (D.32) for \( \hat{\mathbf{E}}_{y, p}^{GM} = \hat{\mathbf{E}}_{y, p}^{GM} \) is reduced to the form (D.61) with the first factor \(-j \omega \mathbf{p}\) replaced by \(j k_x\). The reduced expression for \( \hat{\mathbf{E}}_{y, p}^{GM} \) is then inserted into (D.38), yielding

\[ \hat{\mathbf{F}}_{x, p}^{GM} = -(j \omega \mathbf{p})^{-1} \left( k_x^2 - k_{-p}^2 \right)^{-1} \left( k_x^2 - k_{-p}^2 \right)^{-1} \mathbf{S}_{p}^{(k_x - k_{-p})} \]

\[ - (k_x / \omega \mathbf{p}) \left( k_x^2 - k_{-p}^2 \right)^{-1} \left\{ -j k_x \mathbf{E}_{y, p}^{\mathcal{C}}(x, p) + j \omega \mathbf{p} \mathbf{H}_{z, p}^{\mathcal{C}}(x, p) \right\} \exp(j k_x x_p) \]

\[ + [j k_x \mathbf{E}_{y, p}^{\mathcal{C}}(x_{p-1}) + j \omega \mathbf{p} \mathbf{H}_{z, p}^{\mathcal{C}}(x_{p-1})] \exp(j k_x (x_{p-1})) \].

(D.62)

To completely determine \( \hat{\mathbf{F}}_{x, p}^{GM} \), one should substitute the relevant values of \( \{ \mathbf{E}_{y, p}^{\mathcal{C}}(x, p), \mathbf{H}_{z, p}^{\mathcal{C}}(x, p) \} \) into (D.62). These values are obtained by solving the system of equations (D.49)-(D.51) with \( \alpha_p = -j k_x \) in accordance with (D.43).

In view of (D.3) and (D.4), the expressions (D.61) and (D.62) represent the contributions of the \( p \)-th layer of the waveguide to the tensor elements \( \mathbf{G}_{xy}^{\mathcal{C}} \) and
Thus by summation of these contributions (cf. (D.21)), we obtain as our final results

$$\hat{c}_{xy}(k_x, k_y, k_z) = \sum_{p=1}^{N} \hat{c}_{xy}^{GE}(k_x, p)$$

(D.63)

$$\hat{c}_{MM}(k_x, k_y, k_z) = \sum_{p=1}^{N} \hat{c}_{MM}^{GM}(k_x, p)$$

(D.64)

It remains to evaluate the function $\hat{a}_{p}(k_x, k_y, k_z)$ in (D.61) and (D.62). For $p = 2, ..., N - 1$, it is found from (D.34) that

$$\hat{a}_{p}(k_x, k_y, k_z) = \left\{ \begin{array}{ll}
-j(k_x - k_z)^{-1} \{ \exp(j(k_x - k_z)x_p) - \exp(j(k_x - k_z)x_{p-1}) \} & \text{when } k_x \neq k_z \\x_p - x_{p-1} & \text{when } k_x = k_z
\end{array} \right.$$  

(D.65)

For $p = 1$ and $p = N$ the corresponding integrals (D.35) and (D.38) are divergent for real $k = k_x - k_z$. To overcome this difficulty, the functions $\hat{a}_{1}$ and $\hat{a}_{N}$ are expressed in terms of the Fourier transform of the unit step function, which is well known (Papoulis, 1977, p. 67). In this manner we find

$$\hat{a}_{1}(k_x, k_y, k_z) = j(k_x - k_z)^{-1} \{ \exp(j(k_x - k_z)x_1) + \pi \delta(k_x - k_z) \}$$

(D.66)

$$\hat{a}_{N}(k_x, k_y, k_z) = j(k_x - k_z)^{-1} \{ \exp(j(k_x - k_x)x_{N-1}) + \pi \delta(k_x - k_z) \}$$

(D.67)

In the computations carried out in Section 6.5 the homogeneous media occupying the semi-infinite domains $d_1$ and $d_N$, are identical with permittivity $\varepsilon_1 = \varepsilon_N$ and permeability $\mu_1 = \mu_N$. Then the factors multiplying $\hat{a}_{1}(k_x, k_y, k_z)$ and $\hat{a}_{N}(k_x, k_y, k_z)$ in (D.61)–(D.62) (with $p = 1$ and $p = N$) are the same. By adding the contributions
from the outer domains $d_1$ and $d_N$, it is found that in the expressions for $\hat{G}_{yy}^{EE}$ and $\hat{G}_{xx}^{MM}$ there appears the function

$$S_1(k_x - k_x') + S_N(k_x - k_x')$$

$$= j(k_x - k_x')^{-1} \{ \exp[i(k_x - k_x')x_{N-1}] - \exp[i(k_x - k_x')x_1] \} + 2\pi\delta(k_x - k_x'). \tag{D.68}$$

The same function also comes up as a constituent in the kernels $\hat{G}_{yy}^{EE}(k_x, k_x')$ and $\hat{G}_{xx}^{MM}(k_x, k_x')$, which are obtained by integration of $\hat{G}_{yy}^{EE}$ and $\hat{G}_{xx}^{MM}$ with respect to $k_x$, cf. (D.1). In the integral equations (6.28)–(6.29) the term $\hat{G}(k_x - k_x')$ from (D.68) is handled in the usual manner, i.e., the corresponding integration is carried out by use of the sifting property of the delta function. The first term on the right-hand side in (D.68) gives rise to a kernel that has a removable singularity at $k_x = k_x'$.

D.2. Calculation of the free-space Green's tensors

The Green's tensor elements $\hat{G}_{yy}^{EE}$ and $\hat{G}_{xx}^{MM}$ of a homogeneous medium with permittivity $\varepsilon_1$ and permeability $\mu_1$ for all $\mathbf{r} \in \mathbb{R}^3$ (for shortness referred to as free space), can easily be obtained from (D.62) and (D.64). To that end, we observe that free space may be considered as a waveguide that consists of only one layer $d_1 : -a < x < a$. Then the Fourier Transformation in (D.21) and (D.22) simplifies to

$$\hat{f}(k_x, k_x', k_z) = \int_{-a}^{a} \exp(i k_x x) \tilde{F}(k_x, k_x', k_z) dx. \tag{D.69}$$

Furthermore, the boundary terms $f(x_a)\exp(i k_x x_a)$ and $f(x_b)\exp(i k_x x_b)$ in (D.25) (where $f$ stands for $\mathbf{E}_y$ or $\mathbf{H}_y$, and $x_a = -a$, $x_b = a$ in the present case) will vanish because of the radiation condition. Consequently, the boundary terms $\tilde{F}_{y,0}$ and $\hat{F}_{z,0}$
with \( p = 0, p = 1 \), are equal to zero; cf. (D.44). By taking \( p = 1 \) in (D.61) and (D.62), and by setting the boundary terms \( \{ \mathbf{E}_y, \mathbf{H}_x \}(x_p) \) and \( \{ \mathbf{E}_y, \mathbf{H}_x \}(x_{p-1}) \) equal to zero, we arrive at the following expressions for the tensor elements \( \hat{G}_{yy}^{EB} \) and \( \hat{G}_{xx}^{MM} \):

\[
\hat{G}_{yy}^{EB}(k_x, k_{x}'; k_{x}, k_{x}') = -i\omega \mu_1 (k_x^2 - k_{x}'^2)^{-1} \int_0^\infty \exp(i(k_x - k_{x}')x)dx
\]

\[
= -i\omega \mu_1 (k_x^2 - k_{x}'^2)^{-1}i2\pi\delta(k_x - k_{x}'),
\]

\[
(D.70)
\]

\[
\hat{G}_{xx}^{MM}(k_x, k_{x}'; k_{x}, k_{x}') = -(i\omega \mu_1)^{-1}(k_x^2 - k_{x}'^2)^{-1}(k_x^2 - k_{x}'^2) \int_0^\infty \exp(i(k_x - k_{x}')x)dx
\]

\[
= -(i\omega \mu_1)^{-1}(k_x^2 - k_{x}'^2)^{-1}(k_x^2 - k_{x}'^2) 2\pi\delta(k_x - k_{x}'),
\]

\[
(D.71)
\]

in which \( k_1 = \omega \epsilon_1 \mu_1^{1/2} \), \( k_{x,1} = (k_x^2 - k_{x}^2)^{1/2} \), \( \text{Im}(k_{x,1}) \leq 0 \). In the derivation it is assumed that the medium is slightly lossy, such that \( \text{Im}(k_1) < 0, \text{Im}(k_{x,1}) < 0 \).

Finally, by integration of (D.70) and (D.71) with respect to \( k_z \) (cf. (D.1)), the kernels \( \hat{G}_{yy}^{EB} \) and \( \hat{G}_{xx}^{MM} \) are found to be given by

\[
\hat{G}_{yy}^{EB}(k_x, k_{x}) = -\frac{i}{2\omega \mu_1}(k_x^2 - k_{x}^2)^{-1/2}\delta(k_x - k_{x}'),
\]

\[
(D.72)
\]

\[
\hat{G}_{xx}^{MM}(k_x, k_{x}) = -\frac{i}{2}(\omega \mu_1)^{-1}(k_x^2 - k_{x}^2)^{1/2}\delta(k_x - k_{x}'),
\]

\[
(D.73)
\]

in which \( \text{Im}(k_x^2 - k_{x}^2)^{1/2} \leq 0 \). In the integral equations (6.28)–(6.29) the kernels (D.72)–(D.73) are handled in the obvious manner, i.e., the integration is carried out by use of the shifting property of the delta function.
D.3. Symmetry properties of the tensor elements

From the reciprocity relations (2.32) and (2.33) it follows that

\[ G(x,x',x',x') = G(x',x',x,x), \]  \hspace{1cm} (D.74)

in which \( G \) stands for \( G_{EE}^{XX} \) and \( G_{MM}^{XX} \). In Appendix C it is shown that the Green's tensors depend on \( x \) and \( x' \) through \( z, z' \) only, and that both \( G_{EE}^{YY} \) and \( G_{MM}^{XX} \) are even functions of \( z, z' \). Hence, the left- and right-hand sides of (D.74) can be rewritten as

\[ G(x, z, x', z') = G(x', z, x, z') = G(x', z', x, z). \]  \hspace{1cm} (D.75)

Next, we set \( z = 0 \) in the Green's tensors, then (D.75) transforms into

\[ G(x, x, x', 0) = G(x', x, x, 0) = G(x', x, x, 0). \]  \hspace{1cm} (D.76)

Carrying out the Fourier Transformations with respect to \( z, x' \) and \( x \) (cf. (D.2)), we obtain the symmetry properties of \( G_{YY}^{EE} \) and \( G_{MM}^{XX} \):

\[ \tilde{G}(k_x, k_{x'}, k_y) = \tilde{G}(-k_x', k_y, k_y) = \tilde{G}(-k_x, k_y, k_{y'}). \]  \hspace{1cm} (D.77)

The two rightmost sides of (D.77) imply

\[ \tilde{G}(k_x, k_{x'}, k_y) = \tilde{G}(k_x, k_{x'}, -k_y). \]  \hspace{1cm} (D.78)

Because of this property the (numerical) integration in (D.1) can be restricted to the interval \( 0 \leq k_x < a \).
Applying the inverse Fourier Transformation (D.1) to the left- and right-hand sides of (D.77), we obtain the symmetry properties of the kernels $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$:

$$
\hat{G}^{EE}_{y y}(k_x, k'_x) = \hat{G}^{EE}_{y y}(-k'_x, -k_x), \quad \hat{G}^{MM}_{x x}(k_x, k'_x) = \hat{G}^{MM}_{x x}(-k'_x, -k_x).
$$

(D.79)

D.4. Behaviour in the complex $k_z$-plane

The kernels $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ are determined by integration of $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ with respect to $k_z$, with fixed parameters $k_x$ and $k'_x$, cf. (D.1). Therefore it is necessary to investigate the singularities of $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ in the complex $k_z$-plane.

First, we observe from (D.61) and (D.62) that $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ contain $\{E_y(x_p), H_z(x_p)\}$, i.e., the transformed field distributions at the interfaces due to a transverse (electric or magnetic) current excitation proportional to $\exp(-jk'_x x)$.

Hence, the singularities of $E_y(x_p)$ and $H_z(x_p)$ are singularities of $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ as well. It was found in Section 3.2 that $E_y(x_p)$ and $H_z(x_p)$ have the following singularities in the complex $k_z$-plane: a finite number of simple poles $\{\kappa_n\}$ and, because the media in the outer domains $d_1$ and $d_N$ may be different, two pairs of branch points $k_z = \kappa_1$ and $k_z = \kappa_N$, where $k_1 = \omega/2 \epsilon_1$, $k_N = \omega/2 \epsilon_N$.

All singularities are located in the fourth and second quadrants of the $k_z$-plane. Associated with the branch points, we have the branch cuts $\mathcal{S}^+_N$, $\mathcal{S}^-_N$, on which $\operatorname{Im}(k^2) = 0$, and $\mathcal{S}^+_1$, $\mathcal{S}^-_1$, on which $\operatorname{Im}(k^2) = 0$ (Fig. D.2).

Next, it is seen from (D.61) and (D.62) that $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ apparently have simple poles at $k_z = \kappa_{p,1}$ corresponding to $k_z = \kappa_{p}$ where $\kappa_{p} = (\kappa^2_{p} - k^2_{x})^{1/2}$, $\operatorname{Im}(\kappa_{p}) \leq 0$.

By virtue of the symmetry property (D.77), $\hat{G}^{EE}_{y y}$ and $\hat{G}^{MM}_{x x}$ also have simple poles at $k'_z = \kappa_{p,1}$, corresponding to $k'_z = \kappa'_p$ where $\kappa'_p = (\kappa^2_{p} - k^2_{x})^{1/2}$, $\operatorname{Im}(\kappa'_p) \leq 0$.

Referring to (D.45)-(D.46), the poles at $k_z = \kappa_{T_{p}}$ and $k'_z = \kappa'_{T_{p}}$ with
p = 2, ...N−1, are in fact removable singularities. Thus we conclude that $\mathcal{G}^{\text{EE}}_{\chi \chi}$ and $\mathcal{G}^{\text{MM}}_{\chi \chi}$, when considered as functions of $k_\chi$, have simple poles at $k_\chi = a \xi_1$, $k_\chi = a \xi_N$, $k_\chi = a \xi_1^*,$ $k_\chi = a \xi_N^*$, in addition to the surface-wave poles $\{a \alpha_n\}$. It is easily recognised that the poles $k_\chi = a \xi_1$, $k_\chi = a \xi_1^*$ and $k_\chi = a \xi_N$, $k_\chi = a \xi_N^*$ are located on the branch cuts $\mathcal{B}_1^\chi$ and $\mathcal{B}_N^\chi$, respectively. Considered as functions of $k_\chi$, the poles $k_\chi = a \xi_1$ and $k_\chi = a \xi_N$ coincide with the branch points $k_\chi = a \xi_1$ and $k_\chi = a \xi_N$ when $k_\chi = 0$, and move towards infinity along the branch cuts as $|k_\chi| \to \infty$; see Fig. D.2. Likewise, the poles $k_\chi = a \xi_1^*$ and $k_\chi = a \xi_N^*$ coincide with the branch points $k_\chi = a \xi_1^*$ and $k_\chi = a \xi_N^*$ when $k_\chi^* = 0$, and move towards infinity along the branch cuts as $|k_\chi^*| \to \infty$.

Fig. D.2. Complex $k_\chi$-plane with singularities of $\mathcal{G}^{\text{EE}}_{\chi \chi}$ and $\mathcal{G}^{\text{MM}}_{\chi \chi}$, surface-wave poles $\{a \xi_n\}$, branch points $k_\chi = a \xi_1$ and $k_\chi = a \xi_N$, branch cuts $\mathcal{B}_1^\chi$ and $\mathcal{B}_N^\chi$, and simple poles $k_\chi = a \xi_1^*$, $k_\chi = a \xi_1^{*\dagger}$, $k_\chi = a \xi_N^*$ and $k_\chi = a \xi_N^{*\dagger}$. The poles $k_\chi = a \xi_1$, $k_\chi = a \xi_N$ move along the branch cuts in the direction indicated by the arrows when $|k_\chi|$ increases; the poles $k_\chi = a \xi_1^*$, $k_\chi = a \xi_N^*$ move along the branch cuts in the direction indicated by the arrows when $|k_\chi^*|$ increases.
In principle, the removable singularities of $\hat{G}^{\text{EE}}_{yy}$ and $\hat{G}^{\text{MM}}_{xx}$ at $k_z = s\xi_p$, $k_z = s\zeta_p$, with $p = 2, \ldots, N-1$, present no difficulties. However, they may give rise to inaccuracies in the numerical evaluation of $\hat{G}^{\text{EE}}_{yy}$ and $\hat{G}^{\text{MM}}_{xx}$. In our computations, this inconvenience has been handled by taking the media in the waveguide layers $d_p$, with $p = 2, \ldots, N-1$, slightly lossy as well. Then $k_p$, $\xi_p$ and $\zeta_p$ become complex—valued with $\text{Im}(k_p) < 0$, $\text{Im}(\xi_p) < 0$ and $\text{Im}(\zeta_p) < 0$.

When the media in the outer domains $d_1$ and $d_N$ are identical with constitutive parameters $\epsilon_1 = \epsilon_N$ and $\mu_1 = \mu_N$, we only have one pair of branch points $k_z = s\zeta_1$ with associated branch cuts $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$. Considered as functions of $k_z$, $\hat{G}^{\text{EE}}_{yy}$ and $\hat{G}^{\text{MM}}_{xx}$ then have simple poles at $k_z = s\xi_1$, $k_z = s\zeta_1$, in addition to the surface-wave poles $\{s\zeta_n\}$.

D.5. Transverse Fourier transforms of the modal fields of a multi-step-index planar waveguide

The preceding theory can also be used to determine the transverse Fourier transforms $\{\hat{e}_y, \hat{h}_x\}(k_x)$, with respect to the transverse coordinate $x$, of the modal field components $\{e_y, h_x\}(x)$. We recall that the modal fields satisfy the source-free field equations; cf. (3.30)–(3.32). Therefore, in the analysis of Subsection D.1 all source terms $\hat{g}_p(k_x - k_x^p)$ should be replaced by zero. Through (D.29)–(D.40) and (D.29), the required Fourier transforms are expressed in terms of the values of $\hat{E}_y = e_y$ and $\hat{H}_z = h_z$ at the interfaces $x = x_p$, $p = 1, \ldots, N-1$. These values can again be determined by means of the transfer-matrix formalism. To that end we start from the representations (D.58) and (D.59), in which the source-type terms $I_N^+ = \hat{I}_N^+$ and $I_m^-$ are to be replaced by zero. Then by equating (D.58) and (D.59) for some $p \in \{1, \ldots, N-1\}$, we are led to a system of two linear homogeneous algebraic equations for $\hat{E}_y(x_1)$ and
This system has a non-zero solution only for special values of \( k_z \), which coincide with the eigenvalues \( \{ \kappa_n \} \) of the source-free field equations (3.30)--(3.31); remember that \( \kappa_n \) is also the propagation coefficient of the \( n \)-th surface-wave mode.

After having determined \( k_z = \kappa_n \) as the propagation coefficient of the surface-wave mode under consideration, we solve the system of equations for \( \bar{E}_y(x_1) = e_y(x_1) \) and \( \bar{E}_y(x_{N-1}) = e_y(x_{N-1}) \). The result is inserted into (D.50) and (D.51), whereupon the field quantities \( \{ \bar{E}_y(x_p), \bar{H}_z(x_p) \} = \{ e_y(x_p), h_z(x_p) \}, p = 1, \ldots, N-1 \), can be determined by repeated use of (D.49) and/or (D.57). Here it is understood that the source-type terms \( I_u, I_N, I_p^+ \) and \( I_p^- \) in (D.49) and (D.51) and (D.57) have been replaced by zero, whereas \( k_z = \kappa_n \) should be set in \( k_{\tau_p} \).

Consider next the expressions (D.61) and (D.62) for \( \xi_{y,p}^{GE} \) and \( \xi_{x,p}^{GM} \). In these expressions we substitute the values of \( \{ \bar{E}_y(x_p), \bar{H}_z(x_p) \} \), we replace the source terms \( \xi_{p}^{s}(k_x-k'_x) \) by zero, and we set \( k_z = \kappa_n \); then, \( \xi_{y,p}^{GE} \) and \( \xi_{x,p}^{GM} \) are completely determined. The latter quantities represent the contributions of the \( p \)-th layer of the waveguide to the required Fourier transforms \( \hat{e}_y \) and \( \hat{h}_x \). Thus by summation of these contributions we obtain as our final results

\[
\hat{e}_y(k_x) = \sum_{p=1}^{N} \xi_{y,p}^{GE} \quad \hat{h}_x(k_x) = \sum_{p=1}^{N} \xi_{x,p}^{GM}.
\]
E. EXPRESSIONS FOR THE REFLECTION AND TRANSMISSION COEFFICIENTS OF THE JUNCTION OF TWO OPEN WAVEGUIDES

The configuration of the junction of two planar waveguide sections is shown in Fig. E.1. In waveguide A the \( \text{TE}_m \) mode is incident towards the junction plane \( s = 0 \), and is partly reflected and radiated back into waveguide A, and partly transmitted and radiated into waveguide B. In this appendix expressions are derived for the reflection and transmission coefficients. By representing the field in waveguide A by a modal expansion, the reflection coefficient \( R_m^0 \) is defined as the ratio of the amplitudes of the reflected \( \text{TE}_m \) -modal field and the incident \( \text{TE}_m \) -modal field. Likewise, by representing the field in waveguide B by a modal expansion, the transmission coefficient \( T_m^0 \) is defined as the ratio of the amplitudes of the transmitted \( \text{TE}_m \) -modal field and the incident \( \text{TE}_m \) -modal field. In these definitions, the amplitude of a modal field is the ratio of that field and the Lorentz-normalised field of the same mode.

![Diagram](image)

Fig. E.1. Reflection, transmission and radiation at the junction of two planar waveguide sections; incident \( \text{TE}_m \) -mode; reflected \( \text{TE}_m \) -modes with reflection coefficients \( R_m^0 \); transmitted \( \text{TE}_m \) -modes with transmission coefficients \( T_m^0 \); backward radiated field with radiation mode reflection coefficients \( R_m^K \); forward radiated field with radiation mode transmission coefficients \( T_m^K \).
We start from the general modal expansion for the transverse electromagnetic field in an open waveguide due to transverse electric excitation (cf. (3.14)), viz.

\[ \{E^x_m, H^z_m, T_m\} = \sum_m A_m^+ \{e_m^+, h_m^+, \} \exp(-j\kappa_m z) + \sum_{\xi^+} A_{\xi^+} \{e_{\xi^+}, h_{\xi^+}, \} \exp(-j\omega \xi) \mathrm{d}\kappa \]

\[ + \sum_m A_m^- \{e_m^-, h_m^-, \} \exp(j\kappa_m z) + \sum_{\xi^+} A_{\xi^-} \{e_{\xi^-}, h_{\xi^-}, \} \exp(j\omega \xi) \mathrm{d}\kappa. \]  \hspace{1cm} (E.1)

Here, \( \{e_m^+, h_m^+, \} \) and \( \{e_m^-, h_m^-, \} \) represent the Lorentz-normalised fields of the surface-wave modes and the radiation modes, respectively. The superscript \( \pm \) indicates whether the mode propagates in the positive or negative \( z \)-direction. In view of (3.16), one has \( \{e_m^+, h_m^+, \} = \{e_m, T_m, h_m, \} \), \( \{e_m^-, h_m^-, \} = \{e_{\kappa}, T_{\kappa}, h_{\kappa}, \} \), where \( \{e_m, h_m, \} \) and \( \{e_{\kappa}, h_{\kappa}, \} \) are the Lorentz-normalised modal field constituents introduced in (3.24) and (3.25). The coefficients \( A_m^+ \) and \( A_m^- \) are the complex amplitudes of the surface-wave and radiation modal fields, respectively.

The expression (E.1) is now used to describe the electromagnetic field in the waveguide configuration of Fig. E.1. In waveguide A the total field consists of the incident (Lorentz-normalised) TE\(_m\)—mode travelling in the positive \( z \)-direction, the reflected TE\(_m\)—modes of amplitudes R\(_m\), travelling in the negative \( z \)-direction, and the backward radiated field also travelling in the negative \( z \)-direction. The latter field is represented by a superposition of radiation modes of amplitudes R\(_{n}\), where R\(_{n}\) denotes the radiation mode reflection coefficient. In waveguide B there is no incident field, and all field constituents travel in the positive \( z \)-direction. The total field in waveguide B consists of the transmitted TE\(_m\)—modes of amplitudes T\(_m\), and the forward radiated field represented by a superposition of radiation modes of amplitudes T\(_{n}\), where T\(_{n}\) denotes the radiation mode transmission coefficient. As a result, the transverse electromagnetic field in waveguide A is represented by the expansion...
\[ \{E_n, H_n\} = \left\{ E_n^A, h_n^A \right\} \exp(-j\kappa_n z) + \sum_m R_n^m \left\{ E_m, H_m \right\} \exp(j\kappa_m z) + \int_{\mathbb{R}} R_n^\kappa \left\{ E_n^A, h_n^A, E_n^A, h_n^A \right\} \exp(j\kappa z) \, dz, \quad z \leq 0, \]  \hspace{1cm} (E.2)

In terms of the modal fields of waveguide A. Likewise, the transverse electromagnetic field in waveguide B is represented by the expansion

\[ \{E_n, H_n\} = \sum_m T_n^m \left\{ E_n^B, h_n^B \right\} \exp(-j\kappa_n z) + \int_{\mathbb{R}} T_n^\kappa \left\{ E_n^B, h_n^B, E_n^B, h_n^B \right\} \exp(-j\kappa z) \, dz, \quad z \geq 0, \]  \hspace{1cm} (E.3)

in terms of the modal fields of waveguide B. Notice that both expansions (E.2) and (E.3) remain valid at \( z = 0 \), since the transverse fields are continuous across the junction plane.

Starting from the expansion (E.2) with \( z = 0 \), we form the vector products \( E_n \times h_n^A \) and \( E_n \times h_n \), which are integrated over the entire junction plane. Then, by use of the orthogonality of surface-wave modes and radiation modes (see (B.25) and (B.26)) and of the normalisation conditions (3.24) and (3.25), we find

\[ iA_{E, n}^{m} = \int_{-\infty}^{\infty} (E_n \times h_m^A) \, dz = \begin{cases} \frac{1}{j} R_n^m, & m \neq n, \\ \frac{1}{2} \left( 1 + R_n^m \right), & m = n, \end{cases} \]  \hspace{1cm} (E.4)

\[ iA_{H, n}^{m} = \int_{-\infty}^{\infty} (E_n \times H_m^A) \, dz = \begin{cases} \frac{-1}{j} R_n^m, & m \neq n, \\ \frac{1}{2} \left( 1 - R_n^m \right), & m = n. \end{cases} \]  \hspace{1cm} (E.5)

From (E.4), \( R_n^m \) is obtained as
\[ R^m_n = \begin{cases} 2 \frac{A^m_n}{E^m_n}, & m \neq n, \\ 2 \frac{A^m_n}{E^m_n} - 1, & m = n, \end{cases} \quad (E \text{ result}) \] (E.6)

while (E.5) leads to

\[ R^m_n = \begin{cases} -2 \frac{A^m_n}{H^m_n}, & m \neq n, \\ 1 - 2 \frac{A^m_n}{H^m_n}, & m = n, \end{cases} \quad (H \text{ result}). \] (E.7)

Dividing (E.4) by (E.5) for \( m = n \), we obtain

\[ R^m_n = \left(1 \frac{A^m_n}{E^m_n} - \frac{A^m_n}{H^m_n}\right) / \left(1 \frac{A^m_n}{E^m_n} + \frac{A^m_n}{H^m_n}\right), \quad m = n, \quad (EH \text{ result}). \] (E.8)

For \( m \neq n \), the EH result is defined as the mean value of the E and H results:

\[ R^m_n = \frac{1}{2} \left( \frac{A^m_n}{E^m_n} - \frac{A^m_n}{H^m_n}\right), \quad m \neq n, \quad (EH \text{ result}). \] (E.9)

The transmission coefficients \( T^m_n \) are determined in a similar manner, starting from the expansion (E.3) with \( z = 0 \). By integration of the vector products \( E_T \times h^B_{m,T} \) and \( E^B_{m,T} \times H_T \) over the entire junction plane, we find, by use of the orthonormality properties of the modes,

\[ I^{B^m}_{E,n} = \int_{-a}^{a} (E_T \times h^B_{m,T}) \cdot i^B dx = \frac{1}{2} T^m_n, \] (E.10)

\[ I^{B^m}_{H,n} = \int_{-a}^{a} (E^B_{m,T} \times H_T) \cdot i^B dx = \frac{1}{2} T^m_n \] (E.11)

From (E.10), \( T^m_n \) is obtained as
\[ T_n^m = 2i B_n^m \quad (E \text{ result}) \tag{E.12} \]

while (E.11) leads to

\[ T_n^m = 2i H_n^m \quad (H \text{ result}). \tag{E.13} \]

By adding (E.10) and (E.11), we obtain

\[ T_n^m = i E_n^m + i H_n^m \quad (EH \text{ result}). \tag{E.14} \]

If the integrals \( i E_n^m, i H_n^m, i E_n^m, i H_n^m \) could be evaluated exactly, the E, H and EH results for \( R_n^m \) would be identical, just like the E, H and EH results for \( T_n^m \). Then it follows from (E.6), (E.7), (E.12) and (E.13) that the exact values of the integrals are related by

\[ i E_n^m = -i H_n^m, m \neq n; \quad i E_n^m + i H_n^m = l; \quad i B_n^m = i E_n^m. \]

In practice, the field distributions \( \{E, H, T\} \) and the integrals are to be evaluated numerically, and consequently the E, H and EH results generally differ from each other, due to inaccuracies that arise in the computations. It is pointed out that the (numerically obtained) EH results for \( R_n^m \) with \( m \neq n \), and for \( T_n^m \), are just the mean values of the E and H results for these quantities. Note however, that the EH result from (E.8) will be different from the mean value of the E and H results for \( R_n^m \), since in general \( i E_n^m + i H_n^m \neq 1 \) for the numerically computed field distributions and integrals.

In the actual calculation of the reflection and transmission coefficients, we employ alternative expressions for the integrals \( i E_n^m, i H_n^m, i E_n^m, i H_n^m \) that are obtained by application of Parseval's theorem.
\[ \int_{-w}^{w} f(x)g(x)dx = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{f}(k_x)\hat{g}(-k_x)dk_x \quad (E.15) \]

in (E.4), (E.5), (E.10) and (E.11). These alternative integral expressions involve the transformed field quantities \( \hat{E}_y, \hat{H}_x \) in the \( k_x \)-domain. In this manner the reflection and transmission coefficients can be determined directly from \( \hat{E}_y, \hat{H}_x \), the Fourier transforms that result from the solutions of the equations in Chapter 6.
F. EXPRESSIONS FOR THE DIRECTIVE GAIN OF THE TERMINATING OPEN WAVEGUIDE

The configuration of the terminating planar waveguide radiating into free space, is shown in Fig. F.1. In the waveguide the TE\(_n\) mode is incident towards the terminal plane \(z = 0\), and is partly reflected and radiated back into the waveguide, and partly radiated in forward direction into the half-space \(z > 0\). The medium in this half-space is slightly lossy, with permittivity \(\varepsilon_1\) that is complex with a small negative imaginary part, and permeability \(\mu_1 = \mu_0\). For shortness, this medium is referred to as free space. In this appendix expressions are derived for the directive gain of the terminating waveguide. The directive gain is the normalised far-field power radiation pattern in forward direction as a function of the directive angle.

We consider the electromagnetic field in the half-space \(z > 0\) to be generated by surface-current sources of densities \(J_{\phi r} = i_x \times \mathbf{H}, K_{\phi r} = -i_x \times \mathbf{E}\), located in the terminal

![Diagram](image)

Fig. F.1. Terminating planar waveguide radiating into free space; incident TE\(_n\) mode and reflected TE\(_m\) modes with reflection coefficients \(R_m^n\); observation point \(x\) and source point \(x'\); direction of observation \(i_x\) and angle of observation \(\theta\), with \(-\pi/2 \leq \theta \leq \pi/2\).
plane $z = 0$. Expressed in terms of these surface sources the fields $\mathbf{E}$, $\mathbf{H}$ are given by (De Hoop, 1977)

$$
\mathbf{E} = (j\omega \varepsilon_0 \kappa) \nabla (\nabla \cdot \mathbf{E}) - j\omega \mu_0 \mathbf{H} - \nabla \cdot \mathbf{Q}
$$  \hspace{1cm} (F.1)

$$
\mathbf{H} = (j\omega \mu_0 \kappa) \nabla (\nabla \cdot \mathbf{H}) - j\omega \varepsilon_0 \mathbf{E} + \nabla \cdot \mathbf{Q}
$$  \hspace{1cm} (F.2)

in which, for our two-dimensional case,

$$
\mathbf{E}_{\alpha}(t) = \int_{-\infty}^{\infty} G(t,t') \mathbf{J}_{\alpha}(x') dx',
$$  \hspace{1cm} (F.3)

$$
\mathbf{Q}_{\alpha}(t) = \int_{-\infty}^{\infty} G(t,t') \mathbf{K}_{\alpha}(x') dx'.
$$  \hspace{1cm} (F.4)

Here, $G$ is the two-dimensional free-space Green's function

$$
G(t,t') = -(i/\omega) \mathbf{H}_0^{(2)}(k_0 |t-t'|),
$$  \hspace{1cm} (F.5)

with $k_0 = \omega \epsilon_0 \mu_0^{1/2}$; $\mathbf{x} = \mathbf{x}_x + \mathbf{z}_y$ is the two-dimensional position vector of the observation point; $\mathbf{x}' = \mathbf{x}'_x + \mathbf{z}'_y$ is the two-dimensional position vector of the source point.

In principle, the integration in (F.3) and (F.4) extends over the entire terminal plane. However, in practice the integration may be restricted to some finite interval, since the fields in the terminal plane decrease exponentially with increasing distance from the waveguide. Then for large $r = |r|$ we employ the far-zone approximation.
\[ |z-z'| = z(1-2z \cdot z'/r^2 + |z'|^2/r^2)^{1/2} \approx r - \hat{z}_r \cdot z', \quad (r \to \pm). \] (F.6)

in which \( \hat{z}_r = z/r \) is the unit vector in the direction of observation. Using (F.6) in the large-argument asymptotic expansion of \( \Pi_0^{(2)} \) (Felsen and Marcuvitz, 1973, p. 712) and retaining only the dominant term, we obtain

\[ G(z,z') = -\frac{i}{4} \left( \frac{2 \lambda}{\pi k_1^2 r} \right)^{1/2} \exp(-ik_1 r) \exp(i \hat{z}_r \cdot z'), \quad (r \to \pm). \] (F.7)

In the same approximation, we have

\[ \mp G(z,z') \approx \mp i k_1 G(z,z') \hat{z}_r, \quad (r \to \pm). \] (F.8)

Inserting (F.7) into (F.3) and (F.4), and noting that \( \hat{z}_r \cdot z' = \mp \sin(\theta) \), we find

\[ \mathcal{P}_{\omega_r}(z) = -\frac{i}{4} \left( \frac{2 \lambda}{\pi k_1^2 r} \right)^{1/2} \exp(-ik_1 r) \hat{z}_r \omega_r(z), \] (F.9)

\[ \mathcal{Q}_{\omega_r}(z) = -\frac{i}{4} \left( \frac{2 \lambda}{\pi k_1^2 r} \right)^{1/2} \exp(-ik_1 r) \hat{z}_r \omega_r(z), \] (F.10)

in which \( k_r = k_\perp \sin(\theta) \) and (cf. (6.23))

\[ \{ \hat{z}_r \omega_r \mathcal{K}_{\omega_r} \}(k_r) = \int_{-\infty}^{\infty} \exp(ik_r x') \{ \hat{z}_r \omega_r \mathcal{K}_{\omega_r} \}(x') dx'. \] (F.11)

Using (F.8)–(F.10) in (F.1) and (F.2), we are led to the far-field approximations to the radiated field:

\[ \mathcal{E}(z) = -\frac{i}{4} \left( \frac{2 \lambda}{\pi k_1^2 r} \right)^{1/2} \exp(-ik_1 r) \mathcal{E}(\theta), \] (F.12)
\[ \Pi(r) = -\frac{1}{4}(\frac{\lambda}{\pi k r})^{1/2}(\exp(-jk_1 r)h(\theta)), \]

in which the amplitude radiation characteristics \( e(\theta) \) and \( h(\theta) \) are given by

\[ e(\theta) = j\omega \gamma_1 \left( (i_\gamma \cdot \hat{J}_\varphi(k_\varphi))_{x_\gamma} - \hat{J}_\varphi(k_\varphi) \right) + jk_1 i_\gamma \hat{K}_\varphi(k_\varphi), \]  

\[ h(\theta) = j\omega \gamma_1 \left( (i_\gamma \cdot \hat{K}_\varphi(k_\varphi))_{x_\gamma} - \hat{K}_\varphi(k_\varphi) \right) - jk_1 i_\gamma \hat{J}_\varphi(k_\varphi). \]

The factor multiplying \( e(\theta) \) in (F.12) and \( h(\theta) \) in (F.13) represents the far field (as \( r \to \infty \)) radiated by a scalar isotropic line source. From (F.14) and (F.15) it is readily seen that \( e(\theta) \) and \( h(\theta) \) are related by

\[ e = (\mu_1 / \epsilon_1)^{1/2} (h \ast i_\gamma), \quad h = (\epsilon_1 / \mu_1)^{1/2} (i_\gamma \ast e). \]

Next, we express \( \hat{J}_\varphi(k_\varphi) \) and \( \hat{K}_\varphi(k_\varphi) \) in terms of the Fourier transforms of the fields in the terminal plane, viz.

\[ \hat{J}_\varphi(k_\varphi) = \int_{-\infty}^{\infty} \exp(jk_\varphi x')[H(x')]dx' = \hat{H}_\varphi(k_\varphi), \]

\[ \hat{K}_\varphi(k_\varphi) = \int_{-\infty}^{\infty} \exp(jk_\varphi x')[-i_\gamma \times B(x')]dx' = \hat{B}_\varphi(k_\varphi) \]

Using these expressions in (F.14) and (F.15) and writing \( i_\gamma = \sin(\theta)i_\varphi + \cos(\theta)i_\varrho \), we obtain

\[ e(\theta) = jk_1 [\cos(\theta)\hat{B}_\varphi(k_\varphi) - (\mu_1 / \epsilon_1)^{1/2} \hat{H}_\varphi(k_\varphi)]_{y_\gamma}. \]
\[ h(\theta) = -i\epsilon_1 (\epsilon_1 / \mu_1)^{1/2} \cos(\theta) \hat{E}_y(k_x) = (\mu_1 / \epsilon_1)^{1/2} \hat{H}_x(k_y) \text{[cos(\theta)]}_x - \sin(\theta) z_x]. \quad (F.20) \]

We now determine the leading term of the time–averaged Poynting vector \( S_r(\tau) \). From (F.12) and (F.13) we deduce the far–field approximation
\[ S_r(\tau) = \frac{1}{2} \text{Re}[\vec{E}(\tau) \cdot \vec{H}^* (\tau)] = (8 \pi |k_1|)^{-1} \exp(2i\text{Im}(k_1) \tau) \frac{1}{2} \text{Re}[\vec{g}(\theta) \cdot \vec{h}^* (\theta)], \quad (F.21) \]

where, by use of (F.19) and (F.20),
\[ \text{Re}[\vec{g}(\theta) \cdot \vec{h}^* (\theta)] = |k_1|^{-1} \text{Re}[\epsilon_1 / \mu_1)^{1/2} \] \[ \times \cos(\theta) \hat{E}_y(k_x) = (\mu_1 / \epsilon_1)^{1/2} \hat{H}_x(k_y)]^2 2. \quad (F.22) \]

Since \( S_r(\tau) \) has a radial component only, we shortly write
\[ S_r(\tau) = P(\tau) i_x, \quad (F.23) \]

where \( P(\tau) \) is the far–field power radiation pattern in forward direction. Clearly, \( P(\tau) \)
depends on the directive angle \( \theta \) through the factor \( \text{Re}[\vec{g}(\theta) \cdot \vec{h}^* (\theta)] \), as given by (F.22).

We shall derive two further expressions for \( \text{Re}[\vec{g}(\theta) \cdot \vec{h}^* (\theta)] \), equivalent to (F.22). To that end we observe that the radiated field in the half–space \( z > 0 \) is a TE–field with non–zero components \( \{\hat{E}_y, \hat{H}_x, \hat{H}_z\}(x,s) \), which must satisfy the source–free electro–magnetic field equations. These equations are subjected to a transverse Fourier Transformation with respect to the variable \( x \) (cf. (6.23)). From the resulting system of equations it is readily found that the transformed field quantities \( \{\hat{E}_y, \hat{H}_x, \hat{H}_z\}(k_x,s) \)
depend on \( x \) through the factor \( \exp(-i(k_x^2 - k_y^2)^{1/2} s) \), and that \( \hat{E}_y \) and \( \hat{H}_x \) are related by
\[ \hat{H}_x(k_x,x) = -k_1^{-1}(\epsilon_1/\mu_1)^{1/2} (k_1^2-k_x^2)^{1/2} \hat{E}_y(k_x,x), \quad x \geq 0. \quad (P.24) \]

Notice that the relation (P.24) remains valid at \( z = 0 \), since the transverse field components are continuous across the terminal plane. By setting \( z = 0 \), \( k_x = k_\theta \) in (P.24), and by noting that \((k_1^2-k_\theta^2)^{1/2} = (k_1^2-k_\theta^2\sin^2(\theta))^{1/2} = k_1\cos(\theta)\) in the range \(-\pi/2 \leq \theta \leq \pi/2\) (forward direction), we find

\[ (\mu_1/\epsilon_1)^{1/2} \hat{H}_x(k_\theta) = -\cos(\theta)\hat{E}_y(k_\theta). \quad (P.25) \]

The relation (P.25) is now used in (P.22) to obtain the equivalent expressions

\[ \Re(\hat{E}(\theta) \times \hat{H}^*(\theta)) = 4|k_1|^2 \Re(\epsilon_1/\mu_1)^{1/2}\cos^2(\theta)|\hat{E}_y(k_\theta)|^2 \]

\[ = 4|k_1|^2 \Re((\mu_1/\epsilon_1)^{1/2})|\hat{H}_x(k_\theta)|^2, \quad (P.26) \]

in terms of \( \hat{E}_y(k_\theta) \) only and of \( \hat{H}_x(k_\theta) \) only, respectively.

The angular dependence of the radiated power is usually described by the directive gain \( D(\theta) \), defined by

\[ D(\theta) = P(\theta)/P^{1S}(\theta). \quad (P.27) \]

Here, \( P^{1S}(\theta) \) is the far-field power radiation pattern due to an isotropic radiator with a total power that is equal to the power \( P^1 \) carried by the incident \( TE_{n}\)-mode in the terminal plane. It is known (IEEE, 1983) that \( P^{1S}(\theta) \) is given by

\[ P^{1S}(\theta) = P^1(2\pi r)^{-1}\exp(2\text{Im}(k_1 r)). \quad (P.28) \]
From (F.21), (F.23), (F.27) and (F.28) the directive gain is obtained as

$$D(\theta) = (8|k_1|^{-3}) \hat{P}_r (\text{Re}\{g(\theta) \ast h^*(\theta)\}) / P_i,$$  \hspace{1cm} (F.29)

in which $\text{Re}\{g(\theta) \ast h^*(\theta)\}$ is given by either (F.22) or (F.26).

The forward radiated power from the terminal plane into free space is denoted by $P_f$.

By comparing $P_f$ to the incident power $P_i$, the forward radiated power fraction $P_f / P_i$ is calculated from

$$P_f / P_i = (2\pi)^{-1} \int_{-\pi/2}^{\pi/2} D(\theta) d\theta.$$  \hspace{1cm} (F.30)

The expressions (F.22) and (F.26) for $\text{Re}\{g(\theta) \ast h^*(\theta)\}$ contain the field quantities $(\hat{E}_y, \hat{H}_x)(k_{\theta})$, in which $k_{\theta} = k_x \sin(\theta)$ is complex with a small negative imaginary part. The actual evaluation of $(\hat{E}_y, \hat{H}_x)(k_{\theta})$ is complicated by the fact that in Chapter 6 the transformed field quantities $(\hat{E}_y, \hat{H}_x)(k_x)$ are computed for real $k_x$ only. To overcome this difficulty, we recall that $|\text{Im}(k_{\theta})|$ is small compared to $\text{Re}(k_{\theta})$; hence, we approximate $k_{\theta}$ by $\text{Re}(k_{\theta})$, and $(\hat{E}_y, \hat{H}_x)(k_{\theta})$ by $(\hat{E}_y, \hat{H}_x)(k_{\theta})$, where $k_{\theta} = \text{Re}(k_{\theta}) \sin(\theta)$. As a result we are led to the following approximate expressions for $D(\theta)$, obtained from (F.22), (F.26) and (F.29):

$$D(\theta) \approx \begin{cases} 
\text{Re}(k_{\theta}) \text{Re}(\epsilon_i / \epsilon_1)^{1/2} |\cos(\theta)\tilde{E}_y(k_{\theta}) - (\mu_i / \epsilon_1)^{1/2} \tilde{H}_x(k_{\theta})|^2/(8P_i), \\
\text{Re}(k_{\theta}) \text{Re}(\epsilon_i / \epsilon_1)^{1/2} \cos^2(\theta) |\tilde{E}_y(k_{\theta})|^2/(2P_i), \\
\text{Re}(k_{\theta}) \text{Re}(\epsilon_i / \epsilon_1)^{1/2} \tilde{H}_x(k_{\theta})|^2/(2P_i). 
\end{cases}$$  \hspace{1cm} (F.31)

We briefly discuss the justification of the approximation applied. Let $P$ be a generic notation for the field components $E_y$, $H_x$ in the terminal plane. Then the
approximation of

\[ \hat{F}(k_\theta) = \int_{-\infty}^{\infty} \exp(ik_\theta x)F(x)dx = \int_{-\infty}^{\infty} \exp(ik_\theta x)\exp(-\text{Im}(k_\lambda)x \sin(\theta))F(x)dx \]  \hspace{1cm} (F.32)

by \( \hat{F}(k_\theta) \) amounts to ignoring the factor \( \exp(-\text{Im}(k_\lambda)x \sin(\theta)) \) in the final integrand of (F.32). Now the field component \( F(x) \) decreases exponentially like \( \exp(\text{Im}(k_\lambda)x) \) with increasing distance from the waveguide. For small values of \( \theta \), the decrease of \( F(x) \) dominates the possible increase of the factor \( \exp(-\text{Im}(k_\lambda)x \sin(\theta)) \) as \( x \to \pm \infty \).

Thus the integration in (F.32) may be restricted to some finite interval, and over this interval the factor \( \exp(-\text{Im}(k_\lambda)x \sin(\theta)) \) is close to unity. This justifies the approximation (F.31) to \( D(\theta) \) for small values of \( \theta \). For \( \theta = \pm \pi/2 \), however, the increase of the factor \( \exp(\text{Im}(k_\lambda)x) \) precisely counteracts the decrease of \( F(x) \) as \( x \to \pm \infty \). Then it is not permissible to restrict the integration in (F.32) to a finite interval, and as a result the factor \( \exp(\text{Im}(k_\lambda)x) \) is no longer close to unity over the integration interval. Thus we conclude that the approximation (F.31) to \( D(\theta) \) is not justified for a directive angle close to \( \pm \pi/2 \).

The failure of the approximation (F.31) at \( \theta = \pm \pi/2 \) can also be explained from the behaviour of \( \hat{E}_y(k_\lambda) \) near \( k_\lambda = \pm k_1 \). It is recalled that the transformed field components \( (\hat{E}_y, \hat{H}_x)(k_\lambda) \) are determined as solution of the integral equations (6.28) and (6.29). The tensor elements \( \hat{G}^{EE} \) and \( \hat{G}^{MM} \), which appear as kernels in these integral equations, contain a factor \( (k_1^2 - k_\lambda^2)^{-1/2} \) (see Subsection 6.4.1). We now expect that also \( \hat{E}_y(k_\lambda) \) contains a factor \( (k_1^2 - k_\lambda^2)^{-1/2} \), as it is confirmed by the strongly peaked behaviour of the computed \( \hat{E}_y(k_\lambda) \) at \( k_\lambda = \pm \text{Re}(k_1) \) (cf. Figs. 6.17 and 6.18 in Subsection 6.5.3). Setting \( k_\lambda = k_\theta = k_1 \sin(\theta) \), one has \( (k_1^2 - k_\lambda^2)^{-1/2} = k_1 \cos(\theta) \); hence, \( \hat{E}_y(k_\theta) \) contains a factor \( (\cos(\theta))^{-1} \) which becomes singular at \( \theta = \pm \pi/2 \). To determine \( D(\theta) \) from (F.28) and (F.26), \( \hat{E}_y(k_\theta) \) is to be multiplied by \( \cos(\theta) \) which
annihilates the $(\cos(\theta))^{-1}$-singularity. Thus we find that the true value of $D(\theta)$, calculated from $\tilde{E}_y(k_\theta)$, is finite and non-zero for $\theta = \pm \pi/2$. Next, we approximate $\tilde{E}_y(k_\theta)$ by $\bar{E}_y(k_\theta)$, where $k_\theta = \text{Re}(k_1)\sin(\theta)$. Obviously, $\bar{E}_y(k_\theta)$ is finite over the range $-\pi/2 \leq \theta \leq \pi/2$, and $\cos(\theta)\bar{E}_y(k_\theta) = 0$ at $\theta = \pm \pi/2$. Consequently, the approximate value of $D(\theta)$, calculated from the second expression in (E.31), vanishes for $\theta = \pm \pi/2$. 
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SAMENVATTING

De verbindingen die gebruikt worden voor optische communicatie tussen twee punten, bestaan veelal uit een aantal in serie geschakelde secties van verschillende typen cilindrische open golfgleiders. In het aansluitvlak van twee verschillende secties zijn de elektromagnetische eigenschappen van het medium discontinu, met als gevolg dat daar reflectie, transmissie en uitstraling van elektromagnetische golven optreedt. De quantitatieve analyse van deze verschijnselen vormt het hoofdthema van dit proefschrift.

In de analyse wordt uitgegaan van een mathematische beschrijving van zowel de voortplanting van elektromagnetische golven in een uniforme (oneindige) sectie van een golfgleider, als de interactie van elektromagnetische golven in het aansluitvlak van twee secties. Deze beschrijving is gebaseerd op de vergelijkingen van Maxwell voor het elektromagnetische veld, het reciprocitytheorema voor het frequentiespectrum en de elektromagnetische Grense toestanden. We tonen aan dat de velden in een uniforme (oneindige) sectie van een open golfgleider zijn voor te stellen door een ontwikkeling in termen van oppervlaktegolfmodi en stralinggolfmodi. We bespreken twee methoden voor het berekenen van de oppervlaktegolfmodi en illustreren deze aan de hand van numerieke resultaten voor planaire open golfgleiders.

Vervolgens leiden we integraalvoorstellingen af voor de velden in een eindige sectie van een open golfgleider in termen van de transversale velden in de grensvlakken, en voor de velden in een halfoneindige sectie in termen van het transversale veld in het
eindvlak en van het transversale invallend veld dat zich voortplant naar het eindvlak. Met behulp van deze voorstellingen worden stelsels van integraalvergelijkingen opgesteld voor het veld in het (de) aansluitvlak(ken) van twee (drie) in serie geschakelde secties van open golfgeleiders.

Eén van deze stelsels van integraalvergelijkingen is gekozen en numeriek opgelost, voor een aantal combinaties van twee in serie geschakelde planaire open golfgeleidersecties en voor een halfoneindige golfgeleidersectie eindigend in de vrije ruimte, waarbij een TE-oppervlaktegolfmodus als invallend veld optreedt. Meer in het bijzonder is op het gekozen stelsel van integraalvergelijkingen een ruimtelijke Fouriertransformatie toegepast, waarna het Fouriergetransformeerde stelsel numeriek is opgelost met de momentenmethode. De verkregen oplossing voor de Fouriergetransformeerde van het transversale veld in het aansluitvlak is gebruikt voor de berekening van het transversale veld in het aansluitvlak en van de reflectie van de invallende oppervlaktegolfmodus aan het aansluitvlak. Voorts is de transmissie door het aansluitvlak berekend voor het geval van een serieschakeling van twee golfgeleidersecties, terwijl voor het geval van een halfoneindige golfgeleider eindigend in de vrije ruimte de voorwaartse straling vanuit het eindvlak bepaald is.
CURRICULUM VITAE

De auteur van dit proefschrift is op 21 juli 1964 te Rhoon geboren.


Van december 1978 tot december 1983 was zij als wetenschappelijk medewerker in tijdelijke dienst werkzaam bij de Vakgroep Theoretische Elektrotechniek van de Afdeling der Elektrotechniek van de Technische Hogeschool Eindhoven (thans: Faculteit der Elektrotechniek van de Technische Universiteit Eindhoven). In het kader van deze aanstelling werd het onderzoek verricht dat het onderwerp is van dit proefschrift.

Van maart 1985 af tot heden is zij werkzaam als software ontwikkelaar bij de afdeling Lithography van de Industrial & Electro-acoustic Systems Division van de Nederlandse Philips Bedrijven te Eindhoven.
Stellingen.
behorende bij het proefschrift van
H.M. de Ruijter

Eindhoven, 5 september 1989
1. Voor de genormaliseerde propagatiecoefficiënten $\kappa/k_0$ van de TE-modi in een planaire open golfleider met relatieve permittiviteit $\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ gelden de ongelijkheden:

$$\text{Re}(\kappa/k_0) \leq \max(\varepsilon_r'), \quad \text{min}(\varepsilon_r') \leq -\text{Im}(\kappa/k_0) \leq \max(\varepsilon_r'').$$

Het vermoeden bestaat dat ook voor willekeurige oppervlaktegolfmodi in rechte golfleiders, met willekeurige dwarsdoorsnede en een willekeurig permittiviteitsprofiel dergelijke ongelijkheden zijn op te stellen.

Dit proefschrift, p. 50.

2. Voor rechte open golfleiders is de orthogonaliteit en de volledigheid van de discrete en continue modi coördinatuivaardig af te leiden. Ook voor rechte open golfleiders bestaande uit biaxietrope media is een dergelijke aanduiding op te stellen.

Dit proefschrift, Appendix B.
Shevechenko, V.V. (1980), On the completeness of spectral expansion of the electromagnetic field in the set of dielectric circular rod waveguide eigen waves, Radio Sci., 17(1), 229–231.

3. De gebruikelijke onderverdeling van de oppervlaktegolfmodi van een open kern/maatgolfleider in kern- en maatelmodi is voor een optische golfleider van verbuigende materialen niet bruikbaar.

Dit proefschrift, p. 69.

4. Voor de analyse van verstrooiing aan obstakels in golfleiders is de methode welke gebruik maakt van een benaderde Greense functie waarbij meervoudige reflecties buiten beschouwing worden gelaten, niet bruikbaar voor golfleiders met een hoog brekingsindexcontrast met de omgeving.


5. De bruikbaarheid van de variatieuitdrukking voor de soortelijke weerstand van een metaal ten gevolge van electron–phononinteractie is gering, daar het verkregen resultaat in sterke mate afhangt van de gebruikte proeffunctie.

6. De bijdrage van Umklapp-processen tot de soortelijke weerstand van een metaal als gevolg van electron–fotoninteractie is afhankelijk van de temperatuur en van de vorm van het Fermigebied. Deze afhankelijkheid manifesteert zich in de variatiedrukking voor de soortelijke weerstand. De genoemde bijdrage wordt echter sterk overberekend bij een onjuiste keuze van de proeffunctie in de variatiedrukking.

De Ruiter, H.M. (1978), Modelstudie voor de berekening van de electrische weerstand van een metaal, Laboratory of Electromagnetic Research, Report 1978–9, Delft University of Technology, Delft, Hoofdstuk 5;


7. De gevoeligheid van de genormaliseerde correlatiefunctie, voor irrelevante ruispatronen kan worden gereduceerd door het toevoegen van een constante offset aan de signaalvariantie.


8. Zowel het vele geld, de vele tijd, en de vele energie die gespendeerd worden aan het onderzoek naar kernfusie, als de recente opwinding over het bestaan van een simpele methode voor het tot stand brengen van kernfusie tonen aan, dat er weinig bereidheid is om energieverzuiming en -gebruik te beperken op schoene, onuitputtelijke en natuurlijke bronnen.


10. Het uiteenlopen van misk tijdens het ochtendpittuur op winterdagen met een lage temperatuur wordt veroorzaakt door een verhoogde luchtverontreiniging als gevolg van het gemotoriseerde verkeer.


11. De aanleg van fietspaden langs voor het fietserkeer te onveilig geworden wegen moet, vanuit oorzakelijk standpunt genoeg, niet als fietsvoorziening worden aangemerkt, maar als autovoorziening. Derhalve dient deze aanleg bekostigd te worden uit middelen bestemd voor en bijeengebracht door het autorijverkeer.

12. De begroting van een ministerie is een feitelijke weergave van voorgenoemde beleid dan de woorden van de betreffende minister. Zo geldt het beleid van het Ministerie van Verkeer en Waterstaat op milieugebied duidelijk in gebeke.

"gevleugelde woorden", is misleidend, daar deze Nederlandse uitdrukking een heel
andere inhoud heeft dan de Griekse, die de in de antieke cultuur veronderstelde
wijze van geluidsoverdracht uitbeeldt.

Homerus, Ἰάσων Α, 201;
Mehler, J. (1968), Woordenboek op de Gedichten van Homèros, 1ste druk, Nijgh