Monotonicity Properties of
Infinitely Divisible Distributions

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Bjørn Gårn Hansen

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Bjørn Gårn Hansen

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Chapter 1

PRELIMINARIES

1.1 Introduction and summary

The field of infinitely divisible random variables has grown during the last few decades to take a permanent position in the theory of probability. Almost all standard text books on probability now include at least one chapter devoted to the field of infinite divisibility. This is mainly due to its importance in solving the general central limit problem and to its applications to stochastic processes with stationary independent increments.

Though practical applications do exist (cf. for example Ahmad and Abuammoh (1977), Thorin (1977), Carasso (1987), Kellison and Servi (1987) and Hansen and Willemien (1988)), infinite divisibility does remain a theoretical concept. In this thesis the emphasis is therefore on the theory. Examples and practical applications are, as a rule, not sought.

Our starting point is the Lévy canonical representation of infinitely divisible distributions (cf. Theorem 1.3.2 of this thesis), where the characteristic function of an infinitely divisible distribution $F$ is related to a function $M$, called the Lévy spectral function. We are interested in characterizing the distributions $F$ which have a Lévy spectral function $M$ satisfying some monotonicity requirement.

In Chapter 2 we give a review of known monotonicity results in infinite divisibility and we present a curious connection with analogous results in renewal theory. In Chapters 3 and 4 we consider non-negative infinitely divisible random variables whose Lévy spectral functions are either absolutely continuous or supported by the non-negative integers. Chapter 3, which is based on the article On Moment Sequences and Infinitely Divisible Sequences, Hansen and Steutel (1988), studies these Lévy spectral functions in the context of moment sequences and moment
functions. The main results of On Logconcave and Logconvex Infinitely Divisible Sequences and Densities, Hansen (1988), are given in Chapter 4, where log-concave and log-convex Lévy spectral functions are considered. The set of infinitely divisible distributions with α-unimodal Lévy spectral functions is characterized in Chapter 5. Chapter 6 deviates from the theme of this thesis, as it studies infinitely divisible random variables with α-unimodal Lévy spectral functions as limits of sums of triangular arrays of random variables and as limits of sums of shrunken random variables.

1.2 Notations and conventions

In this section we list notations and conventions, which will be used throughout this thesis, often without further reference.

Random variables will always be one-dimensional and real-valued. They will be denoted by the capitals X, Y, Z, . . . The distribution functions of X and Y will be denoted by F and G; their densities, if they exist, by f and g; their characteristic functions by ϕ and ψ, with

\[ \phi(t) = \int e^{itx} dF(x), \quad t \in \mathbb{R}. \]

If the random variables X and Y are non-negative then we define their Laplace-Stieltjes transforms f and g by

\[ \hat{f}(t) = \int e^{-tx} dF(x), \quad t \in \mathbb{R}_+. \]

If the random variables are non-negative and concentrated on the non-negative integers \( \mathbb{N}_0 = \{0, 1, 2, . . . \} \), then we call their distributions discrete and denote them by \( (p_n)_{n=0}^{\infty}, (q_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}, . . . \). Their probability generating functions will be denoted by P, Q, S, . . . , with

\[ P(z) = \sum_{n=0}^{\infty} p_n z^n, \quad |z| \leq 1. \]

In general, we shall denote the generating function of a sequence \( (a_n)_{n=0}^{\infty} \) by the corresponding capital letter A, where

\[ A(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| \leq r, \]

for some \( r \in \mathbb{R}_+ \). All sequences considered in this thesis will be real-valued and indexed by \( \mathbb{N}_0 \), and henceforth denoted by \( (a_n) \), \( (b_n) \) etc.

A classification of a set \( C \) is a class \( \{ C_t, t \in I \} \) with \( I \subset \mathbb{R} \) an index set, such that the sets \( C_t \) are non-decreasing, i.e., for \( t(1) < t(2) \ldots < t(n) \) with \( t(i) \in I \), the sets
1.2 Notations and conventions

$C_{(1)} \cup C_{(2)} \cup \ldots \cup C_{(m)}$ form a partition of $C$.

A set $C$ of characteristic functions is said to be closed under limits if any characteristic function which is the limit of a sequence of characteristic functions in $C$ is itself a member of $C$.

The letters $M$ and $N$ will always denote Lévy spectral functions (cf. Theorem 1.3.2 of Section 1.3), $H$ will always be a canonical measure (cf. Theorem 1.3.3). The sequences $(p_n)$ and $(r_n)$ will always be assumed to be related through equation (1.2) of Theorem 1.3.4.

1.3 Infinitely divisible distributions

We begin this section with the definition of an infinitely divisible random variable.

**Definition 1.3.1.** A random variable $X$ is said to be infinitely divisible if for every positive integer $n$, there exists independent and identically distributed random variables $X_{1n}, X_{2n}, \ldots, X_{nn}$ such that

\[ X = X_{1n} + X_{2n} + \ldots + X_{nn}, \]

with $d$ denoting equality in distribution.

We say that $\Phi$, $\tilde{\Phi}$, $\tilde{F}$ or $F$ is infinitely divisible if it stems from an infinitely divisible random variable. We state three representation theorems for infinitely divisible distributions on $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_0$. The proof of the first theorem may be found in Lukacs (1970), of the second partly in Feller (1971) and partly in Steutel (1970) and the proof of the last theorem in Feller (1968) and Steutel (1970).

**Theorem 1.3.2.** A function $\Phi$ is an infinitely divisible characteristic function if and only if it can be written in the form

\[ \ln \Phi(t) = i a_q t - \frac{1}{2} \sigma_q^2 t^2 + \int_{\mathbb{R}} k(t, x) \, dM(x), \]

where $a_q \in \mathbb{R}$, $\sigma_q^2 \in \mathbb{R}_+$, $k(t, x) = e^{itx} - 1 - itx (1 + x^2)^{-1}$, and such that the function $M$ (called the Lévy spectral function) satisfies

(i) $M(x)$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$,

(ii) $M(-\infty) = M(\infty) = 0$. 

(iii) The integrals \( \int_{-\infty}^{\infty} x^2 \, dM(x) \) and \( \int_0^\infty x^2 \, dM(x) \) are finite for every \( \varepsilon > 0 \).

The representation is unique.

**Theorem 1.3.3.** A function \( \tilde{f} \) is an infinitely divisible Laplace-Stieltjes transform if and only if it can be written in the form

\[
\ln \tilde{f}(t) = \int_0^\infty (e^{-tx} - 1) x^{-1} \, dH(x),
\]

where the function \( H \) (called the canonical measure) is non-decreasing. Equivalently, \( f \) is infinitely divisible if and only if its distribution \( F \) satisfies

\[
\int_0^x u \, dF(u) = \int_0^x x^{-1} \, dH(x), \quad x \in \mathbb{R}_+.
\]

(1.1)

Necessarily \( \int_0^\infty x^{-1} \, dH(x) < \infty \). The representation is unique.

**Theorem 1.3.4.** A function \( P \) with \( P(0) > 0 \) is an infinitely divisible probability generating function if and only if it can be written in the form

\[
\ln P(x) = -\theta (1 - G(x)),
\]

where \( \theta > 0 \) and \( G \) is a probability generating function with distribution \( (g_n) \) such that \( G(0) = 0 \). Equivalently, \( P \) is infinitely divisible if and only if its distribution \( (p_n) \) satisfies

\[
(n+1)P_{n+1} = \sum_{k=0}^{n} R_k \theta^k, \quad n \in \mathbb{N}_0.
\]

(1.2)

with \( (r_n) \) the canonical measure of \( (g_n) \) non-negative; necessarily \( r_n = \theta (n+1) g_{n+1} \) and \( \sum_{n=0}^\infty r_n / (k+1) : = \theta < \infty \). The representation is unique.

We shall use the following notation throughout this thesis.

**Notation 1.3.5.** Let \( ID(\mathbb{R}) \), \( ID(\mathbb{R}_+) \) and \( ID(\mathbb{N}_0) \) denote the set of infinitely divisible characteristic functions, Laplace-Stieltjes transforms and probability generating functions, respectively.

The last two theorems of this section give some useful properties and another characterization of \( ID(f) \), \( f \in \{ \mathbb{R}, \mathbb{R}_+, \mathbb{N}_0 \} \).
1.3 Infinitely divisible distributions

Theorem 1.3.6. For \( I = \mathbb{R}, \mathbb{R}_+ \) or \( \mathbb{N}_0 \) the sets \( ID(I) \) (cf. Notation 1.3.5) are multiplication semigroups, closed under limits.

Theorem 1.3.7. The following equivalences hold (cf. Notation 1.3.5)

(i) \( o \in ID(\mathbb{R}) \) if and only if \( \phi^{1/n} \) is a characteristic function for all \( n \in \mathbb{N}_+ \);

(ii) \( f \in ID(\mathbb{R}_+) \) if and only if \( \hat{f}^{1/n} \) is a Laplace-Stieltjes transform for all \( n \in \mathbb{N}_+ \);

(iii) \( P \in ID(\mathbb{N}_0) \) if and only if \( P^{1/n} \) is a probability generating function for all \( n \in \mathbb{N}_+ \).

1.4 Self-decomposable and stable distributions

The sets of self-decomposable and stable distributions are two important subsets of the set of infinitely divisible distributions. Stable distributions are widely studied (cf. Lukacs (1970)) as they provide a natural generalization of the normal distribution; self-decomposable distributions are, in turn, a generalization of the stable distributions. In Chapters 5 and 6 we generalize both concepts.

A random variable \( X \) is self-decomposable if there exists sequences \((a_n), (b_n)\) of real numbers with \( a_n \geq 0 \) and \( a_n \to 0 \) as \( n \to \infty \) and a sequence \((X_n)\) of independent random variables such that

\[
a_n S_n + b_n \xrightarrow{w} X \quad \text{as} \quad n \to \infty ,
\]

with \( \xrightarrow{w} \) denoting weak convergence and \( S_n = \sum_{k=1}^{n} X_k \). Let the linear operator \( T \) be defined by \( T x = x \). Then (1.3) can be rewritten as

\[
\sum_{k=1}^{n} T_{a_k} X_k + b_n \xrightarrow{w} X \quad \text{as} \quad n \to \infty .
\]

If \( X_n \) are identically distributed then \( X \) is called stable. Self-decomposability of \( X \) is equivalent to (cf. Feller (1971))

\[
d X = c X' + X_c,
\]

for all \( c \in (0,1) \), with \( = \) denoting equality in distribution and where \( X' \) and \( X_c \) are independent and \( X_c \) is distributed as \( X \). For the corresponding characteristic functions this means that for every \( c \in (0,1) \) there exists a characteristic function \( \phi_c \) such that \( \phi \) satisfies

\[
\phi(t) = \phi(ct) \phi_c(t) , \quad t \in \mathbb{R}_+ .
\]

If $\Phi(t) = \phi_0((1 - t^\delta)^{1/\delta})$ for some $\delta > 0$, then $\Phi$ is stable. Steutel and van Harn (1979) proposed a discrete analogue of self-decomposability and stability. A random variable $X$ is said to be discrete self-decomposable if for every $c \in (0, 1)$ there exists a random variable $X_c$, independent of $X'$, such that

$$X = c \otimes X' + X_c,$$

with $X'$ and $X$ identically distributed. The random variable $c \otimes X$ is defined in distribution by

$$P_{c \otimes X}(z) = P_X((1 - c)(1 - z)),$$

with $P_X$ denoting the probability generating function of $X$. For a probabilistic interpretation of $\otimes$ see Steutel and van Harn (1979). In terms of probability generating functions (1.7) reads

$$P(z) = P((1 - c)(1 - z))P_c(z), \quad 1 \leq z \leq 1, \quad c \in (0, 1),$$

for some probability generating functions $P_c$. If $P_c(z) = P((1 - c^\delta)^{1/\delta}(1 - z))$ for some $\delta > 0$, then $P$ is said to be (discrete) stable.

We now give another notation and list a series of representation theorems for self-decomposable and stable distributions. The proof of Theorem 1.4.1 can be found in Lukacs (1970). This proof contains a minor error, which is corrected in Hall (1981). The proof of Theorem 1.4.5 can also be found in Lukacs. For the proofs for distributions on $\mathbb{R}^n$ we refer to Feller (1971) and those for distributions on $\mathbb{N}_0$ to Steutel and van Harn (1979).

**Theorem 1.4.1.** A function $\phi$ is the characteristic function of a stable distribution if and only if $\phi$ is either normal or $\phi$ can be written in the form

$$\ln \Phi(t) = ita_0 c 2 \pi (1 + i \beta \sigma(t)) w(1t, \delta),$$

where $c \geq 0$, $|\beta| \leq 1$, $\delta \in (0, 2)$ and $a_0 \in \mathbb{R}$. The function $w(1t, \delta)$ is given by

$$w(1t, \delta) = \begin{cases} \tan(\pi \delta/2) & \text{if } \delta \neq 1 \\ -2/\pi \ln |t| & \text{if } \delta = 1 \end{cases}$$

Equivalently, $\phi$ is the characteristic function of a stable distribution if and only if $\phi$ is infinitely divisible and either (cf. Theorem 1.3.2) $\alpha^2 > 0$ and $M(x) = 0$ or $\alpha^2 = 0$ and $M(x) = C_1 |x|^\delta$ for $x < 0$ and $M(x) = -C_2 x^{-\delta}$ for $x > 0$. The parameters satisfy $\delta = (0.2), C_1 \geq 0, C_2 \geq 0$ and $C_1 + C_2 \geq 0$. The parameter $\delta$ is called the exponent of stability of $\phi$.

**Theorem 1.4.2.** A function $\hat{f}$ is the Laplace-Stieltjes transform of a stable distribution on $\mathbb{R}_+$ if and only if it can be written in the form

$$\ln \hat{f}(\tau) = -a_0 \tau - \lambda \tau^\delta, \quad \tau \in \mathbb{R}_+,$$
with $\lambda \geq 0$, $a_j \geq 0$ and $\delta \in (0,1]$. The parameter $\delta$ is called the exponent of stability of $f$.

**Theorem 1.4.3.** A function $P$ is the probability generating function of a stable distribution on $\mathbb{R}_0^+$ if and only if it can be written in the form

$$\ln P(z) = -\lambda(1-z)^{\delta}, \quad 1 \leq z \leq 1,$$

with $\lambda \geq 0$ and $\delta \in (0,1]$. The parameter $\delta$ is called the exponent of stability of $P$.

The degenerate distribution is trivially self-decomposable and stable. It is therefore usual in the literature not to call the degenerate distribution a stable distribution. For our purposes it is however desirable to include the degenerate distributions in the set of stable distributions. We therefore introduce the following notation.

**Notation 1.4.4.** The characteristic function of a stable, possibly degenerate, distribution with exponent $\delta$, will be denoted by $\phi_{\text{stable}(\delta)}$. Similarly, we denote by $P_{\text{stable}(\delta)}$ and $f_{\text{stable}(\delta)}$ the probability generating function and the Laplace-Stieltjes transform of a stable, possibly degenerate, distribution with exponent $\delta$.

**Theorem 1.4.5.** A function $\phi$ is the characteristic function of a self-decomposable distribution if and only if $0$ is infinitely divisible with Lévy spectral function $M$ (cf. Theorem 1.3.2) having left and right hand derivatives and such that $|x|M'(z)$ is non-decreasing on $(-\infty,0)$ and non-increasing on $(0,\infty)$.

**Theorem 1.4.6.** A function $P$ is the probability generating function of a self-decomposable distribution if and only if it can be written in the form

$$\ln P(z) = \int_0^1 \ln Q(1-v(1-z)) v^{-1} dv,$$

with $Q$ a unique infinitely divisible probability generating function. Equivalently, $P$ is a self-decomposable probability generating function if and only if it is infinitely divisible and its canonical measure $(\tau_T)$ (cf. Theorem 1.3.4) is non-increasing.

The analogue of Theorem 1.4.6 for distributions on $\mathbb{R}_0^+$ is proved in van Harn et. al. (1982) and mentioned for distributions on $\mathbb{R}$ in Steutel and van Harn (1979). In Chapter 5 we prove two theorems which include these analogues. We finish this section with a definition, which we use in Chapters 5 and 6.
DEFINITION 1.4.7. The characteristic function \( \phi \) is said to be in the domain of normal attraction of a stable characteristic function with exponent \( \delta \) if and only if for suitable \( (\delta_n) \), \( \lim_{n \to \infty} \phi^{n/(1+\delta_n)}(1/n) = \phi_{\text{STABLE}}(0) \).

1.5 Motivation and methodology

In this thesis we study the relationship between an infinitely divisible distribution and its Lévy spectral function (or canonical measure). We prove that an infinitely divisible distribution whose Lévy spectral function (or canonical measure) possesses some well-known monotonicity property, for example complete monotonicity, log-concavity, log-convexity or \( \alpha \)-unimodality, belongs to a well-known set of distributions, for example the set of mixtures of geometric distributions, strongly unimodal distributions or \( \alpha \)-self-decomposable distributions.

It turns out that in studying monotonicity properties in this context, it is easiest to first consider the discrete case (cf. Theorem 1.3.4). Here the canonical measure can be explicitly expressed in terms of the probabilities, which makes developing and proving hypotheses and constructing counterexamples easier than in the case of distributions on \( \mathbb{R} \) or \( \mathbb{R}_+ \). In many cases we can even prove the equivalent result for distributions on \( \mathbb{N} \) from those on \( \mathbb{N}_0 \) by applying a simple limiting argument (cf. Sections 3.5, 4.3 and 5.5). Also, by studying the discrete case we develop insight which can be helpful in proving the result for \( \mathbb{R} \) (cf. Section 5.4).

As is seen in Chapter 2, many of the monotonicity results obtained in renewal theory and infinite divisibility are quite similar. This observation led us to consider log-convexity and moment sequences in the context of infinite divisibility (cf. Theorems 2.4.2 and 2.4.3 of the next chapter).
Chapter 2

INFINITELY DIVISIBLE SEQUENCES AND RENEWAL SEQUENCES

2.1 Introduction

Much of renewal theory is concerned with determining properties of the renewal function. This includes study of the relationship between the renewal function and its underlying distribution. Similarly, the interplay between an infinitely divisible distribution and its Lévy spectral function plays an important role in the field of infinite divisibility. Many of the results obtained in these two, very different, fields are quite similar. This correspondence, between results in infinite divisibility and in renewal theory, proves to be very useful (cf. Chapters 3 and 4). In this chapter we give a brief review of the results concerning monotonicity properties in these two fields. Section two of this chapter gives a short introduction to renewal theory. The following two sections state the results, with little mention of possible applications. For a more complete description we refer to the references. The last section discusses the interplay between renewal sequences and infinitely divisible sequences.

2.2 A taste of renewal theory

In this section we give a brief introduction to renewal theory. For simplicity we restrict ourselves to renewal sequences on $\mathbb{N}_0$. For a more rigorous introduction to renewal theory we refer to Feller (1968). Let $E$ be an event (for example a renewal) and define the probability distribution $(f_n)$ by

$$f_n := \mathbb{P}(E \text{ occurs for the first time at time } n+1), \ n \in \mathbb{N}_0.$$ 

Define the sequence of probabilities $(u_n)$ by
\[ u_n := \mathbb{P}(E \text{ occurs at time } n), \; n \in \mathbb{N}_0. \]

It then follows that \((f_n)\) and \((u_n)\) are related by

\[ u_{n+1} = \sum_{k=0}^{n} u_{n-k} f_k, \; u_0 = 1, \; n \in \mathbb{N}_0. \tag{2.1} \]

The probability \(f_n\) can be interpreted as the probability that a machine first breaks down at time \(n+1\) given that it broke down at time zero. If a broken down machine gets fixed instantaneously, and after repair is 'as good as new', then \(u_n\) gives the probability that the machine breaks down at time \(n\). In many practical situations the distribution \((f_n)\) is known, or at least some property (for example a monotonicity property) is known, and the sequence \((u_n)\) or its behaviour in some sense, is sought. One of the most important results from renewal theory is:

\[
\mu := \sum_{n=0}^{\infty} (n+1)f_n < \infty \quad \text{and} \quad \langle f_n \rangle \text{ is aperiodic}, \quad \lim_{n \to \infty} u_n = \mu^{-1}.
\]

In the following section we give a review of monotonicity results in renewal theory.

### 2.3 Monotonicity results in renewal theory

To avoid repetition, we only review the results in discrete renewal theory, i.e., on sequences related through (2.1). The analogous results for general renewal functions, related through the so-called renewal equation (cf. Ross (1983)), are also true. In fact in many cases the result for renewal functions can be obtained by applying a limiting argument to the result on renewal sequences (cf. Hansen and Frenk (1988) and Chapter 3). The list is by no means complete and will be given with very few comments to applications.

Kaluzhnik (1928) was the first (to the author's knowledge) to study sequences related through (2.1). Among other results he proved

**Theorem 2.3.1.** Let the sequences \((u_n)\) and \((f_n)\) be related by (2.1). The following implications hold:

(i) If \(f_n \geq 0, \; n \in \mathbb{N}_0\) then \(u_n \geq 0, \; n \in \mathbb{N}_0\).

(ii) If \((u_n)\) is log-convex then \(f_n \geq 0, \; n \in \mathbb{N}_0\).

The definitions of log-convexity and log-concavity are given in Section 4.2. In de Bruijn and Erdös (1953), for part (i), and in Hansen and Frenk (1988), for parts (ii) and (iii), we find the following theorem.
2.3 Monotonicity results in renewal theory

Theorem 2.3.2. Let the sequences \((u_n)\) and \((f_n)\) be related by (2.1). Suppose \(\sum f_n \leq 1\) and \(\sum (n+1)f_n = \mu < \infty\). Let

\[
F(n) := \sum_{k=0}^{n} f_k, \quad F_1(n) := \mu^{-1} \sum_{k=0}^{n} (1 - F(k)).
\]

The following implications hold.

(i) If \((f_n)\) is log-convex then \((u_n)\) is log-convex;
(ii) If \((1 - F(n))\) is log-convex then \((u_n)\) is non-increasing;
(iii) If \((1 - F_1(n))\) is log-convex then \(u_n \geq \mu^{-1}, n \in \mathbb{N}_0\).

For condition (i) in Theorem 2.3.2 it is only necessary to assume that \(f_n \geq 0\) for \(n \in \mathbb{N}_0\). The analogue of Theorem 2.3.2, parts (ii) and (iii), for distributions on \(\mathbb{R}_0\) was first proved in Brown (1980), by coupling methods. Hansen and Frenk (1988) provided a simpler proof by applying a limiting argument to Theorem 2.3.2. The conditions on \((f_n)\) given in (ii) and (iii) are related to distributions with decreasing failure rates and increasing mean residual life-times. Note that

\[
(f_n) \text{ log-convex } \Rightarrow (1 - F(n)) \text{ log-convex } \iff (1 - \mu^{-1}F_1(n)) \text{ log-convex}.
\]

\((u_n)\) log-convex \(\Rightarrow\) \((u_n)\) non-increasing \(\Rightarrow\) \(u_n \geq \mu^{-1}, n \in \mathbb{N}_0\).

Hence Theorem 2.3.2 provides a classification (cf. Section 1.2) of the set of renewal sequences.

Horn (1970) studied (2.1) in the context of moment sequences. For the definition of moment sequences we refer to Notation 3.2.1. Horn (1970) dropped the condition that the sequences are non-negative and convergent.

Theorem 2.3.3. Let the sequences \((u_n)\) and \((f_n)\) be related by (2.1). Then

(i) \((u_n)\) is a Hamburger moment sequence if and only if \((f_n)\) is a Hamburger moment sequence;
(ii) \((u_n)\) is a Stieltjes moment sequence if and only if \((f_n)\) is a Stieltjes moment sequence;
(iii) \((u_n)\) is a Hausdorff moment sequence if and only if \((f_n)\) is a Hausdorff moment sequence with \(\sum f_n \leq 1\).

Kaluzny (1928) proved part (ii) and the 'if' part of (iii), with a proof different from Horn's (1970). Theorem 2.3.3 is stronger than Theorem 2.3.2 in the sense that...
(a_n) Hausdorff moment sequence \rightarrow (a_n) Stieljes moment sequence \rightarrow (a_n) log-convex.

This is easily verified by using Schwarz' inequality. We conclude this section with an easily proved theorem.

**Theorem 2.3.4.** Let the sequences \((u_n)\) and \((f_n)\) be related by (2.1). Then.

(i) \(u_{n+m} \geq u_n u_m\) for \(n \in \mathbb{N}_0, m \in \mathbb{N}_0\);
(ii) If \(u_0 a_1 > 0\) then \(u_n > 0, n \in \mathbb{N}_0\);
(iii) If \(f_0 \in [0,1]\) and \(f_{n+1} / f_n \leq 1 - f_0\) then \((u_n)\) is non-increasing.

### 2.4 Monotonicity results of infinitely divisible distributions

As in Section 2.3 we only review the results for discrete distributions. Most results are here also true for distributions on \(\mathbb{R}_+\) and can usually be obtained by some limiting argument (cf. Chapters 3 and 4) or some other argument (cf. Chapter 5) from those on \(\mathbb{N}_0\). The analogues of Theorems 2.3.2 and 2.3.3 are proved in Chapters 4 and 3, respectively. For completeness we also state them here.

**Theorem 2.4.1.** Let the sequences \((p_n)\) and \((r_n)\) be related by (1.2), i.e., by

\[
(n+1)p_{n+1} = \sum_{k=0}^{n} p_{n-k} r_k, \quad p_0 \geq 0, \quad n \in \mathbb{N}_0 .
\]  

The following implications hold.

(i) If \(r_n \geq 0, n \in \mathbb{N}_0\) then \(p_n \geq 0, n \in \mathbb{N}_0\);
(ii) If \((p_n)\) is log-convex then \(r_n \geq 0, n \in \mathbb{N}_0\).

Part (ii) of Theorem 2.4.1 was first proved in Steutel (1970). For a different proof see Section 4.2. Part (ii) states that all log-convex distributions are infinitely divisible. The absolutely continuous analogue of Theorem 2.4.1 (ii) is also proved in Steutel (1970). The following theorem and its absolutely continuous analogue is proved in Chapter 4.

**Theorem 2.4.2.** Let the sequences \((p_n)\) and \((r_n)\) be related by (2.2). The following implications hold.
2.4 Monotonicity results of infinitely divisible distributions

(i) If \((r_n)\) is log-convex and \(r_0 \leq r_1\) then \((p_n)\) is log-convex;
(ii) If \((r_n)\) is log-concave and \(r_0 \geq r_1\) then \((p_n)\) is log-concave.

In Chapter 3 we prove the analogue of Theorem 2.3.3 for infinitely divisible sequences. This analogue also provides an alternative proof of Theorem 2.3.3 (cf. Section 3.4).

**Theorem 2.4.3.** Let the sequences \((p_n)\) and \((r_n)\) be related by (2.2). Then

(i) \((p_{n+1})\) is a Hamburger moment sequence if and only if \((r_{n/(n+1)})\) is a Hamburger moment sequence with \(\mu \leq \lambda\) (cf. (3.6));
(ii) \((p_n)\) is a Stieltjes moment sequence if and only if \((r_{n/(n+1)})\) is a Stieltjes moment sequence with \(\mu \leq \lambda\) (cf. (3.6));
(iii) \((p_n)\) is a Hausdorff moment sequence if and only if \((r_{n/(n+1)})\) is a Hausdorff moment sequence with \(\mu \leq \lambda\) (cf. (3.6)).

The absolutely continuous analogue of part (iii) is proved in Steutel (1970). Parts (iii) and (iv) of the following theorem are proved in Steutel and van Harn (1979). The first two parts are easily verified.

**Theorem 2.4.4.** Let the sequences \((p_n)\) and \((r_n)\) be related by (2.2). Then,

(i) \((\sum_{n=0}^{\infty} p_{n+m}) p_{n+m} \geq p_n p_m\) for \(n \in \mathbb{N}_0, m \in \mathbb{N}_0;\)
(ii) If \(p_0 p_1 > 0\) then \(p_n > 0\) for \(n \in \mathbb{N}_0;\)
(iii) If \(r_0 \in [0, 1]\) and \((r_n)\) is non-increasing then \((p_n)\) is non-increasing;
(iv) If \((r_n)\) is non-increasing then \((p_n)\) is unimodal.

Condition (iv) implies that all discrete self-decomposable distributions are unimodal (cf. Theorem 1.4.6).

2.5 Renewal sequences and infinitely divisible sequences

As seen in the two previous sections there is a connection between the behaviour of sequences related through (2.1) and sequences related through (2.2). This section is intended to give some understanding to why this is true.
Let \((p_n)\) be a sequence of real numbers and define the two sequences \((r_n)\) and \((f_n)\) through

\[ p_{n+1} = \sum_{k=0}^{n} p_{n-k} f_k, \quad n \in \mathbb{N}_0, \tag{2.3} \]

\[ (n+1)p_{n+1} = \sum_{k=0}^{n} p_{n-k} f_k, \quad n \in \mathbb{N}_0. \tag{2.4} \]

Equation (2.4) is more general than equation (2.3) in the sense that if the sequence \((f_n)\) is non-negative, then \((r_n)\) is also non-negative (cf. Steinel (1970), p.83). In fact, if \((p_n)\) is a probability distribution and \((f_n)\) is non-negative, then \((p_n)\) is a compound geometric distribution and hence infinitely divisible.

Taking generating functions on both sides of (2.3) and (2.4) and eliminating \(P(z)\) yields

\[ R(z) = -\frac{d}{dz} \ln(1-zF(z)). \tag{2.5} \]

Let \(f_\bot := -1\), then (2.5) is equivalent to

\[ (n+1)(-1)^n f_n = \sum_{k=0}^{n} \left( (-1)^{n-k-1} f_{n-k-1} \right) (-1)^k r_k, \quad n \in \mathbb{N}_0. \tag{2.6} \]

One could therefore expect that the relationship between \((p_n)\) and \((r_n)\) is similar to that between \((f_n)\) and \((r_n)\). Hence, if \((r_n)\) has some property which \((p_n)\) inherits, then it is plausible to think that \((f_n)\) will also inherit this property. In a similar way one can start with \((r_n)\). Equation (2.6) does not only provide an idea as to why the theorems in Sections 2.3 and 2.4 are similar, but also provides alternative proofs of some of the results in Section 2.3 (see Chapter 3).
Chapter 3

MOMENT SEQUENCES AND MOMENT FUNCTIONS

3.1 Introduction

In this chapter we consider sequences \((p_n)\) and \((r_n)\) related through equation (1.2), i.e., through

\[(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \quad n \in \mathbb{N}, \tag{3.1}\]

where either \((p_n)\) or \((r_n/(n+1))\) is a moment sequence (cf. Notation 3.2.1), and we consider functions \(f\) and \(h\) related through (cf. (1.1))

\[xf(x) = \int_0^x f(x-u)h(u)du, \quad x \in \mathbb{R}, \tag{3.2}\]

where either \(f\) or \(h(t)/t\) is a moment function (cf. Definition 3.5.1). The discrete case is considered in the same vein as renewal sequences were by Horn (1970) (cf. Theorems 2.3.3 and 2.4.3). The proofs in the discrete case are based on a theorem in monotone matrix function theory by Bendat and Sherman (1955). At first glance it may seem as though this approach uses too powerful tools, but its great advantage is that it results in some very elegant proofs. The somewhat more straightforward, but tedious, approach used to prove Theorems 3.3.2 and 3.3.7 can also be adapted to prove Theorems 3.3.4 and 3.3.8, but fails to prove Theorem 3.3.1. A limiting argument can be used to prove the analogous results for moment functions.

In all but the last section of this chapter we will consider equation (3.1), and its absolutely continuous analogue (3.2), outside its probabilistic context; for (3.1) we drop the conditions that \(r_n \geq 0\) and \(\sum r_n/(n+1) < \infty\) (thus \((p_n)\) need not be non-negative
and \( \sum p_n \) need not be convergent) and take, for convenience, \( p_0 = 1 \); for (3.2) we drop the condition that \( \int h(x) f dx < \infty \) (thus \( f dx \) need not be finite). In Section 3.2 we give some preliminaries. Statements (i), (ii) and (iii) of Theorem 2.4.3 will be proved as separate theorems in Section 3.3. The absolutely continuous analogue is discussed in Section 3.5. Using equation (2.6) and Theorem 2.4.3 we prove Theorem 2.3.3 in Section 3.4. Its absolutely continuous analogue is mentioned in Section 3.6. Our result on Hausdorff moment sequences has applications in probability theory, as sequences of this type are mixtures of geometric distributions. This case and its absolutely continuous analogue will be investigated in more detail in Section 3.7.

3.2 Preliminaries

According to equation (2.1) the functions \( U \) and \( F \) are related by

\[
z F(z) = 1 - \frac{1}{U(z)},
\]

and \( F \) and \( R \), according to (3.1), by

\[
\tilde{R}(z) = z \sum_{n=0}^{\infty} \frac{r_n}{n+1} z^n = \log P(z).
\]

The symbols \( U, F, P \) and \( \tilde{R} \) in (3.3) and (3.4) can be regarded as formal power series. We only use them as functions when \( \mu \) in (3.5) is supported by \([-T, T]\). Then \( U, F, P \) and \( \tilde{R} \) are well-defined functions of the form

\[
f(z) = \int_{[-T,T]} (1-\mu z)^{-1} d\mu(u), \quad z \in (-T^{-1}, T^{-1}).
\]

Hence they are Stieltjes transforms and so the measure \( \mu \) is unique.

As we will become apparent, the essential difference between (3.3) and (3.4) in the context of moment sequences stems from the fact that

\[
1 - \frac{1}{w}
\]

maps the upper half-plane onto the upper half-plane,

\[
\log w
\]

maps the upper half-plane onto the strip \( 0 < \text{Im} w < \pi \).

This difference explains the difference between Theorems 2.3.3 and 2.4.3 and somewhat complicates the proof of Theorem 2.4.3. For ease in notation we introduce the following sets

**Notation 3.2.1.** Let \( MS(\mathbb{R}) \), \( MS(\mathbb{R}_+) \) and \( MS([0, 1]) \) denote the set of Hamburger, Stieltjes and Hausdorff moment sequences, respectively, i.e., for \( I \in \{ \mathbb{R}, \mathbb{R}_+, [0, 1] \} \) let
(3.5) \( (a_n) \in MS(I) \) iff \( a_n = \int_I x^n d\mu(x) , n \in \mathbb{N}_0 \),

where \( \mu \) is a nonnegative measure. Also let the sets \( MS^*(I) \) (cf. (3.5)) be given by

(3.6) \( (a_n) \in MS^*(I) \) iff \( (a_n) \in MS(I) \) with \( \mu \leq \lambda \),

here \( \lambda \) denotes Lebesgue measure and \( \mu_1 \leq \mu_2 \) means that \( \mu_1(\mathcal{B}) \leq \mu_2(\mathcal{B}) \) for all Borel sets \( \mathcal{B} \), i.e., that \( \mu_1 \) is absolutely continuous with respect to \( \mu_2 \) and the Radon-Nikodym derivative \( d\mu_1 / d\mu_2 \leq 1 \). We also define (compare (3.5) and (3.6)) the set \( MS_T(I) \) by

(3.7) \( (a_n) \in MS_T(I) \) iff \( (a_n) \in MS(I) \) and \( \mu \) supported by \([-T,T]\).

and let \( MS_T^*(I) = MS_T(I) \cap MS^*(I) \). In a similar way we define \( MS(\mathbb{R}_+) \) and \( MS^*(\mathbb{R}_+) \). The measure \( \mu \) in (3.5) is called the representing measure of the sequence \( (a_n) \).

We will use the following lemmas describing some properties of moment sequences. The proofs of the first two lemmas can be found in Shohat and Tamarkin (1943). The third lemma follows from Helly’s first theorem and the corollary to Theorem 25.12, p. 292 in Billingsley (1979).

**Lemma 3.2.2.** \( (a_n) \in MS(\mathbb{R}_+) \) iff \( (a_n) \in MS(\mathbb{R}) \) and \( (a_{n+1}) \in MS(\mathbb{R}) \).

**Lemma 3.2.3.** \( (a_n) \in MS([0,1]) \) iff \( (a_n) \in MS(\mathbb{R}_+) \) and \( a_n \) is bounded.

**Lemma 3.2.4.** If \( (a_n(t)) \in MS(I) \) (or \( MS^*(I) \)) for all \( t \geq t_0 \), and \( \lim_{t \to \infty} a_n(t) = a_n \), \( n \in \mathbb{N}_0 \), then \( (a_n) \in MS(I) \) (or \( MS^*(I) \)).

**Remark 3.2.5.** In proving Theorem 2.4.3 we shall use Lemma 3.2.4 together with truncation of integrals. If \( (p_n) \) is given by

\[
p_n = \int_{\mathbb{R}} x^n d\mu(x) , \quad n \in \mathbb{N}_0 ,
\]

and \( (r_n) \) is defined by (3.1), then we define \( (p_n(T)) \) by

\[
p_n(T) = \int_{I_T} x^n d\mu(x) , \quad n \in \mathbb{N}_0 .
\]

If we now define \( (r_n(T)) \) from \( (p_n(T)) \) by means of (3.1), then \( p_n(T) \to p_n \) and \( r_n(T) \to r_n \) as \( T \to \infty \). In a similar way one can start from \( (r_n) \).
Before we proceed we need a definition.

**Definition 3.2.6.** For $T > 0$, let $A_T$ denote the set of real analytic functions on $(-T^{-1}, T^{-1})$ that have an analytic continuation to the upper half-plane, and that are either constant or map the upper half-plane into itself; $A^*_T$ denotes the subset of functions in $A_T$ that map the upper half-plane into the strip $0 < \text{Im} z < \pi$.

The following lemma, proved in Bendat and Sherman (1955), is as basic here as it was in Horn (1970).

**Lemma 3.2.7.** Let $T > 0$ and let $C$ be a real-valued function defined on $(-T^{-1}, T^{-1})$. Then the following statements are equivalent.

(i) $C \in A_T$;

(ii) $C(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-T^{-1}, T^{-1})$ with $(c_{n+1}) \in MS_T(\mathbb{R})$;

(iii) $C(z) = C(0) + \int_{[-T,T]} \frac{z}{1-zt} \, d\mu(t) \quad (\text{Im} z > 0)$.

The next lemma enables us to use Lemma 3.2.7 in the situation of equation (3.6).

**Lemma 3.2.8.** Let $T > 0$ and let $\mu$ be a finite measure on $(-T, T)$. Let $f$ be defined by

$$f(z) = \int_{[-T,T]} \frac{z}{1-zt} \, d\mu(t) \quad (\text{Im} z > 0) \quad (3.8)$$

Then $\mu \leq \lambda$ (cf. (3.6)) iff $0 < \text{Im} f(z) < \pi$ for $\text{Im} z > 0$.

**Proof.** Clearly, $\text{Im} f(z) > 0$ if $\text{Im} z > 0$. If $\mu \leq \lambda$ then for $\text{Im} z > 0$ we have

$$\text{Im} f(z) \leq \int_{[-T,T]} \frac{z}{1-zt} \, dt = \arg(1-zT) - \arg(1-zT) < \pi.$$ 

Now suppose that $0 < \text{Im} f(z) < \pi$. The function

$$g(z) := f(-\frac{1}{z+T}) = \int_{[0,2T]} \frac{1}{z+T} \, d\mu(t-T)$$

is a Stieltjes transform. From the inversion formula for such transforms (cf. Widder (1946), p. 340) we have for any $\xi_1, \xi_2$ with $0 < \xi_1 < \xi_2$, and writing $\overline{\mu(\xi)} = (\mu(\xi + 0) + \mu(\xi - 0))/2$, 

3.2 Preliminaries

\[ \tilde{\mu}(\xi_2) - \tilde{\mu}(\xi_1) = \lim_{\eta \to 0^+} \int_{z_1}^{z_2} \frac{1}{\pi} \frac{1}{s - \eta} \Im f(\frac{1}{s - \eta}) \, ds \leq \xi_2 - \xi_1, \]

since \( \Im(1/z) > 0 \) iff \( \Im z > 0 \). It follows that \( \mu \leq \lambda \). \( \square \)

**Corollary 1.** Let \( f \) be given by (3.8) and let \( c > 0 \). Then \( 0 < \Im f(z) < c \pi \) for \( \Im z > 0 \) iff \( \mu \leq c \lambda \).

Now from Lemmas 3.2.7 and 3.2.8 we obtain

**Lemma 3.2.9.** Let \( T > 0 \) and let \( C \) be a real-valued function on \((-T^{-1}, T^{-1})\). Then the following three statements are equivalent.

1. \( C \in A_T^F \);
2. \( C(x) = \sum_{n=0}^{\infty} c_n x^n \) for \( x \in (-T^{-1}, T^{-1}) \) with \( (c_n) \in MS_T^F(\mathbb{R}) \);
3. \( C(x) = C(0) + \int_{-T}^{T} \frac{x}{1-xt} d\mu(t) \),

for a measure \( \mu \leq \lambda \).

The next lemma is an analogue for our situation of Lemma B in Horn (1979). It reduces to this lemma if all *'s are deleted. We shall need both versions of the lemma, and we shall refer to it as Lemma 3.2.10* if we need it with the *'s and as Lemma 3.2.10 otherwise. These lemmas enable us to study moment sequences by considering the mapping properties of their generating functions. Here we use \( A_T^F \subset A_T \) and \( MS_T^F(\mathbb{R}) \subset MS_T(\mathbb{R}) \).

**Lemma 3.2.10,* (3.2.10.)** Let \( T > 0 \) and let \((c_n)\) be a sequence of real numbers such that \( C(x) = \sum_{n=0}^{\infty} c_n x^n \) converges for all \( x \in (-T^{-1}, T^{-1}) \). Then (cf. Definition 3.2.6 and equation (3.7))

- (a) \( (c_n) \in MS_T^F(\mathbb{R}) \) iff \( C(x) \in A_T^F \);
- (b) \( (c_{n+1}) \in MS_T^F(\mathbb{R}) \) iff \( C(x) \in A_T^F \);
- (c) \( (c_n) \in MS_T^F(\mathbb{R}_+) \) iff \( x C(x) \in A_T^F \) and \( C(x) \in A_T \).
(d) \( (c_{n+1}) \in MS_T^+(\mathbb{R}_+) \) iff \( C(x) \in A_T^+ \) and \( x^{-1}(C(x) - C(0)) \in A_T \).

**Proof.** (a) and (b) follow directly from Lemma 3.2.9; the proofs of (c) and (d) are quite similar. We prove (c). First, let \( (c_n) \in MS_T^+(\mathbb{R}_+) \). Then by (a) we have \( x \in C(x) \in A_T^+ \) and by (c) of Lemma 3.2.10 that \( C(x) \in A_T \). Conversely, if \( x \in C(x) \in A_T^+ \), then \( (c_n) \in MS_T^+(\mathbb{R}) \) by (a), whereas \( C(x) \in A_T \) by (b) of Lemma 3.2.10 implies that \( (c_{n+1}) \in MS_T(\mathbb{R}) \). From Lemma 3.2.2 and the fact that \( MS_T^+(\mathbb{R}) \subset MS_T(\mathbb{R}) \) we conclude that \( (c_n) \in MS_T(\mathbb{R}_+) \). Finally, since moment sequences on \( MS_T(\mathbb{R}) \) have unique representing measures we have \( (c_n) \in MS_T^+(\mathbb{R}) \cap MS_T(\mathbb{R}_+) = MS_T^+(\mathbb{R}_+) \). \( \square \)

The following lemma is immediate from Lemma 3.2.2.

**Lemma 3.2.11.** \( (c_n) \in MS_T^+(\mathbb{R}_+) \) iff \( (c_n) \in MS_T^+(\mathbb{R}) \) and \( (c_{n+1}) \in MS(\mathbb{R}) \).

### 3.3 Moment sequences and infinitely divisible sequences

We are now ready to prove Theorem 2.4.3. We present the statements (i), (ii) and (iii) as separate theorems.

#### 3.3.1 Hamburger moment sequences

**Theorem 3.3.1.** Let \( (p_n) \) and \( (r_n) \) be related by

\[
(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \quad n \in \mathbb{N}_0.
\]

Then (cf. (3.5) and (3.6))

\[
(p_{n+1}) \in MS(\mathbb{R}) \text{ iff } (\frac{r_n}{n+1}) \in MS^+(\mathbb{R}).
\]

**Proof.** In view of Lemma 3.2.4 and Remark 3.2.5 we only have to prove the equivalence for \( MS_T(\mathbb{R}) \) and \( MS_T^+(\mathbb{R}) \). Since (cf. (3.4)) the generating functions of \( (p_n) \) and \( (r_n/(n+1)) \) are related by \( \hat{P}(z) = \log P(z) \), in view of Lemma 3.2.10 (3.2.10) it is sufficient to prove that \( P \in A_T \) iff \( \log P \in A_T^+ \) for some \( T' > 0 \). This last statement is true since (cf. Definition 3.2.6) \( P \) (is constant or) maps the upper half-plane into itself iff \( \log P \) (is constant or) maps the upper half-plane into the strip \( 0 < \text{Im} z < \pi \). Finally, \( P(z) \) is convergent for \( 1 \leq |z| < T \) with \( T' > 0 \) iff \( \log P(z) \) converges for \( 1 \leq |z| < T' \) for some \( T' > 0 \). \( \square \)
3.3 Moment sequences and infinitely divisible sequences

The next theorem, of some interest in its own right, is a preparation for the proof of (ii) in Theorem 2.4.3. Its proof is similar to the proof of Theorem 2.12.1 in Szasz (1970).

**Theorem 3.3.2.** Let \((p_n)\) and \((r_n)\) be related as in Theorem 3.3.1. Then

\[(p_n) \in MS(\mathbb{R}) \iff \frac{r_n}{n+1} = q_n - (-1)^n c_n \text{ with } (q_n), (c_n) \in MS^* (\mathbb{R}_+).\]

**Proof.** Any sequence in \(MS(\mathbb{R})\) can be approximated by sequences in \(MS^2(\mathbb{R})\) as follows. If \((p_n) \in MS^2(\mathbb{R})\) has a discrete representing measure with atoms \(q_i\) at \(t_i (i = 1, 2, \ldots, N)\) with \(-T < t_N < \ldots < t_1 < 0 < t_0 < \ldots < t_1 < T\), then \(P(x)\) takes the form

\[P(x) = \sum_{i=1}^{N} q_i/(1 - x t_i) = Q(x) / \prod_{i=1}^{N} (1 - x t_i),\]

with \(Q\) a polynomial of degree at most \(N - 1\). Observe that

- \(P(x) < 0\) for \(x \in (-\infty, 0)\), for \(k = 1, 2, \ldots, l\),
- \(P(x) > 0\) for \(x \in (0, \infty)\), for \(k = 1, 2, \ldots, l\),
- \(P(x) > 0\) for \(x \in (-\infty, 0)\), for \(k = 1, 2, \ldots, N\),
- \(P(x) < 0\) for \(x \in (0, \infty)\), for \(k = 1, 2, \ldots, N\),

thus, since \(Q\) is continuous on \(\mathbb{R}\) except maybe at \(r_k^\pm, k = 1, 2, \ldots, N\), we know that \(N - 2\) of the \(N - 1\) zeros of \(Q\), denoted by \(s_k^\pm, k = 1, 2, \ldots, l - 1, l + 2, \ldots, N\), are located as follows:

\[-T < t_0 < s_N < \ldots < s_{l+2} < t_1 < 0 < t_1 < s_{l-1} < \ldots < s_1 < t_1 < T.\]

The graph of \(P(x)\) is sketched in figure 3.1 (see next page). If a zero of \(Q\), \(s^{-1}\) say \((s \neq 0)\), is the site of a local extremum of \(Q\), we must have:

\[0 = P(x)|_{s^{-1}} = \int_{[-T,T]} \frac{1}{(1 - s^{-1} t)^2} d\mu(t),\]

which implies that

\[0 = \int_{[-T,T]} \frac{1}{(1 - s^{-1} t)^2} d\mu(t),\]

i.e., that \(\mu = 0\), a.e. Thus no zero of \(Q\) is the site of a local extremum; hence the reciprocal of the \(N - 2\) zero, \(s\) say, must lie in \((t_{l+1}, t_1)\). Let \(s_{l+1} = \min(0, s)\) and \(s_l = \max(0, s)\). If \(Q\) only has \(N - 2\) zeros (this corresponds to \(s = 0\) above), then let \(s_l = s_{l+1} = 0\). In either case;
\[ P(x) = \prod_{k=1}^{N} (1 - x S_k) / \prod_{k=1}^{N} (1 - x L_k). \]  
(3.9)

It follows that \( \hat{R} \) can be written as (cf. (3.4))

\[ \hat{R}(x) = \log P(x) = \int_0^T \frac{x}{1-xL} d\mu_{H,1}(t) - \int_0^T \frac{x}{1-xS} d\mu_{H,2}(t), \]

with \( \mu'_{H,1}(t) = \sum_{k=1}^{T} 1_{(w_k,t)}(t) \), and \( \mu'_{H,2}(t) = \sum_{k=1}^{N} 1_{(-s_k,-w_k)}(t) \), so \( \mu_{H,1} \) and \( \mu_{H,2} \) are bounded by Lebesgue measure, i.e., \( r_n / (n+1) \) has the desired property. A limiting argument completes the proof.

\[ \text{Figure 3.1} \]

Conversely, we approximate the representing measures \( \mu_1 \) and \( \mu_2 \) of \( (b_n) \) and \( (c_n) \), respectively, as follows: For \( k = 1, 2, \ldots, N-1 \), let

\[ L_{k+1} = T(N-k)/N, \]

\[ \mu_{H,2} = \mu_{H,2} - \mu_1(0,\alpha_L) + \mu_1(0,\alpha_{L+1}). \]

\[ \mu'_{H,1}(t) = \sum_{k=1}^{N-1} 1_{\alpha_L,\alpha_{L+1}}(t), \]

\[ \mu_{H,1}(0,\alpha_L) = \mu_1(0,\alpha_L). \]

For \( k = N, N+1, \ldots, 2N-1 \), let

\[ L_{k+1} = T(N-k)/N, \]

\[ \mu_{H,1} = \mu_2(0,-\alpha_L) + \mu_2(0,-\alpha_{L+1}). \]

\[ \mu'_{H,2}(t) = \sum_{k=0}^{2N-2} 1_{(-\alpha_L,-\alpha_{L+1}}(t). \]
3.3 Moment sequences and infinitely divisible sequences

\[ \mu_{W, 2}(0, \tau_{k+1}) = \mu_2(0, \tau_{k+1}) . \]

Since \( \mu_0, 1 \to \mu_1 \) and \( \mu_1, 2 \to \mu_2 \) we have by Helly’s first and second theorems (cf. (3.4))

\[
P(x) = \lim_{N \to \infty} \exp \left\{ \frac{T}{1-\tau} - \frac{X}{1+\tau} \right\}
\]

\[
= \lim_{N \to \infty} \prod_{k=1}^{2N-1} \frac{1-\tau_{k+1}}{1-\tau_k} = \lim_{N \to \infty} \sum_{k=1}^{2N-1} \frac{q_k}{1-\tau_k} = \int_{\tau_{k+1}}^{\tau_k} \frac{1}{1-\tau} d\mu(t) .
\]

\[ \square \]

**COROLLARY 1.** \((p_n) \in MS(\mathbb{R}) \iff (p_n+1/(n+2)) \in MS(\mathbb{R})\) with representing measure \(\mu\) satisfying

\[ \int_{-\infty}^{\infty} 1 \cdot d\mu(t) < \infty \quad \text{and} \quad \mu(B) \leq \int_B 1 dt \quad \text{for all Borel sets } B .\]

**PROOF.** Let \(\mu_b\) and \(\mu_c\) be the representing measures of \((p_n)\) and \((c_n)\) above, and let \(d\mu_1 = d\mu_b\) and \(d\mu_2 = d\mu_c\). Then the measure \(\mu\) defined by

\[ \mu(\infty, 1) = \begin{cases} \mu_2(0, \infty) & \tau \leq 0 \\ \mu_2(0, \infty) + \mu_1(0, \tau) & \tau > 0 \end{cases} \]

satisfies the requirements. \(\square\)

**REMARK 3.3.3.** The proof of Theorem 3.3.2 can be adapted to prove Theorems 3.3.4 and 3.3.7, but it fails to prove Theorem 3.3.1. In Theorem 3.3.2 we have that \(P(x)\) has \(N-1\) zeros (cf. (3.9)), \(N-2\) of the zeros are easily identified by observing the sign of \(P(x)\) as \(x \to \tau_{k+1}\) from the left and the right (cf. figure 3.1). The last zero cannot be a local extremum (since if \(P'(x) = P(x) = 0\) then \(\mu = 0\)) and so must lay outside the interval \([\tau_{k+1}, \tau_{k+2}]\). In Theorem 3.3.1 \(P(x)\) has \(N\) zeros, but still only \(N-2\) of them are easily identified. This method of proof then fails since the last two zeros cannot be confined to the interval \((\tau_{k+1}, \tau_{k+2})\). \(\square\)
3.3.2. Stieltjes moment sequences

**Theorem 3.3.4.** Let \((p_n)\) and \((r_n)\) be related by

\[(n + 1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}, \quad n \in \mathbb{N}_0.\]

Then (cf. (3.5) and (3.6))

\[(p_n) \in MS(\mathbb{R}_+), \text{ iff } \left(\frac{r_n}{n+1}\right) \in MS^*(\mathbb{R}_+).\]

**Proof.** For the first part we restrict attention to \(MS_2(\mathbb{R}_+)\) and \(MS_2^*(\mathbb{R}_+)\) (cf. Remark 3.2.3). If \((p_n) \in MS(\mathbb{R}_+),\) then by Lemma 3.2.2 we have \((p_n) \in MS(\mathbb{R})\) and \((p_{n+1}) \in MS(\mathbb{R})\). Hence, by the proof of Theorem 3.1.1 and the corollary to Theorem 3.3.2 one has \((r_n/n+1) \in MS_2^*(\mathbb{R})\) and \((r_{n+1}/(n+1)) \in MS^*(\mathbb{R}).\) By Lemma 3.2.11 then \((r_n/(n+1)) \in MS_2^*(\mathbb{R}_+).\) Conversely, by Theorem 3.3.2 with \(c_n = 0\) \((n = 0, 1, \ldots)\) we have \((p_n) \in MS(\mathbb{R}).\) Since \(MS^*(\mathbb{R}_+) \subset MS^*(\mathbb{R}),\) by Theorem 3.3.1 we also have \((p_{n+1}) \in MS(\mathbb{R}).\) and therefore, by Lemma 3.2.2, \((p_n) \in MS(\mathbb{R}_+).\)

**Corollary 1.** Let \((u_n)\) and \((v_n)\) be in \(MS(\mathbb{R}_+),\) with generating functions \(U\) and \(V\) and let \(W = U^{a}V^{b}\) be the generating function of \((w_n).\) Then \((w_n) \in MS(\mathbb{R}_+)\) if \(a \geq 0,\)

\[b \geq 0 \text{ and } a + b \leq 1.\]

**Proof.** See (3.4) and (3.6).

**Remark 3.3.5.** Theorem 3.3.4 can also be proved without use of Theorem 3.3.2 in a similar way as its analogue in Horn (1970) is proved. This proof, however, uses Theorem 3.2.3 (ii). As mentioned in Remark 3.3.3, the proof of Theorem 3.3.2 can be adapted to give yet another proof of Theorem 3.3.4.

**Remark 3.3.6.** If \((p_n)\) and \((r_n/(n+1))\) are in \(MS(\mathbb{R})\) then by Theorem 3.3.1 and Lemma 3.2.2, \((p_n) \in MS(\mathbb{R}_+),\) and by the corollary to Theorem 3.3.2 and Lemma 3.2.2, \((r_n/(n+1)) \in MS(\mathbb{R}_+).\) Hence, as in the renewal case (cf. Horn 1970)), \((p_n)\) and \((r_n/(n+1))\) cannot both be in \(MS(\mathbb{R})\) without being in \(MS(\mathbb{R}_+).\)

It is interesting to note the difference between Theorems 3.3.1 and 3.3.4. Theorem 3.3.4 considers \((p_n)\) as a moment sequence, whereas Theorem 3.3.1 considers \((p_{n+1}).\) This shift in indices is necessary to ensure that the sequence \((p_n)\) has the right sign. For example, if \((r_n/(n+1)) \in MS^*(\mathbb{R}),\) then \(r_0 > 0\) and since \(p_0 > 0\)
we must have $p_i > 0$, which $(p_{n+1}) \in MS(\mathbb{R})$ ensures. This observation led us to consider the set $MS(\mathbb{R}_-)$, where the odd terms of the sequence are non-positive.

**Theorem 3.3.7.** Let $(p_n)$ and $(r_n)$ be related by

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}, \quad n \in \mathbb{N}_0.$$ 

Then (cf. (3.5) and (3.6))

$$(p_{n+1}) \in MS(\mathbb{R}_-) \iff \left( -\frac{r_n}{n+1} \right) \in MS^*(\mathbb{R}_-).$$

**Proof.** Any sequence in $MS(\mathbb{R}_-)$ can be approximated by sequences in $MS^*(\mathbb{R}_-)$ as follows. If $(p_n) \in MS^*(\mathbb{R}_-)$ has a discrete representing measure with atoms $q_i$ at $t_i$ with $(i = 1, 2, ..., N)$ and $-T < t_N < ... < t_1 < 0$, then $P(x)$ takes the form

$$P(x) = 1 + \sum_{i=1}^{N} q_i x/(1-x_{t_i}) = Q(x)/\prod_{i=1}^{N} (1-x_{t_i}),$$

where $Q$ is a polynomial of degree at most $N$. Observe that

$P(x) < 0$ for $x < t_1$, for $k = 1, 2, ..., N$,

$P(x) > 0$ for $x > t_k$, for $k = 1, 2, ..., N$,

thus, since $Q$ is continuous on $\mathbb{R}$ except maybe at $t_k$, $k = 1, 2, ..., N$, we know that $N-1$ of the $N$ zeros of $Q$, denoted by $s_k$, $k = 1, 2, ..., N-1$, are located as follows;

$$-T < s_{N-1} < s_{N-2} < ... < s_2 < s_1 < t_1 < 0.$$ 

The graph of $P(x)$ is sketched in figure 3.2.
Since \( P(x) \to -\infty \) as \( x \to y \) and \( P(0) = 1 \), then the last zero, \( z_N \) say, must satisfy 
\[-T < z_N < b_T \] 
for some \( T > 0 \). Hence 
\[
P(x) = \prod_{k=1}^{N} (1 - xz_k) / \prod_{i=1}^{N} (1 - x_i),
\]
and so 
\[
\tilde{R}(x) = \log P(x) = \int_{-T}^{0} \frac{x}{1-xz} d\mu_N(t),
\]
with \( \mu_N(t) = \sum_{k=1}^{N} 1_{(a_k, b_k)}(t) \), so \( \mu_N \) is bounded by Lebesgue measure, i.e., \( r_n / (n+1) \) has the desired property. A limiting argument completes the proof.

The converse is proved as in Theorem 3.3.2. \( \square \)

3.3.3. Hausdorff moment sequences

**Theorem 3.3.8.** Let \( (p_n) \) and \( (r_n) \) be related by

\[
(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}, \quad n \in \mathbb{N}_0.
\]

Then (cf. (3.5) and (3.6))

\[
(p_n) \in MS([0,1]) \text{ iff } \left( \frac{r_n}{n+1} \right) \in MS^*([0,1]).
\]

**Proof.** If \( (p_n) \in MS([0,1]) \), then \( (p_n) \) is non-increasing. By Lemma 3.2.3 \( (p_n) \in MS(\mathbb{R}_+) \). Since \( (p_n) \) is non-increasing and \( p_0 = 1 \), we have \( p_n \leq 1 \). By Theorem 3.3.4 \( (r_n/(n+1)) \in MS^*(\mathbb{R}_+) \) and by (3.1), \( p_n \leq 1 \) implies that \( r_n \leq n+1 \). Hence \( (r_n/(n+1)) \in MS^*([0,1]) \) by Lemma 3.2.3.

Conversely, if \( (r_n/(n+1)) \in MS^*([0,1]) \), then by Lemma 3.2.3 \( (r_n/(n+1)) \in MS^*(\mathbb{R}_+) \) and \( r_n \leq 1 \). By Theorem 3.3.4 \( (p_n) \in MS(\mathbb{R}_+) \) and by (3.1) \( r_n \leq 1 \) implies that \( p_n \leq 1 \) and hence, by Lemma 3.2.3, \( (p_n) \in MS([0,1]) \). \( \square \)

**Corollary 1.** Let \( (u_n) \) and \( (v_n) \) be in \( MS([0,1]) \) with generating functions \( U \) and \( V \) and let \( W = U^{a+b} \) be the generating function of \( (u_n) \). Then \( (w_n) \in MS([0,1]) \) if \( a \geq 0, b \geq 0 \) and \( a+b \leq 1 \).

**Corollary 2.** Let \( (p_n) \) and \( (r_n) \) be related by (3.1). Then (cf. (3.5) and (3.6))

\[
(p_n) \in MS([0,T]) \text{ iff } \left( \frac{r_n}{n+1} \right) \in MS^*([0,T]).
\]
for any \( T > 0 \).

**Proof.** If \((\rho_n) \in MS((0,T))\) then \((T^{-n}\rho_n) \in MS((0,1))\). From (3.1) it follows that
\[
(n+1)(T^{-n-1}\rho_n) = \sum_{k=0}^{n} [T^{-k}\rho_n] [T^{-n-k}r_{n-k}], \quad n \in \mathbb{N}_0.
\]
Hence, by Theorem 3.3.8, \((T^{-n-1}\rho_n \in MS^\infty((0,1))\). Observe that
\[
r_n/(n+1) = T^{n+1} \int_0^T x^n \, d\mu(x) = \int_0^T x^n \, dT \mu(y/T).
\]
Since \(T\mu(y/T)\) is bounded by Lebesgue measure, then \((r_n/(n+1)) \in MS^\infty((0,T))\). The converse is proved similarly. \(\square\)

### 3.4 Moment sequences and renewal sequences

Let the two sequences \((\rho_n)\) and \((r_n)\) be related through (3.1). Define the sequence \((f_n)\) by
\[
p_{n+1} = \sum_{k=0}^{n} \rho_k f_{n-k}, \quad n \in \mathbb{N}_0. \tag{3.10}
\]
In Section 2.5 we proved that for \(f_{-1} := -1\)
\[
(n+1)(-1)^n f_n = \sum_{k=0}^{n} [(-1)^{k-1}f_{k-1}] [(-1)^{n-k}r_{n-k}], \quad n \in \mathbb{N}_0. \tag{3.11}
\]
We now give an alternative proof of Theorem 2.3.3.

**Proof (of Theorem 2.3.3).** From the definitions of \(MS(\mathbb{R})\) and \(MS^\infty(\mathbb{R})\) it is clear that:

(a) \((a_n) \in MS(\mathbb{R})\) (or \(MS^\infty(\mathbb{R})\)) iff \((-1)^n a_n \in MS(\mathbb{R})\) (or \(MS^\infty(\mathbb{R})\));

(b) \((a_n) \in MS(\mathbb{R}_+)\) (or \(MS^\infty(\mathbb{R}_+)\)) iff \((-1)^n a_n \in MS(\mathbb{R}_+)\) (or \(MS^\infty(\mathbb{R}_+)\)).

Part (i) is proved by noting the equivalence of the following statements.

\[(\rho_{n+1}) \in MS(\mathbb{R})\] \(\text{iff} \ (r_n/(n+1)) \in MS^\infty(\mathbb{R})\) \(\text{ iff } ((-1)^n r_n/(n+1)) \in MS^\infty(\mathbb{R})\) \(\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R})\) \(\text{ iff } (f_n) \in MS(\mathbb{R})\).

(c) \(\text{ (cf. (3.1) and Theorem 3.3.1) }\)

\[(r_n/(n+1)) \in MS^\infty(\mathbb{R})\] \(\text{ iff } ((-1)^n r_n/(n+1)) \in MS^\infty(\mathbb{R})\) \(\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R})\) \(\text{ iff } (f_n) \in MS(\mathbb{R})\).

(d) \(\text{ (cf. statement (a)) }\)

\[(r_n) \in MS(\mathbb{R}_+)\] \(\text{ iff } ((-1)^n r_n) \in MS^\infty(\mathbb{R}_+)\) \(\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R}_+)\) \(\text{ iff } (f_n) \in MS(\mathbb{R}_+)\).

(e) \(\text{ (cf. statement (d)) }\)

\[(f_n) \in MS(\mathbb{R})\] \(\text{ iff } ((-1)^n f_n) \in MS^\infty(\mathbb{R})\) \(\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R}_+)\) \(\text{ iff } (f_n) \in MS(\mathbb{R}_+)\).
For the result on Stieltjes moment sequences we note the following.

\[(p_n) \in MS(\mathbb{R}_-)^* \quad \text{iff} \quad (-1)^np_n(n+1) \in MS^*(\mathbb{R}_-) \quad \text{(cf. (3.1) and Theorem 3.3.4)}
\]

\[\text{iff } ((-1)^np_n(n+1)) \in MS^*(\mathbb{R}_-) \quad \text{(cf. statement (b))}
\]

\[\text{iff } ((1)^p f_n) \in MS(\mathbb{R}_-) \quad \text{(cf. (3.11) and Theorem 3.3.7)}
\]

\[\text{iff } (f_n) \in MS(\mathbb{R}_-) \quad \text{(cf. statement (b))}
\]

The result on \(MS((0,1))\) is easiest proved as in Horn (1970).

\[\square\]

The same reasoning for \(MS(\mathbb{R}_-)\) yields

**Corollary 1.** Let \((u_n)\) and \((f_n)\) be related by equation (2.1), i.e., by

\[u_{n+1} = \sum_{k=0}^{n} u_k f_{n-k} \quad n \in \mathbb{N}_0.
\]

Then (cf. (3.5) and (3.6))

\[(u_{n+1}) \in MS(\mathbb{R}_-) \quad \text{iff} \quad (f_n) \in MS(\mathbb{R}_-).
\]

### 3.5 Moment functions and infinitely divisible functions

Let \(F\) and \(H\) be two non-decreasing, right continuous, not necessarily bounded functions on \(\mathbb{R}_+\), related through equation (1.1) of Theorem 1.3.3, i.e., through

\[
\int_0^x u \, dF(u) = \int_0^x F(x-u) \, dH(u) \quad , \quad x \in \mathbb{R}_+ .
\]

(3.12)

Let \(F\) and \(H\) have densities \(f\) and \(h\), then

\[
x \, f(x) = \int_0^x f(x-u) \, h(u) \, du \quad , \quad x \in \mathbb{R}_+ .
\]

(3.13)

In this section we consider (3.13) where either \(f\) or \(h(x)/\lambda\) are moment functions (cf. Definition 3.5.1). In the proof of Theorem 3.5.3 of this section it is essential whether or not the distribution function of \(f\) or of \(h(x)/\lambda\) is bounded. We therefore state the theorem in terms of the densities of the distribution functions in (3.12), instead of in terms of the functions in (3.13). We consider (3.13) in the context of moment functions instead of (3.12), partly because (3.13) is the analogue of (3.1) and partly because \(-xF\) is a bounded moment function for some \(c > 0\) if and only if \(f\) is a moment function with \(\int f = c\).
3.5 Moment functions and infinitely divisible functions

The set $MS([0,1])$ coincides with the set of completely monotone sequences (cf. Feller (1971)). The set of completely monotone functions is equal to the set of Laplace-Stieljes transforms (cf. Feller (1971)). In order to keep the analogy between completely monotone sequences and completely monotone functions we introduce the following definition.

**DEFINITION 3.5.1.** Let $MF(R_+)$ and $MF([0,1])$ denote the sets of Stieltjes and Hausdorff moment functions, respectively, i.e., for $f \in (R_+, [0,1])$ let

$$f \in MF(I) \iff f(t) = \int_{-b(i)} e^{\tau x} d\mu(x), \quad \tau \in R_+,$$

(3.14)

where $\mu$ is a nonnegative measure and $-\text{ln}([0,1]) = [0, \infty]$ and $-\text{ln}(R_+) = R$. Also let the sets $MS^+(I)$ (cf. 3.14)) be given by

$$f \in MF^+(I) \iff f \in MF(I) \text{ with } \mu \leq \lambda.$$  

The measure $\mu$ in (3.14) is called the representing measure of $f$. □

We will need the following lemma (cf. Lemma 3.2.4).

**LEMMA 3.5.2.** If $f_t \in MF(I)$ (or $MF^+(I)$) for all $t \geq t_0$, and if $\lim_{t \to \infty} f_t(t) = f(t)$, $\tau \in R_+$, then $f \in MF(I)$ (or $MF^+(I)$).

Part (ii) of the following theorem was proved by Steutel (1970) for distribution functions. We slightly generalize this result in

**THEOREM 3.5.3.** Let $F$ and $H$ be related by (3.12) and suppose they have densities $f$ and $h$ respectively. Then

(i) $f \in MF(R_+)$ iff $\tau^{-1} h(\tau) \in MF^+(R_+);$  

(ii) $f \in MF([0,1])$ iff $\tau^{-1} h(\tau) \in MF^+(I, 1])$.

**PROOF.** Theorem 2.12.1 of Steutel (1970) proves part (ii) for bounded $F$. Any function $F$ with density $f \in MF([0,1])$ can be written as a limit of bounded functions $F_t$ with densities $f_t \in MF([0,1])$. For each $F_t$ a function $H_t$ can be found satisfying (3.12). By Theorem 3.5.3 (ii) for bounded $F$, $\tau^{-1} h(t) \in MF^+(I, 1])$. Since $F_t \to F$ and $H_t \to H$, then by Lemma 3.5.2, $\tau^{-1} h(t) \in MF^+(I, 1])$. Similarly when starting with $\tau^{-1} h(t)$.
Part (i) can be proved from part (ii) as follows: if \( f \in MF(\mathbb{R}_+ \times \{ 0 \}) \), then by truncation of integrals we have a sequence \( f_n \in MF([0,1]) \) such that \( f_n \to f \). Since \( f_n \in MF([0,1]) \) if and only if \( \tau^{-1} f_n(\tau) \in MF^\#([0,1]) \), then by part (ii) \( \tau^{-1} h(\tau) \in MF^\#([0,1]) \) and hence \( \tau^{-1} h(\tau) \in MF^\#([0,1]) \). By Lemma 3.5.2, on letting \( t \to \infty \) it follows that \( \tau^{-1} h(\tau) \in MF^\#(\mathbb{R}_+) \). Similarly when starting with \( \tau^{-1} h(\tau) \).

Remark 3.5.4. Part (i) of Theorem 3.5.3 can also be proved by applying a limiting argument to Theorem 3.3.8 as in the proof of Theorem 4.3.2. Yet another way to prove Theorem 3.5.3 (i) (or another way to prove Theorem 3.3.8 using Theorem 3.5.3 (i)) is as follows:

Suppose \( f \in MF([0,1]) \). By Steutel (1969) \( f \) is infinitely divisible and therefore satisfies (3.13). For any \( \tau \leq c, c \geq 1 \), \( f \) can be put in the form (cf. (3.14))

\[
\hat{f}(\tau) = \int_0^\infty x (1 + \tau/c)^{-1} d\mu(x) \\
= \int_0^1 (1 - x(1 - \tau/c))^{-1} d\mu_c(x) = P_c(1 - \tau/c) .
\] (3.15)

Let \( (p_n(c)) \) have generating function \( P_c \). Then \( (p_n(c)) \in MS([0,1]) \). Let \( (r_n(c)) \) be defined by using \( (p_n(c)) \) in (3.1) and let \( R_c \) denote its generating function. By Theorem 3.3.8 \( (r_n(c)/(n+1)) \in MS^\#([0,1]) \), and hence

\[
R_c(t) = \int_0^1 (1 - xz)^{-1} dx \mu_c(x) = \int_0^\infty c (y + c(1 - z))^{-1} dy \mu_c^1(y) ,
\] (3.16)

with \( d\mu_c^1(y) = -c ((y/c) + 1)^2 d\mu_c((y/c) + 1)^{-1} \). Hence \( \mu_c^1 \leq \lambda \). Observe that (cf. (3.13))

\[
\hat{h}(\tau) = -\frac{d}{d\tau} \ln f(\tau) = -\frac{d}{d\tau} \ln P_c(1 - \tau/c) = \tau^{-1} R_c(1 - \tau/c) .
\] (3.17)

From (3.16) and (3.17) it follows that \( h(\tau) \in MF([0,1]) \). The converse is shown similarly.

3.6 Moment functions and renewal functions

Let \( F \) be a distribution function on \( \mathbb{R}_+ \) with density \( f \) and Laplace-Stieltjes transform \( \hat{f} \). Suppose \( F \) is a compound geometric distribution, i.e.,

\[
F(x) = (1-p) \sum_{n=0}^{\infty} p^n G^*^n(x) , \quad x \in \mathbb{R}_+ ,
\]

with \( G \) a distribution function, having density \( g \), and \( G^*^n \) being the \( n \)-th convolution power of \( G \). Applying the same type of argumentation as used in Remark 3.5.4 to
3.6 Moment functions and renewal functions

Theorem 2.3.3 or applying a limit argument to Theorem 2.3.3 as in Haken and Frank (1988), it can be shown that (this is partly shown in Smita and Matuda (1987), Theorem 3.3.2, pg. 644 and is proved in Frenk (1988), both with proofs similar to that of Theorem 3.3.2)

\[ f \in MF([0,1]) \text{ iff } g \in MF([0,1]). \]

Let \( U_p := (1-p)^{-1} \) and \( U := \lim_{n \to \infty} U_p. \) If \( u \) is the density of \( U, \) then by Lemma 3.5.2,

\[ u \in MF([0,1]) \text{ iff } g \in MF([0,1]). \quad (3.18) \]

The function \( U \) (called the renewal function (cf. Ross (1983))) is the unique solution of the renewal equation,

\[ U(x) = 1 + \int_0^x U(x-y) \, dU(y), \quad x \in \mathbb{R}_+. \quad (3.19) \]

Using the method of proof of Theorem 3.5.3 on statement (3.18) we obtain:

**Theorem 3.6.1.** Let \( U \) and \( G \) be related by (3.19) and suppose they have densities \( u \) and \( g \) respectively. Then

(i) \[ u \in MF(\mathbb{R}_+) \text{ iff } g \in MF(\mathbb{R}_+); \]

(ii) \[ u \in MF([0,1]) \text{ iff } g \in MF([0,1]) \text{ and } \int_0^\infty g(x) \, dx \leq 1. \]

3.7 Applications and special cases

The main occurrence of (3.1) is in infinite divisibility. In this context \( (\rho_n) \) is a Hausdorff moment sequence if and only if \( (\omega_n) \) is a mixture of geometric distributions, i.e.,

\[ \rho_n = \int_0^\infty x^n (1-x) \, dF(x), \quad n \in \mathbb{N}_0, \quad (3.20) \]

where \( F \) is a distribution function on \([0,1].\) Theorem 3.3.8 can be restated as

**Theorem 3.7.1.** \( (\rho_n) \) in (3.1) is a mixture of geometric distributions iff \( (\omega_n) = (\tau_n / \Theta(n+1)) \) (cf. Theorem 1.3.4) is a mixture of geometric distributions with \( F \) in (3.20) satisfying \( (1-x) \, dF \leq d\lambda \) (cf. (3.6)). Equivalently, the probability generating function \( P \) is of the form

\[ P(x) = \int_0^1 (1-x)^i (1-xz) \, d^F(z), \]
with $F$ a distribution function on $[0,1]$, if and only if $P$ can be represented as

$$\ln P(z) = \int \int_{0}^{1} (1-uv)^{-2} \, d\mu(v) \, du,$$

and $\mu$ bounded by Lebesgue measure. The representation is unique.

The continuous analogue of Theorem 3.7.1 is implicit in Theorem 2.12.1 of Steutel (1970).

As a curiosity we prove that it is possible to have $(\rho_n) = (g_n)$ in Theorem 3.7.1, i.e., $P = G$ in Theorem 1.3.4. Solving for $(\rho_n)$ in $P(z) = P(0) \exp \{iz \cdot P(z) \}$ one finds, e.g. by Lagrange expansion,

$$\rho_n = \frac{(n+1)^n}{(n+1)!} (2e^{-\theta})^n \theta^n, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (3.21)

Since

$$\frac{k^n}{n!} = \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{\sin x}{x} \right)^n \exp(x \cos x) \, dx, \quad n \in \mathbb{N}_0,$$

as has been proved by Bouwkamp (1965), $(\rho_n)$ is a Hausdorff moment sequence. The distribution in (3.21) and its continuous analogue are busy period distributions. This application is discussed in Steutel and Hansen (1988).

We finish this chapter by stating a conjecture, the continuous analogue of which is discussed in Steutel (1970) (p.28, 94). In probabilistic terms the conjecture is that mixtures of negative binomial distributions of order 2, i.e., of probabilities of the form $(n+1)p^n(1-p)^2$ ($n = 0, 1, \ldots$), are infinitely divisible. We state the conjecture more formally as follows.

**Conjecture.** If $(\alpha_n)$ is a Hausdorff moment sequence, then $(\beta_n) := ((n+1)\alpha_n)$ satisfies (3.1) with $\beta_n \geq 0$. 
Chapter 4

LOGCONCAVE AND LOGCONVEX
SEQUENCES AND DENSITIES

4.1 Introduction

Log-concavity and log-convexity of functions and sequences in probability has been of interest to several authors, e.g. Karlin (1968). Khurgin (1956) calls a distribution strongly unimodal if its convolution with any unimodal distribution is unimodal. He proves that the set of strongly unimodal probability densities is equal to the set of log-concave densities. An equivalent result for log-concave discrete probability distributions has been proved by Keilson and Gerber (1971). Much work has been done on the unimodality of infinitely divisible distributions (cf. Wolfe (1971), Yamazato (1978) and Sato and Yamazato (1978)), but little on strong unimodality. The study of log-concave functions and sequences is thus a relatively unknown field in probability, with important applications in the fields of statistics and optimization. Log-convexity is of interest in the study of reliability and of infinitely divisible random variables. Steutel (1970) proves that all log-convex discrete probability distributions are infinitely divisible (Theorem 2.4.1 (i)). The absolutely continuous analogue is also proved in Steutel (1970).

In this chapter we consider distributions of non-negative infinitely divisible random variables whose canonical measures are either absolutely continuous or supported by the integers. We prove that for such distributions to be log-concave (log-convex), it is necessary that their canonical measures be log-concave (log-convex). Our results in the discrete case contain an analogue of Yamazato’s (1982) concavity result (it also provides an alternative proof of this result), and an analogue to the convexity result for renewal sequences in de Bruijn and Erdős (1953) (cf. Theorem 2.3.2 (i)).
4.2 Discrete distributions

In this section we consider infinitely divisible discrete probability distributions \((p_n)\) on \(\mathbb{N}_0\). A sequence \((a_n)\) is log-concave if \((a_n)\) is non-negative and \((\log(a_n))\) is concave, or equivalently if \(a_n \geq 0\) and

\[
\frac{a_n^2}{a_{n+1} a_{n-1}} , \quad n \in \mathbb{N}_0 .
\]  

(4.1)

If the sequence satisfies (4.1) with strict inequality, then the sequence is said to be strictly log-concave. Similarly, \((a_n)\) is log-convex if \(a_n \geq 0\) and the sequence satisfies

\[
\frac{a_n^2}{a_{n+1} a_{n-1}} , \quad n \in \mathbb{N}_0 .
\]  

(4.2)

\((a_n)\) is said to be strictly log-convex if (4.2) is satisfied with strict inequality.

A probability distribution \((p_n)\) on \(\mathbb{N}_0\) with \(p_0 > 0\) is infinitely divisible if and only if it satisfies

\[
(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k} , \quad n \in \mathbb{N}_0 ,
\]  

(4.3)

with non-negative \(r_k\) and, necessarily, \(\sum_{k=0}^{\infty} r_k (k+1) < \infty\) (cf. Theorem 1.3.4). Theorem 2.4.1 part (ii) states that all log-convex distributions are infinitely divisible. This can be proved by induction since

\[
r_n \rho_n \rho_0 = p_n p_{n-1} + \sum_{k=0}^{n-1} r_k (p_{n-k-1} - p_{n-k} - p_k)
\]

is positive if \((p_n)\) is strictly log-convex and noting that any log-convex sequence can be written as a limit of strictly log-convex sequences. Not all log-convex distributions are infinitely divisible (cf. (4.3))

\[
r_1 = p_1^2 \left( 2 p_0 p_0 - p_0^2 \right)
\]

is not necessarily non-negative when \((p_n)\) is log-concave.

The proofs of the main theorems in this section rely on two equations derived from (4.3). Though easily verified by (4.3), the equations were rather hard to find. Because of their importance we state them in a lemma.

**Lemma 4.2.1.** Let \((p_n)\) and \((r_n)\) be related by (4.3) and let \(p_{-1} = 0\). Then

\[
m (m+2) (p_{m+1}^2 - p_m p_{m+2}) = p_{m+1} (r_0 p_m - p_{m+1})
\]

\[
+ \sum_{k=1}^{m} \left[ (p_{m-k} p_{m-k-1} - p_{m-k} p_{m-k+1}) (r_{k-1} r_1 - r_{k+1} r_k) \right] ;
\]

\[
r_{m+1} (m+2) (p_{m+1} p_{m+3} - p_{m+2}^2) = p_{m+1} (r_{m+2} p_{m+2} - r_{m+1} p_{m+3})
\]

\[
+ \sum_{k=0}^{m} (p_{m-k} p_{m+2} - p_{m+1} p_{m-k+1}) (r_{m+2} r_k - r_{m+1} r_{m+1}) .
\]

(4.5)
4.2 Discrete distributions

Relation (4.4) is a discrete analogue of equation (10) in Yamazato (1982), whereas (4.5) is an analogue of formula (7) in de Brujin and Erdős (1953). We will need the following lemma.

**Lemma 4.2.2.** Let \((p_n)\) and \((r_n)\) be related by (4.3) with \(p_0 > 0\). Then

(i) if \(r_n^2 > p_{n-1} - p_{n+1}\) for \(n = 1, 2, \ldots, m\) then \(r_0 p_m - p_{m+1} > 0\);

(ii) if \((r_n)\) is strictly log-convex and \(r_0^2 - r_1 < 0\) then \(r_{m+2} p_{m+2} - r_{m+1} p_{m+3} > 0\).

**Proof.** If \(r_n^2 > p_{n-1} - p_{n+1}\) for \(n = 1, 2, \ldots, m\), then \(p_{n+1}/p_n\) is decreasing for \(n = 1, 2, \ldots, m\), so \(r_0 = p_1/p_0 > p_{m+1}/p_m\).

If \((r_n)\) is strictly log-convex, then \((r_{n+1}/r_n)\) is increasing. Hence,

\[
(m+3)p_{m+2} = p_{m+2} r_0 + \sum_{k=1}^{m-2} p_{m+2-k} r_k - r_{k-1} \frac{r_k}{r_{k-1}} < p_{m+2} r_0 + (m+2)p_{m+2} \max_{1 \leq k \leq m+2} \frac{r_k}{r_{k-1}}
\]

\[
< p_{m+2} r_{m+2} - (m+2)p_{m+2} \frac{r_{m+2}}{r_{m+1}}.
\]

**Theorem 4.2.3.** Let \((p_n)\) and \((r_n)\) be related by

\[
(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \quad n \in \mathbb{N}_0,
\]

with \(r_0 > 0, p_0 > 0\) and let \((r_n)\) be log-convex. Then

\((p_n)\) is log-convex if and only if \(r_0^2 - r_1 \geq 0\).

**Proof.** Suppose that \((r_n)\) is strictly log-convex and \(r_0^2 - r_1 > 0\), then \((p_n)\) is positive and hence \((p_n)\) is positive. Observe that (cf. (4.3))

\[
2(p_1^2 - p_0 p_2) = p_0^2 (r_0^2 - r_1).
\]

By using (4.6), Lemma 4.2.2 (i) and applying induction to (4.4) we see that \((p_n)\) is strictly log-convex. The proof is completed by noting that any log-convex sequence can be written as a limit of strictly log-convex sequences. \(\square\)
**Theorem 4.2.4.** Let \((p_n)\) and \((r_n)\) be related by

\[(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \quad n \in \mathbb{N},\]

with \(r_0 \geq 0, p_0 > 0\) and let \((r_n)\) be log-convex. Then

\((p_n)\) is log-convex if and only if \(r_0^2 - r_1 \leq 0.\)

**Proof.** As in Theorem 4.2.3 except that Lemma 4.2.2 (ii) is used and induction is applied to (4.5).

It is curious to note the difference in equations (4.4) and (4.5). We were not able to find an equation of the form (4.4) to prove Theorem 4.2.4 or one of the form (4.5) to prove Theorem 4.2.3.

**Remark 4.2.5.** The assumption that \((p_n)\) is a probability distribution is not used in the proofs of Theorems 4.2.3 and 4.2.4. These theorems are thus true for arbitrary non-negative sequences related by (4.3).

### 4.3 Absolutely continuous distributions

In this section infinitely divisible probability distributions \(F\) on \(\mathbb{R}\) with absolutely continuous canonical measures are considered. We obtain two results on the log-concavity and log-convexity of the densities of \(F\), which are analogues to those obtained in Section 4.2. The result on log-concave densities is proved in Yamazato (1982). We here propose a proof based on applying a limiting argument to Theorem 4.2.3. This proof can easily be adapted to log-convex densities, thus giving the absolutely continuous analogue of Theorem 4.2.4.

A function \(f\) on \(\mathbb{R}\) is log-concave (log-convex) on an interval \(I\) if \(f\) is positive on \(I\) and \(\log(f)\) is concave (convex) on \(I\). The function \(f\) is said to be log-concave (log-convex) on \(I\) if \(\{x : f > 0\}\) is an interval and \(f\) is log-concave (log-convex) on \(I\). As in the discrete case, \(f\) is strictly log-concave (strictly log-convex) if in \((f)\) is strictly concave (strictly convex).

A distribution function \(F\) on \((0,\infty)\) is infinitely divisible if and only if there exists a non-decreasing measure \(H\) such that

\[
\int_{0}^{s} u \, dF(u) = \int_{0}^{s} F(x-u) \, dH(u), \quad (4.7)
\]
4.3 Absolutely continuous distributions

\[ \int_1^\infty u^{-1} \, dH(u) < \infty, \]  
(4.8)

where \( H \) and \( F \) determine each other uniquely (cf. Theorem 1.3.3). If \( F \) and \( H \) have densities \( f \) and \( h \), then

\[ x f(x) = \left( \int_0^x h(u-u) f(u) \, du \right)^{-1}. \]  
(4.9)

Without loss of generality we assume that \( \inf \{ x \mid f(x) > 0 \} = 0 \). It is shown in Steutel (1970) that all absolutely continuous distributions with log-convex densities are infinitely divisible. As in the discrete case, not all distributions having log-concave densities are infinitely divisible, e.g. \( f(x) = \exp(-x^2) \) for \( x \in (0, \infty) \).

We begin with a lemma.

**Lemma 4.3.1.** Let \( f \) and \( h \) be related by (4.9). Suppose \( h \) is monotone on \((0, \varepsilon)\) for some \( \varepsilon > 0 \) and \( 0 < f(0+) < \infty \). Then \( h(0+) = 1 \).

**Proof.** Suppose \( h \) is non-increasing on \((0, \varepsilon)\), then \( h(0+) > 0 \). By (4.9) the function \( f \) is continuous. From (4.9) it follows that for \( 0 < x < \varepsilon \)

\[ h(0+) \geq x f(x) \left( \int_0^x f(u) \, du \right)^{-1}, \]

\[ h(x) \leq x f(x) \left( \int_0^x f(u) \, du \right)^{-1}. \]

As \( x \to 0 \) the right hand sides tend to one, so \( h(0+) = 1 \). Similarly if \( h \) is non-decreasing. \( \square \)

**Theorem 4.3.2.** (Yamazato) Let \( F \) be an infinitely divisible distribution function on \((0, \infty)\) with an absolutely continuous canonical measure \( H \). Let \( f \) and \( h \) be the densities of \( F \) and \( H \) respectively, and assume that \( h \) is log-concave. Then

\[ f \text{ is log-concave if and only if } h(0+) \geq 1. \]
PROOF. Suppose \( h \) is log-concave and \( h(0^+) > 1 \), then \( h \) must be continuous on \( I \). Define \((r_n(k))\) by
\[
 r_n(k) = h\left(\frac{n+1}{k}\right), \quad n \in \mathbb{N}_0,
\]
and any \( k \in \mathbb{N}_0 \). Then \((r_n(k))\) is log-concave, and since \( h(0^+) > 1 \) we have \((r_n(k))^2 > r_n(k)\), for all sufficiently large \( k \). By (4.8) and the continuity of \( h \) we see that \( \sum r_n(k)/(n+1) < \infty \). For fixed \( k \) define \((p_n(k))\) by
\[
 (n+1)p_{n+1}(k) = \sum_{\ell=0}^{n} p_{n-\ell}(k) r_{\ell}(k), \quad n \in \mathbb{N}_0, \tag{4.10}
\]
\[
p_0(k) = k \exp\left(-\sum_{n=0}^{\infty} r_n(k)/(n+1)\right) > 0, \tag{4.11}
\]
with \( \sum p_n(k) = k \). By Theorem 4.2.3 and Remark 4.2.5 the sequence \((p_n(k))\) is log-concave. Let
\[
 F_k(x) = \sum_{n=0}^{\infty} k^{-1} p_n(k), \tag{4.12}
\]
\[
 H_k(x) = \sum_{n=0}^{\infty} k^{-1} r_n(k). \tag{4.13}
\]
From (4.10) and (4.11) it follows that
\[
 \int_{[0,k] \times [1]} u \, dF_k(u) = \int_{[0,k]} F_k(x-u) \, dH_k(u), \tag{4.14}
\]
\[
 \frac{n+1}{k} p_{n+1}(k) = \int_{[0,n+1]} h\left(\frac{n+1}{k} - u\right) \, dF_k(u). \tag{4.15}
\]
By Helly's first theorem (cf. Feller (1971)) there is a subsequence \((F_{k_{(s)}})\) converging weakly to some distribution function, \( F_{\text{limit}} \), say, as \( s \to \infty \). Hence, since \( H_k \to H \), by Helly's second theorem
\[
 \int_{[0,k]} u \, dF_{\text{limit}}(u) = \int_{[0,k]} F_{\text{limit}}(x-u) \, dH(u),
\]
Since \( H \) uniquely determines \( F \) in (4.7) we must have \( F = F_{\text{limit}} \). Let
\[
 f_k(x) = \langle (p_n(k))^{\mathbb{R}^+}, (p_n(k))^{n+1} \rangle, \quad x \in \left[ \frac{n}{k}, \frac{n+1}{k} \right], \tag{4.16}
\]
then \( f_k \) is a log-concave function of \( x \). Let \( n \to \infty \) and \( k \to \infty \) in such a way that \( k^{-1} (n+1) \to x \); then it follows from (4.9), (4.15) and (4.16) that
\[
 x f_{\text{limit}}(x) = \lim_{k \to \infty \atop n \to \infty} \frac{n+1}{k} f_k\left(\frac{n+1}{k}\right) = \int_{[0,x]} h(x-u) \, dF(u) = x f(x). \tag{4.17}
\]
4.3 Absolutely continuous distributions

Since log-concavity is preserved under convergence, $F$ has a log-concave density $f_{\text{log}}$. As any log-concave function with $h(0+) \geq 1$ can be written as a limit of log-concave functions $(h_n)$ with $h_n(0+) > 1$, this completes the first part of the proof.

Conversely, if $f$ and $h$ are log-concave then $h$ is monotone on $(0, \varepsilon)$ for some $\varepsilon > 0$. If $0 < f(0+) < \infty$ then $h(0+) = 1$ by Lemma 4.3.1. If $f$ is log-concave then $f(0+) = 0$ cannot be infinite. If $f(0+) = 0$, then $f$ is non-decreasing on $(0, \varepsilon)$ and

$$xf(x) \leq f(x) \int_0^x h(u) \, du.$$  

Letting $x \to 0$ yields $h(0+) \geq 1$. \hfill \square

The proof of Theorem 4.3.2 can easily be adapted to log-convex densities by using Theorem 4.2.4 instead of Theorem 4.2.3. We then obtain

**THEOREM 4.3.3.** Let $F$ be an infinitely divisible distribution function with an absolutely continuous canonical measure $H$. Let $f$ and $h$ be the densities of $F$ and $H$ respectively, and assume that $h$ is log-convex. Then

$f$ is log-convex if and only if $h(0+) \leq 1$.

4.4 Applications and counterexamples

In this section we define a set of infinitely divisible distributions in terms of their canonical measures and determine under what conditions a distribution in this class is log-concave or log-convex. An application of this result shows that the reverse statements of our main theorems do not hold. Finally, we characterize the log-convex discrete stable distributions.

Let $D$ denote the set of distributions having canonical measures $(r_n)$ of the form

$$r_n = (n+1) \left\{ \begin{array}{ll}
\int_a^b y^n \, dm(y) + \int_c^a y^n \, dy, & n \in \mathbb{N}_0,
\end{array} \right.$$  

for fixed $a$, $b$, and $c$ with $0 \leq b \leq 1$, $0 \leq c \leq a \leq 1$, $m$ bounded by Lebesgue measure and

$$\int_a^b dm(y) < b - a, \quad \text{if } b > a,$$

$$\int_c^0 dm(y) < c, \quad \text{if } c > 0.$$
The proof of Theorem 4.2 in Yamaizato (1982) can be adapted to prove the following theorem if Theorem 3.3.8 of Chapter 3 and Corollary 2 to Theorem 3.3.8 is used in the same fashion as Lemma 4.2 in Yamaizato (1982).

**Theorem 4.4.1.** Let \( (p_n) \) and \( (r_n) \) be related by
\[
(p_{n+1}) = \sum_{k=0}^{n} r_k p_{n-k}, \quad n \in \mathbb{N}_0,
\]
with non-negative \( r_k \) and \( p_0 > 0 \). Let \( (p_n) \in D \), then

(i) if \( c = 0 \) and \( a \geq b \) then \( (p_n) \) is log-concave;
if \( c \geq 0 \) and \( a < b \) then \( (p_n) \) is not log-concave;
(ii) if \( c = 0 \) and \( a \geq c \geq b \) then \( (p_n) \) is log-convex;
if \( c \geq 0 \) and \( a \geq b > c \) then \( (p_n) \) is not log-convex;
if \( c \geq 0 \) and \( b > a > c \) then \( (p_n) \) is not log-convex;
if \( c \geq 0 \) and \( b < a = c \) then \( (p_n) \) is log-convex.

**Remark 4.4.2.** The absolutely continuous analogue of Theorem 4.4.1 can be obtained by applying the same type of limiting argument as in the proof of Theorem 4.3.2. \( \square \)

**Remark 4.4.3.** Let \( m \) in (4.11) be Lebesgue measure on \( (d, b) \), and zero otherwise. Then \( r_n = b^n - d^n + a^n \) and \( r_n^2 - r_{n+1} r_{n-1} < 0 \) for large \( n \) if \( a > b > d > 0 \), whereas \( (p_n) \) is log-concave by Theorem 4.4.1 (ii). Similarly \( (r_n) \) is asymptotically log-concave if \( 0 = d < b < c < a \), whereas \( (p_n) \) is log-convex by Theorem 4.4.1 (ii). Hence, the reverse statements of Theorems 4.2.3 and 4.2.4 do not hold. \( \square \)

**Remark 4.4.4.** Theorem 4.4.1 characterizes \( D \) in terms of log-convexity and almost completely in terms of log-concavity. We were not able to prove that if \( a \geq b > c > 0 \), then \( (p_n) \) is log-concave (cf. part (i) of Theorem 4.4.1). \( \square \)

A discrete analogue of an absolutely continuous stable distribution was introduced by Steifel and van Harn (1979) and discussed in Section 1.4. Steifel and van Harn (1979) proved that a distribution \( (p_n) \) is discrete stable with exponent \( \delta \) if and only if its generating function is of the form (cf. Theorem 1.4.3)
\[
P(z) = \exp(-\lambda(1-2z)^\delta), \quad \delta \in (0, 1), \quad \lambda \geq 0.
\]
Taking generating functions on both sides of (4.3) and comparing with the Taylor
4.4 Applications and counterexamples

Series expansion of $-\lambda(1-\varepsilon)^\delta$ one sees that $(r_n)$ is strictly log-convex and that $r_n^\delta - r_{n+1} < 0$ if and only if $\delta < 1 - r_n$. Applying Theorem 4.2.4 to these observations gives

**Theorem 4.4.5.** Let $(p_n)$ be discrete stable with exponent $\delta$. Then

$(p_n)$ is strictly log-convex if and only if $\lambda < \delta^{-1} - 1$.

The canonical density $h$ of an absolutely continuous stable distribution on $(0,\infty)$ is of the form $cx^{-\lambda}$, hence $h$ is log-convex and $h(0+) = \infty$. Applying Theorem 4.3.3 we have, rather unexpectedly, that there are no log-convex stable densities on $(0,\infty)$.
Chapter 5

A GENERALIZED SELF-DECOMPOSABILITY

5.1 Introduction

Functional equations have been a helpful tool in representing subsets of the set of infinitely divisible distributions. The definitions of self-decomposable and stable distributions in terms of a functional equation for their characteristic function or probability generating function (cf. Section 1.4) are well-known examples; for other examples see van Harn (1978). O'Connor (1979a) shows that membership in the set of infinitely divisible distributions with unimodal Lévy spectral functions is related with the solutions of the functional equation (cf. equation (1.4))

\[ \phi(t) = e^{\beta \langle \hat{t} \rangle} \hat{\phi}_{\eta}(t) \quad \text{for } t \in (0,1), \beta \in \mathbb{R}, \]

with \( \beta = c \) and where \( \phi \) and \( \hat{\phi}_{\eta} \) are characteristic functions. Jurek (1985) calls such characteristic functions shrinking-self-decomposable, or \( s \)-self-decomposable for short. All self-decomposable characteristic functions are \( s \)-self-decomposable, as follows easily from the fact that self-decomposable characteristic functions are infinitely divisible. Interpolating between (5.1) with \( \beta = c \) and (1.6), O'Connor (1979a) studies equation (5.1) with \( \beta = e^{\alpha \hat{t}} \), \( \alpha \in (0,1) \). In O'Connor (1981) the case \( \alpha = 1,3 \) is considered. Some of the results of Section 5.4 are also proved in Jurek (1988). For a detailed comparison we refer to Remarks 5.4.11 and 5.4.12.

In this chapter we consider random variables \( X \) on \( \mathbb{R} \), on \( \mathbb{R}_+ \), and on \( \mathbb{N} \). We introduce a one parameter family of functional equations, of the form (5.1), satisfied by the characteristic function, Laplace-Stieltjes transform, or probability generating function of \( X \), depending on whether the random variable has support on \( \mathbb{R} \), on \( \mathbb{R}_+ \), or on \( \mathbb{N} \). Our equations for random variables on \( \mathbb{R} \) include O'Connor's and have as special cases the functional equation defining self-decomposable distributions and \( s \)-
5.1 Introduction

self-decomposable distributions. We establish a canonical form for the integral transform satisfying these functional equations, show that these integral transforms are infinitely divisible, and have Lévy spectral functions that are, in an extended sense, α-unimodal. These results include those of O'Connor and Jurek. It is also shown that this one parameter family of functional equations provides a classification of the set of infinitely divisible random variables.

5.2 α-Unimodality

A random variable \( X \) with distribution function \( F \) and density \( f \) is said to be unimodal, with mode at \( x_0 \) (not necessarily unique), if \( f(x) \) is non-decreasing for \( x < x_0 \) and non-increasing for \( x > x_0 \). Throughout this chapter we assume that \( x_0 = 0 \), i.e., if a function is said to be unimodal (or α-unimodal) it is understood that its mode is at the origin. Khintchine (1938) showed that \( X \) is unimodal (at zero) if \( X = UV \), with \( U \) and \( V \) independent and \( U \) uniform on \((0,1)\). Olshen and Savage (1970) generalized this concept; a random variable is said to be α-unimodal (at zero) if it is of the form \( U^\alpha Y \), with \( U \) and \( Y \) independent and \( U \) uniformly distributed on \((0,1)\) and \( \alpha \neq 0 \). If \( Y \) has distribution function \( G \), then

\[
f(x) = \alpha x^{\alpha-1} \int_{-\infty}^{\infty} v^{-\alpha} dG(v), x \in \mathbb{R}_+.
\]

\[
f(x) = \alpha \, 1_{x > 0} \int_{-\infty}^{\infty} v^{-\alpha} dG(v), x \in \mathbb{R}_-.
\]

This result corresponds to Corollary 2 p. 28 in Olshen and Savage (1970). Hence, \( \alpha \)-unimodal if and only if \( x \) is non-decreasing on \((-\infty,0)\) and non-increasing on \((0,\infty)\). We shall use α-unimodality in connection with Lévy spectral functions, so a more general definition is needed.

**Definition 5.2.1.** A function \( C \) (not necessarily non-negative) is said to be α-unimodal for some \( \alpha \in \mathbb{R} \), if \( x \) is bounded from below and non-decreasing on \((-\infty,0)\) and non-increasing on \((0,\infty)\) or equivalently if there exists constants \( \lambda_1, \lambda_2 \in \mathbb{R} \) and a function \( N \) such that

\[
C(x) = \begin{cases} 
\alpha x^{\alpha-1} \left( \int_{-\infty}^{x} v^{-\alpha} dV(v) + \lambda_1 \right) & x > 0 \\
|\alpha|^{\alpha-1} \left( \int_{-\infty}^{x} v^{-\alpha} dV(v) + \lambda_2 \right) & x < 0
\end{cases}
\]

(5.2)

and such that the integrals converge for every \( x \in \mathbb{R} \setminus \{0\} \). \( \square \)
A generalized self-decomposability

An analogue of $\alpha$-unimodality for discrete distributions on $\mathbb{N}_0$ is introduced in Steutel (1988). An equivalent, but different, definition was given by Abouammoh (1987). A random variable is discrete $\alpha$-unimodal (at zero) if $X = U \circ Y$ with $U$ and $Y$ independent and $U$ uniform on $(0,1)$. The multiplication operator $\circ$ is defined in Section 1.4. Since we only consider discrete random variables on $\mathbb{N}_0$, $\alpha$-unimodality at zero is equivalent to $\alpha$-monotonicity. If $X$ and $Y$ are $\mathbb{N}_0$ valued random variables with probability generating functions $G$ and $S$, then

$$G(z) = \alpha \int_0^1 S(1 - v(1 - z)\nu^{v-1}) \, dv.$$ 

Expanding the integral on the right hand side Steutel (1988) shows that $(g_n)$ is $\alpha$-monotone if and only if $(n! \Gamma(n+\alpha)^{-1} g_n)$ is non-increasing or equivalently if $(n + \alpha) g_n \geq (n + 1) g_{n+1}$. We generalize this definition to

**Definition 5.2.2.** A sequence $(r_n)$ is said to be $\alpha$-monotone, for $\alpha > 0$, if $(n! \Gamma(n+\alpha)^{-1} r_n)$ is bounded from below and is non-increasing, or equivalently if there exists $\lambda_1 \in \mathbb{R}$ and a non-negative sequence $(h_n)$ such that

$$r_n = \alpha \frac{\Gamma(n+\alpha)}{n!} \left[ \sum_{k=n}^{\infty} \frac{k!}{\Gamma(k+1-\alpha)} h_k + \lambda_1 \right], \quad n \in \mathbb{N}_0.$$ 

(A 3.3)

A sequence $(r_n)$ is said to be zero-monotone if $r_0 = \lambda_1$ and $r_n = 0$ for $n \geq 1$ (cf. (A 3.3)), i.e., $(r_n)$ is the canonical measure of the Poisson distribution. \qed

**Remark 5.2.3.** It is immediate from Definitions 5.2.1 and 5.2.2 that if a function $f$ or a sequence $(a_n)$ is $\alpha_0$-unimodal then it is $\alpha$-unimodal for every $\alpha \geq \alpha_0$. \qed

### 5.3 Distributions on $\mathbb{N}_0$

The starting point of this chapter was the work on random variables supported by $\mathbb{N}_0$. In this special class of random variables the probabilities themselves can be found explicitly and the corresponding Lévy spectral functions are easily computed. This provided a good source of intuition and insight which was very helpful in establishing the results of Section 5.4. In this section this discrete analogue is discussed. We begin with a definition.

**Definition 5.3.1** Let the random variable $X$ on $\mathbb{N}_0$ have distribution $(p_n)$ and probability generating function $P$. The function $P$ is said to be $\alpha$-self-decomposable for some $\alpha \in \mathbb{R}$ and to belong to the set $S_\alpha(\mathbb{N}_0)$, if for every $c \in (0,1)$ there exists a probability generating function $P_c$ such that

$$P(z) = P_c((1 - c(1 - z)) P_c(z), \quad |z| \leq 1.$$ 

(A 5.4)
If $\alpha=0$ in Definition 5.3.1, then (5.4) provides the functional equation defining discrete self-decomposable probability generating functions as given in Section 1.4.

Let $(r_n)$ be $(\alpha+1)$-monotone and let $\tilde{H}(\gamma) = \sum_{n=0}^{\infty} r_n / \Gamma(n+1) \gamma^{n+1}$. By using

$$
\int_0^1 \psi^\beta (1-\nu)^\beta-1 \, d\nu \equiv \Gamma(\beta) \Gamma(\gamma) \Gamma(\beta+\gamma)^{-1} ,
$$

we obtain

$$
\tilde{R}(\gamma) = \sum_{n=0}^{\infty} r_n / \Gamma(n+1) \gamma^{n+1} = \int_0^1 \left( \tilde{H}(1-\nu(1-\gamma)) - \tilde{H}(1-\gamma) \right) \nu^{\alpha-1} \, d\nu + \lambda_2 - \lambda_2 (1-\gamma)^{-\alpha} ,
$$

(5.5)

where $\lambda_2 = \lambda_1 (1+\alpha) \Gamma(1+\alpha) / (1-\gamma)^{-\alpha}$. Hence $(r_n)$ in (5.3) is a canonical measure, i.e., $r_n \geq 0$ and $\sum_{n=0}^{\infty} r_n / (n+1) < \infty$ (cf. Theorem 1.3.4), if and only if $(h_n)$ is, with $\lambda_2 \geq 0$, $\lambda_1 = 0$ for $\alpha \geq 0$.

Suppose $(r_n)$ has canonical measure $(r_n)$ with $(r_n)$ $(\alpha+1)$-monotone, $\alpha > -1$ and suppose $(q_n)$ is an infinitely divisible distribution with canonical measure $(q_n)$. By (5.5) and Theorem 1.3.4

$$
\ln P(\gamma) = \tilde{R}(\gamma) + \ln P(0) = \tilde{R}(\gamma) - \tilde{R}(1) = \int_0^1 \left( \tilde{H}(1-\nu(1-\gamma)) - \tilde{H}(1-\gamma) \right) \nu^{\alpha-1} \, d\nu - \lambda_2 (1-\gamma)^{-\alpha} = \int_0^1 \ln Q(1-\nu(1-\gamma)) \nu^{\alpha-1} \, d\nu - \lambda_2 (1-\gamma)^{-\alpha} .
$$

We now prove

**Theorem 5.3.2.** Let $\alpha \in (-1, \infty)$ and let $P$ be a probability generating function. The following statements are equivalent:

(i) $P$ is $\alpha$-self-decomposable;

(ii) $\ln P(\gamma) = \int_0^1 \ln Q(1-\nu(1-\gamma)) \nu^{\alpha-1} \, d\nu - \lambda(1-\gamma)^{-\alpha}$, $\lambda \geq 0$ and $Q$ is a unique infinitely divisible probability generating function;

(iii) $(q_n)$ is infinitely divisible and its canonical measure $(r_n)$ is $(\alpha+1)$-monotone.

Furthermore, $\lambda$ in (ii) is zero if $\alpha \geq 0$ and $P_c$ is infinitely divisible for every $c \in (0,1)$ if $P$ is $\alpha$-self-decomposable.
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PROOF. (ii) $\iff$ (iii) has been proved above. We now prove (i) $\iff$ (ii). Let $P$ satisfy (5.4), let $r>0$ and $c_n \in (0,1)$ for $n \in \mathbb{N}$, such that $r(1-c_n)^{-1} \in \mathbb{N}_0$, then

$$
\ln Q_{c_n}(z) := \ln (P_{c_n}(z))^{(1-c_n)^r} = r \left[ \ln P(z) - \ln P(1-c_n(1-z)) + \frac{1-c_n}{1-c_n} \ln P(1-c_n(1-z)) \right]
$$

is the logarithm of a probability generating function. Let $c_n$ be such that $c_n \uparrow 1$ as $n \to \infty$, then

$$
\ln Q_n(z) := \lim_{n \to \infty} \ln Q_{c_n}(z) = r(1-z) P'(z)/P(z) + \alpha \ln P(z). \quad (5.6)
$$

Since $Q_n(z) \to 1$ as $z \to 1$ (cf. Lemma 1, Steutel and van Harn (1979)), by the continuity theorem for probability generating functions, $Q_n$ is a probability generating function for every $r>0$, and thus $Q = Q_1$ is an infinitely divisible probability generating function. (5.6) gives rise to the following differential equation,

$$
\ln Q(z) = -\alpha \frac{d}{dz} \ln P(z) + \alpha \ln P(z),
$$

which has (ii) as unique solution.

Conversely, if $P$ is as in (ii), then $P$ satisfies (5.4) with

$$
\ln P(z) = \int \ln Q(1-v(1-z)) v^{\alpha-1} dv. \quad (5.7)
$$

Let $(p_n(c))$ be the probability distribution corresponding to $P_c$ and define $(r_n(c))$ using $(p_n(c))$ in (1.2). Then

$$
\sum_{n=0}^{\infty} r_n(c) z^n = R_c(z) = \frac{d}{dz} \ln P_c(z)
$$

$$
= \int H(1-v(1-z)) v^{\alpha-1} dv,
$$

which has positive coefficients. Hence $P_c$ is infinitely divisible. \qed

**Corollary 1.** $P$ is discrete stable with exponent $\delta$ (cf. Theorem 1.4.3), then $P = S_\alpha(\mathbb{N}_0)$ for every $\alpha \geq -\delta$.

**Corollary 2.** The Poisson distribution is the only distribution in $S_{\alpha}(\mathbb{N}_0)$ for its canonical measure is $\alpha$-monotone with $\alpha = 0$.

Note that for $\alpha > 0$ the sequence $(r_n)$ is $(\alpha-1)$-monotone if and only if $(p_n)/(n+1)$ is $\alpha$-monotone. Since all log-convex sequences and all Hausdorff moment sequences are non-increasing we have (cf. Theorem 3.3.8 and Theorem 4.2.4)
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**Corollary 3.** If $(p_n)$ is log-convex, then $(p_n)$ is 0-self-decomposable.

**Corollary 4.** If $(p_n)$ is a mixture of geometric distributions, then $(p_n)$ is 1-self-decomposable.

**Remark 5.3.3.** It follows from the proof of Theorem 5.3.2 that the canonical measure $(h_n)$ of the infinitely divisible probability generating function $Q$ in (ii) must satisfy

$$\sum_{k=0}^{\infty} \frac{k!}{\Gamma(k+\alpha)} \frac{h_k}{k+1} < \infty$$

for $\alpha \in (-1, 0)$. If $\alpha = 0$ then

$$\sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{n+1} \frac{h_k}{k+1} < \infty.$$ 

Berg and Forst (1983) introduced the set of $n$-times self-decomposable probability generating functions $L_n(\mathbb{N}_0)$ inductively by letting $L_0(\mathbb{N}_0) = S_0(\mathbb{N}_0)$ and (cf. (5.4))

$$L_{n+1}(\mathbb{N}_0) = \{ P \in L_n(\mathbb{N}_0) | P_c \in L_n(\mathbb{N}_0), c \in (0,1) \}.$$ 

Hence $L_1(\mathbb{N}_0)$ is the subset of $S_0(\mathbb{N}_0)$, where $P_c$ are 0-self-decomposable for every $c \in (0,1)$. We conclude this section with a characterization of the probability generating functions in $S_0(\mathbb{N}_0)$ where $P_c$ are $\alpha$-self-decomposable for every $c \in (0,1)$.

**Theorem 5.3.4.** Let $P \in S_0(\mathbb{N}_0)$ with $P_c$ given by (5.4) and $Q$ by Theorem 5.3.2. Then $Q \in S_0(\mathbb{N}_0)$ if and only if $P_c \in S_0(\mathbb{N}_0)$ for every $c \in (0,1)$.

**Proof.** Suppose $Q \in S_0(\mathbb{N}_0)$. Then $Q$ satisfies (5.4) for some $Q_c$. Since $Q \in S_0(\mathbb{N}_0)$ we have by (5.4) and Theorem 5.3.2

$$\ln P_c(z) = \ln P(z) - c^\alpha \ln P(1-c(1-z))$$

$$= \int_0^1 \ln Q(1-v(1-z)) - c^\alpha \ln Q(1-cv(1-z)) \nu(v) \, dv.$$ 

Hence $P_c \in S_0(\mathbb{N}_0)$.

Conversely, if $P_c \in S_0(\mathbb{N}_0)$, then by (5.7) and Theorem 5.3.2

$$\ln P_c(z) = \int_0^1 \ln Q(1-v(1-z)) \nu(v) \, dv.$$
\[ = \int_0^1 \ln Q_c(1 - v(1 - z)) v^{\alpha-1} \, dv + \ln P_{\text{STABLE}(c)}(z). \]

Letting \( u = 1 - v(1 - z) \) and rewriting yields
\[ \int_0^z \ln Q(u) (1 - u)^{\alpha-1} \, du = \int_0^z \ln Q_c(u) (1 - u)^{\alpha-1} \, du + \ln P_{\text{STABLE}(c)}(z). \]

Differentiating with respect to \( z \) and multiplying by \((1 - z)^{\alpha-1}\) yields
\[ \ln Q(z) - c^\alpha \ln Q(1 - c(1 - z)) = \ln Q_c(z). \]
Hence \( Q \in S_{\alpha}(\mathbb{N}_0) \). \qed

5.4 Distributions on \( \mathbb{R} \)

Throughout this section we will understand by the derivative of a Lévy spectral function \( M \) a right continuous function \( M' \) defined by
\[ M'(x+) = M'_+(x), \]
\[ M'(x-) = M'_-(x), \]
where \( M'_+ \) and \( M'_- \) are respectively the right and left derivatives, 
\[ f(x+) = \lim_{y \uparrow x} f(y) \] and 
\[ f(x-) = \lim_{y \downarrow x} f(y). \] The assumption of right continuity of \( M' \) is non-essential and is assumed for uniqueness only. The results of this section are also true if we were to make \( M' \) left continuous or set \( M' \) equal to any (linear) combination of \( M'_+ \) and \( M'_- \) at points where \( M'_+ \neq M'_- \).

We begin with a definition

**Definition 5.4.1.** Let the random variable \( X \) on \( \mathbb{R} \) have distribution function \( F \) and characteristic function \( \phi \). The function \( \phi \) is said to be \( \alpha \)-self-decomposable for some \( \alpha \in \mathbb{R} \), if for every \( c \in (0, 1) \) there exists a characteristic function \( \phi_c \) such that
\[ \phi(t) = \phi^\alpha(ct) \phi_c(t) \quad , \quad t \in \mathbb{R}. \quad (5.8) \]

For \( \alpha = 0 \) we have the functional equation defining self-decomposable characteristic functions and for \( -2 < \alpha \leq 1 \) the equations corresponding to O'Connor class \( L_{1,\alpha} \). Our results include those of O'Connor, but our proofs differ. For a detailed comparison we refer to Remark 5.4.11.

We shall use the following two lemmas; the proof of the first lemma is similar to the first part of the proof of Theorem 5.11.1 in Lukacs (1970). The discrete counterpart of the second lemma is discussed in the second paragraph of Section 5.3.
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**Lemma 5.4.2.** If \( \phi \) satisfies (5.8), then \( \phi \) has no real zeros.

**Proof.** If \( \phi \) satisfies (5.8), then \( \psi = |\phi|^2 \) is a characteristic function which also satisfies (5.8). Suppose \( \psi \) has zeros. Since \( \psi \) is continuous and \( \psi(0) = 1 \), there exists a \( t_0 \) such that \( \psi(t_0) = 0 \) while \( \psi(t) \neq 0 \) for \( |t| < t_0 \). It follows from (5.8) that \( \psi_y(t_0) = 0 \) while \( \psi_y(t) \neq 0 \) for \( |t| < t_0 \). By Theorem 4.1.2 on p. 69 of Lukacs (1970), with \( n = 1 \) and \( t = t_0/2 \), we have

\[
4(1 - \psi_y(t_0/2)) \geq 1 - \psi_y(t_0) = 1.
\]

Since \( \psi_y(t_0/2) = \psi(t_0/2)/\psi_y(t_0/2) \) is continuous in \( c \), we obtain a contradiction by choosing \( c \) sufficiently close to 1. So \( \psi \) and \( \psi_y \), and hence also \( \phi \) and \( \phi_y \), have no real zeros.

**Lemma 5.4.3.** Let \( \alpha > -2 \) and let \( M \) be such that \( M'(x) \) is \( \alpha \)-unimodal, i.e.,

\[
M'(x) = \begin{cases} 
  x^{\alpha-1} \left( \int \nu^{-\alpha} dN(v) + \lambda_1 \right) & x > 0 \\
  1x^{\alpha-1} \left( \int \nu^{-\alpha} dN(v) + \lambda_2 \right) & x < 0
\end{cases}
\]  

(5.9)

with the integrals converging for every \( \nu \in \mathbb{R} \setminus \{0\} \) and \( \int \nu^{-\alpha} dN(v) < \infty \) if \( \alpha > 0 \). Then \( M \) is a Lévy spectral function if and only if \( N \) can be chosen to be a Lévy spectral function and \( \lambda_1, \lambda_2 \in \{0, \infty\} \) and \( \lambda_1 = \lambda_2 = 0 \) for \( \alpha > 0 \).

**Proof.** First let \( M \) be a Lévy spectral function satisfying (5.9). We have to show that \( N \) satisfies the requirements (ii) and (iii) of Theorem 1.3.2 and that \( \lambda_1 \) and \( \lambda_2 \) satisfy the requirements of the lemma.

By Theorem 1.3.2 (i) and (ii), \( M'(x) \geq 0 \) and \( M(\infty) = 0 \). Since \( \lim_{x \to \infty} x^{1-\alpha} M'(x) = \lambda_1 \) we have \( \lambda_1 \in \{0, \infty\} \). Similarly, \( \lambda_2 \in \{0, \infty\} \). From (5.9) it follows that for \( y > 0 \)

\[
-M'(y) = \int y M'(x) dx = \int x^{\alpha-1} \left( \int \nu^{-\alpha} dN(v) + \lambda_1 \right) dx.
\]  

(5.10)

Since \( 0 \leq -M'(y) \leq \infty \), we must have that \( \int x^{\alpha-1} \nu^{-\alpha} dN(v) + \lambda_1 \) \( < \infty \) and hence that \( \lambda_1 \leq 0 \) for \( \alpha > 0 \). A similar reasoning for \( y < 0 \) yields \( \lambda_2 = 0 \) for \( \alpha > 0 \).

**Requirement (ii) of Theorem 1.3.2.** Let \( \alpha \leq 0 \). By (5.8)

\[
\nu \leq |M'(x)| \leq |x|^{-1} \int x^{\alpha-1} \nu^{-\alpha} dN(v) + \lambda_1 |x|^{\alpha-1} \nu^{-\alpha} dN(v),
\]

hence \( |N(\nu)| \leq \infty \). Similarly \( |N(\nu)| \leq \infty \). Let \( \alpha > 0 \). Observe that for \( y > 0 \) (cf.
(5.10))
\[
-M(y) = \alpha^{-1} \int_y^{\infty} (1 - (y/v)^2) \, dN(v)
\]

By (5.9) \[\int_y^{\infty} (y/v)^{\delta} \, dN(v) < \infty,\] so \[\int_y^{\infty} dN(v)\] converges and hence \[|N(\infty)| < \infty.\] Similarly \[|N(-\infty)| < \infty.\]

**Requirement (iii) of Theorem 1.3.2.** Let \(0 < \delta < \epsilon.\) From (5.9) it follows for \(\alpha > -2,\)

\[
\int_\delta^\epsilon u^2 \, dM(u) = (\alpha + 2)^{-1} \left( (e^{\alpha^2} - e^{\alpha^2}) \left( \int_\epsilon^{\infty} v^{-\alpha} \, dN(v) + \lambda_1 \right) \right.
\]

\[
- e^{\alpha^2} \int_\delta^\epsilon v^{-\alpha} \, dN(v) + \left. \epsilon \int_\delta^\epsilon v^2 \, dN(v) \right].
\]

(5.11)

Since \(e^{\alpha^2} \leq e^{\alpha^2} \) for \(v \in (\delta, \epsilon),\) the right hand side of (5.11) is bounded from below by

\[
(\alpha + 2)^{-1} \left( (e^{\alpha^2} - e^{\alpha^2}) \left( \int_\epsilon^{\infty} v^{-\alpha} \, dN(v) + \lambda_1 \right) \right).
\]

(5.12)

Let \(\delta = 0,\) then the left hand side of (5.11) tends to zero as \(\epsilon \to 0\) and so (5.12) tends to zero as \(\epsilon \to 0.\) By letting \(\delta \to 0 \) in (5.11) it follows that \(N\) satisfies condition (iii) of Theorem 1.3.2. Similarly for \(0 > \delta > \epsilon.\)

Conversely, suppose \(N\) is a Lévy spectral function such that the integrals in (5.9) converge. We have to show that \(M\) satisfies requirements (ii) and (iii) of Theorem 1.3.2. Assume with out loss of generality that \(\lambda_1 = \lambda_2 = 0.\)

**Requirement (ii) of Theorem 1.3.2.** Let \(z > y > 0\) and let \(\alpha \neq 0.\) By (5.9) we have that

\[
0 \leq M(z) - M(y) = \int_y^z M'(x) \, dx
\]

\[
= \alpha^{-1} \left[ z^\alpha \int_y^{\infty} v^{-\alpha} \, dN(v) - y^\alpha \int_y^{\infty} v^{-\alpha} \, dN(v) + N(z) - N(y) \right]
\]

\[
\leq \alpha^{-1} \left[ z^\alpha \int_y^{\infty} v^{-\alpha} \, dN(v) - N(y) \right].
\]

For \(\alpha < 0,\) \(\lim_{z \to 0} z^\alpha \int_y^{\infty} v^{-\alpha} \, dN(v) = 0,\) and so \(|M(\infty)| < \infty.\) If \(\alpha > 0,\) then

\[
\lim_{z \to \infty} z^\alpha \int_y^{\infty} v^{-\alpha} \, dN(v) = \lim_{z \to \infty} z^\alpha \int_y^{\infty} (z/v)^{\alpha} \, dN(v) \leq \lim_{z \to \infty} N(z) = 0,
\]

and hence \(|M(\infty)| < \infty.\) For \(\alpha = 0,\) observe that
5.4 Distributions on \( \mathbb{R} \).

\[
0 \leq M(z) - M(y) = \int_{y}^{z} \ln v \, dN(v) + N(y) \ln y - N(z) \ln z.
\]

By the condition on \( N \) in the lemma for \( \alpha = 0 \), \( \lim_{z \to \infty} N(z) \ln z = 0 \) and hence \( |N(z)| \leq \infty \). A similar argument for \( z < y < 0 \) shows that \( |M(z)| \leq \infty \).

**Requirement (iii) of Theorem 1.3.2.** Let \( 0 < \delta < \epsilon \). The right hand side of (5.11) is bounded from above by

\[
(\alpha + 2)^{-1} \left[ \epsilon^{\alpha/2} \int_{z}^{\infty} v^{-\alpha} \, dN(v) + \int_{\delta}^{\epsilon} v^2 \, dN(v) \right].
\]

(5.13)

Letting \( \delta \to 0 \) in (5.13) and (5.11) we see that \( \int_{0}^{\epsilon} u^2 \, dM(v) < \infty \). A similar argument holds for \( 0 > \delta > \epsilon \).

**Corollary 1.** Let \( N \) be a Lévy spectral function such that the integrals in (5.9) converge for \( \alpha = 0 \) and \( \int_{0}^{\epsilon} \ln v \, dN(v) < \infty \) if \( \alpha = 0 \). Then,

\[
\begin{align*}
(\alpha + 2)^{-1} \int_{z}^{\infty} v^{-\alpha} \, dN(v) & = 0 = \lim_{z \to \infty} \frac{1}{x} \int_{x}^{\infty} v^{-\alpha} \, dN(v) \quad \alpha \in (0, \infty) ; \\
(\alpha + 2)^{-1} \int_{z}^{\infty} v^{-\alpha} \, dN(v) & = 0 = \lim_{z \to \infty} \frac{1}{x} \int_{x}^{\infty} v^{-\alpha} \, dN(v) \quad \alpha \in (-2, \infty). \quad (i)
\end{align*}
\]

**Proof.** This corollary is proved for \( \alpha \in (0, 1) \) in O'Connor (1979b). Part (ii) follows from the proof of Lemma 5.4.3. Part (i) is trivial for \( \alpha < 0 \) since the integrals always converge. For \( \alpha \geq 0 \) part (i) is evident from

\[
\frac{1}{x} \int_{x}^{\infty} v^{-\alpha} \, dN(v) \leq 1 \sup_{x \in \mathbb{R}} N(x).
\]

\[\square\]

**Remark 5.4.4.** If \( M'(x) \) is \( \alpha \)-unimodal then for \( 0 < x \leq \epsilon \),

\[
x^2 M'(x) \geq x^{2\alpha + 1} \int_{\epsilon}^{\infty} v^{-\alpha} \, dN(v) .
\]

Integrating over \((0, \epsilon)\) it follows, by Theorem 1.3.2 (iii), that there are no \( \alpha \)-unimodal Lévy spectral functions with \( \alpha \leq -2 \).

\[\square\]

In Section 1.4 we defined the linear operator \( T_\alpha \) on \( \mathbb{R} \) by \( T_\alpha(x) = x^\alpha \) and showed that it was closely related with the random variables having \( \theta \)-self-decomposable characteristic functions. Let \( T_\alpha \) operate on set functions by
for any Borel set \( B \). The following lemma gives a connection between \( T_\alpha \) and the notion of \( \alpha \)-unimodality.

**Lemma 5.4.5.** Let \( \alpha \in \mathbb{R} \), \( \beta_0 \) be the set of Borel sets on \( \mathbb{R} \), \( \epsilon > 0 \), and \( M \) a Lévy spectral function. The following statements are equivalent.

(i) \( M(B) \geq c^\alpha T_\alpha M(B) \), \( c \in (0,1) \), \( B \in \beta_0 \);

(ii) \( M \) has left and right derivatives on \( \mathbb{R} \setminus \{0\} \) and \( M'(x) \) is \( \alpha \)-unimodal.

**Proof.** First we prove (i) \(\Rightarrow\) (ii). Let \( \alpha \in \mathbb{R} \), fix \( \epsilon > 0 \) and let \( B = (a, b) \), \( 0 < \epsilon \leq a < b \). Then (i) is equivalent to

\[
\epsilon^{-\alpha} (M(b/c) - M(a/c)) \leq M(b) - M(a). \tag{5.14}
\]

If \( \alpha \leq 0 \), then \( M \) is convex and hence by Theorem A, p. 4, Roberts and Varberg (1973), \( M \) is absolutely continuous on \( (\epsilon, \infty) \). Suppose \( \alpha > 0 \) and let \( w(x) := M(x^{1/\alpha}) \). From (5.14) with \( x = (b/c)^\alpha \), \( y = (a/c)^\alpha \), \( x' = b^\alpha \), and \( y' = a^\alpha \), it follows that

\[
\frac{w(x) - w(y)}{x - y} \geq \frac{w(x') - w(y')}{x' - y'}.
\]

Hence \( w \) is convex. By Theorem A, p. 4, Roberts and Varberg (1973), \( w \) and hence also \( M \) is absolutely continuous on \( (\epsilon, \infty) \). Observe that

\[
\int_B M'(x) \, dx = M(B) \geq c^\alpha T_\alpha M(B) = c^{\alpha - 1} \int_B T_\alpha M'(x) \, dx, \tag{5.15}
\]

with \( B = (a, b) \). Differentiating (5.15) with respect to \( a \) and multiplying both sides by \( a^{1-\alpha} \), we see that \( a^{1-\alpha} M'(a) \) is non-increasing on \( \mathbb{R}_+ \). Similarly for \( b < a \leq \epsilon < 0 \) we obtain that \( a^{1-\alpha} M'(a) \) is non-decreasing on \( \mathbb{R}_- \).

The converse is proved for \( B = (a, b) \), \( 0 < \epsilon \leq a < b \), by observing that (cf. (5.9))

\[
M(B) = \int_B M'(x) \, dx = \int_B x^{-\alpha - 1} \left( \int_x^\infty v^{-\alpha} \, dN(v) \right) \, dx = c^\alpha \int_T^\infty \frac{x^{-\alpha - 1} \left( \int_x^\infty v^{-\alpha} \, dN(v) \right) \, dx}{x^{\alpha-1} \beta_0}
\]

Before proving a representation theorem for \( \alpha \)-self-decomposable characteristic functions we will prove a preparatory lemma, whose counterpart for discrete
distributions is discussed in the third paragraph of Section 5.3.

**Lemma 5.4.6.** Let \( \alpha > -2 \) and \( \phi \) be an infinitely divisible characteristic function with Lévy spectral function \( M \) having left and right derivatives on \( \mathbb{R} \setminus \{0\} \) and such that \( M'(x) \) is \( \alpha \)-unimodal. Then

(i) If \( \alpha \in [0, \infty) \) there exists an infinitely divisible characteristic function \( \gamma \) such that
\[
\ln \phi(t) = \int_0^1 \ln \gamma(vt) v^{-\alpha-1} dv;
\]

(ii) If \( \alpha = (-2, 0) \) there exists an infinitely divisible characteristic function \( \gamma \) and a stable characteristic function \( \phi_{\text{STABLE}(-\alpha)}(t) \), possibly degenerate, such that
\[
\ln \phi(t) = \int_0^1 \ln \gamma(vt) v^{-\alpha-1} dv + \ln \phi_{\text{STABLE}(-\alpha)}(t).
\]

**Proof.** We partly follow the proof of Theorem 2 in Alf and O'Connor (1977). The proof of (i) is very similar to that of (ii) for \( \alpha \in (-1, 0) \), so we only prove (ii). For \( \phi \) as given in the lemma, there exists a Lévy spectral function \( N \) and \( \lambda_1, \lambda_2 \in [0, \infty) \) such that \( M \) and \( N \) are related by (5.9). Define the Lévy spectral function \( M_1 \) by
\[
M_1(x) = \begin{cases} 
  M(x) - \alpha^{-1} \lambda_2 x^{\alpha} , & x > 0 \\
  M(x) + \alpha^{-1} \lambda_2 |x|^{\alpha} , & x < 0
\end{cases}.
\]

Hence \( M_1 \) is an \( \alpha \)-unimodal Lévy spectral function with the same \( M \) in (5.9) as \( M \), but with \( \lambda_1 = \lambda_2 = 0 \). Define an infinitely divisible characteristic function \( \gamma \) using \( N \) for \( M \) in Theorem 1.3.2. We now wish to evaluate the integral in (ii) and show, by choosing \( \alpha_\gamma \) and \( \sigma_\gamma \) appropriately, that it is equal to \( \ln \phi(t) \). The manner in which \( \alpha_\gamma \) must be selected is closely related to the proof of Theorem 5.7.3 in Lukacs (1970). We consider two cases.

**Case 1.** \( \alpha \in (-1, 0) \). For ease of notation we define for \( x > 0 \)
\[
L(t,x) = x^\alpha \int_0^1 v^{-\alpha-1} k(v,x) dv = \frac{x^\alpha}{\alpha + 1} \int_0^1 (v^{\alpha+1}-1) v^{-\alpha-1} dv - \frac{x^{\alpha+1}}{\alpha+1}.
\]

From the first equality it follows that \( L(t,x)/x^{\alpha-2} \) is bounded as \( x \to 0 \). Note that
\[
\frac{dL(t,x)}{dx} = x^{\alpha-1} \left\{ k(t,x) + 2tx^2/(\alpha+1)(1+x^2)^2 \right\}.
\]
Observe that
\[
\int_0^1 \ln(\gamma(v)) v^{\alpha-1} dv - (\alpha+1)^{-1} a_{1,2} + 3(\alpha+2)^{-1} \sigma_f^2
\]
\[
= \int_0^1 \int_0^1 v^{\alpha-1} k(v, x) dv\,dV(x)
\]
\[
= \int_0^1 L(t, x) x^{\alpha-1} d\mathcal{V}(x) + \int_0^1 L(-t, x) x^{\alpha-1} d\mathcal{V}(-x)
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{x^{\alpha+2}} \int \frac{x}{x} v^{\alpha-1} d\mathcal{V}(v) \left[ 1 + \int x \frac{\partial L(t, x)}{\partial x} \int v^{\alpha-1} d\mathcal{V}(v) dx \right]
\]
\[
+ \lim_{\varepsilon \to 0} \frac{1}{x^{\alpha+2}} \int \frac{x}{x} v^{\alpha-1} d\mathcal{V}(v) \left[ 1 + \int x \frac{\partial L(-t, x)}{\partial x} \int v^{\alpha-1} d\mathcal{V}(v) dx \right]
\]
\[
= \int k(t, x) dM_1(x) + 2(\alpha+1)^{-1} \int_{\mathbb{R}_0^+} (x^{3/2}(1+x^2)^2) dM_1(x) ,
\]  
(5.18)

where the second equality is obtained by integrating by parts, and the final expression by observing that $L(t, x)/x^{\alpha+2}$ is bounded as $x \to 0$ and using the corollary to Lemma 5.4.3. Let $a_{1,2}$ and $\sigma_f^2$ be defined by
\[
(\alpha+1)^{-1} a_{1,2} + 2(\alpha+1)^{-1} \int_{\mathbb{R}_0^+} x^{3/2}(1+x^2)^2 dM_1(x) = a_{\phi} ,
\]
\[
(\alpha+2)^{-1} \sigma_f^2 = \frac{1}{\alpha} \sigma_f^2 .
\]

From (5.18) we obtain (cf. Theorem 1.4.1 and (5.16))
\[
\int_0^1 \ln(\gamma(v)) v^{\alpha-1} dv - (\alpha+1)^{-1} a_{1,2} + 3(\alpha+2)^{-1} \sigma_f^2
\]
\[
+ 2(\alpha+1)^{-1} \int_{\mathbb{R}_0^+} x^{3/2}(1+x^2)^2 dM_1(x) + \int_{\mathbb{R}_0^+} k(t, x) dM_1(x)
\]
\[
= i a_{1,2} + \frac{1}{\alpha} \sigma_f^2 + \int_{\mathbb{R}_0^+} k(t, x) dM_1(x) - \ln \Theta_{STABLE}(0)(t) .
\]

Hence $\phi$ is of the desired form.

**CASE II, $\alpha \in (-2, -1)$.** For ease of notation we define
\[
L^*(t, x) = \int_0^1 (e^{\theta y} - 1 - \theta y) v^{\alpha-1} dv .
\]

(5.20)

Note that $L^*(t, x)/x^{\alpha+2}$ is bounded as $x \to 0$ and that
\[
\frac{\partial L^*(t, x)}{\partial x} \approx x^{\alpha-1} \{ k(t, x) - i 2 x^2 / (1 + x^2) \} .
\]

Since the integrals in (5.9) converge we can let
5.4 Distributions on \( \mathbb{R} \)

\[
\alpha_x = - \int_{\mathbb{R}^2} \frac{1}{v^3} (1 + v^2) dN(v) .
\]

(5.21)

Also define \( \alpha_x^2 \) by (5.19b). Analogous to Case I, we have that

\[
\int_{\mathbb{R}^2} \frac{1}{v^3} \ln |v^2 + 1| dN(v) + \frac{1}{2} \sigma^2 t^2
\]

\[
= \int_{\mathbb{R}^2} \frac{1}{v^3} \left( e^{ia} - 1 + (ia) \right) v^{a-1} dN(v)
\]

\[
= \int_{\mathbb{R}^2} L^*(t, x) x^{-a} dN(x) + \int_{\mathbb{R}^2} L^*(-t, x) x^{-a} dN(-x)
\]

\[
= \lim_{a \to 0} \int_{\mathbb{R}^2} \frac{L^*(t, x) x^{-a} x^{a-1}}{x^{a+2}} \int_{\mathbb{R}^2} v^{-a} dN(v) \left( \frac{\partial L^*(t, x)}{\partial x} \int_{\mathbb{R}^2} v^{-a} dN(v) dx \right)
\]

\[
+ \lim_{a \to 0} \int_{\mathbb{R}^2} \frac{L^*(-t, x) x^{-a} x^{a-1}}{x^{a+2}} \int_{\mathbb{R}^2} v^{-a} dN(v) \left( \frac{\partial L^*(-t, x)}{\partial x} \int_{\mathbb{R}^2} v^{-a} dN(v) dx \right)
\]

\[
= \int_{\mathbb{R}^2} k(t, x) dM_1(x) - \int_{\mathbb{R}^2} \frac{4v^2}{(1 + v^2)} dM_1(x)
\]

Rewriting as in Case I, we see that \( \phi \) has the desired form.

\[\square\]

Theorem 5.4.7. Let \( \alpha > 0 \) and let \( \phi \) be a characteristic function. The following statements are equivalent.

(i) \( \phi \) is \( \alpha \)-self-decomposable, and \( \phi' \) exists on \( \mathbb{R} \) \([0, \infty) \) with \( \lim_{t \to 0} \phi'(t) = 0 \);

(ii) There exists a unique infinitely divisible characteristic function \( \gamma \) such that

\[
\ln \phi(t) = \int_0^1 \ln \gamma(v^2) v^{a-1} dv ;
\]

(iii) \( \phi \) is infinitely divisible with Lévy spectral function \( M \) having left and right derivatives and such that \( M'(x) \) is \( \alpha \)-unimodal;

(iv) \( \phi \) is \( \alpha \)-self-decomposable and \( \phi_c \) is an infinitely divisible characteristic function for every \( c \in (0, 1) \).

Proof. First we prove (i) \( \Leftrightarrow \) (ii). Let \( r > 0 \) and \( c_n \in (0, 1) \) for \( n \in \mathbb{N} \) such that \( r^2 (1 - c_n)^2 \leq 1 \). By (5.8)

\[
\ln \gamma_n(t) := \ln \phi_n(t)^{r(1 - c_n)} = \left\{ \frac{\ln \phi(t) - \ln \phi_n(t)}{1 - c_n} + \frac{1 - c_n}{1 - c_n} \ln \phi_n(t) \right\} .
\]

(5.22)
is the logarithm of a characteristic function. Let \( c_\alpha \) be such that \( c_\alpha \uparrow 1 \) as \( \alpha \to \infty \), then
\[
\ln \gamma(t) := \lim_{n \to \infty} \ln \gamma_{t/n}(t) = t \phi'(t) + \alpha \ln \phi(t). \tag{5.23}
\]
Since \( \gamma(t) \to 1 \) as \( t \to 0 \), by the continuity theorem for characteristic functions, \( \gamma \) is a characteristic function for every \( r > 0 \), and thus \( \gamma := \gamma_1 \) is an infinitely divisible characteristic function. Equation (5.23) gives rise to the following differential equation:
\[
\gamma^{\alpha-1} \phi(t) = t \phi'(t) + \alpha t^{\alpha-1} \ln \phi(t) = \frac{d}{dt} t^{\alpha} \ln \phi(t). \tag{5.24}
\]
Hence \( \phi \) is given by (ii). Conversely, if \( \phi \) is as in (ii), then \( \phi' \) exists on \( \mathbb{R}\setminus\{0\} \) with \( t \phi'(t) \to 0 \) as \( t \to 0 \) and \( \phi \) satisfies (5.8) with
\[
\ln \phi(t) = \int_0^t \ln \gamma(v) v^{\alpha-1} dv. \tag{5.25}
\]
The characteristic function \( \phi_c \) is infinitely divisible and so (ii) \( \rightarrow \) (iv) is also proved.

Suppose (iv) is satisfied. Since \( \phi = \lim_{c \to 0} \phi_c \), by Theorem 1.3.6, \( \phi \) is infinitely divisible with some Lévy spectral function \( M \). Since \( \phi \) and \( \phi_c \) are related by (5.8), the Lévy spectral function \( M_c \) of \( \phi_c \) is given by
\[
M_c(x) = M(x) - c^\alpha M(x/c)
\]
(cf. Theorem 1.3.2 and Lukacs (1970) p. 163). \( M_c \) is non-decreasing by Theorem 1.3.2 (6), hence from Lemma 5.4.5 we see that (ii) is satisfied.

The proof is completed by applying Lemma 5.4.6 to get (iii) \( \Rightarrow \) (iii).

**Theorem 5.4.8.** Let \( \alpha \in (-2, 0) \) and let \( \phi \) be a characteristic function. The following statements are equivalent.

(i) \( \phi \) is \( \alpha \)-self-decomposable, \( \phi' \) exists on \( \mathbb{R}\setminus\{0\} \) with \( \lim_{t \to 0} t \phi'(t) = 0 \);

(ii) There exists a unique infinitely divisible characteristic function \( \gamma \) and a stable, possibly degenerate, characteristic function \( \phi_{\text{STABLE}(-\alpha)} \) such that
\[
\ln \phi(t) = \int_0^t \ln \gamma(v) v^{\alpha-1} dv + \ln \phi_{\text{STABLE}(-\alpha)}(t);
\]

(iii) \( \phi \) is infinitely divisible with Lévy spectral function \( M \) having left and right derivatives and such that \( M'(x) \) is \( \alpha \)-unimodal;

(iv) \( \phi \) is \( \alpha \)-self-decomposable and \( \phi_c \) is an infinitely divisible characteristic function for every \( c \in (0, 1) \).
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PROOF. Lemma 5.4.6 proves that (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i) is shown as in Theorem 5.4.7. It remains to prove (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). As in the proof of Theorem 5.4.7, (5.8) gives rise to the differential equation (5.24). Integrating on both sides of (5.24) over $(t_0, t)$, $t_0 > 0$ yields

$$
\int_{t_0}^{t} u^{\alpha-1} \ln \gamma(u) du = t^\alpha \ln \phi(t) - t_0^\alpha \ln \phi(t_0),
$$

and hence

$$
\ln \phi(t) = \int_{t_0}^{t} \gamma(u/t)^2 \ln \phi(t) du,
$$

for all $t \geq t_0$. By (5.8) we have

$$
\ln \phi_t(t) = \int_{t}^{\infty} \gamma(v) v^{\alpha-1} dv,
$$

(5.26)

for all $t \geq 0$. Similarly, we obtain that $\phi_t$ is of the form (5.26) for $t \leq 0$. So $\phi_t$ is infinitely divisible and condition (iv) is satisfied.

By Theorem 1.3.7 (i), $\phi_t^{-1}$ is a characteristic function for every $c$ such that $c^{-\alpha} \in \mathbb{N}_+$. From (5.8) it follows that

$$
\phi_t^{-1}(c) = \phi(c) \phi_t^{-1}(c).
$$

So $\phi_t^{-1}(c)$ is a characteristic function for every $c$ such that $c^{-\alpha} \in \mathbb{N}_+$ and therefore, by Theorem 1.3.7 (i), $\phi$ is infinitely divisible. Since $\phi$ and $\phi_t$ are related by (5.8) the Lévy spectral function $M_c$ of $\phi_t$ is given by

$$
M_c(x) = M(x) - c^{\alpha} M(x/c),
$$

(cf. Theorem 1.3.2 and Lukacs (1970), p. 163). $M_c$ is non-increasing by Theorem 1.3.2 (i), hence from Lemma 5.4.5 we see that (iii) is satisfied. $\square$

NOTATION 5.4.9. A characteristic function $\phi$ belongs to the set $S_\alpha(\mathbb{R})$ if $\phi$ satisfies any of the conditions (i) - (iii) of Theorems 5.4.7 and 5.4.8.

If $\phi$ is stable with exponent $\delta$, then, by Theorem 1.4.1, $\phi$ has a Lévy spectral function $M$ such that $M'(x)$ is $(-\delta)$-unimodal and hence (cf. Remark 5.2.3) $\alpha$-unimodal for all $\alpha \geq -\delta$. We thus have the following corollary to Theorem 5.4.8 (iii).

COROLLARY 1. $\phi$ is stable with exponent $\delta$, then $\phi \in S_\alpha(\mathbb{R})$ for every $\alpha \geq -\delta$.

The proof of (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) is also valid for $\alpha \leq -2$. By Remark 5.4.4 and observing that the normal characteristic function is $\alpha$-self-decomposable if and only if
Corollary 2. The normal characteristic function is the only characteristic function in $S_{-2}(\mathbb{R})$ and in the case $\alpha < -2$ $S_{\alpha}(\mathbb{R})$ contains only degenerate characteristic functions.

By (5.26) and Theorem 5.4.8 (iii) we see that
\[ c^\alpha \ln \phi(ct) = \ln \phi(t) - \ln \phi(t) = \int_0^c \ln \gamma(v) v^{\alpha-1} dv + \ln \phi_{STABLE}(\alpha)(t). \]

Let $c = (n+1)^{\alpha/\alpha}$. Letting $n \to \infty$ we have (cf. Definition 1.4.7)

Corollary 3. Let $-2 \leq \alpha < 0$. If $\phi \in S_{\alpha}(\mathbb{R})$, then $\phi$ is in the domain of attraction of a stable characteristic function of order $-\alpha$.

Remark 5.4.10. It follows from the proof of Theorem 5.4.8 that the Lévy spectral function $N$ of the (unique) infinitely divisible characteristic function $\gamma$ in (i) must satisfy
\[ \int_{|v| \geq 2} v^{-\alpha} dN(v) < \infty, \quad \alpha \in (-2, 0), \]
for all $\alpha > 0$. For $\alpha = 0$ the Lévy spectral function $N$ must satisfy
\[ \int_{|v| \geq 2} |v|^{-1} N(v) dv = \int_{|x| \geq 2} \ln |x| dN(x) + N(2) \ln 2 - N(-2) \ln 2 < \infty, \]
and hence
\[ \int_{|x| \geq 2} \ln |x| dN(x) < \infty. \]

Remark 5.4.11. O'Connor (1979b) proves Theorem 5.4.7 for $\alpha = (0, 1)$. In his proof he uses the results of O'Connor (1979a), which proves Theorem 5.4.7 for $\alpha = 1$. We prove Theorem 5.4.7 for any fixed $\alpha > 0$ and we do not use results for other known fixed $\alpha$. O'Connor (1981) does not consider condition (i) of Theorem 5.4.8, but the condition

(i') $\phi$ is $\alpha$-self-decomposable and infinitely divisible.

and proceeds to prove (i') $\Leftrightarrow$ (iii) for $\alpha \in (-2, 0)$ and (ii') $\Leftrightarrow$ (iii) for $\alpha \in (-1, 0)$. We prefer (i) to (i') as it is usually easier to verify that $\phi'$ exists and $t \phi'(t) \to 0$ as $t \to 0$ rather than $\phi$ is infinitely divisible. The proofs are also simpler if condition (i') is used instead of (ii'). O'Connor tried but was unable to prove (ii) $\Leftrightarrow$ (iii) for $\alpha \in (-2, -1)$. 


5.4 Distributions on $\mathbb{R}$

We were able to do so by making the appropriate choices of $a_\gamma$ and $\sigma_\gamma$ in the proof of Lemma 5.4.6.

**Remark 5.4.12.** Jurek (1988) obtains some of the results of this section. Our work, however, was done independently and almost simultaneously of Jurek's. Jurek (1988) defines $S_\alpha(\mathbb{R})$ as the set of characteristic functions $\phi$ which can be obtained as limits of the following kind (cf. Theorem 6.3.3 (ii));

$$\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/j)^2 \ln \phi_j(t/n),$$

where $(\phi_j)$ is a sequence of infinitely divisible characteristic functions. He then proves that $\phi \in S_\alpha(\mathbb{R}) \iff (i')$ and that $(i') \iff (ii)$ of Theorems 5.4.7 and 5.4.8 for $\alpha > -2$.

Jurek however assumes that $\phi$ stems from a symmetric random variable in the case that $\alpha = (-2, -1)$.

**Remark 5.4.13.** The proof of Theorem 5.3.4 can easily be adapted to prove the analogue of Theorem 5.3.4 for distributions on $\mathbb{R}$ (and $\mathbb{R}_+)$.

5.5 Distributions on $\mathbb{R}_+$

In this section we use the results of Section 5.3, to prove the analogue of Theorems 5.3.2, 5.4.7 and 5.4.8 for random variables supported by $\mathbb{R}_+$. The theorem is easily proved by Theorems 5.4.7 and 5.4.8. It is stated separately because it takes a simpler form (as in Theorem 5.3.2), and, as will be shown, the $\alpha$-self-decomposable distributions on $\mathbb{R}$ and on $\mathbb{R}_+$ are closely related.

**Definition 5.5.1.** Let the random variable $X$ on $\mathbb{R}_+$ have distribution function $F$ and Laplace-Stieltjes transform $f$. The function $f$ is said to be $\alpha$-self-decomposable and belong to $S_\alpha(\mathbb{R}_+)$, for some $\alpha \in \mathbb{R}$, if for every $c \in (0, 1)$ there exists a Laplace-Stieltjes transform $f_c$ such that

$$f_c(t) = f^c(t) f(t), \quad t \in \mathbb{R}_+. \quad (5.27)$$

It can be shown (cf. Theorem 2.7.1 of van Harn (1978)) that $X$ is infinitely divisible and supported by $\mathbb{R}_+$ (cf. Theorem 1.3.3) if and only if the characteristic function $\phi$ of $X$ is given by Theorem 1.3.2 with $M \equiv 0$ for $x < 0$, $a_\phi = \sigma_\phi^2 = 0$ and $x M'(x) = H'(x)$ for $x > 0$. Furthermore $f$ is analytic and $\tau f(\tau) \to 0$ as $\tau \to 0$ (cf. Steutel and van Harn (1979)). We have thus proved
THEOREM 5.5.2. Let \( \alpha \in (0, \infty) \) and let \( \hat{f} \) be a Laplace-Stieltjes transform. The following statements are equivalent.

\[ \hat{f}(c) = \int \frac{\ln \hat{G}(c) e^{-\lambda x}}{c - \lambda x} \, dx \quad \text{for all } c \in (0, \infty) \]

(i) \( \hat{f} \) is \( \alpha \)-self-decomposable;

(ii) \( \hat{f}(c) = \int_0^\infty (\ln \hat{G}(c) e^{-\lambda x}) x^{-\alpha - 1} \, dx \) \( \lambda \geq 0 \) and \( \hat{G} \) a unique infinitely divisible Laplace-Stieltjes transform;

(iii) \( \hat{f} \) is infinitely divisible with canonical measure \( H \) having left and right derivatives and such that \( H'(x)/x \) is a unimodal on \( \mathbb{R}_+ \).

Furthermore, \( \lambda \) in (ii) is zero if \( \alpha \geq 0 \) and \( \hat{f}_x \) is infinitely divisible for every \( x \in (0, 1) \) if \( \hat{f} \) is \( \alpha \)-self-decomposable.

Since all log-convex functions and all completely monotone functions are non-increasing we have (cf. Theorems 4.3.3 and 3.5.3)

COROLLARY 1. If \( H'(x)/x \) is log-convex, then \( \hat{f} \in S_0(\mathbb{R}_+) \).

COROLLARY 2. If \( F \) is a mixture of exponential distributions, then \( \hat{f} \in S_1(\mathbb{R}_+) \).

Goldie (1967) proved that there is a correspondence between probability generating functions in \( ID(\mathbb{N}_0) \) and Laplace-Stieltjes transforms in \( ID(\mathbb{R}_+) \). A similar result was proved in Forst (1979) between \( S_0(\mathbb{N}_0) \) and \( S_0(\mathbb{R}_+) \). We now prove a lemma which generalizes these results by giving a correspondence between probability generating functions in \( S_0(\mathbb{N}_0) \) and Laplace-Stieltjes transforms in \( S_0(\mathbb{R}_+) \).

LEMMA 5.5.3. Let \( F \) be a distribution function on \( \mathbb{R}_+ \) and define the probability distribution \( (p_n(s)) \) on \( \mathbb{N}_0 \) by

\[ p_n(s) = \frac{1}{(s+1)^{-1}} \int_0^\infty e^{-t(s+1)} e^{-\lambda x} \, dx \]

The following statements are equivalent

(i) the probability generating function of \( (p_n(s)) \) is in \( S_0(\mathbb{N}_0) \) for all \( s \in \mathbb{R}_+ \);

(ii) the Laplace-Stieltjes transform of \( F \) is in \( S_0(\mathbb{R}_+) \).
5.5 Distributions on \( \mathbb{R}_+ \)

PROOF. The probability generating function \( P_x \) of \( (p_n(x)) \) and the Laplace-Stieltjes transform \( \hat{f} \) of \( F \) are related by

\[
\hat{f}(s(1-e)) = P_x(e).
\]

The lemma now follows from (5.4) and (5.27).

REMARK 5.5.4. Theorem 5.5.2 can also be proved by using Theorem 5.3.2, Lemma 5.5.3, Lemma 4.2.1 of Steutel (1970) and Theorem 1, p. 439 of Feller (1971).

5.6 A classification of ID(1)

From Theorems 5.4.7 and 5.4.8 it is clear that \( S_\alpha(\mathbb{R}) \subseteq ID(\mathbb{R}) \), and that \( S_\alpha(\mathbb{R}) \) is closed under multiplication and limits. If \( \alpha(1) > \alpha(2) \) and \( \phi \in S_\alpha(\mathbb{R}) \), then (cf. (5.8))

\[
\phi(\xi) = \phi^{\alpha(0)}(c\xi) \cdot \phi^{\alpha(0) - \alpha(1)}(c\xi) \cdot \phi(\xi).
\]

Since \( \phi \) is infinitely divisible, \( \phi^{\alpha(0) - \alpha(1)} \) is a characteristic function and hence \( \phi \in S_{\alpha(1)}(\mathbb{R}) \). From the second corollary to Theorem 5.4.8 it follows that \( S_{\alpha}(\mathbb{R}) \), \( \alpha < -2 \), contains only degenerate characteristic functions. Theorem 5.4.7 (ii) implies that for any \( \phi \in ID(\mathbb{R}) \),

\[
\Phi_\alpha(t) := \exp \left( \int_0^1 \ln \phi^{\alpha}(sv) v^{\alpha - 1} \, dv \right) = \exp \left( \int_0^1 \ln \phi(\alpha v) \, dv \right),
\]

is in \( S_{\alpha}(\mathbb{R}) \) for all \( \alpha > 0 \). By Helly's second theorem \( \phi_\alpha \rightarrow \phi \) as \( \alpha \rightarrow \infty \). Hence

\[
\bigcup_{\alpha \in \mathbb{R}} S_{\alpha}(\mathbb{R}) = ID(\mathbb{R}).
\]

Similarly for \( S_{\alpha}(\mathbb{R}_+) \) and \( S_{\alpha}(\mathbb{N}) \). We collect these results in the following theorem.

THEOREM 5.6.1. For \( I = \mathbb{R}, \mathbb{R}_+ \) or \( \mathbb{N}_0 \), the sets \( S_{\alpha}(I) \) are multiplication semigroups, closed under limits and provide a classification of \( ID(I) \), i.e.,

(i) If \( \alpha_2 < \alpha_1 \), then \( S_{\alpha_2}(I) \subseteq S_{\alpha_1}(I) \);

(ii) \( ID(I) = \bigcup_{\alpha \in \mathbb{R}} S_{\alpha}(I) \);

(iii) \( S_{\alpha}(I) \) is closed under limits and multiplication for every \( \alpha \).
Chapter 6

α-SELF-DECOMPOSABILITY AND LIMIT LAWS

6.1 Introduction

Limit distributions of sums of independent random variables has been a central topic in probability theory and statistics for many years. The classical central limit problem and its successive generalizations to stable and self-decomposable random variables are well-known examples. Also, the set of infinitely divisible random variables can be described as the solution of the so-called general central limit problem (cf. Loève (1977), compare Theorem 6.2.4). Recently, Jurek (1981) introduced the set of s-self-decomposable random variables, which are defined as limits of sums of ‘shrunken’ random variables. Jurek (1985) showed that this set coincides with the set of random variables having characteristic functions in $S_1(R)$ (see Notation 5.4.9). The set of 1-self-decomposable random variables satisfying (i) of Theorem 5.4.8 is included in the set of infinitely divisible random variables and in turn includes the sets of self-decomposable and stable random variables.

In contrast to the previous chapters we only study random variables on $R$ in this chapter. We consider random variables with characteristic functions in $S_0(R)$ as limits of sums of independent normed or ‘shrunken’ random variables. In the following section we give some preliminaries. Using the representations of $S_0(R)$ obtained in the previous chapter, we present in Section 6.3 four limit forms of α-self-decomposable random variables. One of these limit forms is the one used by Jurek (1988) to define $S_0(R)$. The method used is easily adapted to $S_0(N)$. In Section 6.4 we introduce two shrinking operators. Both operators are closely connected with α-unimodality. The first operator is a generalization of Jurek’s (1981) shrinking operator and the second operator, a stochastic shrinking operator, is a generalization of the
6.1 Introduction

well-known linear operator \( T \) (cf. Section 1.4). In the same vein as Jurek (1981) we
consider limit distributions of sums of ‘generalized shrunken’ independent random
variables and show that the set of these limit distributions is equal to \( S_{\alpha}(R) \), \( \alpha \geq 0 \). In
Section 6.4 we also characterize the set of distributions obtained as limits of sums of
‘generalized shrunken’ independent identically distributed random variables. In
Section 6.5 we discuss \( S_{\alpha}(R) \) for \( \alpha \in (-2,0] \) in the context of limit distributions in
detail. Among other results we show that \( S_{\alpha}(R) \), \( \alpha \in (-2,0] \), contains limit
distributions of \( T \)-normed sums of blockwise identically distributed random
variables. We conclude in Section 6.6 with a few comments and remarks.

6.2 Preliminaries

An infinitely divisible characteristic function \( \phi \) is uniquely determined by the
triple \( \{a_{\phi}, \sigma^{2}_{\phi}, M\} \) in Theorem 1.3.2. We therefore introduce the notation

**Notation 6.2.1.** An infinitely divisible characteristic function with Lévy spectral
function \( M \) and constants \( a_{\phi} \) and \( \sigma^{2}_{\phi} \) (cf. Theorem 1.3.2) will be denoted by
\( \phi = [a_{\phi}, \sigma^{2}_{\phi}, M] \). □

We now give two definitions and state a theorem. The theorem is a solution to the
general central limit problem and as such is of great importance in probability
theory.

**Definition 6.2.2.** A sequence \( \{X_{k}\} \) of random variables is said to be bounded if
there exists a constant \( c \geq 0 \) such that
\[
P(|X_{k}| \leq c) = 1, \quad \text{for all } k \in \mathbb{N},
\]

**Definition 6.2.3.** By a triangular array of random variables is meant a double
sequence of random variables \( \{X_{k,n}\}, k = 1, 2, \ldots, n, n \in \mathbb{N}, n = \{1, 2, \ldots\} \) (henceforth
denoted \( \{X_{k,n}\} \)), such that the random variables \( X_{1,n}, \ldots, X_{n,n} \) of the \( n \)-th row are
mutually independent.

The triangular array \( \{X_{k,n}\} \) is said to be uniformly asymptotically negligible
\( (u.a.) \) if \( X_{k,n} \to 0 \) in probability, uniformly in \( k \) as \( n \to \infty \), i.e., if for every \( \varepsilon > 0 \)
\[
\lim_{k \to \infty} \sup_{n \geq 1} P(|X_{k,n}| > \varepsilon) = 0.
\]

□
The triangular array of characteristic functions \((\phi_{k,n})\) and the triangular array of distribution functions \((F_{k,n})\) will be called \(\alpha\)-unified if they stem from a \(\alpha\)-unified triangular array of random variables.

**Theorem 6.2.4.** A random variable \(X\) with characteristic function \(\phi\) is infinitely divisible with \(\phi = [a_0, \sigma^2, M]\) if and only if there exists a \(\alpha\)-unified triangular array \((X_{k,n})\), \(k = 1, 2, \ldots, n\), \(n \in \mathbb{N}_+\), of random variables with distribution functions \((F_{k,n})\) such that

\[
\sum_{k=1}^{n} X_{k,n} \xrightarrow{\text{w}} X,
\]

and necessarily

\begin{enumerate}
  \item[(i)] \(M_n = \sum_{k=1}^{n} F_{k,n} \xrightarrow{\text{w}} M\) outside every neighbourhood of the origin;
  \item[(ii)] \(\lim \lim_{\epsilon \to 0} \sum_{k=1}^{n} \int_{|x| < \epsilon} x^2 dF_{k,n}(x) = \sigma^2.\)
\end{enumerate}

A proof of Theorem 6.2.4 can be found in Loève (1977). From this theorem it is evident that all \(\alpha\)-self-decomposable and all stable random variables are infinitely divisible (cf. equations (1.3) and (1.4)). The theorem also shows how the Lévy spectral function \(M\) and the constant \(\sigma^2\) in Theorem 1.3.2 are determined by the sums of the distribution functions of \((X_{k,n})\).

We finish this section with two lemmas, which will be used in the following sections. For a proof we refer to Loève (1977).

**Lemma 6.2.5.** The triangular array of characteristic functions \((\phi_{k,n})\), \(k = 1, 2, \ldots, n\), \(n \in \mathbb{N}_+\), is \(\alpha\)-unified if and only if

\[
\limsup_{n \to \infty} 1_\phi_{k,n}(t) - 1 = 0,
\]

uniformly on every finite interval.

**Lemma 6.2.6.** If \((X_{k,n}^{(1)})\) and \((X_{k,n}^{(2)})\) are \(\alpha\)-unified triangular arrays, then \((X_{k,n}^{(1)} + X_{k,n}^{(2)})\) is a \(\alpha\)-unified triangular array.

### 6.3 Sums of \(\alpha\)-unified triangular arrays

In this section we use equation (5.8) and the results of Section 5.4 to describe random variables \(X\) with characteristic functions \(\phi\) in \(S_\alpha(\mathbb{R})\) as limits of sums of \(\alpha\)-unified
random variables. Our approach is similar to that in Lukacs (1970), Theorem 5.11.1 and its corollary.

Let $\phi_{\text{STABLE}(-\alpha)}(t) = 1$ for $\alpha \geq 0$. Writing the integrals in Theorems 5.4.7 and 5.4.8 as limits of Riemann sums we have (cf. Notation 5.4.9)

**Theorem 6.3.1.** Let $\alpha > -2$ but $\alpha \neq 0$ and let $\phi$ be a characteristic function. The following statements are equivalent.

(i) $\phi \in S_\alpha(\mathbb{R})$;

(ii) There exists an infinitely divisible characteristic function $\gamma$ and a stable characteristic function $\phi_{\text{STABLE}(-\alpha)}(t)$ such that

$$
\ln \phi(t) = \ln \phi_{\text{STABLE}(-\alpha)}(t) + \alpha^{-1} \lim_{n \to \infty} \sum_{j=1}^{\infty} (jn)^{\alpha} \ln \gamma^{(j/n)^{\alpha} t};
$$

(iii) There exists an infinitely divisible characteristic function $\gamma$ and a stable characteristic function $\phi_{\text{STABLE}(-\alpha)}(t)$ such that

$$
\ln \phi(t) = \ln \phi_{\text{STABLE}(-\alpha)}(t) + \lim_{n \to \infty} \sum_{j=1}^{\infty} (jn)^{\alpha} \ln \gamma^{(j/n)^{\alpha} t}.
$$

Conditions (i) and (iii) are also equivalent for $\alpha < 0$.

In Theorem 6.3.1 we see that the logarithm of any characteristic function in $S_\alpha(\mathbb{R})$ can be written as a limit of a weighted sum of logarithms of identical characteristic functions. In the next theorem we obtain a less restrictive limiting form of $\phi$ than in Theorem 6.3.1.

**Theorem 6.3.2.** Let $\alpha > -2$ but $\alpha \neq 0$ and let $\phi$ be a characteristic function. The following statements are equivalent.

(i) $\phi \in S_\alpha(\mathbb{R})$;

(ii) There exist infinitely divisible characteristic functions $(\phi_j)^*\gamma^*$ such that

$$
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{\infty} (jn)^{\alpha} \ln \phi_j(\gamma(j/n)^{\alpha} t);
$$

(iii) There exist infinitely divisible characteristic function $(\phi_j)^*\gamma^*$ such that

$$
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{\infty} (jn)^{\alpha} \ln \phi_j(\gamma(j/n)^{\alpha} t).
$$
Conditions (i) and (iii) are also equivalent for \( \alpha = 0 \).

**Proof.** Let \( \Psi \) be in the domain of attraction of \( \varphi_{\text{STABLE}(\alpha)}(t) \) (cf. Definition 1.4.7). Letting \( \gamma^{j2} = \Omega_j, \ j \geq 2 \) and \( \phi_j = \gamma^{j2} \psi \exp(id_jt) \) in Theorem 6.3.1 parts (ii) and (iii), we see that (i) implies (ii) and (iii). We now prove (iii) \( \Leftrightarrow \) (i). Let \( (k_n) \) be a sequence of real numbers such that \( k_n \to n \) and \( (k_n/n) \to c \in (0, 1) \) as \( n \to \infty \). Observe that

\[
\sum_{j=1}^{n} (j/n)^a \ln \phi_j((j/n)t) = \sum_{j=1}^{n} (j/k_n)^a \ln \phi_j((j/n)(k_n/n)t) + \sum_{j=k_n+1}^{n} (j/n)^a \ln \phi_j((j/n)t).
\]  

(6.1)

Letting \( n \to \infty \) in (6.1), the left hand side tends, by definition, to \( \ln \phi(t) \), the first term on the right hand side tends, by definition, to \( c^a \ln \phi(ct) \) and so the second term tends to a limit, which by Theorems 1.3.7 and 1.3.6, is the logarithm of some infinitely divisible characteristic function, \( \ln \phi_c(t) \) say. By Theorem 5.4.7 (iv) \( \phi \in S_\alpha(\mathbb{R}) \). A similar argument proves (ii) \( \Rightarrow \) (i).

Let \( \ln \tilde{\phi}_j(t) = j \ln \phi_j(j^{-1}t) \) or \( \ln \tilde{\phi}_j(t) = j^a \ln \phi_j(j^{-1}t) \) in Theorem 6.3.2. We then have

**Theorem 6.3.3.** Let \( \alpha > -2 \) but \( \alpha \neq 0 \) and let \( \phi \) be a characteristic function. The following statements are equivalent.

(i) \( \phi \in S_\alpha(\mathbb{R}) \);

(ii) There exist infinitely divisible characteristic functions \( (\phi_j) \) such that

\[
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} 1/n \ln \phi_j((1/n)^{1/\alpha} t);
\]

(iii) There exist infinitely divisible characteristic functions \( (\phi_j) \) such that

\[
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/n)^a \ln \phi_j((1/n)^{1/\alpha} t).
\]

Conditions (i) and (iii) are also equivalent for \( \alpha = 0 \).

Theorem 6.3.2 can also be obtained from Theorem 6.3.3 by letting \( \ln \phi_j(t) = j^{-a} \ln \phi_j(j^{-1/\alpha} t) \) or \( \ln \phi_j(t) = j^{-a} \ln \phi_j(j^{-1} t) \). Jurek (1988) uses condition (ii) of Theorem 6.3.3 to define \( S_\alpha(\mathbb{R}) \).
6.3 Sums of un triangular arrays

REMARK 6.3.4. The implications (i) \( \Rightarrow \) (ii) for \( \alpha > 0 \) and (i) \( \Rightarrow \) (iii) for \( \alpha > -2 \) of Theorem 6.3.2 can also be proved directly by noting that (cf. (5.8))

\[
\sum_{j=1}^{n} (j/n) \ln \phi_{j}( (j/n)^{1/\alpha} t ) = \sum_{j=1}^{n} (j/n) \ln \phi((j/n)^{1/\alpha} t) - \sum_{j=1}^{n} ((j-1)/n) \ln \phi(( (j-1)/n )^{1/\alpha} t) \\
= \sum_{j=1}^{n} (j/n) \ln \phi((j/n)^{1/\alpha} t) - \sum_{j=1}^{n} ((j-1)/n) \ln \phi(( (j-1)/n )^{1/\alpha} t) \\
= \ln \phi(c_{j}),
\]

with \( \alpha > 0 \) and \( c_{j} = ( (j-1)/j )^{1/\alpha} \) and observing that

\[
\sum_{j=1}^{n} (j/n) \ln \phi_{j}( (j/n) t ) = \ln \phi(t), \quad \alpha > 0,
\]

\[
(1/n) \ln \phi(t/n) + \sum_{j=1}^{n} (j/n) \ln \phi_{j}( (j/n) t ) = \ln \phi(t), \quad \alpha \in (-2,0),
\]

where \( c_{j} = (j-1)/j \).

\( \square \)

REMARK 6.3.5. For \( \alpha > 0 \) we can replace the existence of a sequence of infinitely divisible characteristic functions \( \phi_{j} \) in Theorem 6.3.2 (ii) by the existence of a un triangular array \( \left( \phi_{k,n} \right) \), where

\[
\ln \phi_{k,n}(t) = \frac{1}{n} \sum_{j=k}^{n} \ln \phi_{j}( (j/n)^{1/\alpha} t ), \quad k = 1, 2, \ldots, n, \quad n \in \mathbb{N}_{+},
\]

for some sequence \( \phi_{j} \) of characteristic functions. The implication (ii) \( \Rightarrow \) (i) is then proved, as above, by noting that

\[
\sum_{j=1}^{n} (j/n) \ln \phi_{j}( (j/n)^{1/\alpha} t ) = \sum_{j=1}^{n} (j/n) (k_{j}/n) \ln \phi_{j}( (j/k_{j})^{1/\alpha} (k_{j}/n)^{1/\alpha} t ) + \sum_{j=1}^{n} \ln \phi_{j,n}(t) + k_{n} \ln \phi_{k_{n+1},n}(t).
\]

(6.2)

and letting \( n \to \infty \) such that \( (k_{j}/n) \to c \in (0,1) \). Observe that the sum of the last two terms in (6.2) tends to the logarithm of an infinitely divisible characteristic function by the un property of \( \left( \phi_{k,n} \right) \), Theorem 6.2.4, and Lemma 6.2.6. Conversely, by Remark 6.3.4 we need only show that \( \left( \phi_{k,n} \right) \) is un, where

\[
\ln \phi_{k,n}(t) = \frac{1}{n} \sum_{j=k}^{n} \ln \phi_{j}( (j/n)^{1/\alpha} t ), \quad k = 1, 2, \ldots, n, \quad n \in \mathbb{N}_{+},
\]

and \( c_{j} = ( (j-1)/j )^{1/\alpha} \). An infinitely divisible characteristic function \( \gamma \) has no real zeros, hence for any \( T > 0 \) there exists a \( C > 0 \) such that \( \ln \gamma(t) \leq C \), \( t \in [-T, T] \). From (5.25), letting \( a(j) = (j/n)^{1/\alpha} \), we obtain

\[
\left| \ln \phi_{k,n}(t) \right| \leq C \frac{n}{n} \sum_{j=k}^{n} \int_{a(j-1)}^{a(j)} u^{-1} \ln u \, du \leq \frac{C}{n} \ln n \to 0 \quad \text{as} \quad n \to \infty.
\]
uniformly on \([-T, T]\). By Lemma 6.2.5, \((\phi_{h,n})\) is uan. Similarly we can replace the condition that \((\phi_h)\) be infinitely divisible in part (ii) by the condition that the triangular array \((\phi_{h,n})\) be uan (this is done in O'Connor (1979b) for \(\alpha \in (0,1)\)) where
\[
\ln \phi_{h,n}(t) = (1/n)^{\alpha} \sum_{j=1}^{n} \ln \phi_j((j/n) t), \quad k = 1, 2, ..., n, n \in \mathbb{N}_+.
\]

We conclude this section with another limit theorem for characteristic functions in \(S_\alpha(\mathbb{R})\), with \(\alpha \in (-2,0)\). We will consider this theorem and other theorems of the same type concerning the subsets \(S_\alpha(\mathbb{R})\), \(\alpha \in (-2,0)\) of \(S_\alpha(\mathbb{R})\) in more detail in Section 6.5.

**Theorem 6.3.6.** Let \(\alpha \in (-2,0)\). If \(\phi \in S_\alpha(\mathbb{R})\) then there exists a sequence of infinitely divisible characteristic functions \((\phi_j)\) and a non-increasing sequence \((\nu_j)\) with \(\nu_j \to 0\) such that
\[
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} \nu_j \ln \phi_j(\nu_j t).
\]

**Proof.** Let \(\phi \in S_\alpha(\mathbb{R})\). By (5.8) it is easily verified that
\[
\nu_{n+1}^\alpha \ln \phi(\nu_{n+1} t) + \sum_{j=1}^{n} \nu_j^\alpha \ln \phi_j(\nu_j t) = \ln \phi(t),
\]
where \(\nu_j = (\nu_{j+1}/\nu_j)\) and \(\nu_1 = 1\). By Theorem 5.4.8, \(\phi_j\) is infinitely divisible. Applying Corollary 3 to Theorem 5.4.8 yields
\[
\lim_{n \to \infty} \phi_{\alpha n}(\nu_{n+1} t) = \ln \phi_{\text{STABLE}(-\alpha)}(t).
\]
The proof is completed by letting \(\phi_j = \phi_{\nu_j}, j \geq 2\) and \(\phi_1 = \phi_{\nu_1} \cdot \phi_{\text{STABLE}(-\alpha)}\).

**Corollary 1.** Let \(\alpha \in (-2,0)\). If \(\phi \in S_\alpha(\mathbb{R})\) then there exists a sequence of infinitely divisible characteristic functions \((\phi_j)\) such that
\[
\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} \nu_j \ln \phi_j(\nu_j t).
\]

**Remark 6.3.7.** Theorems 6.3.2 and 6.3.3 can be generalized in the same way as Theorem 6.3.6 generalizes its corollary. We leave it in its present form though, for better comparison with Theorem 6.3.1.
6.3 Sums of untriangular arrays

Remark 6.38. We would like to find a suitable definition of the generalized stable laws in $S_\alpha(\mathbb{R})$ analogous to the definition of the stable laws corresponding to the classical self-decomposable laws, i.e., to find the characteristic functions in $S_\alpha(\mathbb{R})$ which can be written as a limit of weighted sums of identical characteristic functions. All the theorems in this section are of the same form as Theorem 6.3.1, where $\phi$ is equal to the limit of weighted sums of identical characteristic functions. In this setting, the stable laws in $S_\alpha(\mathbb{R})$ are just the limit laws in $S_\alpha(\mathbb{R})$. Therefore the approach of this section does not suggest a reasonable definition generalizing classical stability.

6.4 Sums of shrunked random variables

Throughout this section we will study $S_\alpha(\mathbb{R})$ for $\alpha \geq 0$ only. Define the one-parameter family of non-linear shrinking operators $(U_{\alpha,t})$, $\alpha > 0$, by

$$U_{\alpha,t}x = \begin{cases} 0 & \text{if } |x| \leq t \\ (1 - \left(\frac{t}{|x|}\right)^\alpha)^\frac{1}{\alpha} x & \text{if } |x| > t \end{cases}$$

for $t \geq 0$. When $\alpha = 1$, we regain the shrinking operator $U_t$ defined in Jurek (1981). In the following figure we sketch $U_{\alpha,t}$ as a function in $x$ and $\alpha$.
Let $X$ be a random variable defined by $P(Y_i = 1) = r_i = 1 - P(Y_i = 0)$. We then define the one-parameter family of linear stochastic operators $(T_{\alpha})$, $\alpha \geq 0$, by
\[ T_{\alpha}x = X + x, \quad t \in [0, 1]. \] 
(6.4)

If $\alpha = 0$ then $T_{\alpha}$ reduces to the linear operator $T$ (cf. Section 1.4). In this section we show, using the results of the Appendix, that the set of random variables $X$ given by
\[ \sum_{k=1}^{n} U_{\alpha}X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty, \] 
(6.5)
\[ \sum_{k=1}^{n} T_{\alpha}X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty, \] 
(6.6)

for suitable $\{b_n\}$, $(X_k)$ and $(T_{\alpha}X)$, is equal to the set of random variables with characteristic functions in $S_{\alpha}(\mathbb{R})$. We also characterize the random variables $X$ obtained as (6.5) and (6.6) for identically distributed.

6.4.1 $U_{\alpha}$-shrunk random variables.

Let the random variable $X$ have distribution function $F$ and denote by $F_U$ the distribution function of $U_{\alpha}X$. Let $U_{\alpha}$ and $U_{\alpha}^{-1}$ act on Borel sets $B$ by
\[ U_{\alpha}B = \{ y \mid U_{\alpha}x = y, \text{for } x \in B \} \quad \text{and} \quad U_{\alpha}^{-1}B = \{ x \mid U_{\alpha}x \in B \}. \]

Let $I = (a, b)$, $0 < a < b$, then
\[ F_U(I) := P(U_{\alpha}x \in I) = P(X \in U_{\alpha}^{-1}I) = F(U_{\alpha}^{-1}I). \]

We therefore let the operator $U_{\alpha}$ operate on set functions by $U_{\alpha}F(I) = F(U_{\alpha}^{-1}I)$. The following lemma gives a connection between $U_{\alpha}$ and the notion of $\alpha$-unimodality.

**Lemma 6.4.1.** Let $\alpha > 0$, $\epsilon > 0$ and let $\beta_{\epsilon}$ be the set of Borel sets on $\mathbb{R} \backslash \{0\}$. Further let $M$ be a Lévy spectral function. The following statements are equivalent:

(i) $M$ has left and right hand derivatives on $\mathbb{R} \backslash \{0\}$ and $M'(x)$ is $\alpha$-unimodal;

(ii) $M(B) = U_{\alpha}M(B), \quad t \in (0, \infty), \quad B \in \beta_{\epsilon}$.

**Proof.** First we prove (ii) $\Rightarrow$ (i). Let $x \in \mathbb{R}$, fix $\epsilon > 0$ and let $B = (a, b)$, $0 < \epsilon \leq a < b.

(iii) is equivalent to
\[ M((a \wedge t^\alpha)^{1/\alpha} - ((a \wedge t^\alpha)^{1/\alpha} - M(b) - M(a). \]

(6.7)

Let $w(x) = M(x^{1/\alpha})$. From (6.7) with $x = b^\alpha \wedge t^\alpha$, $y = a^\alpha \wedge t^\alpha$, $x' = b^\alpha$ and $y' = a^\alpha$ it follows that
6.4 Sums of shrunken random variables

\[
\frac{w(x) - w(y)}{x - y} \leq \frac{w(x^*) - w(y^*)}{x^* - y^*}.
\]  
(6.8)

Hence \( w \) is convex. By Proposition 16, p. 109, Royden (1968), \( w \) and hence also \( M \) has left and right derivatives on \((e, \infty)\). Letting \( a \to b \) and hence \( y \to x \) and \( y' \to x' \) in (6.8) we obtain

\[
(b^a + a^a)^{-1+a} M'(b^a + a^a) \leq b^{-1+a} M(b).
\]

Let \( c^a = b^a + a^a \), then \( c \geq b \) and

\[
c^{-1+a} M'(c) \leq b^{-1+a} M'(b).
\]

For \( B = (a, b) \), \( 0 < e \leq a < b \) the proof is similar. So \( M' \) is \( \alpha \)-unimodal.

The implication (i) \( \Rightarrow \) (ii) is proved for \( B = (a, b) \), \( 0 < e \leq a < b \), by observing that (cf. (5.9))

\[
M(B) = \int_B M'(x) \, dx = \int_B \left( \int_{x^{-a} - t^a}^{x^{-a}} v^{-a} \, dN(v) \right) \, d\alpha^{-1} x^a
\]

\[
= \int_{\overline{[}a,b]} \left( \int_{x^{-a} - t^a}^{x^{-a}} v^{-a} \, dN(v) \right) \, d\alpha^{-1} (x^{-a} - t^a)
\]

\[
\geq \int_{\overline{[}a,b]} M'(x) \, dx = U_{\alpha,a} M(B).
\]

We are now ready to prove

**Theorem 6.4.2.** Let \( X \) have characteristic function \( \phi \). Then \( \phi \in S_\alpha(\mathbb{R}) \), \( \alpha > 0 \) if and only if there exists a sequence \( (b_n) \) such that

\[
\sum_{k=1}^\infty U_{\alpha,k} X_k + b_n \to X \quad \text{as } n \to \infty,
\]

where \((t_n)\) is a non-negative, non-decreasing sequence and \((X_k)\) is an independent not bounded sequence of random variables such that the triangular array \((U_{\alpha,k} X_k)\) is unan.

**Proof.** In this proof we shall frequently use the results of the Appendix. First we shall show that the operator \( U_{\alpha,t} \) satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5. Fix \( \alpha > 0 \) and let \( f_t := U_{\alpha,t}, t \in [0, \infty) \). By (6.3) we have

\[
f_t * f_s(x) = f_t(\text{sgn}(x)(|x|^{-a} - t^a)^{1/\alpha}) = \text{sgn}(x)(|x|^{-a} - s^{-a})^{1/\alpha} = f_s * f_t(x),
\]

where \( (x)_+) \) is equal to \( x \) if \( x > 0 \) and zero otherwise and \( s \oplus t \) is defined by

\[
s \oplus t = (s^{1/\alpha} + t^{1/\alpha})^{\alpha}.
\]
Obviously \( f_0(x) = x \) and so \( f_0 \circ f = f_1 \). Hence \( S = \{ (f_i)_{i \in \mathbb{R}_+}, \circ \} \) is a composition semigroup with respect to the semigroup \((\mathbb{R}_+, \cdot)\). If \( t \geq 2 \) then
\[
|f_t(x)| = |(1 + x^t - 1)|^{1/2(n-1)} \leq |(1 + x^t - x^t)|^{1/2(n-1)} = |f_1(x)| \leq |x|
\]
and so \( f_t \) in fact does shrink its arguments. From (6.3) it follows that \( f_t(x) \) is continuous in both \( t \) and \( x \) \((f_t(x) \) is differentiable in both \( t \) and \( x \) on \((0, \infty)\), that \( f_t \) is unbounded \((f_t(x) \to \infty \) as \( x \to \infty) \) and that \( f_t \) is one-to-one on \((t, \infty)\).

Suppose \( X \) is obtained by (6.5). Since \((U_{a_0}, X_0)\) is uan, \( X \) is infinitely divisible and hence \( \Phi = [a_0, \sigma_0^2, M] \). By Corollary 1 to Theorem A.3.2 in the Appendix, \( M \) satisfies the inequality of Lemma 6.4.1 (ii). By Lemma 6.4.1 and Theorem 5.4.7, \( \Phi \in S_0(\mathbb{R}) \).

The converse is proved in three steps. First it is shown that the symmetric normal distribution is a limit as described in the theorem; secondly it is shown that a random variable in \( S_0(\mathbb{R}) \) without normal component is of the desired type and finally these two results are combined to prove the theorem.

Let \( X \) be a symmetric normal random variable, i.e., \( \Phi = [0, \sigma^2, 0] \). Let \( F^{(1)} \) and \( F^{(2)} \) be Weibull distributions, defined by
\[
F^{(1)}(x) = 1 - \exp(-\frac{1}{2} \theta x^2) , \ x \in \mathbb{R}_+
\]
\[
F^{(2)}(x) = \exp(-\frac{1}{2} \theta x^2) , \ x \in \mathbb{R}_-
\]
and let \( G \) be given by
\[
G(B) = \frac{1}{2} F^{(1)}(B \cap \mathbb{R}_+) + \frac{1}{2} F^{(2)}(B \cap \mathbb{R}_-)
\]
Choose \( \theta \) such that
\[
\sigma^2 = \int_0^\infty x^2 \exp(-\theta x^2) \, dx.
\]

Define \( (a_n) \) by
\[
a_n = \int_0^\infty x^2 \exp(\frac{1}{2} \theta x^2) \, dx.
\]

Hence \( a_n \to \infty \) as \( n \to \infty \). We shall prove that the triangular array \((F_{k,n})\) with \( F_{k,n} = U_{a_n} G, \ k = 1, 2, \ldots, n \) is uan and satisfies conditions (i) and (ii) of Theorem 6.2.4 with \( M = 0 \) and hence that \( X \) is of the form (6.5).

The uan property. Let \( \epsilon > 0 \) be fixed. Observe that
\[
F_{k,n}(\{ |x| \geq \epsilon \}) = U_{a_n} G(\{ |x| \geq \epsilon \}) = e^{-\frac{1}{2} \theta (\epsilon^2 - \epsilon^2)}.
\]

Hence \((F_{k,n})\) tends to zero outside every neighbourhood of the origin, uniformly in \( k \), as \( n \to \infty \) and so \((F_{k,n})\) is uan.

Condition (i) of Theorem 6.2.4. Let \( \epsilon > 0 \) be fixed. Observe that
6.4 Sums of shrunk random variables

\[ \sum_{k=1}^{n} F_{k,n}(1 \mid x \geq \varepsilon) = nU_{\alpha\lambda}(G(1 \mid x \geq \varepsilon)) = \int_{e^{-\alpha(x+\lambda)} \geq \varepsilon}^{\infty} e^{-\alpha(x+\lambda)} d\lambda, \]

which tends to zero as \( n \to \infty \). Hence \( X \) is infinitely divisible with \( M = 0 \).

**Condition (ii) of Theorem 6.2.4.** Let \( \varepsilon > 0 \) be fixed. Observe that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int_{\varepsilon}^{\infty} x^2 \, dF_{k,n}(x) = \int_{\varepsilon}^{\infty} x^2 \, d(1 - e^{-x}) = \int_{\varepsilon}^{\infty} x^2 \, d(1 - e^{-x}) = o(\varepsilon^2) \text{ as } n \to \infty.
\]

Hence condition (ii) is also satisfied.

Next, let \( X \) be a random variable in \( S_{\alpha}(R) \) without normal component, i.e., \( \phi = [\alpha, \sigma^2, M] \). By Theorem A.3.2 of the Appendix, \( X \) can be obtained as a limit of the form (6.5).

We have now proved that for any \( \phi \in S_{\alpha}(R) \) with \( \phi = [\alpha, \sigma^2, M] \) we can find distribution functions \( F_{k,n}^{(1)} = U_{\alpha\lambda}G \) and \( F_{k,n}^{(2)} = U_{\alpha\lambda}F_{k}^{(2)} \) such that

\[
\prod_{k=1}^{n} F_{k,n}^{(1)} \to \phi^{(1)} := [0, 2\alpha^2, 0],
\]

\[
\prod_{k=1}^{n} F_{k,n}^{(2)} \to \phi^{(2)} := [2\alpha, \sigma, 2M].
\]

Let

\[ F_{k} := \frac{1}{2}(G + F_{k}^{(2)}) \]

It can be verified that the sequence \( (F_{k}) \) is unbounded and that \( (U_{\alpha\lambda}F_{k}) \) is un. Let \( \phi_{k,n} \) be the characteristic functions of \( U_{\alpha\lambda}F_{k} \). In view of Theorem 6.2.4

\[ \prod_{k=1}^{n} \phi_{k,n} \to \phi := [\alpha, \sigma^2, M]. \]

\[ \square \]

**Theorem 6.4.3.** Let \( X \) have characteristic function \( \phi \). Then \( \phi \) is infinitely divisible, i.e., \( \phi = [\alpha, \sigma^2, M] \) with either \( M = 0 \) and \( \sigma^2 \geq 0 \), or \( \sigma^2 = 0 \) and

\[ M(x) = \begin{cases} 
- \frac{C_1}{\alpha} e^{\rho x^2} & x > 0 \\
\frac{C_2}{\alpha} e^{-\alpha |x|^\alpha} & x < 0
\end{cases} \]

with \( \alpha > 0 \), \( \rho > 0 \), \( C_1 > 0 \), \( C_2 > 0 \) and \( C_1 + C_2 > 0 \) if and only if there exists a sequence \( (\theta_n) \) such that

\[ \sum_{k=1}^{n} U_{\alpha\lambda\theta} X_k + \theta_n \to X \text{ as } n \to \infty. \]
where $(t_n)$ is a non-negative, non-decreasing sequence and $(X_k)$ is an independent not bounded sequence of identically distributed random variables such that the triangular array $(U_{n,k}, X_k)$ is uan.

**Proof.** In this proof we shall frequently use the results of the Appendix. From the proof of Theorem 6.4.2, we know that $U_{n,k}$ satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Let $\Phi = [a_1, a_2, M]$ be as in the theorem. If $\Phi = [a_1, a_2, 0]$, then by the proof of Theorem 6.4.2, $X$ is of the form (6.5), with $(X_k)$ identically distributed. Theorem A.4.1 of the Appendix proves the “only if” part for $\Phi = [0, 0, M]$.

Suppose $X$ can be obtained as a limit as described in the theorem. By Theorem 6.4.2, $X$ is infinitely divisible with $\Phi = [a_1, a_2, M]$. By Theorem A.4.1, there exists a semigroup homomorphism $q$ from $(\mathbb{R}_+, \Theta)$ to $(\mathbb{R}_+, \cdot)$, with $\Phi$ defined in the proof of Theorem 6.4.2, such that

$$M(B) = q(1) U_{n,k} M(B).$$

(6.9)

Let $g(x) := q(x^{1/\alpha})$. Then since $q$ is a homomorphism,

$$g(\lambda x)g(y) = q((\lambda x)^{1/\alpha})q(y^{1/\alpha}) = q((\lambda x + y)^{1/\alpha} = g(x + y).$$

Since $g(x) \geq 1$ and $g(0) = 1$ then $g(x) = \exp(px)$ and so

$$q(x) = \exp(px^\alpha),$$

for some $p \geq 0$. If $p = 0$, then by repeated use of (6.9) we have

$$M(B) = U_{n,k} M(B).$$

Letting $n \to \infty$ we see that $M = 0$ and hence $\Phi$ is normal. Now let $p > 0$ and $B = (a^\alpha, b^\alpha)$ with $0 < a \leq b, s \leq t$ and let $w(x) := M(x^{1/\alpha})$. By (6.9)

$$\frac{w(a) - w(b)}{a - b} = e^{px} \frac{w(a + s) - w(b + s)}{(a + s) - (b + s)}.$$

We know from Theorems 6.4.2 and 5.4.7 that $M'$ exists. Letting $b \to a$ we obtain that

$$e^{px} w'(a) = e^{px} w'(a + s) = q,$$

with $q$ a non-negative constant. Hence $w(x) = ce^{-\alpha x}$ and therefore

$$M(x) = w(x^\alpha) = ce^{-\alpha x^\alpha},$$

with $c \in \mathbb{R}_+$. Likewise for $0 > b \geq a$. Hence $M$ is of the desired form.

It remains to be shown that if $M$ is not identically zero then $\alpha \neq 0$. Let $F$ be the distribution function of $(X_k)$ and let $M_n := nU_{n,k} F$. Also define

$$R_n(s) := M_n(\{1 \leq x > s\}) := nU_{n,k} F(\{1 \leq x > s\}).$$
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Let $t$ be such that $R_{\alpha}(s)$ is non-zero in a neighbourhood of $t$ and let $(k(n))$ be a sequence such that $(a_{n}^{\alpha} - s_{n}^{\alpha})^{1/\alpha} \rightarrow t$ as $n \rightarrow \infty$ (this is possible by Lemma A.3.1 of the Appendix). Observe that

$$R_{\alpha}(s) = n (k(n)/n) U_{\alpha, s} F \left( \left\{ x \mid 1 > s \right\} \right)$$

$$= (k(n)/n) M_{\alpha} \left( \left\{ x \mid 1 > (a_{n}^{\alpha} + s_{n}^{\alpha} - s_{n}^{\alpha})^{1/\alpha} \right\} \right) = (k(n)/n) R_{\alpha}(a_{n}^{\alpha} + s_{n}^{\alpha} - s_{n}^{\alpha})^{1/\alpha}.$$

Since $M_{\alpha} \rightarrow M$, $R_{\alpha}(x)$ converges for all $\alpha > 0$ to a not identically zero limit, $R(x)$ say, and we must have $\lim_{n \rightarrow \infty} k(n)/n = c < \infty$. Integrating by parts we have

$$\int_{x \leq \varepsilon} x^{2} dM_{\alpha}(x) = -\varepsilon^{2} R_{\alpha}(\varepsilon) + 2 \varepsilon R_{\alpha}(x^{\alpha} + s_{n}^{\alpha} - s_{n}^{\alpha})^{1/\alpha} ds.$$

Letting $n \rightarrow \infty$ we have

$$\lim_{n \rightarrow \infty} \int_{1 \leq x} x^{2} dM_{\alpha}(x) \leq 2 c \int_{0}^{\varepsilon} s R((s^{a} + t^{b})^{1/\alpha}) ds \leq 2 c R(\varepsilon),$$

which tends to zero as $\varepsilon \rightarrow 0$. Hence, by Theorem 6.2.4, $\alpha_{n}^{2} = 0$. \qed

6.4.2 $T_{\alpha, \tau}$-shrunken random variables.

Let the random variable $X$ have distribution function $F$ and denote by $F_{\tau}$ the distribution function of $T_{\alpha, \tau} X$. Let $I = (a, b)$, $0 < a < b$. We then define $F_{\tau}$ by

$$F_{\tau}(t) = F(T_{\alpha, \tau} X \in I) = t^{a} F(X \in T_{\tau}^{-1}(I)) = t^{a} F(T_{\tau}^{-1}(I)).$$

We therefore let the operator $T_{\alpha, \tau}$ operate on set functions by $T_{\alpha, \tau} F(I) = t^{a} F(T_{\tau}^{-1}(I))$, with $T_{\tau}$ defined in Section 1.4. Hence for any Borel set $B$ of $\mathbb{R} \cap (-\varepsilon, \varepsilon)$, $c > 0$,

$$M(B) \geq c^{2} T_{\alpha, \tau} M(B) \text{ if and only if } M(B) \geq T_{\alpha, \tau} M(B). \quad (6.10)$$

We are now ready to prove

**Theorem 6.4.4.** Let $X$ have characteristic function $\phi$. Then $\phi \in S_{\alpha}(\mathbb{R})$ ($\alpha \geq 0$) if and only if there exists a sequence $(b_{n})$ such that

$$\sum_{k=1}^{n} T_{\alpha, \tau} X_{k} + b_{n} \rightarrow X \quad \text{as } n \rightarrow \infty, \quad (6.6)$$

where $(\alpha_{n})$ is a non-negative, non-increasing sequence with $t_{1} \leq 1$, $(X_{k})$ is an independent unbounded sequence of random variables and the random operators $T_{\alpha, \tau}$ are independent and distributed as $T_{\alpha, \tau}$. 

PROOF. In this proof we shall frequently use the results of the Appendix. First we shall show that \( T_{\alpha, t} \) satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix. Fix \( \alpha > 0 \). Let \( U_t = T_{\alpha, t}, f_t = T_{1, t}, t \in [1, \infty) \) and \( p(t) = t^{-\alpha} \). We have
\[
f_s \circ f_t(x) = f_t(t^{-1} x) = s^{-1} t^{-1} x = f_s \circ t^{-1}(x),
\]
where \( s \circ t \) is defined by
\[
s \circ t = t^{-1} s.
\]
Obviously \( f_0(x) = x \) and so \( f_0 \circ f_t = f_t \). Hence \( S = \{(f_t)_{t \in [1, \infty)}, \circ \} \) is a composition semigroup with respect to the semigroup \( \{\langle 1, \infty \rangle, \circ \} \). If \( t \geq s \) then
\[
|f_t(x)| = t^{-1} x \leq s^{-1} x \leq |f_s(x)| \leq |x|,
\]
and so \( f_t \) in fact does shrink its arguments. From the definition of \( T_t \) (cf. Section 1.4) it follows that \( f_t(x) \) is continuous in both \( t \) and \( x \) \( (f_t(x) \) is differentiable in both \( t \) and \( x \) on \( (0, \infty) \)), that \( f_t \) is unbounded \( (f_t(x) \rightarrow \infty \text{ as } x \rightarrow \infty) \) and that \( f_t \) is one-to-one on \( (0, \infty) \).

The operator \( T_{\alpha, t} \) thus satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Suppose \( X \) is obtained by (6.6). Observe that
\[
\mathbb{P}(|X_{k,n}| \geq \varepsilon) = \mathbb{P}(|T_{\alpha, n} X_k| \geq \varepsilon) = \int_0^\infty \mathbb{P}(|T_{\alpha, n} Y_k| \geq \varepsilon) \leq \mathbb{P}(|T_{\alpha, n} X_k| \geq \varepsilon).
\]
Since the sequence \( (T_{\alpha, n} X_k) \) is uan, \( (T_{\alpha, n} X_k) \) is uan. Hence \( X \) is infinitely divisible and so \( \phi = \{\alpha \sigma_0^2, \alpha \sigma_0^2, M \} \) by (6.10) and Corollary 1 to Theorem A.3.2 in the Appendix. \( M \) satisfies the inequality of Lemma 5.4.5 (ii). By Lemma 5.4.5 and Theorem 5.6.7, \( \phi \in S_\alpha(\mathbb{R}) \).

The converse is proved in three steps. First it is shown that the symmetric normal distribution is a limit as described in the theorem; secondly it is shown that a random variable without normal component is of the desired type and finally these two results are combined to prove the theorem. The last step is identical with the last step of the proof of Theorem 6.4.2 and therefore omitted.

Let \( X \) be a symmetric normal random variable, i.e., \( \phi = \{0, \sigma_0^2, 0\} \). Let \( Y \) be a unbounded random variable on \( \mathbb{R} \) with finite second moment; and let it have distribution function \( G \). Let
\[
\sigma_0^2 = \int_{-\infty}^{\infty} x^2 dG(x).
\]
Let \( (t_\alpha) \) be defined by
\[
t_\alpha = \alpha^{-1}(\alpha + 2).
\]
We shall prove that the triangular array \( (F_{k,n}) \) with \( F_{k,n} = T_{\alpha, t_\alpha} G \), \( k = 1, 2, \ldots, n \) is uan and satisfies conditions (i) and (ii) of Theorem 6.2.4 with \( M = 0 \) and hence that \( X \) is of the form (6.6).
6.4 Sums of shrunken random variables

The un property. Let \( \varepsilon > 0 \) be fixed. Observe that

\[
F_{k,\varepsilon}((- |x| \geq \varepsilon)) = T_{n,\varepsilon} G((- |x| \geq \varepsilon)) \leq n^{-2(\alpha+2)}.
\]

Hence \((F_{k,\varepsilon})\) tends to zero outside every neighbourhood of the origin, uniformly in \( k \), as \( n \to \infty \) and so \((F_{k,\varepsilon})\) is uan.

Condition (i) of Theorem 6.2.4. Let \( \varepsilon > 0 \) be fixed. Observe that

\[
\sum_{k=1}^{n} F_{k,\varepsilon}((- |x| \geq \varepsilon)) = n T_{n,\varepsilon} G((- |x| \geq \varepsilon)) = n^{2(\alpha+2)} G((- |x| \geq n^{1(\alpha+2)} \varepsilon)).
\]

Since \( Y \) has finite second moment, \( \lim_{n \to \infty} x^2 (1 - G(ex)) = 0 \) for \( \varepsilon > 0 \). Hence (6.11) tends to zero as \( n \to \infty \). So \( X \) is infinitely divisible with \( M = 0 \).

Condition (ii) of Theorem 6.2.4. Let \( \varepsilon > 0 \) be fixed. Observe that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int_{|x| \leq \varepsilon} x^2 dF_{k,\varepsilon}(x) = n \int_{0}^{\varepsilon} y^2 dG(y) \rightarrow \int_{0}^{\infty} y^2 dG(y) = \alpha_2 \quad \text{as} \quad n \to \infty.
\]

Hence condition (ii) is also satisfied.

Next, let \( X \) be a random variable without normal component, i.e., \( \phi = [\alpha_2, 0, M] \). By Theorem A.3.2 of the Appendix, \( X \) can be obtained as a limit of the form (6.6).

\[\square\]

Theorem 6.4.5. Let \( X \) have characteristic function \( \phi \). Then \( \phi \) is stable if and only if there exists a sequence \((b_n)\) such that

\[
\sum_{k=1}^{n} T^{(k)}_{1/4} X_k + b_n \rightarrow X \quad \text{as} \quad n \to \infty,
\]

where \((a_n)\) is a non-increasing sequence in \((0,1)\), \((X_k)\) are independent not bounded identically distributed random variables and random operators \(T^{(k)}_{1/4}\) are independent and distributed as \(T_{a_n,1}\).

Proof. In this proof we shall frequently use the results of the Appendix. From the proof of Theorem 6.4.4, we know that \( T_{a_n,1}, \ t \in (0,1) \) satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Let \( \phi = [\alpha_2, \sigma_2^2, M] \) be stable. If \( \phi = [\alpha_2, 0, 0] \), then by the proof of Theorem 6.4.4, \( X \) is of the form (6.6), with \((X_k)\) identically distributed. Theorem A.4.1 of the Appendix proves the "only if" part for \( \phi = [0, 0, M] \).
Suppose X can be obtained as a limit as described in the theorem. Since $(T_{n,x}, X_n)$ is as in (cf. proof of Theorem 6.4.4) X is infinitely divisible with characteristic function $\phi$ of the form $\phi = [\sigma_0, \sigma_1^2, \mu]$. From Theorem 6.2.4 we have
\[
\lim_{n \to \infty} \sum_{k=1}^{n} T_{n,k} F_k = \lim_{n \to \infty} n T_{n} F = M,
\]
outside every neighborhood of the origin. Also
\[
\lim_{n \to \infty} \lim_{k \to \infty} \sum_{k=1}^{n} \int \limits_{[x]} x^2 dN_{T_{n,k}} F_k(x) = \lim_{n \to \infty} \lim_{k \to \infty} \sum_{k=1}^{n} \int \limits_{[x]} x^2 dN_{T_{n,k}} F(x) = \sigma_0^2.
\]
Obviously, if $n T_{n}^2$ is bounded then $M = 0$ and so $\phi$ is normal. Suppose $M$ is not identically zero, then $n T_{n}^2 \to \infty$ as $n \to \infty$. Let $N(n) = n T_{n}^2$. Then
\[
M = \lim_{n \to \infty} N(n) T_{n} F,
\]
outside every neighborhood of the origin and
\[
\sigma_0^2 = \lim_{n \to \infty} \sum_{k=1}^{n} \int \limits_{[x]} x^2 dN_{T_{n,k}} F(x).
\]
By Lemma A.3.1 of the Appendix, $t_{n+1}/t_n \to 1$ as $n \to \infty$. Observe that (cf. (6.12))
\[
(N(n+1) - N(n)) T_{n} F = ((n+1)T_{n+1}^2/nT_{n}^2 - 1)N(n) T_{n} F \to 0
\]
as $n \to \infty$ outside every neighborhood of the origin. Similarly
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int \limits_{[x]} x^2 dN_{T_{n,k}} F(x) = 0.
\]
For every $l \in \mathbb{N}_+$ there exists an $n \in \mathbb{N}_+$ such that $N(n) \leq l < N(n+1)$. Let $(t')_l^{n+1}$ be defined by
\[
t_l' = t_n, \quad N(n) \leq l < N(n+1), \quad l = 1, 2, 3, \ldots.
\]
For any $l \in \mathbb{N}_+$ let $n$ be such that $N(n) \leq l < N(n+1)$. Then (cf. (6.12) and (6.13))
\[
\lim_{l \to \infty} \sum_{k=1}^{l} T_{k} F = \lim_{l \to \infty} l T_{l} F = \lim_{l \to \infty} (N(n) T_{l} F + (l - N(n)) T_{n} F) = M,
\]
outside every neighborhood of the origin. Similarly
\[
\sigma_0^2 = \lim_{l \to \infty} \sum_{k=1}^{l} \int \limits_{[x]} x^2 d l T_{k} F(x).
\]
By Theorem 6.2.4 we see that
\[
\sum_{k=1}^{l} T_{k} X_k \to X \text{ as } l \to \infty,
\]
where $(X_k)$ are independent and identically distributed with distribution function $F$. Hence $X$ is stable (cf. Section 1.4).
Remark 6.4.6. We can also use Theorem A.4.1 to prove the "if" part of Theorem 6.4.5. This proof however must be split into two steps. The first (which give below) shows that \( M \) is of the desired form. In the second step it is shown that if \( M \) is not identically zero then \( \sigma_2^2 = 0 \). This second step is quite involved (cf. proof of Theorem 6.4.3) and therefore omitted.

Suppose \( X \) can be obtained as a limit as described in the theorem. From the proof of Theorem 6.4.4 we know that \( X \) can be normal. By Theorem A.4.1 there exists a semigroup homomorphism \( q \) from \((1, \infty), \oplus \) to \((1, \infty), \cdot \) with \( \oplus \) defined in the proof of Theorem 6.4.4, such that
\[
M(B) = q(s) r^{(q)} T_v M(B).
\]
(6.14)

Since \( q(\gamma) \geq 1 \) and \( q(1) = 1 \) we have that \( q(\gamma) = x^\gamma \) for some \( \gamma \geq 0 \). If \( \gamma = 0 \) then by repeated use of (6.11) we see that \( M \equiv 0 \). Let \( \gamma > 0 \) and \( B = (a, b) \) with \( 0 < a \leq b \). By (6.14)
\[
\frac{M(a) - M(b)}{a - b} = r^{-\alpha} M(at, bt) - M(b).
\]

We know from Theorems 6.4.4 and 5.4.7 that \( M' \) exists. Letting \( b \to a \) we obtain that
\[
a^{-\alpha - 1} M'(a) = (at)^{-\alpha + 1} M'(at) = c_1,
\]
with \( c_1 \) a non-negative constant. Hence
\[
M(x) = c x^{-\alpha - \gamma},
\]
with \( c \in \mathbb{R} \). Since \( M \) is a Lévy spectral function if and only if \( (\alpha - \gamma) < (0, 2) \) we must have \( \gamma = (\alpha, \alpha + 2) \). Likewise for \( b > a \). Hence \( M \) is the Lévy spectral function of a stable distribution.

Remark 6.4.7. The unbounded condition on \( X_\lambda \) in the theorems of this section is assumed, since if \( (X_\lambda) \) were bounded and \( t_n \to 0 \) \( (\lambda \to \infty) \) then the limit in (6.6) ((6.5)) would be degenerate.

6.5 Subsets of the set of self-decomposable limit laws

In this section we consider the subsets \( S_\alpha(\mathbb{R}) \), \( \alpha \in (-2, 0] \) of the self-decomposable limit laws. We show that random variables \( X \) whose characteristic functions are in \( S_\alpha(\mathbb{R}) \) can be written as (1.4), where \( (X_\lambda) \) are independent and blockwise identically distributed random variables. This result in some sense, the counterpart of Theorem 6.4.2 for \( \alpha \in (-2, 0] \).

Let \( Y(t) \) be a stochastic process with independent stationary increments with \( Y(0) = 0 \) with probability one (cf. Feller (1971)) and let \( (X_\lambda) \) be a sequence of independent random variables, all distributed as \( Y(t) \). Then
\[ Y(s) = \sum_{k=1}^{s} X_k, \quad s \in \mathbb{R}_+ , \]

where a non-integer sum of the \( X_k \) is defined by the non-integer power of its characteristic function. Let the stochastic operator \( T^\alpha_{\infty} \) act on infinitely divisible random variables \( Y(1) \) by

\[ d \quad T^\alpha_{\infty} Y(1) = Y(t^\alpha) . \]

Hence, for any infinitely divisible random variable \( X \) we have that

\[ d \quad T^\alpha_{\infty} X = \sum_{k=1}^{\infty} t^\alpha X_k , \]

where \( (X_k) \) are independent random variables distributed as \( X \). If \( \alpha \in \mathbb{R}_+ \) and \( t \in (0, 1] \), then \( t^\alpha \geq 1 \). As \( T^\alpha_{\infty} X \) produces \( t^\alpha \) copies of \( X \) we can interpret \( T^\alpha_{\infty} \) with \( \alpha \in \mathbb{R}_+ \) and \( t \in (0, 1] \), as a stochastic ‘breeding’ operator. Let \( (s_n) \) be a sequence of non-increasing real numbers with \( s_1 = 1 \) and \( s_n \to 0 \) as \( n \to \infty \). Theorem 6.3.6 then states that if \( X \) has a characteristic function in \( S_c(\mathbb{R}) \), \( \alpha \in (-2, 0] \) then there exists a sequence \( X_j \) of independent infinitely divisible random variables such that

\[ \sum_{j=1}^{n} T^\alpha_{s_j} X_j \overset{w}{\to} X . \quad (6.15) \]

Hence \( X \) is the limit of a normed sum of blockwise identically distributed random variables. The size of the first block is \( r^\alpha \), the second block \( r^\alpha \), and so forth. Suppose \( \alpha < 0 \) and \( r^\alpha = j^{2/\alpha} \) (cf. Corollary 1 to Theorem 6.3.6), then (6.15) can be rewritten as

\[ \sum_{j=1}^{n} T^\alpha_{s_j} \left( \sum_{k=1}^{j} X_{k} \right) \overset{w}{\to} X as n \to \infty , \]

where \( (X_{k}) \) are independent and \( X_{k} \), \( k = 1, 2, \ldots, j \), are identically distributed. Theorem 6.3.3 (iii) states that a random variable \( X \) has a characteristic function in \( S_c(\mathbb{R}) \), \( \alpha > -2 \), if and only if there exists a sequence \( (X_j) \) of independent infinitely divisible random variables such that

\[ \sum_{j=1}^{n} T^\alpha_{s_j} X_j \overset{w}{\to} X . \quad (6.16) \]

Thus (6.16) is a limit of a sum of a triangular array as in Theorem 6.2.4.

To eliminate the condition that \((X_k)\) be infinitely divisible we introduce a new stochastic ‘breeding’ operator \( T^\alpha_{\infty} \), as follows.

\[ d \quad T^\alpha_{\infty} X = \sum_{k=1}^{\lfloor x \rfloor} t^\alpha X_k , \quad t \in (0, 1] , \quad \alpha \in (-2, 0] , \]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \) and \( (X_k) \) are independent random variables distributed as \( X \). The following theorem gives us another classification of the \( S_{\alpha}(\mathbb{R}) \) limit laws. For a proof we refer to Theorem A.5.1 of the Appendix.
6.5 Subsets of the set of self-decomposable limit laws

**Theorem 6.5.1.** Let \( \alpha \in (-2,0) \) and let \( X \) have characteristic function \( \phi \). Then \( \phi \in \mathcal{S}_\alpha(\mathbb{R}) \) if and only if there exists a sequence \( (b_n) \) such that

\[
\sum_{k=1}^{n} T_{\alpha, k} X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty,
\]

where \( (t_n) \) is a non-negative, non-increasing sequence and \( (X_k) \) are independent not bounded random variables such that the sequence \( (T_{\alpha, k} X_k) \) is independent and \( \alpha \)-stable.

The set \( \mathcal{S}_\alpha(\mathbb{R}) \) thus contains characteristic functions \( \phi \) whose random variables \( X \) are weak limits of normed sums of block-wise identically distributed random variables. The size of these blocks is \( [\sqrt{n}] \). For \( \alpha = 0 \), the blocksize is one and the limit in Theorem 6.5.1 reduces to the limit defining classical self-decomposable random variables (cf. Section 1.4).

6.6 Remarks and comments

Let \( (Y_k(i)) \) be a sequence of stochastic processes with independent stationary increments such that \( Y_k(0) = 0 \) with probability one (cf. Feller (1971)) and let \( X_k \) be distributed as \( Y_k(1) \). Rewriting (6.15) in terms of stochastic processes with independent stationary increments we get and letting \( t_n = 1/n \),

\[
\frac{Y_1(n^{-\alpha}) + Y_2(n^{-\alpha}) + \ldots + Y_n(n^{-\alpha})}{n} \xrightarrow{w} X \quad \text{as } n \to \infty.
\]

Note that \( n^{-1} Y_i(n^{-\alpha}) = T_{\alpha, i} Y(1) \), with \( T_{\alpha, i} \) defined in Section 6.5. The other theorems of Section 6.3 can also be reformulated in terms of limit distributions of sums of \( T_{\alpha, i} \)-shrunken (or \( T_{\alpha, i}^* \)-shrunken) stochastic processes with independent stationary increments.

The shrinking operators introduced in Section 6.4 do have some practical justification. One could for example imagine a situation where a signal \( X \) must be measured. \( U_{\alpha, i} \) can then be interpreted as follows: if the signal is too small then our instruments can not register the signal and if the signal is registered, then we only measure a fraction of the strength of the true signal. Likewise, we can interpret \( T_{\alpha, i} \) as: with a certain probability we do not receive the signal, and if the signal is received then we only measure a fraction of the strength of the true signal.
APPENDIX

LIMIT DISTRIBUTIONS OF SUMS OF SHRUNKEN RANDOM VARIABLES

A.1 Introduction

In this appendix we consider random variables $X$ obtained as limits of sums of random variables as follows:

$$\sum_{k=1}^{n} U_{i,k}^{(k)} X_{k} \to X \text{ as } n \to \infty,$$

where $(X_{k})$ are independent, unbounded random variables, the triangular array $(U_{i,k}^{(k)} X_{k})$ is i.i.d., and $(U_{i,k}^{(k)})_{i=1}^{\infty}$ are mutually independent and independent of $(X_{k})$ and all distributed as $U_{i}$, with $U_{i}$ a 'generalized stochastic shrinking' operator defined by

$$U_{i}X = Y_{i} f_{i}(X),$$

where $X$ and $Y_{i}$ independent and $\mathbb{P}(Y_{i}=1)=p(i)=1-\mathbb{P}(Y_{i}=0)$. Let $\delta_{0}$ be a distribution function with total mass at zero and let $B$ be a Borel set. The distribution function of the random variable $U_{i}X$ is given by

$$F_{U_{i}X}(B) := p(i) \mathbb{P}(f_{i}(X) \leq B) + (1-p(i)) \delta_{0}(B)$$

$$= p(i) \mathbb{P}(X \leq \{x \mid f_{i}(x) \in B\}) + (1-p(i)) \delta_{0}(B).$$

As special cases of $U_{i}$ we have the shrinking operators $U_{a,i}$ and $T_{a,i}$ introduced in Section 6.4. The assumptions on the 'shrinking functions' $f_{i}$, the norming sequence $(\alpha_{a})$ and the probabilities $p(i)$ are stated in Section A.2. In Section A.3 we characterize the random variables $X$ in (A.1) in terms of an inequality on the Lévy spectral function of $X$. The special case where $(X_{k})$ are identically distributed is studied in Section A.4. In Section A.5 we give a classification of the set of random variables of the form (A.1) and in Section A.6 we list a few examples.
A.2 Preliminaries

Let \( f_t \) and \( f_t^{-1} \) act on Boel sets \( \mathcal{B} \) by
\[
 f_t \mathcal{B} = \{ y \mid f_t(x) = y \text{ for } x \in \mathcal{B} \} \quad \text{and} \quad f_t^{-1} \mathcal{B} = \{ x \mid f_t(x) \in \mathcal{B} \}. \tag{A.4}
\]
We begin by listing the assumptions which we make on \( f_t \), \( t_n \) and \( p(t) \) in (A.1) and (A.2).

ASSUMPTION A.2.1. We make the following assumptions on \( U_k X_k \):

(i) \( (X_k) \) are independent, unbounded random variables;
(ii) \( (U_k)_{k=1}^K \) are mutually independent and independent of \( (X_k) \);
(iii) The triangular array \( (U_k^{(i)} X_k) \) is uan;
(iv) The norming sequence \( (t_n) \) is non-decreasing.

ASSUMPTION A.2.2. We make the following assumptions on \( f_t \):

(i) \( S = (f_t)_{t \in [0,\infty) \setminus \{0\}} \) is a commutative composition semigroup with respect to the semigroup \( (\{0,\infty\}, \oplus) \), i.e.,
   - For all \( t, s \in [0,\infty) \), \( f_t \circ f_s = f_{t \oplus s} \);
   - For all \( s \in (0,\infty) \), \( f_0 \circ f_s = f_s \).

(ii) \( f_t \) are shrinking operators, i.e.,
   - For all \( t, s \in [0,\infty) \) with \( t > s \), \( |f_t(x)| < |f_s(x)| \), \( x \in \mathbb{R} \);
   - For all \( x \in \mathbb{R} \), \( \lim_{t \to \infty} f_t(x) = 0 \).

(iii) \( f_t(x) \) is continuous in both \( t \) and \( x \), unbounded in \( x \) and for any interval \( I \), not containing zero, \( f_t(f_t^{-1}(I)) = I \).

ASSUMPTION A.2.3. We make the following assumptions on \( p(t) \):

(i) \( p \) is a semigroup homomorphism from \( (\{0,\infty\}, \oplus) \) to \( (0,1), \), i.e.,
   \[ p(s \oplus t) = p(s) p(t) \text{ for all } s, t \in [0,\infty). \]
Assumption A.2.1 (iii) ensures that the limit in (A.1) exists, namely an infinitely divisible random variable. Part (i) is assumed, since if \((X_n)\) was bounded and \(t_n \to \infty\) with \(f_{-\infty} = 0\), then (A.1) would have a degenerate limit. Requirement (iv) is equivalent with the assumption that \((t_n)\) be monotone \((t_n)\) non-increasing implies that \((t_n)\) with \(s_n = t_n^{-1}\) is non-decreasing) which is a normal assumption in central limit problems.

Assumption A.2.2 (ii) states that \(f_t\) in fact does shrink its arguments and provides an ordering of \(\{f_t\}\). Conditions (i) and (iii) of Assumption A.2.2 as well as Assumption A.2.3 are essential for solving the limit problem on hand.

As a consequence of our assumptions we have the following lemma.

**Lemma A.2.4.** Let \(\{f_t\}_{t \in [0, \infty)}\) and \(p(t)\) satisfy Assumptions A.2.2 and A.2.3, respectively. Then

\[
\begin{align*}
(1) & \text{ For all } s, t \in [0, \infty) \text{ with } t \geq s \text{ and any interval } I \text{ not containing zero: } \\
& f_t \circ f_s^{-1}(I) = f_{t \oplus s}(I) \text{ with } t \oplus s \in [0, \infty); \\
(2) & \text{ } p(t) \text{ is continuous, non-increasing and non-zero on } [0, \infty) \text{ with } p(0) = 1 \text{ and for all } t, s \in [0, \infty) \text{ with } t \geq s \\
& p(t \oplus s) = p(t)/p(s).
\end{align*}
\]

**Proof.** We first prove part (i). Let \(s, t \in [0, \infty)\) be arbitrary with \(t \geq s\). We can always select a \(y \in [0, \infty)\) such that

\[|f_t(y)| \geq |f_t(x)|,\]

for all \(x \in \mathbb{R}\) and \(x \oplus y \leq t\). By Assumption A.2.2 (ii) it is impossible to have \(w \leq t\) and \(|f_w(x_1)| \geq |f_t(x_1)|\) and \(|f_w(x_2)| < |f_t(x_2)|\) for some \(x_1, x_2 \in \mathbb{R}\). Hence, by the continuity of \(f_t\) in \(t\), there exists a unique \(y \in [0, \infty)\) such that \(f_t \circ f_s^{-1}(x) = f_t(x) := f_{t \oplus s}(x)\), for any \(x \neq 0\).

For part (ii) let \(y = t \oplus s\) and hence \(t = y \ominus s\). Observe that (cf. Assumption A.2.3)

\[p(t \oplus s)p(s) = p(y \ominus s) = p(t).\]

\[\square\]

If \(p(t_n) \to c > 0\) as \(n \to \infty\), then upon replacing \(X_n\) in (A.1) by \(Y_n = ZX_n\), with \(\mathbb{P}(Z = 0) = 1 - \mathbb{P}(Z = 1) = 1 - c\) and \(p(s)\) by \(p^*(s) = 1\), we obtain the same limit problem. We therefore make the following and last assumption.
A.2 Preliminaries

**Assumption A.2.5.** $p(t)$ is either constantly one or $p(t)$ is non-increasing with \( \lim_{t \to \infty} p(t) = 0. \)

We conclude this preliminary section with a lemma and a notation

**Lemma A.2.6.** If \( F_n \to F \) and \( t_n \to t \) as \( n \to \infty \), then \( U_{t_n} F_n \wedge U_t F \) as \( n \to \infty \).

**Proof.** Let \( x_n \to x \), then by the continuity of \( f_t(x) \) in both \( t \) and \( x \), \( U_{t_n} x_n \to U_t x \). The lemma now follows from Theorem 5.5 in Billingsley (1968).

**Notation A.2.7.** Let \( U(S, p) \) denote the set of characteristic functions whose random variables can be described as limits of the form (A.1), under Assumptions A.2.1, A.2.2, A.2.3 and A.2.5.

A.3 A characterization of \( U(S, p) \)

Let the operators \( U_t \) and \( U_t^{-1} \) act on set functions by

\[
U_t F(B) = p(t) F(f_t^{-1} B) \quad \text{and} \quad U_t^{-1} F(B) = p(t)^{-1} F(f_t B).
\]

Let \( t, s \in [0, \infty) \) with \( t \geq s \). From Assumptions A.2.2 (i) and Lemma A.2.4 we have

\[
U_t U_t^{-1} F(B) = p(t) p(s) F((f_t \circ f_s^{-1}) B) = p(t \wedge s) F((f_t \circ f_s^{-1})^{-1} B) = p(t \wedge s) F(f_t f_s^{-1} B) = U_{t \wedge s} F(B),
\]

and that (cf. (A.5))

\[
U_t F(B) = p(t) F(f_t \circ f_s^{-1} \circ f_t^{-1} B) = p(t) p(s) F(f_t \circ f_s^{-1} B) = p(s) U_t U_t^{-1} F(f_t^{-1} B) = p(s) U_t U_t^{-1} = U_t U_t^{-1} F(B) = U_t U_t^{-1} F(B).
\]

Before proving the main theorem of this section, we prove a preparatory lemma.

**Lemma A.3.1.** Let \( \phi \) be infinitesimally divisible with \( \phi = [a_0, a_0^2, M] \). If \( \phi \in U(S, p) \) and if \( M \) is not identically zero, then

(i) \( t_n \to \infty \) as \( n \to \infty \);

(ii) \( \xi_n \equiv a_{n+1} \wedge t_n \to 0 \) as \( n \to \infty \).
PROOF. We first prove part (i). Suppose (i) is not true, then there exists \( t_0 \in (0, \infty) \) such that \( u \leq t_0 < u' \) for all \( u \in \mathbb{N}_+ \). By Assumptions A.2.2 (ii) and A.2.5 we have for each \( \varepsilon > 0 \),

\[
P(\{ U_{j_k}(X) \geq \varepsilon \} \geq p(t_0)P(\{ f_{j_k}(X) \geq \varepsilon \} \geq p(t_0))P(\{ f_{j_k}(X) \geq \varepsilon \} \geq p(t_0)) \cdot
\]

Since \( \{ U_{j_k}(X) \} \) is unif, then

\[
\sup_k P(\{ f_{j_k}(X) \geq \varepsilon \} ) = 0.
\]

From Assumption A.2.2 (iii), \( f_{i} \) is unbounded and hence \( (X_k) \) is bounded. This contradicts Assumption A.2.1 (i). Hence (i) holds.

To prove part (ii) suppose \( \tilde{f}_{i_0} \) has a limit point \( t_0 \in (0, \infty) \). Then there exists a subsequence \( (k(n)) \) such that \( \tilde{f}_{i_0(n)} \rightarrow t_0 \). By Theorem 6.2.1 it follows that outside every neighborhood of the origin,

\[
M_n := \sum_{k \geq 1} U_k F_k \rightarrow M \text{ as } n \rightarrow \infty. \tag{A.7}
\]

Let \( I \) be a continuity set of \( M \), with \( I \) and interval bounded away from the origin. From (A.7) and (A.6) it follows that

\[
M_{k(n)}(I) = U_{k(n)} F_{k(n)}(I) = U_{k(n)} \sum_{k \geq 1} U_{k(n)} F_{k(n)}(I) = U_{k(n)} \tilde{f}_{i_0(n)}(I) = U_{k(n)} M_{k(n)}(I). \tag{A.8}
\]

Letting \( n \rightarrow \infty \) in (A.8) we obtain, by Lemma A.2.6, (A.7) and the unif property of \( \{ U_k F_k \} \) that

\[
M(I) = U_{i_0} M(I) \text{.} \tag{A.9}
\]

By repeated use of (A.9) we see that

\[
M(I) = U_{i_0} M(I) \leq M(f_{i_0}(I)),
\]

for \( k \in \mathbb{N}_+ \). Applying Assumption A.2.2 (ii) yields \( M(I) < 0 \), which contradicts the assumption of the lemma.

\[\square\]

**Theorem A.3.2.** Let \( \phi \) be infinitely divisible with \( \phi = [a, \sigma, \lambda, M] \). Then \( \phi = U(S, \rho) \) if and only if for every \( t \in (0, \infty) \) and for every Borel set \( B \) of \( \mathbb{R}(-\sigma, \sigma) \), for any \( \varepsilon > 0 \), the Lévy spectral function \( M \) satisfies

\[
M(B) \geq U_{i_0} M(B).
\]

**Proof.** We first prove the "only if" part. If \( M \) vanishes identically then the "only if" part is trivial. Suppose \( M \) is not identically zero. From Lemma A.3.1 it follows that for any \( t \in (0, \infty) \) there exists a subsequence \( (k(n)) \) such that \( k(n) \geq n \) and...
A.3 A characterization of $U(S,p)$

\[ \tilde{\eta}_n \equiv \eta_k \otimes \iota_n \rightarrow I \quad \text{as} \quad n \rightarrow \infty. \quad \text{(A.10)} \]

For any continuity set $I$ of $M$, with $I$ an interval bounded away from the origin, we have (cf. (A.7) and (A.6))

\[
M_{\kappa^{(n)}}(I) = \sum_{k=1}^{n} U_{\kappa^{(n)}} F_k(I) \geq \sum_{k=1}^{n} U_{\kappa^{(n)}} \eta_k F_k(I) \\
= \sum_{k=1}^{n} U_k \eta_k F_k(I) = U_k M_k(I). 
\]

Letting $n \rightarrow \infty$ we have by Theorem 6.2.4, Lemma A.2.6 and (A.10) that $M$ satisfies the inequality of the theorem for $B = I$. Since the Borel sets on $\mathbb{R}(-\varepsilon, \varepsilon)$ are generated by the intervals bounded away from zero, the “only if” part is proved.

Conversely, suppose $M$ satisfies the inequality of the theorem. If $M$ vanishes identically then the “if” part is trivial. Suppose $M$ does not vanish identically. Suppose $(\tau_k)$ is a non-decreasing sequence satisfying conditions (i) and (ii) of Lemma A.3.1 and such that $\rho(\tau_k)^{-1} \in \mathbb{N}_e$ (this is possible by Assumption A.2.5). Define $L_\tau$ by

\[ L_\tau = M - U_\tau M. \]

Let $s_k = \tau_k \otimes \tau_{k-1}, k = 2, 3, \ldots \text{ and } s_1 = \infty$. Then $L_{s_1} = M$ and (cf. (A.7) and (A.5))

\[ \sum_{k=1}^{\infty} U_\tau \eta_k L_{s_k} = M. \quad \text{(A.11)} \]

We now approximate $(L_{s_k}^{-1} L_{s_k})$ by distribution functions $(F_k)$ and proceed to show that $(U_s, F_k)$ satisfies the conditions of Theorem 6.2.4.

Define $(\varepsilon_k)$ by

\[ \varepsilon_k = \text{inf}\{\varepsilon > 0 \mid L_{s_k}(\{1 x > \varepsilon\}) \leq 1\}, \ k \geq 1, \]

and $(\eta_k)$ by $\eta_k = L_{s_k}(\{1 x > \varepsilon_k\}, k \in \mathbb{N}_e$. Also let

\[ F_k^s(B) = F_k^s(B \cap \{1 x > \varepsilon_k\}) + (1 - \eta_k) \delta_0(B), \ k \geq 1, \]

for any Borel set $B$ of $\mathbb{R}$, with $\delta_0$ a distribution function with total mass at zero.

Finally, let $k(0) = 0, k(n) = \sum_{k=1}^{n} \rho(\tau_k)^{-1}$ and

\[ t_i = \tau_k \text{ for } k(n-1) < i \leq k(n), \ n \in \mathbb{N}_e; \]

\[ F_i = F_n^s \text{ for } k(n-1) < i \leq k(n), \ n \in \mathbb{N}_e. \]

Observe that $\varepsilon_k \rightarrow 0$ as $n \rightarrow \infty$, since, if $\varepsilon_k \geq \varepsilon_0 > 0$, then $L_{s_k}(\{1 x > \varepsilon_0\}) \geq 1$ and so (cf. Lemma A.3.1 (ii))

\[ 1 \leq \lim_{k \rightarrow \infty} L_{s_k}(\{1 x > \varepsilon_0\}) = M(\{1 x > \varepsilon_0\}) - \lim_{k \rightarrow \infty} U_k M(\{1 x > \varepsilon_0\}) = 0. \]
For every \( n \in \mathbb{N} \), we can choose an \( N \in \mathbb{N} \), such that \( k(N-1) < n \leq k(N) \). Thus

\[
M_n := \sum_{l=1}^{n} U_{n,l} F_{l} = \sum_{l=1}^{k(N)} U_{n,l} F_{l} - \sum_{l=n+1}^{k(N)} U_{n,l} F_{l} \\
= \sum_{k=1}^{N} \rho (r_k)^{-1} U_{n,k} F_{k}^* - (k(N) - n) U_{n,k} F_{k}^*. 
\]

(A.12)

The unan property. For every \( \varepsilon > 0 \), define the sequence \( (j(n)) \) by

\[
\sup_{1 \leq s \leq N} U_{n,s} F_{s}(\{1 \mid x > \varepsilon\}) = U_{n,j(n)} F_{j(n)}(\{1 \mid x > \varepsilon\}).
\]

If \( (j(n)) \) is bounded, then by Lemmas A.2.6 and A.3.1

\[
\lim_{n \to \infty} \sup_{1 \leq s \leq N} U_{n,s} F_{s}(\{1 \mid x > \varepsilon\}) = 0. \quad (A.13)
\]

Suppose \( (j(n)) \) is unbounded. Observe that \( F_{j(n)} = F_{j(n)}^* \) for some \( i(n) \leq N \). Hence by the definition of \( F_k \) and Lemma A.2.4 (i),

\[
U_{n,j(n)} F_{j(n)}(\{1 \mid x > \varepsilon\}) \leq U_{n,\alpha_{j(n)}} F_{\alpha_{j(n)}}(\{1 \mid x > \varepsilon\}) = \rho (r_{\alpha_{j(n)}}) L_{\alpha_{j(n)}}(\{1 \mid x > \varepsilon\}).
\]

From Assumption A.2.2 (ii) and the unboundedness of \( f_r \) we have that

\[
\rho (r_{\alpha_{j(n)}}) \leq C(\{1 \mid x > \varepsilon\}).
\]

Hence

\[
U_{n,j(n)} F_{j(n)}(\{1 \mid x > \varepsilon\}) \leq \rho (r_{\alpha_{j(n)}}) L_{\alpha_{j(n)}}(\{1 \mid x > \varepsilon\}).
\]

Since \( i(n) \to \infty \) as \( n \to \infty \) and so \( \delta_{i(n)} \to \infty \) as \( n \to \infty \), \( L_{\alpha_{i(n)}} \to 0 \). Formula (A.13) is thereby satisfied. The triangular array \( (U_{n,s} F_{s}) \) is therefore unan.

Condition (i) of Theorem 6.2.4. Let \( I \) be a continuity set of \( M \) with \( I \) an interval bounded away from the origin. Since \( r_k \to 0 \) as \( k \to \infty \) there exists a \( k_0 \) such that \( r_k \notin I \) for all \( k > k_0 \). Observe that (cf. (A.12), the definition of \( F_k^n \), (A.5) and (A.6))

\[
M_{k_0}(N)(I) = \sum_{k=1}^{k_0} \rho (r_k)^{-1} U_{n,k} F_{k}^* (I) + \sum_{k=k_0}^{N} \rho (r_k)^{-1} U_{n,k} F_{k}^* (M - L_{\alpha_{k}} M)(I)
\]

\[
= \sum_{k=1}^{k_0} \rho (r_k)^{-1} U_{n,k} F_{k}^* (I) + \sum_{k=k_0}^{N} U_{n,k} U_{n,k}^{-1} M(I) - \sum_{k=k_0}^{N} U_{n,k} U_{n,k}^{-1} U_{n,k} U_{n,k}^{-1} M(I)
\]

\[
= \sum_{k=1}^{k_0} \rho (r_k)^{-1} U_{n,k} F_{k}^* (I) + M(I) - U_{n,k} U_{n,k}^{-1} M(I). 
\]

Letting \( N \to \infty \) we obtain

\[
M_{k_0}(N) \to M \text{ as } N \to \infty 
\]

outside every neighbourhood of the origin. Observe that
A.3 A characterization of $U(S,p)$

$$(k(N) - n)U_rP_{N}(l) \leq p(r_a)^{-1}U_rP_{Na}(l) \leq L_{Na}(l).$$ (A.15)

Since $L_{Na}(l) \to 0$ as $N \to \infty$ we have by (A.12), (A.14) and (A.15), that $M_a \to M$ outside every neighbourhood of the origin. Hence condition (ii) of Theorem 6.2.4 is satisfied.

**Condition (ii) of Theorem 6.2.4.** By the definition of $F_k$ we have (cf. (A.12) and (A.11))

$$\sum_{i=1}^{N} \int_{|z| \leq k} x^2 dU_{r_i} F_k(x) \leq \sum_{k=1}^{N} \int_{|z| \leq k} x^2 dU_{r_k} P_{Na}(x) \leq \sum_{k=1}^{N} \int_{|z| \leq k} x^2 dU_{r_k} U_{r_k}^{-1} L_{Na}(x) = \int_{x \in \mathbb{R}} x^2 dM(x).$$

Letting $r \to \infty$, we see that condition (ii) of Theorem 6.2.4 is met. \qed

In the "only if" part of the proof of Theorem A.3.2 it was not necessary to assume that $\sigma_k^2 = 0$. We therefore have

**Corollary 1.** If $\phi = \sigma_0, \sigma_1^2, M$ and $\phi \in U(S, p)$, then $M$ satisfies the inequality of Theorem A.3.2.

A.4 Stability in $U(S, p)$

In this section we consider (A.1) with the added assumption that the sequence $(X_k)$ is identically distributed. We call such limit distributions $U(S, p)$-stable.

**Theorem A.4.1.** Let $\phi$ be infinitely divisible with $\phi = [\sigma_0, 0, M]$. Then $\phi \in U(S, p)$-stable if and only if there exists a semigroup homomorphism $g$ from $((0, \infty), \ast)$ to $((1, \infty), \ast)$, such that for every $t \in (0, \infty)$ and for every Borel set $B$ of $\mathbb{R}(\ast, \ast)$, for any $\epsilon > 0$, the Lévy spectral function $M$ satisfies

$$M(B) = g(t) U_t M(B).$$ (A.16)

**Proof.** We first prove the "only if" part. If $M$ vanishes identically then the "only if" part is trivial. Suppose $M$ is not identically zero. From Lemma A.3.1 it follows that for any $t \in (0, \infty)$ there exists a subsequence $(k(n))$ such that $k(n) \geq n$ and (A.10) is satisfied. For any continuity set $I$ of $M$, with $I$ an interval bounded away from the origin, we have (cf. (A.6))
\( M_k(\phi)(t) := k(n)U_{\mu_n}F(l)) = k(n)/n U_{\mu_n}M_n(l) \).

Letting \( n \to \infty \) we have by Theorem 6.2.4, Lemma A.2.6 and (A.10) that both \( M_k(\phi) \)
and \( U_{\mu_n}M_n \) converge and hence \( k(n)/n \) must converge. Let \( g(t) = \lim_{n \to \infty} n = f(t) \).

Then \( M \) satisfies (A.16) with \( B = I \). Since the Borel sets on \( \mathbb{R} \) are generated by the intervals bounded away from zero, (A.16) holds for all Borel sets \( B \).

Suppose there is a \( t_0 > 0 \) such that \( g(t_0) \leq 1 \). By repeated use of (A.16) we see that

\[ M(B) \leq \liminf_{n \to \infty} M(B) = \phi(t_0)^{\alpha} \int_{\mathbb{R}} M(B) \]

for \( \alpha \in \mathbb{R} \). Applying Assumption A.2.2 (ii) we have \( M = 0 \). Hence \( g(t) > 1 \) on \( \mathbb{R} \).

Observe that \( M(B) = \int_{\mathbb{R}} g(t) U_{\mu_n}M_n(B) = g(t) \int_{\mathbb{R}} U_{\mu_n}M_n(B) \).

Hence \( g \) is a semigroup homomorphism from \( (0, \infty, \mathbb{R}) \) to \( (1, \infty, \mathbb{R}) \).

Conversely, suppose \( M \) satisfies (A.16). If \( M \) vanishes identically then the 'if' part is trivial. Suppose \( M \) does not vanish identically. Let \( (\alpha_n) \) be a non-decreasing sequence satisfying conditions (i) and (ii) of Lemma A.3.1 and such that \( g(\alpha_n) = \alpha \) (this is possible since \( g \) is unbounded and continuous). Define \( \bar{e} \) by

\[ \bar{e} = \inf \{ \epsilon > 0 \mid M((x > e)) \leq 1 \} \]

and set \( \eta = M((x > \bar{e}) \}. Define the probability distribution \( F \) by

\[ F(B) = M(B \cap (x > \bar{e}) \} + (1 - \eta)\delta_0(B) \]

for any Borel set \( B \) of \( \mathbb{R} \). We now show that \( U_{\mu_n}F \) satisfies the conditions of Theorem 6.2.4.

Condition (i) of Theorem 6.2.4. Let \( l \) be an interval bounded away from the origin. For any \( x \) there exists an \( N \) such that for all \( n \geq N \)

\[ f_{\mu_n}(l) \cap (x > a) = f_{\mu_n}(l) \]

If this were not true then \( f_{\mu_n}(l) \to B \) for all \( n \) large and for some non-empty Borel set \( B \) contained in \( (x > a) \). Hence \( l = f_{\mu_n}(l) \to f_{\mu_n}(B) \). By Assumption A.2.2 (ii) for any \( x \in \mathbb{R} \), \( f_{\mu_n}(x) \to 0 \) as \( n \to \infty \), contradicting the assumption that \( I \) is bounded away from the origin. Hence for \( n \) large

\[ nU_{\mu_n}F(l) = g(\alpha_n)\int_{\mathbb{R}} M((x > \bar{e}) \cap f_{\mu_n}(l)) + (1 - \eta)\delta_0(f_{\mu_n}(l)) \]

\[ g(\alpha_n)U_{\mu_n}M(l) = M(l) \]  \hspace{1cm} (A.17)

Condition (i) of Theorem 6.2.4 is thus satisfied.

The vanishing property. For any \( n \geq N \) and any \( \epsilon > 0 \) we have by (A.17)
A.4 Stability in $U(S,p)$

\[ \sup_{1 \leq |s| \leq n} U_n F((1-x | \geq \varepsilon)) = n^{-1} M((1-x | \geq \varepsilon)). \]

Letting $n \to \infty$ we see that $(U_n F)$ is uan.

Condition (ii) of Theorem 6.2.4. Finally, observe that for $n \geq 1$, (cf. (A.17))

\[ \sum_{k=1}^{n} \int_{1-x | \geq \varepsilon} x^2 \, dU_n F(x) = \int_{1-x | \geq \varepsilon} x^2 \, dM(x). \]

Condition (ii) of Theorem 6.2.4 is now met upon letting $n \to \infty$ and $\varepsilon \to 0$. \hfill \square

A.5 Some subsets of $U(S,p)$

Let the stochastic 'breeding operator' $B_t$ be defined by

\[ B_t X = \sum_{k=1}^{d(t)} \text{U}^{k_{i}} X_k, \]

where $X_1, \ldots, X_{d(t)}$ are independent and distributed as $X$ and $d(t)$ is a semigroup homomorphism from $\langle 0, \infty \rangle$ to $\langle 1, \infty \rangle$. We are interested in the random variables $X$ which have the form

\[ \sum_{k=1}^{n} B_{t_k} X_k \to X \text{ as } n \to \infty, \]  \hfill (A.18)

where $(B_{t_k} X_k)$ are independent and uan, $(X_k)$ are independent unbounded and $(t_k)$ is non-decreasing. We therefore introduce the notation

Notation A.5.1. Let $B(S, p, q)$ denote the set of characteristic functions whose random variables can be described as limits of the form (A.18).

Let $F_k$ be the distribution function of $X_k$. Then

\[ M_n^* := \sum_{k=1}^{n} \sum_{l=1}^{d(t)} U_{t_k} F_k = [q(t)] \sum_{k=1}^{n} U_{t_k} F_k. \]  \hfill (A.19)

As in the proof of Lemma A.3.1 (i) we see that the uan property of $(B_{t_k} X_k)$ implies that $t_k \to \infty$ as $n \to \infty$. If $q(t) \to c < \infty$ as $t \to \infty$, then $c = \lim_{n \to \infty} q([t_{B(n)}]) = cq(t)$ and hence $q(t) = 1$ for all $t$. Hence assume without loss of generality that $q(t) \to \infty$ as $t \to \infty$.

It follows from (A.19) and Theorem 6.2.4, that if $X$ is non-degenerate then $M_n^*$ tends to a Lévy spectral function $M$ not identically zero and so

\[ \sum_{k=1}^{n} U_{t_k} F_k \to 0 \text{ as } n \to \infty. \]
outside every neighbourhood of the origin. Therefore $M^*_n$ and $M_n$ with
\[ M_n := q(I_n) \sum_{k=1}^{n} U_k F_k. \]
have the same limits. The proof of Theorem A.3.2 can now easily be adapted to prove

**Theorem A.5.2.** Let $\phi$ be infinitely divisible with $\phi = [a_0, 0, M]$. Then $\phi \in B(S, p, q)$ if and only if there exists a semigroup homomorphism $q$ from $(\mathbb{R}_+, \mathbb{R}, \cdot)$ to $(\mathbb{R}, \cdot)$, such that for every $t \in [0, \infty)$ and for every Borel set $B$ of \( \mathbb{R}(\mathbb{R}, \mathbb{R}) \), for any $\epsilon > 0$, the Lévy spectral function $M$ satisfies
\[ M(B) \geq q(t) U_M(B). \]

**A.6 Examples**

We conclude this appendix with a few examples. A special case of Example A.6.1 yields the shrinking operator $T_{a,t}$ of Section 6.4.1 and of Example A.6.2, the operator $U_{a,t}$ considered in Section 6.4.2.

**Example A.6.1.**
\[
\begin{align*}
    f_t(x) &= r^\gamma x, \quad t \in (0, 1], \quad \gamma > 0, \\
    p(t) &= t^\alpha, \quad t \in (0, 1], \quad \alpha \geq 0, \\
    s \otimes t &= s \cdot t.
\end{align*}
\]
Hence $U_t = T_{a,t}$. \( \Box \)

**Example A.6.2.**
\[
\begin{align*}
    f_t(x) &= \text{sgn}(x) \left( |x|^{\alpha} - t^{\alpha} \right)_{+}^{\alpha}, \quad t \in (0, \infty), \quad \alpha > 0, \\
    p(t) &= e^{-\alpha t}, \quad t \in (0, \infty), \\
    s \otimes t &= (s^{\alpha} + t^{\alpha})^{\frac{\alpha}{2}}.
\end{align*}
\]
Hence $U_t = U_{a,t}$ if $p = 0$. \( \Box \)

**Example A.6.3.**
\[
\begin{align*}
    f_t(x) &= \text{sgn}(x) \left( |x|^{\alpha} + T^{-1} |x|^{\alpha} \right)_{+}^{\alpha}, \quad t \in (0, 1], \quad \alpha > 0, \\
    p(t) &= r^\gamma, \quad t \in (0, 1], \quad \gamma > 0, \\
    s \otimes t &= s \cdot t.
\end{align*}
\]
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List of symbols

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  \item[$\alpha r^*$] Definition 3.2.6
  \item[$\alpha$] Notation 1.3.5
  \item[$\alpha t$] Definition 3.5.1
  \item[$\alpha t^*$] Definition 3.5.1
  \item[$\alpha s^*$] Notation 3.2.1
  \item[$\alpha s^*$] Notation 3.2.1
  \item[$\alpha f^*$] Notation 3.2.1
  \item[$\alpha f^*$] Notation 3.2.1
  \item[$\alpha f^*$] Definition 5.3.1
  \item[$\alpha f^*$] Definition 5.4.9
  \item[$\alpha f^*$] Definition 5.5.1
  \item[$\alpha f^*$] Section 6.4, equation (6.4)
  \item[$\alpha f^*$] Section 6.4, equation (6.5)
  \item[$\alpha f^*$] Section A.1, equation (A.2)
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Samenvatting

De bestudering van de oneindig deelbare stochastische grootheden heeft zich in korte tijd een vaste plek verworven binnen de kansrekening. In vrijwel ieder inleidend boek over de kansrekening staat te gevaarwoord tenminste één hoofdstuk gewijd aan dit onderwerp. Voor een belangrijk deel is dit gekomen door het gebruik van oneindig deelbaarheid in het oplossen van problemen rondom de centrale limietstelling en door toepassingen op stochastische processen met stationaire onafhankelijke aangroeiingen.


Het uitgangspunt voor deze studie is de Lévy canonische representatie van oneindig deelbare verdelingen (zie Stelling 1.3.2 van het proefschrift), waarin de relatie wordt aangegeven tussen de karakteristieke functie van een oneindig deelbare verdeling $F$ en een functie $M$, die de Lévy spectraalfunctie wordt genoemd. We zijn geïnteresseerd in de karakterisering van die verdelingen $F$, waarvan de Lévy spectraalfunctie $M$ aan bepaalde eisen voldoet die betrekking hebben op de monotone.

Curriculum vitae

Stellingen behorende bij het proefschrift

Monotonicity Properties of
Infinitely Divisible Distributions

B.G. Hansen
Stelling 1.
Zij $(p_n)_{n=0}^\infty$ een discrete kansverdeling op $\{0, 1, 2, \ldots\}$, gegeven door
\[
p_n := \frac{1}{\beta(n+1)} \binom{\beta(n)}{n+1} \left(1-\alpha\right)^{n+1} \left(\log(1-\alpha)\right),
\]
met $\beta \geq 1$ en $0 < \alpha < \beta^{-1}$. Dan is $(p_n)$ strict log-convex en daarom

(i) oneindig deelbaar;

(ii) monoton dalend en dus unimodaal met modus in $n=0$.


Stelling 2.
Zij $(p_n)_{n=0}^\infty$ een kansverdeling zoals in (1) van Stelling 1. Zij verder $(\xi_n)$ de Lévy-maat horende bij $(p_n)_{n=0}^\infty$, d.w.z.
\[
\sum_{n=0}^\infty \xi_n Z^n = \exp(-\lambda \left(1 - \sum_{n=1}^\infty \xi_n Z^n\right)),
\]
met $\lambda = -\log(p_0)$. Dan geldt

(i) $\xi_n \sim \frac{1}{\log(1-\alpha)} p_n \ (n \to \infty)$,

(ii) $\Delta_n := \frac{p_n}{\alpha_n} - \lambda \frac{\log(1-\alpha)}{\log(1-\beta^{-1})} = o(\frac{p_n}{\alpha_n}) \ (n \to \infty)$.

De convergencedienheid in (ii) is exact in de zin dat voor elke rij $(\alpha_n)_{n=0}^\infty$ met $\alpha_n \to \infty$, $\alpha_n \Delta_n(n p_n)^{-1} \to \infty$ als $n \to \infty$.


Stelling 3.
Als een karakteristieke functie $\phi$, reëel is en convex in een omgeving van 0, dan geldt
\[
\lim_{t \to 0} t \phi(t) = 0.
\]
Het vermoeden is gerechtvaardigd dat (2) in het algemeen geldt voor oneindig deelbare karakteristieke functies $\phi$, waarbij $\phi(t)$ bestaat voor $t \neq 0$.

STELLING 4.
Zij X een stochastisch variabelen met inverse Gauss-verdeling, d.w.z. met kansverdeling
\[ f(x) = (2\pi x^2)^{-\frac{1}{2}} e^{-\frac{1}{2} x^{-2}} \exp\left(-\frac{1}{2}(\gamma^2 x + \lambda x^{-1})\right), x > 0, \]
met \( \lambda > 0 \) en \( \gamma > 0 \). Zij verder \((p_n)_{n=0}^{\infty}\) de kansverdeling gegeven door vergelijking (1) in Stelling 1. Dan is
\[ P(|X| = n) = p_n \quad (n \to \infty), \]
met
\[ \gamma^2 = 2 \log \left( \frac{\beta - 1}{\beta (1 - \beta)} \right)^{\beta^{-1} - 1}. \]
\[ \lambda e^{2 \gamma^2} (1 - e^{-\gamma^2}) = (-\log (1 - \beta))^{-1} (\beta (\beta - 1))^{-1/2}. \]
Hieruit blijkt dat de kansverdeling \((p_n)_{n=0}^{\infty}\) (asymptotisch) beschouwd kan worden als een discreet analogon van de inverse Gauss-verdeling.


STELLING 5.
Zij \((f_n)_{n=1}^{\infty}\) een kansverdeling op \(\{1, 2, 3, \ldots\}\) en zij de rij \((u_n)_{n=0}^{\infty}\) gedefinieerd door
\[ u_0 = 1, \quad u_n = \sum_{k=1}^{n} u_{n-k} f_k, \quad n = 1, 2, 3, \ldots. \]
Veronderstel tevens dat \( \mu = \sum_{n=1}^{\infty} n f_n < \infty \) is en noem \( F(n) = \sum_{k=1}^{n} f_k \) en \( F_1(n) = \sum_{k=1}^{n} (1 - F(k)) \). Dan geldt

(i) Als \( (1 - F_1(n))_{n=1}^{\infty} \) log-convex is, dan is \((u_n)_{n=0}^{\infty}\) niet-dalend met limiet \( \mu^{-1} \).
(ii) Als \( (1 - F_1(n))_{n=1}^{\infty} \) log-convex is, dan is \( u_n \geq \mu^{-1} \) voor elke \( n \geq 0 \).

Stelling 6.

Toepassing op de resultaten van Stelling 5 van een limitargument analog aan dat in het bewijs van Stelling 4.3.2 in dit proefschrift, levert een eenvoudig bewijs van Stelling 2 (i) en Stelling 3 (ii) uit Brown (1980).


Stelling 7.

Beschouw het $M/D/1$-wachtrijmodel: klanten komen aan bij een loket volgens een Poisson proces met intensiteit $\alpha < 1$ en worden geholpen gedurende één tijdseenheid. In die tijd leveren zij materiaal in bij het loket, dat later bewerkt moet worden. Wij beschouwen de wachtrij vanaf een moment dat er $k$ klanten in de rij staan, tot het eerste moment dat de loketbediende vrij is. Zij $X(k)$ de totale hoeveelheid tijd die nodig is om het ingeleverde materiaal te bewerken. Als de tijd nodig om het materiaal van één klant te bewerken exponentieel verdeeld is met parameter $\lambda$, dan wordt de kansdichtheid $f_k(x; \lambda, \alpha)$ van $X(k)$ gegeven door

$$f_k(x; \lambda, \alpha) = \sum_{n=1}^{\infty} \frac{(-\alpha/\lambda)^{n-1}}{(n-1)!} \left( e^{-\lambda} \sum_{j=1}^{n} \frac{\lambda^j}{j!} e^{-\lambda x} \right), \quad x > 0.$$ 

Deze dichtheid heeft de volgende eigenschappen:

(i) $f_k(x; \lambda, \alpha)$ is oneindig deelbaar;

(ii) $f_k(x; \lambda, \alpha)$ is volledig monotoon.


Stelling 8.

De omstandigheden dat de gemiddelde trimmer bij goede benadering 10.000 passen per uur doet geeft hem de mogelijkheid om zonder rekenwerk een goede schatting te maken van zijn snelheid, speciaal bij lopen over een betegeld trottoir.