FLEXIBLE SOLUTIONS TO SYSTEMS OF LINEAR EQUALITIES AND INEQUALITIES

A STUDY IN LINEAR PROGRAMMING

PROEFSCHRIFT

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Robert Peter van der Vet

wiskundig ingenieur

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Prof. dr. J.F. Benders
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## CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER I</th>
<th>INTRODUCTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-1</td>
<td>Motivation of the subject</td>
<td>1</td>
</tr>
<tr>
<td>I-2</td>
<td>Summary</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER II</th>
<th>PRELIMINARY MATHEMATICAL CONCEPTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>II-1</td>
<td>Notations and conventions</td>
<td>7</td>
</tr>
<tr>
<td>II-2</td>
<td>Implicit equalities in a consistent system of linear inequalities</td>
<td>9</td>
</tr>
<tr>
<td>II-2.1</td>
<td>Introduction</td>
<td>9</td>
</tr>
<tr>
<td>II-2.2</td>
<td>The concepts of null variables and of implicit equalities</td>
<td>10</td>
</tr>
<tr>
<td>II-2.3</td>
<td>Two algorithms for the identification of null variables</td>
<td>15</td>
</tr>
<tr>
<td>II-3</td>
<td>Redundant inequalities in a consistent system of linear inequalities</td>
<td>21</td>
</tr>
<tr>
<td>II-4</td>
<td>Piecewise linear functions on polyhedral sets</td>
<td>23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER III</th>
<th>THE MATHEMATICAL CONCEPT OF FLEXIBILITY IN POLYHEDRAL SETS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>III-1</td>
<td>Introduction to the concepts of flexibility range and direction of flexibility</td>
<td>26</td>
</tr>
<tr>
<td>III-2</td>
<td>Properties of the direction of flexibility</td>
<td>29</td>
</tr>
<tr>
<td>III-3</td>
<td>Properties of the flexibility range</td>
<td>34</td>
</tr>
<tr>
<td>III-3.1</td>
<td>General properties</td>
<td>34</td>
</tr>
<tr>
<td>III-3.2</td>
<td>Convexity and Lipschitz properties</td>
<td>41</td>
</tr>
<tr>
<td>III-3.3</td>
<td>Piecewise linearity property</td>
<td>43</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

I-1   MOTIVATION OF THE SUBJECT

When applying linear programming to real life problems, one has to deal with the following facts.

a) The mathematical model of this problem does not give a complete description of the real life problem. For instance, model reduction and linearity assumptions cause the model to give only an approximate description of the actual problem.

b) During the course of time between the formulation of the model and the use of the calculated linear programming solution in practice the parameters in the model may have been changed.

c) The calculated linear programming solution must often be adjusted to fulfil certain operational requirements which are difficult to formulate mathematically in a linear programming model. Moreover, it is desirable that such an adjustment can be effected without performing exhaustive calculations.

As linear programming solutions lie on the boundary of the region of feasible solutions, the facts mentioned under points a and b can cause the calculated linear programming solution to be not optimal or, which is more serious, not feasible in the actual situation. If the latter appears to be the case, it can be quite cumbersome to find a solution which can be used in practice.
Even if the model gives an exact representation of the actual problem, however, and, if there are no changes in the parameters, adjustments to the calculated linear programming solution as made under point c can easily make such a solution infeasible.

The above observations have motivated us to investigate operations research methods for finding solutions which are less sensitive to changes in the mathematical model and are easier to adapt. In this thesis we report the results of these investigations. Most of the material has not appeared in the operations research literature before.

The operations research methods under consideration generate solutions in the interior of the feasible region of a linear programming problem with certain distance properties to the boundary of this region. Clearly, such solutions have an objective value deviating from the optimal objective value of the linear programming solution. However, the feasibility of such a solution is not affected, if the actual situation differs slightly from the one described mathematically. On the other hand, this solution can be adjusted in any direction within the feasible region without becoming immediately infeasible. In fact, with these solutions one exchanges optimality for the quality of the solution concerning its feasibility and the ability to adjust such a solution in a simple manner.

Following the economic literature we shall refer to such an interior solution as a flexible solution in the feasible region. Flexible programming will be used as the collective noun for the operations research methods which are directed to finding such solutions.
Most attention will be paid to the mathematical properties of flexible solutions. We shall also propose some methods for finding such solutions. The numerical behaviour of the algorithms in practice will not be considered.

To obtain a clear exposition of the material we shall isolate the problem of flexibility from that of economical optimality. This implies that we shall consider linear programming models without an economical objective function (such as cost or profit). However, a decision maker will in general be more interested in solutions to linear programming problems which have flexibility properties as well as an acceptable value for the economical objective function. In the last chapter we shall therefore pay some attention to the use of the proposed methods in combination with economical objective functions.

We remark that the concept of flexibility has already been treated in the operations research literature. In [1] and [2] linear programming problems are considered in which the parameters of the mathematical model are assumed to be uniformly distributed over a prescribed interval. The optimal solution consists of values for the problem variables and the adjustments which should be made to these variables if the region of feasible solutions is restricted to the most pessimistic case concerning the values of the model parameters. In [3] linear programming problems are considered where it is permitted, at a certain cost, to exceed the boundary of the feasible region. Finally, in [4], [5] and [6] flexibility is used as a measure in multistage decision problems. Flexibility has a slightly different meaning here. It is the range of decisions which remains open after the decision in the first stage of the decision process has been made. The greater this range, the more the decision maker is able to choose an acceptable decision in later stages when the actual model data become available.
Throughout we shall use non-empty polyhedral sets of the form $G := \{ x \in \mathbb{R}^n \mid Ax \leq b \}$. 

In chapter II the conventions and notations to be used in this thesis will be summarized. Further, two algorithms will be developed for the determination of the implicit equalities in the system $Ax = b$. This is important for the subject to be dealt with, as such equalities restrict the directions along which one can move from a point in $G$ without violating its boundary. 

Chapter III contains most of the material which forms the basis to express the flexibility of a point $x \in G$ in a quantitative way. Here we introduce the concept of the flexibility range and of the direction of flexibility. Both concepts are new. We shall further discuss which directions are meaningful to define the flexibility for a point $x \in G$. Finally, a number of properties for this flexibility range will be derived such as concavity, the Lipschitz property and piecewise linearity. 

The concepts introduced in chapter III will be used to formulate two main problems in flexible programming. 

The first problem is a generalization of the well-known problem of finding a point in $G$ for which the minimum distance to the boundary of $G$ is maximal. It is called the weighted distance problem. 

The second problem, although resembling the Nash equilibrium problem in game theory, is entirely new. It is called the equilibrium problem in flexible programming. 

Both problems have in common that they are directed to finding solutions in the interior of $G$. These solutions have certain distance properties to the boundary of $G$ with respect to a set of prescribed directions of flexibility. They differ in the choice
of the directions of flexibility and in the characteristics of the distances with respect to these directions.

We shall study these problems in chapters IV, V and VI. We remark that these studies can be read independently of each other. Hence, the results obtained in chapter IV are not needed in chapters V and VI.

The weighted distance problem will be studied in chapter IV. We shall demonstrate that this problem can be formulated as a linear program. We shall also give some of its properties.

Chapter V deals with a discussion of the equilibrium problem. It will be shown that a solution to this problem always exists, if it is bounded and if the prescribed directions are linearly independent. We shall further show that the equilibrium problem can be formulated as the problem of finding a solution to a system of piecewise linear equations.

An algorithm for the determination of solutions to the equilibrium problem will be developed in chapter VI. Since the equilibrium problem has a non-linear (piecewise linear) structure, a solution cannot be found with a single linear program. However, we demonstrate that a solution can be found by solving a finite sequence of linear programs. Using local information, we select the subsequent linear programs in such a way, that the optimal values of their objective functions decrease monotonically, until a local minimum is possibly found. In the latter case the algorithm proceeds with the selection of another linear program for which the optimal objective value may be higher than the optimal objective value of the preceding linear program. In any case, a solution to the equilibrium problem is found in a finite number of steps, since the number of linear programs is finite.
In the last chapter we shall give some suggestions as to how the optimization and equilibrium problem can be used in cases in which economical optimality is of importance as well.
CHAPTER II

PRELIMINARY MATHEMATICAL CONCEPTS

II-1 NOTATIONS AND CONVENTIONS

The n-dimensional real Euclidean space is denoted by \( \mathbb{R}^n \). Elements in \( \mathbb{R}^n \) are considered to be column vectors, i.e., if \( x \in \mathbb{R}^n \), then 
\[
x := (x_1, \ldots, x_j, \ldots, x_n)^T.
\]
The superscript \( T \) denotes the transpose, the symbol := a definition and \( x_1, \ldots, x_j, \ldots, x_n \) are called the entries or components of \( x \) or sometimes, if \( x \) is an unknown vector, the variables. The vectors \( e_i, i = 1, \ldots, n \) are the unit vectors in \( \mathbb{R}^n \), i.e. \( e_{jj} = 0 \) for \( j \neq i \) and \( e_{ii} = 1 \) for \( i = 1, \ldots, n \). The vector with all entries equal to 1 is denoted by \( e \). The scalar product of two vectors \( x, y \in \mathbb{R}^n \) is expressed by

\[
x^T y := \sum_{j=1}^{n} x_j y_j.
\]

Unless stated otherwise, we shall use the norm \( \|x\| := (x^T x)^{1/2} \) for a vector \( x \in \mathbb{R}^n \). The following convention for vector inequalities will be used:

\[
x \leq y \quad x_j \leq y_j, \quad j=1,\ldots, n,
\]
\[
x \leq y \quad x \leq y \text{ and } x \neq y,
\]
\[
x < y \quad x_j < y_j, \quad j=1,\ldots, n.
\]
The second inequality means that at least one component of the vector \( x \) is smaller than the corresponding component of the vector \( y \). The set of real numbers is denoted by \( \mathbb{R} \) and the set of non-negative real numbers by \( \mathbb{R}_+ \).
Let $Ax \leq b$ be a system of linear inequalities, where $A$ is an $(m \times n)$ matrix and $b$ an $(m \times 1)$ vector. The solution set of this system is denoted by $C$. This system is said to be consistent, if $C \neq \emptyset$. Otherwise, it is called inconsistent. We shall always denote the row index set of the matrix $A$ in such a system by $I := \{1, \ldots, m\}$. The number of indices in $I$ is expressed by $|I|$. Let $(i \in I)$ mean: for all indices in the index set $I$. Then $a_i^T x \leq b_i, (i \in I)$ denotes the equivalent row vector notation of $Ax \leq b$.

Throughout we assume that this system is consistent. In the algorithms, however, we have always included a consistency test.

Let $S$ and $T$ be two non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$. A mapping $f$ from $S$ into $T$ is denoted by $f: S \rightarrow T$. Almost all mappings in the following are real-valued functions. We make a distinction between $f(s)$, denoting the value of the function $f$ in the point $s \in S$ and $f(.)$ or $f$ by which we mean the function itself. The supremum of $f$ over $S$ is denoted by $\text{sup}\{ f(s) \mid s \in S \}$. Similar expressions hold for the infimum, maximum and minimum, abbreviated by $\text{inf}$, $\text{max}$ and $\text{min}$ respectively.

Statements as definitions, assumptions, theorems etc. are consecutively numbered in each chapter. For instance, the second statement in chapter III may be a definition and is denoted by definition 3-2. Remarks and comments, however, are numbered in each section or subsection separately. The end of a proof (a definition, a remark etc.) is denoted by the symbol $\circ$.

The reader is also assumed to be familiar with the notations, the terminology and a number of properties in the following fields of mathematics:
a. the theory of convex sets and convex functions (e.g. [7]),
b. the duality-theory for systems of linear inequalities (e.g. [8]),
c. the theory of linear programming and the simplex method as a tool for solving linear programming problems (e.g. [9]).

II-2  IMPLICIT EQUALITIES IN A CONSISTENT SYSTEM OF LINEAR INEQUALITIES

II-2.1. Introduction

Let $Ax \preceq b$ a consistent system of linear inequalities. If $\text{int } G \neq \emptyset$, it is possible to move over some positive distance from a point $x \in \text{int } G$ in any direction, without violating the boundary of $G$.

However, in general it may happen that the linear system $Ax \preceq b$ contains inequalities which hold as an equality for all solutions of this system. This is the case, for instance, for the linear system:

$$x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 1.$$  

The point $x_1 = 0, x_2 = 1$ is the only solution of this system and all inequalities hold as an equality for this solution. In flexible programming it is important to know the inequalities of this type (later to be defined as implicit equalities in the linear system $Ax \preceq b$), since they restrict the directions in which one can move inside $G$.

This section will deal with the identification of these implicit equalities.

We shall also discuss the concept of null variables in linear systems, since it is closely related to the concept of implicit equalities.
II-2.2. The concepts of null variables and of implicit equalities

Let be given the \((mxm)\) matrix \(A\), the \((mxp)\) matrix \(B\) and the \((mx1)\) vector \(b\). It is assumed that the row vectors of \(A\), denoted by \(a_i^T, (i \in I)\), are non-zero vectors. Let further
\[x := (x_1, \ldots, x_i, \ldots, x_n)^T\] and \(y := (y_1, \ldots, y_k, \ldots, y_p)^T\).

The concept of null variables, introduced by Luenberger [10], is given by the following definition.

**Definition 2.1.** The variable \(y_k, k \in \{1, \ldots, p\}\) is called a null variable with respect to the consistent linear system \(Ax + By = b\), \(y \geq 0\), if \(y_k = 0\) for all feasible solutions \((x, y)\) of this system.

It is called a non-null variable, if this linear system has a solution \((\overline{x}, \overline{y})\) with \(\overline{y}_k > 0\).

For systems of linear inequalities it is more appropriate to work with the concept of implicit equality, which is strongly related to the concept of null variable. It is given by the following definition.

**Definition 2.2.** The inequality \(a_i^T x \leq b_i, i \in I\) is called an implicit equality with respect to the consistent linear system \(Ax \leq b\), if \(a_i^T x = b_i\) for all feasible solutions \(x\) of this system.

Hence, \(a_i^T x \leq b_i\) is an implicit equality with respect to the consistent system \(Ax \leq b\), if, and only if, \(y_i\) is a null variable with respect to the consistent system \(Ax + y = b; y \geq 0\).

For the null variables we have the following duality property. It will play a key role in one of the algorithms to be developed in the following subsection.

**Theorem 2.3.** The variable \(y_k, k \in \{1, \ldots, p\}\) is a null variable with respect to the consistent linear system \(Ax + By = b; y \geq 0\), if, and only if, the dual linear system
\[ A^T u - 0,\ B^T u - v = 0,\ d^T u - 0,\ v \ge 0, \]  
where \( u := (u_1, \ldots, u_k, \ldots, u_M)^T \) and \( v := (v_1, \ldots, v_k, \ldots, v_p)^T \), 
has a solution with \( v_k > 0 \).

Proof. We first observe the following. The variable \( y_k \) is a null variable of the consistent linear system \( Ax + By = b; \ y \ge 0 \), if, and only if, the linear system

\[ Ax + By = b,\ e_{k^T} y > 0,\ y \ge 0 \]

is inconsistent or, equivalently, if, and only if, the system

\[ -Ax - By + bx = 0,\ e_{k^T} y > 0,\ y \ge 0,\ z > 0 \]

is inconsistent. From the duality theorem of Motzkin then follows the consistency of the linear system

\[ A^T u = 0,\ -B^T u + e_{k^T} r + s = 0,\ d^T u + t = 0, \]

\[ (r, t) \ge 0,\ s \ge 0, \]  
(2-2)

where \( u \in \mathbb{R}^M, \ s \in \mathbb{R}^p \) and \( r, t \in \mathbb{R} \). The latter system cannot have a solution \((\bar{u}, \bar{v}, \bar{r}, \bar{t})\) with \( \bar{v} > 0 \), for this would lead to the inconsistency of the system \( Ax + By = b; \ y \ge 0 \), by the theorem of Motzkin. Hence, \( t = 0 \) for all feasible solutions of system (2-2), so that the system has a solution with \( r > 0 \).

The proof of the theorem becomes then obvious, if we substitute \( v = e_{k^T} r + s \).

Let \( p \) be fixed and define \( K := \{1, \ldots, k, \ldots, p\} \). We shall often use this index set in the following partitioned form
\[ X_{eq} := \{ k \in K \mid \gamma_k \text{ is a null variable} \} \]  \hspace{1cm} (2-2.a)

\[ X_{in} := \{ k \in K \mid \gamma_k \text{ is a non-null variable} \} \]  \hspace{1cm} (2-3.b)

Note that for a consistent linear system \( Ax + By = b; y \geq 0 \) this partition always exists and is unique. In the following subsection two algorithms will be developed, which find this partition.

Let \( B \) be the \((nxm)\) unity matrix. Hence, \( p = m \) and \( K = 1 \).

Furthermore, let \( G := \{ x \in \mathbb{R}^m \mid Ax = b \} \). For this particular case we then find

\[ I_{eq} := \{ i \in I \mid \gamma_i \text{ is a null variable} \} \]

\[ = \{ i \in I \mid \forall x \in G \quad a_i^T x = b_i \} \]  \hspace{1cm} (2-4.a)

and

\[ I_{in} := \{ i \in I \mid \gamma_i \text{ is a non-null variable} \} \]

\[ = \{ i \in I \mid \exists x \in G \quad a_i^T x < b_i \} \]  \hspace{1cm} (2-4.b)

Hence, \( I_{eq} \) designates the implicit equalities in the linear system \( Ax = b \). The following topological properties for the solution set \( G \) of this system are now obvious*:

\[ \text{aff } G = \{ x \in \mathbb{R}^m \mid \exists x \in G : a_i^T x = b_i, (i \in I_{eq}) \} \],

\[ \text{ri } G = \{ x \in G \mid a_i^T x = b, (i \in I_{eq}); a_i^T x < b, (i \in I_{in}) \} \].

* The notations \( \text{aff } G \) on \( \text{ri } G \) mean the affine hull and the relative interior of \( G \) respectively. See [7] for a definition of these concepts.
Note that the dimension of aff $\mathcal{C}$ is equal to the maximum number of linear independent vectors among the vectors $a_i, (i \in I_{eq})$. Further, int $\mathcal{C} \neq \emptyset$, if and only if, $I_{eq} = \emptyset$. Let $A_{eq}$ be the submatrix of $A$ with rowvectors $a_{eq}^i, (i \in I_{eq})$ and $b_{eq}$ the subvector of $b$ with entries $b_i, (i \in I_{eq})$.

**Theorem 2.4.** Let the linear system of inequalities $Ax \leq b$ be consistent and let $I_{eq} \neq \emptyset$. Then the linear system

$$A_{eq}^T u = 0, \quad b_{eq}^T u = 0, \quad u > 0$$

is consistent.

**Proof.** According to the duality theorem of Farkas, the consistency of $Ax \leq b$ is equivalent to the inconsistency of the linear system

$$A^T u = 0, \quad b^T u < 0, \quad u > 0. \quad (2-5)$$

Let $I_{eq} \neq \emptyset$. This means that the system $Ax < b$ is inconsistent, which is equivalent to the inconsistency of the linear system

$$Ax - b\theta < 0, \quad \theta > 0,$$

where $\theta \in \mathbb{R}^1$. According to the duality theorem of Gordan we then have the consistency of the linear system

$$A^T u = 0, \quad b^T u + v = 0, \quad (u,v) \geq 0, \quad (2-6)$$

where $v \in \mathbb{R}^1$. If $(u,v)$ is a solution of the system $(2-6)$, then $v = 0$. For let there exist a solution $(\tilde{u}, \tilde{v})$ with $\tilde{v} > 0$. Then $\tilde{u}$ would be a solution of the linear system $(2-5)$, which contradicts the inconsistency of this system. Hence, the linear system
\[ A^T u = 0, \quad b^T u = 0, \quad u \geq 0 \] (2-7)

is consistent.

Let \( \bar{u} \) be a solution of (2-7) and \( \bar{x} \) a solution of \( Ax \leq b \). Then
\[ \bar{u}^T (A\bar{x} - b) = 0 \] (2-8)

Since \( u \geq 0 \) and \( Ax \leq b \) we have that (2-8) holds for each component. Hence,
\[ u_i (a_i^T x - b_i) = 0 \quad (1 \leq i \leq \ell). \]

Let \( a_i^T x < b_i \) be not an explicit equality in the system \( Ax \leq b \).
Then \( i \not\in I_{eq} \) and there exists an \( \bar{x} \in \mathbb{R}^n \) with \( a_i^T \bar{x} < b_i \). Hence, if \( i \not\in I_{eq} \), then \( u_i = 0 \) for any solution of the linear system (2-7).

To complete the proof, let for \( i \in I_{eq} \), \( u(i) \) be a solution of the linear system \( A^T u = 0, \quad b^T u = 0 \), with \( u_i (i) > 0 \).
Then
\[ \bar{u} := \sum_{i \in I_{eq}} u_i (i) \]

is a solution of this system, from which the theorem follows. \( \square \)

Remark 1. Let the consistent linear system \( Ax \leq b \) contain implicit equalities. The theorem then shows that the normals to these inequalities can be linearly combined with positive coefficients to the zero vector. This result implies that the number of implicit equalities is at least two. Hence, \( |I_{eq}| \geq 2 \) if \( |I_{eq}| \neq 0 \). \( \square \)
Two algorithms for the identification of null variables

We shall now develop two finite multistep algorithms for the identification of the null variables in the linear system $Ax + By = b; \ y \geq 0$. Let $p$ be fixed and $K := \{1, \ldots, k, \ldots, p\}$. In each step the index set $K$ is partitioned into three mutually exclusive subsets $K_0$, $K_1$, and $K_2$, where

$$K_0 := \{ k \in K \mid y_k \text{ already identified as null variable} \},$$

$$K_1 := \{ k \in K \mid y_k \text{ already identified as non-null variable} \},$$

$$K_2 := K - (K_0 \cup K_1).$$

The first algorithm can be characterized by the fact that it identifies at least one non-null variable in each step. It proceeds as follows.

Algorithm 2-5. Identification of non-null variables in the linear system $Ax + By = b; \ y \geq 0$.

1 \hspace{1em} \textbf{begin} $K_0 := \emptyset; K_1 := \emptyset; K_2 := \{1, \ldots, k, \ldots, p\};$

\hspace{1em} \text{Check the consistency of the linear system } Ax + By = b; y \geq 0 \text{ by solving the linear program}

\hspace{1em} \text{minimize } \{ e^T z \mid Ax + By + z = b; y \geq 0; z \geq 0 \};

5 \hspace{1em} (\bar{Z}, \bar{Y}, \bar{Z}) := \text{optimal solution};

\hspace{1em} (\bar{U}, \bar{V}) := \text{optimal solution of dual problem};

\hspace{1em} \text{if } e^T \bar{Z} > 0 \text{ then terminate else}

\hspace{1em} \text{begin } K_0 := \{ k \in K_2 \mid \bar{v}_k > 0 \};

\hspace{1em} K_1 := \{ k \in K_2 \mid \bar{y}_k > 0 \};

\hspace{1em} K_2 := K_2 - (K_0 \cup K_1);

10 \hspace{1em} \text{end}
\[\text{while } K_2 \neq \emptyset \text{ do}\]
\[\begin{align*}
&\text{begin for } k := 1 \text{ to } p \text{ do} \\
&\quad \text{begin if } k \in K_2 \text{ then } c_k := 1 \text{ else } c_k := 0; \\
&\quad \text{end} \\
&\quad c := (c_1, \ldots, c_k, \ldots, c_p)^T; \\
&\quad \text{solve the linear program}\]
\[\begin{align*}
&\text{maximize } \{ c^T y \mid Ax + By = b; \ y > 0 \}; \\
&\text{optimal solution } (x^*, y^*); \\
&\text{if } c^T y^* = 0 \\
&\quad \text{then begin } K_0 := K_0 \cup K_2; \\
&\quad K_2 := \emptyset; \\
&\quad \text{end} \\
&\text{else begin } K_1 := K_1 \cup \{ k \in K_2 \mid y_k^* < 0 \}; \\
&\quad K_2 := K_2 \setminus K_1; \\
&\quad \text{end}\]
\end{align*}\]

\text{Comments on algorithm 1-5.}

\text{Comment 1. The statements in the lines 1-11 concern the consistency test of the linear system } Ax + By = b; \ y > 0 \text{ and some exclusions from the index set } K_2 \text{ resulting from this consistency test. Since this part of the algorithm will return in the following algorithm 2-6 in exactly the same way, we have written these statements in such a form, that they can be used in both algorithms without disturbing their structure.}

\text{If } c^T y > 0 \text{ in line 7 of the algorithm, then the linear system } Ax + By = b; \ y > 0 \text{ is inconsistent in which case the identification of the null variables is meaningless. If, however, } c^T y = 0, \text{ then the system is consistent and we can draw the following conclusions.}
a. From the dual linear program follows the consistency of the linear system $A^T u = 0; B^T u = v = 0; b^T u = 0; v \geq 0$. Hence, by theorem 2-3, $y_k$ is identified as a null variable, if $\bar{y}_k > 0$ (line 8).

b. If $\bar{y}_k > 0$, then $y_k$ is a non-null variable (line 9).

We recall that the dual optimal solution ($\bar{u}, \bar{v}$) can be read from the reduced cost row in the optimal simplex tableau of the linear program concerned.

Comment 2. It follows from the construction of the vector $c$ in line 14 that

a. all variables $y_k$, $k \in K_1$ are identified as null variables (line 21), if $c^T \bar{y} = 0$ (line 20). The partition of the variables $\{y_1, \ldots, y_{N_k}, \ldots, y_{N_p}\}$ into null and non-null variables has been completed. The algorithm terminates, due to the setting $K_2 := \emptyset$ (line 22).

b. there exists at least one $k \in K_2$ with $\bar{y}_k > 0$, if $c^T \bar{y} > 0$.

Hence, $y_k$ is a non-null variable and $k$ can be added to $K_1$ and removed from $K_2$ (lines 24 and 25 respectively).

Note that the above conclusions also show that the algorithm will always terminate within $p$ steps.

Comment 3. The initial simplex tableau for the linear program in line 18 is obtained from the optimal simplex tableau of the preceding linear program by changing the reduced cost row in the latter tableau.

Comment 4. The positivity of the optimal objective function $c^T \bar{y}$ is revealed, as soon as a basic feasible solution is found for which there exists at least one $k \in K_1$ with $y_k > 0$. In this case one can alternatively remove this index from $K_2$ and start a new linear program.
The second algorithm identifies at least one null variable in each step. In this algorithm $b_k$ denotes the $k^{th}$ column of the matrix $B$. The algorithm proceeds as follows.

**Algorithm 2-6. Identification of null variables in the linear system $Ax + By = b; y \geq 0$.**

1. $K_0 := \emptyset; K_1 := \emptyset; K_2 := \{1, \ldots, k, \ldots, p\}$
2. execute steps described in lines 2-11 in algorithm 2-5
3. while $K_2 \neq \emptyset$ do
4. begin
5. $a := \sum_{k \in K_2} b_k$
6. solve the linear program
7. maximize $\{ y_0 | Ax + By + a y_0 = b; y \geq 0; y_0 \geq 0 \}$
8. $(\bar{x}, \bar{y}, \bar{y}_0) :=$ optimal solution;
9. $(\bar{u}, \bar{v}) :=$ optimal solution of dual problem;
10. if $\bar{y}_0 > 0$ then begin
11. $K_1 := K_1 \cup K_2$
12. $K_2 := \emptyset$
13. end
14. else begin
15. $K_0 := K_0 \cup \{ k \in K_2 | \bar{y}_k > 0 \}$
16. $K_1 := K_1 \cup \{ k \in K_0 | \bar{y}_k > 0 \}$
17. $K_2 := K_2 \setminus (K_0 \cup K_1)$
18. end

**Comments on algorithm 2-6.**

**Comment 1.** See comment 1 of algorithm 2-5.

**Comment 2.** If $\bar{y}_0 > 0$ in line 9 of the algorithm, then it follows from the construction of the vector $a$ that we have for the optimal solution $(\bar{x}, \bar{y}, \bar{y}_0)$
\[ A\bar{x} + B\bar{y} + y_0 = \sum_{k=1}^{P} \bar{y}_k b_k + \sum_{k \in K_2} \bar{y}_0 b_k \]

with \( \bar{y} \geq 0 \) and \( \bar{y}_0 > 0 \). Thus the linear system \( Ax + By = b; y \geq 0 \) has a solution \((x,y)\) with \( y_k > 0 \) for all \( k \in K_2 \). Hence, all variables \( y_k, (k \in K_2) \) have been identified as non-null variables, in which case the algorithm terminates (by the setting \( K_2 := \emptyset \) in line 10). If \( \bar{y}_0 = 0 \), it follows from the dual linear program that \((\bar{u}, \bar{v})\) satisfies the linear system

\[ A^T u = 0, \quad B^T u - v = 0, \quad b^T u = 0, \quad c^T u \geq 1, \quad v \geq 0. \]

Thus it follows from the construction of the vector \( a \) that there exists at least one \( k \in K_2 \) with \( b_k^T \bar{u} > 0 \). It can then be concluded from \( B^T u - \bar{v} = 0 \) that \( \bar{y}_k > 0 \). Hence, by theorem 2-3, \( y_k \) is a null variable (line 12). If the optimal solution \((\bar{x}, \bar{y}, \bar{y}_0)\) contains a variable \( y_k, k \in K_2 \) with \( \bar{y}_k > 0 \), then a non-null variable is simultaneously identified (line 13). Note that the algorithm will always terminate within \( p \) steps.

**Comment 3.** The initial simplex tableau for a linear program is obtained from the optimal simplex tableau of the preceding linear program by replacing the column vector corresponding to the variable \( y_0 \) in the latter tableau by the newly constructed a-vector.

**Comment 4.** The algorithm can also be terminated, as soon as a basic feasible solution has been found, for which \( y_0 > 0 \) or, as soon as an infinite solution has been found.
For both algorithms we can make the following additional remarks.

a. Apart from the null variables identified in line B of algorithms 2–5, no null variables are known before the termination of this algorithm. Algorithm 2–6, however, identifies at least one null variable and possibly a number of non-null variables in each step with \( y_0 = 0 \) (lines 12 and 13).

b. A less attractive property of algorithm 2–6 is that the addition of the s-vector can disturb the structure in the simplex tableaus. In algorithm 2–5 no elements are introduced which can disturb this structure.

c. Let the algorithms be used for the determination of the implicit equalities in the linear system \( a_i^T x \leq b_i \) (i.e. if \( y \in \mathbb{R}^m \) and if \( B \) is the (aux) unity matrix). If \( y \) has been identified as non-null variable, then there exists an \( x \in \mathcal{G} \), where \( \mathcal{G} := \{ x \in \mathbb{R}^m \mid a_i^T x \leq b_i \} \) (i.e. \( v_i-x \leq 0 \)) with \( a_i^T x = b_i \). All implicit equalities must then be present in the linear subsystem \( a_i^T x \leq b_i \) (i \( \in \mathcal{I} \setminus \{1\} \)). The algorithms can then be made more efficient by removing this inequality from the original system \( a_i^T x \leq b_i \) (i \( \in \mathcal{I} \)). In the simplex procedure this can be effected by preventing pivot operations to be executed in the row associated with this index 1. Note that the removal of an implicit equality from the linear system \( a_i^T x \leq b_i \) (i \( \in \mathcal{I} \)) is prohibited, as this may increase the dimension of \( \text{aff} \mathcal{G} \). It may then happen that the algorithms detect an inequality as being not an implicit equality (identification of non-null variable), whereas it is in fact an implicit equality. It is perhaps worthwhile to recall that \( |I_{\text{eq}}| \geq 2 \) if \( I_{\text{eq}} = \emptyset \) (remark 1).
II.3 REDUNDANT INEQUALITIES IN A CONSISTENT SYSTEM OF LINEAR INEQUALITIES

In the following chapters we sometimes need the concept of (strictly) redundant inequalities in the consistent linear system \( Ax \leq b \).

This concept is defined as follows.

**Definition 2.7.** The inequalities \( a_i^T x \leq b_1, (i \in L) \), with \( L \) a non-empty subset of \( I \), are said to be simultaneously redundant with respect to the consistent linear system \( Ax \leq b \), if \( a_i^T x \leq b_1, (i \in L) \) for all feasible solutions of the reduced linear system \( a_i^T x \leq b_1, (i \in I \setminus L) \). They are called simultaneously strictly redundant, if \( a_i^T x < b_1, (i \in L) \) for all feasible solutions of this reduced system.

We refer to reference [11] for algorithms which can remove this redundancy in linear systems.

The following property will be used in the next chapter.

**Theorem 2.8.** Given the consistent linear system \( Ax \leq b \) and the vector \( d \in \mathbb{R}^n \), \( d \neq 0 \) for which the index set \( I_d := \{ i \in I \mid a_i^T d > 0 \} \) is not empty. Then the inequalities \( a_i^T x \leq b_1, (i \in I_d) \) are not simultaneously redundant with respect to \( Ax \leq b \).

**Proof:** Assume that the inequalities \( a_i^T x \leq b_1, (i \in I_d) \) are simultaneously redundant. Then the system

\[
\begin{align*}
& a_i^T x > b_1 \quad (i \in I_d), \\
& a_i^T x \leq b_1 \quad (i \in I \setminus I_d)
\end{align*}
\]

is inconsistent.
This is equivalent to the inconsistency of the linear system
\[-a_i^T x \leq b_i, \quad i \in I_d,\]
\[a_i^T x - b_i y \leq 0, \quad i \in I_d,\]
\[-y \leq 0,\]
where \(y \in \mathbb{R}\). By the duality theorem of Motzkin, we then have the consistency of the linear system
\[
\sum_{i \in I_d} u_i a_i = \sum_{i \in I_d} u_i b_i = 0, \quad (2-9.a)
\]
\[
\sum_{i \in I_d} u_i b_i - \sum_{i \in I_d} u_i b_i - v = 0, \quad (2-9.b)
\]
\[
u > 0 \quad (i \in I), \quad (2-9.c)
\]
\[
u \geq 0 \quad (2-9.d)
\]
with at least \(v\) or one of the components \(u_i, (i \in I_d)\) strictly positive.

If we take the inner product of (2-9.a) with the vector \(d\) and recall that \(a_i^T d > 0, (i \in I_d)\) and \(a_i^T d \leq 0, (i \in I_d)\), then it follows that the system (2-9) cannot have a solution with one of the components \(u_i, (i \in I_d)\) strictly positive.

On the other hand, the system cannot have a solution with \(u_i = 0,\)
\((i \in I_d)\) and \(v > 0\), for this would lead to the inconsistency of the linear system \(a_i^T x \leq b_i, (i \in I_d)\), by the duality theorem of Farkas.

Hence, we may conclude that the system (2-9) is inconsistent, which is a contradiction.
PIECEWISE LINEAR FUNCTIONS ON POLYHEDRAL SETS

In the chapters III and V we shall deal with real-valued functions which are piecewise linear on a non-empty polyhedral subset $G$ of $\mathbb{R}^n$. This piecewise linearity is of the following form:

a) $G$ is partitioned into a finite family of polyhedral subsets,
b) the real-valued function is linear on each element of this partition.

The objective of this section is to give an exact description of this kind of piecewise linearity.

The concept of a finite polyhedral partition is given by the following definition.

Definition 2-9. The family $\varepsilon := \{G_s, (s \in S)\}$ is called a finite polyhedral partition of the non-empty polyhedral set $G \subset \mathbb{R}^n$, if it satisfies all conditions below:

a. $|S| < \infty$,
b. $G_s$ is a polyhedral subset of $G$ for all $s \in S$,
c. $G = \bigcup_{s \in S} G_s$,
d. $G_s \cap G_t$ is a common face of $G_s$ and $G_t$ for all $s$ and $t$ in $S$.

The interpretation of the conditions a, b and c in this definition is clear.

Condition d restricts the kind of overlap between the elements of the partition. Let $s$ and $t$ be two different indices of the set $S$. Condition d then states that the elements $G_s$ and $G_t$ are either

* For a definition of the concept face of a polyhedral set $G \subset \mathbb{R}^n$ we refer to [7]. Among other things the empty set and $G$ itself are faces of $G$. 

disjoint or meet in a common non-empty face. In the latter case we can distinguish the following exclusive cases.

1) The common face is of lower dimension than \( G_s \) and \( G_t \) (see common face of the elements \( G_1 \) and \( G_2 \) in figure 2-1).

2) The common face coincides with \( G_s \) or \( G_t \) (see elements \( G_s \) and \( G_2 \) in figure 2-1). For instance, let the common face coincide with \( G_s \). This implies that \( G_s \) is a face of \( G_t \). Hence, the face of an element of the partition is itself an element of the partition.

3) The common face coincides with \( G_s \) and \( G_t \). This means that \( G_s \) and \( G_t \) may coincide, although \( s \neq t \).

Such situations as mentioned in points 2 and 3 are allowed in definition 2-9, because they can actually occur in the partitions we shall use in the chapters III and V. Consequently, our definition 2-9 allows a finite polyhedral partition to contain elements which can be omitted without changing the partition of \( G^* \). In general, it is not possible to identify this kind of redundancy. Fortunately, it is not an obstacle to the developments in these chapters.

Note that the empty set is always an element of a polyhedral partition and that \( G := \{ G, G \} \) is also a partition of \( G \).

* An example is given in remark 4 of section IV-4.
The concept of a piecewise linear functional is given by the following definition.

**Definition 3-10.** Let $G$ be a non-empty polyhedral subset of $\mathbb{R}^n$ with polyhedral partition $G := \{ G_s | s \in S \}$. The real-valued function $f : G \to \mathbb{R}$ is called piecewise linear with respect to $G$, if $f$ is linear on each element of this partition.
CHAPTER III

THE MATHEMATICAL CONCEPT OF FLEXIBILITY IN POLYHEDRAL SETS

III-1 INTRODUCTION TO THE CONCEPTS OF FLEXIBILITY RANGE AND DIRECTION OF FLEXIBILITY

The basic quantity which will be used in the following to express the flexibility of a point \( x \) in the polyhedral set \( G := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) mathematically, is the Euclidean distance from \( x \) to the boundary of \( G \) in a certain direction. We therefore introduce the concepts of flexibility range and direction of flexibility.

**Definition 3-1.** The flexibility range of a point \( x \in G \) with respect to a vector \( d \in \mathbb{R}^n \), with \( \|d\| = 1 \), is the maximum distance one can move from that point \( x \) in the direction \( d \) without violating the boundary of \( G \).

Hence, for arbitrary \( x \in G \),

\[
s(x|d) := \sup \{ \sigma \in \mathbb{R} \mid (x + \sigma d) \in G \}
\]

\[
= \sup \{ \sigma \in \mathbb{R} \mid \sigma a_i^T d \leq b_i - a_i^T x, (i \in I) \}.
\]

(3-1)

Let the index set \( I_d \) be defined by

\[
I_d := \{ i \in I \mid a_i^T d > 0 \}.
\]
We note that $s(x|d) = \infty$, if the inequalities $a^T_i x \leq b_i, (i \in I_d)$ would be simultaneously redundant for non-empty $I_d$. This, however, is not possible by theorem 2-8.

It therefore follows from expression (3-1) that $s(x|d) = \infty$, if, and only if, the index set $I_d := \{ i \in I | a^T_i d > 0 \}$ is empty.

Clearly, this happens, if, and only if, $d$ is an element of the polyhedral cone

$$\{ d \in \mathbb{R}^n | A d \leq 0; d \neq 0 \}.$$

Also, if $s(x|d) = \infty$ for some $x \in G$, then this holds for all $x \in G$.

Note that it is easy to verify, whether a given vector $d \in \mathbb{R}^n$ is an element of this cone.

**Definition 3-2.** Let $d$ be a vector in $\mathbb{R}^n$ with $\|d\| = 1$. If $I_d \neq \emptyset$, then $d$ is called a direction of finite flexibility with respect to $G := \{ x \in \mathbb{R}^n | Ax \leq b \}$. It is called a direction of infinite flexibility, if $I_d = \emptyset$.

Hence, $s(.,d)$ is a well-defined function from $G$ into $\mathbb{R}$, if $d$ is of finite flexibility. The flexibility range $s(x|d)$ in a point $x \in G$ with respect to the direction $d$ of finite flexibility is expressed by

$$s(x|d) := \sup \{ \sigma \in \mathbb{R} | (x+\sigma d) \in G \}$$

$$= \max \{ \sigma \in \mathbb{R} | \sigma \leq \frac{b_i - a^T_i x}{a^T_i d}, (i \in I_d) \}$$

$$= \min \{ \frac{b_i - a^T_i x}{a^T_i d}, (i \in I_d) \}.$$  \hspace{1cm} (3-2)
Let $d$ be a direction of finite flexibility. Then it may happen that $s(x|d) > 0$ for some point $x$ in $G$ and $s(x|d) = 0$ for certain points on the (relative) boundary of $G$. However, it may also happen that $s(x|d) = 0$ for all points $x \in G$. This is the case, if the interior of $G$ is empty and if the direction $d$ is pointing outside the affine hull of $G$.

In the following we shall make a distinction between the directions of finite flexibility which allow a positive displacement inside $G$ and the directions for which such a displacement is not possible.

**Definition 3-3.** A direction $d$ of finite flexibility with respect to the polyhedral set $G := \{ x \in \mathbb{R}^n | Ax \leq b \}$ is called proper, if there exists an $x \in G$ with $s(x|d) > 0$. Otherwise, it is called improper.

Clearly, it is of interest to know whether a direction of finite flexibility is proper or not. However, although it is easy to verify that a direction $d$ is of finite flexibility, it is more difficult to verify whether it is proper or improper. The latter involves the identification of the implicit equalities in the linear system $Ax = b$, as will be shown in section 2 of this chapter.

In chapters IV and V it will become clear that neither an identified direction of infinite flexibility, nor an identified improper direction of finite flexibility is interesting for the main problems in flexible programming. In the following we shall therefore pay most attention to proper directions of finite flexibility.
III-2  PROPERTIES OF THE DIRECTION OF FLEXIBILITY

We recall the definition of the index sets \( I_{eq} \) and \( I_{in} \) given by (2-4). The following theorem gives a necessary and sufficient condition for a direction of finite flexibility to be proper.

**Theorem 3-4.** Let \( d \in \mathbb{R}^n \) be a direction of finite flexibility in the non-empty polyhedral set \( G := \{ x \in \mathbb{R}^n | Ax \leq b \} \). Then \( d \) is proper, if, and only if, \( I_d \subseteq I_{in} \).

**Proof.** If we partition the index set \( I \) into the disjoint subsets \( I_{eq} \) and \( I_{in} \), it follows from expression (3-1) that for an arbitrary \( x \in G \)

\[
s(x|d) = \sup \{ a_i^T d : a_i \in I_{eq}, b_i - a_i^T x, (i \in I_{in}) \}.
\]

From this expression it is to be concluded that there exists an \( x \in G \) with \( s(x|d) > 0 \), if, and only if, \( I_d \subseteq I_{in} \).

Note that it is necessary to partition the index set \( I \) into the disjoint subsets \( I_{eq} \) and \( I_{in} \) to verify whether or not a direction of finite flexibility is proper. For this partitioning one of the algorithms developed in subsection 11-2.3 can be used.

Let \( D_G \) be the set of proper directions of finite flexibility with respect to the non-empty polyhedral set \( G \). Then it follows from theorem 3-4 that this set is defined by

\[
D_G := \{ d \in \mathbb{R}^n | \|d\| = 1; I_d = \emptyset; I_d \subseteq I_{in} \} \quad (3-3)
\]

Note that \( D_G = \emptyset \), if \( I_{in} = \emptyset \). The following corollary of theorem 3-4 gives an alternative expression for \( D_G \). Here \( A_{eq} \) is the submatrix of \( A \) with row vectors \( a_i^T (i \in I_{eq}) \), with the agreement
that \( A_{eq} \) is the empty matrix and that every expression containing 
\( A_{eq} \) vanishes, if \( I_{eq} = \emptyset \).

**Corollary 3-5.** Let \( G \neq \emptyset \), then

\[
V_G = \{ d \in \mathbb{R}^n \mid \|d\| = 1; A_{eq}d = 0; \exists_{i \in I_{in}} a_i d > 0 \}. \quad (3-4)
\]

**Proof.** The statement is obvious, if either \( I_{eq} = \emptyset \) or \( I_{in} = \emptyset \). We therefore assume that this is not the case.

a. Let \( d \in V_G \). Then there exists an \( i \in I_{in} \) with \( a_i^T d > 0 \). As \( I_d \subseteq I_{in} \), we have \( I_d \cap I_{eq} = \emptyset \), which implies \( A_{eq}d \leq 0 \).
Since \( A_{eq}d \leq 0 \) is impossible, which follows from the application of the duality theorem of Stiemke to theorem 2-4, we have \( A_{eq}d = 0 \).

b. Conversely, let there exist an \( i \in I_{in} \) with \( a_i^T d > 0 \) and let \( A_{eq}d = 0 \). Then \( I_d \neq \emptyset \) and \( I_d \cap I_{eq} = \emptyset \). Hence, \( I_d \subseteq I_{in} \).

The following corollary gives expressions for \( V_G \), if \( G \) has certain special topological properties.

**Corollary 3-6.** Let \( G \neq \emptyset \) and \( I_{in} \neq \emptyset \).

a. If \( \text{int } G \neq \emptyset \), then \( V_G = \{ d \in \mathbb{R}^n \mid \|d\| = 1; I_d \neq \emptyset \} \).

b. If \( G \) is bounded, then \( V_G = \{ d \in \mathbb{R}^n \mid \|d\| = 1; A_{eq}d = 0 \} \).

c. If \( G \) is bounded and \( \text{int } G \neq \emptyset \), then \( V_G = \{ d \in \mathbb{R}^n \mid \|d\| = 1 \} \).

**Proof.** The proof becomes obvious from the definition of \( V_G \) in (3-3) and the expression for \( V_G \) in (3-4) by observing that
1. \( I_{eq} = \emptyset \), if \( \text{int} \, G \neq \emptyset \) (see definition of \( I_{eq} \) and \( I_{in} \) in (2-4)).

2. there always exists an \( i \in I \) with \( a_i^T \mathbf{d} > 0 \) if \( G \) is bounded. \( \Box \)

Remark 1. Note that a direction of finite flexibility is always proper, if \( \text{int} \, G \neq \emptyset \).

Remark 2. In the cases a and c in the above corollary we always have \( D_G \neq \emptyset \). For instance, \( D_G \) contains at least the normalized vectors \( a_i (i \in I) \). In case b, however, \( D_G \) is empty, if rank \( A_{eq} = n \). The following system of linear inequalities gives such an example in \( \mathbb{R}^2 \):

\[
x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 1.
\]

The solution set of this system consists of the single point \( x := (0,1) \). All inequalities are implicit equalities of this system. \( \Box \)

We have already noted earlier that \( I_{in} \neq \emptyset \) is a necessary condition for \( D_G \) to be non-empty. It is, however, not a sufficient condition, as is shown by the linear system

\[
x_1 + x_2 = 1, \quad x_1 + x_2 \leq 2.
\]

A necessary and sufficient condition for \( D_G \) to be non-empty is given in the following theorem.

Theorem 3-7. Let \( G := \{ x \in \mathbb{R}^p \mid A x \leq b \} \) be non-empty. Then \( D_G \neq \emptyset \) if, and only if, rank \( A \) \( > \) rank \( A_{eq} \).

Proof.

a. Let \( D_G \neq \emptyset \), then \( I_{in} \neq \emptyset \). Furthermore, let \( d \in D_G \), then \( A_{eq} d = 0 \) and there exists an \( l \in I_{in} \) with \( a_l^T d > 0 \). Hence, \( a_l^T \) is
linearly independent of the vectors $a_i^T$, $i \in I_{eq}$, which means that $\text{rank } A > \text{rank } A_{eq}$. Conversely, let $\text{rank } A > \text{rank } A_{eq}$, then $0 \leq \dim N(A) < \dim N(A_{eq})$, where $N(A)$ and $N(A_{eq})$ are the null spaces of $A$ and $A_{eq}$ respectively. From this inequality it follows that there exists a $d \in \mathbb{R}^n$, $\|d\| = 1$ with $a_i^T d = 0$ and $A d \neq 0$. Hence, there exists an $l \in I_{ln}$ with $a_i^T d > 0$. Consequently, $d \in E_0$, if $a_i^T d > 0$ and $-d \in E_0$, if $a_i^T d < 0$. Hence, $E_0 \neq \emptyset$.

In one of the problems to be formulated in the next chapter we need the reversed direction $-d$ in addition to a direction $d$ of flexibility. We shall finally derive two properties for such pairs of directions of flexibility.

If $d \in \mathbb{R}^n$ is a proper direction of flexibility, then $-d$ does not necessarily have the same property. For instance, let $C := \{ x \in \mathbb{R}^2 \mid x_1 \leq 1 \}$, then $d := (1,0)$ is a proper direction of flexibility while $-d$ is a direction of infinite flexibility. We have the following property though.

**Theorem 1.** Let $d$ and $-d$ be two directions of finite flexibility. Then $d$ is proper if, and only if, $-d$ is proper.

**Proof.** We recall the partition of the set $I$ into $I_{eq}$ and $I_{ln}$.

Since $d$ and $-d$ are of finite flexibility, we have

$$a_i^T d > 0 \quad \text{for certain } i \in I,$$

$$a_i^T (-d) > 0 \quad \text{for certain } i \in I.$$

This property shows the equivalence of the following statements:
a. $A_{eq}d = 0$ and there exists an $i \in I_{in}$ with $a_{i}^{T}d > 0$.

b. $A_{eq}(-d) = 0$ and there exists an $i \in I_{in}$ with $a_{i}^{T}(-d) > 0$.

The theorem then follows from corollary 3-5. □

The following theorem gives a necessary and sufficient condition for the existence of a pair $(d, -d)$ of proper directions of finite flexibility. Here $A_{in}$ is the submatrix of $A$ with row vectors $a_{i}^{T}(i \in I_{in})$ and with the same conventions made earlier for $A_{eq}$.

Theorem 3-3. Let $C := \{ x \in R^{n} | Ax \leq b \}$ and $C_{G}$ be non-empty. Then there exists a pair $(d, -d)$ of proper directions of finite flexibility, if, and only if, the linear system

$$A_{eq}^{T}u + A_{in}^{T}v = 0, \quad v > 0$$

(3-5)

is consistent.

Proof. Note that $C_{G} \neq \emptyset$ implies $I_{in} \neq \emptyset$. The expression for $C_{G}$ in (3-4) and the fact that $C_{G} \neq \emptyset$ lead to the conclusion that there exists a pair $(d, -d)$ of proper directions of finite flexibility, if, and only if, the linear system

$$A_{eq}d = 0, \quad A_{in}d \geq 0$$

is inconsistent. By the duality theorem of Tucker, this is equivalent to the consistency of the linear system (3-5). □
III-3 PROPERTIES OF THE FLEXIBILITY RANGE

III-3.1 General properties

The following lemma will often be referred to in this subsection.

Lemma 3.11. Let \( d \in \mathbb{R}^n \) be a direction of finite flexibility in the non-empty polyhedral set \( G := \{ x \in \mathbb{R}^n | Ax \leq b \} \). Then the inequality \( a_i^T x \leq b_1 \), \( i \in I_d \) is strictly redundant with respect to \( Ax \leq b \), if, and only if,

\[
s(x|d) < \frac{b_1 - a_1^T x}{a_1^T d} \quad \text{for all } x \in G. \tag{3-6}
\]

Proof. Let for some \( i \in I_d \), \( a_i^T x \leq b_1 \) be strictly redundant with respect to \( Ax \leq b \). We then have for all \( x \in G \)

\[
s(x|d) = \sup \{ 0 \leq \xi \in \mathbb{R}^n \mid a_i^T (x + \xi d) \leq b_1 \}, \quad i \in I_d.
\]

Since \( a_1^T d > 0 \), it follows that

\[
s(x|d) < \frac{b_1 - a_1^T x}{a_1^T d} \quad \text{for all } x \in G.
\]

The reverse statement is also true, if the inequality (3-6) holds true, then \( a_1^T x = b_1 \) is strictly redundant. For assume that this is not the case. Then there exists an \( x \in G \) with \( a_1^T x \neq b_1 \). Hence, according to (3-6), \( s(x|d) < 0 \), which is a contradiction.

Let \( d \in \mathbb{R}^n \) be a direction of finite flexibility and let \( i \in I_d \). We then define
\[ C_d^1 := \{ x \in \mathbb{C} \mid s(x|d) = \frac{b_1 - a_1^T x}{a_1^T d} \} \]  

(3.7)

From lemma 3.11 it follows that \( C_d^1 = \emptyset \), if, and only if, the inequality \( a_1^T x \leq b_1 \) is strictly redundant with respect to \( A x \leq b \).

The set \( C_d^1 \) can be interpreted as follows. If \( x \in C_d^1 \), then it follows from (3-7) that

\[ a_1^T [x + s(x|d)d] = a_1^T x + a_1^T d \frac{b_1 - a_1^T x}{a_1^T d} = b_1. \]

If \( x \in C \), but \( x \notin C_d^1 \), then

\[ s(x|d) < \frac{b_1 - a_1^T x}{a_1^T d}. \]

Hence, \( C_d^1[x+s(x|d)d] < b_1 \). The set \( C_d^1 \) is thus precisely the set of those points \( x \in C \) with the property that, if we move from \( x \) in the direction \( d \), then the relative boundary* of \( C \) will be met in the hyperplane \( \{ x \in \mathbb{R}^n \mid a_1^T x = b_1 \} \).

For clarification, we have given a geometrical representation of this set in figure 3-1 below (see shaded area).

---

* The relative boundary of \( C \) is the set of points

\[ \{ x \in C \mid x \notin ri \ C \} \]
The following theorem gives a polyhedral representation of $c_i^d$, $l \in 1_d$.

Theorem 3-12. Let $d \in \mathbb{R}^n$ be a direction of finite flexibility and let $l \in 1_d$ be such that $G_i^d \neq \emptyset$. Then $G_i^d$ coincides with the solution set of the linear system

\[
\begin{align*}
\begin{pmatrix}
\frac{a_i}{a_i^d} & \frac{a_1}{a_1^d} \\
\frac{b_i}{a_i^d} & \frac{b_1}{a_1^d}
\end{pmatrix} x & \leq \begin{pmatrix}
\frac{b_i}{a_i^d} & \frac{b_1}{a_1^d}
\end{pmatrix} (i \in 1_d - \{l\}),
\end{align*}
\]

(3.8a)
\[
\begin{align*}
    a_i^T x & \leq b_i \quad (i \in I - \{\ell\}), & (3-8.b) \\
    a_{\ell}^T x & \leq b_{\ell}. & (3-8.c)
\end{align*}
\]

**Proof.** Let \( d \in \mathbb{R}^n \) and \( l \in I_d \) such that \( a_{\ell} d^T \not\in \varnothing \).

a. If \( x \in G_d^d \), it can easily be derived from the expression for \( s(x;d) \) in (3-2) and the definition of \( G_d^d \) in (3-7), that \( x \) satisfies the linear system (3-8).

b. In the converse we first show the consistency of this system. Since \( G_d^d \neq \varnothing \), \( a_{\ell}^T x \leq b_{\ell} \) cannot be strictly redundant (according to (3-7) and lemma 3.11)). This means that there exists an \( \tilde{x} \in \varnothing \) with \( a_{\ell}^T \tilde{x} = b_{\ell} ; a_{\ell}^T \tilde{x} \leq b_{\ell} \quad (i \in I - \{\ell\}) \). Hence, also

\[
\begin{align*}
    \left( \frac{a_{\ell}}{a_{\ell}^T d} - \frac{a_i}{a_{i}^T d} \right)^T \tilde{x} & \leq \frac{b_{\ell}}{a_{\ell}^T d} - \frac{b_i}{a_{i}^T d} \quad (i \in I_d - \{\ell\}), \\
    a_{\ell}^T \tilde{x} & \leq b_{\ell} \quad (i \in (I_d - \{\ell\}) \cup \{\ell\}),
\end{align*}
\]

which shows the consistency of (3-8). Let \( x \) be an arbitrary point satisfying this system, then it satisfies in particular (3-8.a) and (3-8.c). Since \( a_{\ell}^T d > 0, (i \in I_d) \), we have

\[
0 \leq \frac{b_i - a_i^T x}{a_i^T d} \leq \frac{b_{\ell} - a_{\ell}^T x}{a_{\ell}^T d} \quad (i \in I_d - \{\ell\}).
\]

These inequalities, together with (3-8.b), lead to \( x \in C \) and

\[ s(x;d) = \frac{b_{\ell} - a_{\ell}^T x}{a_{\ell}^T d}. \]
Hence, \( x \in G_1^d \). \( \square \)

The following corollary from the above theorem shows that \( G_1^d \) has special topological properties. If \( d \in \mathbb{R}^n \) is an improper direction of finite flexibility.

**Corollary 3-13.** Let \( d \in \mathbb{R}^n \) be an improper direction of finite flexibility. Then \( G_1^d \) is a face of \( G \).

**Proof.** If \( d \) is an improper direction of finite flexibility, then \( s(x|d) = 0 \) for all \( x \in G \). Hence, from the definition of \( G_1^d \) in (3-7) it follows that

\[
G_1^d = \{ x \in G \mid a_1^T x = b_1 \}.
\]

\( \square \)

Let \( G_1^d \neq \varnothing \) for \( l \in I_d \). We can then make the following remarks on additional properties of this set.

**Remark 1.** From the linear system (3-8) it follows that, if \( I_d^{-1}(l) = \varnothing \), then \( G_1^d = G \). Such a situation occurs, for instance, if we choose \( d := (1,0) \) as direction of finite flexibility with respect to the consistent linear system

\[
x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 0.
\]

If \( I_d^{-1}(l) \neq \varnothing \), we then have for all \( i \in I_d^{-1}(l) \)

\[
\left( \frac{a_i}{a_1^d} - \frac{a_l}{a_1^d} \right)^T d = 0.
\]

Hence, for arbitrary \( i \in I_d^{-1}(l) \), the hyperplane
\[ \begin{aligned} &\{x \in \mathbb{R}^n \mid \left( \frac{a_j}{a_j^T d} - \frac{a_{i_1}}{a_{i_1}^T d} \right)^T x = \frac{b_j}{a_j^T d} - \frac{b_{i_1}}{a_{i_1}^T d} \} \\
\end{aligned} \]

has a normal perpendicular to \( d \) and passes through the intersection of the pair of hyperplanes

\[ \{x \in \mathbb{R}^n \mid a_i^T x = b_i, \ i \in I_d \setminus \{1\} \} \]

and

\[ \{x \in \mathbb{R}^n \mid a_1^T x = b_1 \}. \]

This property is represented in figure 3-2 below.

**FIG. 3-2** THE EXPRESSIONS FOR THE BOUNDARY HYPERPLANES OF THE SET \( G_1 \).
Note that the linear system (3-6) may contain strictly redundant inequalities among the inequalities (3-8.a).

Remark 2. Let the linear inequality \( a^T_1x \leq b_1 \), \( i \in 1_d - \{1\} \) be strictly redundant with respect to the linear system \( Ax \leq b \). It then follows from lemma 3-11 that

\[
S(x; d) < \frac{b_1 - a^T_1x}{a^d_1/x}
\]

for all \( x \in G \). Hence, also for all \( x \in G^d_1 \).

From the definition of \( G^d_1 \) in (3-7) it then follows that

\[
\frac{b_1 - a^T_1x}{a^d_1/x} < \frac{b_1 - a^T_1x}{a^d_1/x}
\]

for all \( x \in G^d_1 \), which means that the inequality

\[
\left(\begin{array}{c}
\frac{a_1}{a^d_1} - \frac{a_1}{a^d_1} \\
\frac{a_1}{a^d_1} - \frac{a_1}{a^d_1}
\end{array}\right)T \leq \frac{b_1}{a^d_1/x} - \frac{b_1}{a^d_1/x}
\]

is strictly redundant with respect to the linear system (3-6). \( \Box \)

Remark 3. Let \( \frac{a_1}{a^d_1} = \frac{a_1}{a^d_1} \) for \( i \in 1_d - \{1\} \) in the linear system (3-8).

This means the following. The hyperplanes \( \{ x \in \mathbb{R}^d \mid a^T_1x = b_1 \} \) and \( \{ x \in \mathbb{R}^d \mid a^T_1x = b_1 \} \) are parallel. Since \( G^d_1 \neq \emptyset \), the inequality \( a^T_1x \leq b_1 \) cannot be strictly redundant with respect to \( Ax \leq b \). It can then easily be shown that, either the inequalities \( a^T_1x \leq b_1 \) and \( a^T_1x \leq b_1 \) coincide, or \( a^T_1x \leq b_1 \) is strictly redundant with respect to \( Ax \leq b \). \( \Box \)
III-3.2 Concavity and Lipschitz properties

In this subsection we give two theorems showing that the function \( s(.|d) : G \times R_+ \rightarrow \) is concave and satisfies the Lipschitz condition for each direction of finite flexibility.

We recall that the non-empty set \( S \subseteq R^n \) is said to be convex, if \( x, y \in S \) implies that \( \lambda x + (1-\lambda)y \in S \) for all \( \lambda \in [0,1] \). Furthermore, the function \( f : S \times R \rightarrow \) is said to be concave on \( S \), if

\[
f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)
\]

for all \( \lambda \in [0,1] \). The following theorem shows the concavity of the function \( s(.|d) \).

**Theorem 3-15.** The function \( s(.|d) : G \times R_+ \) is concave for each direction \( d \) of finite flexibility.

**Proof:** Let the direction of finite flexibility \( d \) be fixed and let \( x, y \in G \). Then \( \lambda x + (1-\lambda)y \in G \) for all \( \lambda \in [0,1] \), since \( G \) is convex. It follows that

\[
s(\lambda x + (1-\lambda)y|d) = \\
= \min \left\{ \lambda \frac{b_i - a_i^T x}{a_i^T d} + (1-\lambda) \frac{b_i - a_i^T y}{a_i^T d}, (i \in I) \right\} \\
\geq \lambda \min \left\{ \frac{b_i - a_i^T x}{a_i^T d}, (i \in I) \right\} + (1-\lambda) \min \left\{ \frac{b_i - a_i^T y}{a_i^T d}, (i \in I) \right\} \\
= \lambda s(x|d) + (1-\lambda)s(y|d).
\]

\( \square \)
In order to prove the Lipschitz property of \( s(d) \) we need the following lemma.

**Lemma 3.16.** Let \( \{ u_j, (j \in J) \} \) and \( \{ v_j, (j \in J) \} \) be two finite sequences of real numbers. Then the following inequality holds.

\[
|\min\{u_j, (j \in J)\} - \min\{v_j, (j \in J)\}| \leq \max\{|u_j - v_j|, (j \in J)\}.
\]

**Proof.** We have

\[
\min\{u_j, (j \in J)\} = \min\{u_j - v_j + v_j, (j \in J)\} \geq \\
\geq \min\{u_j - v_j, (j \in J)\} + \min\{v_j, (j \in J)\}.
\]

Hence,

\[
\min\{v_j, (j \in J)\} - \min\{u_j, (j \in J)\} \leq - \min\{u_j - v_j, (j \in J)\} = \\
= \max\{v_j - u_j, (j \in J)\} \leq \max\{|u_j - v_j|, (j \in J)\}.
\]

Similarly, it can be shown that

\[
\min\{u_j, (j \in J)\} - \min\{v_j, (j \in J)\} \leq \\
\leq \max\{|u_j - v_j|, (j \in J)\}.
\]

We recall that a function \( f: S \to \mathbb{R}; S \subseteq \mathbb{R}^n; S \neq \emptyset \) is said to have the Lipschitz property on \( S \), if there exists a real number \( M \geq 0 \), such that

\[
|f(x) - f(y)| \leq M\|x - y\|
\]

for all \( x, y \in S \). This condition implies in particular that \( f \) is uniformly continuous on \( S \).
Theorem 3.17. The function \( s(.|d): G \rightarrow \mathbb{R}_+, G \neq \emptyset \) has the Lipschitz property on \( G \) for each direction \( d \) of finite flexibility.

**Proof.** Let the direction \( d \) of finite flexibility be fixed and let \( x \) and \( y \) be arbitrary points in \( G \). Then it follows from lemma 3.16 that

\[
|s(x|d) - s(y|d)| = \nonumber
\]

\[
= \left| \min \left\{ \frac{b_i - a_i^T x}{a_i^T d}, (i \in I_d) \right\} - \min \left\{ \frac{b_i - a_i^T y}{a_i^T d}, (i \in I_d) \right\} \right| 
\]

\[
\leq \max \left\{ \frac{|a_i^T(x - y)|}{a_i^T d}, (i \in I_d) \right\}. 
\]

If we now apply the Schwarz inequality to the inner product \( |a_i^T(x - y)| \), we find that

\[
|s(x|d) - s(y|d)| \leq M_d \|x - y\|. 
\]

where \( M_d \) the non-negative real number

\[
M_d = \max \left\{ \frac{\|a_i\|}{a_i^T d}, (i \in I_d) \right\}. 
\]

III.3.3 Piecewise linearity property

We finally show that the function \( s(.|d): G \rightarrow \mathbb{R}_+ \) is piecewise linear on \( G \) for each direction \( d \) of finite flexibility. We refer to the definition of the sets \( c_{l}^{d}, (l \in I_d) \) introduced in (3.7). The following lemma is an introduction to theorem 3.19.
Lemma 3-18. Let \( G \neq \emptyset \), then \( G = \bigcup_{d \in D} G_d^d \) for each direction \( d \in D \)
of finite flexibility.

**Proof.** Let the direction \( d \) of finite flexibility be fixed and let \( x \in G \). Then \( s(x|d) \) is well-defined by (3-2). There exists an \( l \in D \) with

\[
s(x|d) = \frac{b_l - a_l x}{d_l}
\]

which, according to the definition of \( G_d^d \) in (3-7), means that \( x \in G_d^d \). The remaining part of the proof is obvious.

In section 3 of chapter II we introduced the concept of a finite polyhedral partition of a polyhedral subset of \( \mathbb{R}^n \) (definition 2-9). The following theorem gives such a partition for \( G \).

**Theorem 3-19.** Let \( G \neq \emptyset \) and \( d \in \mathbb{R}^n \) be a direction of finite flexibility. Then the family \( G^d := \{ G_d^d \} \) is a finite polyhedral partition of \( G \).

**Proof.** Let \( d \in \mathbb{R}^n \) be fixed. To prove the theorem, the conditions in definition 2-9 must be verified. Obviously \( |D| \leq ||d|| < \infty \).

Further, \( G^d \) is a polyhedral subset of \( G \) for all \( i \in D \), according to theorem 3-12 and the fact that the empty set is polyhedral.

Condition c follows from lemma 3-18. To show condition d, let \( l, j \in D \). The proof of the theorem is apparent, if either \( l \neq j \) or \( G_d^d \cap G_j^j = \emptyset \), \( l \neq j \). So we assume that \( G_d^d \cap G_j^j \neq \emptyset \), \( l \neq j \). It must then be shown that \( G_d^d \cap G_j^j \) is a common face of \( G_d^d \) and \( G_j^j \).

It follows from theorem 3-12 that \( G_d^d \cap G_j^j \) is the solution set of the linear system
\[
\begin{align*}
\left( \begin{array}{c}
\frac{a_j}{a_j d} - \frac{a_1}{a_1 d} \\
\frac{a_2 j}{a_2 d}
\end{array} \right)^T x &= \frac{b_3}{a_3 d} - \frac{b_1}{a_1 d}, \\
\left( \begin{array}{c}
\frac{a_j}{a_j d} - \frac{a_1}{a_1 d} \\
\frac{a_2 j}{a_2 d}
\end{array} \right)^T x &\leq \frac{b_j}{a_j d} - \frac{b_1}{a_1 d} \quad (i \in I_d \setminus \{1,j\}), \\
\left( \begin{array}{c}
\frac{a_j}{a_j d} - \frac{a_1}{a_1 d} \\
\frac{a_2 j}{a_2 d}
\end{array} \right)^T x &\leq \frac{b_1}{a_1 d} - \frac{b_1}{a_1 d} \quad (i \in I_d \setminus \{1,j\}) \quad (3-9.4)
\end{align*}
\]

If
\[
\frac{a_j}{a_j d} - \frac{a_1}{a_1 d} = 0, 
\quad (3-10)
\]

we have (see remark 4 in subsection III-3.1) either:

a. \( a_j = a_1 \) and \( b_j = b_1 \), in which case it follows from (3-9) and the polyhedral representations for \( C^d_1 \) and \( C^d_j \) in theorem 3-12 that \( C^d_1 = C^d_j \). Hence, \( C^d_1 \cap C^d_j = C^d_1 = C^d_j \), in which case condition \( d \) is satisfied for the index pair \( (1,j) \).

or

b. \( a_j^T x \leq b_j \) is strictly redundant with respect to the linear system \( Ax \leq b \). Then, \( C^d_j = \emptyset \) by lemma 3-11 and the expression for \( C^d_j \) in (3-7). Hence, \( C^d_1 \cap C^d_j = \emptyset \), in which case condition \( d \) is satisfied as well.

Hence, we assume that the equality (3-10) does not hold.

Let the affine space, represented by (3-9.a), be denoted by \( \mathbb{H} \).

Since the system (3-9) is consistent, it follows from theorem 3-12 and the expressions (3-9.a), (3-9.b) and (3-9.d) that \( C^d_1 \cap \mathbb{H} \) is a non-empty face of \( C^d_1 \). Similarly, \( C^d_j \cap \mathbb{H} \) is a non-empty face of \( C^d_j \). We shall now show that
\[ \mathcal{C}_1^d \cap \mathcal{C}_2^d = \mathcal{C}_1^d \cap H = \mathcal{C}_2^d \cap H. \]

Obviously,

\[ \mathcal{C}_1^d \cap \mathcal{C}_2^d \subseteq \mathcal{C}_1^d \cap H. \]

Let \( x \in \mathcal{C}_1^d \cap H \), then \( x \in \mathcal{C}_1^d \) and, by theorem 3-12, \( x \) satisfies in particular (3-9.b). Since \( x \in H \), we can draw the following conclusions.

a. From the property that \( x \) satisfies (3-9.a) and (3-9.b) it can be derived that \( x \) also satisfies (3-9.c).

b. From (3-9.a) it follows that

\[
\begin{pmatrix}
\frac{a_{i,1}^T}{a_{j,1}^T} \\
\frac{a_{i,2}^T}{a_{j,2}^T}
\end{pmatrix} x = \frac{b_{i,1} - a_{i,2}^T x}{a_{j,1}^T} \leq \frac{b_{j,2}}{a_{j,2}^T}
\]

since \( a_{j,1}^T > 0; a_{j,2}^T > 0 \) and \( b_{i,1} - a_{i,2}^T x \geq 0 \). Hence, \( a_{j,2}^T x \leq b_j \).

These observations lead to the conclusion that \( x \in \mathcal{C}_2^d \). Hence, \( x \in \mathcal{C}_1^d \cap \mathcal{C}_2^d \) and \( \mathcal{C}_1^d \cap \mathcal{C}_2^d = \mathcal{C}_1^d \cap H \).

A similar reasoning can be used to show that \( \mathcal{C}_1^d \cap \mathcal{C}_2^d = \mathcal{C}_2^d \cap H \), which completes the proof of the theorem.

The concept of a piecewise linear function, defined on a non-empty polyhedral set, was introduced in section 3 of chapter II (definition 2-10). We are now able to show that \( s(\cdot | d) \) is piecewise linear on \( \mathcal{C} \).

**Theorem 3-20.** Let \( \mathcal{C} \neq \emptyset \). Then \( s(\cdot | d) : \mathcal{C} \to \mathbb{R} \) is piecewise linear on \( \mathcal{C} \) for each direction \( d \) of finite flexibility.

**Proof.** Let \( d \) be a direction of finite flexibility. We then have that the family \( \mathcal{C}^d := \{ \mathcal{C}_i^d \mid i \in I_d \} \) is a finite polyhedral
partition of $G$ and

$$s(x; d) = \frac{b_i}{a_i^d} - \left( \frac{a_i}{a_i^d} \right)^T x$$

for all $i \in I_d$ with $G^d_i \neq \emptyset$ and for all $x \in G^d_i$ (by (3-7)). \qed
CHAPTER IV

THE WEIGHTED DISTANCE PROBLEM IN FLEXIBLE PROGRAMMING

IV-1 INTRODUCTION

The problem of finding a point in the non-empty polyhedral set $G$, for which the minimum Euclidean distance to the boundary hyper-planes of $G$ is maximal, is a well-known problem in linear programming*. In terms of our concepts of direction of flexibility and flexibility range, this problem can be formulated as follows. Let the directions of finite flexibility be

$$d_k := \frac{a_k}{\|a_k\|} \quad (k \in I).$$

Then find the point $x \in G$ for which

$$\min \{ s(x|d_k), \ (k \in I) \}$$

is maximal.

The weighted distance problem in flexible programming is a generalization of the above problem. We shall use directions which do not necessarily coincide with the normals of the constraints of the linear system $Ax \leq b$. Moreover, factors will be used to weight the flexibility ranges.

* This problem is sometimes called the inscribed sphere problem. This designation is, however, not consistent with the one normally used for figures inscribed in polyhedral sets. See, for instance, reference [12].
Let \( K := \{ 1, \ldots, p \} \) be a finite index set. The problem to be considered in this chapter then reads as follows.

**Problem 4-1**. (weighted distance problem in flexible programming)

\[
\begin{align*}
\text{maximize} & \quad \rho(x) & (4-1.a) \\
\text{subject to} & \quad Ax \leq b, & (4-1.b) \\
\text{where} & \quad \rho(x) := \min \{ \phi_k \rho(x|d_k), (k \in K) \} & (4-1.c)
\end{align*}
\]

and \( \phi_k \in (0,1) \) a weighting factor and \( d_k \) a direction of finite flexibility for all \( k \in K \).

Let \( \bar{x} \in G \) be a solution of this problem. One can then move from \( \bar{x} \) with respect to the directions \( d_1, \ldots, d_k, \ldots, d_p \) at least over the distance \( \rho(\bar{x}) \) without violating the boundary of \( G \).

We shall use the following notations in this chapter:

\[
\begin{align*}
I_k := \{ i \in I \mid a_i^T d_k > 0 \} & \quad (k \in K), & (4-2) \\
K_i := \{ k \in K \mid a_i^T d_k > 0 \} & \quad (i \in I), & (4-3) \\
I_0 := \bigcup_{k \in K} I_k. & \quad (4-4)
\end{align*}
\]

Let \( D \) be the \((n \times p)\) matrix with column vectors \( d_1, \ldots, d_k, \ldots, d_p \). The interpretation of the above index sets is then as follows. \( I_k \) is the set of row indices of the matrix \( AD \), for which the entries in column \( k \) are positive; \( K_i \) is the set of column indices of \( AD \), for which the entries in row \( i \) are positive; \( I_0 \) is the set of indices, for which the rows of \( AD \) contain at least one positive entry. Note that, by the definition of the direction of finite
flexibility (definition 3-2), \( I_k \neq \emptyset \) for all \( k \in K \). Hence, each column in \( AD \) contains at least one positive entry. Note also that \( I_0 \) can equivalently be represented by

\[
I_0 = \{ i \in I \mid \exists k \in K \text{ s.t. } a_{ik}^T d_k > 0 \} \quad (4-5.a)
\]

\[
= \{ i \in I \mid K_i \neq \emptyset \}. \quad (4-5.b)
\]

We always have \( I_0 \neq \emptyset \). This would otherwise lead to

\[
a_i^T d_k \leq 0, (i \in I; k \in K),
\]

which contradicts the condition that \( I_k \neq \emptyset, (k \in K) \).

IV-2 A LINEAR PROGRAMMING FORMULATION OF THE WEIGHTED DISTANCE PROBLEM

In the proof that problem 4-1 can be formulated as a linear program we need the replacement of index pairs of the set

\[
U_1 := \{ (i,k) \mid (k \in K); (i \in I_k) \}
\]

by index pairs of the set

\[
U_2 := \{ (i,k) \mid (i \in I_0); (k \in K_1) \}.
\]

The following lemma shows that these sets coincide.

**Lemma 4-2.** \( U_1 = U_2 \).

**Proof.** From the definitions of \( I_k; (k \in K) \); \( K_1; (i \in I) \) and \( I_0 \) in (4-2); (4-3) and (4-4) respectively, it follows that the following statements are equivalent:

...
a. \( k \in K \) and \( i \in I_k \).

b. \( i \in I_0; a_{lk}^T d_k > 0 \) and \( k \in K \).

c. \( i \in I_0 \) and \( k \in K_i \).

A linear programming formulation of the weighted distance problem will be given in the following theorem.

**Theorem 4-3.** Consider problem 4-1 and the following linear program.

**Problem 4-4.** (linear program).

\[
\text{maximize } \quad \rho \\
\text{subject to } \quad a_{lx}^T x + w_i \rho \leq b_i \quad (i \in I_0), \\
\quad a_{lx}^T x \leq b_i \quad (i \in I-I_0),
\]

where

\[
w_i := \max \{ \frac{a_{lk}^T d_k}{\phi_k}; (k \in K_i) \}
\]

for all \( i \in I_0 \).

The following two statements are then equivalent:

a. \( x \) is a solution to problem 4-1 with \( \rho := \rho(x) \),

b. \((\rho, x)\) is a solution to problem 4-4.

**Proof.** From (4-1.c) and the expression for the flexibility range in (3-2) it follows that
\[ p(x) = \min \left\{ \phi_k s(x|d_k), (k \in K) \right\} \]

\[ = \min \left\{ \phi_k \min \left\{ \frac{b_i - a_i^T x}{s_i^2 d_k}, (i \in I_k) \right\}, (k \in K) \right\} \]

\[ = \min \left\{ [b_i - a_i^T x] \frac{\phi_k}{s_i^2 d_k}, (k \in K); (i \in I_k) \right\}. \]

Hence, by lemma 4-2:

\[ p(x) = \min \left\{ [b_i - a_i^T x] \frac{\phi_k}{s_i^2 d_k}, (i \in I_0), (k \in K_1) \right\} \]

\[ = \min \left\{ [b_i - a_i^T x] \min \left\{ \frac{\phi_k}{s_i^2 d_k}, (k \in K_1) \right\}, (i \in I_0) \right\}. \]

If we now substitute the variable \( \omega_1 \), defined in (4-6.d), into the latter expression, we find that

\[ p(x) = \min \left\{ \frac{b_i - a_i^T x}{\omega_1}, (i \in I_0) \right\}. \]

The theorem then follows from the equivalence of the following problems:

1. \[ \text{maximize } \left\{ p(x) \mid A x \leq b \right\}, \]

2. \[ \text{maximize } \left\{ \min \left\{ \frac{b_i - a_i^T x}{\omega_1}, (i \in I_0) \right\} \mid A x \leq b \right\}, \]
3. \[ \text{maximize } \left\{ \rho \mid a_i^T x - b_i / \omega_i, (i \in I_0); Ax \leq b \right\}, \]

4. \[ \text{maximize } \left\{ \rho \mid a_i^T x + \omega_i \rho \leq b_i, (i \in I_0); Ax \leq b \right\}, \]

5. \[ \text{maximize } \left\{ \rho \mid a_i^T x + \omega_i \rho \leq b_i, (i \in I_0) \right\}. \]

This completes the proof of this theorem. 

The following remarks deal with some properties of the weighted distance problem.

**Remark 1.** Since the directions \( d_1, \ldots, d_K, \ldots, d_d \) are of finite flexibility, we have \( \rho(x) \leq \epsilon \) for all \( x \in G \). Of course, this does not necessarily imply that program 4-1 (and therefore also linear program 4-4) has a finite solution (see also remark 2).

If the direction \( d_k \) of finite flexibility is improper, \( s(x|d_k) = 0 \) for all \( x \in G \), which implies that \( \rho(x) = 0 \) for all \( x \in G \) and \( \rho = 0 \) for the optimal solution of linear program 4-4. Hence, the weighted distance problem in flexible programming is not interesting, if at least one of the directions of finite flexibility is improper.

If all directions are proper, there exists an \( x \in G \) with \( s(x|d_k) > 0 \) for all \( k \in K \). Since also \( \delta_k > 0 \) for all \( k \in K \), we may conclude that \( \rho(x) > 0 \), if \( x \) is a solution to the weighted distance problem. Note that this does not necessarily imply that such a solution lies in the interior of \( G := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \). For instance, it can be shown that the weighted distance problem has the solution \( x = 0 \) with \( \rho(0) = \frac{1}{2} \sqrt{2} \), when applied to the consistent linear system.
\[ x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 0 \]

with proper direction \( d := (1,1) \).

**Remark 2.** The dual problem of the linear programming problem 4-4 reads:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in I_0} u_i b_i + \sum_{i \in I-I_0} v_i b_i \\
\text{subject to} & \quad \sum_{i \in I_0} u_i a_i + \sum_{i \in I-I_0} v_i a_i = 0, \quad (4-7.b) \\
& \quad \sum_{i \in I_0} u_i w_i = 1, \quad (4-7.c) \\
& \quad u_i \geq 0 \ (i \in I_0), \quad (4-7.d) \\
& \quad v_i \geq 0 \ (i \in I-I_0). \quad (4-7.d)
\end{align*}
\]

If the primal problem 4-4 has a solution, so does the dual problem. Since the weighting factors \( w_i \ (i \in I_0) \) are strictly positive, we have the consistency of the linear system

\[
\begin{align*}
\sum_{i \in I_0} u_i a_i + \sum_{i \in I-I_0} v_i a_i = 0, \\
& \quad u_i \geq 0 \ (i \in I_0), \quad (4-8) \\
& \quad v_i \geq 0 \ (i \in I-I_0)
\end{align*}
\]
with at least one of the variables \( u_i, (i \in I_0) \) strictly positive.

Conversely, the consistency of the linear system (4-7.b), (4-7.c) and (4-7.d) in the dual problem follows from the consistency of the linear system (4-8). Moreover, the consistency of the linear system (4-6.b) and (4-6.c) in the primal problem 4-4 follows from the consistency of the linear system \( Ax \leq b \).

The primal problem cannot have an unbounded solution, since this would lead to the inconsistency of the linear system (4-7.b), (4-7.c) and (4-7.d) in the dual problem. Hence, we may conclude that the weighted distance problem 4-1 has a (bounded) solution, if and only if, the linear system (4-6) is consistent.

Remark 3. Let the directions \( d_1, \ldots, d_k, \ldots, d_p \) be proper and let \( K_i \neq \emptyset \) for all \( i \in I \). Then \( I_0 = I \) (see (4-5.b)). For this particular case we have the following property. Let \((\tilde{x}, \tilde{p})\) be a solution of the linear programming problem 4-4. Then, \( \tilde{p} > 0 \) by theorem 4-3 and remark 1. Since also \( \omega_i > 0 \) for all \( i \in I \), we have that

\[
a_i^T \tilde{x} < b_i \quad (i \in I).
\]

Hence, a solution to the weighted distance problem lies in the interior of \( G : = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \), if \( K_i \neq \emptyset \) for all \( i \in I \) and if the directions of finite flexibility are proper.

Remark 4. Let \( \text{int} G \neq \emptyset \). If we choose

\[
\theta_k = \frac{b_i}{c} \quad \text{and} \quad d_k = \frac{b_i}{\| b \|}
\]

for all \( k \in K \), the linear program 4-4 is transformed into
maximize \quad p

subject to \quad a_i^T x + \|a_i\| p \leq b_i \quad (i \in I).

This linear program solves the well-known problem of finding a
point in G, for which the minimum Euclidean distance to the
boundary of G is maximal. It can be shown that a solution of this
linear program is the centre of an n-dimensional sphere which
lies entirely in G and has maximum radius. \qed
CHAPTER V
THE EQUILIBRIUM PROBLEM IN FLEXIBLE PROGRAMMING

V-1 INTRODUCTION

Let $d_1, \ldots, d_p, \ldots, d_p$ be directions of finite flexibility. In the equilibrium problem we always consider the reversed directions $-d_1, \ldots, -d_k, \ldots, -d_p$ as well and assume that these are also finite. Contrary to the weighted distance problem, however, we shall impose requirements on the distance properties for each pair of directions $\{d_k, -d_k\}$ separately. This is achieved as follows. Let the line segment $L_k(x)$ with $x \in G$ and $k \in \{1, \ldots, p\}$ be given by

$$L_k(x) := \{ y \in \mathbb{R}^d \mid y := x + \lambda d_k, \quad \lambda \in \mathbb{R} \}. \quad (5-1)$$

Note that the length of this line segment is equal to

$$s(x|d_k) + s(x|-d_k).$$

We then choose a point $\bar{x} \in L_k(x)$ for which

$$\min \{ \phi_k s(x|d_k), (1-\phi_k) s(x|-d_k) \},$$

where $\phi_k \in (0,1)$ is a weighting factor, is maximal.

For instance, if we choose $\phi_k = \frac{1}{2}$, the point $\bar{x}$ will lie halfway the line segment $L_k(x)$. Other choices of the weighting factor will lead to a balance between the distances over which the point $\bar{x}$ can be moved along $L_k(x)$.

*Note that this is also possible in the weighted distance problem.*
The equilibrium problem in flexible programming is directed to finding a point in $G$ which has the above properties for all $k = 1, \ldots, p$. For instance, the point $\bar{x}$ in figure 5-1 below is a solution to the equilibrium problem with directions $e_1^*= (1, 0)$, $e_2^* = (0, 1)$, $-e_1$, $-e_2$ and weighting factors $\phi_1 = \phi_2 = 1$.

![Figure 5-1: Example in $\mathbb{R}^2$ of a solution to the equilibrium problem.](image)

The equilibrium problem reads as follows.

**Problem 5-1.** (equilibrium problem in flexible programming)

Find a point $\bar{x}$ in the non-empty polyhedral set $G := \{x \in \mathbb{R}^n | Ax \leq b\}$, such that for all $k = 1, \ldots, p$

$$\rho_k(\bar{x}) := \max \{ \rho_k(x) | x \in f_k(\bar{x}) \}, \quad (5-2.a)$$
where
\[ \rho_k(x) := \min \{ \phi_k s(x|d_k), (1-\phi_k)s(x|-d_k) \}, \quad (5.2.2) \]
\( \phi_k \in (0,1) \) a weighting factor and \( \{d_k, -d_k\} \) a pair of directions of finite flexibility for \( k = 1, \ldots, p \).

Later in this section we shall demonstrate that the above equilibrium problem resembles the Nash equilibrium problem in game theory.

Let \( \tilde{x} \in D \) be a solution to problem 5-1. In section 2 of this chapter we show that
\[ \rho_X(\tilde{x}) = \phi_k s(\tilde{x}|d_k) = (1-\phi_k)s(\tilde{x}|-d_k) \quad k = 1, \ldots, p. \]

Hence, a solution to the equilibrium problem is a point in \( D \), from which one can move in the directions \( d_k \) and \( -d_k \) over the distances
\[ s(\tilde{x}|d_k) = \frac{\rho_k(\tilde{x})}{\phi_k} \quad \text{and} \quad s(\tilde{x}|-d_k) = \frac{\rho_k(\tilde{x})}{1-\phi_k} \]
respectively, without violating the boundary of \( D \).

Moreover, the flexibility ranges are in the proportion of \( \phi_k \) to \( (1-\phi_k) \). These properties hold for each \( k \in \{1, \ldots, p\} \) separately.

There is a resemblance between the above equilibrium problem and the Nash equilibrium point problem in game theory. To see this resemblance compare the equilibrium problem in flexible programming having the directions \( \{e_1, -e_1, \ldots, e_k, -e_k, \ldots, e_p, -e_p\} \) and the weighting factors \( \phi_k = \frac{1}{k}, k = 1, \ldots, p \) with the following non-cooperative game. There are \( p \) players labelled 1, \ldots, \( k, \ldots, p \). The decision variable of player \( k \) is \( x_k \). The decision variables of
all players are restricted to the common non-empty set
\[ G := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \]. Let \( x_1, \ldots, x_k, \ldots, x_p \) be a set of feasible decisions of the players in \( G \) and let
\[ x := (x_1, \ldots, x_k, \ldots, x_p)^T \]. Then \( L_k(x) \) is the range of decisions which remains open to player \( k \) if the other players keep their decisions fixed. The objective function for this player is
\[ f_k(x) = \min \{ s(x|e_k), s(x|e_k) \}, \]
\[ \text{i.e. the possible deviation from his decision within the range of admissible decisions } L_k(x). \]
The point \( \tilde{x} \in G \) is then a Nash equilibrium point for this game, if no player finds it to his advantage to change to a different decision, so long as he believes that the other players will not change their decisions. Hence,
\[ f_k(\tilde{x}_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_p) \leq f_k(\tilde{x}_1, \ldots, y_k, \ldots, \tilde{x}_p) \]
for all \( k = 1, \ldots, p \) and for all \( y_k \in L_k(\tilde{x}) \). This solution coincides with a solution of the considered equilibrium problem in flexible programming.

The objective of this chapter is to derive a number of mathematical properties for the equilibrium problem. The following points summarize the main topics of this chapter.

a. Sufficient conditions will be given for the existence of a solution to the equilibrium problem (section 3).

* Note that we can omit \( f_k \) from the objective function \( f_k \), since \( f_k = f_k \) for \( k = 1, \ldots, p \).
b. It will be shown that the equilibrium problem can also be formulated as one of finding a solution to a system of piecewise linear equations (section 4).

In the next chapter we shall develop a finite multistep algorithm for finding a solution to the equilibrium problem.

The following index notations will be used in this chapter and the next:

\[ I_{k}^{+} := \{ i \in I \mid a_{i}^{T}d_{k} > 0 \} \tag{5-3.a} \]

\[ I_{k}^{-} := \{ i \in I \mid a_{i}^{T}(-d_{k}) > 0 \} \]

\[ = \{ i \in I \mid a_{i}^{T}d_{k} < 0 \} \tag{5-3.b} \]

for \( k = 1, \ldots, p \).

Note that \( I_{k}^{+} \neq \emptyset \) and \( I_{k}^{-} \neq \emptyset \), \( k = 1, \ldots, p \), since the directions in problem 5-1 are assumed to be of finite flexibility.

Finally, let \( x \) be an arbitrary point in \( G \), then, according to (3-2), we also have the following expressions for the flexibility ranges in \( x \in G \):

\[ s(x|d_{k}) = \min \left\{ \frac{b_{i} - a_{i}^{T}x}{a_{i}^{T}d_{k}}, (i \in I_{k}^{+}) \right\} \tag{5-4.a} \]

\[ s(x|-d_{k}) = \min \left\{ \frac{b_{i} - a_{i}^{T}x}{-a_{i}^{T}d_{k}}, (i \in I_{k}^{-}) \right\} \tag{5-4.b} \]
A REFORMULATION OF THE EQUILIBRIUM PROBLEM

It will be shown first that the equilibrium problem 5-1, can be reformulated as one of finding a solution to the system of equations

$$G_k s(x|d_k) - (1-\theta_k) s(x|-d_k) = 0 \quad k = 1, \ldots, p.$$  

For the derivation of this result we need the following observation. Let \( x \) be an arbitrary point in \( G \), \( d \in \mathbb{R}^n \) an arbitrary direction of finite flexibility and \( t \) a real number, such that \( (x+td) \in G \). It then follows from the expression for the flexibility range in (3-2) that

$$s(x+td|d) = \min \left\{ \frac{b_i - a_i^T(x+td)}{a_i^T d}, \{i \in I_d\} \right\}$$

$$= \min \left\{ \frac{b_i - a_i^T x}{a_i^T d} - t, \{i \in I_d\} \right\}.$$  

Hence,

$$s(x+td|d) = s(x|d) - t. \quad (5-5)$$

This result will be used in the following key theorem.

Theorem 5-2. The point \( \xi \in G \) is a solution of the equilibrium problem, if, and only if, \( \xi \) is a solution to the system of equations*

* We recall that the domains of the functions \( s(.|d_k) \) and \( s(.|-d_k) \), \( k = 1, \ldots, p \) are restricted to \( G \). Hence, if the system (5-6) has a solution \( \xi \), then \( \xi \in G \).
\[ \phi_k s(x|d_k) + (1-\phi_k)s(x|-d_k) = 0 \quad k = 1, \ldots, p. \] (5-6)

**Proof.** Let the point \( \xi \in G \) be fixed and let the real numbers \( f_k, k = 1, \ldots, p \) be given by:

\[ f_k := \phi_k s(\xi|d_k) - (1-\phi_k)s(\xi|-d_k) \quad k = 1, \ldots, p. \] (5-7)

We further consider the points \( \xi + f_k d_k, k = 1, \ldots, p, \) for which we can write, after the substitution of \( f_k, \)

\[ \xi + f_k d_k = \xi + \phi_k s(\xi|d_k)d_k - (1-\phi_k)s(\xi|-d_k)d_k \]
\[ = \phi_k \left[ \xi + s(\xi|d_k)d_k \right] + (1-\phi_k) \left[ \xi + s(\xi|-d_k)(-d_k) \right]. \]

Since we have,

\[ \xi + s(\xi|d_k)d_k \in bd G, \]
\[ \xi + s(\xi|-d_k)(-d_k) \in bd G, \]
\[ 0 < \phi_k \leq 1 \]

for all \( k = 1, \ldots, p, \) the fact that \( G \) is closed and convex leads to the conclusion that \( \xi + f_k d_k \in G, k = 1, \ldots, p. \) It further follows that

\[ \xi + f_k d_k \in L_k(\xi) \quad k = 1, \ldots, p, \]

where \( L_k(\xi) \) the line segment given by (5-1).

Note that the flexibility ranges in the points \( \xi \) and \( \xi + f_k d_k, k = 1, \ldots, p \) are well-defined, since these points lie in \( G. \) Hence, \( \rho_k(\xi) \) and \( \rho_k(\xi + f_k d_k), k = 1, \ldots, p \) are well-defined. We shall now derive some expressions for them. For \( k = 1, \ldots, p \) we have
\( \rho_k(\xi) := \min \{ \varphi_k s(\xi|d_k), (1-\varphi_k)s(\xi|-d_k) \} \)
\[
= \min \{ \varphi_k s(\xi|d_k), \varphi_k s(\xi|d_k) - f_k \}, \quad (5.9) 
\]

- \( k = 1 \): 
  \[
  s(\xi,f_k d_k|d_k) = s(\xi|d_k) - f_k \quad (5.9.a) 
  \]
  \[
  s(\xi,f_k d_k|-d_k) = s(\xi|-f_k) + d_k = s(\xi|d_k) + f_k \quad (5.9.b) 
  \]

Hence,
\[
\rho_k(\xi+f_k d_k) := 
\]
\[
= \min \{ \varphi_k s(\xi+f_k d_k|d_k), (1-\varphi_k)s(\xi+f_k d_k|-d_k) \} = 
\]
\[
= \min \{ \varphi_k s(\xi|d_k) - \varphi_k f_k, (1-\varphi_k)s(\xi|-d_k) + (1-\varphi_k) f_k \}. 
\]

By using (5.7) this results in
\[
\rho_k(\xi+f_k d_k) = \varphi_k s(\xi|d_k) - \varphi_k f_k \quad (5.10) 
\]

I. If \( \xi \) is a solution to the equilibrium problem, we have
\[
\rho_k(\xi+f_k d_k) \leq \rho_k(\xi) \quad k = 1, \ldots, p \quad (5.11) 
\]

Assume that \( \xi \) is not a solution to the system of equations (5.6). Then, by (5.7), there exists at least one \( k \in \{1, \ldots, p\} \) with \( f_k \neq 0 \). There are two possibilities.
b-1. $f_k > 0$. Then, by (5-8),

$$p_k(\xi) = \phi_k s(\xi|d_k) - f_k.$$ 

Since $\phi_k \in (0,1)$, it follows that

$$\phi_k s(\xi|d_k) - f_k < \phi_k s(\xi|d_k) - \phi_k f_k.$$ 

Hence, by (5-10), $p_k(\xi) < p_k(\xi + f_k d_k)$, which contradicts (5-11).

b-2. $f_k < 0$. It then follows from (5-8) that

$$p_k(\xi) = \phi_k s(\xi|d_k).$$ 

This, together with $\phi_k > 0$, results in

$$p_k(\xi) < \phi_k s(\xi|d_k) - \phi_k f_k = p_k(\xi + f_k d_k).$$ 

which is also a contradiction.

Hence, $\xi$ is a solution to the system (5-6).

II. Conversely, let $\xi \in G$ be a solution to this system. We then have for $k = 1, \ldots, p$:

$$p_k(\xi) = \phi_k s(\xi|d_k) = (1-\phi_k) s(\xi|d_k). \quad \text{(5-12)}$$ 

Let $r_k \in \mathbb{R}$, such that $\xi + r_k d_k \in G$ for all $k = 1, \ldots, p$. Hence, also $\xi + r_k d_k \in L_k(\xi)$. Then, by using the relations (5-9), (5-12) and the assumption that $\phi_k \in (0,1)$ respectively, it follows that for $k = 1, \ldots, p$:

...
\[\rho_k(\xi^*+r_kd_k) = \]
\[= \min \{ \phi_k s(\xi^*+r_kd_k|d_k), (1-\phi_k)s(\xi^*+r_kd_k|-d_k) \} \]
\[= \min \{ \phi_k s(\xi|d_k) - \phi_k r_k, (1-\phi_k)s(\xi|-d_k) + (1-\phi_k)r_k \} \]
\[= \min \{ \rho_k(\xi) - \phi_k r_k, \rho_k(\xi) - \phi_k r_k + r_k \} \]
\[\leq \rho_k(\xi).\]

Hence, \(\xi\) is a solution to the equilibrium problem, which completes the proof of the theorem. \(\blacksquare\)

**Remark 1.** Let \(\xi \in G\) be a solution to the equilibrium problem. Since \(\phi_k \in (0,1)\), \(k = 1, \ldots, p\), we have by (5-6), that \(s(\xi|d_k) > 0\), if, and only if, \(s(\xi|-d_k) > 0\).

**Remark 2.** Let the directions \(d_k\) and \(-d_k\) be of finite flexibility. We then have that \(s(x|d_k) = s(x|-d_k) = 0\) for all \(x \in G\). Hence, the \(k^{th}\) equation in (5-6) is satisfied for all \(x \in G\). Improper directions of flexibility are therefore not interesting in the equilibrium problem. However, even if \(d_k\) and \(-d_k\) are proper, the system (5-6) may have a solution \(\xi \in G\) with \(s(\xi|d_k) = s(\xi|-d_k) = 0\) for certain \(k \in \{1, \ldots, p\}\). For instance, if we choose in \(\mathbb{R}^2\)

\[a=1, \text{ the polyhedral set} \]
\[G := \{ x \in \mathbb{R}^2 \mid x_1 - 2x_2 \leq 0; \ -2x_1 + x_2 \leq 0 \}, \]

which has a non-empty interior.
a-2. the proper directions of finite flexibility $d_1 := (1,0), -d_1$, $d_2 := (0,1)$ and $-d_2$.

It can be shown that $s(\xi | d_k) = s(\xi | -d_k) = 0$ for $k = 1, 2$ in the point $\xi := (0,0)$. Although in this case $\xi = (0,0)$ is a solution to the equilibrium problem, it is not possible to move from this point in the coordinate directions or the reversed directions without violating the boundary of $G$. It is therefore appropriate to call a solution $\xi$ to the equilibrium problem (5-1) degenerated, if there exists a $k \in \{1, \ldots, p\}$ with $s(\xi | d_k) = s(\xi | -d_k) = 0$. For such degenerated solutions to the equilibrium problem we can make a distinction between

b-1. completely degenerated solution, if $s(\xi | d_k) = s(\xi | -d_k) = 0$ for all $k = 1, \ldots, p$,

b-2. partially degenerated solutions, if they are degenerated but not completely degenerated. 

Remark 3. Let $\xi$ be a solution to the equilibrium problem. We then have that

$$\phi_k = \frac{s(\xi | -d_k)}{s(\xi | d_k) + s(\xi | -d_k)}$$

for all $k = 1, \ldots, p$ with $s(\xi | d_k) > 0$ and $s(\xi | -d_k) > 0$. We recall that the length of the line segment $L_k(\xi)$ is equal to $s(\xi | d_k) + s(\xi | -d_k)$. Hence, the solution $\xi$ divides the length of this line segment according to the weighting factor $\phi_k$. 

\[\Box\]
In this section it will be shown that boundedness of 
\( G := \{ x \in \mathbb{R}^n | Ax \leq b \} \) and linear independence of the directions 
\( d_1, \ldots, d_k, \ldots, d_p \) of finite flexibility are sufficient conditions 
for the existence of a solution to the equilibrium problem (5-1) 
or, which is implied by theorem 5-2, to the system (5-6). 
At this stage it is appropriate to introduce the following functions:

\[
f_k(.) := s_k s(.|d_k) - (1-s_k) s(.|-d_k), \quad k = 1, \ldots, p. \tag{5-13}
\]

Note that \( f_k \) is a real-valued function defined on \( G \) and has the 
Lipschitz property for each fixed value of \( \varphi_k \), since \( s(.|d_k) \) and 
\( s(.|-d_k) \) have these properties (see theorem 3-17).

We further introduce for \( k = 1, \ldots, p \) the function \( g_k: G \rightarrow \mathbb{R}^n \) given by

\[
g_k(x) := x + f_k(x)d_k. \tag{5-14}
\]

The following lemma gives a property of these functions.

**Lemma 5-3.** For all \( k = 1, \ldots, p \), \( g_k \) is a function from \( G \) into \( G \) 
and has the Lipschitz property.

**Proof.** Let \( x \in G \) and let \( k \in \{1, \ldots, p\} \) be fixed. We define

\[
y_k^+ := x + s(x|d_k)d_k, \tag{5-15}
\]

\[
y_k^- := x - s(x|-d_k)d_k.
\]

It then follows from the definition of the flexibility range 
(definition 3-1) that \( y_k^+ \in \text{bd} G \) and \( y_k^- \in \text{bd} G \). Since \( G \) is closed 
and convex and \( \varphi_k \in (0,1) \), it follows that
\( y_k \in G, \) where \( y_k = \phi_k y_k^* + (1-\phi_k) \hat{y}_k \).

After substitution of (5-15) we find for \( y_k \)
\[
y_k = x + \left[ \phi_k x(x|d_k) - (1-\phi_k) x(x|\hat{d}_k) \right] d_k
\]
\[
= x + f_k(x) d_k.
\]
Hence, by (5-14), \( y_k = g_k(x) \) and \( g_k(x) \in G \). The Lipschitz property of \( g_k \) can be derived by using (5-13) and theorem 3-17.

We also introduce the composed functions \( g^k := g_k \circ g^{k-1} \) for \( k = 1, \ldots, p \) given by
\[
g^k(x) := g_k \left( g^{k-1}(x) \right), \tag{5-16}
\]
where \( x \in G \) and, by convention, \( g^0(x) := x \). The following lemma shows that these functions are well defined. We also give a recurrent expression for them.

**Lemma 5-4.** For all \( k = 1, \ldots, p \), \( g^k \) is a function from \( G \) into \( G \) and has the Lipschitz property. Moreover, let \( x \in G \), then
\[
g^k(x) = x + \sum_{j=1}^{k} f_j \left( g^{j-1}(x) \right) d_j \quad k = 1, \ldots, p. \tag{5-17}
\]

**Proof.** Let \( k \in \{1, \ldots, p\} \). From the definition of \( g^k \) in (5-16) and the preceding lemma it follows that

\begin{enumerate}
\item \( g^k(x) \in G \), if \( g^{k-1}(x) \in G \).
\item \( g^k \) satisfies the Lipschitz property, if \( g^{k-1} \) does.
\end{enumerate}

We can prove the first part by induction. Since \( x \in G \) implies that \( g^k(x) \in G \) for all \( k = 1, \ldots, p \), it follows from the definition of \( g_k \) in (5-14) that
\[ g^k(x) := g_k\left(g^{k-1}(x)\right) = g^{k-1}(x) + \frac{f_k\left(g^{k-1}(x)\right)}{d_k}. \]  

(5-18)

Then the expressions (5-17) also easily follow by induction.  

Sufficient conditions for the existence of solutions to the equilibrium problem 5-1 are given in the following theorem.

**Theorem 5-5.** There exists a solution to the equilibrium problem 5-1, if

a. \( G \) is bounded,

b. the directions \( d_1, \ldots, d_k, \ldots, d_p \) of finite flexibility are linearly independent.

**Proof.** According to theorem 5-2 and (5-13), it is sufficient to show that the system of equations \( f_k(x) = 0, \ k = 1, \ldots, p \) has a solution under the conditions in the theorem. From lemma 5-4 it follows that \( g^p \) is a continuous function from \( G \) into \( G \), where \( G \) is a closed, bounded and convex subset of \( \mathbb{R}^n \).

Hence, by Brouwer's fixed point theorem, there exists a \( \xi \in G \) with \( g^p(\xi) = \xi \). According to (5-17) this implies that

\[ \sum_{j=1}^{p} f_j\left(g^{j-1}(\xi)\right) d_j = 0. \]  

(5-19)

Since the vectors \( d_1, \ldots, d_j, \ldots, d_p \) are linearly independent, it follows from (5-19) that

\[ f_j\left(g^{j-1}(\xi)\right) = 0 \quad j = 1, \ldots, p. \]  

(5-20)
We then have, by (5-18),
\[ g^j(\xi) = g^{j-1}(\xi) \quad j = 1, \ldots, p. \]

Since, by convention, \( g^0(\xi) = \xi \), it follows that \( g^j(\xi) = \xi \) for all \( j = 1, \ldots, p \). Hence, by (5-20), \( f_j(\xi) = 0 \), \( j = 1, \ldots, p \).

So \( \xi \) is a solution to the equilibrium problem, which had to be shown.

The following remarks show that boundedness of \( G \) and linear independence of the directions \( d_1, \ldots, d_k, \ldots, d_p \) are not necessary for the existence of a solution to the equilibrium problem.

**Remark 1.** Let be given

a-1. the unbounded polyhedral set
\[ G := \{ x \in \mathbb{R}^2 | x_1^,* x_2^, \leq 1; -x_1^, x_2^, \leq 1 \} \]
with a non-empty interior,

a-2. the proper directions \( d_1 := (1,0)^T, -d_2, d_2 := (0,1)^T, -d_2 \) of finite flexibility,

a-3. the weighting factors \( \alpha_1 = \beta_2 = 1/2 \).

For this problem it can be derived that the point \( \xi := (0,0) \) is a solution to the equilibrium problem with
\[ s(\xi | d_k) = s(\xi | -d_k) = 1 \quad k=1,2. \]

Hence, also non-degenerated solutions to the equilibrium problem can exist in unbounded polyhedral sets (see figure 5-2 below for clarification). Note that all points of the set \( \{ x \in \mathbb{R}^2 | x_1^, x_2^, = 0 \} \) are solutions to the equilibrium problem in this example.
For the following remark we need the concept of radially symmetric sets. The non-empty set $S \subseteq \mathbb{R}^n$ is called radially symmetric with respect to the point $y \in S$, if $y + d \in S$ implies and is implied by $y - d \in S$ for all vectors $d \in \mathbb{R}^n$.

**Remark 2.** Let the polyhedral set $C := \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ have a non-empty interior and be radially symmetric with respect to the point $\| \in C$. Let furthermore $d \in \mathbb{R}^n$ be an arbitrary direction of finite flexibility. Then
\[ s(\|d\|) := \sup \{ \sigma \in \mathbb{R} \mid (\xi + \sigma d) \in G \} \]
and, since \( G \) is radially symmetric,
\[ \sup \{ \sigma \in \mathbb{R} \mid (\xi + \sigma d) \in G \} = \sup \{ \sigma \in \mathbb{R} \mid (\xi - \sigma d) \in G \} \]
Hence, \( s(\|d\|) = s(\|\xi - d\|) \). So \( \xi \) is a solution to the equilibrium problem with \( \phi_k = h_k \), \( k = 1, \ldots, p \), whatever the value of the integer \( p \) may be and whatever is chosen for the directions \( d_1, \ldots, d_k, \ldots, d_p \). Hence, for the existence of a solution to the equilibrium problem it is not necessary that the directions \( d_1, \ldots, d_k, \ldots, d_p \) of finite flexibility are linearly independent. \( \square \)

V-4 A PIECEWISE LINEAR REPRESENTATION OF THE EQUILIBRIUM PROBLEM

In subsection 3.3 of chapter III we demonstrated that the function \( s(\|d\|) : G \times \mathbb{R}^p \) is piecewise linear on \( G \) for each direction of finite flexibility (Theorem 3-20). It has also been shown (Theorem 5-2) that the equilibrium problem 5-1 is equivalent to the problem of finding a solution to the system of equations
\[
\phi_k s(x|d_k) - (1-h_k)s(x|-d_k) = 0 \quad k = 1, \ldots, p.
\]
These two results enable us to show that the solutions to the equilibrium problem are the solutions to a system of piecewise linear equations. The derivation of this property will be the subject of this section. The material to be developed is partly a generalization of the properties derived in subsection 3.3 of chapter III.
We define \( Q \) as the set of index multiples

\[
Q := I_1^+ \times I_1^- \times \ldots \times I_k^+ \times I_k^- \times \ldots \times I_p^+ \times I_p^-.
\]

Let \( q := (i_1^+, i_1^-, \ldots, i_k^+, i_k^-, \ldots, i_p^+, i_p^-) \in Q \) with \( i_k^+ \in I_k^+ \) and \( i_k^- \in I_k^- \) for \( k = 1, \ldots, p \). Then \( q \) will be referred to as a selection from the index sets \( I_k^+ \) and \( I_k^- \), \( k = 1, \ldots, p \). In the following we shall reserve the symbol \( q \) to denote the selection \((i_1^+, i_1^-, \ldots, i_k^+, i_k^-, \ldots, i_p^+, i_p^-)\). Other selections, when used, will be written out completely. Note that \( 0 < |q| < a \). Let \( x \) be an arbitrary point in \( C \), then \( S(x) \) is a subset of \( Q \) defined by

\[
S(x) := \left\{ q \in Q \mid \begin{bmatrix}
    b_+ - a^T x \\
    i_k^+ \\
    a^T d \\
    i_k^-
\end{bmatrix} \right\}_{i_k = k}^{i_k^+} \quad \text{for } k = 1, \ldots, p.
\]

(5.21)

From the assumption \( I_k^+ \neq \emptyset \) and \( I_k^- \neq \emptyset \) for \( k = 1, \ldots, p \) and the expressions (5.4) it follows that \( S(x) \neq \emptyset \) for all \( x \in G \).

Let \( q \in Q \), then \( G_q \) is a subset of \( G \) defined by
The sets $G$ and $S(x)$ can be interpreted as follows. Let $x \in G_q$, the boundary of $G$ will then be reached in the boundary hyperplanes with index numbers $i^+_k$ and $i^-_k$. If we move from $x$ in the directions $d_k$ and $-d_k$ respectively for $k = 1, \ldots, p$. The set of selections $S(x)$ has a similar interpretation.

For instance, let $G$ be the polyhedral set in $\mathbb{R}^2$ as drawn in figure 5-3 below (the numbers denote the index numbers of the boundary hyperplanes of $G$).
Let the directions of flexibility be \( e_1, -e_1, e_2, -e_2 \). For the index sets we find

\[
\begin{align*}
I^+_1 &= \{ 3, 4 \}; \quad I^-_1 = \{ 1, 2 \}, \\
I^+_2 &= \{ 4, 5 \}; \quad I^-_2 = \{ 2 \}.
\end{align*}
\]

Let us choose \( q := (3,1,5,2) \). The boundary of \( G \) will then be reached in the hyperplanes with index numbers 3, 1, 5 and 2 respectively, if we move from an arbitrary point in the shaded area in the directions of flexibility. For the point \( x \) on the boundary of \( G_q \) we find that \( S(x) \) contains the selections \( q \) and \( r \).

Note that \( G_q \) is empty for the selection \( t := (4,2,5,2) \).

**Definition 5-6.** A selection \( q \in Q \) is called proper, if \( G_q \neq \emptyset \).
Otherwise, it is called improper.

Later it will become clear how it can be verified whether a selection \( q \in Q \) is proper or not. The set of proper selections in \( Q \) will be denoted by \( Q^* \). The following theorem gives some properties of the quantities introduced above.

**Theorem 5-7.** The following statements hold true.

a. \( G \neq \emptyset \), if, and only if, \( G \neq \emptyset \).

b. \( G = \bigcup_{x \in G} S(x) \) and \( G = \bigcup_{q \in Q} G_q \).

c. \( x \in G_q \), if, and only if, \( q \in S(x) \).

**Proof.** The proof of the above statements becomes apparent if we use the expressions (5-21), (5-22) and the property that for arbitrary \( x \in G \) there exist, by (5-4), indices \( I^+_k \in I^+_k \) and
\[ i_k^+ \in I_k^- \] for \( k = 1, \ldots, p \), such that \( x \) satisfies the formulas between brackets in (5-21) and (5-22).

Remark 1. From property c in the above theorem it follows that the selections in the set \( S(x) \) are always proper.

The following theorem gives a polyhedral representation of \( G_q \).

Theorem 5-8. \( G_q \), \( q \in Q \) is a polyhedral set described by the following system of linear inequalities.

\[
\begin{align*}
\left( \frac{a}{s^d} - \frac{a^+}{s^k} \right) x & \leq \frac{b}{s^d} - \frac{b^+}{s^k} \quad (i \in i_k^+ - \{i_k^+\}) \quad (5.23.a) \\
\left( \frac{a}{s^d} - \frac{a^-}{s^k} \right) x & \geq \frac{b}{s^d} - \frac{b^-}{s^k} \quad (i \in i_k^- - \{i_k^-\}) \quad (5.23.b)
\end{align*}
\]

\[
\begin{align*}
\alpha^T x & \leq b^+ \quad (i_k^-) \quad (5.23.c) \\
\alpha^T x & \leq b^- \quad (i_k^+) \quad (5.23.d)
\end{align*}
\]

for \( k = 1, \ldots, p \), where

\[ i := \{ i \in I \mid i \notin i_k^+; i \notin i_k^-; k = 1, \ldots, p \}. \quad (5.24)\]

Proof. The proof of the theorem can easily be obtained by using the expression for the flexibility range and the defining expression for \( G_q \) (see (5-8) and (5-22)).
Remark 2. Similar comments as have been made in the remarks 1, 2 and 3 in subsection III-3.1, hold for this theorem.

Remark 3. This theorem enables us to determine, whether a selection \( q \in Q \) is proper or not, by testing the consistency of the linear system (5-23) with the simplex method in linear programming.

We recall that all directions in the equilibrium problem are assumed to be of finite flexibility.

Remark 4. Let the directions \( d_1, \ldots, d_k, \ldots, d_p \) be proper. This implies that \( -d_1, \ldots, -d_j, \ldots, -d_p \) are proper (theorem 3-8).

Hence, by theorem 3-4,

\[
i_k^+ \subset I_{in} \quad k = 1, \ldots, p
\]

\[
i_k^- \subset I_{in}
\]

we then have the following two properties for the index set \( I \).

a. \( I := \{ i \in I \mid i \notin i_k^+, i \notin I_k^-, k = 1, \ldots, p \} \)

\[
i = I - \bigcup_{k=1}^{p} (I_k^+ \cup I_k^-)
\]

\[
\supseteq I - I_{in} = I_{eq}.
\]

Hence, all implicit equalities are contained in the system of linear equalities (5-23.e).

b. In the particular case that \( \text{int} \ G \neq \emptyset \), \( p = n \) and \( d_1, \ldots, d_k, \ldots, d_p \) is a basis of \( \mathbb{R}^n \), we have \( I = \emptyset \). For let \( I \neq \emptyset \) and let \( i \in I \), then, by (5-24), \( i \notin I_k^+ \) and \( i \notin I_k^- \) for \( k = 1, \ldots, n \). Hence, \( a_i d_k = 0 \) for all \( k = 1, \ldots, n \). However,
since \([d_1,\ldots,d_k,\ldots,d_n]\) is a basis of \(\mathbb{R}^n\), this would imply that \(a_1 \parallel \mathbb{R}^n\), which is a contradiction.

In chapter II we introduced the concept of a finite polyhedral partition of a non-empty polyhedral subset of \(\mathbb{R}^n\) (see definition 2-9). The following theorem gives such a partition for \(G\). It is a generalization of theorem 3-19.

**Theorem 5-9.** The family \(G := \{ G_q : (q \in \mathcal{Q}) \}\) is a finite polyhedral partition of \(G\).

**Proof.** We have to verify the conditions in definition 2-9. Obviously \(|\mathcal{Q}| < \infty\). According to theorem 5-8, \(G_q\) is polyhedral for all proper selections \(q \in \mathcal{Q}\) and the empty set is polyhedral by definition. Condition \(c\) follows from theorem 5-7. In order to show condition \(d\), let \(q,r \in \mathcal{Q}\); \(r := (\ell_1^q,\ell_2^q,\ldots,\ell_k^q,\ldots,\ell_p^q)\) such that \(G_q \cap G_r \neq \emptyset\). We have to show that \(G_q \cap G_r\) is a common face of \(G_q\) and \(G_r\). This is obviously the case, if \(q = r\). So we assume that \(q \neq r\). From the inequalities (5-23) in theorem 5-9 it can be derived that \(G_q \cap G_r\) is a subset of the affine space:

\[
\begin{align*}
\left( \frac{a_1^q}{1_k^q} - \frac{a_2^q}{1_k^q} \right) \mathbb{T} \left( \begin{array}{c} x_k \\frac{b_1^q}{1_k} \\frac{b_2^q}{1_k} \\
1_k^q \\frac{a_2^q}{1_k^q} \\frac{a_1^q}{1_k^q} \\frac{1_k}{1_k} 
\end{array} \right) = \left( \begin{array}{c} 0 \\
0
\end{array} \right), \quad \text{if } a_1^q \neq a_2^q, \\
\left( \frac{a_1^q}{1_k^q} - \frac{a_2^q}{1_k^q} \right) \mathbb{T} \left( \begin{array}{c} x_k \\frac{b_1^q}{1_k} \\frac{b_2^q}{1_k} \\
1_k^q \\frac{a_2^q}{1_k^q} \\frac{a_1^q}{1_k^q} \\frac{1_k}{1_k} 
\end{array} \right) = \left( \begin{array}{c} 0 \\
0
\end{array} \right), \quad \text{if } a_1^q = a_2^q,
\end{align*}
\]

and

\[
\begin{align*}
\left( \frac{a_1^r}{1_k^r} - \frac{a_2^r}{1_k^r} \right) \mathbb{T} \left( \begin{array}{c} x_k \\frac{b_1^r}{1_k} \\frac{b_2^r}{1_k} \\
1_k^r \\frac{a_2^r}{1_k^r} \\frac{a_1^r}{1_k^r} \\frac{1_k}{1_k} 
\end{array} \right) = \left( \begin{array}{c} 0 \\
0
\end{array} \right), \quad \text{if } a_1^r \neq a_2^r, \\
\left( \frac{a_1^r}{1_k^r} - \frac{a_2^r}{1_k^r} \right) \mathbb{T} \left( \begin{array}{c} x_k \\frac{b_1^r}{1_k} \\frac{b_2^r}{1_k} \\
1_k^r \\frac{a_2^r}{1_k^r} \\frac{a_1^r}{1_k^r} \\frac{1_k}{1_k} 
\end{array} \right) = \left( \begin{array}{c} 0 \\
0
\end{array} \right), \quad \text{if } a_1^r = a_2^r,
\end{align*}
\]

for \(k = 1,\ldots,n\). Let \(H\) denote the solution set of this space. From the same inequalities (5-23) it follows that \(G_q \cap H\) and \(G_r \cap H\) are non-empty faces of \(G_q\) and \(G_r\) respectively. It can then
be derived that $Q_q \cap C_{r} = C_q \cap H = G_q \cap H$, if we follow the same line of reasoning as we did in the verification of condition d in theorem 3-19. This completes the proof of the theorem.

Remark 5. The partition $\mathcal{E}$ may contain coinciding elements. An element may also be the face of another element. Both aspects will be demonstrated in the following example. We consider the consistent system of linear inequalities:

1. $x_1 + x_2 \leq 1$.
2. $x_1 - x_2 \leq 1$.
3. $-x_1 - x_2 \leq 1$.
4. $-x_1 + x_2 \leq 1$.

The figures on the left hand side are the inequality indices. The directions of finite flexibility are $e_1 = (1,0)^T$ and $-e_1$. Using the expressions (5-3) and (5-4), we find for this particular case

$$I_1^+ = \{ 1, 2 \}; s(x|e_1) = \min \{ 1-x_1-x_2; 1-x_1+x_2 \},$$

$$I_1^- = \{ 3, 4 \}; s(x|-e_1) = \min \{ 1+x_1+x_2; 1+x_1-x_2 \}.$$

The selections are denoted as follows:

$q_1 := (1,4); q_2 := (2,3); q_3 := (1,3); q_4 := (2,4).$

For the elements of the partition $G := \{ G_q, q = q_1, q_2, q_3, q_4 \}$ we can now derive the following expressions (see also figure 5-4 below)
a. \[ G_{q_1} = \{ x \in G \mid s(x|e_1) = 1 - x_1 - x_2, s(x|e_0) = 1 - x_1 - x_2 \} \]

= \{ x \in G \mid 1 - x_1 - x_2 = \min \{ 1 - x_1 - x_2, 1 - x_1 + x_2 \} \} \\
= \{ x \in G \mid 1 + x_1 - x_2 = \min \{ 1 + x_1 + x_2, 1 + x_1 - x_2 \} \} \\
= \{ x \in G \mid x_2 \geq 0 \}

and, similarly,

b. \[ G_{q_2} = \{ x \in G \mid x_2 \leq 0 \} \]
\[ G_{q_3} = \{ x \in G \mid x_2 = 0 \}. \]

\[ G_{q_4} = \{ x \in G \mid x_2 = 0 \}. \]

The example shows that both \( G_{q_3} \) and \( G_{q_4} \) are faces of \( G_{q_3} \) and \( G_{q_2} \) and that \( G_{q_3} \) and \( G_{q_4} \) coincide for \( q_3', q_4'. \)

In chapter II we introduced the concept of a piecewise linear function defined on \( G \) (see definition 2-10). By theorem 3-20, \( s(.|d_k) \) and \( s(.|-d_k) \) are piecewise linear on \( G \) for \( k = 1, \ldots, p \). We further refer to the functions \( f_k: G \to \mathbb{R} \) given by (5-13). The following theorem shows that they are piecewise linear on \( G \).

**Theorem 5-10.** The functions \( f_k: G \to \mathbb{R}, k = 1, \ldots, p \) are piecewise linear on \( G \).

**Proof.** According theorem 5-9. \( G := \{ G_q \mid q \in Q \} \) is a finite polyhedral partition of \( G \). Let \( q \in G \). Hence, \( G_q \neq \emptyset \). It then follows from (5-22) and (5-13) that

\[
f_k(x) = \\
= \left( \begin{array}{c}
\frac{b_+}{a^+_k} + \frac{b_-}{a^-_k} \\
\frac{1}{a^+_k} + \frac{1}{a^-_k}
\end{array} \right) - \left( \begin{array}{c}
\frac{a_+}{a^+_k} + \frac{a_-}{a^-_k} \\
\frac{1}{a^+_k} + \frac{1}{a^-_k}
\end{array} \right) x
\]

for all \( k = 1, \ldots, p \) and for all \( x \in G_q \). This proves the theorem.

We introduce the following notations for arbitrary \( q \in Q \):
a. \( \mathbf{c}_q \) is the \((p \times n)\) matrix with row vectors

\[
\begin{pmatrix}
    \frac{a^+_{1k}}{a^+_{k}} & \frac{a^-_{1k}}{a^-_{k}} \\
    \frac{a^+_{2k}}{a^+_{k}} & \frac{a^-_{2k}}{a^-_{k}} \\
    \vdots & \vdots \\
    \frac{a^+_{pk}}{a^+_{k}} & \frac{a^-_{pk}}{a^-_{k}}
\end{pmatrix}
\quad \text{for } k = 1, \ldots, p.
\tag{5.25.a}
\]

b. \( \mathbf{b}_q \) is the \((p \times 1)\) vector with entries

\[
\begin{pmatrix}
    b^+_{1k} \\
    b^+_{2k} \\
    \vdots \\
    b^+_{pk}
\end{pmatrix} + \begin{pmatrix}
    b^-_{1k} \\
    b^-_{2k} \\
    \vdots \\
    b^-_{pk}
\end{pmatrix}
\quad \text{for } k = 1, \ldots, p.
\tag{5.25.b}
\]

It is now obvious that we can write for all proper selections

\[
\begin{bmatrix}
    f_1(x) \\
    f_2(x) \\
    \vdots \\
    f_p(x)
\end{bmatrix} =
\begin{bmatrix}
    \phi_1 \varphi(x|d_1) - (1-\phi_1) \varphi(x|-d_1) \\
    \phi_2 \varphi(x|d_2) - (1-\phi_2) \varphi(x|-d_2) \\
    \vdots \\
    \phi_p \varphi(x|d_p) - (1-\phi_p) \varphi(x|-d_p)
\end{bmatrix} = \mathbf{A}_q x - \mathbf{b}_q
\tag{5.26}
\]

for all \( x \in G_q \).

The piecewise linear representation of the equilibrium problem is now given by the following theorem.

**Theorem 5.11.** The point \( \xi \in G \) is a solution to the equilibrium problem 5-1, if, and only if, there exists a proper selection \( q \in G \), such that \( \mathbf{A}_q \xi = \mathbf{b}_q \) and \( \xi \in G_q \).

**Proof.** Let \( \xi \in G \). There exists a proper selection \( q \) with \( \xi \in G_q \) (see property b in theorem 5.7). Then, by theorem 5.2 and the definition of \( f_k \) in (5.13), \( \xi \) is a solution to the equilibrium problem, if, and only if, \( \xi_k(\xi) = 0 \), \( k = 1, \ldots, p \) and by (5.26), if, and only if, \( \mathbf{A}_q \xi = \mathbf{b}_q \) and \( \xi \in G_q \). \( \Box \)
The above theorem shows that a solution to the equilibrium problem can be found by testing the consistency of the linear system

$$L_q x = t_q, \quad x \in C_q$$

for the selections \( q \in Q \). For instance, the simplex method in linear programming can be used for this purpose. If a selection has been found for which this system has a solution \( \xi \), then \( \xi \) is a solution to the equilibrium problem. If this system is inconsistent for all selections \( q \in Q \), then there does not exist a solution to the equilibrium problem.

In the next chapter we shall give more details of methods of finding a solution to the equilibrium problem.
CHAPTER VI

A FINITE MULTISTEP ALGORITHM FOR FINDING A SOLUTION TO THE EQUILIBRIUM PROBLEM

VI-1 INTRODUCTION

In the last section of the previous chapter we showed that a solution to the equilibrium problem can be found by testing the consistency of the linear system \( b_q \cdot x = b_q \cdot x \in G_q \) for the selections \( q \in Q \). For this consistency test the first phase of the simplex method in linear programming can be used. However, the coefficients in these linear systems can mutually differ considerably. Hence, a complete new initial simplex tableau has to be constructed in passing from one linear program to the other. Another thing is that this consistency test must be performed for proper selections only. These selections, however, are difficult to identify.

In this chapter we shall develop a finite multistep algorithm for finding a solution to the equilibrium problem. This algorithm also consists of solving a sequence of linear programs. However, it has the advantage that the initial simplex tableau of a linear program can be obtained from the final simplex tableau of the preceding linear program by changing only the reduced cost row. Moreover, an attempt is made to choose only proper selections and to maintain monotony in the values of the objective functions of two subsequent linear programs.

This finite multistep algorithm will be developed in the sections VI-2 and VI-3. In section VI-4 we illustrate it with an example.
We first refer to two other methods which seem to be competitive with our multistep algorithm.

A. We recall theorem 5-2 to note that \( \xi \) is a solution to the equilibrium problem, if, and only if, it is a solution to the system of equations

\[
\phi_k s(x|d_k) - (1-\xi_k)s(x|-d_k) = 0 \quad k = 1, \ldots, p.
\]

In [13] the iterative non-linear Gauss-Seidel method has been proposed for solving this system. It concerns the following case:

a. the polyhedral set \( G \) is bounded,

b. the set of directions of flexibility is

\[
\{e_1, -e_1, \ldots, e_n, -e_n\}.
\]

Note that, under these conditions, a solution to the equilibrium problem exists (theorem 5-5). We further remark that the choice of the standard coordinate system \( \{e_1, \ldots, e_n\} \) in \( \mathbb{R}^n \) is not a restriction. Let \( \{d_1, \ldots, d_k, \ldots, d_n\} \) be another set of mutually independent vectors in \( \mathbb{R}^n \) and let \( D \) be the \((nxn)\) matrix with column vectors \( d_k \), \( k = 1, \ldots, n \). The linear transformation \( x = Dy \) then transforms the linear system \( Ax \leq b \) into the linear system \( A'y \leq b \). Let \( \tilde{y} \) satisfy the latter system and let \( \tilde{x} = D\tilde{y} \). It can then easily be verified that \( s(\tilde{y}|e_k) = s(\tilde{x}|d_k) \) and 

\[
\tilde{s}(\tilde{y}|e_k) = s(\tilde{x}|d_k)
\]

for \( k = 1, \ldots, n \). Hence, there is a one to one correspondence between the solutions of the systems

\[
\phi_k s(y|e_k) - (1-\xi_k)s(y|-e_k) = 0 \quad \text{and} \quad \phi_k s(x|d_k) - (1-\xi_k)s(x|-d_k) = 0
\]

for \( k = 1, \ldots, n \).

The major advantage of the application of the non-linear Gauss-Seidel method to the problem under consideration is that it is numerically very simple and that it does not require a transformation of the constraint matrix \( A \) during the execution of the
algorithm. In fact, it is only necessary to store the non-zero elements of this matrix. A drawback, however, is that convergence of the algorithm is not always ensured, which can be shown as follows. Let \( \bar{x} \in G \) be a solution to the equilibrium problem, such that \( \bar{x} \in \text{int } G_q \) for certain proper selection \( q \in G \), where \( G_q \) is given by (5-20). Then, by theorem 5-11, \( \bar{x} \) is a solution to the linear system \( A_q x = b_q \). If \( N(\bar{x}) \) is an neighbourhood of \( \bar{x} \), then the non-linear Gauss-Seidel procedure will behave in \( N(\bar{x}) \cap \text{int } G_q \) like the Gauss-Seidel procedure for the solution of the linear system \( A_q x = b_q \). Hence, local convergence of the algorithm will only occur, if the value of the spectral radius of the \((nxn)\) matrix \( (D_q - L_q)^{-1}U_q \) is less than one, where \( D_q \), \( L_q \) and \( U_q \) are the diagonal-, lower triangular- and upper triangular part of the matrix \( A_q \), respectively.\( ^* \)

**B.** In the literature attention has also been paid to algorithms for finding a solution to a system of \( n \) piecewise linear equations in \( n \) unknown variables (see, for instance, the references 14-17). These algorithms have in common that they generate a piecewise linear path through the elements of the partition.

Let us consider the equilibrium problem with respect to a set of \( n \) directions of finite flexibility and the reversed directions. Hence, \( f_\eta \) is an \((n \times n)\) matrix, \( b_q \) an \((n \times 1)\) vector and the equilibrium problem a solution to the system of \( n \) piecewise linear equations \( f_\eta(x) = 0 \), \( k = 1, \ldots, n \). A basic step of these algorithms is then as follows (see figure 6-1 below for clarification)

\( ^* \) Local convergence to a flexible point is, however, always ensured in any non-empty convex set in \( \mathbb{R}^2 \). See statement VI appended to this thesis.
Let $x^F$ be a point on the boundary of the element $G_x$ and let $t^F$ be a solution of the linear system $A_x x + b_x = 0$. The algorithm then proceeds along the line segment

$$\{ y \in G_x \mid y = x^F + \lambda(t^F - x^F) ; \lambda \in \mathbb{R} \}.$$ 

Let $x^S$ be the point where this line segment meets the intersection of $G_x$ with its neighbouring element $G_y$. If $t^F \in G_x$, then $t^F$ is a solution of the equilibrium problem and the algorithm terminates. Otherwise, the procedure is repeated in $G_y$ starting from the point $x^S$.

Difficulties arise with these algorithms when the path meets the intersection of three or more elements and when it meets an element $G_{q'}$ where the rank of the matrix $A_q$ is less than $n$. In both cases there is no unique direction in which the path can be continued. These difficulties are, however, overcome by perturbing the starting point of the algorithm in such a way that the path avoids such situations.
The basic problem for these algorithms is to find an appropriate starting point from which a path can be constructed which leads to a solution of the equilibrium problem. In any case, the algorithms require the starting point to be situated in the interior of one of the elements of the partition (allowing perturbations of the starting point in any direction within the element) and the associated matrix to be regular (which uniquely determines the initial direction of the path).

Although it may be easy to find such a starting point for certain special applications, this is not the case for the equilibrium problem in flexible programming. Because this means finding a selection \( q \in \mathcal{Q} \) with \( \operatorname{int} C_q \neq \emptyset \) and rank \( A_q = n \). Moreover, a point in the interior of \( C_q \) must be constructed.

But even when such a starting point has been found, the algorithm will find a solution to the system of piecewise linear equations for certain specific problems only.

We therefore find these algorithms not appropriate for finding a solution to the equilibrium problem.

VI-2 PRELIMINARY PROPERTIES

We first derive a linear system in which the equalities and inequalities are independent of the selections \( q \). This system written down in (6-1) below, forms the basis of our multistep algorithm (in remark 3 below we give a geometrical interpretation).

Let \( \xi \in \mathcal{G} \) be a solution of the equilibrium problem. Then, by theorem 5-12, there exists a proper selection \( q \in \mathcal{Q} \) with

\[
A_q \xi - b_q = 0 \quad \text{and} \quad \xi \in C_q.
\]

We recall that the system of equalities \( A_q \xi - b_q = 0 \) can be replaced by (see (5-28))
\[ \phi_k \mathcal{E}(\xi|d_k) - (1-\phi_k)\mathcal{E}(\xi|-d_k) = 0 \quad k = 1,\ldots,p. \]

The expression of the flexibility range in (5-4) and the expression for \( \mathcal{G}_q \) in (5-22) lead to the conclusion that \( \xi \in \mathcal{G}_q \) implies the following.

a. For \( k = 1,\ldots,p \)

\[ \phi_k \hat{s}(\xi|d_k) = \phi_k \frac{b_i - a_i \xi}{a_i^2 d_k} - y_{i,k} \quad (i \in I_k^+) \]

with \( y_{i,k} \geq 0, (i \in I_k^+) \) and in particular \( y_{i,k}, k = 0 \).

\[ (1-\phi_k)\hat{s}(\xi|-d_k) = (1-\phi_k) \frac{b_i - a_i \xi}{a_i^2 d_k} - z_{i,k} \quad (i \in I_k^-) \]

with \( z_{i,k} \geq 0, (i \in I_k^-) \) and in particular \( z_{i,k}, k = 0 \).

b. \( a_{i,k}^+ \leq b_i \quad (i \in I) \)

where again (see (5-24)

\[ I := \{ i \in I \mid i \notin I_k^+ \cup I_k^- \}, \quad k = 1,\ldots,p \} \]

We introduce the new variables \( s_k, k = 1,\ldots,p \) given by

\[ s_k := \phi_k \mathcal{E}(\xi|d_k) - (1-\phi_k)\mathcal{E}(\xi|-d_k) \quad k = 1,\ldots,p \]

and define the vectors

\[ z := (s_1,\ldots,s_k,\ldots,s_p)^T \]
\[ y := \left( y_{i_1, k} (i \in I_1^k), \ldots, y_{i_p, k} (i \in I_p^k) \right) \]
\[ z := \left( z_{i_1, k} (i \in I_1), \ldots, z_{i_p, k} (i \in I_p) \right) \]

It then follows from points a and b above that \( \xi \) is a solution of the following linear system.

a. for \( k = 1, \ldots, p \)

\[ \Phi_k \left( \frac{a_i}{1 - a_{i_1} d_k} \right) x + s_k + y_{i_1, k} = \Phi_k \left( \frac{b_i}{1 - a_{i_1} d_k} \right) (i \in I_1^k), \quad (6-1.a) \]
\[ (1 - \Phi_k \left( \frac{a_i}{1 - a_{i_1} d_k} \right)) x + s_k + z_{i_1, k} = (1 - \Phi_k \left( \frac{b_i}{1 - a_{i_1} d_k} \right)) (i \in I_1^k). \quad (6-1.b) \]

b.

\[ a_i x \leq b_i \quad (i \in I), \quad (6-1.c) \]

\[ (s, y, z) \geq 0 \quad (6-1.d) \]

with

\[ y_{i_1, k} = z_{i_1, k} = 0 \quad k = 1, \ldots, p. \quad (6-2) \]

Conversely, it can also be shown that \( \xi_0^k - b = 0 \) and \( \xi \in C_q \) if \( (\xi, s, y, z) \) is a solution to the linear system (6-1) with property (6-2).

We shall summarize the above results in a theorem.

**Theorem 6-1.** The following three statements are equivalent.

1. $\xi \in G$ is a solution to the equilibrium problem.

2. $A_{q}^{i} - b_{q}$ and $\xi \in G_{q}$ for some proper selection $q \in Q$, where $A_{q}, b_{q}$ and $G_{q}$ given by (5-25.a), (5-25.b) and (5-23) respectively.

3. There exist vectors $s, y$ and $z$, such that $(\xi, s, y, z)$ is a solution to the linear system (6-1) with property (6-2) for some proper selection $q \in Q$.

In the following we shall denote the solution set of the linear system (6-1) by $\Pi$.

Remark 1. Since, for $k = 1, \ldots, p$,

\[ \xi_{k} \in (0, 1), \]
\[ a_{i}^{T}d_{k} > 0 \text{ for all } i \in I_{+}^{k}, \]
\[ a_{i}^{T}d_{k} < 0 \text{ for all } i \in I_{-}^{k}, \]

it follows from the linear system (6-1) that $H \neq \emptyset$, if, and only if, $G \neq \emptyset$. In particular, if $x \in G$ then there exist vectors $s, y$ and $z$, such that $(x, s, y, z) \in H$. Conversely, if $(x, s, y, z) \in H$, then $x \in G$.

Remark 2. Note that the coefficient matrix of this linear system has a special structure owing to the value of the coefficients of the variables $s_{1}, \ldots, s_{k}, \ldots, s_{p}$ (see example in subsection 4.2 of this chapter).

Remark 3. The following geometrical interpretation can be given to the system (6-1) for an arbitrary $k \in \{1, \ldots, p\}$ and for $a_{k} = \frac{y}{2}$ (see figure 6-2 below for clarification in $R^{2}$).
Let $\mathbf{x} \in G$. Then $\mathbf{x} + \sigma_k \mathbf{d}_k$ and $\mathbf{x} - \theta_k \mathbf{d}_k$ are boundary points of $G$ for certain non-negative values of $\sigma_k$ and $\theta_k$. Let the point $\mathbf{x} + \sigma_k \mathbf{d}_k$ be in the boundary hyperplane $a_i^T \mathbf{x} = b_i$, $i \in I^+_k$ of $G$ and $\mathbf{x} - \theta_k \mathbf{d}_k$ in the boundary hyperplane $a_j^T \mathbf{x} = b_j$, $j \in I^-_k$.

We write the variables $\sigma_k$ and $\theta_k$ as

$$\sigma_k := \sigma_k^+ y_{i,k} \quad \text{and} \quad \theta_k := \theta_k^+ z_{j,k}.$$
We then obtain

\[ a_i^T x + s_{k_i} a_i^T d_k = y_{i,k} \quad i \in \mathbb{I}_k, \]
\[ a_j^T x - s_{k_j} a_j^T d_k = z_{j,k} \quad j \in \mathbb{I}_k', \]

which are two equations of the system (6-1) for \( x \in \mathbb{Q} \). If we can find a point \( x \in \mathbb{Q} \), such that \( y_{i,k} = z_{j,k} = 0 \), the distances from this point to the boundary of \( \mathbb{Q} \) with respect to the directions \( d_k \) and \(-d_k\) are both equal to \( s_k \). This is precisely what is required for a solution of the equilibrium problem with \( x \in \mathbb{Q} \).

\[ \square \]

For the calculation of a solution to the equilibrium problem we introduce the linear program \( \mathcal{P}_q \), \( q \in \mathbb{Q} \) given by

**Problem 6-2.** (linear program \( \mathcal{P}_q \), \( q \in \mathbb{Q} \))

\[
\text{minimize} \quad \sum_{k=1}^{P} (y_{i_k,k} + z_{i_k,k})
\]

subject to \( (x, y, z) \in \mathbb{H} \),

where \( \mathbb{H} \) the solution set of the linear system (6-1).

\[ \square \]

**Remark 4.** Let \( (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{H} \) be an optimal solution of the linear program \( \mathcal{P}_q \), \( q \in \mathbb{Q} \). It then follows from the conditions \( y \geq 0 \) and \( z \geq 0 \), that \( \bar{x} \) is a solution to the equilibrium problem if, and only if,

\[
\sum_{k=1}^{P} (\bar{y}_{i_k,k} + \bar{z}_{i_k,k}) = 0.
\]
Moreover, \( \bar{x} \) is a point in the element \( G_q \) of the partition \( G \) of \( G \). We can also conclude that
\[
\sum_{k=1}^{p} \left( \bar{y}_{i_k^+,k} + \bar{z}_{i_k^-,k} \right) > 0,
\]
if and only if, \( G_q \) does not contain a solution of the equilibrium problem. This does not necessarily mean that \( \bar{x} \) is not a solution of the equilibrium problem, since the value of this objective function may be zero for another selection.

**Remark 5.** The constraints in the linear programs are independent of the selections \( q \in G \). The selection \( q \) only affects the objective function. Hence, the initial simplex tableau for a linear program is obtained from the optimal simplex tableau of the preceding linear program by replacing the reduced cost row in the latter tableau by the reduced cost row of the objective function of the new linear program.

We recall the defining expression for the set \( S(x) \) in (5-21) for an arbitrary point \( x \in G \). The next theorem shows that the set \( S(x) \) can also be determined in terms of the associated point \( (x,y,z) \in S \) (see remark 1 of this section).

**Theorem 6-3.** Let \( (\bar{x},\bar{y},\bar{z},\bar{w}) \) be an arbitrary feasible point in \( S \). Then
\[
S(\bar{x}) = \left\{ q \in G \mid \begin{cases}
\bar{y}_{i_k^+,k} = \min \left\{ \bar{y}_{i_k^+,k} \mid i_k^+, \ i_k^- \right\} \\
\text{for } k=1, \ldots, p
\end{cases}, \right\}
\]
\[
\left\{ q \in G \mid \begin{cases}
\bar{z}_{i_k^-,k} = \min \left\{ \bar{z}_{i_k^-,k} \mid i_k^-, \ i_k^+ \right\} \\
\text{for } k=1, \ldots, p
\end{cases} \right\}
\]

(6-3)
Proof. Note that if \((\bar{z}, \bar{y}, \bar{z}_1, \bar{z}_2) \in R\), then \(\bar{z}_1 \in Q\) (see remark 1 of this section). Thus \(S(\bar{x})\) is well-defined by expression \((6.21)\). The proof of the theorem then easily follows since, by \((6.1)\), we have for arbitrary \(k \in \{1, \ldots, p\}\)

\[
\bar{y} = \Omega^k \frac{\bar{a}^T \bar{x}}{\bar{a}^T_{1k} \bar{d}_k} = \bar{z}_k,
\]

\[
b \bar{y}_i \leq \bar{y} \leq b_i,
\]

b. \(\min \{ \bar{y}_{i, k} \mid i \in I^+_k \} = \phi_k \min \left\{ \frac{b_i - \bar{a}^T \bar{x}}{\bar{a}^T_{1k} \bar{d}_k} \mid i \in I^+_k \right\} - \bar{z}_k
\]

\[
= \phi_k S(\bar{x} | \bar{d}_k) - \bar{z}_k.
\]

Similar relations hold for \(\bar{z}_1\) and \(\min \{ \bar{z}_1 \mid i \in I^-_k \}\). \(\Box\)

Remark 6. We recall that the selections in the set \(S(\bar{x})\) are always proper (see remark 1 in section 4 of chapter V). This property and the fact that this set can easily be determined for each feasible solution of the linear system \((6.1)\) is of great importance in our algorithm. \(\Box\)

We finally give a lemma which will be used for the explanation of the algorithm to be developed in the next section.

Lemma 6.4. Let \(q \in Q\) and let \((\bar{x}, \bar{y}, \bar{z}, \bar{z}_1)\) be an optimal solution of the linear program \(P_q\). Then for each \(k = 1, \ldots, p\)

\[
\min \{ \bar{y}^+_i, \bar{z}_1^- \mid i \in I^+_k, I^-_k \} = 0
\]
Proof. Assume that there exists a \( j \in \{1, \ldots, p\} \) with

\[
\min \{ \hat{y}_{i_{j}^{+}, j}, \hat{z}_{i_{j}^{-}, j} \} > 0.
\]

Without loss of generality we may assume that \( 0 < \hat{y}_{i_{j}^{+}, j} \leq \hat{z}_{i_{j}^{-}, j} \).

We now construct the point \((\hat{x}, \hat{y}, \hat{z})\) as follows:

\[
\hat{x}_{j} := \hat{x}_{j}^{+} + \hat{y}_{i_{j}^{+}, j} \quad \hat{y}_{i_{j}^{+}, j} := 0; \quad \hat{z}_{i_{j}^{-}, j} := \hat{z}_{i_{j}^{-}, j} - \hat{y}_{i_{j}^{+}, j}.
\]

The remaining variables are kept the same as in \((\hat{x}, \hat{y}, \hat{z})\). It can then be derived from the linear system (6-1) that \((\hat{x}, \hat{y}, \hat{z}) \in H\). However, for this point we have, by (6-4), that

\[
\sum_{k=1}^{p} (\hat{y}_{i_{k}^{+}, k} + \hat{z}_{i_{k}^{-}, k}) < \sum_{k=1}^{p} (\hat{y}_{i_{k}^{+}, k} + \hat{z}_{i_{k}^{-}, k}).
\]

This contradicts the property that \((\hat{x}, \hat{y}, \hat{z})\) is an optimal solution of the linear program \(p_{q}\).

VI-3 DESCRIPTION OF THE ALGORITHM

The multistep algorithm to be developed in this section is based on successively solving the linear programs \(p_{q}(q \in Q)\) formulated in problem 6-2. However, by choosing the selections in a special order, an attempt is made to fulfill the following conditions:

a. only proper selections are chosen,
b. the optimal objective values of two subsequent linear programs are monotonically decreasing.
Whether or not these conditions can be fulfilled becomes clear by considering a basic step of the algorithm. Let for arbitrary 
$r \in Q$, where $r := (r_1, r_2, \ldots, r_k, r_k, \ldots, r_k)$, $(\tilde{x}, \tilde{s}, \tilde{y}, \tilde{z}) \in H$ be an optimal solution of the linear program $F_Q$. Let furthermore \(\delta(\tilde{a})\) be calculated by (4-3) (or alternatively by (5-21)). We recall that all selections in $S(\tilde{x})$ are proper (see remark 6 in the preceding section). Finally, let the reduced cost row of the linear program $F_Q$ in the basic feasible solution $(x, s, y, z)$ of the linear system (5-1) be denoted by $C_{\tilde{x}}(x, s, y, z)$.

If

$$\sum_{k=1}^{p} (\tilde{y}_{j_k, k}^+ + \tilde{z}_{j_k, k}^-) = 0,$$

then $\tilde{x}$ is a solution to the equilibrium problem which lies in the element $G_k$ of the partition. Furthermore, let $q \in S(\tilde{x})$. Then, by (6-3),

$$\tilde{y}_{i_k, k}^+ = s_{i_k, k} \quad (i \in I_k^-),$$

$$\tilde{z}_{i_k, k}^- = z_{i_k, k} \quad (i \in I_k^+),$$

for $k = 1, \ldots, p$. Hence, $\tilde{y}_{i_k, k}^+ = \tilde{z}_{i_k, k}^- = 0$ for $k = 1, \ldots, p$. This means that all linear programs $F_{Q'}(q \in S(\tilde{x}))$ will lead to a solution of the equilibrium problem.

If, however,

$$\sum_{k=1}^{p} (\tilde{y}_{j_k, k}^+ + \tilde{z}_{j_k, k}^-) > 0,$$
then the element $G_r$ of the partition will not contain a solution to the equilibrium problem (see remark 4 in the preceding section). In this case we distinguish the following two cases c and d.

c. $r \not\in S(\bar{x})$, which implies that $\bar{x} \not\in G_r$ (property c in theorem 5-7). Let $q$ be an arbitrary selection in $S(\bar{x})$. Hence, $\bar{x} \in C_q$.

Since $(\bar{x}, \bar{y}, \bar{z}) \in H$ is an optimal solution of the linear program $F_\bar{z}$, we have, by lemma 6-4, that

$$\min \left\{ \frac{\bar{y}_i^+}{\bar{j}_i^+}, \frac{\bar{z}_j^-}{\bar{j}_j^-} \right\} = 0 \quad k = 1, \ldots, p.$$ 

Without loss of generality we may assume that

$$\frac{\bar{y}_i^+}{\bar{j}_i^+} = 0 \quad k = 1, \ldots, p.$$ 

Since $q \in S(\bar{x})$ and $r \not\in S(\bar{x})$, it can be derived from the expression (6-3) that

$$\begin{align*}
\frac{\bar{y}_i^+}{\bar{j}_i^+} & = \frac{\bar{y}_i^+}{\bar{j}_i^+} = 0 \\
\frac{\bar{z}_j^-}{\bar{j}_j^-} & \leq \frac{\bar{z}_j^-}{\bar{j}_j^-} \quad k = 1, \ldots, p
\end{align*}$$

(6-6.a)

(6-6.b)

with a strict inequality for at least one index $k \in \{ 1, \ldots, p \}$ in (6-6.b).

Therefore,

$$0 \leq \sum_{k=1}^{p} \left( \frac{\bar{y}_i^+}{\bar{j}_i^+} \cdot \frac{\bar{z}_j^-}{\bar{j}_j^-} \right) < \sum_{k=1}^{p} \left( \frac{\bar{y}_i^+}{\bar{j}_i^+} \cdot \frac{\bar{z}_j^-}{\bar{j}_j^-} \right).$$

Hence, in the optimal solution $(\bar{x}, \bar{y}, \bar{z})$ of the linear program $F_\bar{z}$, the objective value of the linear program $F_q$ is strictly smaller
than the optimal objective value of $P_q$. The objective value of $P_q$

in this point may even turn out to be equal to zero. This means

that $\bar{x}$ is a solution to the equilibrium problem, although this

has not been noticed by the linear program $P_\tau$. Of course, this

will be noticed immediately, when the selections in the set $S(\bar{x})$

are analyzed. Anyhow, it is appropriate to proceed with the

linear program $P_q$ for $q \notin S(\bar{x})$, in order to obtain monotony in

the values of the objective functions of two successive linear

programs.

Remark 1. Geometrically, the case $x \notin G_\tau$ can be clarified as

follows. In figure 6-3 below we have drawn a part of a partition

of a polyhedral set $G$ in $\mathbb{R}^2$.

![FIG.6-3. GEOMETRICAL CLARIFICATION OF THE CASE $x \notin G_\tau$.]
The point \((\bar{x}, \bar{z}, \bar{y}, \bar{z})\) is an optimal solution of the linear program \(P_\bar{x}\), while \(\bar{x}\) does not belong to the element \(G_x\) of the partition. The objective value of \(P_q\) in \((\bar{x}, \bar{z}, \bar{y}, \bar{z})\) is then strictly smaller than the optimal objective value of \(P_{\bar{x}}\). \(\square\)

d. \(x \in S(\bar{x})\), which implies that \(x \in G_x\). Since \(G_x\) does not contain a solution to the equilibrium problem, \(x\) will not be such a solution. Let \(q\) be another selection in \(S(\bar{x})\). It then follows from the relations (6-5) that

\[
\sum_{k=1}^{P} (\bar{y}_{+}^k + \bar{y}_{-}^k) \leq \sum_{k=1}^{P} (\bar{y}_{+}^k + \bar{y}_{-}^k).
\]

This means that, if we choose \(P_q\) with \(q \in S(\bar{x})\) as the next linear program to be solved, the optimal value of the objective function of \(P_q\) will not be greater than that of \(P_{\bar{x}}\). We can now distinguish the following two cases.

d-1. There exists a selection \(q \in S(\bar{x})\) for which the reduced cost row \(c_q^q(\bar{x}, \bar{z}, \bar{y}, \bar{z})\) of \(P_q\) contains a positive element. Then \(P_q\) will be the next linear program to be solved. The optimal objective value of \(P_q\) will then be smaller (lexicographically anyhow) than that of \(P_{\bar{x}}\).

d-2. For all selections \(q \in S(\bar{x})\) we have that \(c_q^q(\bar{x}, \bar{z}, \bar{y}, \bar{z}) \leq 0\). This means that \((\bar{x}, \bar{z}, \bar{y}, \bar{z})\) is an optimal solution for all linear programs \(P_q\) (\(q \in S(\bar{x})\)). The selections in \(S(\bar{x})\) will thus not lead to a solution of the equilibrium problem and therefore need not be considered anymore. In the next subsection we shall give an example which shows that such a situation can actually occur. In the present case we are then forced to choose a selection in the set \(Q = S(\bar{x})\). This selection is chosen according to an enumeration of \(Q\).
Remark 2. The case $x \notin G_r$ can be clarified geometrically as follows (see figure 6-4 below).

Let $S(\bar{x})$ contain the elements $q, r$ and $t$. This means that the point $\bar{x}$ lies in the intersection of the elements $G_q$, $G_r$, and $G_t$. The element $G_r$ does not contain a solution of the equilibrium problem. It is then further investigated whether the elements $G_q$ or $G_t$ contain such a solution. If $c_q(\bar{x}, \bar{s}, \bar{y}, \bar{z}) > 0$, we proceed to solve the linear program $F_q$. If, however, $c_q(\bar{x}, \bar{s}, \bar{y}, \bar{z}) \leq 0$ and $c_t(\bar{x}, \bar{s}, \bar{y}, \bar{z}) \leq 0$, then $(\bar{x}, \bar{s}, \bar{y}, \bar{z})$ is also an optimal solution of the linear programs $F_q$ and $F_t$. This implies that none of the elements $G_q$, $G_r$, and $G_t$ surrounding $\bar{x}$ contain a solution of the equilibrium problem. \[ \diamond \]
It may be clear from cases c and d that, in the attempt to fulfill the conditions a and b, it is logical to choose the selections for subsequent linear programs from the set $S(x)$.

The algorithm we propose for finding a solution to the equilibrium problem now proceeds as follows:

**Algorithm 6-5. Finding a solution to the equilibrium problem 5-1**

**Comment 1.** The following new quantities are used in the algorithm.

a. $U$ is the set in which the selections are stored, which do not lead to a solution of the equilibrium problem.

b. The objective function of the linear program $P_q$ in the point $(x, z, y, z) \in H$ will be denoted by

$$g_q(x, z) := \sum_{k=1}^n \left( y_k^+ + z_k^+ \right).$$

c. The boolean variable "success 1" indicates whether a subsequent linear program will be solved and "success 2" indicates whether a solution to the equilibrium problem has been found. Hence,

$$\text{success1} := \text{true, if } c_q(x, s, y, z) \neq 0 \text{ and}$$

$$\text{success2} := \text{true, if } c_q(x, y, z) = 0.$$  

1. **begin**
   test the consistency of the linear system (5-1);
   if this system is consistent then
   begin
   5. $(x, z, y, z) := \text{basic feasible solution found in the consistency test of (5-1)}$;
   $U := \emptyset$; $\text{success1} := \text{true}$; $\text{success2} := \text{false}$;
while success1 and not success2 do
begin
    calculate $S(x)$;
    success1 := false;
    repeat select next $q \in S(x)$;
        if $q \notin U$ then
            begin
                $U := U \cup \{q\}$;
                if $g_q(y,z) = 0$ then success2 := true else;
                if $c_q(x,s,y,z) \neq 0$ then success1 := true;
            end
        until success1 or success2 or last $q \in S(x)$;
    if not success1 and not success2 then
        repeat select next $q \in Q$;
            if $q \notin U$ then
                begin
                    $U := U \cup \{q\}$;
                    if $c_q(x,s,y,z) \neq 0$ then
                        success1 := true;
                end
            until success1 or last $q \in Q$;
    if success1 then
        begin
            solve linear program $P_q$;
            $(x,s,y,z) :=$ optimal solution;
            if $g_q(y,z) = 0$ then success2 := true;
        end
    end
    if success2 then print $(x)$
        else print (no solution);
end
else print (inconsistent);
Comment 2. (Order in which selections are chosen). The algorithm requires an order in which the selections are chosen, either from $Q$ or from $S(x)$. A general way of doing this is as follows. Let $J$ be an arbitrary non-empty subset of $Q$ defined by

$$J := J_1^+ x J_1^- x J_2^+ x J_2^- x \ldots x J_p^+ x J_p^-,$$

where for $k = 1, \ldots, p$

$$J_k^+ \subseteq I_k^+, \quad J_k^- \subseteq I_k^-, \quad J_k^+ \neq \emptyset, \quad J_k^- \neq \emptyset.$$

Hence, $J$ may either coincide with $Q$ or it may be the set $S(x)$ in which case $J$ is constructed by using expression (5.3). Let furthermore $q := (i_1^+, i_1^-, \ldots, i_k^+, i_k^-, \ldots, i_p^+, i_p^-)$ be an arbitrary selection in $J$. The following selection is then $(i_1^+, i_1^-, \ldots, i_k^+, i_k^-, i_p^+, i_p^-)$, where the index $i_p^-$ comes after $i_p^+$ in the index set $J_p$. If, on the other hand, $i_p^+$ is the last index in $J_p$, the following selection is $(i_1^+, i_1^-, \ldots, i_p^-, i_p^+)$, where $j_p^-$ is the first index in $J_p^-$ and $j_p^+$ comes after $j_p^-$ in $J_p^+$ and so on.

Comment 3. Obviously the algorithm can be used for finding more than one solution to the equilibrium problem by restarting the algorithm after a solution has been found. This may, for instance, be desirable, when the algorithm has found a degenerated solution to the equilibrium problem. Alternatively, one can proceed as follows in the latter case. Let $(x, s, y, z) \in H$ be an optimal solution of the linear program $F_q$, where $x$ is a degenerated solution to the equilibrium problem. Then proceed by solving the linear program.

maximize \( \sigma \)

subject to \( (x, s, y, z) \in H, \)
\[ \begin{align*}
    y_\ast &= 0 \\
    z_\ast &= 0 \\
    \sigma - \eta_k &\leq 0
\end{align*} \tag{6-7.a} \tag{6-7.b} \tag{6-7.c}
\]

Let \((\bar{x}, \bar{z}, \bar{y}, \bar{z}, \bar{\sigma})\) be an optimal solution of this linear program. Then \((\bar{x}, \bar{z}, \bar{y}, \bar{z}, \bar{\sigma})\) is also an optimal solution of the linear program \(P_q\) (by the constraints \((6-7.a)\) and \((6-7.b)\)) and \(\bar{x}\) is a solution to the equilibrium problem. Moreover, if \(\bar{\sigma} > 0\) then \(\bar{\eta}_k > 0\) for all \(k = 1, \ldots, p\) (by \((6-7.c)\)). It can then be shown that \(\bar{x}\) is a non-degenerated solution to the equilibrium problem. \(\square\)

VI-4  AN ILLUSTRATIVE EXAMPLE

VI-4.1  Introduction

We consider the system of linear inequalities

\[ \begin{align*}
    x_1 - 2x_2 &\leq 1, \\
    x_1 + x_2 &\leq 7, \\
    -2x_1 + x_2 &\leq 1, \\
    -x_1 &\leq 0, \\
    -x_2 &\leq 0.
\end{align*} \tag{6-8} \]

The solution set of this system is denoted by \(G\) and the row index set is denoted by \(I := \{1, 2, 3, 4, 5\}\). Note that \(x := (1, 1) \in \text{int } G\). Thus \(\text{int } G \neq \emptyset\). As directions of finite flexibility we choose \(e_1 := (1, 0)^T\); \(e_2 := (0, 1)^T\); \(-c_1\); \(-c_2\) and the weighting factors are \(\phi_1 = \phi_2 = \frac{1}{2}\). For these directions of finite flexibility we have
\[ i_k^+ := \{ i \in I \mid a_{ik} > 0 \} = \{ i \in I \mid a_{1k} > 0 \} \text{ and } i_k^- = \{ i \in I \mid a_{ik} < 0 \} \]

for \( k = 1, 2 \). Hence, according to (6-8),

\[ I_1^+ = \{1, 2\}, I_1^- = \{3, 4\}, I_2^+ = \{2, 3\}, I_2^- = \{1, 5\}. \]

The set of selections \( Q \) is given by \( Q = I_1^+ \times I_1^- \times I_2^+ \times I_2^- \). The selections in \( Q \) are enumerated as given in table 6-5 below.

<table>
<thead>
<tr>
<th>n(q)</th>
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<th>n(q)</th>
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<td>9</td>
<td>(2, 3, 1)</td>
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<td>2</td>
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</tr>
<tr>
<td>5</td>
<td>(1, 4, 1)</td>
<td>13</td>
<td>(2, 4, 1)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 4, 5)</td>
<td>14</td>
<td>(2, 4, 5)</td>
</tr>
<tr>
<td>7</td>
<td>(1, 4, 3)</td>
<td>15</td>
<td>(2, 4, 3)</td>
</tr>
<tr>
<td>8</td>
<td>(1, 4, 3)</td>
<td>16</td>
<td>(2, 4, 3, 5)</td>
</tr>
</tbody>
</table>

**Table 6-5. Selections in the set \( Q \) with their enumerated values.**

The set \( G \) with its partition \( \mathcal{G} \) (indicated with dotted lines) has been drawn in figure 6-6. This figure shows that the selections with enumerated values 2, 6, 10, 13, 14, 15, 16 are improper.
VI-4.2 The associated linear system (6-1)

Since \( \{e_1, e_2\} \) is a basis in \( \mathbb{R}^2 \), we have that \( I = \emptyset \) (remark 4 section V-4) Since \( \psi_k = 0 \), \( k = 1, 2 \) in the present case, we shall make the following modification in the system (6-1). The equalities (6-1.a) and (6-1.b) are multiplied by 2 for \( k=1,2 \) and the resulting variables \( Z_{2k}, Z_{2k+1}, Z_{2k+2} \) and \( Z_{2k}, Z_{2k+1}, Z_{2k+2} \) are replaced by the new
variables $s_k$, $y_{1,k}$ and $z_{4,k}$. This modification simplifies the coefficients in the system (6-1), which then becomes:

\[
\begin{align*}
    x_1 - 2x_2 + s_1 + y_{1,1} &= 1, \\
    x_1 + x_2 + s_1 + y_{2,1} &= 7, \\
    -x_1 + 3x_2 + y_{2,1} + z_{3,1} &= 4, \\
    -x_1 + s_1 + y_{1,1} + z_{4,1} &= 0, \quad (6-9) \\
    x_1 + x_2 + s_2 + y_{2,2} &= 7, \\
    -2x_1 + x_2 + s_2 + y_{3,2} &= 2, \\
    3x_1 - x_2 + s_2 + y_{2,2} &= 4, \\
    -x_2 + s_2 + z_{5,2} &= 0.
\end{align*}
\]

VI-4.3 Application of the algorithm 6-5

We start with $U := \emptyset$ and with the point $(x^0, y^0, s^0, z^0)$ given by

\[
(x_1^0, x_2^0) := (0,0), \quad (s_1^0, s_2^0) := (0,0)
\]

and for $k = 1$

\[
(y_{1,1}^0, y_{2,1}^0) := (1,7), \quad (z_{3,1}^0, z_{4,1}^0) := (7,1) \quad (6-10)
\]

and for $k = 2$

\[
(y_{2,2}^0, y_{3,2}^0) := (4,0), \quad (z_{1,2}^0, z_{5,2}^0) := (5,0).
\]

The initial simplex tableau $T^0$ is given in table 6-7.a (see page 112). It follows from this tableau that the point given by (6-10) is a basic feasible solution to the linear system (6-9). Hence, it is not necessary to execute the consistency test in the algorithm. From the tableau $T^0$ and the expression for $S(x)$ in (6-3) it follows that $S(x^0) = \{(1,4,3,5)\}$. The enumerated value of this selection is $n(q) = 8$.

**Step 1.** The value of the objective function of the linear program
\( \mathbf{P}_8 \) in the point \((x^0, e^0, y^0, z^0)\) is

\[ y_{1,1}^0 + y_{4,1}^0 + y_{3,2}^0 + y_{5,2}^0 = 2 \not= 0 \]

The reduced cost row for the linear program \( \mathbf{P}_8 \) in the point \((x^0, e^0, y^0, z^0)\) can be constructed from the tableau \(T^0\). It becomes

\[ c_8^0(x^0, e^0, y^0, z^0) := (-2, 2, 2, 0, ... , 0) \not= 0 \]

Hence, \( \text{success1 := true; } U := [8] \) and linear program \( \mathbf{P}_8 \) is solved. The optimal simplex tableau \( T^1 \) is given in Table 5-7.b. Note that the optimal solution of \( \mathbf{P}_8 \) is the same point as given by (6-10) and that the optimal value of the objective function is greater than zero. We define \((x^1, e^1, y^1, z^1) := (x^0, e^0, y^0, z^0)\).

**Step 2.** From the preceding step it follows that \( S(x^1) = S(x^0) := \{(1, 4, 3, 5)\} \). Since this selection has already been excluded, we will not find "success1" or "success2" in that part of the algorithm which analyzes the selections in \( S(x) \) (lines 9 - 19). We recall case d-2 in section VI-3, where such a situation has been announced. The algorithm now proceeds to analyze the selections in \( Q \) (lines 20 - 26). The first selection in \( Q \) is, by Table 6-5, \( q := (1, 3, 2, 1) \) with \( n(q) = 1 \). This selection has not been excluded yet. For this selection we have the following.

a. The value of the objective function of the linear program \( \mathbf{P}_1 \) in the point \((x^1, e^1, y^1, z^1)\):

\[ y_{1,1}^1 + y_{3,1}^1 + y_{2,2}^1 + y_{1,2}^1 = 9 \]

* For convenience we shall use the enumerated values of the selections in the subscripts for the linear program \( \mathbf{P}_q \) and the reduced cost row \( c_q^r(\cdot) \).
Note that this value is greater than the optimal value of the objective function of the preceding linear program \( \mathbf{f}_3 \).

b. The reduced cost row is

\[ \mathbf{c}_2'(x^1, y^1, z^1, s^1) := (3/2, -3/2, 2, 2, 0, 0, 0, 0) \not\leq 0. \]

Hence, \textit{success} \( := \text{true} \) (line 26); \( U := \{1, 8\} \) and linear program \( \mathbf{f}_2 \) is solved. The optimal simplex tableau \( \mathbf{T}^2 \) is given in table 6-7.c. The optimal solution is denoted by \((x^2, s^2, y^2, z^2)\).

\textbf{Step 3.} From the tableau \( \mathbf{T}^2 \) it can be derived that \( S(x^2) := \{(1,3,2,1);(2,3,2,1)\} \) with enumerated values 1 and 9 respectively. Since the selection \((1,3,2,1)\) has already been excluded in the preceding step, we concentrate on the selection \((2,3,2,1)\). For this selection we have the reduced cost row

\[ \mathbf{c}_3'(x^2, s^2, y^2, z^2) := (0, 0, 0, 0, 5/8, -15/8, 3/4, 0, 0, 0, -2, 0) \not\leq 0 \]

Hence \( \mathbf{f}_3 \) is the next linear program to be solved. The optimal simplex tableau \( \mathbf{T}^3 \) is given in table 6-7.d. It shows that \( x := (13/5, 13/5) \) is a solution to the equilibrium problem.

We finally note that if we omit the constraint with index number 2 from the linear system (6-8), then \( G \) becomes unbounded (see also figure 6-6). It follows from table 6-5 that \( G \) consists of the selections with enumerated values 3, 4, 7 and 8. It can then be shown that none of the linear programs \( \mathbf{f}_q' \), \( q = 3, 4, 7, 8 \) provide a solution to the equilibrium problem. Hence, a solution to the changed problem does not exist.
### Table 6-1a: Initial Simplex Tableau $T_0$ of the Linear Program $p_0$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$x_3$</th>
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### Table 6-1b: Optimal Simplex Tableau $T_f$ of the Linear Program $p_0$

<table>
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<th>$x_4$</th>
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### Table 6-1c: Optimal Simplex Tableau $T_f$ of the Linear Program $p_1$

<table>
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### Table 6-1d: Optimal Simplex Tableau $T_f$ of the Linear Program $p_2$

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CHAPTER VII

CONCLUDING REMARKS

In this thesis we have studied a novel approach to finding so-called flexible solutions to linear programming problems. In this approach the flexibility of a point in the solution set G of the consistent linear systemAx ≤ b is defined as the Euclidean distance from this point to the boundary of G with respect to a prescribed direction. Two problems, a weighted distance - and an equilibrium problem, have been defined for finding solutions in G which have these flexibility properties with respect to a set of these directions. Both problems can be solved by finite algorithms.

The weighted distance problem is a max-min problem. A solution is found by solving a single linear program. This linear program has one more variable and the same number of constraints as the linear system Ax ≤ b.

The equilibrium problem, resembling the Nash equilibrium problem in game theory, requires the solution of a finite sequence of linear programs and is therefore more difficult to solve. The number of variables and constraints of these linear programs is greater than the number in the original linear system Ax ≤ b. Apart from this exact algorithm there exists a heuristic iterative algorithm. The major advantage of this algorithm is that it is numerically very simple and does not require a transformation of the constraint matrix A.

For the application of both methods to practical problems it is necessary to identify the implicit equalities in the linear system Ax ≤ b. Two finite algorithms have been developed to perform this identification.
In order to isolate the problem of flexibility from that of economical optimality we have left an economical objective function out of consideration. However, a decision maker will in general be more interested in solutions to linear programming problems which have both flexibility properties and an acceptable value for the original objective function of the linear program (such as cost or profit). For example, the following multi-objective programming problem gives such a solution (see also the linear program 4-4).

\[
\begin{align*}
\text{maximize} & \quad (\rho, c^T x) \\
\text{subject to} & \quad a_i^T x + w_i \rho \leq b_i \quad (i \in I_0), \\
& \quad a_i^T x \leq b_i \quad (i \in I-I_0),
\end{align*}
\]

where \( \rho \) reflects the flexibility of a solution to the linear system \( Ax \leq b \) and where \( c^T x \) is an economical objective function. The interactive multi-objective programming method of Zionts and Wallenius [18] seems to be the most appropriate for this approach, since it enables the decision maker to find his own balance between economical optimality and flexibility. It should be noted, however, that the objective functions \( \rho \) and \( c^T x \) in the above problem need not be conflicting. For instance, this is obviously the case if we consider the consistent linear system

\[
x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 0
\]

and choose \( d := (-1,0)^T \) as direction of flexibility and \( c := (1,0)^T \) for the cost vector in the economical objective function.

Another possible approach for finding a balance between economical optimality and flexibility is the following. Let \( \bar{x} \) be the optimal solution to the linear program...
\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
\end{align*}
\]

and let \( \bar{\tilde{z}} := c^T \tilde{z} \). The methods described can then be applied to find a flexible point with respect to the consistent linear system

\[
Ax \leq b, \quad c^T x < \bar{\tilde{z}} - \delta,
\]

where \( \delta \) is a positive real number to be specified by the decision maker. For instance, if we take \( \delta \) as a fraction of \( \bar{\tilde{z}} \), say \( \delta = 0.1 \bar{\tilde{z}} \), then a flexible point is generated which belongs to \( \tilde{G} \) and for which the economical objective value is at least 90% of the optimal objective value \( \bar{\tilde{z}} \). We note, however, that this optimal value ought to be found first.
References

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4. J. Rosenhead, M. Elton, S.K. Gupta; "Robustness and optimality as criteria for strategic decisions"; Operational Research Quarterly 23 (1972) 413-431.


CURRICULUM VITAE

1936 Geboren te Amsterdam.

1955 Diploma HBO-b aan het Groen van Prinsterer lyceum te Vlaardingen.


1958 Beëdiging tot technische dienst officier bij de Koninklijke Marine.

1958-1964 Diverse technische functies aan boord van marineschepen vervuld.


1971 Kersvol ontslag uit de Koninklijke Marine.

1971- Research medewerker op het Koninklijke Shell Laboratorium te Amsterdam.
STELLINGEN

I

Zij gegeven het veelvlak \( G := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) met niet-leeg inwendige. De volgende uitspreken zijn equivalent.

a. De grootste bol die in \( G \) beschreven kan worden heeft een eindige straal.

b. Het systeem \( A^* u = 0; u \geq 0 \) heeft een oplossing met tenminste één van de componenten van de vektor \( u \) groter dan nul.

II

Zij gegeven het veelvlak \( G := \{ x \in \mathbb{R}^n \mid Ax \geq b \} \) met niet-leeg inwendige. Het punt \( \bar{x} \in \text{int} G \) voldoet aan het stelsel vergelijkingen

\[
\phi_k(x|x_k) - (1-\phi_k)x(x_k - e_k) = 0 \quad k = 1, \ldots, n.
\]

\( \phi_k \in (0,1) \)

Zij verder \( Ax = b \) het in stelling 5-11 van dit proefschrift bedoelde stelsel lineaire vergelijkingen waarvan \( \bar{x} \) een oplossing is.

De elementen \( a_{i,j} ; i, j = 1, \ldots, n \) van de \((nxn)\) matrix \( A \) voldoen dan aan de ongelijkheid

\[
|a_{i,j}| \leq \frac{\phi_j s(\bar{x}|e_j)}{\phi_j s(\bar{x}|e_j)}
\]
Hieruit volgt dat voor deze elementen in het bijzonder geldt

\[ |a_{i_1, i_2}, a_{i_2, i_3}, \ldots, a_{i_{k-1}, i_k}, a_{i_k, i_1}| = 1 \]

voor ieder eindig stelsel indices \( \{ i_1, i_2, \ldots, i_k \} \).

M.F. ter Horst en R.P. van der Vat;  
"On the existence and convergence aspects of the flexible programming method";  
Proc. of the second European Congress on  
Operations Research (Stockholm, Sweden,  
Nov 29 - Dec 1, 1976).

III

Zij \( C \) een gesloten convex deelverzameling van \( \mathbb{R}^2 \). Het punt \( x^* := 0 \) behoort tot het inwendige van \( C \) en voldoet aan het stelsel vergelijkingen

\[ s(x|e_k) - s(x|-e_k) = 0 \quad k = 1, 2, \]

met \( s(x|e_k) = s(x|-e_k) = 1 \) voor \( k = 1, 2 \). De Gauss-Seidel methode toegepast op bovengenoemd stelsel convergeert naar een oplossing van dit stelsel voor ieder startpunt in de verzameling

\[ C \cap \{ x \in \mathbb{R}^2 | -1 \leq x_k \leq 1; \; k = 1, 2 \}. \]

Deze eigenschap is niet te generaliseren tot convex deelverzamelingen in \( \mathbb{R}^n \) met \( n \geq 3 \).
Zij gegeven de reëelwaardige $(mn)$ matrix $A$ met tenminste één kolomvektor met een positief en negatief element. Dan is in $n$ stappen een reguliere $(mxn)$ matrix $B$ te construeren met de eigenschap dat iedere kolomvektor van $AB$ een positief en negatief element bevat.

De bewering van Chien en Kuh, dat het door hen ontwikkelde algoritme voor het oplossen van stuksgewijs lineaire vergelijkingen convergent is, is onvoldoende geïnformeerd.


Bij het oplossen van lineaire programmeringsproblemen met de simplex methode, is het weinig zinvol vooraf de冗多余的不等式 te verwijderen.
VII

Het is in het belang van de geloofwaardigheid in het gebruik van simulatie voor het oplossen van praktijkproblemen, dat bij de rapportage van simulatieoplossingen meer aandacht wordt besteed aan de kwalitatieve en kwantitatieve gebruikswoorden van het simulatiemodel.

VIII

De in de operations research literatuur beschreven wiskundige modellen van het vehicle scheduling probleem dragen nauwelijks bij tot het analyseren en oplossen van dit probleem in de praktijk.

IX

De naamgeving voor de in gebruik zijnde muziekinstrumenten in de klarinetfamilie is onvolledig, niet eenduidig en onsystematisch. Het is daaron gewenst voor deze klarinetten eenzelfde naamgeving in te voeren, zoals die bij de saxofoons wordt gehanteerd.

X

Om de investeringen in gebieden met hoge werkloosheid te stimuleren, zou de overheid de mogelijkheid moeten openen deze investeringen te laten financieren met obligaties waarvan de interest is vrijgesteld van inkomstenbelasting.

24 juni 1980

R.P. van der Vet