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0.1. Introductory Remarks

The first reference to uniformly distributed sequences of real numbers on the segment \([0,1]\) is to be found in a paper by Weyl [14].

One may consider uniformly distributed sequences from two different points of view.

On the one hand, a uniformly distributed sequence \(\xi = \{x_n\}_{n=1}^{\infty}\) \((0 \leq x_n \leq 1 \text{ for all } n)\) is characterized by the relation

\[
(1.1) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(x_n) = b - a
\]

for all \(a,b \ (0 \leq a < b \leq 1)\), where \(\chi_{[a,b]}\) denotes the characteristic function of the interval \([a,b]\).

On the other hand, a uniformly distributed sequence \(\xi = \{x_n\}_{n=1}^{\infty}\) \((0 \leq x_n \leq 1 \text{ for all } n)\) may be characterized by the fact that

\[
(1.2) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx
\]

for all continuous functions \(f\) on the interval \([0,1]\).

The studies made of the aspects of uniformly distributed sequences may be roughly divided into quantitative and qualitative ones. Quantitative questions are concerned with the order of convergence of the mean values in (1.1) and (1.2) to their respective limits, and have been investigated e.g. by Van der Corput [3] and Koksma [12] (for more information on the literature about this and the following subjects see Cigler and Helmberg [2]).
study of the qualitative aspects of uniform distribution was further developed particularly by Hlawka [8], [9], who, starting from a paper by Sierpinski [4], introduced the concept of uniformly distributed sequences in a compact topological space with a normalised Borel-measure \( \mu \). Generalising the situation on the unit interval, it appears that every Riemann-integrable function on a compact space may be integrated by taking its asymptotic mean value on the points of a uniformly distributed sequence (cf. (1.2)).

All these investigations concerned bounded functions. Helmberg [7] studied the case of non-bounded continuous functions on a locally compact normalised measure-space. Such a function may also be regarded as a function on a compact space (the one-point compactification of the original space), which is continuous everywhere, one single point excepted, and may be approximated (pointwise) by a sequence of bounded continuous functions. Helmberg, amongst others, proved that for any uniformly distributed sequence \( \xi = \{x_n\}^\infty_{n=1} \) and for any non-negative continuous function \( f \) on a locally compact space \( X \) we have the following inequality:

\[
(1.3) \quad \int_X f(x) d\mu(x) \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(x_n).
\]

He also gave an example of a situation where \( \int_X f(x) d\mu(x) < \infty \) and

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \infty.
\]

In accordance with Helmberg [7] we shall define a function \( f \) to be \((\xi, d\mu)\)-summable, if \( f \) is integrable and

\[
(1.4) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_X f(x) d\mu(x).
\]

In this thesis we continue the above mentioned investigations.
about the behaviour of unbounded functions with respect to a uniformly distributed sequence.

In Chapter I we shall prove that, given any uniformly distributed sequence $\xi = \{x_n\}_{n=1}^{\infty}$ without repetitions and any three numbers $\alpha, \beta, \gamma \ (0 < \alpha \leq \beta \leq \gamma < \infty)$, there exists a non-negative continuous function $f$ such that $\int f \, d\mu = \alpha$, $\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \beta$ and $\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \gamma$. In Chapter II some sufficient conditions for $(\xi, d\mu)$-summability are discussed (Section 2.3). We shall sketch here some background considerations, which gave rise to the investigation of these particular conditions:

(i) The $(\xi, d\mu)$-summability of a continuous function $f$ on a locally compact space depends on the behaviour of $f$ near the point $\infty$, i.e., the point which has to be added in order to compactify our space.

(ii) The topological and measure-theoretical situation near the point $\infty$ is related to the system of neighbourhoods of $\infty$.

(iii) We are able to construe a special sequence of neighbourhoods of $\infty$, related to $\xi$ (Section 2.2), such that the global behaviour of a function $f$ with respect to this particular sequence of neighbourhoods is decisive for the $(\xi, d\mu)$-summability of $f$. These arguments are found to apply also to a larger class of functions than continuous functions only (Section 2.3).

In Section 2.4 we shall exhibit some examples which show that the results of Section 2.3 are in a certain sense best-possible.

In Chapter III we shall illustrate the results of Chapter II, mainly for the special case of a sequence introduced by von Neumann [13]. One of the arguments used here also applies to almost all sequences $\{ n\beta \}_{n=1}^{\infty} \mod 1$ (where $\beta$ is an irrational number), as will be shown in the final Section.
0.2. NOTATIONS

In order to avoid repetition of definitions of some notions which are used in all Chapters of this thesis we here give a survey of some concepts and notations used.

X is given to be a locally compact, non-compact Hausdorff-space with a countable base \( \mathcal{W} \) of open neighbourhoods \( F \). Its one-point compactification \( X \cup \{ \infty \} \) is denoted by \( X^* \). If \( A \) is a subset of \( X \) we write \( A^c, A^\circ, A^\partial, \) and \( \partial A \), respectively, for the complement, closure, interior, and boundary of \( A \).

\( \mu \) is defined to be a normalised non-atomic Borel-measure on \( X \) with a non-compact support \( S \). This measure \( \mu \) induces a normalised non-atomic Borel-measure \( \mu^* \) on \( X^* \) if we define \( \mu^*(A) = \mu^*(A \cup \{ \infty \}) = \mu(A) \) for all Borel-sets \( A \subset X \).

Without loss of generality we may assume that all neighbourhoods \( E \subset \mathcal{W} \) have compact closures and zero-boundaries, i.e., \( \mu(\partial E) = 0 \) for all \( E \subset \mathcal{W} \) (cf. Helmsberg [7]).

\( \xi = \{ x_n \}_{n=1}^{\infty} \) denotes a \( d \mu \)-uniformly distributed sequence of points \( x_n \in X \, (n = 1, 2, 3, \ldots) \), in other words we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{X} f(x) \, d\mu(x)
\]

for all continuous complex-valued functions \( f \) on \( X \) with compact support. It is well known that for all Borel-sets \( F \) with zero-boundaries we have

\[
\lim_{N \to \infty} \frac{1}{N} A(\xi, F; N) = \mu(F)
\]

where \( A(\xi, F; N) = \sum_{n=1}^{N} \chi_{F}(x_n) \) denotes the number of indices \( n = 1, \ldots, N \) for which \( x_n \in F \) (cf. Helmsberg [7], p. 172 Hilfssatz 2).

For every finite collection of sets \( \{ E_i \}_{i=1}^{K} \), the following inclusion may be verified by straightforward calculation:
\( \text{a}(\bigcup_{j=1}^{k} E_j) \subset \bigcup_{j=1}^{k} (\partial E_j) \), \( \text{a}(\bigcap_{j=1}^{k} E_j) \subset \bigcup_{j=1}^{k} (\partial E_j) \)

(cf. Kelley [10]). These relations imply that the union and the intersection of a finite number of sets \( E_j \) with zero-boundaries are sets with zero-boundaries themselves.
CHAPTER 1

1.1. PRELIMINARY REMARKS - AUXILIARY PROPOSITIONS

In the present Section we shall derive some lemmas concerning \(\mu\)-uniformly distributed sequences and set-theoretical topology.

Lemma 1.1.1. If \(F\) is an open set with compact closure and zero-boundary, such that \(\mu(F) > 0\), if \(\xi = \{x_k\}_{k=1}^{\infty}\) is \(\mu\)-uniformly distributed in \(X\), and if \(\{k_i\}_{i=1}^{\infty}\) is the sequence of all indices \(k\) such that \(x_k \in F\), then \(\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1\).

Proof. Obviously \(A(\xi, F, k_{i+1}) = A(\xi, F, k_i) + 1\) (\(i = 1, 2, 3, \ldots\)) and this expression tends to infinity if \(i \to \infty\). Now we have

\[
1 = \frac{\mu(F)}{\mu(F)} = \lim_{i \to \infty} \frac{A(\xi, F, k_i)}{k_i} = \lim_{i \to \infty} \frac{k_{i+1}}{k_i}.
\]

Lemma 1.1.2. If \(x_0 \in S_\mu\), and if \(\xi = \{x_k\}_{k=1}^{\infty}\) is \(\mu\)-uniformly distributed, then there exists a sequence of indices \(\{i_k\}_{k=1}^{\infty}\) such that \(\lim_{i \to \infty} x_{i_k} = x_0\) and \(\lim_{i \to \infty} \frac{i_{k+1}}{i_k} = 1\).

Proof. Let \(\{F_k\}_{k=1}^{\infty}\) be a countable base of open neighbourhoods of \(x_0\), \(E_k \in \mathcal{G}^k\) \((k = 1, 2, 3, \ldots)\). Then \(E_k\) is compact, \(\mu(\partial E_k) = 0\) \((k = 1, 2, 3, \ldots)\) and \(\cap_{k=1}^{\infty} E_k = \{x_0\}\) (cf. Ch.0). If \(F_k\) is defined by \(F_k = \cap_{j=1}^{k} E_j\) \((k = 1, 2, 3, \ldots)\), then \(F_k\) is compact, and \(\mu(\partial F_k) = 0\) \((k = 1, 2, 3, \ldots)\), \(F_1 \supset F_2 \supset \ldots \supset F_k \supset \ldots\), and \(\cap_{k=1}^{\infty} F_k = \{x_0\}\). Since \(\{E_k\}_{k=1}^{\infty}\) is a base of neighbourhoods of \(x_0\)
we may find for any neighbourhood \( U \) of \( x_0 \) a set \( F_k \), such that \( F_k \in U \). As \( \mu \) is non-atomic we obtain \( \lim_{k \to \infty} \mu(F_k) = 0 \), and \( x_0 \in \mu \) implies \( \mu(F_k) > 0 \) \( (k = 1, 2, 3, \ldots) \).

For each of the sets \( F_k \) lemma 1.1.1 applies and we obtain a chain of sub-sequences of indices \( \{k_j, j \}_{j=1}^{\infty} \) \( (k = 1, 2, 3, \ldots) \) such that for every fixed \( k \) we have \( x_{k_j} \in F_k \) for all \( j \) and

\[
\lim_{j \to \infty} \frac{k_{j+1}}{k_j} = 1. \quad \text{Let} \quad \frac{k_{j+1}}{k_j} < 1 + 2^{-k} \quad \text{for all} \quad j \text{ satisfying} \quad j, k_j \gg_k \kappa_0.
\]

Without loss of generality we may assume that \( k_j = \lambda_{k_j} \) for some index \( j \), and that \( k_{j+1} > k_j \) \( (k = 1, 2, 3, \ldots) \). If \( F_k \) is the (non-empty) finite sequence of all \( \lambda_{k_j} \) for which \( k_k \ll \lambda_{k_j} < k_{k+1} \), and \( \{\lambda_k\}_{k=1}^{\infty} \) is the sequence formed by writing down all blocks \( R_k \) in succession, we see that \( 1 < \frac{k_{j+1}}{k_j} < 1 + 2^{-k} \) for all \( j \in F_k \).

Hence \( \lim_{j \to \infty} \frac{k_{j+1}}{k_j} = 1. \) The construction of \( \{F_k\}_{k=1}^{\infty} \) implies that

\[
\lim_{j \to \infty} x_{k_j} = x_0.
\]

**Corollary 1.1.2.1.** There exists a sub-sequence \( \{x_{k_j}\}_{j=1}^{\infty} \) of \( \xi \) such that \( \lim_{i \to \infty} x_{k_i} = \infty \) and \( \lim_{i \to \infty} \frac{k_i}{k_i} = 1 \).

**Lemma 1.1.3.** If \( \{k_j\}_{j=1}^{\infty} \) and \( \{m_j\}_{j=1}^{\infty} \) are two monotonically increasing sequences of integers such that \( \lim_{j \to \infty} \frac{k_j + 1}{k_j} = \lim_{j \to \infty} \frac{m_{j+1}}{m_j} = 1 \),

then there exists a sub-sequence \( \{k_{j_i}\}_{i=1}^{\infty} \) of \( \{k_j\}_{j=1}^{\infty} \) such that \( \lim_{i \to \infty} \frac{k_{j_{i+1}}}{k_{j_i}} = 1 \) and \( k_j \gg m_j \) for all \( j \).

**Proof.** We define \( k_0 = \min \{k_j | k_j \gg m_1\} \) and by induction \( k_{j+1} = \min \{k_j | k_j \gg m_{j+1} \} \) and \( k_j \gg k_{j_i} \) \( (i = 1, 2, 3, \ldots) \).

For all \( c > 0 \) there exists a number \( \tilde{N} \) such that for all \( k_j \gg \tilde{N} \).
and for all $j \geq N$

$$1 < \frac{k_{i+1}}{k_i} < (1 + \varepsilon)^\frac{1}{2}, \quad 1 < \frac{m_{i+1}}{m_i} < (1 + \varepsilon)^\frac{1}{2}.$$ 

Let $m_i = N$. Then we distinguish two cases:

1. $k_i \geq m_{i+1}$. In this case $\frac{k_{i+1}}{k_i}$ is the quotient of two consecutive $k_j$, both of which are greater than $N$. Hence $1 < \frac{k_{i+1}}{k_i} < (1 + \varepsilon)^\frac{1}{2} < 1 + \varepsilon$.

2. $m_i < k_i < m_{i+1}$. Then $k_{i+1}$ is the smallest $k_j \geq m_{i+1}$, say $k_{i+1} = k_{n_i} > m_{i+1} > k_{n_i-1}$. Hence

$$1 < \frac{k_{i+1}}{k_i} < \frac{k_{n_i}}{k_{n_i-1}} < \frac{m_{i+1}}{m_i} < (1 + \varepsilon)^\frac{1}{2},$$

so that $\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1$.

Lemma 1.1.4. If $\mathbb{V}$ is a closed subset of $X^*$ with zero-boundary, $x_0 \notin \mathbb{V}$, and $\kappa$ and $\lambda$ are real numbers such that $0 < \kappa < \lambda < 1 - \mu^*(\mathbb{V})$, then there exists an open neighbourhood $U$ of $x_0$ in $X^*$ such that $\overline{U} \cap \mathbb{V} = \emptyset$ and $\kappa < \mu^*(U) = \mu^*(\overline{U}) < \lambda$.

Proof. The normality of $X^*$ (cf. Kelley [10]) implies that there exist disjoint open sets $W_1$ and $W_2$ such that $x_0 \in W_1$ and $\overline{W} \subset W_2$. $\mu^*$ is a regular measure (cf. Halmos [5]), hence $W_2$ contains an open set $W_3$ with zero-boundary, $\overline{W} \subset W_3 \subset W_2$, satisfying $1 - \mu^*(W_3) > \lambda$. Again by the regularity of $\mu$, for any point $x \in X^*$ there exists an open neighbourhood $F_x$ such that $\mu^*(F_x) = \mu^*(\overline{F_x}) < \lambda - \kappa$. By Heine-Borel's lemma a finite number of these neighbourhoods already covers $X^*$, say, $\bigcup_{k=1}^{n} F_k = X^*$. Let $x_0 \in F_1$. The intersections $F_k \setminus \overline{W_3} = F_k \cap (\overline{W_3})'$ ($k=1, \ldots, n$) cover $X^* \setminus \overline{W_3}$ and all sets $F_k \setminus \overline{W_3}$ are open and have zero-boundaries. Since $\mu^*(F_k \setminus \overline{W_3}) < \lambda - \kappa$ for all $k$ it is possible to choose the number
such that the open set \( U = \bigcup_{k=1}^{\infty} (\mathbb{R} \setminus \overline{W}_k) \) has the desired properties.

Remark. In Chapter II we shall prove a lemma (lemma 2.7.1), which is both a sharpening and a generalisation of lemma 1.1.4. Since the proof of this lemma is fairly more complicated and in this Section only the weaker results of lemma 1.1.4 are needed, we do not give the stronger results here.

Lemma 1.1.5. If \( \{x_i\}_{i=1}^{\infty} \) is a sequence of distinct points of \( X \) converging to the point \( \infty \) in \( X^* \), and if \( \{\alpha_i\}_{i=1}^{\infty} \) and \( \{\beta_i\}_{i=1}^{\infty} \) are sequences of real numbers, \( 0 < \alpha_i < \beta_i \) (\( i = 1, 2, 3, \ldots \)), \( \Sigma_{i=1}^{\infty} \beta_i < \infty \), then there exist open neighbourhoods \( \overline{U}_k \) of \( x_i \) (\( i = 1, 2, 3, \ldots \)) with compact closures and zero-boundaries in \( X \) such that \( \overline{U}_1 \cap \overline{U}_j = \emptyset \) (\( i \neq j \)) and \( \alpha_i < \mu(\overline{U}_k) < \beta_i \) (\( i = 1, 2, 3, \ldots \)).

Proof. The construction may be performed by induction.

(1) \( i = 1 \). Let \( \overline{V}_i \) be an open neighbourhood of \( \infty \) in \( X^* \) with zero-boundary and \( \mu(\overline{V}_i) < \frac{1}{2}(1 - \beta_i) \), such that \( x_1 \not\in \overline{V}_i \). Since \( \overline{V}_i \) covers \( \{x_i\}_{i=2}^{\infty} \) except for a finite number of points, say \( x_1, \ldots, x_k \), we may construct open neighbourhoods \( \overline{V}_1, \ldots, \overline{V}_k \) of \( x_1, \ldots, x_k \) respectively, with zero-boundaries, such that \( x_1 \not\in \bigcup_{i=1}^{k} \overline{V}_i \) and \( \mu(\overline{V}_1) + \cdots + \mu(\overline{V}_k) < \frac{1}{2}(1 - \beta_i) \). If we choose \( \overline{V}(1) = \overline{V}_1 \cup \overline{V}_2 \cup \cdots \cup \overline{V}_k \), then \( \overline{V}(1) \), \( x_1 \), \( \alpha_i \) and \( \beta_i \) may be substituted for \( \overline{V} \), \( x_0 \), \( \lambda \) and \( \lambda \) in order to make lemma 1.1.4 applicable, so that there exists an open neighbourhood \( \overline{U}_1 \) of \( x_1 \) with the property that \( \overline{U}_1 \) is compact, \( \alpha_i < \mu(\overline{U}_1) = \mu(\overline{U}_1) < \beta_i \) and \( \overline{U}_1 \cap \overline{V}(1) = \emptyset \). As a consequence, we have \( x_k \not\in \overline{U}_1 \) (\( k = 2, 3, 4, \ldots \)).

(2) The inductive step. Let \( \overline{U}_1, \ldots, \overline{U}_n \) be open neighbourhoods of \( x_1, \ldots, x_n \) respectively, such that \( \overline{U}_1 \cap \overline{U}_j = \emptyset \) (\( i \neq j \)), \( \overline{U}_1 \) is .
compact, \( \alpha_i \leq \mu(U_i) < \beta_i \) \((i=1, \ldots, r)\) and let \( x_k \not\in U_i \) \((i \neq k)\). Since \( \sum_{i=1}^{\infty} \beta_i < 1 \) and \( \beta_{r+1} > 0 \) there exists an open neighbourhood \( V_{r+1} \) of \( \infty \) in \( X^* \) with zero boundary and \( \mu(V_{r+1}) < \frac{1}{2}(1 - \sum_{i=1}^{r+1} \beta_i) \), \( x_k \not\in V_{r+1} \). Because \( x_k \to \infty \) as \( i \to \infty \), this neighbourhood contains all \( x_i \) \((i \neq r+1)\) except for a finite number, say \( x_i^{(r+1)}, \ldots, x_k^{(r+1)} \), which in their turn have open neighbourhoods \( V_{i}^{(r+1)}, \ldots, V_{k}^{(r+1)} \), respectively, with zero boundaries, such that \( x_k^{(r+1)} \not\in \bigcup_{i=1}^{k} V_{i}^{(r+1)} \) and \( \mu(\bigcup_{i=1}^{k} V_{i}^{(r+1)}) < \frac{1}{2}(1 - \sum_{i=1}^{r+1} \beta_i) \).

Now we define \( r^{(r+1)} = U_1 \cup \ldots \cup U_k \cup \bigcup_{i=1}^{r+1} V_{i}^{(r+1)} \cup \ldots \cup \bigcup_{i=1}^{k} V_{i}^{(r+1)} \) and obtain

\[
\mu(r^{(r+1)}) = \mu(V_{r+1}) < \sum_{i=1}^{\infty} \beta_i + 2 \cdot \frac{1}{2}(1 - \sum_{i=1}^{r+1} \beta_i)^r.
\]

Again lemma 1.1.4 applies when we substitute \( V = V_{r+1} \), \( x_0 = x_{r+1} \), \( x = x_{r+1} \), \( \lambda = \beta_{r+1} \), so that there exists an open neighbourhood \( U_{r+1} \) of \( x_{r+1} \) with compact closure, \( \alpha_{r+1} \leq \mu(U_{r+1}) < \beta_{r+1} \), \( U_{r+1} \cap V_{r+1} = \emptyset \). As a consequence, we have \( x_k \not\in \overline{U}_{r+1} \) \((k \neq r+1)\), so that lemma 1.1.5 is proved.

1.2. CONSTRUCTION OF A FUNCTION WITH PRESCRIBED MEAN AND INTEGRAL VALUES

Helmberg [7] proved that for any continuous non-negative function \( f \) on \( X \) and any \( d\mu \)-uniformly distributed sequence \( \xi = \{x_k\}_{k=1}^{\infty} \), we have

\[
(2.1) \quad \int_X f(x) d\mu \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k).
\]

He also gave an example of an unbounded non-negative function \( f \) such that \( \int_X f(x) d\mu = \infty \) and \( \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \infty \).

Our intention is to prove the existence of a non-negative con-
continuous function \( f \) with prescribed values of \( \int f \, d\mu \), 
\[ \lim \inf \frac{1}{n} \sum_{k=1}^{n} f(x_k) \text{ and } \lim \sup \frac{1}{n} \sum_{k=1}^{n} f(x_k). \] The proof shows that we must impose an additional condition on \( \xi \), viz. that all points \( x_k \in \xi \) are distinct, in other words, that \( \xi \) has no repetitions. A discussion of the consequences which may arise when we delete this condition will follow at the end of this Chapter.

**Definition 1.2.1.** A continuous function \( g \) is called a Urysohn-function if \( 0 \leq g(x) \leq 1 \) for all \( x \in X \), \( g(x) = 1 \) for some point \( x = x_0 \), the support of \( g \) is compact, and if the sets \( A_g = \{ x \mid g(x) > 0 \} \) and \( B_g = \{ x \mid g(x) = 1 \} \) both have zero-boundaries.

Obviously, if \( g \) is a Urysohn-function and if \( \sigma \) is a positive number, then \( h = g^\sigma \) is also a Urysohn-function, and \( A_g = A_h, B_g = B_h \).

**Lemma 1.2.1.** If \( g \) is a Urysohn-function then \( I_{\sigma} = \int_X g^\sigma \, d\mu \) is a non-increasing continuous function of \( \sigma \), \( \lim_{\sigma \to 0} I_{\sigma} = \mu(A_g) \) and \( \lim_{\sigma \to \infty} I_{\sigma} = \mu(B_g) \).

**Proof.** This lemma follows from an application of Lebesgue's theorem on monotone convergence (cf. Zaanen [15]).

**Lemma 1.2.2.** Let \( U \) be an open neighbourhood of \( x_0 \) with compact closure \( \overline{U} \) and zero-boundary, and let \( 0 < \delta < \mu(U) \). Then there exists a Urysohn-function \( h \) with support \( S_h = \overline{U} \) such that \( A_h = \{ x_0 \} \) and \( \int_X h \, d\mu = \delta \).

**Proof.** From a result mentioned by Halmos (cf. [5], p. 217) it follows that there exists a non-negative continuous function \( f_1 \) which vanishes at the point \( x \) if and only if \( x = x_0 \), and a non-negative continuous function \( f_2 \), vanishing at the point \( x \) if and only if \( x \notin U \). We define \( f \) by

\[
 f(x) = \begin{cases} 
 \exp(-f_1 f_2^{-1}(x)) & \text{if } x \in U \\
 0 & \text{if } x \notin U
\end{cases}
\]
so that $f$ is a Urysohn-function with support $\overline{U}$ and $B_r = \{x_0\}$. According to lemma 1.2.1 the integral $I_\alpha = \int_X f^\alpha \, d\mu$ is a continuous function of $\alpha$ and $\lim_{\alpha \to \infty} I_\alpha = \mu(U) = \mu(\overline{U})$, $\lim_{\alpha \to 0} I_\alpha = 0$ so that there exists a positive number $\beta$ satisfying $I_\beta = \delta$. The corresponding Urysohn-function $h = f^\beta$ has the desired properties.

Notations. If $\xi = \{x_k\}_{k=1}^\infty$ is $d\mu$-uniformly distributed and $f$ is a continuous non-negative function we write by way of abbreviation $\overline{f}[N] = \frac{1}{N} \sum_{k=1}^N f(x_k)$ and $\mu(f) = \int_X f(x) \, d\mu$. If $f$ is bounded, then we have (cf. Holmberg [?])

\[(2.2) \quad \lim_{N \to \infty} \overline{f}[N] = \mu(f) .\]

In general we do not know anything about the behaviour of $\overline{f}[N]$ for "small" $N$, i.e. those $N$ for which $\overline{f}[N]$ is not yet "close" to $\mu(f)$ in some specified sense. This behaviour depends on the one hand on the topological properties of $X$, which have a connection with the values of $f$ since $f$ is continuous; on the other hand, it depends on the measure-theoretical and topological structure of $X$, which relate to the properties of $\xi$. The idea of the following investigations is to construct functions for which we have this behaviour under control. Under the assumption that $\xi$ has no repetitions we shall construct a continuous function $f$ with prescribed integral for which the support is contained in a given open set and for which the value in one single given point $x_k \in \xi$ already determines the behaviour of $\overline{f}[N]$ up to a given index and for "small" $N$. In fact, $\overline{f}[N] = 0$ for all $N < K$ and $\overline{f}[N] = C$ (C being constant) for all $N \geq K$ up to a given index and until $\overline{f}[N]$ is "close" to $\mu(f)$. More precisely, we shall prove the following lemma:

**Lemma 1.2.2.** Suppose $\xi = \{x_k\}_{k=1}^\infty$ is a $d\mu$-uniformly distributed sequence without repetitions, $L > K > 0$ are given indices, $U$ is
an open neighbourhood of $x_k$ with compact closure, $\mu(U) = \mu(\overline{U}) > 0$, and $\varepsilon > 0$ and $\sigma > \rho > 0$ are given, such that $K_0 \mu(U) > \rho$.

Then there exists a non-negative continuous function $f$ with compact support $S_f \subset \overline{U}$, $\mu(\partial S_f) = 0$, and an index $M \geq L$ such that

- (a) $\overline{F}[N] = 0 \quad (1 \leq N < K)$
- (b) $\overline{F}[N] = \frac{K_0}{N} \quad (K \leq N < M)$, in particular $\overline{F}[K] = \sigma$
- (c) $\frac{K_0}{N} < \overline{F}[N] < \rho + \varepsilon \quad (N \geq M)$
- (d) $\mu(f) = \rho$.

Proof. According to Lemma 1.2.2 there exists a Urysohn-function $h$ such that $S_h = \overline{U}$, $R_h = \{x_k\}$ and $\mu(h)$ has a given value smaller than $\mu(U)$, for example $\frac{K_0}{N} < \mu(h) < \min\left(\frac{\rho + \varepsilon}{K_0}, \mu(U)\right)$. Since $K_0 \overline{F}[N]$ tends to $K_0 \mu(h)$ (cf. (2.2)) as $N \to \infty$ and $K_0 \mu(h) < \rho + \varepsilon$, there exists an index $M_0$ such that $K_0 \overline{F}[N] < \rho + \varepsilon$ for all $N \geq M_0$. Take $M = \max(M_0, N_0)$. Now let $V_q$ be open neighbourhoods of $x_q$ ($q = 1, \ldots, M-1 ; q \notin \mathcal{K}$) with compact closures and zero-boundaries, and let $V$ be an open neighbourhood of $x_k$ with compact closure and zero-boundary such that $\mu(U) < \frac{K_0}{N}$ and $x_q \notin \overline{V}$ ($q = 1, \ldots, M-1 ; q \notin \mathcal{K}$).

This is possible since $\xi$ has no repetitions. The set $M^{-1}
\hat{\mathbb{V}} = [\bigcup U \cup (\bigcup_{q=1}^{M-1} V_q)] \setminus \overline{V}$ is open. $\hat{\mathbb{V}}$ is compact and $\mu(\partial\hat{\mathbb{V}}) = 0$. Moreover, we observe that $\int_{X \setminus \hat{\mathbb{V}}} K_0 h \mu(x) \, dx = \int_{X \setminus \overline{V}} K_0 h \mu(x) \, dx < \sigma$. Lemma 1.2.2 enables us to construct Urysohn-function $h_q$ with respect to $\hat{\mathbb{V}}$ as $S_h$ and $\{x_k\}$ as $R_h$ ($q = 1, \ldots, M-1 ; q \notin \mathcal{K}$), their integrals being immaterial. If we define $f = \int_{X \setminus \hat{\mathbb{V}}} \Pi_{q=1}^{M-1} (1-h_q) \alpha_{q \notin \mathcal{K}}$ ($\alpha > 0$), then $f(x_q) = 0$ ($q = 1, \ldots, M-1 ; q \notin \mathcal{K}$) and $f(x_k) = K_0$ so that $\overline{F}[N]$ has the properties (a), (b) and (c), no matter what value of $\alpha$ is given. Since $\lim_{\alpha \to 0} \int_X f \mu = \int_X K_0 h \mu = K_0 \mu(h) > \rho$ and
\[
\lim_{\alpha \to \infty} \int_X f \, d\nu = \int_X f \, d\nu \setminus W \quad \text{Kohde} \mu < \rho \text{ there exists an exponent } \alpha \text{ for which}
\int_X f \, d\nu = \rho \text{ (cf. Lebesgue's theorem on monotone convergence (Zaanen \cite{15})))}. \quad \text{Q.E.D.}
\]

**Theorem 1.2.1.** If \( \xi = \{x_k\}_{k=1}^\infty \) is \( d\mu \)-uniformly distributed and has no repetitions, and if \( \alpha, \beta, \gamma \) are generalised real numbers, \( 0 < \alpha < \beta < \gamma < \infty \), then there exists an unbounded, non-negative, continuous function \( f \) such that \( \mu(f) = \alpha \), \( \liminf_{N \to \infty} \mathbb{I}[N] = \beta \) and \( \limsup_{N \to \infty} \mathbb{I}[N] = \gamma \).

**Proof:**

1. The case \( 0 < \alpha = \beta = \gamma = \infty \) is trivial: we take a non-negative continuous function \( f \) such that \( \mu(f) = \infty \).

2. We shall now prove the four cases \( 0 < \alpha < \beta < \gamma < \infty \). By corollary 1.1.2.1 there exists a sub-sequence \( \{x_{k_j}\}_{j=1}^\infty \) of \( \xi \) tending to \( \infty \) as \( j \to \infty \) and such that \( \lim_{j \to \infty} \frac{k_{j+1}}{k_j} = 1 \). According to lemma 1.1.3 this sequence contains a sub-sequence \( \{x_{k_{j_i}}\}_{i=1}^\infty \) for which still \( \lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1 \) and moreover

\[
\Sigma_{i=1}^\infty \sum_{i=1}^j \frac{1}{i} \leq \frac{1}{2} \tag{2.3}
\]

(in lemma 1.1.3 we choose e.g. \( m_j = (3+1)^j \) \( j = 1, 2, 3, \ldots \) for some suitable exponent \( p > 0 \)).

In view of (2.3), using lemma 1.1.5, we may construct disjoint neighbourhoods \( W_{k_i} \) of \( x_{k_i} \) with compact closures such that

\[
L_i \leq \mu(W_{k_i}) = \mu(W_{k_i}) < E_i^{i-1} + 2^{i-1} \quad (i = 1, 2, 3, \ldots)
\]

For the construction of the required function \( f \) we shall make use of a sub-sequence \( \{x_i\}_{i=1}^\infty = \{E_i\}_{i=1}^\infty \) of \( \{k_i\}_{i=1}^\infty \) by means of a
repeated application of lemma 1.2.3 in the following inductive process:

(a) We start by defining

\[ n_1 = K_1 = \xi_1, \quad L_1 = \xi_1 + 1, \quad U_1 = W_1, \quad \rho_1 = 2^1 \alpha, \quad \sigma_1 = \gamma, \quad \epsilon_1 = \frac{1}{2}. \]

Notice that \( K_1 \sigma_1 \mu (U_1) > \xi_1 \gamma \epsilon_1^{-1} \approx \sigma > \rho_1 \), therefore lemma 1.2.3 is indeed applicable. Let \( f_1 \) be the corresponding function, the existence of which is proved by this lemma.

(b) Now let \( \{n_t, \rho_t, \sigma_t, f_t\}_{t=1}^{r-1} \) be determined by lemma 1.2.3 so that \( \rho_t = 2^{-t} \alpha, \quad \sigma_t = 2^{-t}, \quad S_{n_t} \subset n_t \) \((t = 1, \ldots, r-1)\). Let \( \nu_s = \sum_{i=1}^{s} f_t \) if \( 1 \leq s < r - 1 \). Then \( \nu_{s+1} [N] = \sum_{i=1}^{s+1} f_t [N] \) tends to \( \mu (\xi_{r-1}) = \alpha (1 - 2^{1-r}) \) as \( N \to \infty \). Obviously, \( \alpha (1 - 2^{1-r}) < \beta (1 - 2^{-r}) < \beta \), which makes it possible to define the data for application of lemma 1.2.3 for the construction of \( f_r \), viz.

\[
\begin{align*}
\begin{cases}
  n_r = K_r = \min \{ \lambda | \lambda > n_{r-1}, V_r > \lambda^{-1} \zeta [N] < \beta (1 - 2^{-r}) \} \\
  L_r = \max (K_r, M_{r-1}) + 1 \\
  U_r = W_r \\
  \rho_r = 2^{-r} \alpha \\
  \sigma_r = \gamma - \zeta_{r-1} [n_r] \\
  \epsilon_r = 2^{-r}
\end{cases}
\end{align*}
\]

Notice that \( K_r \sigma_r \mu (U_r) > n_r \sigma_r \epsilon_r^{-1} > \gamma - \beta (1 - 2^{-r}) \approx \beta 2^{-r} \approx \alpha 2^{-r} = \rho_r \)

(cf. the hypothesis of lemma 1.2.3).

Finally, \( f \) is defined as

\[ f = \sum_{r=1}^{\infty} f_r. \]
Some of the properties of $f$ are immediately clear:

(i) $f$ is continuous since all $f_x$ are continuous and have disjoint supports;

(ii) $\mu(f) = \sum_{x=1}^{\infty} \mu(f_x) = \alpha$ by Lebesgue's theorem since all $f_x$ are positive;

(iii) $\limsup_{N \to \infty} \overline{F}[N] \geq \gamma$ since for all $r$ we obtain

$$\overline{F}[n_r] = \sum_{t=1}^{r} \overline{F}(n_r) = \overline{F}_{r-1}[n_r] + \overline{F}_r[n_r] =$$

$$= \sum_{i=1}^{r-1} [n_r] + \sigma_r = \gamma$$

(iv) $\liminf_{N \to \infty} \overline{F}[N] = \beta$, because for all $r$ we have

$$\overline{F}[n_r - 1] = \overline{F}_{r-1}[n_r - 1] < \beta(1 - 2^{-r}) < \beta.$$ 

As $\liminf_{N \to \infty} \overline{F}[N] = \mu(f)$ (cf. Helmsberg [7]) we may conclude that in the case $\alpha = \beta$ we already have proved

$$\liminf_{N \to \infty} \overline{F}[N] = \beta.$$ 

To prove the required equality signs in the remaining cases let us assume that $\eta$ is an arbitrary positive real number.

Consider the following arguments (a) and (b)

(a) If $r_1 \geq r_0(\eta)$ then for all $s > r_1$ we have

$$\sum_{r=r_1+1}^{s} (\rho_r + \epsilon_r) < \sum_{r=r_1+1}^{\infty} (\rho_r + \epsilon_r) = (\alpha + 1)2^{-r_1} < \eta.$$ 

(b) If $r_1$ is fixed and $N > N_0(r_1, \eta)$ then for all $r < r_1$ we obtain

$$|\overline{F}_r[N] - 2^{-r} \alpha| < \eta 2^{-r} \alpha | < \eta.$$ 

We take $r_1 = r_0(\eta)$ in (a). Then the integer $N_0(r_1, \eta)$ is a constant which only depends on $\eta$ say, $N_0(r_1, \eta) = N_0(\eta)$. Suppose that $s$ satisfies the following conditions...
\[
\begin{align*}
\text{a. } & \quad n_s > R_1(\eta) \\
\text{b. } & \quad \beta \cdot 2^{-2^s} < \eta \\
\text{c. } & \quad 1 - \frac{\delta_{k+1} - 2}{\lambda_k} < \frac{\eta}{\gamma + \eta} \quad \text{for all } \lambda_k > n_s \\
\text{d. } & \quad s \geq r_1.
\end{align*}
\]

(2.8)

The condition (2.8) may be satisfied since \( \lim_{k \to \infty} \frac{\delta_{k+1}}{\lambda_k} = 1 \). In the interval \( n_s \leq N < n_{s+1} \), the functions \( F[N] \) and \( \mathcal{E}_s[N] \) are identical. Restricting ourselves to the interval \( n_s \leq N < n_{s+1} \), we may therefore consider \( \mathcal{E}_s[N] \) instead of \( F[N] \). This is useful since the behavior of \( \mathcal{E}_s[N] \) for \( N = n_s \) determines the value of \( n_{s+1} \) (cf. (2.4)). We obtain

\[
\mathcal{E}_s[N] = \sum_{r=1}^{n_1} F_r[N] + \sum_{r=r_1+1}^{n_2} F_r[N] + \sum_{r=r_2+1}^{r_3} F_r[N] 
\]

where \( r_2 \) is defined as the maximum of \( r_1 \) and the greatest index \( r \) for which \( M_r = n_s \) (it may occur that some of these sums are empty). We consider the sums in (2.9) separately, calling them \( S_1, S_2, S_3 \) respectively:

(1) \( S_1 = \sum_{r=1}^{n_1} F_r[N] \) differs from \( \alpha(1 - 2^{-n_1}) \) by an amount less than \( \eta \) (cf. (2.7), (2.8)).

(2) \( 0 \leq S_2 = \sum_{r=r_1+1}^{n_2} F_r[N] < \sum_{r=r_1+1}^{n_2} (\rho_1 + \varepsilon_r) \) since for these values of \( r \) we have \( M_r = N \) (cf. lemma 1.2.3 (c)).

(3) \( S_3 = \sum_{r=r_2+1}^{r_3} F_r[N] \). For these values of \( r \) we have \( n_s < M_r \) so that by lemma 1.2.3 (b), (c) we may write \( F_r[N] = \frac{K_r}{N} + \delta_{N,N_r} \), where \( 0 \leq \delta_{N,N_r} < \rho_1 + \varepsilon_r \), and therefore \( S_2 = \sum_{r=r_2+1}^{r_3} \frac{K_r}{N} + S_1' \), where \( 0 \leq S_1' = \sum_{r=r_2+1}^{r_3} (\rho_1 + \varepsilon_r) \).

Combining these results we obtain estimates for \( \mathcal{E}_s[N] \) in the in-
torval \( N \geq n_b \):

\[
\begin{align*}
\alpha (1 - 2^{-r_0}) - \eta + \sum_{r=r_2+1}^{\rho \sigma} \frac{K_{r, s}}{N} < \bar{g}_s[N] < \\
< \alpha (1 - 2^{-r_0}) + \eta + \sum_{r=r_2+1}^{\rho \sigma} \frac{K_{r, s}}{N} + \sum_{r=r_1+1}^{\rho \sigma} (\bar{e}_r + \bar{e}_s) < \\
< \alpha (1 - 2^{-r_0}) + \sum_{r=r_2+1}^{\rho \sigma} \frac{K_{r, s}}{N} + 2\eta
\end{align*}
\]

(2.6)

In other words we have

\[
\begin{align*}
\alpha (1 - 2^{-r_0}) + \frac{1}{N} \sum_{r=r_2+1}^{\rho \sigma} K_{r, s} - \eta < \bar{g}_s[N] < \\
< \alpha (1 - 2^{-r_0}) + \frac{1}{N} \sum_{r=r_2+1}^{\rho \sigma} K_{r, s} + 2\eta \quad (N \geq n_b),
\end{align*}
\]

which may symbolically be written in the form

\[
\begin{align*}
(2.10^a) \quad & L_{N, s} < \bar{g}_s[N] < R_{N, s} \quad (N \geq n_b).
\end{align*}
\]

It should be remarked that the sum \( \sum_{r=r_2+1}^{\rho \sigma} K_{r, s} \) depends only on \( s \) and \( \eta \) (cf. the definition of \( r_0 \) in (2.9)). Therefore we may conclude

\[
(2.10^b) \quad R_{N, s} \text{ is a non-increasing function of } N \quad (N \geq n_b).
\]

Moreover, \( R_{N, s} - L_{N, s} \) has the constant value \( 5\eta \), so that

\[
(2.10^c) \quad \bar{g}_s[N] > L_{N, s} = 3\eta \quad (N \geq n_b).
\]

An explicit estimate for \( \sum_{r=r_2+1}^{\rho \sigma} K_{r, s} \) follows from (2.10) if we substitute in it \( N = n_b \) and consequently \( \bar{g}_s[n_b] = \gamma \). Then we obtain

\[
(2.11) \quad \gamma - \alpha (1 - 2^{-r_1}) - 2\eta < \frac{1}{n_b} \sum_{r=r_2+1}^{\rho \sigma} K_{r, s} < \gamma - \alpha (1 - 2^{-r_0}) + \eta.
\]

Combination of the right hand inequalities of (2.10) and (2.11) yields for \( N \geq n_b \)

\[
\bar{g}_s[N] < \alpha (1 - 2^{-r_1}) + \frac{n_b}{N} (\gamma - \alpha (1 - 2^{-r_0}) + \eta) + 2\eta.
\]

Bearing in mind that \( \frac{n_b}{N} \leq 1 \) we conclude

\[
\bar{g}_s[N] < \gamma + 3\eta \quad (N \geq n_b),
\]
\[ \overline{F}[N] < \gamma + 3\eta \quad (n_s \leq N \leq n_{s+1}) \]

This holds for all sufficiently large values of \( s \) (cf. (2.10')). Hence \( \limsup_{N \to \infty} \overline{F}[N] \leq \gamma \) so that in view of (iii) \( \limsup_{N \to \infty} \overline{F}[N] = \gamma \).

For the proof that \( \liminf_{N \to \infty} \overline{F}[N] \geq \beta \) we distinguish two cases:

(a) \( R_{N_0,s} \) attains no value \( \leq \beta - \eta \) \( (n_s \leq N \leq n_{s+1}) \). Then, by (2.10'),

\[ \overline{F}[N] = \overline{\alpha}_s[N] > \beta - 4\eta \quad (n_s \leq N \leq n_{s+1}) \]

(b) \( R_{N_0,s} < \beta - \eta \) for some \( N \) \( (n_s \leq N \leq n_{s+1}) \). Let \( N_0 \) be the greatest value of \( N \) in this interval such that \( R_{N_0,s} \geq \beta - \eta \). By (2.10)^{(b, 1)} it follows that for all \( N \) such that \( n_s \leq N \leq N_0 \)

\[ (2.12) \quad \overline{\alpha}_s[N] = \overline{F}[N] > R_{N_0,s} - 3\eta \geq \beta - 4\eta \]

From (2.9^b) we deduce that for all \( N > N_0 \)

\[ \overline{\alpha}_s[N] < R_{N_0,s} < \beta - \eta < \beta(1 - \varepsilon^{-1}) \]

so that by (2.4) we obtain

\[ n_{s+1} < \min\{k_1, k_2, N_0 + 2, 2\} \]

Suppose \( k = \min\{k_1, k_2, N_0 + 2, 2\} \). Then we have the inequality

\[ n_s < k_{s+1} < N_0 + 2 < k \]

and

\[ R_{k,s} = R_{N_0,s} - R_{k,s} \geq \beta - \eta - \left( \frac{1}{N_0} - \frac{1}{k} \right) \Sigma_{r=1}^{k} R_{r,s} \]

We may write for the last term

\[ \left( \frac{1}{N_0} - \frac{1}{k} \right) \Sigma_{r=1}^{k} R_{r,s} \leq \frac{1}{N_0} \left( 1 - \frac{N_0}{k} \right) \Sigma_{r=1}^{k} R_{r,s} \]

\[ = \frac{1}{n_s} \left( 1 - \frac{k-1}{k} \right) \Sigma_{r=1}^{k} R_{r,s} \leq \quad (\text{cf. (2.9^c), (2.11)}) \]

\[ \leq \frac{n}{\gamma + \eta} (\gamma + \eta) = \eta \]
Therefore $R_{g, \theta} > \beta - 2\eta$ and consequently by the monotonicity property (2.10b) we obtain

$$R_{g, \theta} > \beta - 2\eta,$$

which implies (cf. 2.10b) \(0 < N < n_{g+1}\).

$$N_0 < N < n_{g+1}.$$ \(2.13\)

Combination of (2.12) and (2.13) yields

$$R_{g, \theta} > \beta - 5\eta$$

so that both in case (a) and (b) we have, under the hypotheses \(N_0 < N < n_{g+1}\), the inequality \(R[N] > \beta - 5\eta\), which proves \(\inf \lim_{N \to \infty} R[N] = \beta\). Therefore (cf. (iv)) \(\inf \lim_{N \to \infty} R[N] = \beta\),

which completes the proof of theorem 1.2.1 in the cases

$$0 < a < \beta < \gamma < \infty.$$ \(N = \infty\)

III. The remaining three cases \((0 < a < \infty, \gamma = \infty, a = \beta = \gamma)\) may be proved by slight modifications of the foregoing proof. We shall restrict ourselves to the most important definitions and consequences:

A. If \(\beta < \infty\) we define a sequence of functions \(\{f_t\}_{t=1}^{\infty}\) as in II by an inductive method, using lemma 1.2.3 as follows:

(a) We start by defining

$$n_t = K_t, L_t = L_t + 1, \beta_t = \beta_t, \alpha_t = \frac{1}{2}\beta_t, \sigma_t = \beta t + 1, \epsilon_t = \frac{1}{2}\beta$$

and construct the corresponding function \(f_t\) by lemma 1.2.3,

(b) If \(\{n_t, \beta_t, \epsilon_t, f_t\}_{t=1}^{R-1}\) are determined by lemma 1.2.3 so that \(n_t = 2^{-t}\alpha, \beta_t = 2^{-t}, \beta_t = 2^{-t}, \epsilon_t = \frac{1}{2}\beta_t\) \((t = 1, \ldots, R-1)\) and \(g_{\beta}\) is defined by \(g_{\beta} = \sum_{t=1}^{R-1} f_t\) if \(1 < s < R-1\), then we define

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\[
\begin{align*}
\eta_r &= K_r = \min \{ \lambda_1 \mid \lambda_1 > n_{r-1}, \forall N > \lambda_1 \bar{E}_{r-1}[N] < \beta(1 - 2^{-r}) \} \\
L_r &= \max \{ \lambda_r, n_{r-1} \} + 1 \\
U_r &= \eta_r \\
\rho_r &= 2^{-r} \alpha \\
\sigma_r &= \beta + r = \bar{E}_{r-1}[n_r] \\
\varepsilon_r &= 2^{-r}
\end{align*}
\]

Now we put \( f = \sum_{r=1}^{\infty} f_r \) and immediately conclude that \( \mu(f) = \alpha \) and \( \limsup_{r \to \infty} \bar{F}[n] = \lim_{r \to \infty} \bar{F}[n_r] = \lim (\beta + r) = \infty \). The proof that \( \liminf_{N \to \infty} \bar{F}[N] = \beta \) may be given in a manner quite analogous to the foregoing proof, as follows: obviously, \( \liminf_{N \to \infty} \bar{F}[N] < \beta \).

Now let us assume \( \eta > 0 \) and consider the same arguments (a), (b), (2.6) and (2.7), taking \( r_1 = r_0(\eta) \), \( E_0(r_1, \eta) = N_0(\eta) \). Suppose that \( s \) has the following properties (cf. (2.8))

\[
\begin{align*}
&\text{a.} \quad n_s > N_0(\eta) \\
&\text{b.} \quad \beta \cdot 2^{-s-1} < \eta \\
&\text{c.} \quad 2 (1 - \frac{\lambda_{s-1} - 2}{k}) (\beta - \alpha(1 - 2^{-r_1}) - 3\eta) < \eta \\
&\text{d.} \quad s > r_1
\end{align*}
\]

We may apply exactly the same argumentation as in the lines as far as \( (2.10^a) \). In particular, \( (2.10^b), (2.10^c), (2.10^d) \) and \( (2.10^b) \) remain valid (the reasoning then following, which leads to the statement \( \limsup_{N \to \infty} \bar{F}[N] = \gamma \) is irrelevant to the present case, since we have already shown \( \limsup_{N \to \infty} \bar{F}[N] = \infty \). The discussion...
of the cases (a) and (b) remains also valid without change up to the point where in case (b) we find
\[
R_{k,s} = R_{0,s} - (R_{0,s} - R_{k,s}) > \beta - \eta - \left( \frac{1}{N_0} \right) \sum_{r=2}^{\infty} K_r \sigma_r > \\
\geq \beta - \eta - \frac{N_0 + 1}{N_0} \frac{1}{N_0} \sum_{r=2}^{\infty} K_r \sigma_r > \\
\geq \beta - \eta - \frac{N_0 + 1}{N_0} \frac{1}{N_0} \sum_{r=2}^{\infty} K_r \sigma_r.
\]

Since \( R_{0,s+1} > \beta - \eta \), the last factor in the right hand side of this inequality may be majorized by (2.10)
\[
\frac{1}{N_0 + 1} \sum_{r=2}^{\infty} K_r \sigma_r < \beta - \eta - (1 - 2^{-r_1}) - 2\eta = \beta - \eta - (1 - 2^{-r_1}) - 2\eta.
\]

Hence
\[
R_{k,s} > \beta - \eta - 2(1 - \frac{l_{k-1} - 2}{l_k})(\beta - \eta - (1 - 2^{-r_1}) - 2\eta) > \\
\geq \beta - 2\eta > 0 \quad \text{(cf. (2.15))}.
\]

Now the monotonicity of \( R_{k,s} \) implies
\[
R_{s+1,s} > R_{k,s} > \beta - 2\eta
\]
so that by (2.10c)
\[
\bar{\gamma}_a[N] = \bar{F}[N] > \beta - 5\eta \quad \text{for} \quad n_s \leq N < n_{s+1}
\]
and
\[
\lim_{N \to \infty} \inf \bar{F}[N] = \beta.
\]
Since the converse inequality has already been shown we obtain \( \lim_{N \to \infty} \inf \bar{F}[N] = \beta \).

B. The case \( \alpha < \infty, \beta = \gamma = \infty \) may be treated as follows:
We take \( n_r = K_r - l_r, l_r = l_r + 1, U_r = W_r, \rho_r = 2^{-r} a, \beta_r = 2^{-r}, \)
and \( \sigma_r > \rho_r \) and so large that for all \( r \)
\[
\sum_{j=1}^{r} t_j \bar{F}[N] > r \quad \text{for} \quad n_r \leq N < n_{r+1}.
\]
Then we define $f = \sum_{k=1}^{\infty} f_k$ and immediately verify that $\mu(f) = \alpha$,
\[ \lim_{N \to \infty} \mu(F[N]) = \infty. \quad \text{Q.E.D.} \]

1.3. **Uniformly Distributed Sequences with Repetitions**

In Section 1.2 we proved theorem 1.2.1 under the assumption that the sequence $F$ contained no repetitions. It remains an open question whether a similar statement holds for uniformly distributed sequences with repetitions. This Section will show that the case of sequences with repetitions cannot simply be reduced to the case of sequences without repetitions.

Let $\eta = \{y_k\}_{k=1}^{\infty}$ be a $\mu$-uniformly distributed sequence. Consider the sub-sequence $\eta' = \{z_j\}_{j=1}^{\infty} = \{y_{k_j}\}_{j=1}^{\infty}$ of all points $y_{k_j}$ ($j = 1, 2, 3, \ldots$) which do not coincide with any of the preceding points $y_{k_j}$ ($r < k_j$) of the sequence $\eta$. (If $\eta$ has no repetitions then $\eta' = \eta$; for every $\eta$ the sequence $\eta'$ has no repetitions.) One might expect that $\eta'$ is $\mu$-uniformly distributed too, and then try to apply theorem 1.2.1. This conjecture, however, turns out to be false, as may be shown by the following example.

**Example:** Let $X$ be the unit interval $[0,1]$ and let $\mu$ be the Lebesgue-measure on the Borel-subsets of $X$. The sequence

$$(3.1) \quad \zeta = \{z_j\}_{j=1}^{\infty} = \{1, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}, \ldots \} ,$$

where $z_1 = 1$, $z_{2^{p+L}} = \frac{2^{p+1} - 1}{2^{p+1}}$ ($p = 0, 1, 2, \ldots; L = 1, \ldots, 2^p$), is a sequence without repetitions. It is easily seen that $\zeta$ is not $\mu$-uniformly distributed, since $A(\zeta, [0, \frac{1}{2}], \frac{2^p + 2^{p-1}}{2^p + 2^{p-1}}) = \frac{1}{2}$ for all $p = 1, 2, 3, \ldots$. We shall construct a $\mu$-uniformly distributed sequence $\eta$ such that $\eta' = \zeta$. To this end we define $\eta = \{y_k\}_{k=1}^{\infty}$, where
\[
\begin{align*}
\begin{cases}
  y_1 &= 1 \\
  y_{2^{p+1} + \ell} &\equiv \frac{2^p - 1}{2^{p+1}} \pmod{1} \quad (p=0,1,2,\ldots; \ell=1,\ldots,2^p) \\
  y_{2 \cdot 2^{p+1} + \ell} &\equiv \frac{4}{2^{p+1}} \pmod{1} \quad (p=0,1,2,\ldots; \ell=1,\ldots,2 \cdot 2^p)
\end{cases}
\end{align*}
\]

The structure of \( \eta \) becomes clear, when we arrange its elements into a scheme of rows which has to be read in the order of its rows from top to bottom to generate \( \eta \); the total number \( k \) of elements in \( \eta \) up to and including each row is placed at the end of each row, and the elements of \( \eta' \) are placed in parentheses:

\[
\begin{array}{ccccccccc}
\hline
k & (1) & 1 \\
(\frac{1}{2}) & \frac{1}{2} & 1 & 4 \\
(\frac{1}{4}) & \frac{1}{4} & \frac{3}{4} & 8 \\
(\frac{1}{8}) & \frac{1}{8} & \frac{5}{8} & \frac{7}{8} & 32 \\
(\frac{1}{16}) & \frac{1}{16} & \frac{9}{16} & \frac{11}{16} & \frac{13}{16} & \frac{15}{16} & 64 \\
\end{array}
\]
It should be remarked that among the first $4^P$ (resp. $2 \cdot 4^P$) elements of $\eta$ each of the rational numbers $\frac{n}{2^p}$ ($n = 1, \ldots, 2^p$) (resp. $\frac{m}{2^{p+1}}$ ($m = 1, \ldots, 2^{p+1}$)) occurs exactly $2^p$ times ($p = 0, 1, 2, \ldots$), so that, if $\beta$ is a real number, $0 < \beta < 1$, we obtain

$$\left\{ \begin{array}{l}
A(\eta, \lambda, 0, \beta) = [2^p \beta] 2^p , \\
A(\eta, \lambda, 0, \beta) = [2^{p+1} \beta] 2^p .
\end{array} \right. \tag{3.3}$$

The total number of elements in $[0, \beta]$ occurring in each row between $k = 4^P$ and $k = 2 \cdot 4^P$ is equal to $[2^p \beta + \frac{1}{2}]$, and if $2 \cdot 4^P < k < 4^{p+1}$ this row-score attains the value $[2^{p+1} \beta]$. In view of (3.3) this implies for all $p = 1, 2, 3, \ldots$

$$\left\{ \begin{array}{l}
[2^p \beta] 2^p + (k-1) [2^p \beta + \frac{1}{2}] \leq A(\eta, \lambda, 0, \beta) = \leq [2^p \beta] 2^p + [2^p \beta + \frac{1}{2}] \\
\text{if } 4^P < k \leq 4^P + 2^P + \frac{1}{2} ;
\end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{l}
[2^{p+1} \beta] 2^p + (k-1) [2^{p+1} \beta + \frac{1}{2}] \leq A(\eta, \lambda, 0, \beta) = \leq [2^{p+1} \beta] 2^p + [2^{p+1} \beta + \frac{1}{2}] \\
\text{if } 2 \cdot 4^P < k \leq 2 \cdot 4^P + 2^P + 2^{p+1} \\
\text{or } (k-1) 2^{p+1} < 2 \cdot 4^P + 2^P + 2^{p+1} ;
\end{array} \right. \tag{3.5}$$

From the inequalities (3.4), (3.5) it follows from elementary estimates that

$$\beta - 2^{1-p} \leq \frac{A(\eta, \lambda, 0, \beta) - k}{k} \leq \beta + 2^{1-p} , \tag{3.6}$$

provided that $4^P < k \leq 4^{p+1}$.

Hence

$$\lim_{k \to \infty} \frac{A(\eta, \lambda, 0, \beta) - k}{k} = \beta ,$$

and, since $\beta$ is chosen arbitrarily between 0 and 1, we obtain for each interval $I = [\alpha, \beta]$

$$\lim_{N \to \infty} \frac{\Lambda(\eta, \lambda, \alpha, \beta) - N}{N} = \mu([\alpha, \beta]) , \tag{3.7}$$
which implies that $\eta$ is $d\mu$-uniformly distributed. It is easily seen that $\eta' = \zeta$, so that the assertions in the example are proved.
CHAPTER II

2.1. $(\xi, d\mu)$-SUMMABILITY - INTRODUCTION

Definition 2.1.1. A Borel-measurable function $f$ is called $(\xi, d\mu)$-summable, if $f$ is integrable and
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \mu(f). \]

It is well known that every bounded continuous function (in fact every bounded $\mathbb{R}$-integrable function on $X$, cf. e.g. Helnberg [7]) is $(\xi, d\mu)$-summable.

Our intention is to investigate the $(\xi, d\mu)$-summability for a larger class of Borel-measurable functions. Chapter I shows that there are limitations to the possibility of extending the range of functions, for which $(\xi, d\mu)$-summability can be established, even when we consider continuous functions only.

It turns out that the concept of local $\mathbb{R}$-integrability, which will be defined now, is suitable for our purpose.

Definition 2.1.2. A real-valued Borel-measurable function $f$ is called locally $\mathbb{R}$-integrable, if for every compact set $V$ with zero-boundary and for every $\varepsilon > 0$ there exist two continuous functions $f_1$ and $f_2$ with compact supports, such that $f_1 x_v = f x_v = f_2 x_v$ and
\[ \mu(f_2 - f_1) x_v < \varepsilon. \]

Remark. Every continuous function is locally $\mathbb{R}$-integrable.

Section 2.2 gives the topological and measure-theoretical background for some theorems, which will be proved in Section 2.3. In Section 2.4 we shall discuss that the results of Section 2.3 are best possible in a certain sense.

In order to avoid unnecessarily complicated proofs we shall assume
from now on that the support $S(\mu)$ coincides with $X$ itself.

Besides, we make the notational convention to write $\mu$ both for the measure defined on $X$ and for its natural extension on $X^*$.

2.2. TOPOLOGICAL AND MEASURE-THEORETICAL PRELIMINARIES

The following lemma both sharpens and generalises lemma 1.1.4 (cf. the remark at the end of lemma 1.1.4).

Lemma 2.2.1. If $A$ and $B$ are disjoint closed subsets of $X^*$, $\mu(A) + \mu(B) < 1$, $\mu(\partial A) = \mu(\partial B) = 0$, and if $v$ is a real number such that $\mu(A) < v < 1 - \mu(B)$, then there exists an open set $C \supset A$ such that $\overline{C} \cap B = \emptyset$ and $\mu(C) = \mu(\overline{C}) = v$.

Proof. We shall use the notation $\Delta = \overline{U^{(1)}}$, $\Delta = \overline{U^{(2)}}$. Since $X^*$ is a compact Hausdorff-space, it is normal, and therefore there exist open neighbourhoods $Z^{(1)}$ and $Z^{(2)}$ of $\overline{U^{(1)}}$ and $\overline{U^{(2)}}$ respectively, such that $Z^{(1)} \cap Z^{(2)} = \emptyset$. As $\mu$ is a regular measure, there exist open sets $U_0^{(1)}$ and $U_0^{(2)}$ with zero-boundaries, such that $V^{(1)}_0 \subset Z^{(1)}$ and $V^{(2)}_0 \subset Z^{(2)}$, and that, moreover,

\[ \mu(V^{(1)}_0) < v < 1 - \mu(V^{(2)}_0). \]

Suppose that $\mathcal{V}^{(1)}_0 = \{V^{(1)}_k\}_{k=1}^\infty$ is an open covering of $\overline{W^{(2)}_0} = X^* \setminus \overline{W^{(2)}_0}$, consisting of $V^{(1)}_0$ and a sequence of open sets $\{V^{(1)}_k\}_{k=1}^\infty$, each of which has measure $< 2^{-1}$ and zero-boundary.

According to Heine-Borel's lemma a finite subcollection $\{V^{(1)}_k\}_{k=1}^{N_1}$ already covers $\overline{W^{(2)}_0}$ and we consider the expanding system of open sets $\{U^{(1)}_k\}_{k=0}^{N_1}$ defined by

\[ U^{(1)}_k = \bigcup_{\ell=0}^k (V^{(1)}_\ell \setminus \overline{W^{(2)}_0}) \quad (0 \leq k \leq N_1). \]
For exactly one index $k_1$, we have
\begin{equation}
\mu(U_{k_1}^{(1)}) < v < \mu(U_{k_1+1}^{(1)}) < \mu(U_{k_1}^{(1)}) + 2^{-1},
\end{equation}
and the corresponding set $U_{k_1}^{(1)}$ has zero-boundary (see Fig. 1). If
$\mu(U_{k_1}^{(1)}) = v$, the proof is completed by taking $C = U_{k_1}^{(1)}$, since
$U_{k_1}^{(1)} \subset V_0^{(2)}$, which is a closed set disjoint with $U^{(2)}$, so that
$C \cap S = \emptyset$.

![Figure 1](image)

Therefore, let us assume that $\mu(U_{k_1}^{(1)}) < v$. We define $U^{(3)} = U_{k_1}^{(1)}$.
Because $U^{(3)} \subset W_0^{(2)}$, we obtain $U^{(3)} \cap U^{(3)} = \emptyset$ and $\mu(U^{(3)}) < 1 - v < 1 - \mu(U^{(3)})$. Following the same method as before, we may state the existence of open sets $V_0^{(2)}$ and $W_0^{(3)}$ with zero-boundaries such that $\overline{U^{(2)}} \subset V_0^{(2)}$, $\overline{U^{(3)}} \subset W_0^{(3)}$, $V_0^{(2)} \cap W_0^{(3)} = \emptyset$, and
\begin{equation}
\mu(V_0^{(2)}) < 1 - v < 1 - \mu(W_0^{(3)}).
\end{equation}
Now let $\phi^{(2)} = \{V_0^{(2)}\}_{k=1}^{\infty}$ be an open covering of $W_0^{(3)} = X \setminus W_0^{(3)}$, consisting of $V_0^{(2)}$ and a sequence of open sets $\{V_0^{(2)}\}_{k=1}^{\infty}$, each of
which has measure $< 2^{-2}$ and zero-boundary. By Heine-Borel's lemma a finite subcollection \( \{ \gamma_k^{(r)} \}_{k=0}^{N_2} \) already covers \( \gamma_0^{(s)} \), and from the expanding system \( \{ \Omega_k^{(2)} \}_{k=0}^{N_2} \), where

\[
U_k^{(2)} = U_{k_0}^{(2)} (\gamma_k^{(r)} \setminus \gamma_0^{(s)}) \quad (0 \leq k \leq N_2),
\]

exactly one net, say \( U_k^{(2)} \), satisfies the inequalities

\[
\mu(U_k^{(2)}) \leq 1 - \nu < \mu(U_{k_2}^{(2)}) < \mu(U_k^{(2)}) + 2^{-2},
\]

and has zero-boundary (see Fig. 2).

Fig. 2

If \( \mu(U_k^{(2)}) = 1 - \nu \) then the proof is completed by taking \( C = \overline{U_k^{(2)}} \).

(Obviously \( C \) is open and has zero-boundary. \( C' = \overline{U_k^{(2)}} \setminus \gamma_0^{(s)} \setminus \gamma_k^{(r)} \) \( \supset B \), so that \( C \) is contained in a closed set \( \gamma_0^{(s)} \) disjoint with \( B \). Hence \( C \cap B = \emptyset \). On the other hand \( C \supset \gamma_0^{(s)} \supset A \), so that all conditions for \( C \) are satisfied). Assuming that \( \mu(U_k^{(2)}) < 1 - \nu \), we define \( U_A^{(s)} = U_k^{(2)} \) and observe that we are in the same position as when we started the proof, but with each index \( (j) \) increased.
by 2. In the same way we continue this process by induction, choosing in the \(j\)th step an open covering \(\mathcal{U}(j) = \{U_k(j)\}_{k=0}^{\infty}\) of \(\mathcal{V}_0(j+1)\) consisting of \(V_0(j)\) and a sequence of open sets \(\{U_k(j)\}_{k=1}^{\infty}\) each of which has measure \(< 2^{-j}\) and zero-boundary. If this process results (in the manner just described) in a set \(\mathcal{U}(j+2)\) of measure \(\nu(j)\) (if \(j\) is odd), or in a set \(\mathcal{U}(j+2)\) of measure \(1 - \nu(j)\) (if \(j\) is even), then the proof of the lemma is completed by taking \(C = \overline{\mathcal{U}(j+2)}\) and \(C = \overline{\mathcal{U}(j+2)}\), respectively. Otherwise, we obtain two expanding sequences of open sets
\[
(A \cup U^{(3)}) \subset U^{(5)} \subset U^{(7)} \subset \ldots, \quad \text{and}
(B \cup U^{(4)}) \subset U^{(6)} \subset U^{(8)} \subset \ldots,
\]

such that \(\mathcal{U}(k) \cap \mathcal{U}(j) = \emptyset\) (\(k + j\) odd), \(\left| \mu(U^{(2i+1)}) - \nu \right| < 2^{-2i+1}\) \((i = 1, 2, 3, \ldots)\) and \(\left| \mu(U^{(2i+2)}) - 1 + \nu \right| < 2^{-2i}\) \((i = 1, 2, 3, \ldots)\).

Now \(\bigcup_{i=1}^{\infty} U^{(2i+1)}\) and \(\bigcup_{j=1}^{\infty} U^{(2j+2)}\) are open disjoint sets of measure \(\nu(j)\) and \(1 - \nu(j)\) respectively, and neighbourhoods of \(A\) and \(B\) respectively. Hence the set \(C\), defined by \(C = \bigcup_{i=1}^{\infty} U^{(2i+1)}\), has the desired properties, which proves lemma 2.2.1.

Lemma 2.2.2. (Interpolation lemma). If \(A\) and \(D\) are subsets of \(X\) with zero-boundaries, \(A\) is compact, \(D\) is open, and \(D \supset A\), and if \(\nu\) is a real number, such that \(\mu(A) < \nu < \mu(D)\), then there exists an open set \(C\) with zero-boundary and compact closure, such that \(A \subset C \subset \overline{C} \subset D\) and \(\mu(C) = \nu\).

Proof. Consider \(A\) and \(D\) as subsets of \(X^*\). We distinguish two cases:
(1) \(D' = X^* \setminus D\) contains the point \(\infty\) as an interior point (i.e. ...
$D$ is compact in $X^*$. In this case lemma 2.2.1 applies if we choose
$U = X^* \setminus D$, and the lemma is proved.

(2) $D^* = X^* \setminus D$ contains the point $\infty$ as a boundary-point. Now
we construct an open neighbourhood $E$ of $\infty$ in $X^*$ with zero-boundary,
such that $E \cap A = \emptyset$ and $\mu(E) < \mu(D) - \nu$. Then the set $B$ de-
defined by $B = (X^* \setminus D) \cup E$ is closed in $X^*$ and has zero-boundary,
$E \cap A = \emptyset$, and

$$\mu(A) < \nu < \mu(D) - \mu(E) = 1 - \left( \mu(X^* \setminus D) + \mu(E) \right) \leq 1 - \mu(D),$$
so that lemma 2.2.1 may be applied. Q.E.D.

**Lemma 2.2.2.** If $\{\alpha_n\}_{n=1}^\infty$ is a sequence of real numbers satisfying
$0 < \alpha_1 < \alpha_2 < \ldots$ and $\lim_{n \to \infty} \alpha_n = 1$, then there exists a sequence
of open sets $\{C_n\}_{n=1}^\infty$ in $X$ with compact closures and zero-boundaries,
such that

(i) $C_1 \subseteq C_2 \subseteq \ldots$

(ii) $\bigcup_{n=1}^\infty C_n = X$

(iii) $\mu(C_n) = \alpha_n \quad (n=1,2,3,\ldots)$.

**Proof.** The normality of $X^*$ implies that there exists a strictly
contracting sequence of open neighbourhoods $\{P_i\}_{i=1}^\infty$ of $\infty$, such
that $X^* = P_1 \supset P_2 \supset P_3 \supset \ldots$, $\mu(P_i \setminus P_{i+1}) = 0 \quad (i=1,2,3,\ldots)$,
$\bigcap_{i=1}^\infty P_i = \{\infty\}$, and $\lim_{i \to \infty} \mu(P_i) = 0$. Consider the sequence $\{A_i\}_{i=1}^\infty$,
defined by $A_i = X^* \setminus P_i \quad (i=1,2,3,\ldots)$ and the sequence $\{B_i\}_{i=1}^\infty$,
defined by $B_i = X^* \setminus P_{i+1} \quad (i=1,2,3,\ldots)$. Then the pair $\{A_i,B_i\}$
satisfies the hypotheses of lemma 2.2.1 $(i=1,2,3,\ldots)$. If $B_1$
is defined as $\beta_i = \mu(A_i)$ ($i = 1, 2, 3, \ldots$), then $\mu(\mathcal{C}_i) = \beta_i$ ($i = 1, 2, 3, \ldots$), and $0 = \beta_1 < \beta_2 < \beta_3 < \ldots$, $\lim_{i \to \infty} \beta_i = 1$.

It follows that for all $\alpha_n$ ($n = 1, 2, 3, \ldots$) there exists exactly one $\beta_i$, such that $\beta_i < \alpha_n < \beta_{i+1}$, and that for fixed $\beta_i$ there are at most finitely many $\alpha_n$ satisfying $\beta_i < \alpha_n < \beta_{i+1}$ (such a "block" of elements $\alpha$ between two consecutive elements $\beta$ may be empty). Let us consider a non-empty block

$$\beta_i \ll \alpha_{n_{i,1}} < \alpha_{n_{i,2}} < \ldots < \alpha_{n_{i,k(i)}} < \beta_{i+1}.$$

We define $C_{n_{i,1}} = A_i$ if $\alpha_{n_{i,1}} = \beta_i$, otherwise $C_{n_{i,1}}$ is defined as a set of measure $\alpha_{n_{i,1}}$, obtained by interpolation between $A_i$ and $D_i$ in Lemma 2.2.2. In either case $C_{n_{i,1}}$ and $D_i$ satisfy the hypotheses of the interpolation lemma, so that the sets $C_{n_{i,k}}$ with measure $\alpha_{n_{i,k}}$ ($k = 2, \ldots, k(i)$) may be constructed by induction as interpolatory sets between $C_{n_{i,k-1}}$ and $D_i$ (Lemma 2.2.2).

Finally, the sequence $\{\mathcal{C}_n\}_{n=1}^\infty$ is obtained by ordering the blocks $\{C_{n_{i,j}}\}_{i,j=1}^{k(i)}$.

The required properties (i) and (iii) of the sequence $\{\mathcal{C}_n\}_{n=1}^\infty$ are clear. (ii) follows from the fact that for all $\beta_j$ ($j = 1, 2, 3, \ldots$) there exists an element $\alpha_n > \beta_j$. In other words for all $A_j$ there exists a set $C_n > A_j$. Since $\bigcup_{j=1}^\infty A_j = \bigcup_{j=1}^\infty (X \setminus F_j)$,

$$= X \setminus \bigcap_{j=1}^\infty F_j = X,$$

we obtain $\bigcup_{n=1}^\infty C_n = X$.

The following lemma is concerned with finite expanding sequences of open sets with compact closures and zero-boundaries. The nota-
section used in this lemma is chosen in order to facilitate its application in theorem 2.2.1.

**Lemma 2.2.1.** Let \( \{G_i\}_{i=m+1}^d \) be a sequence of open sets with compact closures and zero-boundary, satisfying
\[ G_{i+1} \subseteq G_i \subseteq G_{i+2} \subseteq \cdots \subseteq G_2, \quad \text{and} \quad \mu(G_i) = \gamma_i \ (i=m+1, \ldots, d) \]. Let \( H \) be a closed subset of \( G_s \) with zero-boundary and suppose
\[ H_{m+1} = H \cap G_{m+1}, \quad H_i = H \cap (G_i \setminus G_{i-1}) \ (i=m+2, \ldots, d), \]
\[ \mu(H_i) = \eta_i \ (i=m+1, \ldots, d) \]. Let \( \{\theta_i\}_{i=m+1}^d \) be a sequence of real numbers, satisfying
\[ (a) \quad \eta_i < \theta_i < \gamma_i - \eta_{i-1} \ (i=m+2, \ldots, d) \]
\[ (b) \quad \eta_{m+1} < \theta_{m+1} < \gamma_{m+1} . \]

Then there exists an open set \( E \) with zero-boundary, \( E \subseteq E' \subseteq G_s \), such that for the sets \( F_i \) defined by
\[ F_{m+1} = E \cap G_{m+1}, F_i = E \cap (G_i \setminus G_{i-1}) \ (i=m+2, \ldots, d) \], we have \( \mu(F_i) = \theta_i \ (i=m+1, \ldots, d) \).
(see Fig. 3).
Proof. Let \( \delta \) be defined as \( \delta = \frac{1}{2} \min \{ \alpha_i - \eta_i \mid m+1 \leq i \leq \ell \} \). Obviously we have \( \mu(\{i\}) + \delta < \eta_i \), so that in view of the interpolation lemma there exists an open set \( \mathcal{L} \) with zero-boundary, \( \mathcal{H} \subset \mathcal{L} \subset \mathcal{G} \), and \( \mu(\mathcal{L}) = \mu(\mathcal{H}) + \delta \). Since \( \mu(\mathcal{L} \cap \mathcal{G}_{i+1}) < \mu(\mathcal{G}_{i+1}) \delta \) \( \leq \), \( \mathcal{G}_{i+1} \subset \mathcal{G}_{i-1} \) and \( \mu(\mathcal{L} \cap (\mathcal{G}_i \setminus \mathcal{G}_{i-1})) \leq \mu(\mathcal{H}_i) + \delta < \theta_i + \gamma_i = \gamma_{i-1} \) \( (i=m+2, \ldots, \ell) \), neither of the sets \( \mathcal{G}_{m+1} \setminus \mathcal{L} \) and \( (\mathcal{G}_i \setminus \mathcal{G}_{i-1}) \setminus \mathcal{L} \) \( (i=m+2, \ldots, \ell) \) is empty, and because all these sets have zero-boundaries and positive measures, the interpolation lemma applies when we take \( \lambda = \emptyset \), \( \psi = (\mathcal{G}_i \setminus \mathcal{G}_{i-1}) \setminus \mathcal{L} \) and \( \gamma = \theta_i + \mu(\mathcal{L} \cap (\mathcal{G}_i \setminus \mathcal{G}_{i-1})) \) \( (i=m+1, \ldots, \ell) \); we put \( \mathcal{G}_i = \emptyset \). Let \( \mathcal{M}_i \) \( (i=m+1, \ldots, \ell) \) be the interpolatory sets. Then the set \( \mathcal{E} = \mathcal{L} \cup \left( \bigcup_{i=m+1}^{\ell} \mathcal{M}_i \right) \) has the required properties.

We shall now deduce a theorem, which will serve as a basis for the theorems on \( (\xi, d\mu) \)-summability of unbounded locally \( R \)-integrable functions as formulated and proved in the next Section. It will state the existence of an infinite expanding sequence of open sets with compact closures and zero-boundaries, which behave in a specified manner with respect to a given \( d\mu \)-uniformly distributed sequence. The significance of the measure-theoretical requirements, stated in theorem 2.2.1, will become clear in the sequel (cf. Section 2.3).

**THEOREM 2.2.1.** Let \( \xi = \{ \xi_n \}_{n=1}^{\infty} \) be a \( d\mu \)-uniformly distributed sequence in \( X \), and let \( \varepsilon \) be any real positive number. Then there exists a sequence \( \{ \mathcal{E}_k \}_{k=0}^{\infty} \) of open sets with compact closures and zero-boundaries, such that

(a) \( \mathcal{E}_0 = \emptyset \neq \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \ldots \)

(b) \( \bigcup_{k=1}^{\infty} \mathcal{E}_k = X \)

(c) \( \frac{\lambda(\xi, \mathcal{E}_{k+1} \setminus \mathcal{E}_k; N)}{N} < (1 + \varepsilon)\mu(\mathcal{E}_{k+1} \setminus \mathcal{E}_k) \) \( (k=0,1,2,\ldots; N=1,2,3,\ldots) \)
\( \mu(\mathcal{E}_k \setminus \mathcal{E}_{k-1}) / \mu(\mathcal{E}_{k+1} \setminus \mathcal{E}_k) = c \), where \( c \) is a constant (as a consequence we have \( c = \frac{1}{1 - \mu(\mathcal{E}_k)} \)).

**Proof.** Let \( \delta \) and \( p \) be fixed real numbers such that

\[
\begin{align*}
&\delta > 0, \quad p > 3 \\
&1 + \frac{1}{2} \delta < p \\
&(1 + \delta) \cdot \frac{p}{p - 1} < 1 + \varepsilon.
\end{align*}
\]

It is not difficult to find numbers \( \delta \) and \( p \) with these properties. We choose for example \( \delta = \frac{1}{2} \varepsilon \) and \( p \) sufficiently large.

By lemma 2.2.3 there exists an expanding sequence of open sets

\[ \{c_k\}_{k=1}^\infty \]

with compact closures and zero-boundaries \( (k=1,2,3,\ldots) \), such that

\[
\begin{align*}
&c_1 \subset c_2 \subset \cdots \\
&\cup_{k=1}^\infty c_k = \mathcal{X} \\
&\mu(c_k) = 1 - \frac{1}{p}(1 + \frac{1}{2} \delta) \quad (k=1,2,3,\ldots).
\end{align*}
\]

Our intention is to extend each \( c_k \) in an appropriate way to an open set \( \mathcal{E}_k \) with the required properties, and satisfying \( \mu(\mathcal{E}_k) = 1 - \frac{1}{p}(1 + \frac{1}{2} \delta) \) \( (k=1,2,3,\ldots) \). We use an inductive method:

1. **The construction of \( \mathcal{E}_1 \)**

Since \( \mathcal{E} \) is \( \mu \)-uniformly distributed we have (cf. (2.9))

\[
\lim_{N \to \infty} \frac{A(\mathcal{E},c_1;N)}{N} = \mu(c_1) = 1 - \frac{1}{p}(1 + \frac{1}{2} \delta),
\]

from which it follows that

\[
\frac{A(\mathcal{E},c_1;N)}{N} > 1 - \frac{1}{p}(1 + \delta) \quad (N > N_1).
\]
let us consider the finite set \( \{x_n\}_{n=1}^{N} \), and determine the smallest index \( k' \geq 3 \) such that \( \{x_n\}_{n=1}^{N} \subset C_{i_{k'}} \). We now apply Lemma 2.2.4, replacing in it the sequence \( \{c_{i_k}\}_{k=1}^{N} \) by the sequence \( \{c_{i_k}\}_{k=2}^{N} \), and defining \( \mathcal{H} = \bigcup_{k=1}^{N} c_{i_k} \). By this lemma there exists a set \( \mathcal{E}_k \) such that

\[
\begin{align*}
E_k & \text{ is open and has zero-boundary} \\
\mathcal{H} + \{x_n\}_{n=1}^{N} & \subset E_k \subset E_k \subset C_{i_k} \\
\mu(E_k \cap (C_{i_k} \setminus C_{i_{k-1}})) &= \frac{5}{2p}(1 - \frac{1}{2^k}) \quad (i = 3, \ldots, l_1) \\
\mu(E_k) &= 1 - \frac{1}{p}.
\end{align*}
\]

All conditions in the hypothesis of Lemma 2.2.4 are indeed satisfied, since we have (cf. (2.6))

\[\gamma_1 = \gamma_{i-1} = \mu(C_{i} \setminus C_{i_{i-1}}) = (1 + \frac{1}{2} \delta) (\frac{1}{p} - \frac{1}{p}) = \]

\[
= \frac{1 - \frac{1}{p}}{p} (1 + \frac{1}{2} \delta) > \frac{\delta}{2p} (1 - \frac{1}{2}) = \theta_1 > 0 =
\]

\[\mu(\mathcal{H} \cap (C_{i} \setminus C_{i_{i-1}})) = \eta_i \quad (i = 3, \ldots, l_1)\]

\[\gamma_2 = \mu(C_2) = 1 - \frac{1}{p} (1 + \frac{1}{2} \delta) > 1 - \frac{1}{p} \cdot p >
\]

\[> 1 - \frac{1}{p} - \sum_{i=3}^{l_1} \frac{\delta}{2p} (1 - \frac{1}{2}) = \theta_2 >
\]

\[> 1 - \frac{1}{p} - \frac{\delta}{4} \sum_{i=3}^{\infty} \frac{1}{p} = 1 - \frac{1}{p} - \frac{\delta}{4p} (p - 1) >
\]

\[> 1 - \frac{1}{p} - \frac{\delta}{2p} = \mu(\mathcal{H} \cap C_2) = \eta_2.
\]

As a consequence of this construction and (2.10) we observe that

\[\gamma_{i+1} \geq \frac{\mathcal{A}(E_k, E_{i+1})}{N} > 1 - \frac{1}{p} (1 + \delta) \quad (N = 1, 2, 3, \ldots) . \]
(2) The construction of $E_{k+1}$ by induction from $E_k$

Let the set $E_k$ and the index $N_k$ satisfy the following conditions:

\[
\begin{align*}
E_k & \text{ is open, } \mu(\partial E_k) = 0 \\
& \exists x_{k+2} \in \Omega_k \cup \{x_n \mid n = 1, \ldots, N_k\} \subset E_k \subset \overline{E}_k \cup \partial E_k \\
& \mu(E_k \cap (C_1 \setminus C_{i-1})) = \frac{5}{2p} \left(1 - \frac{1}{z_k^i}\right) \quad (i = k+2, \ldots, n_k) \\
& \mu(E_{n_k}) = 1 - \frac{1}{p^k}
\end{align*}
\]

If $k = 1$, then (2.13) is identical with (2.11).

Consider the set $D_k$ defined by $D_k = C_{k+1} \cup N_k$ (see Fig. 4). $D_k$ is open and has zero boundary. Lower and upper estimates for $\mu(D_k)$ are given by the following chain of inequalities (cf. (2.9), (2.13)):

\[
1 - \frac{1}{p^k}(1 + \frac{5}{z_k}) = \mu(C_{k+1}) \leq \mu(D_k) =
\]
\[
\mu(C_{k+1}) + \Sigma_{i=k+2}^{N} \mu(C_{i} \setminus C_{i-1}) < 1 - \frac{1}{p^{k+1}}(1 + \frac{1}{3} \cdot b) + \frac{1}{p^{k+1}} \frac{1}{2} (1 - \frac{1}{2^{k+1}}) \cdot \frac{1}{p^{k+1}} < 1 - \frac{1}{p^{k+1}}.
\]

so that

\[(2.14) \quad 1 - \frac{1}{p^{k+1}}(1 + \frac{1}{3} \cdot b) \leq \mu(C_{k}) < 1 - \frac{1}{p^{k+1}}.\]

From the left hand inequality of (2.14) we deduce that there exists an index \(N_{k+1} > N_{k}\) such that

\[(2.15) \quad \frac{\Delta_{k+k+1}}{N} > 1 - \frac{1}{p^{k+1}}(1 + b) \quad (N > N_{k+1}).\]

The set \(\overline{C}_{k} \cup \{x_{n}\}_{n=1}^{N_{k+1}}\) is a closed subset of \(C_{k+1}\), where \(A_{k+1} = \min\{A_{k+1} - 1\} \{x_{n}\}_{n=1}^{N_{k+1}} \subset C_{j}\) (so that obviously \(A_{k+1} \geq k+1\)), and this set has zero-boundary. Therefore, we apply lemma 2.2.4, replacing in it the sequence \(\{C_{i}\}_{i=k+2}^{A_{k+1}}\) by the sequence \(\{C_{i}\}_{i=k+2}^{N_{k+1}}\) and the set \(\overline{C}_{k} \cup \{x_{n}\}_{n=1}^{N_{k+1}}\). By this lemma there exists an open set \(E_{k+1}\) with zero-boundary, such that

\[(2.16) \quad \begin{cases} 
\overline{C}_{k+1} \subset \overline{C}_{k} \cup \{x_{n}\}_{n=1}^{N_{k+1}} \subset E_{k+1} \subset \overline{E}_{k+1} \subset \overline{C}_{k+1} \\
\mu(\overline{E}_{k+1} \cap C_{i} \setminus C_{i-1}) = \frac{b}{2p^{k+1}}(1 - \frac{1}{2^{k+1}}) \\
\mu(\overline{E}_{k+1}) = 1 - \frac{1}{p^{k+1}}. 
\end{cases} \quad (l = k+2, \ldots, A_{k+1})
\]

Indeed, all conditions in the hypothesis of lemma 2.2.4 are really satisfied, because (cf. (2.8))
(a) \[ \gamma_i - \gamma_{i-1} = \mu(C_i \setminus C_{i-1}) = \frac{2}{p} \left( 1 - \frac{1}{2'} \right) \]

\[ > \frac{5}{2p} \left( 1 - \frac{1}{2^{k+1}} \right) = \theta_k > \frac{5}{2p} \left( 1 - \frac{1}{2^k} \right) = \]

\[ = \mu(H \cap (C_i \setminus C_{i-1})) = \eta_i \]

\[ (i = k + 3, \ldots, \ell^*) \]

(this statement may be empty, viz. if \( \ell^* = k + 2 \),)

and in the same way

\[ \gamma_i - \gamma_{i-1} > \theta_i > 0 = \mu(H \cap (C_i \setminus C_{i-1})) = \eta_i \]

\[ (i = \ell^* + 1, \ldots, \ell^*) \]

(b) \[ \gamma_{k+2} = \mu(C_{k+2}) = 1 - \frac{1}{p} (1 + \frac{2}{p} \delta) > 1 - \frac{1}{p} \]

\[ > 1 - \frac{1}{p} + \frac{\ell^*+1}{2p} (1 - \frac{1}{2^{k+1}}) = \theta_{k+2} > \]

\[ > \eta_{k+2} = \mu(\overline{\mathcal{E}}_k \cap C_{k+2}) \]

where the last inequality in this chain of inequalities is motivated by the fact that (cf. (2.13))

\[ \overline{\mathcal{E}}_k \cap C_{k+2} = \overline{C}_{k+1} \cup (\overline{\mathcal{E}}_k \cap (C_{k+2} \setminus C_{k+1})) \]

and therefore \( \mu(\overline{\mathcal{E}}_k \cap C_{k+2}) = 1 - \frac{1}{p} \left( 1 + \frac{2}{p} \delta \right) + \frac{5}{2p} \left( 1 - \frac{1}{2^k} \right) \),

which implies

\[ \theta_{k+2} - \eta_{k+2} > 1 - \frac{1}{p} \left( 1 + \frac{2}{p} \delta \right) + \frac{5}{2p} \left( 1 - \frac{1}{2^k} \right) - \]

\[ - \left\{ 1 - \frac{1}{p} \left( 1 + \frac{2}{p} \delta \right) + \frac{5}{2p} \left( 1 - \frac{1}{2^k} \right) \right\} = \]

\[ = \frac{b}{2p} \left( 1 - \frac{1}{2^{k+1}} \right) + \frac{5}{2p} \left( 1 - \frac{1}{2^k} \right) = \]

\[ = \frac{b}{2p} \left( \frac{2^{k+1} - b - 1}{2^{k+1} - 1} \right) > \frac{b}{2p} \left( \frac{2^{k+2} - 2^{k+1} - b}{2^{k+2} - 2^k} \right) > 0 \].
As a consequence of this construction of $E_{k+1}$ we have, since
\[ \{x_n^{k+1}\}^N_{n=1} \subset E_{k+1} \] (cf. (2.15) and the definition of $D_k$; $E_{k+1}$)

\[ A\left(\{E_{k+1}\}^N_{N=1}; N\right) > 1 - \frac{1}{p^{k+1}(1 + \delta)} \quad (N = 1, 2, 3, \ldots) , \]

and $E_{k+1} \times X_{k+1}$ satisfy the induction conditions again (cf. (2.13),
(2.16)).

(3) Verification of the properties (a), (b), (c) and (d) in theorem 2.2.1:

(a) is a consequence of the first line of (2.16) and the definition of $\overline{B}_k : \overline{E}_k \subset \overline{C}_{k+1} \cup \overline{E}_k = \overline{D}_k \subset E_{k+1}$ ($k = 1, 2, 3, \ldots$).

(b) follows from (2.9) and the fact that $E_k \supset C_k$ ($k = 1, 2, 3, \ldots$).

(c) may be proved as follows: on the one hand, since $E_{k+1} \supset E_k$
for all $k$, we have $\mu(E_{k+1} \setminus E_k) = \frac{p-1}{p^{k+1}}$. On the other hand, by
(2.17) we obtain

\[ A\left(\{E_{k+1}\}^N_{N=1}; N\right) > 1 - \frac{1}{p^{k+1}}(1 + \delta) \quad (k, N = 1, 2, 3, \ldots) . \]

Taking complements, we conclude

\[ A\left(\{E_{k+1} \setminus E_k\}^N_{N=1}; N\right) < \frac{1}{p^{k+1}}(1 + \delta) \quad (k, N = 1, 2, 3, \ldots) , \]

so that a fortiori

\[ A\left(\{E_{k+1} \setminus E_k\}^N_{N=1}; N\right) < \frac{1}{p^{k+1}}(1 + \delta) \quad (k, N = 1, 2, 3, \ldots) . \]

Combination of these results yields (cf. (2.9))

\[ \frac{A\left(\{E_{k+1} \setminus E_k\}^N_{N=1}; N\right)}{\mu(E_{k+1} \setminus E_k)} < \frac{1 + \delta}{p^{k+1}} \cdot \frac{p^{k+1}}{p-1} = (1 + \delta) \cdot \frac{p}{p-1} < 1 + \varepsilon , \]

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which holds for \( k, N = 1, 2, 3, \ldots \), but also for \( k = 0 \ (E_0 = \emptyset) \).

(d) Obviously we have \( \frac{\mu(A, E_{k+1} \setminus E_k)}{\mu(A, E_{k+1} \setminus E_k)} = p > 0 \).

This completes the proof of theorem 2.2.1.

2.3. SUFFICIENT CONDITIONS FOR \((\xi, \nu)\)-SUMMABILITY

Let \( \xi = \{x_n\}_{n=1}^\infty \) be a \( \xi \)-uniformly distributed sequence. For later use we note that a locally \( R \)-integrable function with a compact support is \( \xi \)-integrable in the sense as defined in [7] and hence \((\xi, d\mu)\)-summable.

Let \( \{E_k\}_{k=0}^\infty \) be a sequence of open sets with compact closures and zero-bounded, such that

\[
\begin{align*}
E_0 &= \emptyset, \quad E_1 \subset E_1' \subset E_2 \subset E_2' \subset \ldots \\
\bigcup_{k=1}^\infty E_k &= X.
\end{align*}
\]

Sometimes we shall require additional properties of this sequence of sets, namely

\[
\begin{align*}
\text{for some fixed } b, \text{ and } k = 0, 1, 2, \ldots; N = 1, 2, 3, \ldots, \text{ and }
\end{align*}
\]

\[
\begin{align*}
\mu(E_k \setminus E_{k-1}) / \mu(E_k \setminus E_{k+1}) &< c \\
\text{for some fixed } c, \text{ and } k = 1, 2, 3, \ldots.
\end{align*}
\]

It should be remarked that \((3.1^b)\) tacitly implies \( \mu(E_k \setminus E_{k-1}) > 0 \) \( (k = 1, 2, 3, \ldots) \) in view of the fact that \( E_0 = \emptyset, E_1 \neq \emptyset, S(\mu) = X \).

In theorem 2.2.1 we have established the existence of sequences \( \{E_k\}_{k=0}^\infty \) with the properties \((3.1), (3.1^a)\) and \((3.1^b)\).
Let $f$ be a real-valued function on $X$ which is bounded on every compact subset of $X$ (in particular every locally $R$-integrable function has this property). We introduce the following notations ($\chi_E$ denotes the characteristic function of the set $E$):

$$
\chi_k(x) = \chi_{E_k \setminus E_{k-1}}(x) \quad (k=1,2,3,\ldots)
$$

Notations:

$$
\|f\|_k = \sup_{x \in X} |f(x)\chi_k(x)|
$$

The following theorems state sufficient conditions for the $(\xi, \text{d}\mu)$-summability of locally $R$-integrable functions. Applications of these theorems to some special $\text{d}\mu$-uniformly distributed sequences in $[0,1]$ will be discussed in Chapter III.

**Theorem 2.3.** If $\{\xi_k\}_{k=1}^\infty$ satisfies (3.1) and (3.1$^\alpha$), and if $f$ is a locally $R$-integrable function such that $\sum_{k=1}^\infty \|f\|_k \mu(E_k \setminus E_{k-1}) < \infty$, then $f$ is $(\xi, \text{d}\mu)$-summable.

**Proof.** The integrability of $f$ follows from the fact that $f$ is measurable and may be majorised by the integrable function $\sum_{k=1}^\infty \|f\|_k \chi_k$ (cf. Halmos [5]).

Now let us first assume that $f(x) \neq 0$ for all $x \in X$. As in Chapter I, we shall denote $\int_X f(x) \text{d}\mu(x)$ by $\mu(f)$. Suppose that $\varepsilon$ is any given positive number. We consider the following arguments:

(a) Since $\mu(f) = \sum_{k=1}^\infty \mu(f \chi_k) < \infty$ we may find an index $P_0(\varepsilon)$ such that

$$
\sum_{k=P_0+1}^\infty \mu(f \chi_k) < \varepsilon \quad (P_0 \gg P_0(\varepsilon)).
$$

(b) The convergence of $\sum_{k=1}^\infty \|f\|_k \mu(E_k \setminus E_{k-1})$ implies that there exists an index $P_1(\varepsilon)$ such that (cf. (3.1$^\alpha$))
\( (3.3) \quad \sum_{k=P^n B}^{\infty} \|f_k \|_\mu(\mathcal{E}_k \setminus \mathcal{E}_{k-1}) < C \quad (P > P_1(e)) \)

(c) Let \( P \) be fixed, \( P = \max(P_0(e), P_1(e)) \) (cf. (a), (b)). Since all functions \( f \chi_k \) have compact supports and are locally \( R \)-integrable they are \((\xi, d\mu)\)-summable (cf. the second paragraph of this Section), and therefore

\( (3.4) \quad \left| \frac{1}{N} \sum_{n=1}^{N} f_k(\chi_n) - \mu(f_k) \right| < \frac{\varepsilon}{P} \)

for all \( k = 1, \ldots, P \), provided that \( N > N(P, e) = N_0(e) \). Now let us assume that \( N > N_0(e) \) according to (a), (b), and (c). Then we have (cf. (3.1'), (3.2), (3.3) and (3.4))

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(\chi_n) - \mu(f) \right| = \left| \frac{1}{N} \sum_{n=1}^{N} f(\chi_n) - \sum_{k=1}^{\infty} \mu(f_k) \right| \\
\quad \leq \left| \sum_{k=1}^{P-1} \left( \frac{1}{N} \sum_{n=1}^{N} f_k(\chi_n) - \mu(f_k) \right) \right| + \\
\quad + \sum_{k=P}^{\infty} \left( \frac{1}{N} \sum_{n=1}^{N} f_k(\chi_n) + \sum_{k=P+1}^{\infty} \mu(f_k) \right) \\
\quad < \frac{1}{P} + \sum_{k=P}^{\infty} \frac{1}{N} \|f_k \|_\mu(\mathcal{E}_k \setminus \mathcal{E}_{k-1}) \|_\mu + \varepsilon \\
\quad < \varepsilon + \sum_{k=P+1}^{\infty} \frac{1}{N} \|f_k \|_\mu(\mathcal{E}_k \setminus \mathcal{E}_{k-1}) + \varepsilon \\
\quad < \varepsilon + \frac{c}{b} + \varepsilon = 3\varepsilon ,
\]

which proves theorem 2.3.1 for non-negative functions. If \( f \) attains any negative values, we introduce the locally \( R \)-integrable functions \( f_+ = \max(f, 0) \) and \( f_- = -\min(f, 0) \). Then \( f_+ \) and \( f_- \) are both non-negative, \( \|f_+ \|_K \leq \|f \|_K \), \( \|f_- \|_K \leq \|f \|_K (k = 1, 2, \ldots) \), and \( f = f_+ - f_- \). Because of the first part of the proof, both \( f_+ \) and \( f_- \) are \((\xi, d\mu)\)-summable, and therefore \( f \) is a \((\xi, d\mu)\)-summable function, which completes the proof of theorem 2.3.1.
Corollary to theorem 2.3.1. Every bounded locally \( R \)-integrable function is \((\xi, d\mu)\)-summable.

Remark. In fact one may prove that a bounded locally \( R \)-integrable function is also \( R \)-integrable, so that the \((\xi, d\mu)\)-summability also may be deduced in this way (cf. Helberg [7], p. 172).

We now define a property of real-valued functions on \( X \), which may be regarded as a monotonicity-property.

Definition 2.3.1. Let \( \{E_k\}_{k=0}^{\infty} \) satisfy (3.1). Then the real-valued function \( f \) on \( X \) is said to be a **monotonically increasing function** with respect to \( \{E_k\}_{k=0}^{\infty} \), if there exists a monotonically increasing sequence of real numbers \( \{\rho_k\}_{k=0}^{\infty} \), such that

\[
\rho_{k-1} x_k \leq f x_k \leq \rho_k x_k \quad (k = 1, 2, 3, \ldots).
\]

**Theorem 2.3.2.** If \( \{E_k\}_{k=0}^{\infty} \) satisfies (3.1), (3.1^a), (3.1^b), and if \( f \) is a locally \( R \)-integrable, monotonically increasing function with respect to \( \{E_k\}_{k=0}^{\infty} \), and if \( \mu(f) < \infty \), then \( f \) is \((\xi, d\mu)\)-summable.

**Proof.** Without loss of generality we may assume \( f \geq 0 \) and consequently \( 0 \leq \rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots \) (cf. def. 2.3.1).

Since \( f \) is a Lebesgue-integrable majorant of \( \Sigma_{k=1}^{\infty} \rho_{k-1} x_k \), the function \( \Sigma_{k=1}^{\infty} \rho_{k-1} x_k \) itself is Lebesgue-integrable. Therefore we have

\[
\Sigma_{k=1}^{\infty} \rho_{k-1} \mu(E_k \setminus E_{k-1}) \leq \Sigma_{k=1}^{\infty} \rho_k \mu(E_k \setminus E_{k-1}) \leq \Sigma_{k=1}^{\infty} \rho_{k+1} \mu(E_{k+1} \setminus E_k) \leq \Sigma_{k=1}^{\infty} \rho_{k-1} \mu(E_k \setminus E_{k-1}) \leq \infty,
\]

so that according to the monotonicity-property...
\[ E_{k=1} \|f\|_k \mu(E_k \setminus E_{k-1}) < \infty \] and by theorem 2.3.1 \( f \) is \((L, d\mu)\)-summable.

2.4. DISCUSSION OF BEST-POSSIBLE RESULTS

It is natural to ask whether in the theorems 2.3.1 and 2.3.2 the hypotheses may be weakened. In this Section we shall show that the results of Section 2.3 are in a certain sense best-possible.

The following theorem demonstrates that the conclusion of theorem 2.3.1 breaks down if the hypothesis (3.1') in this theorem is violated.

**Theorem 2.4.1.** If \( \{E_k\}_{k=0}^\infty \) satisfies (3.1), \( \mu(E_k \setminus E_{k-1}) \neq 0 \) for all \( k = 1, 2, 3, \ldots \), and if for all \( N > 0 \) there exists an index \( k \geq 1 \) and an index \( N > 1 \), such that

\[
\frac{A(E_k, E_k \setminus E_{k-1}; N)}{N} = M_k(E_k \setminus E_{k-1})
\]

then there exists a non-negative locally \( L \)-integrable function \( f \), such that \( E_{k=1} \|f\|_k \mu(E_k \setminus E_{k-1}) < \infty \), and that \( f \) is not \((L, d\mu)\)-summable.

**Proof.** Let \( \{a_{k, N}\}_{k, N=1}^\infty \) be the double-infinite matrix with elements

\[
a_{k, N} = \frac{A(E_k, E_k \setminus E_{k-1}; N)}{M_k(E_k \setminus E_{k-1})} \quad (k, N = 1, 2, 3, \ldots). \]

This matrix has the following properties:

(1) In each column \( (N \text{ is fixed}) \) all elements vanish with a finite number of exceptions.

(2) In each row \( (k \text{ is fixed}) \) we have \( \lim_{N \to \infty} a_{k, N} = 1 \), since \( L \) is \( d\mu \)-uniformly distributed.

Combining these results we conclude that both \( \{a_{k, N}\}_{N=1}^\infty \) for fixed
$k_i$ and $\{a_{k_i,N}\}_{k_i=1}^{\infty}$ for fixed $N$ are bounded, and therefore the set

$$\{a_{k_i,N}\}_{k_i=1}^{\infty} \setminus \{a_{k_i,N} N > k_0, N > N_0\}$$

is a bounded set of real numbers, its upper bound being a function of $k_0$ and $N_0$.

On the other hand, (4.1) implies that the set $\{a_{k_i,N}\}_{k_i=1}^{\infty}$ is not bounded. Hence there exist two sequences $\{k_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ of positive integers such that

$$k_i < k_{i+1}, \quad N_i < N_{i+1} \quad (i = 1, 2, 3, \ldots)$$

and such that $a_{k_i,N_i} > i^3$ \((i = 1, 2, 3, \ldots)\), in other words

$$\mathcal{A}(k_i, E_{k_i} \setminus E_{k_i-1} ; N_i) > i^3 N_i \mu(E_{k_i} \setminus E_{k_i-1})$$

\((i = 1, 2, 3, \ldots)\).

Now let $f$ be defined by

$$f = \sum_{i=1}^{\infty} \frac{1}{i^2 \mu(E_{k_i} \setminus E_{k_i-1})} \chi_{k_i}$$

Then $f$ is locally $\mathbb{R}$-integrable and

$$\mu(f) = \sum_{k=1}^{\infty} \|f\|_k \mu(E_k \setminus E_{k-1}) = \frac{1}{4} \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

On the other hand, we have for all $i = 1, 2, 3, \ldots$

$$\frac{1}{N_i} \sum_{n=1}^{N_i} f(x_n) > \frac{\mathcal{A}(k_i, E_{k_i} \setminus E_{k_i-1} ; N_i)}{N_i} \|f\|_{k_i} >$$

$$> i^3 \mu(E_{k_i} \setminus E_{k_i-1}) \cdot \frac{1}{i^2 \mu(E_{k_i} \setminus E_{k_i-1})} = 1$$

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so that $\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \infty$, and clearly $f$ is not $(\xi, d\mu)$-summable, which completes the proof.

Now we shall show that the conclusion of theorem 2.3.1 breaks down, if the hypothesis on the convergence of

$\sum_{k=1}^{\infty} ||f||_{k} \mu((E_k \setminus E_{k-1}))$ is violated.

**Theorem 2.4.2.** If $\{E_k\}_{k=0}^{\infty}$ satisfies (3.1), (3.1) and if $\{c_k\}_{k=1}^{\infty}$ is a sequence of positive numbers, such that $\sum_{k=1}^{\infty} c_k \mu((E_k \setminus E_{k-1})) = \infty$, then there exists a locally $\mu$-integrable function $f$ such that $||f||_{k} < c_k$, $\mu(f) < \infty$, and that $f$ is not $(\xi, d\mu)$-summable.

**Proof.** Since $\sum_{k=1}^{\infty} c_k \mu((E_k \setminus E_{k-1})) = \infty$ there exists a monotonically increasing sequence of indices $M_0 = 0 < M_1 < M_2 < \ldots$, such that

$$\sum_{k=M_{j-1}+1}^{M_j} c_k \mu((E_k \setminus E_{k-1})) > j \quad (j=1, 2, 3, \ldots).$$

Since $\xi$ is $d\mu$-uniformly distributed we have

$$\lim_{N \to \infty} \frac{\mu((E_k \setminus E_{k-1}) \cap N)}{N} = \mu((E_k \setminus E_{k-1}) \cap N) \quad (k=1, 2, 3, \ldots)$$

so that for all $k=1, 2, 3, \ldots$ there exists an index $N_k$ such that

$$\frac{\mu((E_k \setminus E_{k-1}) \cap N)}{N} > \frac{1}{2} \mu((E_k \setminus E_{k-1}) \cap N) \quad (N > N_k).$$

Combining (4.2) and (4.4) we obtain for $N > \max\{N_k\}_{k=M_{j-1}+1}^{M_j}$

$$\sum_{k=M_{j-1}+1}^{M_j} c_k \mu((E_k \setminus E_{k-1}) \cap N) > \frac{1}{2} j \quad (j=1, 2, 3, \ldots).$$

Let $f$ be defined as follows

$$f = \frac{1}{E_{j}} \sum_{k=M_{j-1}+1}^{M_j} c_k \chi_{E_k} \chi_{x \in E_{j}} \{x\}.$$
Then the function $f$ has the following properties:

1. For every $k = 1, 2, 3, \ldots$ the intersection of $E_k \setminus E_{k-1}$ and the support of $f$ is a finite point-set, and in these points $f$ attains a non-negative value $\leq c_k$.

2. For all values of $j$ we may write

$$\frac{1}{P_j} \sum_{n=1}^{P_j} f(x_n) > \frac{1}{P_j} \sum_{n=1}^{P_j} f(x_n) \chi_{E_j \setminus E_{j-1}}(x_n),$$

$$> \frac{1}{P_j} \sum_{k=K_{j-1}+1}^{P_j} \frac{A(E_k \setminus E_{k-1}, f)}{P_j} \geq c_j,$$

so that $\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \infty$.

On the other hand, it follows from (1) that $f$ is locally R-integrable and $u(x) = 0$, so that $f$ is not $(l, d\mu)$-summable.

**Remark.** One may even prove that the hypotheses of theorem 2.4.2 imply the existence of a continuous function $\tilde{f}$ with the required properties. The basic idea of the construction of such a function is the following: the results of the foregoing proof as far as (4.5) remain essentially the same, when we replace each set $E_k \setminus E_{k-1}$ by its interior $(E_k \setminus E_{k-1})^o$ $(k = 1, 2, 3, \ldots)$. The corresponding function $\tilde{f}$ defined by (4.6) vanishes outside the points of a discrete sequence. We replace this function by a non-negative, continuous function $f$, which is integrable, assumes the same values as $f$ in the points mentioned above, and vanishes outside sufficiently small neighbourhoods of these points (each of these neighbourhoods being a subset of some $E_k \setminus E_{k-1}$).

Finally we shall exhibit an example which shows that the conclusion of theorem 2.5.2 may become false, if $(3.1^b)$ is violated.
THEOREM 2.4.3. Let \( \xi = \{x_n\}_{n=1}^{\infty} \) be a d\(\mu\)-uniformly distributed sequence. Then there exist a sequence \( \{E_k\}_{k=1}^{\infty} \), satisfying (3.1) and (3.1' a), for which the sequence \( \left\{ \frac{\mu(E_k \setminus E_{k+1})}{\mu(E_k)} \right\}_{k=1}^{\infty} \) is not bounded, and a locally B-integrable function \( f \), monotonically increasing with respect to \( \{E_k\}_{k=0}^{\infty} \), such that \( \mu(f) < \infty \) and
\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \infty.
\]

Proof. Let \( \{E_k\}_{k=0}^{\infty} \) have the properties (a), (b), (c) and (d) mentioned in the conclusion of theorem 2.2.1. We define \( E_k = F_k^{(k-1)^2} \) (\( k = 0, 1, 2, \ldots \)). Then \( \{E_k\}_{k=0}^{\infty} \) obviously satisfies (3.1) and (3.1' a).

As a consequence of theorem 2.2.1 we have \( \mu(E_k) = 1 - c^{k^2} \) (\( k = 0, 1, 2, \ldots \)), and the following asymptotic relations hold:
\[
(4.7) \quad \mu(E_k \setminus E_{k-1}) \sim c^{-(k-1)^2} \quad \text{(} k \to \infty \text{)}
\]
\[
(4.8) \quad \frac{\mu(E_k \setminus E_{k-1})}{\mu(E_{k+1} \setminus E_k)} \sim c^{2k^2} \quad \text{(} k \to \infty \text{)}.
\]
As a consequence of (4.8) the quotients \( \mu(E_k \setminus E_{k-1}) / \mu(E_{k+1} \setminus E_k) \) constitute an unbounded sequence of real numbers (\( k = 1, 2, 3, \ldots \)).

Let \( (d_k)_{k=1}^{\infty} \) be defined by
\[
(4.9) \quad d_k = c^{(k-1)^2} + k \quad \text{(} k = 0, 1, 2, \ldots \text{)}.
\]
Then \( \{d_k\}_{k=1}^{\infty} \) is a monotonically increasing sequence, and from (4.7) and (4.9) it follows that
\[
(4.10) \quad d_k \mu(E_k \setminus E_{k-1}) \sim c^k \quad \text{(} k \to \infty \text{)}.
\]
\[
(4.11) \quad d_k \mu(E_{k+1} \setminus E_k) \sim c^{1-k} \quad \text{(} k \to \infty \text{)}.
\]
Since \( \xi \) is \( \mu \)-uniformly distributed we have
\[
\lim_{N \to \infty} \frac{\mu(E_k \setminus E_{k-1}; N)}{N} = \mu(E_k \setminus E_{k-1}) \quad (k = 1, 2, 3, \ldots).
\]
Consequently for every \( k = 1, 2, 3, \ldots \) there exists an index \( N_k \), such that
\[
\frac{\mu(E_k \setminus E_{k-1}; N_k)}{N_k} \geq \frac{1}{2} \mu(E_k \setminus E_{k-1}) \quad (k = 1, 2, 3, \ldots).
\]
Finally \( f \) is defined as follows
\[
\begin{align*}
\text{if} \quad & x \in (E_k \setminus E_{k-1}) \cap \left( \bigcup_{n=1}^{N_k} \{x_n\} \right) \\
\text{then} \quad & f(x) = d_k \quad (k = 1, 2, 3, \ldots) \\
\text{if} \quad & x \in (E_k \setminus E_{k-1}) \setminus \left( \bigcup_{n=1}^{N_k} \{x_n\} \right) \\
\text{then} \quad & f(x) = d_{k-1} \quad (k = 1, 2, 3, \ldots).
\end{align*}
\]
Obviously \( f \) is locally \( \mu \)-integrable since for all \( x \in E_k \setminus E_{k-1} \) with a finite number of exceptions we have \( f(x) = d_{k-1} \quad (k = 1, 2, 3, \ldots) \). Moreover \( f \) is a monotonically increasing function with respect to \( \{E_k\}_{k=0}^{\infty} \) since
\[
d_{k-1} \leq f \leq d_k \quad (k = 1, 2, 3, \ldots).
\]
From (4.11) and (4.13) it follows that
\[
\mu(f) = \sum_{k=1}^{\infty} d_k \mu(E_k \setminus E_{k-1}) = \\
= \sum_{k=1}^{\infty} c \lambda(E_k \setminus E_{k-1}) < \infty.
\]
On the other hand, by (4.10), (4.12) and (4.13) we have
\[
\frac{1}{N_k} \sum_{n=1}^{N_k} f(x_n) = \frac{1}{N_k} \sum_{n=1}^{N_k} f(x_n) >
\]
\[
A \left( \mathbb{L}_k \left( \mathbb{R}_k \setminus K_{k-1} \right) \right) \cdot d_k \geq \frac{1}{2} c_k (\mathbb{R}_k \setminus K_{k-1}) \sim \frac{1}{2} c^k \quad (k \to \infty),
\]
so that
\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \infty
\]
and \( f \) is proved to be not \((\mathbb{L}_i, d_\mu)\)-summable.

Remark. As in the remark on theorem 2.4.2 one may even prove that the hypotheses of theorem 2.4.3 allow of the existence of a continuous function \( f \) with the same properties.
CHAPTER III

3.1. INTRODUCTION

In this Chapter we shall consider some special situations, where the theorems of Chapter II apply. In order to formulate our theorems in a more natural way we shall make use of the following lemma (cf. Baayen and Helmberg [1], lemma 3).

Lemma 3.1.1. Let \( \mu \) be a non-atomic measure and \( \xi = \{x_n\}_{n=1}^{\infty} \) an \( \mu \)-uniformly distributed sequence in \( X \). Let \( \xi' = \{x'_n\}_{n=1}^{\infty} \) be the sequence obtained from \( \xi \) by cancelling all elements equal to a fixed element \( y_0 \in X \). Then \( \xi' \) is \( \mu \)-uniformly distributed too. If \( f \) is a measurable integrable function, such that \( |f(y_0)| < \infty \), then

\[
\lim \sup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x'_n) = \lim \sup_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} f(x'_n),
\]

and

\[
\lim \inf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x'_n) = \lim \inf_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} f(x'_n) .
\]

Proof. For every open set \( E \) with compact closure and zero-boundary we have

\[
\Lambda(E; N) = \Lambda(E \setminus \{y_0\}; N) + \Lambda(E \cap \{y_0\}; N) =
\]

\[
= \Lambda(E \setminus \{y_0\}; N) + \Lambda(E \cap \{y_0\}; N) =
\]

\[
= \Lambda(E; N) - \Lambda(E \cap \{y_0\}; N) + \Lambda(E \cap \{y_0\}; N) =
\]

\[
= \Lambda(E; N) - \Lambda(E \setminus \{y_0\}; N) + \Lambda(E \setminus \{y_0\}; N),
\]
so that
\[
A(\xi, E; N) = \frac{A(\xi, E; N) - A(\xi, \{y_o\}; N)}{N - A(\xi, \{y_o\}; N)} \cdot \left(1 - \frac{A(\xi, \{y_o\}; N)}{N}\right) + \frac{A(\xi, E \cap \{y_o\}; N)}{N} \quad (N = 1, 2, 3, \ldots).
\]

If \(N \to \infty\) then \(A(\xi, E; N) \to \mu(E)\) and \(\frac{A(\xi, \{y_o\}; N)}{N} \to \mu(\{y_o\}) = 0\), so that \(\frac{A(\xi, E \cap \{y_o\}; N)}{N} \to 0\). Since \(M = N - A(\xi, \{y_o\}; N)\) runs through all natural numbers we get
\[
\lim_{M \to \infty} \frac{A(\xi, E; M)}{M} = \mu(E),
\]
so that \(\xi\) is \(\mu\)-uniformly distributed. Furthermore we have the identity (\(M\) being defined as above)
\[
\sum_{n=1}^{N} f(\xi_n) = \sum_{n=1}^{M} f(\xi_n) + A(\xi, \{y_o\}; N) f(y_o) \quad (N = 1, 2, 3, \ldots).
\]

It follows that
\[
\frac{1}{N} \sum_{n=1}^{N} f(\xi_n) = \frac{1}{M} \sum_{n=1}^{M} f(\xi_n) \cdot \left(1 - \frac{A(\xi, \{y_o\}; N)}{N}\right) + \frac{A(\xi, \{y_o\}; N)}{N} f(y_o) \quad (N = 1, 2, 3, \ldots),
\]
and the desired equalities are evident. \(\Box\).

In this Chapter we consider the closed unit-interval \(Y = [0, 1]\) with Lebesgue-measure \(\mu\). Let \(\theta = \{t_n\}_{n=1}^{\infty}\) be the sequence of all rational numbers \(\frac{k}{m} (1 \leq m < \infty, 1 \leq k < m)\), arranged in their lexicographic ordering, viz.
\[
(1.1) \quad \theta = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots\right\}.
\]
This sequence was introduced and proved to be $d_\mu$-uniformly distributed by von Neumann [13]. For reasons of completeness we shall give a proof of this statement here.

The index $n$ of $t_n = \frac{k}{k}$ in $\Theta$ turns out to be $\frac{1}{2}k(k+1)$. Now let $\alpha$ be any real number, $0 < \alpha \leq 1$. We count the number $\Lambda(\Theta, [0, \alpha]; N)$ of elements $t_n \in [0, \alpha]$, such that $1 \leq n < N = \frac{1}{2}k(k+1) + \ell$

$(0 \leq \ell \leq k)$, and obtain

$$(1.2) \quad \Lambda(\Theta, [0, \alpha]; N) = \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \ldots + \lfloor k\alpha \rfloor + \min(\mathcal{A}, [(k+1)\alpha]).$$

A simple bilateral estimate yields

$$(\alpha - 1) + (2\alpha - 1) + \ldots + (k\alpha - 1) < \Lambda(\Theta, [0, \alpha]; N) < \alpha + 2\alpha + \ldots + (k+1)\alpha,$$

so that

$$(1.3) \quad \frac{1}{2}k(k+1) - k < \Lambda(\Theta, [0, \alpha]; N) < \frac{1}{2}k(k+1)(k+2).$$

On the other hand $N$ satisfies the inequality

$$(1.4) \quad \frac{1}{2}k(k+1) < N < \frac{1}{2}(k+1)(k+2).$$

Combining (1.3) and (1.4) we have

$$\alpha(1 - \frac{2}{k+2}) - \frac{2k}{(k+1)(k+2)} < \frac{\Lambda(\Theta, [0, \alpha]; N)}{N} < \alpha(1 + \frac{2}{k}),$$

so that, if $N \to \infty$ (and consequently $k \to \infty$)

$$\lim_{N \to \infty} \frac{\Lambda(\Theta, [0, \alpha]; N)}{N} = \alpha,$$

which shows that $\Theta$ is $d_\mu$-uniformly distributed in $Y$.

By lemma 3.1.1, for every real number $\beta$ ($0 \leq \beta \leq 1$) the sequence $\tilde{\Theta}_\beta$, obtained by cancellation of all $t_n = \beta$, is also $d_\mu$-uniformly distributed. Notice that for all irrational $\beta$ this cancellation does not affect $\Theta$, so that $\tilde{\Theta}_\beta = \Theta$ in this case.
In order to apply the results of Chapter II to the sequence \( \tilde{\theta}_n \) and to functions on \([0,1]\), which are locally \( R \)-integrable on \([0,1] \setminus \{\beta\}\) (for instance functions \( f \), which are continuous on \([0,1] \setminus \{\beta\}\) and satisfy \( f(x) \to \infty \) as \( x \uparrow \beta \), \( x \downarrow \beta \), or \( x \to \beta \) respectively), we consider three cases:

1. \( X \) is disconnected and consists of the two connected components \([0,\beta]\) and \([\beta,1]\), each of these intervals in its natural topology. The one-point compactification \( X^* = X \cup \{\beta^*\} \) is also disconnected and consists of the connected components \([0,\beta]\) and \([\beta^*,1]\).

2. Similarly as in (1), \( X \) consists of the two connected components \([0,\beta[\) and \([\beta,1]\), and \( X^* = X \cup \{\beta^*\} = [0,\beta^-] \cup [\beta,1] \).

3. \( X \) is the subspace \([0,1] \setminus \{\beta\}\) of \([0,1]\) in its relative topology, \( X^* \) coincides with \([0,1]\).

In the following number-theoretical lemma the symbol \((p,q)\) denotes the greatest common divisor of the natural numbers \( p \) and \( q \).

Lemma 3.1.2. If \((p,q) = 1\), then \((pk+1,qk) = q\) for all \( k = 1,2,3,\ldots \)

Proof. Let \((pk+1,qk) = d\) and suppose \( d \mid q \). Then \( d \) contains a prime factor \( t \) which divides both \( k \) and \( pk+1 \), which is evidently impossible.

Lemma 3.1.3. If \( \beta = \frac{p}{q} \), \((p,q) = 1\), \( 0 < \beta < 1 \) and if the sequence of sets \( \{V_k\}_{k=0}^{\infty} \) is defined by \( V_0 = [0,1] \), \( V_k = [\beta, \beta + \frac{1}{(k+1)q}] \) \((k = 1,2,3,\ldots)\), then there exists a positive number \( C \) (depending on \( q \) only), such that

\[
\frac{A(\theta, V_{k-1} \setminus V_k; N)}{N\mu(V_{k-1} \setminus V_k)} < C \quad \text{for all } k = 1,2,3,\ldots, \quad N = 1,2,3,\ldots.
\]
Proof. The definition of $\mathcal{V}_k$ yields

$$\mathcal{V}_{k-1} \setminus \mathcal{V}_k = \left\{ \frac{p(k+1)+1}{q(k+1)}, \frac{pk+1}{qk} \right\} \quad (k = 2, 3, 4, \ldots).$$

We make use of the fact that $\Theta = \{t_k\}_{k=1}^{\infty}$ is lexicographically ordered to estimate $\frac{1}{N} A(\theta, v_{k-1} \setminus v_k, N)$ as follows:

If $t_N = \frac{L}{M}$ then $\frac{3}{2} M(M-1) < N < \frac{3}{2} M(M+1)$, and an upper bound for $A(\theta, \mathcal{V}_{k-1} \setminus \mathcal{V}_k, N)$ is given by the number $B(M)$ of all fractions $\frac{r}{s}$ such that $0 < r < s \leq M$ and $\frac{r}{s} \in \mathcal{V}_{k-1} \setminus \mathcal{V}_k$. The estimate of $B(M)$ is made as follows: Any of the numbers $\frac{r}{s}$ is made to correspond with the lattice point $(s, r)$ in a euclidean $(x, y)$-plane. $B(M)$ is equal to the number of lattice points within the triangle $\Delta_k(M)$ given by (cf. (1.5), see Fig. 5)

$$\Delta_k(M) : \begin{cases} x \leq M \\ \frac{p(k+1)+1}{q(k+1)} x \leq y \leq \frac{pk+1}{qk} x \end{cases}.$$
It is a well-known fact that two lattice points \((x_1, y_1), (x_2, y_2)\) form an integral basis for the set of all lattice points if and only if \(x_1 y_2 - x_2 y_1 = \pm 1\) (cf. Hardy and Wright [6] § 3). Let \(\mathcal{L}_0\) denote the half-line \(y = \frac{p(k+1)+1}{q(k+1)} - x (x > 0)\). The lattice points on \(\mathcal{L}_0\) are equidistant and the distance of their projections upon the \(x\)-axis is \((k+1)d^{-1}\), \(d\) being the g.c.d. of \(p(k+1)+1\) and \(q(k+1)\), so that by lemma 3.1.2 these projections have a mutual distance \(k+1\) at the least. On the other hand, since \((0,0)\) and \((q(k+1), p(k+1)+1)\) are both lattice points on \(\mathcal{L}_0\), the distance of the projections mentioned above is at most \(q(k+1)\), and we may write for the lattice points \(\{P_j\}_{j=0}^\infty\) on \(\mathcal{L}_0\) (cf. lemma 3.1.2)

\[
(1.7) \quad \begin{cases} 
  P_j = (jpq(k+1), jq(p(k+1)+1)) & (j = 0, 1, 2, \ldots) \\
  \text{where} \quad q^{-1} \leq d^{-1} \leq 1 
\end{cases}
\]

The lattice points in \(A_k(\mathcal{M})\) are arranged on equidistant lines \(\mathcal{L}_1, \mathcal{L}_2, \ldots\) parallel to \(\mathcal{L}_0\) and the mutual distance of the lattice points on each \(\mathcal{L}_1 (\lambda = 1, 2, 3, \ldots)\) equals \(|OP_1|\).

An upper bound for \(S(\mathcal{M})\) may be found when we assume that the points of intersection \(O_1\) of the line \(y = \frac{pk+1}{qk}x\) and the lines \(\mathcal{L}_1 (\lambda = 1, 2, 3, \ldots)\) are lattice points themselves (this need not really be the case). Since \(OP_1\) and \(OQ_1\) are then edges of a parallelogram of unit area and have the same orientation as the positive \(x\)- and \(y\)-axes, we write \(Q_1 = (pq, op(k+1))\) and conclude that

\[
(1.8) \quad \det(0P_1, OQ_1) = \begin{vmatrix} 
  pq(k+1) & op(p(k+1)+1) \\
  qk & op(k+1) 
\end{vmatrix} = +1.
\]

Elementary calculations show that \((1.8)\) yields \(pq \equiv 1\).

This equality together with \((1.7)\) yields
\( (1.9) \quad \begin{cases} Q_i = (\rho qk, \rho (pk + 1)) & (i = 1, 2, 3, \ldots), \\ \text{where} \quad q \to x < x < 1. \end{cases} \)

We shall now majorise \( B(M) \) by counting lattice points on \( L_1, L_2, \ldots \), respectively, and obtain the following results:

(i) \( B(M) = 0 \) if \( M < k \), since \( Q \) has an abscissa \( \rho qk \geq k \).

(ii) Note that on \( L_i \), there lie at most \( \left[ \frac{M - \rho qk}{\rho q(k+1)} \right] + 1 \) lattice points. Therefore

\[
B(M) \leq \left[ \frac{M - \rho qk}{\rho q(k+1)} \right] + 1 + \left[ \frac{M - 2\rho qk}{\rho q(k+1)} \right] + 1 + \left[ \frac{M - 3\rho qk}{\rho q(k+1)} \right] + 1,
\]

where \( t = \left[ \frac{M}{\rho qk} \right] \). Using the fact that \( a - 1 < [a] \leq a \) for all \( a \), we majorise this expression as follows

\[
B(M) \leq \left( \frac{M}{\rho q(k+1)} + 1 \right) \frac{M - \rho qk}{\rho q(k+1)} - \frac{\rho qk}{\rho q(k+1)} - \frac{1}{2} \frac{M}{\rho qk} \left( \frac{M}{\rho qk} - 1 \right)
\]

\[
= \frac{1}{2} \frac{M^2}{\rho qk^2(k+1)} + \frac{M}{\rho qk} + \frac{1}{2} \frac{N}{\rho q(k+1)}. \]

Since \( \rho q \gg 1 \) and \( pq \gg 1 \) (cf. (1.7), (1.9)) we obtain

\[
B(M) \leq \frac{1}{2} \frac{M^2}{k(k+1)} + \frac{3}{2} \frac{M}{k}
\]

We shall use this estimate in the case \( M \gg k \gg 2 \) (cf. (1.5), (ii)).

Then

\[
\frac{B(M)}{\mu(\mathcal{V}_{k-1} \setminus \mathcal{V}_k)} < \frac{\frac{1}{2} M^2 + 3M(k+1)}{k(k+1)} \cdot qk(k+1) \leq
\]

\[
< \frac{1}{2} q(M^2 + 3M(k+1)) = \frac{1}{2} qM(4M+1)
\]

Summing up the results sub (i) and (ii) we have

\[
(1.10) \quad \frac{B(M)}{\mu(\mathcal{V}_{k-1} \setminus \mathcal{V}_k)} \begin{cases} = 0 & (M < k) \\ < \frac{1}{2} qM(4M+1) & (M \geq k) \end{cases} \quad (k \geq 2).
\]
Our final intention is to give an estimate of the expression
\[
\frac{A(\theta, \bar{V}_{k-1} \setminus \bar{V}_k; N)}{N \mu(\bar{V}_{k-1} \setminus \bar{V}_k)}
\] (cf. the first lines of this proof).

Since \( \frac{3}{2} M(M-1) < N \leq \frac{3}{2} M(M+1) \) it follows from (1.10) that
\[
\frac{A(\theta, \bar{V}_{k-1} \setminus \bar{V}_k; N)}{N \mu(\bar{V}_{k-1} \setminus \bar{V}_k)} \begin{cases} = 0 & (k < N) \\ \leq \frac{3qM(M+1)}{M-1} = q \frac{4M+3}{M-1} & (k \geq N) \end{cases}
\] for \( k \geq 2 \).

The expression \( \frac{4M+3}{M-1} \) attains its maximum value \((M \geq 2)\) if \( M = 2 \), and this value equals \( 11 \). Hence
\[
(1.11) \quad \frac{A(\theta, \bar{V}_{k-1} \setminus \bar{V}_k; N)}{N \mu(\bar{V}_{k-1} \setminus \bar{V}_k)} \leq 11q \quad (k \geq 2, N \geq 1),
\]
and we only have to investigate the case \( k = 1 \):

Now \( \frac{A(\theta, \bar{V}_{k-1} \setminus \bar{V}_k; N)}{N} \geq 1 \), and \( \mu(\bar{V}_0 \setminus \bar{V}_1) = 1 - \frac{1}{2q} = \frac{2q-1}{2q} \geq \frac{1}{2q} \),
so that (1.11) is valid in this case too, which proves lemma 3.1.3.

In the same way we may prove lemma 3.1.3\( ^a \):

**Lemma 3.1.3\( ^a \).** If \( \beta = \frac{p}{q}, (p,q) = 1, 0 < \beta < 1 \) and if the sequence of sets \( \{W_k\}_{k=0}^\infty \) is defined by \( W_0 = [0,1] \),
\[ W_k = \left[ \beta \frac{1}{(k+1)q}, \beta \right] (k = 1,2,3,\ldots) \) then there exists a positive number \( C \) (depending on \( q \) only), such that
\[
\frac{A(\theta, \bar{W}_{k-1} \setminus \bar{W}_k; N)}{\mu(\bar{W}_{k-1} \setminus \bar{W}_k)} \leq C \text{ for all } k = 1,2,3,\ldots, N = 1,2,3,\ldots.
\]

Now let \( \tilde{\theta}_\beta \) be the sequence obtained from \( \theta = \{t_n\}_{n=1}^\infty \) by cancellation of the number \( \beta \) every time it occurs in \( \theta \). In particular

we shall study the behaviour of the expression
\[
\frac{A(\beta; \overline{U}_{k-1} \setminus \overline{U}_k; N)}{\bar{N}_k(\overline{U}_{k-1} \setminus \overline{U}_k)}, \quad \text{where } \{\overline{U}_k\}_{k=0}^\infty \text{ is written for } \{V_k\}_{k=0}^\infty \text{ and } \\
\{\overline{U}_k\}_{k=0}^\infty, \text{ these sequences of sets being defined in lemma 3.1.3} \\
\text{and lemma 3.1.3a, respectively. Splitting up } \theta \text{ into finite sections, each of which corresponds with one and the same denominator, viz.} \\
\theta = \{ \frac{1}{n}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots \}, \\
\text{so that the } n^{th} \text{ section consists of the elements } \frac{1}{n}, \ldots, \frac{n}{n}; \text{ we see that each section contains the given number } \beta \text{ only once at most. Moreover, none of the given intervals } \overline{U}_{k-1} \setminus \overline{U}_k \text{ contain } \beta. \\
\text{Therefore, if } D(N) \text{ denotes the number of elements } \tau_m (1 \leq m \leq N) \text{ such that } \tau_m = \beta, \text{ we have} \\
(1.12) \quad \frac{A(\beta; \overline{U}_{k-1} \setminus \overline{U}_k; N)}{\bar{N}_k(\overline{U}_{k-1} \setminus \overline{U}_k)} = \frac{A(\beta; \overline{U}_{k-1} \setminus \overline{U}_k; N-D(N))}{(N-D(N)) \bar{N}_k(\overline{U}_{k-1} \setminus \overline{U}_k)} \cdot \frac{N-D(N)}{N}. \\
\text{Our intention is to obtain an upper bound for the first factor in the right hand member of (1.12). Since an upper bound for the left hand member of (1.12) is given by lemma 3.1.3 and lemma 3.1.3a it suffices to find a lower bound for the factor } \frac{N-D(N)}{N}. \\
\text{Therefore we make a lower estimate of the factor } \frac{N-D(N)}{N} \text{ for } \\
D(N) < N: \text{ if } \frac{1}{2}M(M-1) < N \leq \frac{1}{2}M(M+1) \text{ then } 0 < D(N) \leq M, \text{ so that} \\
D(N) \leq \frac{M}{2}M(M-1) = \frac{2}{N-1} < \frac{2}{3}, \text{ provided that } M \geq 4. \text{ We only have to majorise } \frac{D(N)}{N} \text{ for } D(N) < N \text{ and } M \leq 5 (N \leq 6). \text{ Obviously, in} \\
\text{this case we obtain } \frac{D(N)}{N} \leq \frac{5}{6}. \text{ Hence } \frac{N-D(N)}{N} \geq \frac{1}{6} \text{ and we have proved the following lemmas.} \\
\text{Lemma 3.1.4. Under the hypotheses of lemma 3.1.3 there exists a} \\
\text{positive number } C \text{ (depending on } q \text{ only), such that}
\[
\frac{A(y, \overline{V}_{k-1} \setminus \overline{V}_k ; k)}{N(y, \overline{V}_{k-1} \setminus \overline{V}_k)} < C \quad (k \geq 1, \ N \geq 1).
\]

Lemma 3.1.4. Under the hypotheses of lemma 3.1.3, there exists a positive number \(C\) (depending on \(q\) only), such that
\[
\frac{A(y, \overline{W}_{k-1} \setminus \overline{W}_k ; N)}{N(y, \overline{W}_{k-1} \setminus \overline{W}_k)} < C \quad (k \geq 1, \ N \geq 1).
\]

Remark. Since in none of the proofs the particular arrangement of the rationals with the same denominator in their natural order of magnitude is used, the case lemmas 3.1.3, 3.1.5, 3.1.4 and 3.1.4 hold when we permute the rational fractions with equal denominators in the underlying sequence \(\theta\).

Straightforward calculations show that the sequences
\[
\{ \mu(\overline{V}_{k-1} \setminus \overline{V}_k) / \mu(\overline{V}_k \setminus \overline{V}_{k+1}) \}_{k=1}^{\infty} \quad \text{and} \quad \{ \mu(\overline{W}_{k-1} \setminus \overline{W}_k) / \mu(\overline{W}_k \setminus \overline{W}_{k+1}) \}_{k=1}^{\infty},
\]
where \(\{Y_k\}_{k=0}^{\infty}\) and \(\{W_k\}_{k=0}^{\infty}\) are the intervals defined in lemma 3.1.3 and lemma 3.1.5, are bounded sequences. With respect to the cases (1), (2) and (3) mentioned before lemma 3.1.2 we now introduce the sequences \(\{E_k\}_{k=0}^{\infty}\) which, in view of the arguments above, will easily be checked to satisfy the conditions (2.3.1), (2.3.15) and (2.3.1b):

\[
(1') \quad E_0 = X = [0, \beta] \cup [\beta, 1]; \quad E_k = [0, \beta] \cup [\beta, 1] = [0, \beta] \cup (k+1)^{-1} (k+1)^{-1}, 1 = X \setminus \overline{V}_k \quad (k = 1, 2, 3, \ldots), \text{if } X \text{ is defined as in (1)};
\]

\[
(2') \quad E_0 = X = [0, \beta] \cup [\beta, 1]; \quad E_k = [0, \beta] \cup [\beta, 1] = X \setminus \overline{W}_k \quad (k = 1, 2, 3, \ldots), \text{if } X \text{ is defined as in (2)};
\]

\[
(3') \quad E_0 = X = [0, \beta] \cup [\beta, 1];
\]
\[ E_k = \left[ 0, \beta - \frac{1}{(k+1)q} \right] U \left[ \beta + \frac{1}{(k+1)q}, 1 \right] \]

\[ = x \setminus (\overline{W}_k U \overline{V}_k) \quad (k=1,2,3,\ldots), \]

if \( x \) is defined as in (3).

### 3.2. APPLICATIONS OF \((\mathfrak{G}, d_\mu)\)-SUMMABILITY

Since the sequences \( \{E_k\}_{k=0}^\infty \) defined sub (1'), (2') and (3') at the end of Section 3.1 satisfy the conditions (2.3.1), (2.3.1') and (2.3.1'') in the appropriate spaces \( X \), we may now reformulate the theorems 2.3.1 and 2.3.2 using these special systems of neighbourhoods; this leads to the theorems 3.2.1 and 3.2.2.

These results may be interpreted in terms of \( \mathfrak{G}_\beta \), but also, by means of lemma 3.1.1, in terms of the underlying sequence \( \{t_n\}_{n=1}^\infty \), provided the function \( f \) which is considered attains a finite value at \( \beta \).

**THEOREM 3.2.1.** Let \( \beta = \frac{p}{q} \) be a rational number, \( 0 \leq \beta \leq 1 \). If \( f \) is a locally \( R \)-integrable function on \([0,1]\) such that

\[ \sum_{k=1}^{\infty} \frac{\|f\|_k}{k^2} < \infty, \]

where \( \|f\|_k \) is defined by

\[ \|f\|_k = \text{Sup} \{ |f(x)| \mid \frac{1}{q(k+1)} < |x - \beta| < \frac{1}{qk} \} \quad (k=1,2,3,\ldots), \]

and if \( f(x) \) is bounded for \( |x - \beta| > \frac{1}{q} \), then \( f \) is \((\mathfrak{G}_\beta, d_\mu)\)-summable. If, moreover, \( |f(\beta)| < \infty \), then \( f \) is \((\mathfrak{G}, d_\mu)\)-summable.

Since the sets \( W_k \) \((k=1,2,3,\ldots)\) and \( V_k \) \((k=1,2,3,\ldots)\) defined in lemma 3.1.3 and lemma 3.1.3' respectively are intervals, a function \( f \) which vanishes for all \( x \in [0,\beta] \) and which tends mo-
notonically to \( \infty \) as \( x \downarrow \beta \), is a monotonically increasing function with respect to \( \{ x_k \}_{k=0}^{\infty} \) (cf. (1')). We may consider in the same way any function that vanishes for all \( x \in [\beta, 1] \) and tends monotonically to infinity as \( x \downarrow \beta \) (cf. (2')). Combination of these arguments yields (cf. theorem 2.3.2)

**Theorem 3.2.2.** Let \( \beta = \frac{p}{q} \) be a rational number, \( 0 < \beta < 1 \). If \( f \) is locally \( \mu \)-integrable on \( [0,1] \), \( \mu(f) < \infty \), and if \( f(x) \) tends monotonically to \( \infty \) as \( x \downarrow \beta \) in some interval \([a, \gamma]\) containing \( \beta \), \( f(x) \) being bounded outside of this interval, then \( f \) is \((\mathcal{C}_S, \mu)\)-summable. If, moreover, \( |f(\beta)| < \infty \), then \( f \) is \((\mathcal{S}, \mu)\)-summable.

The \((\mathcal{S}, \mu)\)-summability in theorem 3.2.2 may also be proved directly, i.e. without making use of the results proved in Chapter II. We shall now sketch the proof, which is based upon two lemmas.

**Lemma 3.2.1.** If \( \frac{p}{q} \) and \( \frac{t}{s} \) are two distinct rational numbers, then

\[
|\frac{p}{q} - \frac{t}{s}| > \frac{1}{qs}.
\]

**Proof.** Trivial.

**Lemma 3.2.2.** Let \( f \) be a non-negative, integrable function, and suppose that \( f(x) \) is monotonically increasing, both as \( x \uparrow \beta \) and as \( x \downarrow \beta \). Then

\[
\lim_{x \rightarrow \beta} (x - \beta)f(x) = 0.
\]

**Proof.** Let us assume the contrary, so that there exists a positive number \( \rho \) and a sequence \( \{ x_i \}_{i=1}^{\infty} \) such that \( \lim_{i \rightarrow \infty} x_i = \beta \) and

\[
x_i - \beta |f(x_i)| > \rho \quad (i = 1, 2, 3, \ldots).
\]

Without loss of generality we may assume that \( |x_{i+1} - \beta| < \frac{1}{2} |x_i - \beta| \quad (i = 1, 2, 3, \ldots) \), and that
all $x_i - \beta$ have the same, say, positive sign. Then

$$
\mu(r) \geq \sum_{i=1}^{\infty} (x_i - x_{i+1} + f(x_i) > \frac{1}{\varepsilon} \sum_{i=1}^{\infty} (x_i - \beta) f(x_i) = \infty,
$$
in contradiction with the integrability of $f$.

Direct proof of the $(\theta, q, \mu)$-summability in theorem 3.2.2. Without loss of generality we assume that $f > 0$, $f(\beta) = 0$ and that $f(x)$ is monotonically increasing both as $x \uparrow \beta$ and as $x \downarrow \beta$. Then for fixed integers $s$ and $t$ $(1 \leq s \leq t)$ we have

$$
\sum_{x=t}^{s} f(x) \leq \sum_{x=s}^{t} f(x) = s \sum_{i=1}^{s} \frac{1}{s} f(x) =
$$

$$
\leq s \int_{0}^{\beta} f(x) \mu(x) + f(x_1) + s \int_{\beta}^{1} f(x) \mu(x) + f(x_2),
$$

where $x_1 = \max \{ \frac{t}{s} \mid \frac{t}{s} < \beta \}$ and $x_2 = \min \{ \frac{t}{s} \mid \frac{t}{s} > \beta \}$.

In view of lemma 3.2.1 and the monotonicity of $f$ we obtain

$$
\sum_{x=t}^{s} f(x) \leq s \int_{0}^{1} f(x) \mu(x) + f(\beta - \frac{1}{qs}) + f(\beta + \frac{1}{qs}),
$$

which reduces to (cf. lemma 3.2.2)

$$
(2.1) \quad \sum_{x=t}^{s} f(x) \leq s(\mu(f) + \varphi(s))
$$

where $\varphi(s) \to 0$ for $s \to \infty$.

Now suppose $\frac{1}{2}M(M-1) < K \leq \frac{1}{2}M(M+1)$. From the lexicographic ordering of $\theta$ and from (2.1) it follows that
\[
\frac{1}{N} \sum_{n=1}^{N} f(t_n) = \frac{1}{N} \sum_{s=1}^{N} \sum_{n=1}^{S} f(s) + \frac{1}{N} \sum_{n=1}^{N} \sum_{x=1}^{M-\frac{1}{2}M(M-1)} f(x) \leq
\]
\[
\leq \frac{1}{N} \sum_{s=1}^{N} \sum_{n=1}^{S} f(s) \leq \frac{1}{N} \sum_{s=1}^{N} \mu(x) + \varphi(s)) <
\]
\[
< \frac{2}{M(M-1)} \frac{M(M+1)}{2} \mu(x) + o(1) \quad (\text{as } M \to \infty),
\]
and therefore \( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(t_n) \leq \mu(x) \),

so that \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(t_n) = \mu(x) \) \( \quad \text{Q.E.D.} \)

While theorem 3.2.2 could have been proved without reference to lemma 3.1.3 and lemma 3.1.5, these lemmas are illustrative for two reasons.

(a) They provide a common basis for both theorem 3.2.1 and theorem 3.2.2.

(b) They demonstrate that the neighbourhoods of \( \infty \), \( X \setminus \mathbb{E} \), and \( X \setminus \mathbb{R} \), the existence of which has been proved in Chapter II, may be chosen to be intervals containing \( \beta \) (which represents \( \infty \) in the cases of the spaces \( X \) under consideration), if \( \beta \) is rational. This is not necessarily the case if \( \beta \) is irrational, as will be demonstrated in the sequel.

The following theorem is due to Khintchine [11] ((cf. Hardy and Wright [6], theorem 199):

**Theorem.** If \( \psi(\beta) \) is an increasing function and \( \sum_{s=1}^{\infty} \frac{1}{\psi(s)} = \infty \), then, for almost all irrational \( \beta \), the inequality \( |x - \beta| < \frac{1}{\psi(x)} \)

has infinitely many solutions \( x, s \) being integers.
Substituting \( s \log s \) for \( \psi(s) \) in this theorem, we see that for almost all irrational \( \beta \) there are infinitely many rational numbers \( \frac{r_k}{s_k} \) such that

\[
(2.2) \quad \left| \frac{r_k}{s_k} - \beta \right| < \frac{1}{s^2 \log s} .
\]

Let \( \beta \) be such an irrational number \((0 < \beta < 1)\) and let \( \left\{ \frac{r_k}{s_k} \right\}_{k=1}^{\infty} \) be a sequence of rational numbers satisfying (2.2), and let

\[
\begin{align*}
& s_1 < s_2 < s_3 < \cdots \\
& s_k > \exp(s^k) \quad (k=1,2,3,\ldots).
\end{align*}
\]

We define a function \( f(x) \) in the following way:

\[
(2.3) \quad
\begin{cases}
  f(x) = 0 & \text{if } |x - \beta| > \frac{1}{s^2 \log s} , \\
  f(x) = s_k^2 & \text{if } \frac{1}{s^2 \log s_{k+1}} < |x - \beta| \leq \frac{1}{s_k^2 \log s_k} \\
               & (k=1,2,3,\ldots).
\end{cases}
\]

Then \( f(x) \) tends monotonically to infinity as \( x \to \beta \) (both \( x \uparrow \beta \) and \( x \downarrow \beta \)) and \( f \) is locally \( s \)-integrable. Moreover we have

\[
\mu(f) \leq 2 \sum_{k=1}^{\infty} \frac{s_k^2}{s_k^2 \log s_k} < 2 \sum_{k=1}^{\infty} e^{-k} = \frac{2}{e-1} .
\]

On the other hand, it follows from (2.2) and (2.3) that \( \frac{r_k}{s_k} \approx s_k^2 \), so that, if \( N = \frac{1}{2} s_k(s_k + 1) \) (note that the original sequence \( \theta \) does not contain the irrational number \( \beta \)), we obtain

\[
\frac{1}{N} \sum_{n=1}^{N} f(n) \approx \frac{2}{s_k(s_k + 1)} f \left( \frac{r_k}{s_k} \right) \approx \frac{2s_k^2}{s_k(s_k + 1)} .
\]
Since the last member of this inequality tends to 2 as \( s_k \to \infty \) we conclude

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(t_n) \geq 2 \geq \frac{2}{e-1} > f(t)
\]

and \( f \) is proved to be not \((\theta, \delta_\mu)\)-summable.

In view of theorem 2.3.2 we have therefore proved the following theorem:

**Theorem 2.3.3.** For almost all irrational \( \beta \) \((0 < \beta < 1)\) there exists an integrable non-negative function \( f \) on \([0,1]\) such that \( f(x) \) tends monotonically to \( \infty \) as \( x \uparrow \beta \) and as \( x \downarrow \beta \), and such that \( f \) is not \((0, \delta_\mu)\)-summable. Hence for almost all irrational \( \beta \) there does not exist a contracting sequence of closed intervals \( \{ I_k \}_{k=0}^\infty \) around \( \beta \) such that both

\[
\left\{ \frac{A(\theta, u_{k-1}, u_k)}{N_k(u_{k-1}, u_k)} \right\}_{k=0}^\infty \quad \text{and} \quad \left\{ \frac{\mu(u_{k-1}, u_k)}{\mu(u_k)} \right\}_{k=1}^\infty
\]

are bounded sequences.

The same result of Khintchine also allows of deducing a similar result for the behaviour of the sequence \( \eta_\beta = \{ \eta_n \}_{n=1}^\infty \) defined by \( \eta_n = n\delta \pmod{1} \) \((n=1, 2, 3, \ldots)\), where \( \beta \) is an irrational number.

Weyl [14] proved that \( \eta_\beta \) is \( \delta_\mu \)-uniformly distributed on \( Y = [0,1] \) for every irrational number \( \beta \).

Applying the result obtained by Khintchine mentioned above ([14], cf. Hardy and Wright [6], theorem 199), for almost all irrational \( \beta \) there exist infinitely many rationals \( \frac{r}{s} \) satisfying (2.2), i.e.

\[
\left| \frac{r}{s} - \beta \right| < \frac{1}{s^2 \log s}
\]
which may be reduced to
\[ |x - \beta s| < \frac{1}{n \log s} \quad . \]

Putting \( s = n - d \), \( d \) fixed, we get
\[ |x + \beta d - \beta n| < \frac{1}{(n - d) \log(n - d)} \quad , \]

or, reduced mod 1,
\[ (2.4) \quad |n\beta - dp| < \frac{1}{(n - d) \log(n - d)} \quad (\text{mod} 1) \quad . \]

Now let us assume that \( \{n_k\}_{k=1}^{\infty} \) is a sequence of positive integers
\( n_k \), each of which satisfies (2.4) and such that
\[
\begin{align*}
&\{ n_1 < n_2 < n_3 < \cdots \\
&\text{and } n_k - d > \exp(s^{2k}) \quad \text{(mod 1)}
\end{align*}
\]

Then we define a function \( f \), such that \( f(x) \) monotonically increases as \( x \to \beta d \) (from both sides), as follows:

\[ (2.5) \quad \begin{cases}
  f(x) = 0 & \text{if } |x - \beta d| > \frac{1}{(n_1 - d) \log(n_1 - d)} \\
  f(x) = n_k - d & \text{if } \frac{1}{(n_k - d) \log(n_k - d)} < |x - \beta d| < \frac{1}{(n_{k+1} - d) \log(n_{k+1} - d)}
\end{cases} \]

From (2.5) it follows that
\[ \mu(f) \leq \sum_{k=1}^{\infty} \frac{2}{n_k} \leq \sum_{k=1}^{\infty} \frac{2}{s_k} = \sum_{k=1}^{\infty} \frac{2}{s_k - 1} < 1 \quad . \]

On the other hand the average value of \( f \) on \( \eta_\beta \) satisfies
\[ \mathbb{E}[f] = \frac{1}{n_k} f(\beta n_k \mod 1) > \frac{1}{n_k} (n_k - d) \text{, which tends to 1 as } k \to \infty \quad . \]

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Hence \( \mu(f) < \limsup_{N \to \infty} \overline{f}[N] \), so that \( f \) is integrable, locally \( \mathbb{R} \)-integrable, but not \( (\eta, d\mu) \)-summable. In view of theorem 2.3.3 we may conclude

**Theorem 3.2.4.** For almost all irrational \( \beta \) and for every integer \( d \) there exists an integrable non-negative function \( f \) such that \( f(x) \) tends monotonically to \( \infty \) as \( x \to \beta d \mod 1 \), and such that \( f \) is not \( (\eta, d\mu) \)-summable. Hence for almost all irrational \( \beta \) there does not exist a contracting sequence of closed intervals \( \{U_k\}_{k=0}^\infty \) around \( \beta d \mod 1 \) (\( d \) any integer), such that both

\[
\left\{ \frac{A(\eta, f \vert U_{k+1} \setminus U_k)}{\mu(U_{k+1} \setminus U_k)} \right\}_{k=N}^\infty \quad \text{and} \quad \left\{ \frac{\mu(U_{k+1} \setminus U_k)}{\mu(U_{k+1} \setminus U_k)} \right\}_{k=N}^\infty
\]

are bounded sequences.
REFERENCES


SAMENVATTING

Als uitgangspunt van de theorie der gelijkverdeelde rijen kan worden beschouwd het artikel van Weyl [14] over gelijkverdeling op het eenheidsinterval. Eén van de resultaten van deze theorie is het feit dat de integraal van een willekeurige Riemann-integereerbare (dus begrensd) functie over het eenheidsinterval gelijk is aan de limiet van het functiewaardengemiddelde over de punten van een gelijkverdeelde rij. Dit resultaat is gegeeneraliseerd voor gelijkverdeelde rijen in compacte en locaal-compacte genormeerde ruimten (Nalwa [3], [9]; Helberg [7]).

Onbegrensd continue functies op een locaal-compacte ruimte hebben de bovengenoemde eigenschap in het algemeen niet. Helberg [7] gaf hiervan voorbeelden en bewees dat voor niet-negatieve continue functies de integraal niet groter is dan de limiet inferior van het functiewaardengemiddelde over een gelijkverdeelde rij.

In dit proefschrift wordt bewezen dat in locaal-compacte, niet-compacte ruimten bij elke gelijkverdeelde rij zonder repetities een continue niet-negatieve functie bestaat met voorgeschreven integraal en voorgeschreven limieten superior en limieten inferior van het functiewaardengemiddelde (Chapter I).

Verder wordt bij een willekeurige gelijkverdeelde rij een stelsel open verzamelingen geconstrueerd met bijzondere regulariteits-eigenschappen ten opzichte van die rij. Met behulp van dit stelsel worden classes van onbegrensd continue functies beschreven, waarvoor integraal en asymptotisch functiewaardengemiddelde wél overeenstemmen (Chapter II).

Tenslotte worden de bereikte resultaten toegepast op enkele bekende gelijkverdeelde rijen op het eenheidsinterval (Chapter III).
CURRICULUM VITAE

De samensteller van dit proefschrift werd geboren op 11 februari 1933 te Haarlem.
Van 1955 tot 1958 was hij assistent aan de Leidse Universiteit. Na gedurende een jaar leraar te zijn geweest bij het middelbaar onderwijs is hij sedert augustus 1959 als wetenschappelijk medewerker verbonden aan de Technische Hogeschool te Eindhoven.
STELLINGEN

I.

Een begrensd, Lebesgue-integreerbare functie $f$ op het interval $[0,1]$ is dan en slechts dan Riemann-integreerbaar, als voor elke gelijktijdig gelijke verzameling $\{x_n\}_{n=1}^{\infty}$ op $[0,1]$ geldt

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1]} f(x) \, dx,$$

(Probleem, gesteld door M.C. de Bruijn)

II.

De meetbare, loocaal begrensd, functies $f$, gedefinieerd voor $x > 0$, die voldoen aan de integraalvergelijking

$$f(x) = \int_{x-1}^{x} k(x,y)f(y) \, dy \quad (x \geq 1),$$

waarbij $k = k(x,y)$ een gegeven, meetbare, over elke eindige rechte hoek kwadratisch integreerbare functie is, zijn door hun restrictions op het interval $[0,1]$ eenduidig bepaald.

III.

De enige loocaal-integreerbare functies $f$, die voldoen aan de integraalvergelijking

$$f(\frac{x}{a}) = \frac{1}{x} \int_0^x f(t)dt \quad (x > 0, \ a \ constant, \ 1 \leq a < e),$$

en waarvoor $f(x) = O(1) (x \to \infty)$, zijn constanten $(x > 0)$.

Voor $a > e$ bestaan er ook niet-constante oplossingen $f$ van deze integraalvergelijking met de eigenschap $f(x) = O(1) (x \to \infty)$.

(n.a.v. probleem, gesteld door W.A.J. Luxemburg)

IV.

Voor elke loocaal-integreerbare functie $f$, die voldoet aan de integraalvergelijking

$$f(\frac{x}{a}) = \frac{1}{x} \int_0^x f(t)dt \quad (x > 0, \ a \ constant, \ 1 \leq a < e)$$

bestaat $\lim f(x)$ en geldt

$$\lim_{x \to 0} f(x) = \frac{f(1) - \int_1^x f(t)dt}{1 - \log a}.$$

(n.a.v. probleem, gesteld door W.A.J. Luxemburg)

V.

Een convex, begrensd, gesloten veelvlak in $\mathbb{R}^3$ is dan en slechts dan een gelijkzijdig viervlak, als het een oneindig lange goedeet zonder dubbelpunten bevat.
VI.

In $\mathbb{R}^n$ bestaat het inwendige van twee overstaande hoeken (de zgn. "double-wedge"), gevormd door het hypervlakkenpaar $(y, x) = 0$, $(v, x) = 0$, uit de vectoren $x$ waarvoor geldt $(y, x) \cdot (v, x) < 0$. De openingshoek van deze double-wedge bedraagt $\arccos \left( \frac{(y, v)}{(y, x) \cdot (v, x)} \right)$. 

We beschouwen in $\mathbb{R}^n$ n-tallen double-wedges met gelijke openingshoek, die samen $\mathbb{R}^n$ opvullen (op de vector $0$ na). Een dergelijk n-tal noemen we "compleet". Dan geldt

(a) In $\mathbb{R}^3$ bestaat er bij elke $\delta > 0$ een compleet drietal double-wedges met openingshoek $< \frac{\pi}{3} + \delta$.

(b) Bij iedere $\epsilon > 0$ bestaat er voor alle voldoend grote $n$ een compleet n-tal double-wedges in $\mathbb{R}^n$ met openingshoek $< \epsilon$.

(n.a.v. probleem, gesteld door D. Gale)

VII.

Mazur stelde het volgende probleem:

"In the three-dimensional Euclidean space there is given a convex surface $W$ and a point $O$ in its interior. Consider the set $V$ of all points $P$ defined by: the length of the interval $OP$ is equal to the area of the plane section of $W$ through $O$ and perpendicular to $PO$. Is the set $V$ convex?"

De door Mazur in dit probleem gestelde vraag is door een elementair voorbeeld in negatieve zin te beantwoorden.


VIII.

De beide uitspraken, vervat in een lemma van Kolchin en Sinai (zie Jacobs, Lecture Notes on Ergodic Theory) kunnen worden ge-
combinerend en uitgebreid tot de volgende algemene bewering:
Voor willekeurige deel-$\sigma$-algebra $R_0$ en $R_1$ van een gegeven $\sigma$-algebra $R$ geldt:
Al $\mathbb{H}(R_0 \mid (R_0 \cup R_1)(t,\infty)) < \infty$ en $\mathbb{H}(R_0 \cup R_1 \mid R_1(t,\infty)) < \infty$, dan is
$$\lim_{t \to \infty} \frac{1}{t} \mathbb{H}(R_0(0,t-1) \mid R_1(t,\infty)) = \mathbb{H}(R_0 \mid R_1(t,\infty)).$$


IX.

De wijze, waarop in het Handboek der Wiskunde de vergelijking
$\Delta \varphi + \lambda \varphi = 0$ voor de trillingsvorm $\varphi$ van een membraan bij de laagste positieve eigenwaarde $\lambda$ wordt afgeleid uit het variatieprincipe van Rayleigh, is onjuist.


X.

De stelling, genoemd in M.A. Naimark, Normed Rings, § 15.2.1 is onder de gegeven vooronderstellingen fout. Wel geldt deze stelling in een normale topologische ruimte, maar ook in dit geval is het door Naimark gegeven bewijs fout, evenals het door Köthe en Tillmann geredigeerde bewijs in de Duitse editie van genoemd boek.

XI.

Woordelijk geschreven taal werkt vaak remmend bij het definitieën van mathematische begrippen en bij de toepassing er van. In dit verband is het wenselijk ook toekomstige ingenieurs in niet-wiskundige studierichtingen gedurende hun eerste studiejaar vertrouwd te maken met symbolen uit de propositiologica, zoals kwantoren en het negatiesymbool, bij wijze van verkorte notatie.

XII.

De uitvoering van barokmuziek wint aan expressie, wanneer, tenzij door de componist uitdrukkelijk anders is aangegeven, maat en maatvoedeling als articulatie-unidades worden geïnterpreteerd. Deze wijze van uitvoeren wordt door historische bronnen ondersteund.

XIII.

Voor een doeltreffende geluidsanalyse van bestaande orgelpijpen en de ontwikkeling van nieuwe pijputsen is een verdergang van de theorie der geluidssverstrijken bij orgelpijpen gewenst. Gezien de gegevens, verkregen met de tot heden gebruikte meetapparaat voor deze verschijnselen, moet men betwijfelen of er ooit een volwaardig electronisch equivalent van het bestaande pijnpool zal kunnen worden ontwikkeld.
Stellingen behorende bij het proefschrift van
K.A. Post

Eindhoven, 20 juni 1967