GENERAL GROUP EXTENSIONS

PROEFSCHRIFT

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Index of notation

\[ \emptyset \] empty set
\{a,b,\ldots\} set whose members are a,b...
\{x|P\} set of all x such that P is true
\mathbb{N} set of natural numbers
\mathbb{Z} integers
\mathbb{Q} rational numbers
\mathbb{R} real numbers
A\times B cartesian product
\circ(A) cardinal number of A
\setminus set difference
\langle a,b \rangle greatest common divisor of a and b
a|b a divides b
iff if and only if
\langle 1-1 \rangle one to one
\phi:S\rightarrow T \phi is a function from S into T
\phi(a) the image of a under the function \phi
\phi(x) the image of x under the function \phi
\phi\circ(a,b) the product of a and b
\phi|D function \phi restricted to D
\phi|D:D\rightarrow K if \phi:S\rightarrow T and DCS, KCT then \phi|D:D\rightarrow K is \phi|D with range restricted to K
K^\phi if \phi:S\rightarrow T and KCT then K^\phi = \{a|a\in S, a\in K\}
\langle \Gamma, e \rangle neargroup consisting of a set \Gamma and a binary operation e
G_a subgroup of the permutation group G fixing the letter a
G/H quotient group
[G:H] index of K in G
<table>
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<th>Symbol</th>
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<tr>
<td>$G'$</td>
<td>commutator subgroup of $G$</td>
</tr>
<tr>
<td>$(S)$</td>
<td>subgroup generated by $S$</td>
</tr>
<tr>
<td>$S_{\Gamma}$</td>
<td>symmetric group on a set $\Gamma$ also if $\Gamma$ is infinite</td>
</tr>
<tr>
<td>Core($A$)</td>
<td>greatest normal subgroup contained in $A$</td>
</tr>
<tr>
<td>Ker($a$)</td>
<td>kernel of $a$</td>
</tr>
<tr>
<td>$N_G(A)$</td>
<td>normalizer of $A$ in $G$</td>
</tr>
<tr>
<td>$C_G(A)$</td>
<td>centralizer of $A$ in $G$</td>
</tr>
<tr>
<td>Aut($G$)</td>
<td>group of automorphisms of $G$</td>
</tr>
<tr>
<td>Inn($G$)</td>
<td>group of innerautomorphisms of $G$</td>
</tr>
<tr>
<td>Lat($N,G$)</td>
<td>lattice of all subgroups of $G$ containing $N$</td>
</tr>
<tr>
<td>Ch($g$)</td>
<td>character of a permutation $g = c((a</td>
</tr>
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One

Introduction and summary

1.1. If $G$ is a group and $K$ a normal subgroup of $G$, a uniquely determined quotient group $Q$ corresponds to $G$ and $K$. It may be asked if from the groups $K$ and $Q$ the group $G$ may be found back. The problem may be formulated somewhat differently as follows.

Let two groups $K$ and $Q$ be given. Determine all the groups $G$ containing $K$ as a normal subgroup, so that $G/K$ is isomorphic with $Q$.

O. Schreier [1] was the first to give a solution of this problem, so that from now on a group $G$, which has been constructed in the above-mentioned way from $K$ and $Q$, will be called a Schreier-extension of $K$ with $Q$.

Generally speaking every group containing a group $A$ as a subgroup is called an extension of $A$. R. Baer [1] has amply studied these extensions. They will be called general extensions.

Let a group $A$ and a set $\Gamma$ be given, so that $A \cdot \Gamma = \{a \cdot \gamma \mid a \in A, \gamma \in \Gamma\}$ (where $\cdot$ is the identity of $A$). The purpose is to construct all the groups which have $A$ as a subgroup and $\Gamma$ as a system of representatives of the right cosets of $A$ in $G$.

So the final result is to become a group $G$ on the set $A \cup A_0$; $a \in A \setminus \{e\}$ $e a$ is also indicated by $a$. Therefore we require both $A$ and $\Gamma$ to be subsets of $G$.

The construction as described by J. Szép [14] is given, without entering into details. In it an element of $A$ is indicated by a small Latin letter and an element of $\Gamma$ unequal to $e$ by a small Greek letter. Put $A \cdot \Gamma = T$. If it is possible for every pair as with $\omega \in \Gamma$ and $\sigma \in T$ to write $\omega \sigma = b\alpha$ with $b \in A$ and $\alpha \in \Gamma$ a definition of a binary operation in $G$ becomes possible.
Pairs at with aEA and bEA are multiplied as if they were elements of A and pairs are as are left as formal products.

It stands to reason to introduce two functions $\alpha^a: \mathbb{Z} \times T \rightarrow A$ and $\alpha^b: \mathbb{Z} \times T \rightarrow A$ and to write $a^s = \alpha^a_s \cdot \alpha^b_s$. The point between $\alpha^a$ and $\alpha^b$ is written for clarity.

These functions are called S-functions. It then still remains to impose such conditions on these S-functions that G becomes a group. The requirements which are to be made for the S-functions in order that $\varepsilon$ becomes the identity of G and that every element of G has an inverse, J. Szép calls the initial conditions of the S-functions. The requirement that the operation is associative, results in 6 functional equations.

The outcome as published in J. Szép [14] is reproduced here.

Initial conditions:

\[ a^\varepsilon = a, \quad \text{for } a \in G, \]

\[ a^\varepsilon = b \mapsto a = b, \quad \text{for } a \in G, b \in G, \]

\[ \forall a \in G, \forall b \in G: a^b = e \]

and $e$ is uniquely determined,

\[ (aa)(bb) = (a(ab)^b) = a((ab)b), \quad \text{for } a \in G, b \in G, a \in A, b \in A, \]

\[ ea = a, \quad \text{for } a \in G, \]

\[ as = be = a = b \text{ and } a = e, \quad \text{for } a \in A, b \in A, a \in G, b \in G. \]

**THEOREM (SZÉP).** A set satisfying the above mentioned conditions is a group, iff for the functions $\alpha^a$ and $\alpha^b$ the following conditions hold:

\[ \alpha^a(ab) = \alpha^a(a) \cdot \alpha^a(b), \quad \text{for } a \in G, a \in A, b \in A, \]

\[ \alpha^b(ab) = \alpha^b(a) \cdot \alpha^b(b), \quad \text{for } a \in G, a \in A, b \in A, \]

\[ \alpha^a(\varepsilon a) = (\alpha^a(a)) \cdot \varepsilon, \quad \text{for } a \in G, b \in G, a \in A, \]

\[ \alpha^b(\varepsilon b) = (\alpha^b(b)) \cdot \varepsilon, \quad \text{for } a \in G, b \in G, a \in A, \]

\[ \alpha^a(a^b) = \alpha^a(Fa) \cdot \alpha^a(Fb), \quad \text{for } a \in G, b \in G, a \in A, \]

\[ \alpha^b(a^b) = \alpha^b(Fa) \cdot \alpha^b(Fb), \quad \text{for } a \in G, b \in G, a \in A, \]
1.2. In section 2 the above-mentioned theorem is proved by choosing another construction. See W. Peremans [8]. In his construction $S \in \bar{G}$ starts with the fact that the elements of $G$ are of the form $aa^b$ with $a \in A$ and $a \in T$, then defines a product with use of the $S$-functions and finally derives conditions in order that the group axioms are satisfied, of which the axiom of associativity is the most difficult. We turn the other way round and take a starting point where the associativity is guaranteed from the beginning, whereas the fact that the elements of $G$ may be written uniquely in the form $aa^b$ is achieved by imposing conditions on the $S$-functions.

The group $G$ is generated by $A$ and $\Gamma$, so it is plausible to start with the free group generated by the set $T$. For technical reasons it is preferable to choose a free semigroup $F$ instead of a free group.

Again the $S$-functions $a^0$ and $a^8$ are introduced and in $F$ a set of relations is chosen to the effect that a word of the form $W\epsilon_{U}$ becomes equivalent to $W_{\epsilon}$ if $ab = c$ in $A$ and a word of the form $W_{\epsilon_{U}}$ becomes equivalent to $W \cdot ^{a}_{b} \cdot ^{a} \cdot U$ ($W$ and $U$ denote arbitrary words of $F$).

The quotient semigroup with respect to these relations is formed. As a matter of course this semigroup is associative.

The requirement that every element of this quotient semigroup is an equivalence class containing exactly one word of the form $aa^b$ together with the requirement that the semigroup is a group gives in a natural way the initial conditions and functional equations found by $S \in \bar{G}$.

1.3. In section 3 the functions $a^0$ and $a^8$ are examined. The base-theorem of this section says that the mapping $\pi: G \rightarrow \Sigma_{T}$ defined by $\pi(gT) = (a^b)^{a^c}$ for $a \in \Sigma_{T}$, $g = b\epsilon_{T}$, $b \in A$, $b \in T$, is a permutation representation of $G$ on $\Gamma$, so that the mapping $a \rightarrow a^8$ for any $a \in \Sigma_{T}$ is a permutation of the set $\Gamma$.

1.4. The function $a^8: \Gamma \times \Gamma \rightarrow \Gamma$ is a binary operation on $\Gamma$ with $e$ as identity and having the property that any element has a
unique left inverse. This operation however, need not be
associative because \((a \beta)^{\gamma} \neq (a^{\gamma} \beta)^{\gamma}\).
A structure consisting of
the set \(\Gamma\) and an operation with the properties mentioned above
is called a neargroup. In the case of a Schreier-extension the
neargroup is the quotient group \(G/A\).
Similarly as in the case of the Schreier-extensions, the general
extension problem can now be considered to be the construction
of all the groups \(G\) with a given subgroup \(A\) and a given near-
group \(\Gamma\).
In section 4 this is worked out further.

1.5. The conception "equivalent S-extensions", which logically
results from the introduction of another system of represen-
tatives is discussed in section 5. First of all it is noticed
that, in contrast with Schreier-extensions in which equivalent
extensions have the same quotient group, equivalent S-extensions
can have non-isomorphic neargroups.
Passing on to a suitable equivalent extension the S-functions
can sometimes be greatly simplified.
A necessary and sufficient condition for an S-extension to be
equivalent to an extension for which \(\forall \alpha \in \Gamma, \forall \beta \in \Gamma : \alpha \beta = e\)
is derived. Analogous to the case of Schreier-extensions these
S-extensions will be called splitting. It appears that in this
case the neargroup \(\Gamma\) is a group and \(G = AF\) with \(A \cap F = \{e\}\).
From this it follows that the extensions studied by G. Zappa ([16])
form a special case of the S-extensions discussed here.

1.6. If \(A\) is a subgroup of \(G\) and \(\Gamma\) a system of representativ-
es of the right cosets (s.r.c.) of \(A\) in \(G\), \(\Gamma\) is — in general —
no system of representatives of the left cosets (s.l.c.) of \(A\)
in \(G\). If, however, \(\Gamma\) is a common system of representatives of
the right and left cosets, \(\Gamma\) is called a s.t.r.
Because as is derived in §16, with every \(\alpha \in F\) an \(\overline{\alpha}\) \(\Gamma\) is
associated, so that \(\overline{\alpha}^{\gamma} = e\), a mapping \(\phi\) of \(\Gamma\) into \(\Gamma\) may be
defined by \(\alpha \phi = \overline{\alpha}\).
The mapping \(\alpha : A - A\) defined by \(\alpha(a) = a\) also plays a part in
In 6.11, \( T \) is shown to be an s.t.r. iff \( \delta \) is \((1-1)\) and onto or formulated differently iff every element of the neargroup \( T \) has a right inverse.

Moreover, it is proved, that the condition that, \( \omega \) is a permutation of \( A \) for all \( \alpha \in T \), is equivalent with the condition that \( \delta \) is \((1-1)\) and onto.

It is immediately clear that both with Schreier and with Zappa-extensions this condition is fulfilled because in these extensions the neargroup \( T \) is a group. In the case of Schreier-extensions \( \omega \) is even an automorphism of \( A \).

Furthermore, it is investigated whether for a given group \( G \) and a given subgroup \( A \) there exists an s.t.r. It will be proved in 6.8, that for the existence of an s.t.r. it is necessary and sufficient that the order of the \( \alpha \)-stabilizer of \( A \) is equal to the order of the \( \delta \)-stabilizer of \( A \), or, expressed differently, that in any double coset \( \alpha \omega A \) there are as many right as left cosets of \( A \).

The theorem proved in 6.2 Zassenhaus [17] that \( A \) has an s.t.r. if the index of \( A \) in \( G \) is finite, follows from the result of 6.8. Moreover, an almost trivial consequence of 6.8, is that any finite subgroup of a group has an s.t.r.

In 6.14, an example of a subgroup without an s.t.r. is given. According to 6.16 and 6.17, "\( \delta \) is \((1-1)\)" and "\( \delta \) is onto" are both sufficient conditions for the existence of an s.t.r. In 6.18 and 6.19, it is shown that, "\( \forall \alpha \in \Gamma: \omega \) is \((1-1)\)" and "\( \forall \alpha \in \Gamma: \omega \) is onto" are both of them also sufficient in order that \( A \) has an s.t.r.

1.7. In section 7 the connection between the transfer of \( G \) in \( A \) and the \( S \)-functions is discussed.

If \( G = AB \) with \( AB = \{e\} \) in which \( A \) is a subgroup and \( B \) a normal subgroup of \( G \), \( B \) is called a normal complement of \( A \). Some theorems will be proved for \( S \)-extensions of a finite Abelian group \( A \), with a finite set \( \Gamma \), for which \( (o(\Gamma), o(A)) \) = 1.
(So A is an Abelian Hall subgroup of G).
7.5. If $\forall a \in A: a^{-1}A = A$, $A$ has a normal complement.
7.8. If $A$ is a $p$-group and $N_p(A) = C_p(A)$, $A$ has a normal complement (Burnside).
7.11. If $A$ is a cyclic $p$-group and $N_p(A) = A$, $G'$ is a normal complement of $A$.
7.11. If $p(A)$ is a prime and $N_p(A) = A$, $G$ is solvable.

1.8. While in section 7 attention is paid to finite $S$-extensions, in section 8 some special $S$-extensions are considered in which other restrictions then finiteness are supposed.

8A contains a short discussion of those $S$-extensions for which $\text{Core}(A)$ is as great as possible, i.e. $\text{Core}(A) = A$. These are clearly the Schreier-extensions. In order to illustrate the connection between the $S$-functions and the factorsets and automorphisms occurring in a Schreier-extension, some theorems on Schreier-extensions will be proved.

In 8B the other extreme case viz $\text{Core}(A) = \{e\}$ will be treated. These extensions are called pure-extensions.

Seeing that in this case $\tau$ is an automorphism we conclude that a pure-extension is isomorphic with a transitive subgroup of $S_\tau$. Therefore it is possible to describe all the pure-extensions belonging to a given neargroup $\Gamma$.

Another consequence of the above is, that the $S$-functions have to fulfill fewer conditions. Also $S$-extensions for which the neargroup $\Gamma$ is a group will be discussed (8C). Although in these group-extensions the group $\Gamma$ is not necessarily a subgroup of $G$, still it is possible to show that some theorems, about the solvability of $G$, holding for factorizable groups, also hold for these group-extensions.

In 8D it is not only supposed that $\text{Core}(A) = \{e\}$ but the much stronger condition is made that $\forall a \in N(A): (a^\sigma = a \Rightarrow a = e)$, so that for $a \not= e$ the $\sigma$-stabilizer of $A$ consists of one element. Limitation to finite groups results in $G$ to be a Frobenius group with a Frobenius-kernel, which is a normal complement of $A$. 
The above assertion will only be proved in the case of $A$ being solvable. R. Shaw [13] proves the same theorem in another way. The theorem holds also good if $A$ is a non-solvable group, but of this fact no character-free proof is known. For a proof with the use of group characters see R. Huppert [6].

1.9. A generalisation of the construction of section 2 will be discussed in section 9. In this generalisation it is supposed that $A$ is given by a generating set $T$, and a number of defining relations. $G$ is an $S$-extension of the group $A$ with the set $T$.

The purpose is to construct the group $G$ with subgroup $A$ from a generating set $T, \omega$ instead of $\omega^{-1}$.

This construction will be used in section 10 and 11.

1.10. Now a general $S$-extension will be reduced to a Schreier-extension and a pure-extension, which elucidates the fundamental character of these two extensions.

If $G$ is an $S$-extension of $A$ with $\Gamma$, $A$ is a Schreier-extension of Core$(A)$ with $A^\pi$.

Core$(A)$, $A^\pi$ and the $S$-function belonging to the construction of $A$ from Core$(A)$ and $A^\pi$ are supposed to be given.

$G^\pi$ is a pure-extension of $A^\pi$ with $\Gamma^\pi$. Also the $S$-functions belonging to the construction of $G^\pi$ from $A^\pi$ and $\Gamma^\pi$ are supposed to be given. As a generating set of the semigroup, of which $G$ is a quotient group, $G = \text{Core}(A)A^\pi$ will be taken.

It will be shown that in addition to the functions given before only one function: $\pi^\pi : \tau \rightarrow T - \text{Core}(A)$ is necessary to construct $G$. The fundamental equations for this function will be derived in section 10.

1.11. A group $A$, a group $B$ and a common subgroup $C$ of $A$ and $B$ are supposed to be given. The aim of the construction described in section 11 is to construct all the groups $G$, so that $G = A'B'$ where $A'$ is isomorphic with $A$, $B'$ isomorphic with $B$ and $A' \cap B'$ isomorphic with $C$.  

The method used in this thesis is based on section 9.
Also by L. Rédei and J. Szép [10] a construction is described.
G. Casadio [2] describes a construction for the case that $G$
is a normal subgroup of $H$. 
Two
Construction

2.1. DEFINITION. Let $S$ be a set. Let $W$ be the set of all non-empty finite strings ("words") of elements of $S$ ("letters"). The free semigroup $F$ generated by $S$ is the structure with $W$ as set and juxtaposition as operation.

2.2. REMARKS.
1. The associativity of the operation of $F$ is trivial.
2. The elements of $S$ are identified with the corresponding one-letter words of $F$.

In $F$ a set $I$ of relations $X : Y$, with $X \in F$ and $Y \in F$, may be chosen. In fact $I$ is a subset of $F \times F$.

2.3. CONSTRUCTION OF A CONGRUENCE RELATIONS $R_I$ WITH RESPECT TO $I$.

\[ R_I = \{(U,V) \mid U \in F, V \in F, s_i \in S_i \} \quad \text{and} \quad V \in F \] with origin $x_i \in F$ with origin $y_i \in F$ with origin $z_i \in F$ with origin $w_i \in F$

\[ \text{such that } (X_i, Y_i) \in I \] or \[ (Y_i, X_i) \in I, \quad U_{i-1} = W_iX_iZ_i \quad \text{and} \quad U_i = W_iY_iZ_i. \]

2.4. REMARK. It is clear that $I \in K$ and that $R_I$ is an equivalence relation. Moreover if $U \in F$, $V \in F$, $X \in F$, $Y \in F$,

\[ (U, V) \in R_I, \quad (U, V \in F), \quad (U, V \in F), \quad (U, V \in F), \quad \text{i.e. } R_I \text{ is a congruence relation. So in the quotient set of } F \text{ with respect to } R_I \text{ multiplication may be defined by multiplication of representatives of the equivalence classes with respect to } R_I. \text{ The quotient set with this multiplication is the quotient semigroup of } F \text{ with respect to } R_I. \]
2.5. DEFINITION. The quotient semigroup $G$ of $F$ with respect to (the congruence relation) $\mathcal{R}_1$ is called the quotient semigroup of $F$ with respect to (the set of relations) $I$.

2.6. REMARKS.
1. Different elements of $S$ may happen to be identified in $G$.
   So we cannot say that $G$ is generated by $S$.
2. This construction must not be confused with the case that the elements of $S$ are considered as indeterminates, where in "relations" substitution is allowed. In our case, if $ab = ba$ is written, this permutability is only required for those elements, for which the relation is given.
3. Every semigroup with generating set $S$ is isomorphic with a quotient semigroup of $F$.

Let $A$ be a semigroup with identity $e$. Let $ab$ denote a two-letter word but $(ab)$ the one-letter word consisting of the letter, which is the product in $A$ of the elements $a$ and $b$ of $A$.
Let $I_1$ be a set and $\cap A = \emptyset$. Put $I = \cup I_1 \cup \{e\}$ and $T = I_1 \cup A$.
Let the following mappings be given:
  $\delta : I \times T \rightarrow A$ with $a \in I_1$, $\varepsilon \in T$ and $a \cdot \varepsilon A$,
  $a : I \rightarrow T \times I$ with $a \in I_1$, $\varepsilon \in T$ and $a \cdot \varepsilon I_1$.
Let $F$ be the free semigroup with $T$ as generating set.
The following set of relations $I$ in $F$ is taken:
  $V_{m \in A}, V_{n \in A} : a \cdot (ab) = (a \cdot b)$,
  $\forall \varepsilon \in T, \forall a \in A : e \cdot e = a \cdot a$,
  $\forall \varepsilon \in T, \forall a \in A : e \cdot a = a \cdot e$.
Let $G$ be the quotient semigroup of $F$ with respect to $I$.
We are going to determine necessary and sufficient conditions in order that every element of $G$ contains exactly one word of the form $a \cdot b$ with $a \in A$ and $\varepsilon \in T$.
For that purpose a standard reduction will be defined.
We first extend the functions $\delta$ and $a$ as follows
  $\delta : I \times T \rightarrow A$ with $a \in I_1$, $\varepsilon \in T$ and $a \cdot \varepsilon A$,
  $a : I \rightarrow T \times I_1$ with $a \in I_1$, $\varepsilon \in T$ and $a \cdot \varepsilon I_1$,
by putting
  $a = a$, $\varepsilon = e$, $e \cdot a = a$, $e \cdot e = a$ for $a \in A$ and $\varepsilon \in T$.  

2.7. DEFINITION of a reduction-step.
Let \( \mathcal{S} = \{ a \mathcal{S} | a \in \mathcal{A}, \mathcal{S} \in \mathcal{F} \} \).
A reduction-step \( \mathcal{S}: F \rightarrow F' \) is defined by:
- \( sWf = csW \) for \( a \mathcal{S}, \mathcal{S} \in \mathcal{F} \),
- \( sf = ae \) for \( a \in \mathcal{A} \),
- \( abWf = (ab) W \) for \( a \mathcal{S}, b \mathcal{S} \in \mathcal{F} \),
- \( asWf = (a^*s)^*W \) for \( a \mathcal{S}, c \mathcal{S}, c \mathcal{S} \in \mathcal{F} \).

2.8. REMARKS.
1. Taking \( n = e \) in the last line of 2.7 we get
   - \( abWf = (ab)eW \) for \( a \mathcal{S}, b \mathcal{S} \in \mathcal{F} \),
   - \( aWf = asW \) for \( a \mathcal{S}, c \mathcal{S} \in \mathcal{F} \).
2. If \( \mathcal{S} \geq \mathcal{E} \), \( \mathcal{S} \geq \mathcal{G} \), then \( \mathcal{S} \geq \mathcal{G} \).
3. Let \( f^n = f^n \) and \( f^n = f^n \), then \( \mathcal{S} \geq \mathcal{E} \), \( \mathcal{S} \) with \( f^n \).

So every \( \mathcal{S} \in \mathcal{G} \) contains at least one element of \( \mathcal{N} \).

2.9. DEFINITION OF STANDARD REDUCTION.
A standard reduction \( \mathcal{S}: F \rightarrow F' \) is defined as follows:
For \( \mathcal{S} : F \rightarrow F' \),
for \( \mathcal{S} : F \rightarrow F' \) with the \( n \) for which \( f^n \).

2.10. REMARKS.
1. If \( \mathcal{S} \geq \mathcal{E} \) and \( \mathcal{S} \geq \mathcal{G} \) then \( \mathcal{S} \geq \mathcal{G} \).
2. \( WS = WS \) for \( \mathcal{S} \in \mathcal{F} \).

2.11. THEOREM.
1. \( US \) = \( US \) for \( \mathcal{S} \in \mathcal{F} \) and \( \mathcal{S} \in \mathcal{F} \).
2. \( \mathcal{S} = \mathcal{S} \) for \( \mathcal{S} \in \mathcal{F} \) and \( \mathcal{S} \in \mathcal{F} \).

PROOF. From the definition of \( \mathcal{S} \) it follows that it suffices to prove:
1. \( US = US \) for \( \mathcal{S} \in \mathcal{F} \) and \( \mathcal{S} \in \mathcal{F} \).
2. \( \mathcal{S} = \mathcal{S} \) for \( \mathcal{S} \in \mathcal{F} \) and \( \mathcal{S} \in \mathcal{F} \).

The only case in which the first of these two is not trivial is \( U = a \) with \( a \mathcal{A} \). We then apply mathematical induction with respect to the number of letters in \( V \). If \( V = \phi \) we get
\[US = \text{UG} \ (2.10.2) \]. If \( V \neq \emptyset \) we put \( V = eV \), with \( \mu e, V, \text{EFU}(\emptyset) \) or \( V = bV \), with \( \mu b \neq \emptyset, V, \text{EFU}(\emptyset) \).

\[ aV, S = aS, V, S = \text{EFU}(\emptyset), \]
\[ abV, S = (ab)V, S = (\text{by induction})(ab)V, S = (ab)eV, S = aebV, S = \]
\[ = aSEbV, S. \]

Before proving the second equality we first remark that
\[ abWS = (ab)WS \text{ for } a, b, a, b \in \text{EFU}(\emptyset), \text{ for } b \neq e \text{ this follows} \]

from 2.10.2 and for \( b = e \) from 2.11.1. Now 2.11.2 follows from the associativity in \( A \) and 2.10.2:

\[ caWS = caWS = c(aSE)bS, \]
\[ caS = (ca)S = (ca)eS = cSEaS = c(ab)S, \]
\[ cSEbWS = (ca)bWS = (ca)bWS = (c(ab))WS = c(ab)WS = \]
\[ = c(ab)S = c(ab)S = c(ab)S, \]
\[ caSEbWS = (ca)SEbWS = (ca)SEbWS = (c(ab))SEbWS = \]
\[ = c(ab)SEbWS = c(ab)SEbWS. \]

2.12. THEOREM. Every \( \text{EG} \) contains exactly one element of \( a \)

iff \( \forall a \in A \), \( \forall (X, Y) \in I : aXS = aYS. \)

PROOF. Every \( \text{EG} \) contains at least one element of \( a \) (2.8.)

Suppose every \( \text{EG} \) contains only one element of \( a \) then
\[ aXS = aYS \text{ because } (X, Y) \in I = (aX, aY) \in I = (aXS, aYS) \in I. \]

Now suppose \( \forall a \in A \), \( \forall (X, Y) \in I : aXS = aYS. \)

\[ \forall (X, Y) \in I : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]
\[ \forall (X, Y) \in I, \forall a \in A : aXS = aYS. \]

2.13. THEOREM. Every \( \text{EG} \) contains exactly one element of \( a \)

iff \( \forall a \in A \), \( \forall (u, v) \in I, \forall (u, v) \in I : (a^a) = (a(ab)). \)

\[ (a^a) = (a(ab)), (a^a) = (a(ab)), \]
\((a^b)^c = (a^c)^b\), \((a^b)^c = (a^c)^b\),
\((a^b)^c = a^{bc}\), \((a^b)^c = a^{bc}\).

**Proof.** By straightforward calculation from 2.12.

Suppose that every \(g \in G\) contains exactly one element of \(H\) and let \(\sigma : T \rightarrow G\) be defined by: \(\sigma(a) = ab\), for \(a \in T\). Then \(\sigma\) is \((1-1)\), \(\sigma|A\) is a monomorphism, because \(((ab), ab) \in \mathcal{F}_1\).

Identifying \(T\) and its image in \(G\) we may consider \(A\) as a sub-semigroup of \(G\) and \(F\) as a subset of \(G\). From now on we suppose this identification to be made. \(G\) now is generated by \(T\) and the elements of \(G\) are products of elements of \(T\). This product is written as juxtaposition, the use of brackets for the product in \(A\) no longer is necessary.

Every element of \(G\) may be written in the form \(a^b\) with \(a \in A\), \(b \in F\).

Multiplication in \(G\) is given by:
\[ a \cdot b = a^b, \quad (a^b)^c = a^{bc} \] with \(a \in A\), \(b \in F\), \(a \in A\), \(b \in F\).

In \(G\) \(e\) is a left identity because \(a \cdot e = a\).

It is easy to verify that the equalities in 2.13 are also valid if \(a \in A\) and \(b \in F\) (instead of \(F\)).

2.14. **Theorem.** If \(e\) and \(a^b\) satisfy the conditions of 2.13 then \(G\) is a group iff \( \langle a \rangle \) is a group and \( \bar{\mathcal{F}}_{1'} = \mathcal{G}_{1'} \), such that \(e = e\).

**Proof.** Suppose \(G\) is a group, \(a^b\) and \(b^c\) with \(b^c\) and \(b^c\) is the inverse of \(a\) in \(G\). Then \(a \cdot b = e\), so \(a = e\) and \(e = e\), so every \(a^b\) has a right inverse in \(A\). Therefore \(A\) is a group.

If \(a \in A\), and \(b \in F\) and \(b^c\) is the inverse of \(a\) in \(G\) then \(b \cdot a = e\), \(a \cdot b = e\), \(a \cdot a = e\).

Conversely, suppose \(A\) is a group and \(\bar{\mathcal{F}}_{1'} = \mathcal{G}_{1'}\), such that \(e = e\). Then \((a)^{-1} = b = e\), so every element of \(F\) has a left inverse. Because also every element of \(A\) has a left inverse, it follows that every element of \(G\) has a left inverse. So \(G\) is a group.
2.15. COROLLARY. If \( G \) is a group then \( a, a^2 = e \) \( a \in G \) for \( a \), so \( a^2 = e \) and \( a^2 \) = \( e \). But \( a^2 = e \) and \( a^2 = e \) imply the last line of 2.11.

2.16. MAIN THEOREM. If \( A \) is a group with identity \( e \), \( F \) a set with \( A \cap F = \{ e \} \), \( T = A \cup F \), \( a \in A \), \( a \in F \), \( a^F \) (\( a^F \) is \( A \) and \( a^F \) is \( F \)) functions, then there exists a group \( G \) with \( A \) as subgroup, \( F \) as system of representatives \( (s, x, r) \) of the right cosets of \( A \) in \( G \) and \( a^G = a^F \) iff \( \forall a \in A, b \in A, y \in F, \begin{align*}
\forall a \in A, b \in A: & \quad a^G = a^F \\
\forall a \in A, b \in A, y \in F: & \quad a^G = a^F
\end{align*} \) and \( b^G = b^F \).

\begin{align*}
& a^a, b^a = a(ab), \quad (a^b)^c = a(ab), \\
& a^a, b^a = a(ab), \quad b^a, \quad (a^b)^c = (a^b)^c, \\
& e^a = e, \quad e^a = e, \\
& e^a = e, \quad e^a = e, \\
\forall a \in F, b \in F & \quad a^F = e.
\end{align*}

A group \( G \) satisfying the conditions above is determined by \( A, F, a^F \) and \( a^F \) up to isomorphism.

\( G \) is isomorphic with the quotient semigroup of the free semigroup generated by \( T \) with respect to the relations

\begin{align*}
ab & \in \{ ab \} \quad aA, bA, \\
ab & \in \{ ab \} \quad bA \setminus \{ e \}, seT, \\
ab & \in \{ ab \} \quad ab \setminus \{ e \}, seT.
\end{align*}
Three 
S-functions

In this section a definition of permutation isomorphism and of similar permutation representations is given.

3.5. contains a criterion to determine the similarity of two permutation representations. In 3.7. a permutation representation \( \pi \), playing a fundamental role in the remaining part of this dissertation, is introduced. An examination into the orbit and stabilizers of \( \pi \) is made.

If \( G \) is an \( H \)-extension of \( A \) with \( F \) it is proved in 3.26. that 
\[
\left( \left( \sigma \right) \sigma \right) A = \left( \left( \sigma \right) \sigma \right) G \text{ for } \sigma \in G,
\]
\( (\sigma) \) is the subgroup generated by \( \sigma \).

Let \( \mathcal{G}_H \) be the set of all permutations of the set \( H \).

A permutation representation of a group \( G \) is a homomorphism \( \psi \) of \( G \) onto a group \( \mathcal{G}_H \) of permutations of some set \( H \).

Such a representation will be called a representation of \( G \) on \( H \). A representation on \( H \) is called transitive (primitive) iff \( H \) has this property. A representation is called faithful iff the mapping from \( G \) on \( \mathcal{G}_H \) is an isomorphism.

3.1. DEFINITION. If \( P \) is a permutation group on \( M \) and \( Q \) a permutation group on \( N \), then \( \phi : P \to Q \) is a permutation isomorphism from \( P \) onto \( Q \) iff:
1. \( \phi \) is an isomorphism from \( P \) onto \( Q \),
2. \( M : H \cdot \Phi \) which is \( (1-1) \), onto and such that
\[
\forall \phi \in \mathcal{G}_M, \forall \phi P: \phi (\phi P) = \phi \phi P.
\]

3.2. REMARK. As a matter of fact \( \psi \) is determined by \( \phi \).

3.3. DEFINITION. If \( \phi \) and \( \psi \) are permutation representations of the group \( G \) on \( M \) resp. \( N \) then \( \phi \) is similar to \( \psi \) iff:
$\forall g \in G, \forall g_1 \in G : g \cdot g_1 = g_1 \cdot g = g_1$ and \\
$\psi : G \rightarrow G$ defined by $g \cdot \psi = \psi g$ for $g \in G$ is a permutation isomorphism.

3.4. REMARK. Similarity is an equivalence relation in the class of permutation representations of a group $G$.

3.5. THEOREM. If $\tau$ is a permutation representation of $G$ on $N$ and $\sigma : M \rightarrow N$ is $(1-1)$ and onto, then $\tau \circ \sigma : M \rightarrow N$ defined by $\sigma(g) = \sigma(g_1) \cdot \tau$ for $g \in M$ and $g \in G$ is a permutation representation of $G$ on $N$, which is similar to $\tau$.

PROOF. $g \mapsto \sigma(g) \in M \mapsto N$ defined by $\sigma(g) = \sigma(g_1) \cdot \tau$ is a permutation of $N$.

$\tau : G \rightarrow G_M$ is a homomorphism, because

$$\sigma(g \cdot h) = \sigma(g_1 \cdot h_1) \cdot \tau = \sigma(g_1) \cdot \sigma(h) \cdot \tau = \sigma(g_1) \cdot \sigma(h_1) \cdot \tau = \sigma(g_1) \cdot \sigma(h_1) \cdot \tau = \sigma(g_1) \cdot \sigma(h_1) \cdot \tau$$

So $\tau$ is a permutation representation of $G$ on $N$.

$\psi : G \rightarrow G_M$ defined by $\psi = g \cdot \sigma \cdot \tau$ is an isomorphism as follows immediately from $g \cdot \psi = \psi^{-1}(g) \cdot \tau$ for $g \in G$.

Then $\psi$ is a permutation isomorphism from $G$ onto $G_M$ because $\psi(g) = a(g) \cdot \psi$ for $g \in M$ and $g \in G$.

3.6. REMARK. If $\tau$ and $\sigma$ are similar permutation representations of the group $G$ on $M$ and $N$, it follows immediately from definition 3.3. that there exists $\gamma : M \rightarrow N$ such that: $\tau \circ \gamma = \sigma$ for $\gamma \in G_M$.

It is well known that:

If $A$ is a subgroup of the group $G$ and $M$ the set of the right cosets of $A$, then $\tau : M \rightarrow M$ defined by $\tau(g) = g \cdot \gamma A$ for $g \in M$ and $g \in G$ is a permutation representation of $G$ on $M$.

$\tau$ is called the natural representation of $G$ on the right cosets of $A$ in $G$.

$\tau$ is transitive and $\text{Ker}(\tau) = \text{Core}(A)$ = the greatest normal subgroup of $G$ which is contained in $A$.
3.7. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \), then

\[
\tau : S \to Aut(G) \text{ defined by } \tau(g) = (a^b)^\beta \text{ for } a \geq b, \ b \leq c, \ \beta \in \Gamma
\]

is a permutation representation of \( G \) on \( \Gamma \) similar to the natural representation \( \tau \) on the right cosets of \( A \) in \( G \).

Proof. \( M \) is the set of the right cosets of \( A \) in \( G \).

\[
\sigma : M \to T \text{ defined by } Aa = a \text{ for } a \in \tau \text{ is } (1-1) \text{ and onto.}
\]

\[
\sigma(Aa) = (a^b)^\beta \Rightarrow A(a^b)^\beta = Aab = \sigma(a)
\]

Now the theorem follows from 3.2.

\( \tau \) is called the natural representation of the \( S \)-extension \( G \) on \( \Gamma \).

3.6. THEOREM. If \( \psi \) is an \( S \)-extension of \( A \) with \( \Gamma \), the natural representation of \( \psi \) on \( \Gamma \) and \( A = \psi^{-1}(A) \) is the normalizer of \( A \) in \( G \), then \( \psi : S \to Aut(G) \) (in which \( G \) is the set of conjugate subgroups of \( A \) in \( G \)), defined by

\[
\psi^{-1}(Aa) = a^{-1}(a^b)^\beta a \text{ for } a \in G \text{ and } a \in \Gamma
\]

is a permutation representation of \( G \) on \( C \) similar to \( \tau \).

Proof. \( \psi : S \to Aut(G) \) defined by \( a \mapsto a^{-1}Aa \) for \( a \in \tau \) is \((1-1)\) and onto because \( A = \psi^{-1}(A) \).

\[
\sigma(g) = a^{-1}Aa(g) = g^{-1}a^{-1}Aag = ((a^b)^\beta)^{-1}A(a^b)^\beta = \sigma(g)
\]

Now apply theorem 3.5.

3.5. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \) and \( c \) a homomorphism of \( G \) with \( Ker(c) \cap \text{Core}(A) \) then \( c \) is a system of representatives of the right cosets of \( Ac \) in \( G \) and the image \( \gamma \) of the natural representation \( \tau \) of \( G \) on \( \Gamma \) is permutation isomorphic with the image \( \gamma \) of the natural representation \( \tau \) of \( G \) on \( \Gamma \).

Proof. \( A \sigma = G \) and \( a \neq b \Rightarrow Aa = Ac \neq Ab \) for \( a \in \Gamma \) and \( b \in \Gamma \) because \( Aa = Ab \Rightarrow Aa = ab \Rightarrow a = 0 \).
So \( \sigma \) is a W.R.R. of \( A \) in \( G \), \( \sigma \upharpoonright \Gamma = (1-1) \).

If \( a \in A \) and \( a \in \Gamma \) then: \( (a^n) o ((a^n)^c) = (a^n o a^n) o = o(a^n o a^n) = o(a^n o a^n)^{a^n} \) in which \( o(a^n o a^n) \) are the \( \sigma \)-functions of \( G \) with respect to the group \( A \) and the W.R.R. \( \sigma \).

So \( (a^n) o ((a^n)^c) = o(a^n o a^n) \). A repeated application results in \( ((a^n)^{a^n})^{a^n} = (a^n o a^n)^{a^n} \) for \( a \in A \), \( a \in \Gamma \), \( b \in \Lambda \).

So if \( g = b^n \) then:

\[ a(o(g_{n+1})) = a(o(b^n) o ((a^n)^{a^n})^{n+1}) = (a^{b^n} o a^n) o = a(g^n) o. \]

Therefore \( g_{n+1} = g_n \).

So \( \psi : g_{n+1} \rightarrow g_n \) defined by \( g_{n+1} = g_n \) for \( g \in G \) is a mapping.

Let \( g_{n+1} = g_n \) (identity of \( G \)) then \( g_{n+1} = g_n = g \in \text{Core}(A) \) in \( G \).

Therefore \( \psi \) is a monomorphism.

It has been shown already that \( \psi \upharpoonright \Gamma = (1-1) \) and \( o(a^n o a^n) = a(g^n o) \).

Therefore \( \upsilon : \Gamma \rightarrow \Gamma \) is a permutation isomorphism.

3.8b. COROLLARY. The natural representation of \( G \) on \( \Gamma \) is permutation isomorphic with the natural representation of \( G \) on \( \Gamma \) as follows from 3.8a. if we take \( c = \upsilon \).

3.9. THEOREM. If \( \tau \) is the natural representation of \( G \) on the set \( M \) of right cosets of the subgroup \( H \) and \( \tau \) the natural representation of \( G \) on the set \( N \) of the right cosets of the subgroup \( K \), then \( \tau \) is similar to \( \upsilon \) iff \( H \) and \( K \) are conjugate subgroups of \( G \).

PROOF. Let \( K = l^{-1} H K \) with \( l \in G \) and \( \Gamma \) be a system of representatives of the right cosets of \( H \) in \( G \). It follows that \( \phi : M \rightarrow N \) defined by \( H \phi = l^{-1} H a = K l^{-1} K \) for \( a \in \Gamma \) is (1-1) and onto.

\[ \text{Edi}(\tau) = l^{-1} H a \phi (g) = (H a)(g a) \phi \] for \( a \in \Gamma \) and \( g \in G \).

Let \( \tau \) be similar to \( \upsilon \) and \( \upsilon \). The \( H \)-stabilizer of \( \upsilon \) is \( a^{-1} H a \) because \( g \mid a \phi (g) = H a \) = \( g \phi (H a) = g \phi (g a^{-1} H a) = a^{-1} H a \).

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By the similarity of \( \pi \) and \( \tau \) there exists a \((\cdot, 1)\) mapping \( \phi \) of \( M \) onto \( H \) such that \( H \circ \phi(g) = H \cdot \phi(g) \).

\[ K = \{ g | K \cdot \phi(g) = K \} = \{ g | K \cdot \phi^{-1}(g) = K \} = \{ g | K \cdot \phi^{-1}(g) = K \cdot \phi^{-1} \} \]

If the representative of \( K \cdot \phi^{-1} \) is called \( \gamma \), so it follows

\[ \gamma \cdot \gamma^{-1} \cdot \gamma = K \text{ or } H \text{ is conjugate to } K. \]

3.10. COROLLARY. Let \( G \) be an \( S \)-extension of \( A \) with \( \Gamma \) and \( \pi \) the natural representation of \( G \) on \( \Gamma \). \( B \) is a subgroup of \( G \) and \( \zeta \) a system of representatives of the right cosets of \( B \) with \( e \) as representative of \( B \).

\( \zeta \) is the natural representation of \( G \) on \( \zeta \). Then \( \gamma \) is similar to \( \gamma \cdot \zeta \) iff \( B \) is conjugate to \( A \).

Let \( H \) be an \( S \)-extension of \( A \) with \( \Gamma \) and \( \cdot \) the natural representation of \( G \) on \( \Gamma \). The following symbols are introduced:

For \( H \circ \Gamma \) and \( \cdot \Gamma \) put:

\[ A \cdot H = \{ a(h) \cdot r | a \in A \text{ and } h \in H \}. \]

If \( A = \{ a \} \), \( a \cdot H \) is written and if \( H = \{ h \} \), \( a \cdot h \) is written.

By this notation \( a \cdot H \) for \( a \in A \cdot H \) obtains a new meaning, but this is allowed because \( a \cdot h \) in the new meaning is the set which contains \( a \cdot h \) in the old meaning as a unique element.

\( A \cdot H \) is the \( a \)-orbit of \( H \), so \( A \cdot H \cdot a \).

3.11. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \) and \( H \circ \Gamma \) then \( A \cdot H = A \cdot h \) for \( a \in H \).

PROOF. Because \( A \cdot h \cdot A \cdot H \) is evident we have to prove \( A \cdot h \cdot A \cdot H \).

If \( a \in A \), \( a \in H \), \( b \in A \) and \( b \in H \) then:

\[ a \cdot (b) \cdot a \cdot h = (a \cdot a \cdot h)^{-1} \cdot (a \cdot b)^{-1} \cdot a \cdot b \cdot (a \cdot b) \cdot a \cdot h \cdot A \cdot H. \]

3.12. COROLLARY. If \( V \) is the set of right cosets of \( A \) contained in \( A \cdot h \) then \( o(V) = o(A \cdot H) \).

For \( H \) is a subgroup of \( G \) and \( \cdot \Gamma \) put:

\[ H \cdot A = \{ h \cdot a | h \in H \text{ and } a \in A \}; \]

\[ H \cdot a = \{ h \cdot a | h \in H \text{ and } a \in A \}; \]

For \( \gamma \) a right coset of \( A \) in \( A \cdot h \), then \( o(V) = o(A \cdot H) \).

\[ \gamma \cdot \gamma^{-1} \cdot \gamma = K \text{ or } H \text{ is conjugate to } K. \]
So \( H^2 a = (H a) a \). (The brackets are used for clarity).

If \( \Delta = (a) \), \( H_0 \) and \( H^1 a \) is written.
\( G_{ae} = A \Delta \) because \( a(g) a = e = g \Delta A \) for \( g \in G \).

\( G_0 a = (\Delta)^{-1} A \Delta \Delta = a^{-1} A a \Delta \) because \( G_{ae} \) is the \( a \)-stabilizer of \( G_0 \).

\( H_0 a = a^{-1} A a \Delta H \).

According to Hall, corollary 5.2.1., \([5]\) \( \sigma(a^H) = [H_0 : H^0 a] \).

\( H_0 = (H_0 a)^+ \).

3.13. THEOREM. If \( G \) is an \( S \)-extension of a finite group \( A \) with \( \Gamma \) and \( H \) is a finite subgroup of \( G \) then
\( \sigma(a^H) = [H : a^{-1} A a \Delta H] \) for \( a \in H \).

PROOF.
\[
\sigma(a^H) = \sigma(AaH) = \sigma(a^{-1} AaH) = \frac{\sigma(a^{-1} Aa)}{\sigma(a)} = \frac{\sigma(H)}{\sigma(a^{-1} Aa) \sigma(A) \sigma((a^{-1} Aa) \cap H)}.
\]

3.14. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \), \( H \) is a subgroup of \( G \) and \( \text{Core}(A) CH \), then for \( a \in H \),
1. \( H_a = (H^a)_0 \).
2. \( H_0 = a^{-1} A a \Delta H \).
3. \( \sigma(a^H) = [H : H_0 a] \).

PROOF.
1. \( \text{Core}(A) C H^a \) because \( \ker \text{Core}(A) = \ker H \) and \( a(\ker H) = a \).
2. \( (H^a)_0 = H_0 = (a^{-1} A a) \Delta H \) because \( \text{Core}(A) C H \) and \( \text{Core}(A) C a^{-1} A a \).

So \( H_0 = a^{-1} A a \Delta H \) because \( \text{Core}(A) C a^{-1} A a \).

3. \( \sigma(a^H) = [H : H_0 a] = [H : H_a] \) because \( \text{Core}(A) C H \).

3.15. COROLLARY.
Taking \( H = a \) we get:
1. \( A^a = (A^a)^* \),
2. \( A_0 = a^{-1} A a \Delta A \),
3. \( \sigma(a^A) = [A : A^a] = [A : A^a] \).

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3.16. THEOREM. If $G$ is an $S$-extension of $A$ with $\Gamma$, and for $h \in G$, $h : G \rightarrow G$ is the innerautomorphism of $G$ defined by $g(h) = h^{-1}gh$ for $g \in G$ then:

- $(A_a)(a) = A_a(a)$ for $a \in A$ and $a \in \Gamma$,
- $(A_a)(a) = A_a$ for $a \in \Gamma$.

PROOF.

- $(A_a)(a) = a^{-1}a^{-1}A_a \sigma_a a^{-1}A_a = (a^A)^{-1}A_a a^{-1}A_a = A_a(a^A)$,
- $(A_a)(a) = a^{-1}(a^{-1}A_a \sigma_a a^{-1}A_a a^{-1}A_a = A_a a^{-1}A_a = A_a$.

3.17. COROLLARY.

1. $A_a = o(A_a)$.
2. $A_a = A_a = o(A_a)$ for $a \in \Gamma$.

This condition is evidently fulfilled if $A_a = Core(A)$.

3. If $G$ is an $S$-extension of $A$ with $\Gamma$ and $A_a = Core(A)$ for all $a \in \Gamma \backslash \{e\}$ then $\forall e \in \Gamma : o(A_a) = o(A)^A$ for:

- $o(a^A) = |A : A_a| = |A : A_a| = o(A^A)$ and $o(a^A) = o(A^A) = 1$.

These extensions are called Frobenius extensions of $A$ with $\Gamma$.

3.18. THEOREM. If $G$ is an $S$-extension of $A$ with $\Gamma$ and $|A : Core(A)|$ is finite then $\forall e \in \Gamma : o(A^A) = o(A^A)$.

PROOF. $o(A^A) = |A : Core(A)|$ is finite.

From $(A_a)^A = (a^A)A_a = (a^A)A_a(a^A)^{-1} = (A_a)^A(a^A)^{-1}$ it follows that $o((A_a)^A) = o((A_a)^A)$.

So: $o(a^A) = |A : (A_a)^A| = |A : (A_a)^A| = o(A^A)$.

3.19. COROLLARY.

1. If $G$ is an $S$-extension of $A$ with a finite $\Gamma$, then $\forall e \in \Gamma : o(A^A) = o(A^A)$ for:

- $\Gamma$ is finite $\Rightarrow S_\Gamma$ is finite $\Rightarrow G$ is finite $\Rightarrow A^A$ is finite
- $|A : Core(A)|$ is finite.

2. If $G$ is an $S$-extension of a finite group $A$ with $\Gamma$, then $\forall e \in \Gamma : o(A^A) = o(A^A)$ for:

- $A$ is finite $\Rightarrow |A : Core(A)|$ is finite.
3.20. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \), it follows:
1. if \( \Delta^A \cap S^A \) is not empty, \( \Delta^A = \gamma^A \) for \( \gamma \in \Gamma \) and \( \gamma \not\in \Gamma \).
2. \( \Delta^A = \alpha \Delta^A \cap S^A \) so \( \Delta^A = \alpha \Delta^A \) for \( \alpha \in \Gamma \).
3. \( \Delta^A = \alpha^A \) so \( \Delta^A \cap \alpha^A \) for \( \alpha \in \Gamma \) and \( \alpha \notin \Gamma \).
4. \( \Delta^A = \{ \gamma \mid \gamma \in \Gamma \} \) and \( \gamma \notin \Gamma \) for \( \gamma \in \Gamma \).

PROOF.
1. evident because \( \Delta^A \) and \( S^A \) are orbits of \( A^A \).
2. \( \alpha = \alpha^A = (\alpha^A)^0 = (\alpha^A)^0 \Delta^A = \alpha^A \).
3. \( e = (e^A)^0 \).
4. \( \gamma \Delta^A \cap \gamma \Delta^A = (\gamma \Delta^A \cap \gamma \Delta^A) \).
   \( \gamma \Delta^A = \gamma \Delta^A \) and \( \gamma \Delta^A \).

3.21. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \) and \( \gamma \in \Gamma \), then \( \Delta^A = \alpha^A \) iff \( 3 g_\Gamma^A \) with \( g_\Gamma^A \).

PROOF.
\( \alpha^A = \alpha^A \cap \Delta^A = \Delta^A \cap \alpha^A = \gamma \Delta^A \).
\( \gamma \in \Gamma \) and \( \gamma \not\in \Gamma \).

3.22. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with a finite \( \Gamma \), then \( \exists g_\Gamma^A \) with \( \Delta^A = \alpha^A \) iff \( 2 \mid \{ e \in \text{Core}(A) \} \).

PROOF. \( \Gamma \) is finite \(- \{ e \in \text{Core}(A) \} \) is finite because \( \{ e \in \text{Core}(A) \} = \{ e \} \).
2. \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) is the identical permutation and \( \gamma \in \text{Core}(A) \).

Suppose \( (\gamma \in \Gamma) \).
Then \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).

Suppose \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) then \( \gamma \in \Gamma \).
Take \( \gamma \in \Gamma \) such that \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).

So \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \).
From this it follows that \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).
Conversely let \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) then \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).
So \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).
From this it follows that \( \gamma \in \Gamma \) with \( \gamma \in \Gamma \) and \( \gamma \in \Gamma \).
and \( e(g^n) = e \in G \setminus \{e\} \).
From this it follows that the order of the element \( g^n \) of \( G^* \) is even, therefore \( o(g^n) = |G:Core(A)| \) is even.

3.23. **THEOREM.** If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \) and \( \alpha \in \Gamma \) then \( \alpha \in E_G(A) \) iff \( \alpha^A = \{a\} \) and \( \alpha^A = \{a\} \).

**PROOF.**
\( \alpha \in E_G(A) \) iff \( \alpha A = \alpha A \) iff \( \alpha A = \alpha A \).
\( \alpha \in E_G(A) \) iff \( \alpha \in E_G(A) \) iff \( \alpha \in E_G(A) \).
Conversely
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \) iff \( \alpha \in \Gamma \).
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \) iff \( \alpha \in \Gamma \).

3.24. **REMARK.** From the above proof it follows that:
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \).

3.24. **THEOREM.** If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \), \( \alpha \in \Gamma \), and \( |A:Core(A)| \) is finite then \( \alpha^A = \{a\} \) iff \( \alpha A = \alpha A \) iff \( \alpha \in E_G(A) \).

**PROOF.**
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \).
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \).
\( \alpha^A = \{a\} \) iff \( \alpha \in \Gamma \).

3.25. **THEOREM.** If \( G \) is an \( S \)-extension of a finite Abelian group \( A \) with a finite set \( \Gamma \), then \( \alpha^A = \{a\} \) iff \( \alpha A = \alpha A \) iff \( \alpha \in E_G(A) \).

**PROOF.** Let \( A \) be Abelian, \( \alpha^A = \{a\} \) and \( \beta \in E_G(A) \) for \( \beta \in \Gamma \).
It follows that \( \beta \in E_G(A) \) for \( \beta \in \Gamma \).
\( \beta \in E_G(A) \) for \( \beta \in \Gamma \).
\( \beta \in E_G(A) \) for \( \beta \in \Gamma \).

Let conversely \( A \) be Abelian and finite, \( \alpha A = \alpha A \) and \( \alpha^A = \{a\} \). It follows that \( \alpha A = \alpha A \) because \( A \) is finite. So:
\[ a^A = \{a\} \rightarrow a^G(A) = a^G = \{a^G\} = \{a\} = a = a^A = a^3_a. \]

\( G \) is a group and \( D \) is a subset of the set \( G \). The subgroup of \( G \) generated by \( D \) is written \( \langle D \rangle \).

3.26. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \tau \), it follows:
1. \( \langle \tau \rangle = \{a^\mathbb{A} | a^\mathbb{A}, \tau \mathbb{A}\}, \)
2. \( \langle \tau \rangle \mathbb{G} \mathbb{A} = \{a^\mathbb{A}, \tau \mathbb{A}\} \).

PROOF.
1. Obvious.
2. Put \( \langle \tau \rangle \mathbb{A} = K \mathbb{A} \) and \( \{a^\mathbb{A} | a^\mathbb{A}, \tau \mathbb{A}\} = K \).

K is because \( a^\mathbb{A} = a^\mathbb{A}(a^\mathbb{A})^{-1} \). What follows shows that all \( K \).

\[ \beta^\gamma = a(\beta^\gamma) \text{ for all } a \mathbb{A}, \beta \mathbb{A} \text{ and } \gamma \mathbb{A}. \]

\[ a^\mathbb{A} \beta^{-1} \mathbb{A} = a(\beta^{-1}) \text{ for all } a \mathbb{A}, \beta \mathbb{A} \text{ and } \gamma \mathbb{A}. \]

\[ a^\mathbb{A} \beta^{-1} \mathbb{A} = a(\beta^{-1}) \text{ for all } a \mathbb{A}, \beta \mathbb{A} \text{ and } \gamma \mathbb{A}. \]

Suppose \( g \mathbb{K} \) then \( g \mathbb{A} \mathbb{G} \mathbb{A} \) is a product of elements of \( \mathbb{F} \mathbb{G}^{-1} \).

From \( a^\mathbb{A} = a^\mathbb{A} \mathbb{G} \) for \( a \mathbb{A}, \beta \mathbb{A} \),

\[ a^\mathbb{A} \beta^{-1} \mathbb{A} = a(\beta^{-1}), \mathbb{F}^{-1} \text{ for all } a \mathbb{A} \text{ and } \beta \mathbb{A}. \]

\[ a^\mathbb{A} \beta^{-1} \mathbb{A} = a(\beta^{-1}) \text{ for all } a \mathbb{A} \text{ and } \beta \mathbb{A}. \]

it follows that \( g \mathbb{A} \mathbb{K} \mathbb{A} \) with \( a \mathbb{K} \mathbb{A} \mathbb{G} \mathbb{A} \).

Because \( g \mathbb{K} \mathbb{A} \mathbb{G} \mathbb{A} \), \( a = e \),

so \( g \mathbb{K} \mathbb{A} \mathbb{G} \mathbb{A} \).

3.27. DEFINITION. A group \( G \) is factorizable iff \( G = AB \) in which \( A \) and \( B \) are subgroups of \( G \) and \( A \) and \( B \) proper subgroups of the set \( G \).

3.28. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \tau \neq \{e\} \) and \( \{a^\mathbb{A} | a^\mathbb{A}, \tau \mathbb{A}\} \neq A \mathbb{A} \mathbb{A} \) then \( G \) is factorizable in \( G = A \mathbb{A} \mathbb{A} \).

PROOF. \( \tau \neq \{e\} \rightarrow A \neq G \) and \( \{a^\mathbb{A} | a^\mathbb{A}, \tau \mathbb{A}\} \neq A \mathbb{A} \mathbb{A} \) = \( \tau \mathbb{A} \neq G \).
Neargroups

In order to come to an agreement between the S-extensions and the classical Schreier-extensions in which A is a normal subgroup of G, the neargroup-idea is introduced. A neargroup is a non-associative structure, indeed possessing an identity and in which each element has a left-inverse. In the case of an S-extension the neargroup takes over the part played by the quotientgroup in the Schreier-extension theory.

4.1. DEFINITION. A neargroup is an ordered pair \( (M, \circ) \) such that \( M \) is a set and \( \circ \) is a binary operation on \( M \) so that:

a. \( \forall a \in M \) such that \( \forall a \in M : a \circ a = a \),
b. \( \forall a \in M, \forall b \in M, \exists \gamma \in M : a \circ b = \gamma \),
c. \( \forall a \in M, \forall \gamma \in M, \forall \alpha \in M : a \circ \gamma = \alpha \rightarrow a = \beta. \)

The element \( e \) is called the identity of \( (M, \circ) \) and the unique element \( a \) such that \( a \circ e = e \) is written as \( \gamma. \)

A neargroup is a group iff the neargroup is associative.

4.2. DEFINITION. A homomorphism of a neargroup \( (M_1, \circ) \) into a neargroup \( (M_2, *) \) is a \( \phi : M_1 \rightarrow M_2 \) such that

\( \forall a \in M_1, \forall b \in M_1 : (a \circ b) \circ \phi = (a \circ b) \phi. \)

If \( \phi \) is (1-1) the homomorphism is called monomorphism.

If \( \phi \) is onto the homomorphism is called epimorphism.

4.3. THEOREM. If \( \phi \) is a homomorphism of \( (M_1, \circ) \) into \( (M_2, *) \), \( e_1 \), the identity of \( (M_1, \circ) \) and \( e_2 \), the identity of \( (M_2, *) \), then \( e_1 \phi = e_2 \) and \( \exists \gamma : \phi = \gamma \) for \( a \in M_1. \)
4.4. THEOREM. Let \((X, \varnothing)\) be a neargroup and for each \(a \in N\), let \(a^P : M \rightarrow M\) be defined by \(b(a^P) = \varnothing a\) for \(a \in M\), then \(a \in N\), 
\(e(a^P) = e(e^P) = a\) and \(P:M \rightarrow M\) is \((1-1)\).

PROOF:
1. \(a = a^P\) is onto and \(1.1. a = a^P\) is \((1-1)\) so \(a \in N\).
2. \(a = e(a^P) = e(e^P) = a\).

From \(aP = \varnothing P = e(aP) = e(e^P) = a = \varnothing\) it follows that \(P\) is \((1-1)\).

4.5. THEOREM. If \(M\) is a set with \(e \in M\), \(P:M \rightarrow M\) is such that 
\(\forall\theta \in M : a(e^P) = e(e^P) = \varnothing\) and \(e\) is defined by \(e\theta = e(\theta)\) then
\((X, \varnothing)\) is a neargroup.

PROOF:
1. \(\varnothing = \varnothing \circ \varnothing = \varnothing\).
2. Since \(a \in M\), \(\forall \alpha \in M\), \(\forall \gamma \in M\), \(\gamma \alpha = \beta\).
3. Since \(a \in M\), \(a(\gamma \varnothing) = \beta(\gamma \varnothing) = \alpha = \beta\) so \(\alpha \gamma = \beta \gamma = \alpha = \beta\) 
for \(a \in M\), \(\beta \in M\), \(\gamma \in M\).

4.6. THEOREM. If \(G\) is an S-extension of \(A\) with \(\varnothing\) then \((G, \varnothing)\)
with \(e \varnothing = a^\theta\) for \(a \in A\) and \(e \varnothing\) is a neargroup.

PROOF. If \(\varnothing\) is the natural representation of \(G\) on \(A\) then
\(a \varnothing = a \varnothing\) for \(a \in A\) and \(a(e^\varnothing) = e(a^\varnothing) = a\) so according to 4.5. \((G, \varnothing)\)

is a neargroup.

4.7. DEFINITION. By an S-extension of \(A\) with a neargroup 
\((G, \varnothing)\) is to be understood an S-extension of \(A\) with the
set \(G\) where \(a^\theta = a \varnothing\) for \(a \in A\) and \(e \varnothing\). In this case the near-
group should be considered as given.
Equivalent S-extensions

Let $G$ be an S-extension of $A$ with $r$. For $a \in A$ and $\pi \in \pi(A)$ are $\pi^a$ and $\pi^\gamma$ the S-functions belonging to $G$.

Let $f : T - A$ be a mapping of $T$ into $A$ with $ef = e$.

$s : G - G$ is defined by $sa = a(af)a$ for $a \in A$ and $s \in \pi(A)$.

$\pi$ is (1-1) and onto and $a \pi = a$ for $a \in A$. $s^{-1} = a(af)^{-1} a$.

A new operation on the set $G$, which is indicated by $*$ is defined by $g_1 \ast g_2 = g_1 (g_2 \pi)^{-1}$ for $g_1 \in G$ and $g_2 \in G$.

$\pi$ is an isomorphism of $G^\pi$ onto $G$ where $G^\pi$ is the structure in which $G$ is the set and $*$ the operation. For $g_1 \ast g_1 \pi = g_1 (g_1 \pi)$.

It is evident that for $a \in A$, $\pi(a)$, $af$ and $\pi(af)$:

$\pi^a = a$,  
$\pi^a = a$,  
$\pi^a = a (a)^{-1}, a$,  
$\pi^a = (a)^{-1}, (a)^{\pi}$, 
$\pi^a = (a)^{\pi}$, 

$\pi$ is a system of representatives of the right cosets (s.r.c.) of $A$ in $G^\pi$ because:

$A \pi = A \pi = G$ and $a \pi = b \pi \Rightarrow a = b \Rightarrow a = b$.

The S-functions of $G^\pi$ with reference to the subgroup $A$ and the s.r.c. $\pi$ are called $\pi^a$ and $\pi^\gamma$ so that $a \pi^a = a \pi^a = a \pi^a = a \pi^a$ for $a \in A$ and $\pi \in \pi(A)$.

$\pi^a \pi^b = \pi^a \pi^b = \pi^a \pi^b$ for $a \in A$ and $\pi \in \pi(G)$.

From this it follows, because $\pi = a \pi$ and $a \pi = a \pi$ for $a \in A$ and $\pi \in \pi(A)$ that $\pi^a \pi = a \pi^a = (a)^{-1}$.

$\pi^{a \pi} = a \pi$, 

$\pi^{a \pi} = a \pi$.

$\pi^a \pi = a \pi^a = (a)^{-1}$,  
$\pi^{a \pi} = (a)^{-1} \pi^a$. 

$\pi^{a \pi} \pi = (a)^{-1} \pi^a \pi$.
G and \( G' \) are called equivalent S-extensions of \( A \) with \( \Gamma \) by means of \( \pi \cdot \varphi ^{-1} \cdot A \) or by means of \( \pi \).

5.1. THEOREM. Let \( G \) and \( G' \) be equivalent S-extensions of \( A \) with \( \Gamma \) by means of \( \pi : G \rightarrow G' \) then \( G \) is also an S-extension of \( A \) with \( \Gamma \) and for \( a \in G \) and \( \gamma \in \Gamma \),
\[
\alpha^{\pi \cdot \gamma} = \alpha \cdot \gamma \quad \text{and} \quad \alpha^{\pi \cdot \gamma} = \alpha^{\gamma} \quad \text{in which} \quad \alpha^{\pi \cdot \gamma} \quad \text{and} \quad \alpha^{\gamma} \quad \text{are the S-functions of} \ G \ \text{with respect to the group} \ A \ \text{and the s.r.r.} \ \gamma.
\]

PROOF. \( \pi \) is an isomorphism of \( G' \) onto \( G \) and \( \alpha = \pi \) for \( \pi \cdot A \).
\[
\alpha^{\pi} \cdot \alpha^{\pi \cdot \gamma} = \alpha \gamma = \alpha \cdot \gamma = \alpha^{\gamma} = \alpha^{\gamma} \cdot \alpha^{\pi \cdot \gamma} = \alpha^{\gamma} \left( \alpha^{\pi \cdot \gamma} \right)
\]

Let \( \pi : G \rightarrow G' \) be the natural representation of \( G \) on \( \Gamma \) and
\( \pi : G' \rightarrow G' \) be the natural representation of \( G' \) on \( \Gamma \).
\[
\left( a^{\pi b} \right)^{\pi} = \left( a^{\pi b} \right)^{\pi} = \left( a^{\pi b} \right)^{\pi} = \left( a^{\pi b} \right)^{\pi} \ \text{for} \ b \in A, \ a \in G \ \text{and} \ b \in G'.
\]
So \( \alpha \gamma \) = \( \alpha \gamma \) for \( \alpha \gamma \) and \( \gamma \in \Gamma \), and \( \gamma = \gamma \).

5.2 REMARK. Two S-extensions of a group \( A \) with \( \Gamma \) can be isomorphic as a group without being equivalent as S-extensions. We may get this for instance from two subgroups \( A \) and \( B \) of \( G \), which are isomorphic, while \( A \) is normal and \( B \) is not. A simple example follows.

5.3. EXAMPLE. Let the dihedral group \( D_{2n} \) be generated by \( a \) and \( t \), \( a^{2n} = 1 \), \( t^2 = 1 \) and \( tst = a^n \).
The centre of \( D_{2n} \) is \( \{ e, a^n \} \). So \( D_{2n} \) is isomorphic with an S-extension of \( C_2 \) with the set \( \Gamma \) of the natural numbers \( 2n \) with S-functions \( \alpha^\gamma \) satisfying \( \alpha^\gamma = \alpha \) for \( \alpha \in A \) and \( a \in G \).
Put \( A = \{ e \} \). So \( A \) is a subgroup of \( D_{2n} \) isomorphic with \( C_2 \), \( \alpha \) as s.r.r. of \( A \) in \( D_{2n} \) can be taken. From \( \alpha \cdot t = t \cdot \alpha^{-1} \) it follows \( D_{2n} \) is isomorphic with an S-extension of \( C_2 \) with \( \Gamma \), where the S-function \( \alpha^\gamma \) satisfies \( \alpha^\gamma = \alpha \) (mod \( 2n \)) for \( \alpha, \ a \in C_2 \) with \( \alpha \neq a \). Because \( \alpha \in \Gamma \) with \( \alpha \neq a \) (mod \( 2n \)), the two S-functions are different. So there are two S-extensions of \( C_2 \) with \( \Gamma \) which are isomorphic as a group and not equivalent as S-extension.
5.4. THEOREM. If $G$ is an $S$-extension of $A$ with $r$ and if $a$ is an element of $r$, for which $A_a = (e)$, then there is an $S$-extension $G'$ which is equivalent to $G$ and for which:

\[ \forall a \in A, \forall b \in A, \forall c \in A, \forall \gamma \in r : (r \gamma a) = (r \beta b). \]

**Proof.** $A_a = (e)$ implies $a^A = a^B = a = b$ for all $a \in A$, $b \in A$. $e \in A$ implies $A = (e)$. But then $A_a = (e)$ means $A = (e)$. So the theorem is trivial.

Now suppose $e \in A$. Therefore we may define $f : r \rightarrow A$ by $bf = (b^2)^{-1}$ for $b = a^B$ and $b \in A$, $bf$ is an arbitrary element of $A$ for $A$ and $\gamma \neq e$ and $e \neq e$. Then for $\gamma = a^B$ with $b \in A$ and for $a \in A$:

\[ a^B = \gamma (a^B) = (a^B a)^{-1} = (a^B a)^{-1} = (a^B a)^{-1} = e \]

and

\[ (\gamma \beta a) = (\gamma \beta a) = (a^B a)^{-1} \]

5.5. REMARK. For the above $S$-extensions Core(A) = (e) because Core(A) = (e).

5.6. THEOREM. If $G$ is an $S$-extension of $A$ with $r$ and if $a$ is an element of $r$ for which $A_a = (e)$, then there is an $S$-extension $G'$, equivalent to $G$ and for which:

\[ \forall a \in A, \forall b \in A, \forall c \in A, \forall \gamma \in r : (r \gamma a) = (r \beta b). \]

**Proof.** By the same arguments as in the proof of 5.4, we may define $f : r \rightarrow A$ by $bf = b^{-1}(a^B b)$ for $b = a^B$ and $b \in A$, $bf$ is an arbitrary element of $A$ for $b \in A$ and $\gamma \neq e$ and $e \neq e$. Then it follows that for $\gamma = a^B$ with $b \in A$ and $a \in A$.
\[ a^{-b} = (a^{-b})^{-1} \]

\[ (a^{-b})^{-1} = (a^{-b})^{-1} = a^{-b} \]

5.7. **Definition.** An S-extension of A with \( \gamma \) is called a split S-extension iff \( \forall \in \Gamma, \forall \in \Gamma : o\, \rho = e \).

5.8. **Theorem.** An S-extension of A with \( \gamma \) is equivalent to a split S-extension iff \( \exists \phi : \gamma \Rightarrow A \) with:

\[ \forall \in \Gamma, \forall \in \Gamma : o(\phi), \phi^{-1} \gamma = (\phi^{-1}) \gamma \phi. \]

**Proof.**

\[ a^{-b} = a^{-b} \phi^{-1} \phi = a^{-b} \phi^{-1} \phi^{-1} = (a^{-b})^{-1} \phi^{-1} \phi. \]

5.9. **Theorem.** If \( G \) is an S-extension of A with \( \gamma \) and \( \exists \phi : \gamma \Rightarrow A \) such that \( \phi = (a^{-b}) \phi \) is a subgroup of \( G \) then for the S-extension \( G' \) which is equivalent to \( G \) by means of \( \phi \) \( \forall \in \Gamma, \forall \in \Gamma : o\, \rho = e \) and \( \Gamma \) is a subgroup of \( G' \).

If moreover \( H \) is a normal subgroup of \( G' \) then \( \forall \in \Gamma, \forall \in \Gamma : o\, \rho = e \) in this case \( \Gamma \) is a normal subgroup of \( G' \) and is called a normal complement of A in \( G' \).

**Proof.** \( \forall \in \Gamma \Rightarrow G \) defined by \( \forall \in \Gamma \Rightarrow G \) for \( \forall \in \Gamma \) and \( \forall \in \Gamma \) is an isomorphism for which \( \Gamma = H \) (see the beginning of this section). Therefore \( H \) is a subgroup of \( G = \Gamma \) is a subgroup of \( G' \). So \( \forall \in \Gamma, \forall \in \Gamma : o\, \rho = e \) or \( o\, \rho = e \).

5.10. **Theorem.** If \( G \) is an S-extension of A with \( \gamma \), \( G' \) equivalent to \( G \), \( \exists \phi : \gamma \Rightarrow A \) defined by \( o\, \phi \) and \( \phi : \gamma \Rightarrow \phi \) defined by \( o\, \phi \) then \( o\, \phi \) = \( o\, \phi \) \( o\, \phi \).

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PROP. \((a\bar{\pi})^a = e = (a\bar{\pi})^{-a} = ((a\bar{\pi})^{a\bar{\pi}})^a \Rightarrow a\bar{\pi} = (a\bar{\pi})^{a\bar{\pi}} \Rightarrow a\bar{\pi} = (a\bar{\pi})^{a\bar{\pi}^{-1}}\).
A system of two-sided representatives

In this section $A:G \Rightarrow \Gamma$ is the mapping defined by $(aa)^\Gamma = e$, so $a\Gamma = \Gamma$ for $a \in A$.

A system of representatives of the right cosets of $A$ in $G$ is written as s.r.r. of $A$ in $G$ and a system of representatives of the left cosets as s.l.r. With s.t.r. is meant a common system of representatives of the left and the right cosets of $A$ in $G$.

In 6.1, it is proved that in an $S$-extension of $A$ with $T$, $\Gamma$ is a s.t.r. iff $\phi$ is $(1-1)$ and onto and in 6.7 we shall prove that $\Gamma$ is a s.t.r. iff $\phi \mapsto \phi^G$ the mapping $a^G \mapsto \phi^G$ is $(1-1)$ and onto. Van der Waerden proved by using a combinatorial theorem that if $G$ is finite every subgroup of $G$ has a s.t.r. and Zassenhaus [71] extended this proof to the case that $|G:A|$ is finite.

In 6.10, we shall prove the more general theorem that a subgroup $A$ of $G$ has a s.t.r. if $|A : \text{Core}(A)|$ is finite. This theorem includes obviously the case that $A$ itself is finite.

In the remaining part of this section we furnishes a group with a subgroup without a s.t.r.

6.1. THEOREM. If $G$ is an $S$-extension of $A$ with $T$, then $\Gamma$ is a s.t.r. of $A$ in $G$ iff $\phi$ is $(1-1)$ and onto.

PROOF. Let $\Gamma$ be a s.t.r. of $A$ in $G$.

$\Gamma$ is a s.l.r. so $\{\phi^{-1} a \phi \mid a \in A\}$ is a s.r.r. $\{\phi^{-1} a \phi \mid a \in A\}$ is a s.r.r. $\{\phi \phi^{-1} a \phi \mid a \in A\}$ is a s.r.r. $\Gamma \phi = \Gamma$ because $\Gamma \phi \phi^{-1}$ is evident.

So: $\phi \Gamma \Rightarrow \Gamma$ is onto.

Furthermore if $\phi = \phi \phi^{-1} \Rightarrow \phi = \phi$ (for $\phi \phi^{-1} = (\phi \phi^{-1})$). Hence $\phi = \phi$ so $\phi^{-1} \phi = e$ for $\Gamma$ is a s.l.r.

So: $\phi \Gamma \Rightarrow \Gamma$ is $(1-1)$. 
Conversely, let \( \psi : \Gamma \to \Theta \) be \((1,1)\) and onto.

\( \Gamma \) is a s.f.r. \( \Rightarrow \) \( \Gamma \) is a s.f.r. \( \Rightarrow \) \( \{ \alpha \in A : \exists \gamma \in \Theta \} \) is a s.f.r. \( \Rightarrow \) \( \{ \alpha \in A : \exists \gamma \in \Theta \} \) is a s.f.r. \( \Rightarrow \) \( \Gamma \) is a s.f.r. \( \Rightarrow \) \( \Gamma \) is a s.f.r.

So \( \Gamma \) is a s.f.r. of \( A \) in \( \Theta \).

The following example gives an affirmative answer to the questions "can \( \phi \) be \((1,1)\) without being onto?" and "can \( \phi \) be onto without being \((1,1)\)?".

6.2. EXAMPLE. \( \Theta \) is the set of the real numbers and \( R \subseteq R \), with the operation \((a, b)(c, d) = (a + bc, bd)\).

\( A \) is the subgroup \( \{(0, a) : a \in \mathbb{R}_1 \} \), \( F \) is the set of all \( f : R \to R \).

\( \forall a \in A, \forall b \in B \) because \( \forall a \in A, \forall b \in B \) and \( \forall a \in A, \forall b \in B \) and \( \forall a \in A, \forall b \in B \).

\( (0, 1) \) is the representative of \( A \).

The following computation, for \( \phi \), can be made.

\( (t(tf), tf) \cdot (-t(tf)) = (-t(tf)) \cdot (t(tf), tf), (-t(tf), tf) \).

Hence: \( (t(tf), tf) \phi = (t(tf), tf), (-t(tf), tf) \phi = (0, 1), \).

\( \phi : \Gamma \to \Theta \) is defined by \( t \phi = (t(tf), tf) \) for \( t \in \mathbb{R}_1 \), and \( 0 \phi = (0, 1) \).

Obviously \( \phi \) is \((1,1)\) and onto.

\( g : R \to R \) is defined by \( t \phi = (t(tf), tf) \) for \( t \in \mathbb{R}_1 \), and \( 0 g = 0 \).

Hence \( t \phi = t g \) for \( t \in \mathbb{R}_1 \), so \( t \phi = t g \).

It follows that \( \phi \) is onto iff \( g \) is onto and \( \phi \) is \((1,1)\) iff \( g \) is \((1,1)\).

By a suitable choice of \( \phi \) it is easy to make \( g \) \((1,1)\) and not onto or onto and not \((1,1)\).

Let \( \text{CCA} \) and \( \text{DCB} \), then \( \phi \vert C \) is the restriction of \( \phi : A \to B \) to \( C \) and \( \phi \vert C : C \to D \) is the restriction of \( \phi : A \to D \) to \( C \).

6.3. THEOREM. If \( C \) is an \( S \)-extension of \( A \) with \( \Gamma \) and \( \sigma : A \to A \) is defined by \( \sigma (\alpha) = \alpha \) for \( \alpha \in \Gamma \) and \( \sigma (a) = a \), then

\( \phi \mid A \circ \sigma = \sigma \) is onto iff \( \sigma \) is onto.
PROOF. Let \( a \subseteq a \cup a \rightarrow a \) be onto. So \( \forall \, a \in \mathcal{A}, \exists \mathcal{B} \in \mathcal{A} : a = \overline{a} \).

For these \( a \) and \( b \) the following conclusions can be made:
\( b \subseteq a \cup b \rightarrow a \subseteq a \cup b \) and \( a = b \subseteq \mathcal{A} \cup \mathcal{B} = \mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \), hence \( a \) is onto.

Conversely let \( a \) be onto. So \( \forall \, a \in \mathcal{A}, \exists \mathcal{B} \in \mathcal{A} : \mathcal{B} \subseteq a \).

It follows that: \( \overline{a} \subseteq \overline{a} \subseteq a \). Hence \( \phi \mid a \subseteq a \cup a \rightarrow a \) is onto.

6.4. THEOREM. If \( \mathcal{G} \) is an \( S \)-extension of \( \mathcal{A} \) with \( \Gamma \) and \( \mathcal{S} \) is defined as in 6.3, then \( \phi \mid a \subseteq a \cup a \rightarrow a \) is \( (1-1) \) iff \( \mathcal{S} \) is \( (1-1) \).

PROOF. Let \( \phi \mid a \subseteq a \cup a \rightarrow a \) be \((1-1)\) and \( \mathcal{S} \subseteq \mathcal{A} \), \( \mathcal{S} \subseteq \mathcal{A} \) then \( \overline{a} \subseteq b \subseteq a \cup b \rightarrow \overline{a} \subseteq b \subseteq a \cup b \).

Then \( a \subseteq b \subseteq a \cup b \rightarrow \overline{a} \subseteq b \subseteq a \cup b \).

For \( a \subseteq b \subseteq a \cup b \rightarrow \overline{a} \subseteq b \subseteq a \cup b \), put \( a \subseteq b \subseteq a \cup b \rightarrow \overline{a} \subseteq b \subseteq a \cup b \).

6.5. THEOREM. If \( \mathcal{G} \) is an \( S \)-extension of \( \mathcal{A} \) with \( \Gamma \), then for \( \phi \subseteq \mathcal{G}, (\phi) \subseteq a \).

PROOF. From 3.21, it follows that \( \phi \subseteq \mathcal{G}, (\phi) \subseteq a \).

Let \( \overline{\mathcal{B}} \subseteq \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A} \) for \( \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A} \).

From \( \mathcal{B} \subseteq a \subseteq a \) (3.21.) it follows that \( \mathcal{B} \subseteq a \).

Hence \( (\phi) \subseteq \mathcal{A} \).

\( \square \)
6.6. THEOREM.
1. \( \psi: \Gamma \rightarrow 1 \) iff \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is \( (1-1) \).
2. \( \psi: \Gamma \rightarrow \Gamma \) is onto iff \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is onto.

PROOF.
1. If \( \psi: \Gamma \rightarrow 1 \) then \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) obviously \( (1-1) \).
   Conversely, \( \overline{\overline{\sigma}} = \overline{\sigma} \Rightarrow \overline{\sigma} = \overline{\overline{\sigma}} = \overline{\overline{\overline{\sigma}}} = \overline{\overline{\sigma}} = \overline{\sigma} \) (3.20) \( \exists \in \sigma^A \).
   So \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is \( (1-1) \) \( \Rightarrow \psi: \Gamma \rightarrow 1 \) is \( (1-1) \).
2. \( \psi: \Gamma \rightarrow \Gamma \) is onto \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is onto. This follows from 6.5.
   Conversely, \( \overline{\overline{\sigma}} = \overline{\sigma} \) making use of the fact that
   \( \phi \rceil \overline{\sigma} : \overline{\sigma} \rightarrow \overline{\overline{\sigma}} \) is onto, it follows that \( \exists \in \sigma^A \) with \( \overline{\sigma} = \overline{\overline{\sigma}} = \sigma \).
   So \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is onto \( \Rightarrow \psi: \Gamma \rightarrow \Gamma \) is onto.

6.7. COROLLARY.
1. That \( \Gamma \) is a s.t.r. of \( A \) in \( G \) iff \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is \( (1-1) \)
   and onto follows from 6.1. and 6.6.
2. That \( \Gamma \) is a s.t.r. of \( A \) in \( G \) iff \( \forall \sigma \in \Gamma, \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is \( (1-1) \) and onto
   (\( \phi \) is defined as in 6.3.) follows from 6.3., 6.4. and 6.7.1.

6.8. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with \( \Gamma \) then \( \psi\rceil G^S \)
eq equivalent to \( G \) and having \( \Gamma \) as a s.t.r. of \( A \) in \( G^S \) iff
   \( \forall \sigma \in \Gamma, o(\sigma^A) = o(\overline{\sigma^A}) \).

PROOF. Let \( G^S \) be equivalent to \( G \) and \( \Gamma \) a s.t.r. of \( A \) in \( G^S \).
   \( \Psi: \Gamma \rightarrow \Gamma \) is defined by \( (\overline{\sigma})^A = \overline{\sigma} \).
   From 6.6, it follows that \( \Psi \rceil \sigma^A : \sigma^A \rightarrow \overline{\sigma^A} \) is \( (1-1) \) and onto.
   Since \( \Psi \rceil \sigma^A \) is onto, \( \sigma^A = \sigma^A \) (section 5), and \( o(\sigma^A) \) \( (5.10) \)
   it follows that
   \( \sigma^A \) = \( (\overline{\sigma^A})^A = (\sigma^A)^A = (\sigma^A)^A \).
   Hence, because \( \overline{\sigma} \) is \( (1-1) \), \( \forall \sigma \in \Gamma, o(\sigma^A) = o(\overline{\sigma^A}) \).
   Conversely let \( \forall \sigma \in \Gamma, o(\sigma^A) = o(\overline{\sigma^A}) \).
   Then \( \exists \overline{\sigma} \in \Gamma \) with \( \phi \rceil \sigma^A : \sigma \rightarrow \overline{\sigma} \) is \( (1-1) \) and onto.
   Suppose \( \sigma^A = \overline{\sigma^A} = \exists \in \overline{\sigma^A} \), \( \exists \sigma^A \overline{\sigma^A} = \overline{\sigma^A} = \overline{\overline{\sigma^A}} = \overline{\overline{\sigma^A}} = \overline{\sigma^A} \).
   Hence, because \( \overline{\sigma} \) is \( (1-1) \), \( \forall \sigma \in \Gamma, o(\sigma^A) = o(\overline{\sigma^A}) \).

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$f^{-1}$ is onto because $a^{\sigma A}$.

Because $a \in a^{\sigma A}$, it is possible to take for every $a \in A$ an element $a \in A$ such that $a = a^{(af)^{-1}}$. Because $e = e$, we can choose $ef = e$. So $f$ becomes a mapping from $F$ into $A$.

Let $G'$ be the $S$-extension of $A$ with $F$ equivalent to $G$ by means of $f$.

From 5.10, it follows $af = (a^f)^{-1} = a^f$. So $G' = G$.

From 6.1, it follows that $F$ is a s.t.r. of $A$ in $G'$ because $F$ is $(1-1)$ and onto.

6.9. REMARKS.
1. $o(a^A) = o(Aa^A) = |A : (Aa)\tau| = |A : (Aa)\tau| = |A : Aa| = |A : Aa|$

This follows from 3.18.

2. $o(a^A) = o(Aa) = \text{The set of right cosets of } A \text{ in } AaA \text{ has the same order as the set of right cosets of } A \text{ in } AaA$.

This follows from 3.13.

3. $o(a^A) = o(Aa) = \text{The set of right cosets of } A \text{ in } AaA \text{ has the same order as the set of left cosets of } A \text{ in } AaA$.

PROOF. $(Aa)^{-1} = AaA$.

6.10. THEOREM. Let $G$ be a group and $A$ a subgroup of $G$.

If $[A : \text{Core}(A)]$ is finite, a s.t.r. of $A$ in $G$ exists.

PROOF. This follows from 3.20. and 6.8.

6.11. THEOREM. If $A$ is a finite subgroup of $G$, a s.t.r. of $A$ in $G$ exists.

PROOF. This follows from 6.10.

6.12. THEOREM. If $A$ is a subgroup of $G$ with finite index, a s.t.r. of $A$ in $G$ exists.

PROOF. This follows from 3.21. and 6.8.

6.13. THEOREM. If $G$ is a Frobenius extension of $A$ with $F$, an $S$-extension $G'$ equivalent to $G$ exists, having $F$ as a s.t.r. of $A$ in $G$.
PROOF. This follows from 3.19.

The following furnishes a group having a subgroup without a s.t.r.

6.14. EXAMPLE.
Q is the set of the rational numbers and 2 the set of the integral rational numbers Q, = Q \ {0}.
G is a group on QxQ, with the operation:
(a,b)(c,d) = (a+bc, ad).
A is the subgroup \{(n,1) | n \in Z\}.
Take a = (0,2) and \bar{a} = (0,1), then \bar{a}a = (0,1).
a^{-1}Aa = \{(n,1) | n \in Z\},
Aa = \{(n,1) | n \in Z\}.
So: o(a^A) = |A:a| = 1.
A^{-1}Aa = \{(2n,1) | n \in Z\},
Aa = \{(2n,1) | n \in Z\}.
So: o(a^A) = |A:a| = 2.
Hence o(a^A) ≠ o(A^a).

6.15. REMARK. The group of 6.14 is a simple example of a group, having a subgroup A, shrinking by conjugation with an element of G, because a^{-1}AaCA.

6.16. THEOREM. If G is an S-extension of A with \Gamma and \phi: \Gamma \rightarrow \Gamma is onto then a s.t.r. of A in G exists.

PROOF. 6.3. and the fact that \phi is onto imply a^A \phi = a^A for \phi \in \Gamma. Using a^A = \phi a^A we get in the same way \phi a^A = a^A.
So \forall \phi \in \Gamma: o(a^A) = o(A^a).
The theorem now follows from 6.8.

6.17. THEOREM. If G is an S-extension of A with \Gamma and \phi: \Gamma \rightarrow \Gamma is (1-1) then \phi is a s.t.r. of A in G exists.

PROOF. Let \phi \in \Gamma. From \phi(a^A) = a^A, \phi(a^A) = a^A and the fact that \phi is (1-1) it follows that \forall \phi \in \Gamma: o(a^A) = o(A^a) (Schröder - Bernstein).
The theorem now follows from 6.8.
6.18. THEOREM. If G is an S-extension of A with \( \Gamma \) and \( \forall a \in \Gamma, a: A \to A \) as defined in 6.3 is onto then a s.t.r. of A in G exists.

PROOF. Follows from 6.3., 6.6. and 6.16.

6.19. THEOREM. If G is an S-extension of A with \( \Gamma \) and \( \forall a \in \Gamma, a: A \to A \) as defined in 6.3 is (1-1) then a s.t.r. of A in G exists.

PROOF. Follows from 6.4., 6.6. and 6.17.
The connection between the $S$-functions and the transfer

The commutator group of the group $G$ is called $G'$. It is evident that $g_i g_j G' = g_j g_i G'$ for $g_i \in G$ and $g_j \in G$, so that $\prod g_i G'$ with $g_i \in G$ is independent of the order of factors.

The transfer $T$ of a group $G$ in a subgroup $A$ is a homomorphism of $G$ into $A/A'$. $T$ is one of the means that are used to show that a given group is not simple or sometimes also to show that a given group is solvable. Then Ker$(T)$ is shown not to be a trivial subgroup of $G$. This method can only be used if $A \neq A'$. Moreover, the normal subgroups of $G$ which can thus be found, have an Abelian quotient group.

In 7.2. $T$ is defined by means of $S$-functions and 7.4. gives the connection between $T$ of $G$ in $A$ and $T$ of $G^\sim$ in $A$.

7.5. and 7.6. contain sufficient conditions for being equivalent of an $S$-extension to a split $S$-extension and 7.11. gives a sufficient condition that $G$ be equivalent to a $G'$ for which $G^\sim = A(G^\sim)$. In 7.12. we use 7.11. to show that any group of the order $p(p+1)$ ($p$ is prime) is solvable if any group of the order $p+1$ is solvable.

7.1. THEOREM. If $G$ is an $S$-extension of $A$ with the finite set $T$ then:

1. $\gamma_{a', A'} = \bigcup_{y \in T} \gamma_{y, A'}$ for $a' \in A$, $b \in A$ and $y \in T$.

2. $\gamma_{(ab), A'} = \bigcup_{y \in T} \gamma_{a', b, A'}$ for $a \in A$ and $b \in A$. 

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PROOF.
1. bßß is defined by γ(bßß) = (γßß) is a permutation of Γ according to 3.7.

2. \( \Pi \gamma(a\beta).A' = \Pi \gamma_a.\gamma_b.\beta'.A' = \Pi \gamma_a.\gamma_b.\beta'.A' \),
   \( \gamma \in \Gamma \)
   \( \gamma \in \Gamma \)

7.2. THEOREM. If \( G \) is an \( S \)-extension of \( A \) with a finite set \( \Gamma \) and \( T: G-A/A' \) is defined by \( a\alpha T = \Pi \gamma_a.\gamma_a.A' \) for \( \alpha \in \Lambda \) and \( a \in \Gamma \), then \( T \) is a homomorphism.

PROOF. For \( \epsilon_1 = aa \) and \( \epsilon_2 = bß \) with \( \epsilon_1 \in \Lambda \), \( \epsilon_2 \in \Gamma \), we have:

\[ \epsilon_1 \epsilon_2 T = aabßT = a.a.b.b.a.b.(a.b)ßT = \]
\[ = \Pi \gamma(a.a.b.b).\gamma((a.b)ß).A' = \gamma \in \Gamma \]
\[ = \Pi \gamma_a.\gamma_a.\gamma_b.\gamma((a.b)ß).A' = \gamma \in \Gamma \]
\[ = \Pi \gamma_a.\gamma_a.\gamma_b.\gamma((a.b)ß).A' = \gamma \in \Gamma \]
\[ = \Pi \gamma_a.\gamma_a.\gamma_b.\gammaß.A' = \gamma \in \Gamma \]
\[ = \Pi \gamma_a.\gamma_a.\gammaß.A' = \gamma \in \Gamma \]
\[ = \epsilon_1 T(\epsilon_2 T). \]

7.3. REMARKS.
1. \( T \) is called somewhat inaccurately the transfer of \( G \) in \( A \).

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2. If $A$ is Abelian, then $T$ is an endomorphism.
3. If the $S$-extension of $A$ with $T$ splits, then $T \text{C Ker}(T)$.
4. $G' \text{C Ker}(T)$ because $G'$ is Abelian.

7.4. THEOREM. If $G$ is an $S$-extension of $A$ with a finite $T$, $G'$ an $S$-extension equivalent with $0$ and derived from $G$ by means of $T$ (section 5), $T'\text{ the transfer of } G\text{ in } A$ and $T'$ the transfer of $G'$ in $A$, then $T' = T$.

PROOF. The notations of section 5 are used.

For $a \in A$ and $a \in F$: $a \circ T = \Pi \gamma \cdot a \cdot \gamma$.

$= \Pi \gamma \cdot a \cdot \gamma \cdot a \cdot (\gamma \cdot a \cdot \gamma)^{-1} \cdot a = \gamma \cdot a \\
= \Pi \gamma \cdot a \cdot \gamma \cdot a \cdot \gamma$.

$= a \circ T = a \circ a T$.

7.4. COROLLARY. From 7.4. it follows that $G$ for $g \in G$ is independent of the choice of the r.v. of $A$ in $G$.

7.5. THEOREM. If $G$ is an $S$-extension of a finite Abelian group $A$ with a finite set $T$, $(o(A), o(T)) = 1$ and $V \in AA$, for $a = a$, then there is an extension $G'$, equivalent to $G$, for which holds:

1. $V \in AA$, $V \in AA$, $a = a$,
2. $V \in AA$, $V \in AA$, $a = a$.

PROOF. $a T = a \circ (T)$ for $a \in A$ and $T$ the transfer of $G$ in $A$.

Let $k$ be a natural number satisfying $k \cdot o(T) = 1 \pmod{o(A)}$

$f: A$ is defined by $a f = (a T)^k$ so $a f = e$.

$G'$ is equivalent to $G$ by means of $f$.

$a^S = (a^T)^{-1} a^S = a^{-1} a$, so $a^S T = a T$ for $a \in A$ and $a^T$.

$a^S = (a^T)^{-1} a^S$, so $a^S T = (a^S)^{-1} a^T(a T)$ for $a \in A$ and $a^T$.

$a^S a = a f$. $a^S f = (a T)^{k} e (a T)^{k} e = a$ (because $A$ is Abelian).
\[a^g = af.\beta(f), \beta((a^g)f)^{-1} = af.\beta(f), \beta((a^g)f)^{-1} = \]

\[= (aT)^k.(Tg)^k, \beta(\gamma^{-1} \circ \gamma).\gamma^{-1}(a^gT).\gamma^{-1} = e.\]

7.6. COROLLARY. \( \Gamma \) is a normal subgroup of \( G \) because
\( \Gamma = \text{Ker}(\gamma): a^\gamma = a^g, \) and \( \text{from } \langle \gamma(\Gamma), \gamma(A) \rangle = 1 \) it follows that \( a^\gamma(\Gamma) = e = a = e.\)

7.7. REMARK. The condition \( \forall a \in A, \forall \gamma \in \Gamma : a^\gamma = a \) of 7.5. is
equivalent to \( \forall a \in A : a^{-1}Ta = T.\)

7.8. THEOREM. (Burnside) If \( p \) is a prime, \( G \) an \( S \)-extension
of a finite Abelian \( p \)-group \( A \) with a finite set \( T \), \( p \mid |G| \) \( \) and
\( g^A = a^\gamma = a \) for \( \gamma \in \Gamma, \) then there is an extension \( G' \),
equivalent to \( G \) for which holds:
1. \( \forall \gamma \in \Gamma, \forall a \in A : a^\gamma = a,\)
2. \( \forall a \in A, \forall \gamma \in \Gamma : a^{-1}g^{-1} = e.\)

PROOF. From 3.25 it follows that the above data are equi-

evalent to \( N_G(A) = C_G(A).\)

We first prove that \( a^\gamma = a^g = a \) for \( \gamma \in \Gamma \) and \( \gamma \in \Gamma.\)
If \( \gamma \in \Gamma \) then \( a^A = a^\gamma \), so \( a^\gamma = a \) for all \( \gamma \in \Gamma.\)
So we may suppose \( \gamma \in \Gamma \) and \( a^\gamma = a \) for a certain \( \gamma \in \Gamma.\)

Then \( a a^\gamma = a^\gamma = a \), \( a \in C_G(a^\gamma) \) because \( A \) is Abelian and \( a a^\gamma \in C_G(A).\)

Because \( \forall \gamma \in \Gamma, a^\gamma = a \), \( a = a a^{-1} = a^{-1} a = a^{-1} g^{-1} = g \cdot a \cdot g^{-1} = a.\)

So \( A \) and \( A \) are both Sylow-subgroups of \( N_G(a^\gamma), \) consequent-
ly they are conjugate in \( N_G(a^\gamma), \) so \( \exists \gamma \in \Gamma : a^\gamma = \gamma \) with \( a a^{-1} g^{-1} = \gamma \)

So \( \gamma \in \Gamma \) and \( a = a a^{-1} g^{-1} = a \cdot g^{-1} = a.\)

We have proved \( \forall \gamma \in \Gamma, \forall a \in A : a^\gamma = a, \) for \( \gamma \in \Gamma \) and \( \gamma \in \Gamma.\)

For every orbital of \( A \) (being the natural representation of \( G \) on \( T \)),
so \( \gamma \) is one and only one element occurs from every
orbit of \( A \) (being the natural representation of \( G \) on \( T \)).

So \( \gamma \) is one and only one element occurs from every
orbit of \( A \) (being the natural representation of \( G \) on \( T \)).

If \( \gamma = a \), then \( a^\gamma = a\gamma = a, \) so \( (ab^{-1}) = ab^{-1} \) and
\( a^{-1} a = a^{-1} b^{-1} \cdot (ab^{-1})b = a^{-1} b^{-1} b^1 = b^{-1} b.\)
So we may define \( f : \Gamma \rightarrow A \) by \( a^\gamma = a^{-1}a \) for \( a \in I \) and \( a \in A \).
We have \( c' = c \).
Let \( \theta \) be an \( S \)-extension of \( A \) with \( \Gamma \) equivalent with \( \theta \) by means of \( f \).
Then \( \forall a \in I, \exists b \in A, \exists e \in E : a = b \cdot b \).
So \( \forall a \in A, \exists b \in I : a = b \cdot b \cdot (b \cdot b)^{-1} = b^{-1} \cdot b \cdot b \cdot (b \cdot b)^{-1} = ab = a \).
So \( \theta \) is an \( S \)-extension of \( A \) with \( \Gamma \) satisfying the conditions of 7.5. and the theorem follows.

7.9. REMARK. Also in this case \( \theta = \operatorname{AKer}(\hat{T}) \) with \( A \cap \operatorname{Ker}(\hat{T}) = \{ e \} \).

7.10. LEMMA. If \( G = \hat{A} \), \( A \cap E = \{ e \} \), \( A \) a cyclic subgroup, \( E \) a finite normal subgroup \( \neq \{ e \} \) and \( G(A) = A \) then \( B = G' \).
If, moreover, the order of \( A \) is a prime then \( G \) is solvable.

PROOF. \( G' \subseteq B \) for \( A \) is Abelian. Let \( A = \{ a \} \), \( a \in B \) and suppose \( a^{-1}a = a \).
Then \( \forall a \in E \) (the set of integers) \( a^{-1}a = a \).
So \( \forall a \in E : a^{-1}a = a \).

For \( a \in E \) and \( b \in G : ab^{-1}a^{-1} = b^{-1}ab^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}a = a^{-1}b^{-1}a = b^{-1}a \).

So \( \forall a \in E : b^{-1}a = a = e \).
For \( a \in E \) and \( b \in G : ab^{-1}a^{-1} = b^{-1}ab^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}a = b^{-1}a \).
Now let the order of \( A \) be a prime, then \( \varphi : B \rightarrow B \) defined by \( a \rightarrow a^{-1}a \) is \( (1-1) \) and consequently onto. Because \( (a \cdot a^{-1}a^{-1}) \in \varphi(\hat{A}) \) it follows \( G' = \hat{B} \).

Now let the order of \( A \) be a prime, then \( \varphi : B \rightarrow B \) defined by \( a \rightarrow a^{-1}a \) is an automorphism of \( B \), leaving only the \( e \)-element fixed and the order of \( \varphi \) in \( \operatorname{Aut}(B) \) is a prime. So \( B \) is solvable.
Because \( B = G' \) also \( G \) is solvable.

7.11. THEOREM. If \( p \) is a prime, \( G \) is an \( S \)-extension of a cyclic \( p \)-group \( A \) with a finite \( \Gamma \), \( p \in (\Gamma) \) and \( a^\Gamma = (a^{-1}a) \) then \( G' \) is equivalent with \( G \) and \( \Gamma = (G')^{-1} \).
If \( o(A) = p \) then \( G \) is solvable.

PROOF. The statement follows from 7.8. (Burnside) and 7.10.
7.12. THEOREM. If $p$ is a prime and $G$ is a group with $\phi(G) = p(p+1)$ then $G$ possesses a normal subgroup of order $p$ or a normal subgroup of order $p+1$. In the latter case $G$ is solvable.

PROOF. Take $P$ to be a Sylow-subgroup of $G$ of order $p$ and $n$ the number of conjugates of $P$. According to Sylow's theorem $n \equiv 1 \pmod{p}$ and $n|p+1$, so $n = 1$ or $n = p+1$. If $n = 1$ then $P$ is normal in $G$. If $n = p+1$ then $[G:N_G(P)] = p+1$ so $N_G(P) = P$, consequently $G$ is solvable according to 7.11.

7.13. THEOREM. If $p$ is a prime and any group of order $p+1$ is solvable then any group of order $p(p+1)$ is solvable.

PROOF. The statement follows from the theorem that $G$ is solvable if it has a normal subgroup $H$ such that both $H$ and $G/H$ are solvable and 7.10.
Eight

Schreier-extensions - Pure extensions
S-extensions with a group - Frobenius-extensions

This section contains a discussion of some special S-
extensions.
In 8a a short discussion of Schreier-extensions ($\text{Core}(A) = A$)
is given, in order to illustrate the connection between the
S-functions and the factorsets and automorphisms occurring in
Schreier-extensions.
8b gives the other extreme case via $\text{Core}(A) = \{e\}$. These exten-
sions are called pure extensions. Because in this case $x$ is a
monomorphism of $G$ in $S_p$, pure extensions are isomorphic with
a transitive subgroup of $S_p$.
Therefore it is possible to describe all the pure extensions
belonging to a given neargroup $T$. Another consequence of the
above is that the S-functions have to fulfill fewer conditions.
S-extensions for which the neargroup is a group are discussed in 8c. Some theorems about the solvability of group extensions
are proved.
In 8d we suppose that: $\forall a \in G \setminus \{e\} : a^n = e = a = e$. Limitation
to finite groups results in $G$ being a Frobenius-group. This
assertion will only be proved if $A$ is solvable.
This theorem holds also good, if $A$ is non-solvable, but of
this no group-character free proof is known.

A. SCHREIER-EXTENSIONS

8.1. DEFINITION. $G$ is a Schreier-extension of $A$ with $T$ iff
$G$ is an $S$-extension of $A$ with $T$ and $\forall a \in T, \forall a \in A : a^n = a$. 

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8.2. THEOREM. \( G \) is a Schreier-extension of \( A \) with \( I \iff \text{Core}(A) = A \).

8.3. THEOREM. There exists a Schreier-extension of \( A \) with \( (I, \circ) \iff (I, \circ) \) is a group.

PROOF. Let \( G \) be a Schreier-extension of \( A \) with \( (I, \circ) \) then \( G \) is a group for \( \forall \alpha \in A \forall \gamma \in \Gamma \gamma \circ (\alpha \circ \gamma) = (\alpha \gamma) \circ \gamma = (\alpha \gamma) \gamma \). So \( (I, \circ) \) is associative, hence \( (I, \circ) \) is a group for a near-group possesses an identity and every element of a neargroup has a left inverse.

If \( (I, \circ) \) is a group, the direct product of \( A \) with \( I \) is a Schreier-extension of \( A \) with the neargroup \( (I, \circ) \).

8.4. COROLLARY. If \( G \) is a Schreier-extension of \( A \) with \( (I, \circ) \) then \( (I, \circ) \) is isomorphic with \( G/A \).

PROOF. This follows from 3.7.

Every Schreier-extension of a group \( A \) with a group \( (I, \circ) \) can be considered as a quotient group of the free semigroup on \( A^\Gamma \), with \( \Gamma = \Gamma(a) \) with respect to the identities:

\[
\begin{align*}
ab &= (ab) & \text{for } a \in A \text{ and } b \in A, \\
ab &= a.b & \text{for } a \in A \text{ and } \forall \gamma \in \Gamma, \\
as &= (a \gamma)(as) & \text{for } a \in A, \forall \gamma \in \Gamma, \\
as &= a & \text{for } a \in A, \\
s^a &= a & \text{satisfy:} \\
s^a.s^b &= (ab), \\
s^a.(a \circ b)_a &= a.(b \circ a).a, \\
s^a.(a \circ b)_a &= a.(b \circ a).a, \\
s^a &= e, \\
s^a &= a, \\
s^e &= e & \text{for } a \in A, b \in A, a \circ b, a \in \Gamma, b \in \Gamma, \forall \gamma \in \Gamma.
\end{align*}
\]

8.5. REMARK. \( (\circ, a \in A, b \in A) \) is called the factorset of the Schreier-extension.
8.6. THEOREM. If $G$ is a Schreier-extension of $A$ with $(r,\alpha)$ then $c: A \rightarrow A$ defined by $c(\alpha) = \alpha a$ for $\alpha \in A$ and $cA$ is an automorphism and $c: \text{Aut}(A)/\text{Inn}(A)$, defined by $c = \overline{c} \in \text{Inn}(A)$ for $c \in G$ is a homomorphism.

PROOF. $c$ is an automorphism of $A$ because $a^c = a^c a^{-1}$ and $A$ is a normal subgroup of $G$.

Let $a$ be the inner automorphism of $A$ with $a$ and $\overline{a}$ be the inverse of $a$ in $(r,\alpha)$. So $\overline{a}a = (a\overline{a})$.

From $\overline{a}, (a\overline{a})a = (a\overline{a})a$, it follows that

$v_0 \in A : a(\overline{a}a) = (a\overline{a})^{-1}(a(\overline{a}a))(a\overline{a})^{-1} = a(\overline{a}a)(a\overline{a})^{-1}.

\text{so } a(\overline{a}a) \text{ Inn}(A) = (a(\overline{a}a))g(\overline{a}a)g(A)\text{ Inn}(A).

Hence $(a(\overline{a}a))g(\overline{a}a)g(A)\text{ Inn}(A)$ or $a(\overline{a}a) = \alpha G\text{ Inn}(A)$.

The investigation for the Schreier-extensions which are possible if $A$, $(r,\alpha)$ and $g$ are given, forms the subject of the Schreier-extension theory.

B. PURE EXTENSIONS

8.7. DEFINITION. $G$ is a pure extension of $A$ with $r$ iff $G$ is an S-extension of $A$ with $r$ and $\forall \in A : (a^n = a \Rightarrow a = e)$ for $a \in A$.

8.8. THEOREM. $G$ is a pure extension of $A$ with $r$ iff $\text{Core}(A) = (e)$.

8.9. COROLLARY. If $G$ is a pure extension of $A$ with $r$, then the natural representation $\phi$ of $G$ on $r$ is a monomorphism because $\text{Core}(A) = (e)$.

8.10. THEOREM. Let $G$ be a permutation-group on $r$, $\in G$ and $G_0$ be the $a$-stabilizer of $G$. If $\forall : T \rightarrow G$ such that $\forall \in G : (aU) = \phi(aU) = a$ then
1. $G$ is transitive.
2. $\Gamma$ with the operation $\circ$ defined by $\alpha \circ \beta = \alpha(tU)$ for $\alpha, \beta \in \Gamma$ and $tU$ is a neargroup.
3. $\Gamma U$ is a s.r.r. of $G_e$ in $G$.
4. $G$ is a pure extension of $G_e$ with $\Gamma U$.
5. If $(\Gamma U, \circ)$ is the neargroup appearing in the above extension then $U: \Gamma \rightarrow \Gamma U$ is an isomorphism of $(\Gamma, \circ)$ on $(\Gamma U, \circ)$.

**Proof.** $U$ is $(1, 1)$ for $aU = bU \Rightarrow e(aU) = e(bU) \Rightarrow a = b$.
1. $e(\Gamma U) = \Gamma$ so $G$ is transitive.
2. This follows from $e(aU) = e(aU) \circ e$ and 4.5.
3. $(G_e(aU) = \{g \in G | a \circ g(aU) = a\}$ so $(G_e)(\Gamma U) = G$ and $a \neq b \Rightarrow (G_e)(\Gamma U) \cap (G_e)(U) = \emptyset$.
4. $Core(G_e) \in G = \bigcap_{\alpha \in G} (G_e)^{-1} = \bigcap_{\alpha \in G} (\alpha U)^{-1} (G_e)(\alpha U) = \bigcap_{\alpha \in G} G_e = \{identity of S_1\}$.
5. $e(aU e(U)) = e(aU) e(U) = e(a) e(U) = a e(U) = e(aU e(U))$.
But $e(U) = e(U) \Rightarrow U = \gamma$ for $\gamma \in \Gamma$ and $\gamma \Gamma = \Gamma$.
So $aU e(U) = e(aU U)$.
Furthermore $U$ is $(1, 1)$.

8.11. **Definition.** If $G$ is a group and $H$ a subgroup of $G$ then $\text{Let}(H, e) = (K | K$ is a subgroup of $G$ and $eK eG)$.

8.12. **Theorem.** Let $(\Gamma, \circ)$ be a neargroup and $\Gamma: \Gamma \rightarrow \Gamma$ defined by $\alpha(\beta) = \alpha \circ \beta$, then a group $G$ contains a subgroup $A$, such that $G$ is a pure extension of $A$ with $(\Gamma, \circ)$ (4.7.) iff $\text{Let}(\Gamma \Gamma, S_\Gamma)$ contains an element isomorphic with $G$.

**Proof.** If $G$ is a pure extension of $A$ with $(\Gamma, \circ)$ and $\gamma$ the natural representation of $G$ on $\gamma$, then $\gamma \Gamma = \gamma \Gamma$ (4.6.) and $\gamma$ is a monomorphism of $G$ in $S_\Gamma$ (8.9) so $G \in \text{Let}(\Gamma \Gamma, S_\Gamma)$.
Conversely let $H \in \text{Let}(\Gamma \Gamma, S_\Gamma)$ then $\Gamma \chi H$.
$\forall e \in \Gamma: \chi(e) = e(\chi(e)) = \alpha (\gamma_4, \gamma_6)$.
So $H$ contains a subgroup $\chi$ such that $H$ is a pure extension of $H e$ with a neargroup isomorphic with $(\Gamma, \circ)$ (8.10). It follows that $H$ is isomorphic with a group which is a pure extension.
8.13. REMARK. From 8.12, it follows, that if \( (\Gamma, \sigma) \) is given all the pure extensions, in which \((\Gamma, \sigma)\) appears as neargroup, are known up to isomorphism.

8.14. THEOREM. If \( A \) is a group, \((\Gamma, \sigma)\) a neargroup and \( P \) defined as in 8.12, then \( \exists \) a pure extension of \( A \) with \((\Gamma, \sigma)\) iff \( \exists \) a monomorphism \( \phi : A \to (\Gamma, \sigma) \) with:

1. \( (TP) \subseteq CA^\sigma \),
2. \( \forall a \in A, \forall \gamma \in \Gamma : \gamma (aP) \sigma = A \gamma \) and \( \forall \gamma \in \Gamma : \gamma (aP) \sigma = \gamma (aP) \).

PROOF. Let \( G \) be a pure extension of \( A \) with \((\Gamma, \sigma)\) and \( \gamma \) the natural representation of \( G \) on \( I \) then \( \gamma \) is a monomorphism and \( aP = \sigma a \) for \( \forall a \in A \). So \( (TP) \subseteq (\Gamma P) \subseteq (\Gamma P)^\prime = A \gamma \) and \( \forall \gamma \in \Gamma : \gamma (aP) \sigma = \gamma (aP) \).

Conversely let \( \phi : A \to (\Gamma, \sigma) \) be a monomorphism satisfying 1. and 2. Because \( \forall a \in A : \phi (aP) = \phi (aP) \sigma = \sigma a \), \( (TP) \) is a transitive permutation group on \( P \) and \( T P \) is a s.r.r. of \( (TP) \) in \( (\Gamma P) \) \((6.10)\).

So any element of \( (TP) \) may be written uniquely as a product of an element of \( A \) and an element of \( \Gamma P \). In particular:
3. \( \forall \gamma \in \Gamma P, \exists \gamma \in \Gamma P : \gamma = \gamma (aP) \).

Let \( H \) be the subgroup of \( S_P \) generated by \( A \cup TP \). An element of \( H \) is a product of elements of \( A \cup TP \) and their inverses. Using 3. and 2. it may be put into the form \( b \sigma c \) with \( b \in A \) and \( c \in (\Gamma P) \) and then by the remark on elements of \( (TP) \) into the form \( \phi (aP) \) with \( \phi (aP) \in \Gamma P \).

Because \( \phi (aP) \sigma = \phi (aP) \), we have \( \phi (A) = \phi (A) \gamma = \phi (A) \). By 8.10, \( A \) is a pure extension of \( A \) with \( (\Gamma, \sigma) \), whereas the corresponding neargroup is isomorphic with \((\Gamma, \sigma)\). The groups \( A \) and \( A \gamma \) are isomorphic. So there also exists a pure extension \( G \) of \( A \) with the neargroup \((\Gamma, \sigma)\).

8.14.a. REMARK.
1. The \( S \)-functions constructing \( G \) from \( A \) and \( \Gamma \) may be made explicit as follows.
\(a^g = a(a^g),\quad a^g = a(gP),\)
\((a^g)^g = aP(a^g)(a^gP)^{-1},\quad (a^g)_g = aP(gP)(a(gP)^{-1})^{-1},\)
for \(g \in G, \ a \in \Gamma, \ g \in G.

Where the formula for \((a^g)_g\) is motivated by:
\(eP(e^gP)(e^gP)^{-1} = eP(e^gP)^{-1} = e.\)

2. In any pure extension the function \(a^g\) determines uniquely the function \(a^g\)'s.

8.15. DEFINITION. If \(G\) is a transitive permutationgroup on \(\Gamma\),
then \(\Gamma G\) is a block iff
1. \(1 < (aB)\).
2. \(\Gamma \neq \Gamma\).
3. \(\forall \sigma \in \Gamma : B = \sigma B\) or \(\sigma B \cap B\) is empty.

8.16. DEFINITION. A primitive permutationgroup is a transitive permutationgroup without blocks.

8.17. THEOREM. A transitive permutationgroup \(G\) on \(\Gamma\) with \(G\) transitive is primitive iff \(^G\sigma\) is a maximal subgroup of \(G\).

PROOF. Theorem 5.6.1. Hall [5].

8.18. THEOREM. If \(H\) is a permutationgroup, \(G \subseteq H\) and \(G\) is transitive and primitive then \(H\) is primitive.

PROOF. A block of \(H\) is also a block of \(G\).

8.19. DEFINITION. \((\Gamma, \sigma)\) is a primitive neargroup iff \((\Gamma P)\) is a primitive permutationgroup. \(P\) is defined as in 8.12.

8.20. THEOREM. If \((\Gamma, \sigma)\) is a primitive neargroup and \(G\) is a pure extension of \(A\) with \((\Gamma, \sigma)\) then \(G\) is isomorphic with a primitive permutationgroup.

PROOF. \(G\) is isomorphic with an element of \(\text{Lat}((\Gamma P), \sigma)\) (8.12) and \((\Gamma P)\) is primitive. Now the statement follows from 8.18.
C. S-EXTENSIONS WITH A GROUP

8.21. THEOREM. The neargroup \((\Gamma, e)\) of an S-extension of \(A\) with \(\Gamma\) is a group iff \(\forall a \in \Gamma, \forall e \in : \theta \in \Core(A)\) or iff \(\text{AK} \in \mathbb{C} \text{Core}(A)\).

PROOF. Suppose \(\forall a \in \Gamma, \forall e \in : \theta \in \Core(A)\) then \((\theta^\beta)^{\gamma} = \theta^{(\beta \gamma)}\) so the neargroup \((\Gamma, e)\) is associative and therefore a group.

Suppose \((\Gamma, e)\) is a group then
\[\forall a \in \Gamma, \forall b \in \Gamma, \forall e \in : a^{(b \gamma)} = (a^{\beta})^{\gamma} = (a \gamma)^{\beta}.\]

So \(\forall a \in \Gamma, \forall b \in \Gamma, \forall e \in : a^{(b \gamma)} e = a^{\gamma} e = \forall e \in : \theta \in \Core(A)\).

8.22. COROLLARY. A pure extension of \(A\) with a neargroup \((\Gamma, e)\) splits iff \((\Gamma, e)\) is a group.

8.23. THEOREM. The neargroup \((\Gamma, e)\) of an S-extension of \(A\) with \(\Gamma\) is a group iff \(\text{AK}(\Gamma)\) is a normal subgroup of \((\Gamma)\).

PROOF. Let \((\Gamma, e)\) be a group. Then \(\text{AK}(\Gamma) \subseteq \Core(A)\). So
\(\text{AK}(\Gamma)\) is a normal subgroup of \((\Gamma)\) and \((\Gamma, e)\) is isomorphic with \((\text{AK}(\Gamma), e)\) because \((\Gamma) = (\text{AK}(\Gamma), e, \Gamma)\).

Suppose \(\text{AK}(\Gamma)\) is a normal subgroup of \((\Gamma)\). Then
\[\forall a \in \Gamma, \forall b \in \Gamma, \forall e \in : a^{(b \gamma)} e^{-1} e a = e.\]

So \(\forall a \in \Gamma, \forall b \in \Gamma, \forall e \in : a^{(b \gamma)} e = e\) or \(\forall e \in : \theta \in \Core(A)\).

8.24. THEOREM. An S-extension \(G\) of a nilpotent group \(A\) with a finite nilpotent group \(\Gamma\) is solvable.

PROOF. Let \(\pi\) be the natural representation of \(G\) on \(\Gamma\).
\(G\pi\) is a pure extension of \(A\pi\) with \(\Gamma\pi\) and the neargroup \((\Gamma\pi, e')\)
is isomorphic with the group \(\Gamma\).
So \(G\pi = A\pi \times \Gamma\pi\) with \(e'\) (identity of \(S\pi\)) \(A\pi\) and \(\Gamma\pi\) are both finite and nilpotent.
Therefore \(G\pi\) is solvable. See O. Kegel [7].
\(\Core(A) = \text{Ker}(\pi)\) is nilpotent so \(\text{Ker}(\gamma)\) is solvable.
Hence \(G\pi\) is solvable.
8.25. THEOREM. An S-extension $G$ of a cyclic group $A$ with a
cyclic group $\Gamma$ is supersolvable.

PROOF. Let $\pi$ be the natural representation of $G$ on $\Gamma$.
$G\pi = A\pi \times \pi$ and $A\pi \Gamma \pi = \{e\}$ (identity of $S_\Gamma$).
$\pi$ is isomorphic with $\Gamma$ so $\Gamma$ is cyclic, also $A\pi$ is cyclic.
So $G\pi$ is supersolvable. See P. Cohn [3].
Core($A\pi$) = Ker($\pi$) is cyclic.
What follows shows that $G$ is supersolvable.
$B_0 = e$, $B_1$, $\ldots$, $B_n = G\pi$ is an invariant sequence with cyclic
factors of $G\pi$. So $B_1 (G\pi/G^0)$ is a normal subgroup of $G\pi$ and
$B_{i+1}/B_i$ is cyclic.
The sequence $e, \text{Core}(A\pi), B_1 \Gamma, \ldots, B_n \Gamma = G$ is an invariant
sequence with cyclic factors of $G$ so $G$ is supersolvable.

D. FROBENIUS-EXTENSIONS

8.26. DEFINITION. An S-extension $G$ of $A$ with $\Gamma$ is a
Frobenius-extension (F-extension) iff $G$ is finite and
$\forall a \in \Gamma \setminus \{e\}: Aa = \{e\}$.

8.27. REMARK. According to this definition, in contrast
with the usual definition, any finite group $\neq \{e\}$ is an F-
extension of the subgroup $\{e\}$.

8.28. REMARK. For an F-extension:
1. Core($A\pi$) = $\{e\}$ if $\Gamma \neq \{e\}$.
2. $N_{G\pi}(A\pi) = A$ if $A \neq \{e\}$.
3. For $gA\pi: A\pi g^{-1}Ag = A\pi a^{-1}Aa = \{e\}$. (g = $a\pi$ with $a\in A$, $\omega \in \Gamma$)

8.29. DEFINITION. If $p \in S_\Gamma$, then $\chi(p) = \sum_{a \in \gamma} a \pi p = a, \omega \pi \Gamma$.

8.30. THEOREM. If $G$ is an F-extension of $A$ with $\Gamma$, $\pi$ the
natural representation on $\Gamma$ and $g \in G$, then $\chi(g\pi) = 0$, 1 or
$\chi(\pi)$.
PROOF. \( \text{Ch}(\sigma) = \sigma(\Gamma) \).
Suppose \( g \in G \) and \( \text{Ch}(g) > 1 \). Then:
\[ \exists \gamma \in \Gamma, \exists \delta \in \Gamma \text{ with } \sigma(g) = \sigma, \delta(g) = \delta \text{ and } \gamma \neq \delta. \]
Moreover
\[ \forall \gamma \in \Gamma : \gamma(g) = \gamma = g \gamma^{-1} A \gamma. \]
Therefore \( g \sigma^{-1} A \sigma \sigma^{-1} A \delta = \sigma^{-1} (A \sigma \delta^{-1} A \sigma^{-1}) \sigma = (e) \).

6.31. DEFINITION. A regular permutation is a permutation all the cycles of which (1-cycles included) are of equal length.

6.32. THEOREM. If \( G \) is an \( F \)-extension of \( A \) with \( \Gamma \), \( g \in G \) and \( \sigma \) the natural representation on \( \Gamma \), then every \( g \) with \( \text{Ch}(g) = 0 \) is regular and the length of a cycle is a divisor of \( \text{Ch}(g) \). A permutation \( g \) with \( \text{Ch}(g) > 1 \) consists of a 1-cycle and cycles of equal length; this length is a divisor of \( \text{Ch}(g) \).

PROOF. Suppose \( g \) has an 1-cycle and an \( m \)-cycle with \( 1 < m \), then \( 1 < \text{Ch}(g) < \text{Ch}(g)^m \).

6.33. THEOREM. If \( G \) is an \( F \)-extension of \( A \) with \( \Gamma \), then
\[ \sigma(\Gamma) = k \sigma(A) + 1 \]
in which \( k \) is a natural number or zero.

PROOF. For \( n \in \Gamma \setminus \{e\} \) is \( \sigma^k = \{A, \sigma A\} = \sigma(A) \) and \( \sigma(A) \sigma(A) = 1 \).
So \( \sigma(\Gamma) = k \sigma(\Gamma) + 1 \).

6.34. THEOREM. If \( G \) is an \( F \)-extension of \( A \) with \( \Gamma \),
\[ H = \{g \in G \mid \text{Ch}(g) = 0, g \in G\} \]
then:
1. \( (H) \) is a characteristic subgroup of \( G \),
2. \( (H) \) is an \( F \)-extension of \( (H) \cap A \).

PROOF. If \( \text{Ch}(g) = 0 \), \( (g)^k \sigma(A) = \sigma^{-1} \), so \( g^k \sigma(A) = g^{-1} \)
\( k \) from 8.32. If \( \text{Ch}(g) = 1 \), \( (g)^\sigma(A) = e \) (identical permutation)
so \( \sigma(A) = e \). Therefore \( (H) = (g^k \sigma(A) g \in G) \) and consequently
a characteristic subgroup of \( G \).
Suppose \( h \in H \setminus (H) \cap A \). Then \( (H) \cap A \sigma^{-1} \{H \cap A\} h = (e) \).
So \( (H) \) is an \( F \)-extension of \( (H) \cap A \).

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8.35. THEOREM. If \( G \) is an \( F \)-extension of a solvable group \( A \) with \( F \) and \( H \) is defined as in 8.34, then \( H \cap H \leq \{e\} \).

PROOF. Without loss of generality we may assume that
\( H = 0 \), so we have to prove that \( A = \{e\} \).
Suppose \( A \neq \{e\} \), then \( A \neq A' \) because a subgroup of a solvable group is solvable.
Let \( T \) be the transfer of \( G \) in \( A \).
From the datum it follows:
\( \forall a \in \{e\} : A^a = \{e\} \) and so from 5.4, it follows:
\( A \leq A' \), equivalent with \( A = A' \).
Because \( A \leq A' \), it follows \( A = A' \).
So \( A' = A/A' = A \).
\( H \leq \langle h \rangle \rangle \langle h \rangle \rangle \). Because \( \langle h \rangle \rangle \langle h \rangle \rangle = 1 \).
\( \forall h : \langle h \rangle \rangle \langle h \rangle \rangle = 1 \).
So \( \forall h : \langle h \rangle \rangle \langle h \rangle \rangle = 1 \).
Then \( \forall h : \langle h \rangle \rangle \langle h \rangle \rangle = 1 \), contradicting \( A = A' \) and \( A \neq A' \). So \( A = \{e\} \).

8.36 REMARK. \( H \leq \{e\} \) is a characteristic subgroup of \( G \) called the Frobenius Kernel of \( G \).
In section 2 we have constructed a semigroup $G$ generated by a subsemigroup $A$ and a subset $T$. For section 10 and section 11 a generalisation of this construction is needed.

Let $A$ itself have a generating set $T$. We suppose that a set $I_i$ of defining relations for $A$ in terms of $T$ has been given. We now require a construction of a semigroup $G$ with subsemigroup $A$ from a generating set $T_i \cup I_i$ instead of $A \cup T_i$.

Because $A$ is isomorphic with a quotient semigroup of a free semigroup generated by $T_i$, in the construction it is supposed that $A$ itself is this quotient semigroup.

The construction follows the same pattern as the construction in section 2, but is somewhat more complicated.

Let $A$ be a quotient semigroup with identity of the free semigroup $F_i$ with generating set $T_i$ and with respect to the set of relations $I_i$.

Let $G_i$ be a set of representatives in $F_i$ for the elements of $A$ and let $E$ denote an element of the identity of $A$. We do not suppose $E G_i$.

9.1. DEFINITION. $S_i : F_i \rightarrow G_i$ is defined by:

$ZS_i, aG_i$, and $(ZS_i, Z) \in T_i$ for $Z \in F_i$ (see 2.3.)

9.2. LEMMA. $\forall Z \in F_i$, $\forall W \in G_i : Z_i ZZ_i = Z_i ZZ_i = Z_i (Z_i Z_i)Z_i$.

PROOF. Obvious.

Let $F_i$ be a set satisfying $F_i \cap T_i = \emptyset$.

Put $T_i \cup \emptyset_i = T$ and $F_i \cup \emptyset = \emptyset_i$.
Let \( P \) be the free semigroup generated by \( \sigma \). Because \( T, \sigma \) it is allowed to consider \( F_1 \) as a subsemigroup of \( P \).

The following mappings are supposed to be given:
\[
\begin{align*}
(a, s) & \mapsto T^s, \text{ with } a \in T^s \\
\sigma & \mapsto 1, \text{ with } \sigma \in T^1 \\
\sigma^2 & \mapsto 1, \text{ with } \sigma^2 \in T^1.
\end{align*}
\]

The following set of relations I in \( Y \) is taken:
\[
\begin{align*}
V \in F_1^{*}, & \, V \in F_1^{*}, \text{ with } (X, Y) \in I, X : Y, \\
Y & \in F_1^{*}, \text{ with } a \in T^a, \text{ for } a \in T. \\
Y & \in F_1^{*}, \text{ with } a \in T^a, \text{ for } a \in T.
\end{align*}
\]

Let \( G \) be the quotient semigroup of \( P \) with respect to \( I \).

We shall determine necessary and sufficient conditions in order that every element of \( G \) contains exactly one word of the form \( Z a \) with \( Z \in T^Z \), \( a \in T > 0 \).

For that purpose a standard reduction \( \mathcal{B} \) will be defined.

We first extend the functions \((a,s)0\) and \( a^\sigma \) as follows:
\[
\begin{align*}
(a, s) & \mapsto a^s, \text{ with } a \in T^s \\
\sigma & \mapsto 1, \text{ with } \sigma \in T^1 \\
\sigma^2 & \mapsto 1, \text{ with } \sigma^2 \in T^1 \\
0 & \mapsto 0, \text{ with } 0 \in T^0
\end{align*}
\]

by putting
\[
\begin{align*}
(E, s) & = a^s, \text{ for } a \in T^s, \\
(E, 0) & = 0, \text{ for } a \in T^0.
\end{align*}
\]

In order to avoid complications we impose the following condition.

9.3. CONDITION.
\[
\begin{align*}
V_1, & \in F_1^{*}, V_2, \in H_1^{*}, V_3, \in F_1^{*}, \text{ with } \omega(\emptyset): Z \neq Z_1, Z_2, \\
V_1, & \in H_1^{*}, V_2, \in H_1^{*}, V_3, \in F_1^{*}, \text{ with } \omega(\emptyset): E = F, E_1 = Z_1, \# Z_1, \emptyset.
\end{align*}
\]

9.4. DEFINITION of a reduction-step.

Let \( \mathcal{B} = \{ [Z_1, Z_2] \mid Z \in T^Z \} \).

A reduction-step \( f : \mathcal{B} \) is defined by
\[
\begin{align*}
z f = Z E & \text{ for } Z \in T^Z, \\
\sigma f = Z E & \text{ for } \sigma \in T^1, \\
\sigma^2 f = Z E & \text{ for } \sigma^2 \in T^1, \\
z f = Z E & \text{ for } Z \in T^Z, \text{ with } \omega(\emptyset): Z \neq Z_1, Z_2.
\end{align*}
\]
\[ Z_{\text{WS}} = Z_{\text{WS}}^{\text{E}} \text{ for } Z_{\text{EF}}, Z_{\text{WS}}^{\text{E}}, \text{wef}(\phi), V, \in, \text{e}, \ldots, \]
\[ V_{\text{Z}} \in F, \text{we}(\phi); 2 \neq 2, \text{E}, \]
\[ Z_{\text{WS}} = 2((s,a)g)_n, Z_{\text{WS}}^{\text{E}} \text{ for } Z_{\text{E}}, \text{wep}, \text{we}(\phi). \]

9.5. REMARKS.
1. By 9.3, it is guaranteed that \( f \) is determined uniquely by
9.4, and that the domain of \( f \) is \( F \setminus \text{E} \).
2. Taking \( \rho = E \) in the last line of 9.4, and using 9.2, we get
\[ Z_{\text{E}} = Z_{\text{E}}^{\text{E}}, \text{wef} \text{ for } Z_{\text{E}}, \text{wep}, \text{we}(\phi), \]
\[ Z_{\text{E}} = Z_{\text{E}}^{\text{E}} \text{ for } Z_{\text{E}}, \text{wep}, \text{we}(\phi). \]
3. If \( g \in G, \text{we} g \) and \( \text{we} g \), then \( \text{we} g \).
4. \( \forall \text{wef} \setminus \text{E}, Z_{\text{WS}}^{\text{E}} \text{wef} \) with \( \text{we} g \).
5. Every element of \( G \) contains at least one element of \( G \).

9.6. DEFINITION OF A STANDARD REDUCTION \( S \) IN \( F \).
\( S:F^+S \) is defined by:
\[ WS = W \text{ for } \text{we} g, \]
\[ WS = W^n \text{ for } \text{wef} \setminus \text{E}, \text{we} g. \]

9.7. REMARKS.
1. If \( g \in G \) and \( \text{we} g \), then \( \text{we} g \).
2. \( WS = W S \) for \( \text{we} g \).

9.8. LEMMA.
1. \( Z_S = Z_{\text{S}}^S \) \( E \) for \( Z_{\text{EF}}, \)
2. \( Z_{\text{WS}} = Z_{\text{S}}^S, \text{wef} \) \( \text{for } Z_{\text{EF}}, \in, \text{wef}, \text{we}(\phi). \)

PROOF. If \( Z_{\text{E}} \), then \( Z_S = Z_{\text{S}}^S, \text{E} \) and \( Z_{\text{WS}} = Z_{\text{S}}^S, \text{wef} \).
If \( Z_{\text{E}}, V, Z_{\text{E}}, Z_{\text{E}}, Z_{\text{E}}^S, \text{we}(\phi); Z \neq 2, \text{E}, \)
then \( Z_S = Z_{\text{S}}^S, \text{E} \) and \( Z_{\text{WS}} = Z_{\text{S}}^S, \text{wef} \).
If \( Z = 2, \text{E}, Z_{\text{E}}, Z_{\text{E}}, Z_{\text{E}}, Z_{\text{E}}^S, \text{we}(\phi) \), we apply
mathematical induction with respect to the number of letters
in \( Z \).
If \( Z = 2, \text{E}, Z_{\text{E}}, Z_{\text{E}}, Z_{\text{E}}, Z_{\text{E}}^S, \text{we}(\phi) \),
then \( Z_S = Z_{\text{S}}^S, \text{E} \) and \( Z_{\text{WS}} = Z_{\text{S}}^S, \text{wef} \).
If \( Z_{\text{S}} = Z_{\text{S}}^S, \text{E} \), put \( Z_{\text{E}} = Z_{\text{E}}^S, \text{wef} \).
then \( Z_{\text{E}} = Z_{\text{E}}^S, \text{E} = Z_{\text{S}}^S, \text{E} \) and \( Z_{\text{WS}} = Z_{\text{S}}^S, \text{wef} \).
by induction \( Z_{\text{S}}^S, \text{E} = Z_{\text{S}}^S, \text{wef} \).
then \( Z_{\text{S}}^S, \text{E} = Z_{\text{S}}^S, \text{wef} \).
9.9. **Lemma.** \( Z \subseteq \text{VS} \) for \( Z \in \mathbb{E}_I \). \( \mathcal{V}_I \).

**Proof.** If \( Z \in \mathbb{E}_I \), then (by 9.8.1) \( Z \subseteq \text{VS} \), i.e.

\[ Z \subseteq \text{VS} \]

If \( Z \subseteq \text{VS} \) with \( Z \cap \mathcal{V}_I \neq \emptyset \), then (by 9.8.2)

\[ Z \subseteq \text{VS} \]

9.10. **Theorem.**

1. \( \mathcal{V}_I \subseteq \text{VS} \) for \( \mathcal{V}_I \) and \( \mathcal{V}_I \).

2. \( \text{VS} \subseteq Z(\mathcal{C}_I) \subseteq \mathcal{V}_I \).

**Proof.** From the definition of \( \mathcal{C}_I \) it follows that it suffices to prove:

1. \( \mathcal{V}_I \subseteq \text{VS} \) for \( \mathcal{V}_I \) and \( \mathcal{V}_I \).

2. \( \text{VS} \subseteq Z(\mathcal{C}_I) \subseteq \mathcal{V}_I \).

The only case in which the first of these two is not an immediate consequence of the definition of \( \mathcal{C}_I \) is where \( \mathcal{V}_I \) and \( \mathcal{V}_I \) is true, whereas \( \mathcal{V}_I \) is true or \( \mathcal{V}_I \) is false. In both cases the result follows from 9.9.

The only case in which the second equality is not an immediate consequence of 9.9. is the last case of definition 9.4.

\[ Z(\mathcal{C}_I) = \text{VS} \]

9.11. **Theorem.** Every \( \mathcal{E}_I \) contains exactly one element of \( \mathcal{E} \) if \( \forall \mathcal{E}_I \subseteq \mathcal{V}_I \).

**Proof.** Completely similar to the proof of 2.12.

9.12. **Remark.** Specifying the relations of \( \mathcal{I} \) we get for all \( \mathcal{E}_I \);

\[ \sigma \times \sigma = \alpha \times \alpha \quad \text{for} \quad (\mathcal{I}, \mathcal{I}) \subseteq \mathcal{V}_I \]

\[ \alpha \times \alpha = \alpha \times \alpha \quad \text{for} \quad \mathcal{E}_I \subseteq \mathcal{V}_I \]

\[ \alpha \times \alpha = \alpha \times \alpha \quad \text{for} \quad \mathcal{E}_I \subseteq \mathcal{V}_I \]

9.13. **Theorem.** If every element of \( \mathcal{E}_I \) contains exactly one element of \( \mathcal{E}_I \), \( Z \in \mathbb{E}_I \), and \( Z \in \mathbb{E}_I \), then \( (Z, Z) \in \mathbb{E}_I \).

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PROOF. \((z_1, s_1) \in H_1 \Rightarrow (z_2, z_2) \in E_1\) is an immediate consequence of \(1,\langle 1\rangle\).

Suppose conversely that \((z_1, z_2) \in H_1\). Then \((z_1, s_1, s_1) \in E_1\)
\(\Rightarrow z_1, s_1 = z_2, s_1 = z_2, s_1 = z_1, s_1 = (z_1, z_2) \in E_1\).

If every element of \(G\) contains exactly one element of \(H\), we define \(\sigma : A \cup \Gamma \leftrightarrow G\) as follows:
for \(a \in A, s \in \Sigma, a \in \Sigma, a \in G\),
for \(a \in \Gamma, s \in \Sigma, a \in G\).

Then \(\sigma\) is \((1-1)\) and \(\sigma|A\) is a nonomorphism.
Identifying \(A \cup \Gamma\) and its image in \(G\) we may consider \(A\) as a subsemigroup of \(G\) and \(\Gamma\) as a subset of \(G\). If this identification is made, \(G\) is generated by \(A \cup \Gamma\). Every element of \(G\) may be written in the form \(a^m \sigma^n\) with \(a \in A, \sigma \in \Sigma, u(e)\), where \(e\) denotes the identity of \(A\).

9.14. THEOREM. If \(G\) satisfies the condition of 9.11, \(G\) is a group iff \(A\) is a group and \(\forall a \in A, \exists \sigma \in \Gamma\) such that \(a^o = 1\).

PROOF. Similar to the proof of 2.14.
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A construction of an S-extension from a given Schreier-extension and a given pure extension

10.1. Let $G$ be an $S$-extension of $A$ with $\Gamma$, $K = \text{Core}(A)$ and $\pi$ the natural representation of $G$ on $\Gamma$, then:

$$K = \text{Ker}(\pi), \pi|\Gamma \text{ is (1-1)}.$$ 

$C = G\pi$ is a pure extension of $B = A\pi$ with $\pi\Gamma$.

We try to construct $G$ if $A$ and the pure extension $C$ of $B$ and $\pi\Gamma$ are given. We may consider $A$ as a Schreier-extension of $K$ and $B$.

We may reformulate the problem as follows:

Let the groups $K$ and $B$, the set $\Gamma$, a Schreier-extension $A$ of $K$ with $B$ and a pure extension $C$ of $B$ with $\pi\Gamma$ be given.

Construct all $S$-extensions $G$ of $A$ with $\Gamma$ with natural representation $\pi$ of $G$ on $\Gamma$, such that $K = \text{Core}(A)$ in $G$.

$\pi|\Gamma$ may be extended to an isomorphic mapping of $C$ onto $\pi\Gamma$.

We shall make use of the method of section 9. Therefore the given extensions will be considered as quotient semigroups of free semigroups.

10.2. Let $K$ and $B$ be groups and $\Gamma$ a set. Let $g$ be the identity of $K$, $e$ the identity of $B$. The sets $K$, $B$ and $\Gamma$ are assumed pairwise disjoint.

10.3. Let $A$ be a Schreier-extension of $K$ with $B$: a quotient group of the free semigroup $F$, with generating set $F_1 = K\Gamma B_1$, where $B_1 = B\{e\}$. Put $B' = B\cup\{g\}$, $B'$ is made in the obvious way into a group with identity $g$, such that the identical mapping of $B_1$ may be extended to an isomorphic mapping of $B'$ onto $B$. In order to avoid confusion with functions, which will be introduced later, we use $a^*$ for the $S$-function.
$\mathbb{Z}^\infty \times T_i \times K$, with $z \in \mathbb{Z}^\infty$, $w \in T_i$, $c \in \mathbb{Z}^\infty$, which satisfies:

$z^g = g$, $z^a = z$,
$a^z = a^g$, $b^z = a^{(gb)}$, $(ab)$ denotes a product in $K$.

$a^z. (ab)^z = a^z(b^w.g), a^z b$, $(ab)$ denotes a product in $\mathbb{Z}^\infty$.

$a^z. (ab)^z = (a^z b)^z$, $a^z (bc)$,

for all $g \in \mathbb{Z}^\infty, b \in \mathbb{Z}^\infty, a \in T_i$.

$A$ is the quotient group of $T_i$ with respect to the following set $I_i$ of relations:

$ab \equiv (ab)$,

$ab \equiv a b$, $a b = a b$,

$ab = a b, (ab)$,

$ab = a, ga = a$,

for all $g \in \mathbb{Z}^\infty, b \in \mathbb{Z}^\infty, a \in T_i$.

10.4. Let $C$ be a pure extension of $\mathbb{Z}^\infty$ with $\mathbb{Z} = \mathbb{Z}^\infty \cup \{g\}$ a quotient group of the free semigroup $F_2$ with generating set $T_2 = \mathbb{Z}^\infty \cup T_1$. We have functions

$\sigma : T_1 \to \mathbb{Z}$, with $a \in T_1, b \in T_1$,
$c : T_2 \to \mathbb{Z}$, with $a \in T_1, b \in T_1$,

which satisfy:

$\sigma a = a, \sigma a = g, a^g = g, a^g a = a$,

$a, b \mapsto a \sigma (ab), (a^b) \mapsto a \sigma (ab)$,

$a \sigma = g, a^g = a$,

for all $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $a \in T_1$,

$d : \mathbb{Z} \to \mathbb{Z}^\infty$,

if $d(a) : a^g = a$, then $a = g$.

$C$ is the quotient semigroup of $F_2$ with respect to the following set $I_i$ of relations:

$ab \equiv (ab)$,

$ab \equiv a b$, $a b = a b$,

$ab = a, ga = a$,

for all $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $a \in T_1$.
10.5. We are going to apply the construction of section 9. 
Let $T$, $F$, $I$, and $\lambda$ of section 9 are taken from 10.3. We define 
$S' = \{ e \in E \mid \exists e \in E \}$.
$E = \emptyset$.
$S$, as in section 9.
Then $E_1 = S$.
As in section 9 we put $T = T \cup T_1 = T \cup T_1\{g\}$, and $F = F \cup \{g\}$, 
the latter in accordance with 10.4. $F$ is defined as the free 
semigroup generated by $T$, $F$, $CF$.
We have to define mappings
$s: F \times T \to K$, with $s \in E$, $\forall k \in F$, $(s, v) \in E_1$,
$s': F \times T \to E$, with $s', s \in E$, $\forall k \in F$.
(the star $\ast$, which does not occur in section 9, is introduced 
in order to avoid confusion with the $S$-functions in 10.4.), 
satisfying:
$(g, 0) = 0$, $s = s$, $s = s$
$(g, s) = g$, $s = s$
$(g, a) = g$, $s = s$
$(g, 0) = 0$, $s = s$
for all $g \in E$, $s \in E_1$, $a \in F$.
From the definition of $A$, it follows that we may write 
$(s, v) = e \ast v$ with $e \in E$, $\forall k \in E_1$.
The conditions for $e$ then split as follows:
$s = g$, $s = g$
$s = g$, $s = g$
$s = g$, $s = g$
for all $g \in E$, $s \in E_1$, $a \in F$.
Condition 9.3. is satisfied because all words in $S$, have the 
same length.

10.6. REMARK. We have to investigate whether there is no 
danger of confusion between 
$s': F \times T \to K$ of 10.3, and 
$s: F \times T \to E$ of 10.5.
$B = \{g\}$ and $T \to T = T_1$. For both functions we have 
s' = $g$ and $s' = s$ for $g \in E$, $s \in E_1$.
so in the cases where both functions are defined they coincide.
10.7. As in section 9 we define $G$ as the quotient semigroup of $F$ with respect to the set $I$ of relations, where $I$ is defined as in section 9. We now discuss the conditions which are to be satisfied in order to meet the requirements of 10.1.

10.8. Suppose for a moment that the construction has been accomplished and the identifications made, which imply:

$K CA$, $B'CC$, $ACG$, $B'CAG$, $FCC$, $FGC$

The following conditions then are to be satisfied:
1. $K$ is a normal subgroup of $G$.
2. The mapping $\phi: G \to G/K$, defined by $a\phi = KanK$ for $a \in E$, $\rho \in G$ is an isomorphism.

Because $K$ is normal in $A$, it follows that $K$ is normal in $G$ iff $YaE, YeK; s\rho \rho^{-1}E K$.

Because $s\rho = \rho^{-1} s\rho \rho$, we get the condition

$YaE, YeK; s\rho = \rho^{-1} s\rho \rho$.

Take $s \in E, s \in T$. In $G$ we have:

$s \rho = \rho^{-1} s \rho \rho$ with $s \in E\setminus E'$, $\rho \in E$.

If $\phi$ is an isomorphism we have:

$K s \rho \rho = s \rho \rho = \rho \rho \rho = K s \rho \rho$.

We have found the following necessary condition for 2:

$YaE, YeT; s \rho = \rho^{-1} s \rho \rho = \rho \rho$.

This condition however, also is sufficient. It is trivial that $\phi$ is (1-1) and onto. If the condition is satisfied we have for $a \in E\setminus E'$, $b \in E\setminus E'$, $c \rho, \rho \in G$:

$a \rho (b \rho) = KanKb = Kan[b, b] (a \rho b, a \rho b \rho) = Kan[b, b] (a \rho b, a \rho b \rho) = a \rho b \rho$.

So $\phi$ is an isomorphism.

It has to be remarked, that the conditions found are compatible with the definitions for $\phi = g$ in 10.5., and that for $s = g$ the second condition is compatible with the first.

10.9. We return to the construction of $G$. From 10.8. it follows that the functions $[s, w]$ and $s \rho w$ of 10.5. are already
completely determined. The set \( I \) of relations now reads as follows.

\[
\begin{align*}
ab &= (ab), \\
ag &= a^b g, \\
ab &= a^b (ab), \\
fa &= a, \\
ag &= a^g, \\
as &= a^{s_g}, \\
g = a \\
\text{for all } g \in K, b \in E, s \in E_1, \sigma \in E_1, \sigma_{E_1}, \sigma_{E_1} \cup F_1.
\end{align*}
\]

Defining \( a \) as in 9.4 we apply theorem 9.11.

10.10. Theorem. Every \( g \in G \) contains exactly one element of

\[ G = \{ g \in G | g \in K, b \in E, s \in E_1 \} \text{iff} \]

\[
\begin{align*}
agb &= a(ab)b, \\
as &= a^b g, \\
ab &= a^{b(a)b}, \\
fa &= a, \\
ag &= a^g, \\
as &= a^{s_g}, \\
ag &= a g, \\
as &= a^{s_g}, \\
g &= a \\
\text{for all } g \in K, b \in E, s \in E_1, \sigma \in E_1, \sigma_{E_1}, \sigma_{E_1} \cup F_1.
\end{align*}
\]

10.11. Remark. The group condition 9.14 for \( G \) is satisfied because \( a^{a} = a \), where \( a \) is taken from 10.4.

Straightforward calculation shows that from the equalities in 10.10 the fourth and the seventh give rise to \( \sigma = a \) for \( \sigma_1 \).

The result of the calculation of the others may be found in 10.12.

We make the identification of 9.13, by which \( A \) becomes a subgroup of \( G \) and \( F_1 \), a subset of \( G \) and an identification making \( K \) a subgroup of \( A \) and \( E_1 \), a subset of \( A \). The elements of
G now are simply products $ga^e$ with $g \in K, a \in B', a \in G$.

We remark that

\[ \rho^a = \sigma^a \circ \rho, \quad \rho^a = \sigma^a \circ \rho^a, \]

which are valid by the definitions for $\sigma \in G, a \in B_i, \rho \in G$, also hold for $a \in B', a \in G$, because $\rho^a = \rho$ and $\rho^a = \rho$.

We are going to resume the results of this section.

10.12. MAIN THEOREM. Let $K$ and $B'$ be groups with the same identity $g$ and $G$, be a set, $B'_1 = B' \setminus \{g\}$, $G = G \setminus \{g\}$. Let $K, B'$ and $G$ be pairwise disjoint.

Let $A$ be a Schreier-extension of $K$ with $B'$ with $S$-function:

\[ c^u \text{ with } a \in B'_1, u \in K \cup B'_1, c \in K \cup G. \]

Let $C$ be a pure extension of $B'$ with $G$ with $S$-functions

\[ \rho^a \text{ with } c \in G, a \in B'_1 \cup G, \rho \in G, \]

\[ \rho^c \text{ with } c \in G, a \in B'_1 \cup G, \rho \in G. \]

Let the following function be given:

\[ \rho^c : \rho(K \cup B'_1, G) \rightarrow \rho(K \cup B'_1, G), \rho \in G. \]

Define $G$ as the set of triples $ga^e$ with $g \in K, a \in B', \rho \in G$. A multiplication in $G$ is defined by

\[ ga^e \rho b^\prime = gc^y, \]

with $g \in K, a \in K, a \in B', b \in B', c \in G, \rho \in G$ and

\[ g = g, \quad a = a, \quad a = (a, b), \quad (a, b) = (a, b), \quad (a, b) = (a, b), \]

\[ c = a, \quad b = b, \quad \rho = (a, b). \]

The following conditions I and II are equivalent:

I. $G$ is a group which is an $S$-extension of its subgroup

\[ \Lambda^* = \{ga^e \mid g \in K, a \in B'\} \]

with the set $G^* = \{g \in G \mid a \in G\}$. In $G$ we have $Core(A^*) = K^* = \{g \in K \mid a \in K\}$. $\Lambda^*$ is canonically isomorphic with the given Schreier-extension $A$ of $K$ and $B'$ where $B^* = \{ga^e \mid a \in B'\}$ corresponds to $B'$.

$G^*/K^*$ as an $S$-extension of the group $\Lambda^*/K^*$ with the set $g^*/K^*$ is canonically isomorphic with the given pure extension $G$ of $B'$ and $G$. 

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II. \( u^\alpha = \xi, \quad \xi^\beta = \xi \),
\[
\left( u^\alpha \right)^\beta = u^\alpha (\xi^\beta) \, ,
\]
\[
a = a u^\alpha (a^\beta \xi^\gamma) = a u^\alpha (a^\beta), \quad a = a \, ,
\]
\[
a = a u^\alpha (a^\beta \xi^\gamma) = a u^\alpha (a^\beta) = a u^\alpha (a^\beta), \quad a = a \, ,
\]
\[
a = a u^\alpha (a^\beta \xi^\gamma) = a u^\alpha (a^\beta) = a u^\alpha (a^\beta), \quad a = a \, ,
\]
\[
a = a u^\alpha (a^\beta \xi^\gamma) = a u^\alpha (a^\beta) = a u^\alpha (a^\beta), \quad a = a \, ,
\]
\[
\text{for all } \alpha \in \xi, \beta \in \xi, \gamma \in \xi, \, \tau \in \xi, \, \sigma \in \xi, \, \nu \in \xi, \, \mu \in \xi, \, \nu \in \xi, \, \mu \in \xi.
Eleven

Groups products

11.1. Let $G$ be a group, $G = AB$, where $A$ and $B$ are subgroups of $G$. Put $C = A \cap B$. The problem is to construct all possible $C$, given $A, B, C$.

If $a_1 b_1 = a_2 b_2, a_1 \in A, a_2 \in A, b_1 \in B, b_2 \in B$, then $b_1 b_2^{-1} = a_2^{-1} a_1 = c \in C$, so $c = a_1 c b_1 c^{-1} b_1$. By an element of $G = AB$ the left coset of $C$ in $A$ and the right coset of $C$ in $B$ are determined.

If we choose a s.l.r. $\Gamma'$ of $C$ in $A$ and a s.r.r. $\Gamma$ of $C$ in $B$, every element of $C$ may be put in a unique way in the form $a c a'$ with $a \in F', a \in C, a' \in F$.

We assume that $B$ is given as an $S$-extension of $C$ with $\Gamma$, but $A$ as an $S$-extension with left cosets of $C$ with $\Gamma'$.

We shall make use of the method of section 9. Therefore the given extensions will be considered as quotient semigroups of free semigroups.

11.2. Let $C$ be a group and $\Gamma$, $\Gamma_1$ and $\Gamma_1'$ sets. Let $e$ be the identity of $C$. The sets $C, F_1$ and $\Gamma_1'$ are assumed pairwise disjoint.

11.3. Let $B$ be an $S$-extension of $C$ with $\Gamma = \Gamma, \{e\}$: a quotient group of the free semigroup $F_0$ with generating set $T_0 = C \cup F_1$.

We have functions

$\sigma: T_0 \rightarrow C$, with $e \in \Gamma, a \in T_0, \sigma(a) \in C$,

$\rho: T_0 \rightarrow \Gamma$, with $e \in \Gamma, a \in T_0, \rho(a) \in \Gamma$,

which satisfy:

$e^2 = e, e a = a e, e e = e, e^2 = e$,

$\sigma(a) \rho(a) = \rho(a) \sigma(a)$, $(\sigma^{a} b) = \sigma(a b)$.
\[ a, b = a, (a^b)^c = (a^c)^b, \]

\[ a^c = a, a^b = a, \]

for all \( a, b, c \in \mathcal{C} \), \( a, b, c \in \mathcal{T}_s \), \( a, b, c \in \mathcal{T}_r \).

\( \mathcal{S} \) is the quotient semigroup of \( \mathcal{P}_v \) with respect to the following set \( I \) of relations:

\( ab = (ab), \)

\( a^c = a, \)

\( ca = a, \)

for all \( a, b, c \in \mathcal{C} \), \( a, b, c \in \mathcal{T}_s \), \( a, b, c \in \mathcal{T}_r \).

11.4. Let \( A \) be an \( \mathcal{S} \)-extension with left cosets of \( \mathcal{C} \) with

\( a^c = a, a^b = a, a^c = a, a^b = a, \)

\( a^c = a, a^b = a, a^c = a, a^b = a, \)

for all \( a, b, c \in \mathcal{C} \), \( a, b, c \in \mathcal{T}_s \), \( a, b, c \in \mathcal{T}_r \).

\( \mathcal{S} \) is the quotient group of \( \mathcal{P}_v \) with respect to the following set \( I \) of relations:

\( ab = (ab), \)

\( a^c = a, \)

\( ca = a, \)

for all \( a, b, c \in \mathcal{C} \), \( a, b, c \in \mathcal{T}_s \), \( a, b, c \in \mathcal{T}_r \).

11.5. REMARK. We have to investigate whether there is no danger of confusion between
\[ \rho : T \to T \] and \[ \delta : T \to T \] of 11.3, and
\[ \omega : T \to T ' \] and \[ \nu : T \to T ' \] of 11.4.
\[ T \cap T ' = \{ e \} \] and \[ T \cap T ' = \{ e \} \]. For both functions we have \[ e = e \]
and \[ e = e \] so in the cases where both functions are defined
they coincide.

11.6. We are going to apply the construction of section 9.
\[ T_T, F_T, \Gamma_T \] and \( \Lambda \) of section 9 are taken from 11.4. We define
\[ \Omega = \{ (g, \phi) \mid \phi \in \Gamma, \alpha \in C \} \],
\[ \Sigma = \{ e \} \],
\[ \mathcal{S} = \text{as in section 9}. \]
Then \( \mathcal{S} = \{ e \} \).
As in section 9 we put \[ T = T_T \cup \Gamma_T = \Omega \cup \Gamma_T \],
and \[ \Gamma = \Gamma_T \cup \{ e \} \],
the latter in accordance with 11.3. \( F \) is defined as the free
semigroup generated by \( T, \Gamma, \phi \).
We have to define mappings
\[ (\rho, \tau) : T \to \Omega, \] with \( \phi \in \Gamma, \omega \in T, \phi \in \Omega \],
\[ \phi : T \to \Gamma, \] with \( \phi \in \Gamma, \omega \in T, \phi \in \Gamma \],
satisfying:
\[(e, \alpha) \circ (\rho, \tau) = (\rho, \tau), e^n = e, \]
\[(e, \alpha) \circ (\rho, \tau) = (\rho, \tau), e^2 = e, \]
\[(e, \alpha) \circ (\rho, \tau) = (\rho, \tau), e^e = e, \]
for all \( \alpha \in C, \phi \in \Gamma, \alpha \in C \).
From the definition of \( \Omega \), it follows that we may write
\[(\rho, \tau) = (\rho, \tau), (\rho, \tau) \in \Omega \],
with \( \phi \in \Gamma, (\rho, \tau) \in C \), for the sake of
symmetry we write \( \rho \) instead of \( \phi \).
The conditions for \( \circ \) then split as follows
\[ e^{\alpha} \circ e = e, \]
\[ e^{\alpha} \circ e = e, \]
\[ e^{\alpha} \circ e = e, \]
\[ e^{\alpha} \circ e = e, \]
for all \( \alpha \in C, \phi \in \Gamma, \alpha \in C \).
Condition 9.3 is satisfied because all words in \( \Omega \) have the
same length.

11.7. As in section 9 we define \( \phi \) as the quotient semi-
group of \( F \) with respect to the set \( I \) of relations, where
I is defined as in section 3. We now discuss the conditions which are to be satisfied in order to meet the requirements of 11.1.

11.8. Suppose for a moment that the construction has been accomplished and the identifications made, which imply:

- \( \text{CCB} \)
- \( \text{CCA} \)
- \( \text{ACG} \)
- \( \text{I'CA} \)
- \( \text{I'CG} \)

The following condition is to be satisfied:
The mapping \( \psi : \mathcal{B} \to \mathcal{C} \) defined by \( \psi \psi = \psi \) for \( \alpha \in \mathcal{C} \), \( \rho \in \Gamma \), is a monomorphism.

Take \( \psi \in \mathcal{C} \), \( \rho \in \Gamma \). In \( \mathcal{B} \) we have

\[ \rho^\alpha = \rho^\alpha \rho^\beta \text{ with } \rho^\alpha \in \mathcal{C} \rho \in \Gamma. \]

If \( \psi \) is a monomorphism we have

\[ \rho^\alpha \rho^\beta = \rho^\alpha \rho^\beta = \rho^\alpha \rho^\beta = (\rho^\alpha \rho^\beta)^\gamma. \]

We have found the following necessary condition:

\[ \forall \rho \in \Gamma, \forall \rho \in \mathcal{C}: \rho \psi = \rho \psi = \rho \psi = \rho \psi = \rho \psi. \]

This condition, however, also is sufficient. It is trivial that \( \psi \) is (1-1). If the condition is satisfied we have for \( \alpha \in \mathcal{C}, \beta \in \mathcal{B}, \gamma \in \Gamma \):

\[ \rho \psi(b \cdot c) = \rho \psi(b \cdot c) = a \cdot b \cdot c \cdot \rho \cdot b \cdot c \cdot (\rho^\beta)^\gamma = \]

\[ = a \cdot \rho \cdot b \cdot (\rho^\beta)^\gamma = a \cdot \rho \cdot b \cdot (\rho^\beta)^\gamma = \rho \psi(b \cdot c). \]

So \( \psi \) is a monomorphism.

It has to be remembered that the condition found is compatible with the definition for \( \rho = e \) in 11.6.

11.9. We return to the construction of \( \mathcal{C} \). From 11.8. it follows that the functions \( \rho^{\alpha} \cdot \beta, \rho \cdot \beta \) and \( \rho \cdot \alpha \) for \( \alpha \in \mathcal{C}, \beta \in \Gamma \), are already completely determined. We may restrict ourselves to \( \rho^{\alpha} \cdot \beta, \rho \cdot \beta \) and \( \rho \cdot \alpha \) for \( \alpha \in \mathcal{C}, \beta \in \Gamma \). If we do so, the stars may be omitted without danger of confusion.

In order to see this we have to compare the following functions
\( e^G \) and \( e^\Gamma \) defined on \( \Gamma \times \Gamma \),
\( e^S \) and \( e^\Sigma \) defined on \( \Gamma \times \Sigma \),
\( e^\theta \) and \( e^\Theta \) defined on \( \Sigma \times \Sigma \).

\( \Gamma \times \Gamma = \{ (e) \} \) and \( \Gamma \times \Sigma = \{ (e) \} \). We have \( e^\varphi - e = e = e^{\varphi} - e \)
for \( \varphi \in \Gamma \), \( e^g = g = g^e \), \( e^g = e = e^g \) for \( g \in \Gamma \), so in the cases
where two functions are both defined they coincide.

From now on we shall omit the stars. The set \( X \) of relations
now reads as follows:

\[
\begin{align*}
ab &= (ab), \\
ug &= u_{g} \cdot u_{g}, \\
gg &= g, \\
ag &= a \cdot g \cdot \{ a, g \} \cdot s, \\
as &= e \cdot s \cdot a, \\
se &= s, \\
\end{align*}
\]

for all \( a \in G, b \in G, \sigma \in \Gamma ', \alpha \in \Gamma , \beta \in \Sigma , \omega \in \Sigma '. \)

Defining \( \xi \) as in 9.4, we apply theorem 9.11:

11.10. THEOREM. Every \( g \in G \) contains exactly one element of
\( G = (g, a \in G, \alpha \in \Gamma , \beta \in \Gamma ) \) iff

\[
\begin{align*}
abg &= a \cdot (ab) g, \\
ugg &= a \cdot u_{g} \cdot u_{g}, \\
agg &= a \cdot g \cdot \{ a, g \} \cdot s, \\
ags &= a \cdot s \cdot a \cdot s, \\
ase &= a \cdot s, \\
\end{align*}
\]

for all \( a \in G, b \in G, \sigma \in \Gamma ', \alpha \in \Gamma , \beta \in \Sigma , \omega \in \Sigma '. \)

11.11. REMARK. The group condition 9.14. for \( 0 \) is satisfied
because \( a^n = e \), where \( n \) is taken from 11.3.

Straightforward calculation shows that from the equalities
in 11.10. the first, third, fifth and sixth follow from the
definition of \( \Theta \) in 11.3.

The result of the calculation of the others may be found in 11.12.
We make the identification of 9.13, by which \( A \) becomes a subgroup of \( G \) and \( \Gamma \), a subset of \( G \) and an identification making \( G \) a subgroup of \( A \) and \( \Gamma \) a subset of \( A \). The elements of \( G \) now are simply products \( g \cdot \Gamma', \omega \cdot \Gamma', \omega \cdot \Gamma \).

We remark that 
\[ \omega = \alpha \cdot \beta \cdot \lambda \cdot \delta \]
which is valid by the definitions for \( \alpha, \beta, \gamma \), also holds for \( \gamma \cdot \delta \), because \( \gamma = \epsilon \) and \( |\epsilon| = \gamma \).

We are going to resume the results of this section.

11.12. MAIN THEOREM. Let \( G \) be a group with identity \( e \), \( \Gamma \), and \( \Gamma' \) sets, \( \Gamma = \Gamma \cup \{e\} \) and \( \Gamma' = \Gamma' \cup \{e\} \). Let \( C, \Gamma, \) and \( \Gamma' \) be pairwise disjoint.

Let \( B \) be an \( S \)-extension of \( G \) with \( \Gamma \) with \( S \)-functions:
\( u \) with \( \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \),
\( v \) with \( \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \).

Let \( A \) be an \( S \)-extension with left cosets of \( G \) with \( \Gamma' \) with \( S \)-functions:
\( u \) with \( \omega \cdot \gamma', \omega \cdot \gamma', \omega \cdot \gamma' \),
\( v \) with \( \omega \cdot \gamma', \omega \cdot \gamma', \omega \cdot \gamma' \).

Let the following functions be given
\( \omega ; \gamma \cdot \delta \cdot \gamma' = \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \),
\( \omega ; \gamma \cdot \delta \cdot \gamma' = \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \),
\( \omega \cdot \gamma' \cdot \gamma \cdot \delta \cdot \gamma' = \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \).

Define \( G \) as the set of triples \( g \cdot \Gamma', \omega \cdot \Gamma', \omega \cdot \Gamma \). A multiplication in \( G \) is defined by
\( \omega \cdot \gamma, \omega \cdot \gamma' = \omega \cdot \gamma' \), with
\( \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma', \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma \) and
\( g = \omega (\gamma' \delta) \).

\( c = \omega \cdot \gamma, \omega \cdot \gamma' - \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \),
\( \omega \cdot \gamma, \omega \cdot \gamma' - \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \),
\( \omega \cdot \gamma, \omega \cdot \gamma' = \omega \cdot \gamma, \omega \cdot \gamma', \omega \cdot \gamma' \).

The following conditions I and II are equivalent:
I. \( G \) is a group containing the subgroup \( A^* = \{ \varphi g \varphi' \mid g \varphi' \in G, \varphi, \varphi' \in \Gamma \} \) which is canonically isomorphic with \( A \) and the subgroup 
\( B^* = \{ \varphi g \varphi, \varphi \in \Gamma \} \) which is canonically isomorphic with \( B \). 
\( G = A^* B^* \) and \( A^* \Gamma B^* = C^* = \{ \varphi g \mid \varphi \in \Gamma \} \) which is canonically isomorphic with \( C \).

**XI.** \( \{ e, g \} = \{ a, e \} = \{ a, e \} = e, \)

\[
3_e \{ a, g \} (a^g_h) = a (h^g), 
\]

\[
\left( a^g \right) \left( a, g \right) (a^h_b) = \left( a^h \right) \left( a, g \right), \left( a^h \right) (a, g) \left( a^h_b \right) = \left( a, g^h \right), \left( a, g \right) \left( a^h_b \right) = \left( a, g^h \right), 
\]

\[
\left( a^h_b \right) = \left( a, g \right) ^h \left( a, g \right), 
\]

\[
\left( a^h \right) ^g = \left( a, g \right) ^g, \left( a, g \right) ^g = \left( a, g \right) ^g, 
\]

\[
\left( a^h \right) \left( a, g \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), 
\]

\[
\left( a^h \right) \left( a, g \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), 
\]

\[
\left( a^h \right) \left( a, g \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), \left( a, g \right) \left( a^h \right) = \left( a, g \right), 
\]

for all \( g \in \Gamma, h, \psi \in \Gamma, \varphi, \varphi' \in \Gamma, \varphi, \varphi' \in \Gamma, \). 

11.3. REMARK. A quite different approach to the problem of this section may be found in Redei and Szép [10]. These authors do not use systems of representatives for cosets of \( C \). They work with products \( \psi A B \), an equivalence relation for such products and functions \( b_A : A \times A \) and \( b_B : B \times A \) (translated in a notation corresponding to our notation). An isomorphism \( \psi \) of a subgroup \( C \) of \( A \) and a subgroup \( C' \) of \( B \) is given. They derive a number of equalities for their functions, which
are necessary and sufficient for the construction of the
desired group product.
Their result may be derived by a modified version of the theory
of section 9; we shall not go into this question.
Bibliography


Samenvatting

In dit proefschrift wordt een onderzoek ingesteld naar alle groepen $G$, waarvan een gegeven groep $A$ een ondergroep is en een gegeven verzameling $\Gamma$ een representatief stelsel van de rechter nevenklassen van $A$ en $G$. Hierbij wordt gebruik gemaakt van de vrije semigroep $F$ voortgebracht door de verzameling $AWG$. Met behulp van twee functie’s ($S$-functie’s) worden relaties tussen de elementen van $F$ gedefinieerd. Er worden nodige en voldoende voorwaarden voor de $S$-functie’s afgeleid, opdat de quotiëntgroep $G$ van $F$ ten opzichte van deze relaties aan de gestelde eisen voldoet. Het probleem om alle groepen $G$ te construeren, die $A$ tot ondergroep hebben en $\Gamma$ tot representatief stelsel van de rechter nevenklassen van $A$ wordt dus teruggedraaid tot het probleem om alle $S$-functie’s te vinden, die aan deze voorwaarden voldoen.

In par. 3 worden de voorwaarden waaraan de $S$-functie’s moeten voldoen bestudeerd en worden deze voorwaarden voor zover dat mogelijk is geformuleerd met begrippen, die in de groepentheorie gebruikelijk zijn. De rol, die bij een normale ondergroep $A$ de quotiëntgroep $G/A$ speelt, wordt bij deze algemene uitbreidingen overgenomen door een structuur, die neergroep wordt genoemd.

In par. 5 wordt onderzocht welke veranderingen de $S$-functie’s ondergaan, door het invoeren van een ander representatief stelsel. Vervolgens wordt bewezen dat $\Gamma$ dan en slechts dan
een gemeneenschappelijk representanter stelst is van de
rechter en de linker nevenklassen van A in G, indien ieder
element van de neergroep een rechter inverse bezit. Ook is
bevestig dat voor het bestaan van zo'n gemeneenschappelijk
representanten stelst het nodig en voldoende is dat in
eedere dubbelnevenklasse A a A het aantal linker neven-
klassen gelijk is aan het aantal rechter nevenklassen.
Het verband tussen de S-functie's en de transfer van G in
A wordt onderzocht in par. 7.
De veronderstelling dat A een normale ondergroep is van G
en dat de quotiëntgroep G / A gegeven is, is gelijkwaardig
met het bekend veronderstellen van een van de S-functie's.
De voorwaarden waaraan de nog resterende S-functie moet
voldoen blijken juist de voorwaarden te zijn, die O. Schreier
heeft afgeleid voor de factorsets en de automorfismen, die
optreden bij de Schreier-extensions.
Het andere extreme geval, waarbij ook maar een S-functie
optreedt wordt verkregen door aan te nemen dat de grootste
normale ondergroep van G in A het e-element van A is. Deze
klasse uitbreidingen worden pure-extensions genoemd. Zij
zijn isomorf met een transitiieve ondergroep van S₂.
Paragraaf 9 geeft een herhaling van de constructie van
par. 2, met dien verstande, dat nu wordt verondersteld, dat
A gegeven is door een voortbringen de verzameling T, en een
aantal definierende relatie's. In dit geval is het mogelijk
om G te construeren als een quotiëntgroep aan de vrije semi-
groep voortgebracht door T, UT.
Het doel van par. 10 is een onderzoek in te stellen naar
alle groepen G, die een gegeven groep A tot ondergroep, een
gegeven verzameling T tot representanten stelst van de
rechter nevenklassen van A in G en een gegeven normale on-
dergroep K van A tot grootste normale ondergroep van G in A
hebben. Behalve de nu als gegeven beschouwde S-functie
waarmee A is geconstrueerd uit K (Schreier-extension) en
de eveneens als gegeven beschouwde S-functie waarmee G / K
uit A / K is geconstrueerd (pure - extension) kan men nu
volstaan met het invoeren van één nieuwe functie. De nodige en voldoende voorwaarden waaraan deze nieuwe functie moet voldoen worden afgeleid.

In de laatste paragraaf is een groep $A$, een groep $B$ en een gemeenschappelijke ondergroep $C$ van $A$ en $B$ gegeven. Er wordt een onderzoek ingesteld naar alle groepen $G$, zodanig dat $G = A'B'$ waarbij $A$ isomorf is met $A'$, $B$ isomorf is met $B'$ en $C$ isomorf is met $A' \cap B'$. 
Curriculum vitae

De schrijver van dit proefschrift werd geboren in 1914 te Arle-Rixtel, volgde de middelbare school aan het St. Carolus Borromeus college te Helmond en behaalde zijn doctoraal examen wis- en natuurkunde aan de Gemeentelijke Universiteit te Amsterdam.

Tot 1 september 1960 was hij leraar bij het V.H.M.O. voornamelijk aan de H.K.Lyceum te 's-Hertogenbosch.

Vanaf 1 september 1950 is hij als wiskunde docent verbonden aan de Katholieke Leerlingen te Tilburg en in 1958 werd hij benoemd tot leider van de opleidingen voor de middelbare akten MO-A en MO-B. Wiskunde van dit instituut.

Vanaf 1 september 1961 is hij verbonden aan de Katholieke Hogeschool te Tilburg met als leeropdracht Algebra. Deze leeropdracht werd in 1968 uitgebreid met een leeropdracht tot het geven van wiskunde colleges in de faculteit der sociale wetenschappen.
Getypt door Meij. R. Hornman
Lay-out en typografische verzorging: Katholieke Hogeschool, Tilburg
STELLINGEN

I De verwondering van G. Ringel uitgedrukt in de zin "It is very surprising that the both triangle systems S and T, though constructed in quite different manner, contain the same number of triangles" is op niet al te ingewikkelde wijze weg te nemen.

G. Ringel.
Theory of Graphs and its Applications.
Extremal problems in the theory of graphs.

II Als $J_p(x)$ en $J_q(x)$ de Besselfuncties van de orde p resp. q zijn, $B_n$ de getallen van Bernoulli, $0 < x < 1$ en

$$R = \sum_{n=1}^{\infty} \frac{J_p(nx)J_q(nx)}{n}$$

dan volgt uit:

1. $p + q$ oneven dat $R$ onafhankelijk is van $x$.

2. $p + q$ even en $p \neq q$ dat $R = \sum_{s=0}^{\infty} a_s(p,q)x^{2s+p+q}$ waarin $a_s(p,q) = \frac{(-1)^s}{2^{p+q}} \sum_{s=0}^{p+q} \frac{B_{p+q+2s}(p+q+2s-1)!}{s!(p+s)!(q+s)!(p+q+s)!2^{p+q+2s+1}}$.

3. $p = q$ dat $R = \frac{(-1)^{p-1}}{p} + \sum_{s=0}^{\infty} a_s(p,p)x^{2s+2p}$.
III In het proefschrift van R. Nottrot wordt beweerd, dat sekeren differentieerbareheidsdissen, die voor de definitie van toersie in de differentiaal-meetkunde gewoonlijk worden gesteld, niet wazelijk zijn voor het begrip toersie. Aiahezal het voorbeeld, dat wordt aangevoerd om deze bewering aan te tonen, niet voldoet, is de bewering zelf toch juist.

IV Als het product van alle elementen van een eindige groep G niet behoort tot de commutatorondergroep van G dan is de orde van G even en de 2-Sylow-ondergroep van G cyclisch.

V Indien de normalisator van een ondergroep H van een groep G wordt gedefinieerd zoals in J. Rotman "The theory of groups" dan is het niet zeker dat deze normalisator een ondergroep is van G.


VI De oplossing van het 3de vraagstuk van het schriftelijk examen wiskunde K.O.B. 1965 zoals die gegeven wordt in het Nieuw tijdschrift voor wiskunde is fout.

Nieuw tijdschrift voor wiskunde 50ste jaargang 1965/1966-No.11, 78-79.
VII Het onderwijs in de natuurwetenschappen bij het V.H.M.O. heeft ondermeer tot doel om de leerlingen belangstelling bij te brengen voor de verschijnselen van de natuur. Dit doel kan voor leerlingen, die voor het eerst kennis maken met dit vakgebied, het best worden bereikt door het invoeren van het vak natuurwetenschap. Eerst in de bovenbouw van het V.H.M.O. dient dit vak gesplitst te worden in natuurgeneeskunde, scheikunde en biologie.

VIII Aangezien het, volgens het voorstel leerploen rijks- scholen voor natuurgeneeskunde en scheikunde, wenselijk is dat in alle leerjaren praktische oefeningen worden uitgevoerd is het niet logisch om op de examens de theoretische kennis wel te testen en geen onderzoek in te stellen naar de ervaring in het uitvoeren van proeven en het maken van een verslag van de gevonden resultaten.

Voorstel Leerploen Rijkscholen 1. Natuurgeneeskunde (v.w.o. en h.a.v.o.) natuurgeneeskunde (m.a.v.o.).

IX Bij de eindexamens V.W.O. moet het onmogelijk worden gemaakt om "verlengde" examens af te leggen.
X Aangiften de meerderheid van de wiskunde leraren bij het V.H.M.O. hun bevoegdheid ontleent aan een middelbare akte wiskunde zal als het ontwerp-leerplan van de "Commissie Modernisering Leerplan Wiskunde" voor de bovenbouw V.W.O. wordt aangenomen, het nodzakelijk zijn de waarschijnlijkheidscorrecties als examenvak voor het examen H.O.W. wiskunde op te nemen.


XI Het zou een groot verlies zijn indien bij de dagopleidingen nieuwe stijl van leraren geen gebruik werd gemaakt van de grote ervaring, die de bestaande erkende opleidingsinstituten met deze opleidingen bezitten.

XII Het verdient uit een oogpunt van verkeersveiligheid aanbeveling verkeerslichten alleen van groen en rood licht te voorzien. Vóór het verkeerslicht dient een lichtbron geplaatst te zijn, die een geel flinkerend licht uitslaat alleen op de tijden, dat het rode licht van het verkeerslicht brandt, zodat een automobilist die dit "voorsien" passeert zonder dat het brandt, kan doorrijden.

Eindhoven, 28 Juni 1968

H.P.J. van de Kerkhof.