THE DIFFERENTIAL-DIFFERENCE EQUATION

\[ \alpha x f'(x) + f(x-1) = 0 \]

PROEFSCHRIFT

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GENERAL INTRODUCTION

Analytic number theory is a source of a great variety of highly interesting asymptotic problems. The subject of this thesis arises from a problem studied by several mathematicians, such as A.A. Buchstabl [10], N.G. de Bruijn [6], V. Ramaswami [15], and others. They discuss the number, \( \Psi(x,y) \), of positive integers \( \leq x \) which are free from prime factors \( > y \) \( (x > 0, y \geq 2) \). Part of the problem of the asymptotic behaviour of \( \Psi(x,y) \) for \( x \) and \( y \) tending to infinity has been solved. If \( u \) is a fixed positive number, and if \( x = y^u \), then we have

\[
(1) \quad \lim_{y \to \infty} y^{-u} \Psi(y^u, y) = \rho(u),
\]

where \( \rho(u) \) is continuous for \( u > 0 \) and satisfies the equation

\[
(2) \quad up'(u) + \rho(u-1) = 0 \quad (u > 1)
\]

together with the initial condition \( \rho(u) = 1 \) \( (0 \leq u \leq 1) \).

In 1952 N.G. de Bruijn [7] determined the behaviour of \( \rho(u) \), for large values of \( u \), by means of techniques which are frequently used in the theory of linear differential equations.

Another problem, analogous to the one formulated above, deals with the number of positive integers \( \leq x \) which have no prime factors \( < y \) \( (x > 0, y \geq 2) \). If we denote this number by \( \Phi(x,y) \), then we have for any fixed value of \( u > 1 \) (see [11])

\[
(3) \quad \lim_{y \to \infty} y^{-u} \log y \cdot \Phi(y^u, y) = u^{-1} \rho(u-1).
\]
The function \( \rho(u) \) is continuous for \( u > 0 \), and uniquely defined by

\[
(4) \quad u \rho'(u) - \rho(u-1) = 0 \quad (u > 1); \quad \rho(u) = 1 \quad (0 < u < 1).
\]

It was proved in [5] that \( \lim_{u \to \infty} u^{-\gamma} \rho(u) = e^{-\gamma} \), where \( \gamma \) is Euler's constant.

In the present thesis the more general differential-difference equation

\[
(5) \quad \alpha x f'(x) + F(x-1) = 0
\]

will be investigated both for positive and for negative values of the parameter \( \alpha \). We shall study (5) for positive values of \( x \), and in particular the behaviour of the solutions if \( x \) tends to infinity.

Since there is very close similarity between the theory of linear differential-difference equations and that of linear differential equations, many of the methods useful for deriving information about the solutions of linear differential equations (such as Green function, adjoint equation, biorthogonal system, Laplace transform and saddle-point analysis) can be extended in such a way as to be helpful in analysing linear differential-difference equations.

N.G. de Bruijn, in a paper [6] essentially based upon these methods, treated the equation \( F'(x) = e^{\alpha x + \beta} F(x-1) \). We shall deal with the equation (5) in a similar way.

In this thesis an infinite set of analytic solutions will be constructed. These solutions constitute a fundamental system, i.e. they are linearly independent, and any other solution can be expressed, in a unique way, in terms of these by finite or infinite linear combinations.

From this representation the asymptotic behaviour of a solution can be deduced, expressed in terms of the initial values of that solution.
Chapter 1 contains some preliminary investigations. The initial value problem is formulated, and the concepts of adjoint equation, inner product and Green function are introduced.

In chapter 2, sets of analytic solutions of the equation (5) and of the adjoint equation respectively, are constructed by means of the Laplace transform. In order to find these solutions we have to deal with the cases $\alpha > 0$ and $\alpha < 0$ separately.

The next five chapters are devoted to a discussion of the equation with $\alpha$ a fixed positive number.

In chapter 3 saddle-point analysis is applied to the Laplace integrals representing the special solutions, in order to obtain information concerning the asymptotic behaviour of these functions for large values of $|x|$. The principal result in this chapter is theorem 3.2, which states that the sets of analytic solutions of the equation (5) and of the adjoint equation form a biorthogonal system.

In chapter 5 we prove that the Green function can be expressed in terms of the functions of the biorthogonal system in the form of an infinite series (the convergence of which is investigated in chapter 4).

This result can be extended in a simple way to arbitrary solutions (chapter 6).

The discussion of the asymptotic properties of the solutions is the subject of the next chapter.

The case $\alpha < 0$ is treated in chapter 8. A number of details are omitted here since no new arguments are needed.

This thesis ends with a chapter on the application of the theory to a certain space-filling problem.
CHAPTER I

PRELIMINARY INVESTIGATIONS

1.1 Introduction

In this thesis we deal with the linear differential-difference equation

\[(1.1) \quad \alpha x f'(x) + f(x-1) = 0\]

where \(\alpha\) is a constant, either positive or negative.

We shall study (1.1) for positive values of \(x\), and in particular the behaviour of the solutions if \(x\) tends to infinity.

The following examples illustrate that this linear functional equation has a great variety of solutions, one of which can be singled out by specifying its values over an interval of length one.

Example 1

Let \(g(x)\) be a continuous function for \(1 \leq x \leq 2\). We try to find a function \(f(x)\) which is continuous for \(x = 1\), which equals \(g(x)\) for \(1 \leq x < 2\), and which is a solution of (1.1) for all \(x > 2\).

Since the values of \(f'(x)\) for \(2 < x \leq 3\) are determined by (1.1) and \(f(x)\) is required to be continuous at \(x = 2\), we find

\[f(x) = g(2) - \frac{1}{\alpha} \int_{1}^{x} \frac{g(t)}{t+1}\, dt \quad (2 \leq x \leq 3),\]

Since \(f(x)\) is now known for \(2 \leq x \leq 3\), equation (1.1) determines \(f(x)\) for \(3 \leq x \leq 4\). We can proceed in this fashion, extending the definition of \(f(x)\) from one interval to the next.

The solution so obtained has a continuous first derivative for \(x > 2\).

A somewhat different situation arises in the following case.
Example 2

Suppose that \( f(x) = 0 \) \((-1 < x < 0)\), \( f(0) = 1 \). We try to continue \( f(x) \) to a continuous solution of (1.1) for \( x > 0 \).

We have \( f'(x) = 0 \) \((0 < x < 1)\) and therefore, because of the continuity to the right at \( x = 0 : f(x) = 1 \) \((0 \leq x < 1)\). If (1.1) is to be satisfied for \( 1 < x < 2 \), we must therefore have \( f'(x) = -1/\alpha x \) \((1 < x < 2)\) and so \( f(x) = 1 - \frac{1}{\alpha} \log x \) in that interval. By repeating this process, we can continue \( f(x) \) as far as we please. This function satisfies (1.1) for \( x > 0 \) except at \( x = 1 \). Equation (1.1) is satisfied at \( x = 1 \) only in the sense of left-hand and right-hand limits.

A boundary condition which prescribes the solution in an initial interval of values of \( x \), from which the solution can be continued, will be called an initial condition. Henceforth, we shall ordinarily impose initial conditions on the solutions of (1.1).

We now introduce the following definition.

Definition 1.1

Assume \( y > 0 \). Let \( f(x) \) be a real- or complex-valued function defined for \( x \geq y - 1 \). Suppose, moreover, that the ratio \( f(x)/(x + 1) \) is bounded in \( y - 1 < x < y \). Then \( f(x) \) is said to be "a solution of (1.1) for \( x \geq y \)" if it is continuous for \( x \geq y \), piecewise continuous for \( y - 1 < x < y \), and satisfies the equation (1.1) for all \( x > y \) except those for which \( x = 1 \) is a point of discontinuity.

It has to be understood that \( f(x) \) is continuous to the right at \( x = y \).

Evidently, any function \( f(x) \) that is continuous by parts in \( y - 1 < x < y \), and for which the quotient \( f(x)/(x + 1) \) is bounded in that interval, can be continued uniquely to a solution of (1.1) for \( x \geq y \). In the points of discontinuity the values of \( f(x) \) are irrelevant except at \( x = y \).

We notice that a solution for \( x \geq y \) has \( n \) successive continuous derivatives for \( x > y + n \).
1.2 The adjoint equation

In the theory of linear differential equations a certain part is played by what is called the adjoint equation. An adjoint for differential-difference equations was introduced at an early date by R.E. Borden [2]. The concept is also used by N.G. de Bruijn [8]. We consider the linear differential-difference operator \( L \) of the form

\[
(1.2) \quad If(x) = f'(x) + \frac{1}{\alpha x} f(x-1)
\]

and the adjoint operator \( L^* \) defined by

\[
(1.3) \quad L^* h(x) = - h'(x) + \frac{h(x+1)}{\alpha (x+1)}
\]

Then we have, formally,

\[
(1.4) \quad \int_a^b \{ h(x) L f(x) - f(x) L^* h(x) \} dx = \varphi(b) - \varphi(a)
\]

where

\[
(1.5) \quad \varphi(x) = f(x)h(x) + \frac{1}{\alpha} \int_{x-1}^x \frac{f(t)h(t+1)}{t+1} dt.
\]

If \( f \) and \( h \) satisfy \( Lf = 0 \) and \( L^* h = 0 \) respectively, then, by (1.4), \( \varphi(b) = \varphi(a) \) and so \( \varphi(x) \) is independent of \( x \). Since, obviously, there are connections between the solutions of \( Lf = 0 \) and \( L^* h = 0 \), we shall also consider the equation \( L^* h = 0 \), henceforth to be called the adjoint equation, in order to get information about the solutions of (1.1). The adjoint equation reads

\[
(1.6) \quad a(x+1)h'(x) - h(x+1) = 0.
\]

It is not hard to see that \( h(x) \) can be taken equal to an arbitrary piecewise continuous function over an initial closed interval of length one entirely to the left of \( x = -1 \). Once this has been done, the function \( h(x) \) can be continued backwards in a unique way to a continuous solution of (1.6) by the same process as was used for
equation (1.1) if the initial interval is taken to the right of
the point \( x = -1 \), then it may happen that the process of continu-
ation will come to an end at \( x = -1 \) because of the possible singu-
larities of \( h'(x) \) at this point. Since, however, we are mainly in-
terested in the equation (1.6) in order to deduce something about
the solutions of (1.1) for positive values of \( x \), it suffices to
consider (1.6) for \( x > -1 \) only. This means that we can confine our-
selves to the following definition.

**Definition 1.2**

Assume \( y > 0 \). A real- or complex-valued function \( h(x) \), defined for
\(-1 < x < y + 1\), is said to be "a solution of (1.6) for \(-1 < x < y\)"
if it is continuous for \(-1 < x < y\), continuous by parts for
\( y < x < y + 1\), and satisfies the equation (1.6) for all \(-1 < x < y\)
except those for which \( x+1 \) is a point of discontinuity. It has to
be understood that \( h(x) \) is continuous to the left at \( x = y \).

Suppose now that \( f(x) \) is a solution of (1.1) for \( x \geq a \), and \( h(x) \)
is a solution of (1.6) for \(-1 < x < b\), where \( 0 < a < b \). Then the expression

\[
(1.7) \quad f(x)h(x) = \int_a^x \frac{f(t)h(t+1)}{t+1} \, dt
\]

is a constant for all \( x \) in the closed interval \([a,b]\). This invariant of the functions \( f \) and \( h \) will henceforth be called the inner product \([f,h]\), as it is bilinear over the linear solution spaces
of (1.1) and (1.6).

1.3 The Green function \( G_a(y,x) \)

We now introduce a special solution of the equation (1.1), to
be called the Green function, which is of major importance in the
following chapters. Let \( \eta \) be a real non-negative number. Then we
denote by \( G_a(\eta,x) \) the solution of (1.1) for \( x \geq \eta \), subject to
the initial condition

\[
(1.8) \quad G_a(\eta,x) = \begin{cases} 
0 & (\eta - 1 < x < \eta) ; \\
1 & \end{cases} 
\]

\( G_a(\eta,\eta) = 1 \).
Moreover, we define \( G_\alpha(\eta,x) = 0 \) for \( x < \eta - 1 \). By the process of
continuation described in sec. 1.1, we can continue \( G_\alpha(\eta,x) \) indefinitely to the right. By this process we might establish a for-

mula which gives the value of \( G_\alpha(\eta,x) \) in any interval
\( \eta < x \leq \eta + n + 1 \) \( (n = 0, 1, \ldots) \). At any rate, such a for-

mula would not be particularly helpful in finding the behaviour of
this function as \( x \) becomes indefinitely large. For this reason we shall derive in section 1.5 an integral representation by means of
the Laplace transform.

We now proceed to the adjoint equation. We assume \( \xi \) to be a non-
negative number. The Green function \( G_\alpha^*(x,\xi) \) is the solution of
(1.6) for \( -1 < x < \xi \), subject to the initial condition

\[
(1.9) \quad G_\alpha^*(x,\xi) = 0 \quad (x > \xi); \quad G_\alpha^*(\xi,\xi) = 1.
\]

It is not difficult to prove that \( G_\alpha(\eta,\xi) = G_\alpha^*(\eta,\xi) \) for \( \eta > 0 \),
\( \xi = 0 \). This is trivial if \( \xi < \eta \). Now suppose \( \xi \geq \eta \). Then we infer
from (1.7) that the inner product

\[
(1.10) \quad \{ G_\alpha(\eta,x), G_\alpha^*(x,\xi) \} =
\]

\[
= \frac{1}{\alpha} \int_{x-1}^{x} \frac{G_\alpha(\eta,t) G_\alpha^*(t+1,\xi)}{t + 1} \, dt
\]

is defined and constant throughout the interval \( \eta < x < \xi \). If we
insert the values \( x = \eta \) and \( x = \xi \) respectively, the integral in
the above expression vanishes both times and the result constitutes
the proof.

In this way we have shown that the Green function \( G_\alpha(\gamma,x) \), which
is a solution of (1.1) with respect to \( x \), is a solution of (1.6) with
respect to \( \gamma \). We notice that \( G_\alpha(\gamma,x) = 1 \) for \( y < x < y + 1 \)
(\( \gamma > 0 \)).

1.4 A solution expressed in terms of its initial values

The Green function enables us to express arbitrary solutions
in terms of their initial values. In order to get an idea how this
can be done we use the well-known method of superposition. We consider a function \( f(x) \) which is defined and has a continuous first derivative for \( y - 1 \leq x \leq y \) \( (y > 0) \). In this interval \( f(x) \) can be approximated by the finite sum of Green functions

\[
f(t_0) G_a(t_0,x) + \sum_{k=1}^{n} \{ f(t_k) - f(t_{k-1}) \} G_a(t_k,x) \text{ with}
\]

\[
y - 1 = t_0 < t_1 < \ldots < t_n = y \text{ and } t_k - t_{k-1} = \frac{1}{n} \text{ for } k = 1, 2, \ldots, n,
\]

n large.

Since \( f(x) \) has a continuous differential coefficient we may even write \( (y - 1 \leq x \leq y) \)

\[
(1.11) \quad f(x) = f(y-1) G_a(y-1,x) + \int_{y-1}^{y} f'(t) G_a(t,x) \, dt.
\]

This relation expresses \( f(x) \) as an "infinite linear combination" of Green functions. We can continue \( f(x) \) in the usual way as a solution of (1.1) for \( x > y \). Since the equation is linear and homogeneous, we may hope that the right-hand side of (1.11) also represents \( f(x) \) for all \( x > y \). In order to get rid of the differential coefficient \( f'(t) \), we integrate by parts and obtain for the right-hand member of (1.11), provided that \( x > y \)

\[
f(y) G_a(y,x) = \frac{1}{\alpha} \int_{y-1}^{y} \frac{f(t) G_a(t+1,x)}{t+1} \, dt.
\]

This seems to be the expression we were looking for. We recognize it as a special case of the expression (1.7). It is easy to get rid of the differentiability condition imposed upon \( f \). We have the following theorem:

**Theorem 1.1**

Assume \( y > 0 \). Any function \( f(x) \) which is a solution of (1.1) for \( x > y \), can be expressed in terms of its initial values by the relation, which holds for all \( x > y \),

\[
(1.12) \quad f(x) = f(y) G_a(y,x) - \frac{1}{\alpha} \int_{y-1}^{y} \frac{f(t) G_a(t+1,x)}{t+1} \, dt.
\]
Proof

Let \( \xi \) be any number \( \geq y \). Then \( G^\#_a(x, \xi) \) is a solution of (1.6) for 
\(-1 < x < \xi \). The inner product 
\[
(f(x), G^\#_a(x, \xi)) = f(x) G^\#_a(x, \xi) - \frac{1}{a} \int_{x-1}^{x} \frac{f(t)G^\#_a(t+1, \xi)}{t+1} dt 
\]
is defined and constant throughout the interval \( y < x < \xi \). If 
\( x = \xi \), the integral in (1.13) vanishes and we find 
\[
(1.14) \quad f(\xi) = (f(x), G^\#_a(x, \xi)) = 
\]
\[
f(y) G^\#_a(y, \xi) - \frac{1}{a} \int_{y-1}^{y} \frac{f(t)G^\#_a(t+1, \xi)}{t+1} dt . 
\]

The proof now clearly follows from the fact that \( G^\#_a \) and \( G^\#_a \) are 
identical for non-negative values of the arguments. The theorem is 
important as it enables us to derive certain properties of arbitrary 
solutions from similar properties of the Green function.

1.5 An integral representation of the Green function

Using the Laplace transform, we shall derive in this section an 
integral representation of the Green function. If \( f(x) \) is 
defined for \( x > 0 \), then we write 
\[
(1.15) \quad F(t) = \int_0^\infty e^{-tx} f(x) dx 
\]
for those \( t \) for which the integral converges absolutely. \( F \) is 
called the Laplace transform of \( f \). With (1.15) we can transform 
the linear functional equation (1.1) into a linear differential equation 
involving \( F(t) \) and its derivative \( F'(t) \) only.
In order to establish the convergence of the above integral if 
\( f(x) = G^\#_a(y, x) \), we must have some a priori estimate of the order 
of magnitude of \( G^\#_a(y, x) \) for large values of \( x \). First we assume 
\( \alpha < 0 \). Since \( G^\#_a(y, x) = 1 \) if \( y < x < y + 1 \), we infer from (1.1)
that this function increases for \( x > y + 1 \). Hence we have
\[
\frac{d G_a(y, x)}{dx} < \frac{1}{\sqrt{|x|}} G_a(y, x) \quad (x > y + 1),
\]
and thus
\[
(1.16) \quad |G_a(y, x)| \leq \left( \frac{x}{y + 1} \right)^{1/|x|} (x > y + 1).
\]
This upper bound is pretty close to the asymptotic behaviour as will be proved later on. The above relation still holds if \( a > 0 \).
In this case, however, the upper bound will turn out to be a very rough one. It follows from (1.16) that the Laplace transform of the Green function exists for \( R_a(t) > 0 \). Moreover,
\[
F(t) = \int_0^\infty e^{-xt} G_a(y, x) dx \quad \text{is an analytic function in the right half-plane \( R_a(t) > 0 \).}
\]
Now we shall evaluate \( F(t) \). For convenience we repeat the definition of \( G_a(y, x) \):
\[
(1.17) \quad G_a(y, x) = 0 \quad (x < y), \quad G_a(y, y) = 1 \quad (y > 0),
\]
\[
\frac{d}{dx} G_a(y, x) = \frac{1}{x} G_a(y, x - 1) \quad (x > y, x \neq y + 1).
\]
If we multiply the equation by \( e^{-xt} \) and integrate from \( y \) to \( \infty \), we get
\[
(1.18) \quad \int_y^\infty e^{-xt} \frac{d}{dx} G_a(y, x) dx = - \frac{1}{x} \int_y^\infty e^{-xt} G_a(y, x - 1) dx.
\]
The integral on the left can be expressed in terms of \( F(t) \) by means of integration by parts
\[
(1.19) \quad \int_y^\infty e^{-xt} \frac{d}{dx} G_a(y, x) dx = - e^{-yt} + tF(t).
\]
We denote the right-hand side of (1.18) by \( \varphi(t) \). Then
\[
(1.20) \quad \varphi'(t) = \frac{1}{\sqrt{|x|}} \int_{y+1}^\infty e^{-xt} G_a(y, x - 1) dx = \frac{1}{\sqrt{|x|}} e^{-t} F(t).
\]
Hence, \( P(t) \) satisfies the linear differential equation

\[
(1.21) \quad a t^\alpha P(t) + (a - e^{-t}) P(t) = -c y e^{y t}.
\]

For convenience, we shall deal with the cases \( y = 0 \) and \( y > 0 \) separately.

(i) \( y = 0 \).

The general solution of (1.21) is given by

\[
(1.22) \quad P(t) = c \exp \left\{ \frac{1}{\alpha} \int_0^t \frac{e^{s - 1}}{s} \, ds \right\} t^{1/\alpha - 1}
\]

where \( c \) is an arbitrary constant. The function \( t^{1/\alpha - 1} \) is defined by \( t^{1/\alpha - 1} = \exp((1/\alpha - 1) \log t) \), \( \log t \) denotes the principal value of the logarithm of \( t \), i.e., the value whose imaginary part lies between \(-\pi \) and \( \pi \).

The constant \( c \) can be determined by evaluating, in two different ways, the asymptotic behaviour of \( P(t) \) for large positive values of \( t \). First we deduce from

\[
P(t) = \int_0^\infty e^{-xt} \alpha (0,x) \, dx = \int_0^1 e^{-xt} \, dx + \int_1^\infty e^{-xt} \alpha (0,x) \, dx
\]

that

\[
(1.23) \quad P(t) \sim 1/t \quad (t \to \infty),
\]

Secondly, we have by formula (1.22)

\[
P(t) = \frac{c}{t} \exp \left\{ \frac{1}{\alpha} \int_0^1 \frac{e^{s - 1}}{s} \, ds + \frac{1}{\alpha} \int_1^\infty \frac{e^{s - 1}}{s} \, ds = \frac{1}{\alpha} \int_1^\infty \frac{e^{s - 1}}{s} \
\]

Since

\[
\int_0^1 \frac{1 - e^{-s}}{s} \, ds = \int_1^\infty \frac{e^{-s}}{s} \, ds = \gamma,
\]
where \( \gamma \) denotes Euler's constant, it follows that

\[
(1.24) \quad F(t) = \frac{\alpha}{t} e^{-\gamma} \left[ 1 + O(e^{-t}) \right] \quad (t > 1).
\]

Comparing (1.23) and (1.24), we find \( c = e^{-\gamma} \). We can state the result as the following theorem.

**Theorem 1.2**

The Laplace transform \( F(t) = \int_0^\infty e^{-xt} g(x) dx \) of the Green function \( g(x) \) is given by

\[
(1.25) \quad F(t) = \exp \left\{ \gamma/\alpha + 1/\alpha \int_0^t \frac{s^{\alpha-1}}{s} ds \right\} t^{1/\alpha-1} \quad (\Re(t) > 0),
\]

where \( \gamma \) is Euler's constant.

We notice that the right-hand side of (1.25) is an analytic function in the complex \( t \)-plane cut along the negative real axis from the origin to infinity and so it provides the analytic continuation of \( F(t) \) in that region.

\( (ii) \quad y > 0. \)

The general solution of (1.21) is now given by

\[
(1.26) \quad F(t) = F_0(t) \ast [c + y \int_0^\infty e^{-y} \frac{1}{\alpha} \int_0^t \frac{s^{\alpha-1}}{s} ds] u^{1/\alpha} du,
\]

where \( c \) is an arbitrary constant and \( F_0(t) \) is defined by

\[
F_0(t) = t^{1/\alpha-1} \exp \left\{ \frac{1}{\alpha} \int_0^t \frac{s^{\alpha-1}}{s} ds \right\}.
\]

The multi-valued functions \( t^{1/\alpha} \) and \( u^{-1/\alpha} \) are defined by \( \exp((1/\alpha-1) \log t) \) and \( \exp(-1/\alpha \log u) \) respectively, all logarithms having their principal values. Finally, we note that the path of integration \( (t, \infty) \) should not intersect the negative real axis.
The constant \( c \) can be determined by the method used previously in the case \( y = 0 \). First we deduce from

\[
\psi(t) = \int_0^\infty e^{-xt} g_\alpha(y, x) dx = \\
= \int_0^{y+t} e^{-xt} dx + \int_0^\infty e^{-xt} g_\alpha(y-t, x) dx
\]

that

(1.27) \( \psi(t) = \frac{y+t}{t} (1 + O(t^{-1})) \quad (t > 1) \).

Next, putting \( u = t - y \) \((t > 1)\) in (1.26), we obtain

(1.28) \( \psi(t) = c \phi \left( \frac{y}{t} \right) + y^{1/\alpha} \int_0^\infty \exp \left\{ -yu - \frac{y-t}{t} \int_0^{s+1} ds \right\} dv \).

In case (i) we prove that the first term of the right-hand side of (1.28) is \( \frac{y^{1/\alpha}}{t} (1 + O(t^{-1})) \). Since the second term is \( \frac{y^{1/\alpha}}{t} (1 + O(t^{-1})) \), it follows that

(1.29) \( \psi(t) = \frac{y^{1/\alpha}}{t} (1 + O(t^{-1})) (1 + O(t^{-1})) \quad (t > 1) \).

Comparing the asymptotic formulae (1.27) and (1.29) we find \( c = 0 \).

We state the result as the following theorem.

**Theorem 1.**

Assume \( y > 0 \). Then the Laplace transform \( \Phi(t) = \int_0^\infty e^{-xt} g_\alpha(y, x) dx \)

of the Green function \( g_\alpha(y, x) \), is given by

(1.30) \( \psi(t) = \psi_0(t) \int_0^\infty \exp \left\{ -yu - \frac{1}{2} \int_0^{s+1} ds \right\} u^{-1/\alpha} du, \)

where

\[
\psi_0(t) = t^{1/\alpha} e^{-w} \exp \left\{ \frac{1}{2} \int_0^t \frac{u}{\alpha} du \right\}.
\]
It should be remarked that under the conditions previously mentioned the right-hand side of (1.30) is an analytic function in the complex $t$-plane cut along the negative real axis from 0 to $-\infty$, and so it provides the analytic continuation of $F(t)$ in that region.

Since the Green function is continuous for $x > y$ and of bounded variation on any finite interval, the inversion formulas for the Laplace transform yields (see [17], §17)

\[(1.31) \quad G_{\alpha}(y,x) = \frac{1}{2\pi i} \lim_{p \to -\infty} \int_{\beta - ip}^{\beta + ip} e^{xt} F(t) dt \quad (x > y > 0),\]

where $\beta$ is an arbitrary positive number.

Replacing $t$ by $-t$ we find by means of (1.25) and (1.30)

Theorem 1.4

Let $\beta$ be any positive number. Then we have for $x > 0$

\[(1.32) \quad G_{\alpha}(y,x) = \frac{\frac{\gamma}{\alpha}}{2\pi i} \lim_{p \to -\infty} \int_{\beta - ip}^{\beta + ip} e^{xt + \frac{1}{\alpha} \int_{0}^{t} \frac{s^{\alpha-1}}{s^{\alpha}} ds} \{(-t)^{1/\alpha} - dt.\}

The multi-valued function $(-t)^{1/\alpha - 1}$ is defined by $\exp\{(t/\alpha - 1)\log(-t)\}$, where $\log(-t)$ denotes the principal value of the logarithm of $-t$.

Theorem 1.5

Let $\beta$ be any positive number. Then we have for $x > y > 0$

\[(1.33) \quad G_{\alpha}(y,x) = \frac{1}{2\pi i} \lim_{p \to -\infty} \int_{\beta - ip}^{\beta + ip} A(t) e^{xt + \frac{1}{\alpha} \int_{0}^{t} \frac{s^{\alpha-1}}{s^{\alpha}} ds} \{(-t)^{1/\alpha - 1} - dt,\}

where

\[(1.34) \quad A(t) = \int_{-\infty}^{t} \exp\{yv - \frac{1}{\alpha} \int_{0}^{v} \frac{s^{\alpha-1}}{s^{\alpha}} ds\} (-v)^{-1/\alpha} dv.\]
The multi-valued functions \((-t)^{1/\alpha}\) and \((-\Gamma)^{-1/\alpha}\) are defined in the usual way. Moreover, the path of integration \((-\infty, t)\) should not intersect the positive real axis.

We note that the integrands in (1.32) and (1.33) are analytic functions in the complex t-plane cut along the positive real axis from the origin to infinity.
CHAPTER II

SYSTEMS OF SPECIAL SOLUTIONS

2.1 Introduction

It is well known that any function satisfying a linear homogeneous differential equation can be written as a linear combination of a finite number of particular solutions. In the present thesis we shall see that the solutions of the differential-difference equation (1.1) can likewise be written as sum of particular solutions. In this connection, however, our equation has to be regarded as a differential equation of \( \alpha \) order. In the following sections we shall first construct infinite sets of analytic solutions of (1.1) and (1.6) respectively, where we have to distinguish between positive and negative values of \( \alpha \). It is not difficult to see that the case \( \alpha < 0 \) can be reduced to the case \( \alpha > 0 \). The reasoning is as follows. The substitution \( h(x) = f(-x-1) \) transforms the adjoint equation (1.6) into the equation \(-\alpha x f'(x) + f(x-1) = 0\). If \( h(x) \) is a solution of (1.6), then \( f(x) = h(-x-1) \) satisfies the linear functional equation (1.1) where \( \alpha \) is replaced by \(-\alpha\). Conversely, if \( f(x) \) is a solution of (1.1), then \( h(x) = f(-x-1) \) satisfies the adjoint equation where \( \alpha \) is replaced by \(-\alpha\).

2.2 The case \( \alpha > 0 \)

We try to solve (1.1) by a Laplace integral \( f(x) = \int_{\mathbb{W}} e^{-xZ}g(z)dz \).

Integrating by parts, we have, formally,

\[
(2.1) \quad xf'(x) = -x \int_{\mathbb{W}} e^{-xZ} z g(z)dz = \\
= e^{-xZ} z g(z) \bigg|_{\mathbb{W}} - \int_{\mathbb{W}} e^{-xZ} \{z g'(z) + g(z)\} dz.
\]
Suppose now that the path \(\mathcal{W}\) and the function \(g(z)\) have been chosen in such a way that the integrated term vanishes, then \(f(x)\) satisfies (1.1) if

\[
(2.2) \quad \int_{\mathcal{W}} e^{-x z} \left\{ az g'(z) + (a - e^z) g(z) \right\} \, dz = 0.
\]

The integrand is identically zero if \(g(z)\) is a solution of the linear differential equation \(az g'(z) + (a - e^z) g(z) = 0\). Clearly,

\[
g(z) = z^{1/a-1} \exp \left( \frac{1}{a} \int_0^z \frac{e^s - 1}{s} \, ds \right),
\]

\(z^{1/a-1}\) having its principal value, is an analytic solution of this equation throughout the \(z\)-plane cut along the negative real axis from the origin to \(-\infty\). Since this function tends to zero very rapidly as \(z\) tends to infinity along the half-lines \(z = \pm \pi i + \tau\) \((\tau > 0)\), we choose \(\mathcal{W}\) as a curve starting at \(-\pi i + \infty\) and tending to \(\pi i + \infty\), avoiding the negative real axis. For this choice of \(g\) and \(\mathcal{W}\), the integrated term in (2.1) vanishes, and the same thing is true for the contours \(\mathcal{W} + 2n\pi i\), where \(n\) is any integer \((\mathcal{W} + 2n\pi i\) is the curve described by \(z + 2n\pi i\) if \(z\) describes \(\mathcal{W}\)). Now, one can easily verify that the functions \(F_n(x), n = 0, \pm 1, \pm 2, \ldots\), defined by

\[
(2.3) \quad F_n(x) = \frac{1}{2\pi i} \int_{\mathcal{W} + 2n\pi i} z^{1/a-1} \exp \left\{ -xz + \frac{1}{a} \int_0^z \frac{e^s - 1}{s} \, ds \right\} \, dz
\]

are holomorphic and satisfy (1.1) for all complex values of \(x\). Because of the discontinuity of the integrand on the negative real axis we must demand that the path \(\mathcal{W} + 2n\pi i\) avoids this half-line.

Now let \(\mathcal{W}_1\) and \(\mathcal{W}_2\) be contours starting at the origin and tending to \(\pi i + \infty\) and \(-\pi i + \infty\) respectively, both contours avoiding the negative real axis. For this choice of the path of integration the integrated term in (2.1) vanishes in either case. The functions \(F_+(x)\) and \(F_-(x)\), defined by
\[ F_+ (x) = \frac{1}{2\pi i} \int_{\gamma_1} z^{1/\alpha} a^{-\alpha} \exp \{-xz + \frac{1}{\alpha} \int_{0}^{z} \frac{s^{a-1}}{s} ds \} dz, \]

and

\[ F_- (x) = \frac{1}{2\pi i} \int_{\gamma_2} z^{1/\alpha} a^{-\alpha} \exp \{-xz + \frac{1}{\alpha} \int_{0}^{z} \frac{s^{a-1}}{s} ds \} dz \]

are entire functions satisfying (1.1) for all values of \( x \).

It should be noted that

\[ F_0 (x) = F_+ (x) - F_- (x) \]

In what follows we shall also use the solution \( \overline{F}(x) \) where

\[ \overline{F}(x) = F_+ (x) + F_- (x) \]

Still another solution of (1.1), in the half-plane \( \Re (x) < 0 \), is

\[ N(x) = \frac{1}{2\pi i} \int_{-\infty}^{0} (-z)^{1/\alpha} a^{-\alpha} \exp \{-xz + \frac{1}{\alpha} \int_{0}^{z} \frac{s^{a-1}}{s} ds \} dz. \]

The multi-valued function \((-z)^{1/\alpha} a^{-\alpha}\) is made definite by choosing \((-z)^{1/\alpha} a^{-\alpha}\) real on the negative real axis.

Dealing similarly with the adjoint equation, we find that the functions \((n = 0, \pm 1, \pm 2, \ldots)\)

\[ \varphi_{2n \pm 1} (x) = -\frac{1}{\alpha} \int_{\gamma_{2n \pm 1}} z^{1/\alpha} a^{-\alpha} \exp \{(x \mp 1)z - \frac{1}{\alpha} \int_{0}^{z} \frac{s^{a-1}}{s} ds \} dz \]

are analytic everywhere and satisfy (1.6) throughout the whole complex \( x \)-plane. The multi-valued function \( z^{1/\alpha} a^{-\alpha}\) is defined in the usual way. Because of the discontinuity of the integrand on the negative real axis the path of integration, which leads from \(\infty + 2n \pi \text{i}\) to \(\infty + 2(n+1) \pi \text{i}\), is required to avoid this half-line.
Moreover, it can easily be verified that the function

\[
\psi_0(x) = -\frac{1}{\alpha} \int_P (\zeta)^{-1/\alpha} \exp \{(x+1)\zeta - \frac{1}{\alpha} \int_0^\zeta \frac{\zeta^s - 1}{s} \, ds \} \, d\zeta
\]

is an analytic solution of (1.6) in the whole complex \(x\)-plane.

The contour \(P\) leads from \(z = \infty\) to \(1\) in the lower half-plane, encircles the origin once in the negative direction, and returns to \(\infty\) in the upper half-plane. The function \((-\zeta)^{-1/\alpha}\) is defined by its principal value.

We add to the set \(\{\psi_{2n+1}, \psi_0\}\) the functions \(\psi_+(x)\) and \(\psi_-(x)\) which are solutions of the adjoint equation for \(P_\sigma(x) > -1\) only. We have

\[
\psi_+(x) = -\frac{1}{\alpha} \int_{P_1} z^{-1/\alpha} \exp \{(x+1)z - \frac{1}{\alpha} \int_0^z \frac{z^s - 1}{s} \, ds \} \, dz
\]

and

\[
\psi_-(x) = -\frac{1}{\alpha} \int_{P_2} z^{-1/\alpha} \exp \{(x+1)z - \frac{1}{\alpha} \int_0^z \frac{z^s - 1}{s} \, ds \} \, dz
\]

\(P_1\) being a curve lying entirely in the upper half-plane \(\Im(z) > 0\) and leading from \(z = -\infty\) to \(z = \infty\), while \(P_2\) is the complex conjugate of \(P_1\).

Since the principal values of \((-\zeta)^{-1/\alpha}\) and \(z^{-1/\alpha}\) are related by

\[
(-z)^{-1/\alpha} = e^{\pi i/\alpha} \frac{1}{(-z)^{-1/\alpha}} \quad (\Im(z) > 0),
\]

\[
(-z)^{-1/\alpha} = e^{-\pi i/\alpha} \frac{1}{(-z)^{-1/\alpha}} \quad (\Im(z) < 0),
\]

we have

\[
\psi_0(x) = -e^{\pi i/\alpha} \psi_+(x) + e^{-\pi i/\alpha} \psi_-(x) \quad (P_\sigma(x) > -1).
\]

Omitting the proof, we now state the following lemma about the behaviour of the integrands in the above formulae.
Lemma 2.1

Assume $\alpha > 0$. Further, suppose that $n$ is a fixed integer, and that $\varepsilon$ is any positive number $< \pi/2$.

If $|L_{b}(z) - (2n + 1)\pi| < \pi/2 - \varepsilon$, $u = R_{q}(\varepsilon)$, then for $u = \infty$

$$\frac{1}{\alpha} \exp \left\{ \frac{1}{\alpha} \int_{0}^{z} \frac{s^{3} - 1}{s} \, ds \right\} = 0 \left\{ \exp \left( -\frac{s^{n}}{\alpha} \sin \varepsilon \right) \right\}.\tag{2.15}$$

If $|L_{b}(z) - 2n\pi| < \pi/2 - \varepsilon$, $u = R_{q}(\varepsilon)$, then for $u = \infty$

$$\frac{1}{\alpha} \exp \left\{ \frac{1}{\alpha} \int_{0}^{z} \frac{s^{3} - 1}{s} \, ds \right\} = 0 \left\{ \exp \left( -\frac{s^{n}}{\alpha u} \sin \varepsilon \right) \right\}.\tag{2.16}$$

The two constants implied in the $O$-symbols depend on $\alpha$, $n$ only. Furthermore, the two functions occurring on the left-hand side of the above relations, are bounded in the part of the half--plane $R_{q}(z) < 0$ where $|z| > 1$.

It follows from Cauchy's theorem and lemma 2.1 that a vertical shift of the path $\gamma + 2\pi i$ over a distance less than $\pi/2$ has no influence upon the value of the integral on the right-hand side of (2.3). Needless to say, the new path should not intersect the cut along the negative real axis. Moreover, we notice that a vertical shift of the contour in (2.9) over a distance less than $\pi/2$ does not affect the value of the integral.

Since the equation (1.1) is linear, we can generate new solutions by forming linear combinations of known solutions. In this way we can construct an infinite set \( \{P_{n+1}\}, \ n = 0, \pm 1, \ldots \) of solutions by means of the relations

$$\frac{\alpha}{\alpha} N(x) = \frac{\alpha}{\alpha} N(x) + F_{+}(x) + \sum_{1 \leq k \leq n} P_{k}(x) \ (n \geq 0),\tag{2.17}$$

$$\frac{\alpha}{\alpha} N(x) = \frac{\alpha}{\alpha} N(x) + F_{-}(x) + \sum_{n+1 \leq k \leq -1} P_{k}(x) \ (n \leq -1).\tag{2.18}$$
If $n = 0$, the sum $\sum_{1 \leq k \leq n}$ is vacuous, and its value has to be interpreted as 0. The same convention holds for $\sum_{n+1 \leq k \leq -1}$, if $n = -1$. These functions, which are analytic solutions of (1.1) for $R_0(x) \neq 0$, play an important part in the case $\alpha < 0$. By means of lemma 2.1 and formula (2.13) one easily verifies that

\begin{equation}
(2.19) \quad \varphi_{2n+1}(x) = \frac{1}{2\pi i} \int_{-\infty-i(2n+1)\pi}^{\infty-i(2n+1)\pi} z^{1/\alpha} e^{-\frac{1}{\alpha} \int_0^z \frac{e^{s-1}}{s} ds} dz.
\end{equation}

Dealing similarly with the adjoint equation, we define a set \{\(R_n, \tilde{R}\), \(n = 0, \pm 1, \pm 2, \ldots\), of solutions by the relations

\begin{equation}
(2.20) \quad R_0(x) = \frac{1}{2} \{\varphi_+(x) + \varphi_-(x)\}, \quad \tilde{R}(x) = \frac{1}{2} \{\varphi_+(x) - \varphi_-(x)\},
\end{equation}

\begin{equation}
(2.21) \quad R_n(x) = \varphi_+(x) + \sum_{0 \leq k \leq n-1} \varphi_{2k+1}(x) \quad (n \geq 1),
\end{equation}

\begin{equation}
(2.22) \quad R_n(x) = \varphi_+(x) - \sum_{n \leq k \leq -1} \varphi_{2k+1}(x) \quad (n \leq -1).
\end{equation}

These functions are analytic in the half-plane $R_0(x) > -1$.

If $n \neq 0$, we have

\begin{equation}
(2.23) \quad R_n(x) = -\frac{1}{\alpha} \int_{-\infty-i(2n+1)\pi}^{\infty-i(2n+1)\pi} z^{1/\alpha} e^{-\frac{1}{\alpha} \int_0^z \frac{e^{s-1}}{s} ds} dz.
\end{equation}

In the following chapters we shall prove that the solutions $\varphi(x)$, $P_n(x)$, $P_n(x)$, constitute a fundamental system of the equation (1.1), that is to say they are linearly independent, and any function $\mathbf{f}(x)$ that is a solution for $x > y > 0$, can be expressed, in a unique way, in terms of the functions of the system by means of finite or infinite linear combinations. We have (see theorem 6.1)
\( f(x) = \left\{ f_n \text{e}^{-x} \right\} \phi(x) + \sum_{-\infty}^{\infty} \left\{ f_n \text{e}^{-x} \right\} \phi_n(x) \quad (x > y) \)

The coefficients in the above series are defined by the expression (1.7) for the inner products.

It should be noted that the function \( N(x) \) does not appear in (2.24); it is needed only if we should develop solutions of (1.1) for \( x < 0 \). Although \( N(x) \) is originally defined for \( \Re(\phi) < 0 \) only, it can be continued analytically to the right half-plane \( \Re(\phi) > 0 \) cut along the positive real axis from the origin to infinity. We shall discuss this problem here as it is a very interesting one in several respects. Our method for establishing the continuation of \( N(x) \) depends on Cauchy's theorem which enables us to deform the path of integration in (2.8) in such a way that we obtain formulae which hold in a wider region.

**Theorem 2.1**

The analytic continuation of \( N(x) \) (which is also denoted by \( N(x) \)) throughout the upper half-plane \( \Re(\phi) > 0 \) is given by

\[ N(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (-z)^{1/\alpha - 1} \exp \left( \frac{1}{\alpha} \int_{0}^{z} s^{-1/\alpha} ds \right) dz, \]

and this relation still holds for any real \( x < 0 \). Again, \((-z)^{1/\alpha - 1}\) is defined by its principal value.

**Proof**

Let \( v \) be a positive number \( \neq 1 \). From lemma 2.1 we infer that the path of integration \((-\infty, 0)\) in (2.9) may be replaced by a curve consisting of the parts

(i) the half-line \( z = u + iv \quad (-\infty < u \leq 0) \)

(ii) the vertical line segment \( z = -iu \quad (v \geq u \geq 0) \).

It is easily seen that the integral along the horizontal line (i) is, in absolute value, less than a constant multiplied by
\((vR(x))^{-1} e^{-vIm(x)}\). Then, making \(v \to \infty\), it follows that the equality (2.25) holds whenever \(Im(x) > 0\), \(R(x) < 0\). Obviously, the right-hand side of (2.25) is an analytic function in the half-plane \(Im(x) > 0\), and this completes the proof.

**Theorem 2.2**

The analytic continuation of \(N(x)\) to the lower half-plane \(Im(x) < 0\) is given by

\[[2.26] \quad N(x) = \frac{1}{2\pi i} \int_{i\infty}^{0} (-e)^{1/\alpha - 1} \exp \left\{ -xz + \frac{1}{\alpha} \int_{0}^{z} \frac{-s-1}{s} \, ds \right\} \, dz,

and this relation also holds for any real \(x < 0\).

This theorem can be proved in the same way as theorem 2.1. Both theorems together establish the continuation of \(N(x)\) over the whole complex \(x\)-plane cut along the positive real axis from 0 to \(\infty\). Of course, \(N(x)\) satisfies the linear differential-difference equation (1.1) in that region. The behaviour on the upper and the lower sides of the cut is very remarkable. In order to see this we substitute, in (2.25), \(z = -iu\). After some trivial calculations we obtain

\[[2.27] \quad (-e)^{1/\alpha - 1} \exp \left\{ -xz + \frac{1}{\alpha} \int_{0}^{z} \frac{-s-1}{s} \, ds \right\} = \frac{c}{u} \exp \left\{ iux - \frac{1}{\alpha} \int_{0}^{\infty} e^{-is} \, ds \right\} \quad (u > 1),

where \(c\) is a constant (depending on \(\alpha\) only).

Partial integration yields

\[[2.28] \quad \frac{1}{\alpha} \int_{0}^{\infty} \frac{-s}{u} \, ds = \frac{1}{\alpha u} e^{-1u} + O(\frac{1}{u^2}) \quad (u \gg 1),\]
Hence,

\[(2.29) \quad (-z)^{1/\alpha - 1} \exp \left\{ -xz + \frac{1}{\alpha} \int_0^z \frac{s^{\alpha - 1}}{s} \, ds \right\} = \]

\[= \frac{\hat{u}}{u} e^{ixu} \left\{ 1 + \frac{1}{\alpha u} e^{-iu} + O\left( \frac{1}{u^2} \right) \right\} \quad (u \gg 1).\]

Then it can easily be verified that the integral on the right-hand side of (2.25) is a continuous function in the region \( I_B(x) > 0 \) \( (x \neq 0) \), satisfying (1.1) on the positive real axis for \( x > 1 \). In the same way it can be shown that the integral on the right of (2.26) is continuous for \( I_B(x) < 0 \) \( (x \neq 0) \), and satisfies (1.1) for \( x > 1 \).

The jump of \( N(x) \) at the cut - which is defined by its value on the upper side minus its value on the lower side - equals

\[(2.30) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} (-z)^{1/\alpha - 1} \exp \left\{ -xz + \frac{1}{\alpha} \int_0^z \frac{s^{\alpha - 1}}{s} \, ds \right\} \, dz \quad (x > 0).\]

If we now combine this result with theorem 1.4, we find that we have established the following theorem.

**Theorem 2.3**

The jump of \( N(x) \) at the positive real axis is equal to \( e^{-\gamma} q_{0} (0, x) \), where \( \gamma \) is Euler's constant.

In the half-plane \( \mathbb{R} \) \( (x) > 0 \) the function \( N(x) \) can be expressed in terms of the solutions \( F_+ \), \( F_- \), \( F_n \) \( (|n| > 1) \).

**Theorem 2.4**

If \( F_0(x) > 0 \), \( I_B(x) > 0 \), then

\[(2.31) \quad N(x) = e^{\pi \frac{1}{2 \alpha}} F_+(x) - e^{\pi \frac{1}{2 \alpha}} \sum_{n = -1}^{B} F_n(x),\]
and this relation also holds for \( x > 0 \) with the value of \( N(x) \) corresponding to its value in the upper half-plane.

If \( R_{1}(x) > 0, \ I_{1}(x) < 0 \), then

\[
(2.32) \quad N(x) = e^{-\pi d/a} F_{+}(x) + e^{-\pi d/a} \sum_{n \geq 1} F_{n}(x),
\]

and this equality is still true for \( x > 0 \) with the value of \( N(x) \) taken from the lower half-plane.

Proof

We first take \( x \) in the region \( R_{1}(x) > 0, \ I_{1}(x) > 0 \), for any integer \( m \geq 1 \) we have

\[
(2.33) \quad -F_{-}(x) + \sum_{n \in \mathbb{Z}, n \neq 0} F_{n}(x) =
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \exp \left( \frac{\psi(z)}{z} \right) \frac{dz}{z} - \frac{1}{\pi} \int_{-(2m+1)i}^{-(2m+1)i} \exp \left( \frac{\psi(z)}{z} \right) \frac{dz}{z},
\]

where, with the principal value of the logarithm,

\[
\psi(z) = \frac{1}{a} \log z - z + \frac{1}{a} \int_{0}^{z} \frac{s^{-1}}{s} ds.
\]

Since the function \( z^{1/a} \exp \left( \frac{1}{a} \int_{0}^{z} \frac{s^{-1}}{s} ds \right) \) is uniformly bounded with respect to \( m \) on the half-lines \( z = -(2m+1)i + u \) \((m > 0, u > 1)\), the second term on the right-hand side of \(2.33\) is, in absolute value, less than a constant multiplied by \( \pi^{-1} \) (the constant depending on \( x \) and \( a \) only). Then, keeping \( x \) fixed and making \( m \to \infty \), we obtain

\[
(2.34) \quad -F_{-}(x) + \sum_{n \leq 1} F_{n}(x) = \frac{1}{\pi} \int_{-\infty}^{0} \exp \left( \frac{\psi(z)}{z} \right) \frac{dz}{z}.
\]
Finally, (2.31) follows from (2.13), (2.25) and (2.34). The second part of the theorem can be proved in the same way.

2.3 The case $\alpha < 0$

In order to obtain an infinite set of solutions of (1.1), we start from the system \( \{ \Xi_{2n+1}, \Pi_0, \Pi_+ \} \) defined in the previous section. Replacing $x$ by $-x-1$ and $\alpha$ by $-\alpha$, we find as solutions of (1.1) the functions $(n = 0, \pm 1, \pm 2, \ldots)$

\[
(2.35) \quad f_{2n+1}(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^{1/\alpha - 1}}{z + (2n+1)\alpha} \exp \left\{ \frac{-x}{\alpha} \int_0^{2n+1} \frac{z}{s} ds \right\} dz,
\]

\[
(2.36) \quad f_0(x) = \frac{1}{2\pi i} \int_{\Gamma} (-z)^{1/\alpha - 1} \exp \left\{ -x \int_0^{2n+1} \frac{z}{s} ds \right\} dz,
\]

\[
(2.37) \quad n(x) = \frac{1}{2\pi i} \int_{\Gamma} (-z)^{1/\alpha - 1} \exp \left\{ -x \int_0^{2n+1} \frac{z}{s} ds \right\} dz.
\]

The multiplicative factor $\sqrt{2\pi i}$ has been chosen in accordance with formula (2.3); $\Xi$, $\Pi$ and $\Pi_+$ are defined in sec 2.2. It should be observed that $f_{2n+1}$, $f_0$ are entire functions, while $n(x)$ is analytic for $R_0(x) < 0$.

Dealing similarly with the adjoint equation, we obtain, starting from the set \{ $\eta_{2n+1}(x)$, $N(x)$ \} defined in the previous section, a system \{ $h_0$, $h_{2n+1}$ \}, $n = 0, \pm 1, \ldots$, of analytic solutions for $R_0(x) > -1$. We have, $n = 0, \pm 1, \ldots$

\[
(2.38) \quad h_{2n+1}(x) = -\frac{1}{\alpha} \int_{-\infty}^{-x+(2n+1)\alpha i} \left( x+1 \right)^{-1/\alpha - 1} \exp \left\{ \frac{1}{\alpha} \int_0^{2n+1} \frac{z}{s} ds \right\} dz,
\]

\[
(2.39) \quad h_0(x) = \frac{1}{\alpha} \int_0^{x} \left( z \right)^{-1/\alpha - 1} \exp \left\{ \frac{1}{\alpha} \int_0^{2n+1} \frac{z}{s} ds \right\} dz.
\]
It will be shown later on that the functions \( f_0, f_{2n+1} (n = \infty, \pm 1, \pm 2, \ldots) \) constitute a fundamental system of (1.1). Any function \( f(x) \) that is a solution of (1.1) for \( x > y > 0 \), can be expressed in terms of these solutions by means of finite or infinite linear combinations. We have (see theorem 3.4)

\[
(2.40) \quad f(x) = \{f_0 h_0\} f_0 (x) + \sum_{-\infty}^{\infty} \{f_n h_{2n+1}\} f_{2n+1} (x) \quad (x > y).
\]

The function \( n(x) \) does not appear in the above formula; it is needed only if we develop solutions of (1.1) for negative values of \( x \). It should be noted that \( n(x) \), which is originally defined for \( R_0 (x) < 0 \) only, can be continued analytically over the whole \( x \)-plane cut along the positive real axis from the origin to infinity, the jump at the positive real axis being equal to \( e^W \delta_0 (0, x) \) (see also theorem 2.7). Finally, we note that \( n(x) \) can be expanded in terms of the solutions \( f_0, f_{2n+1} (|n| \gg 0) \), if \( x \) lies in the half-plane \( R_0 (x) > 0 \) (see also theorem 2.4).
CHAPTER III

THE CASE $a > 0$

3.1 Introduction

So far we have only carried out some preliminary investigations needed for a further detailed discussion of the linear functional equation. In the previous chapter we had to distinguish between a positive and a negative in order to obtain sets of special solutions. For this reason it is convenient to deal from now on with the two cases separately. Throughout the next chapters we assume $a$ to be a fixed positive number. We already mentioned that the most important theorems to be derived states that any function $f(x)$ constituting a solution of (1.2) for $x, y > 0$, can be written as an infinite linear combination of the form

$$\left\{ f, \delta \right\} \phi(x) + \sum_{n=-\infty}^{\infty} \left\{ f, \gamma_n \right\} F_n(x) \quad (x > y).$$

From this representation the asymptotic properties of $f(x)$ for large positive values of $x$ can be deduced. This gives rise to the following problems. First, in order to establish the convergence of the above series, we must know the behaviour of $F_n(x)$ and $N_n(x)$ for large values of $|n|$ when $x$ is any positive real number. Secondly, we have to determine the asymptotic behaviour of the special solutions as $x$ approaches infinity. Both problems can be solved by means of saddle point analysis. Satisfactory results can be obtained by using the saddle points of the function $\phi(z)$, where

$$\phi(z) = \frac{1}{\alpha} \log z - z + \frac{1}{\alpha} \int_{0}^{z} \frac{\log s - 1}{s} ds;$$

$\log z$ is defined by its principal value.
3.2 The saddle points of \( \varphi(z) \)

The saddle points of \( \varphi(z) \) are the roots of the equation 
\( \varphi'(z) = 0 \). We have \( \varphi'(z) = \omega + e^z/(\omega z) \), and therefore \( \varphi'(z) = 0 \)
if \( e^z = \omega z \). Let \( S(\epsilon, \delta) \) be the sector: \( \epsilon \leq |z| < \infty \),
\[ |\arg z| < \pi/2 - \delta, \]
where \( \epsilon \) and \( \delta \) are small positive numbers
\( (\delta < \pi/2) \). For each integer \( k \), denote by \( T_k \) the half-strip
\[ (2k-1)\pi \leq \arg z \leq (2k+1)\pi, \quad R_\varphi(z) \geq 1. \]

Now we shall prove the following theorem concerning the distribution of the saddle points of \( \varphi(z) \) over the half-plane \( R_\varphi(z) \geq 1 \).

**Theorem 3.1**

There exists a large positive number \( p_0 \), depending on \( \epsilon, \delta \) only, such that the equation \( \varphi'(z) = 0 \) has for each \( x \in S(\epsilon, \delta) \)
one and only one solution in every horizontal strip \( T_k \) with
\[ |2k\pi + \log \omega x| \geq p_0, \]
Further, this root in the \( k \)-th strip, denoted by \( z_k \), is the sum of an absolutely convergent double power series
\[
(3.2) \quad z_k = \rho + \log \rho + \tau \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} c_{lm} \rho^l \tau^m
\]
where
\[
(3.3) \quad \rho = 2k\pi + \log \omega x, \quad \tau = 1/\rho, \quad \tau = \log p/\rho,
\]
all the logarithms having their principal values; the \( c_{lm} \) are constants, and we note that \( c_{00} = c_{10} = 1, c_{01} = -\frac{1}{2} \).

**Proof**

Let \( P \) be the set consisting of the complex numbers
\( \rho = 2k\pi + \log \omega x \), where \( k \) runs through all integer values and
\( x \in S(\epsilon, \delta) \). If \( \rho \in \rho \) and \( |\rho| \) is sufficiently large, say, \( |\rho| \geq c_1 \),
then \( |\arg \rho| < \pi/2 + \delta/2 \). Now choose \( x \) in the sector \( S(\epsilon, \delta) \) and,
keeping \( x \) fixed, choose \( k \) such that \( |p| \geq c_1 \), \( |p| \geq c_2/|x| \). The linear transformation

\[(3.4) \quad u = x - p - \log p\]

maps the part of \( T_k \) where \( R_\alpha(z) = 1 + \log |ax| + \log |p| \), into the infinite strip

\[(3.5) \quad |\tau(u)| \leq 2\kappa - \delta/2, \quad R_\alpha(u) \geq -1.\]

Under this transformation the equation \( f'(z) = 0 \) takes the form

\[(3.6) \quad e^u - 1 - cu - \tau = 0\]

where \( c \) and \( \tau \) are defined by (3.3).

For the time being, we ignore the relation existing between \( c \) and \( \tau \), and we shall consider them as small independent complex parameters. N. H. de Bruijn proved (see [2], § 2.4) that there exists a small positive number \( \eta \), such that, if \( |c| < \eta \), \( |	au| < \eta \), the equation (3.6) has just one solution in the infinite strip (3.5), and that this solution lies in the domain \( |u| < \delta/2 \). Further, this solution, denoted by \( u \), is the sum of an absolutely convergent double power series

\[(3.7) \quad u = \tau \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} c_{lm} \frac{1}{l!} \tau^m \quad (|c| < \eta, \ |	au| < \eta),\]

where the \( c_{lm} \) are constants.

We now return to the special values of \( c \) and \( \tau \) given in formula (3.3), viz. \( c = p^{-1} \), \( \tau = \log p/p \). For \( |p| \) sufficiently large, say, \( |p| \geq c_2 \), we have \( |c| < \eta \), \( |	au| < \eta \). Keeping \( x \) fixed, choose \( k \) large enough in order to guarantee that

\[|p| = |2k \pi i + \log ax| \geq \max(c_1, c_2, e^2/|x|).\]

For those values of \( k \), each half-strip \( I_{\alpha}(z) = 2k \pi i \leq \kappa, \ R_\alpha(z) \geq -1 + \log |ax| + \log |p| \) contains just one saddle point of the function \( f(z) \), and this saddle point is given by (3.2).
It still remains to be proved that \( \varphi'(z) \) does not vanish throughout the part of \( \Gamma_k \) where \( \Re \varphi'(z) \approx -1 + \log |ax| + \log |p| \).

In this region, for any fixed value of \( \overline{\Gamma}_m(z) \), the modulus of \( \varphi'/z \) is an increasing function of \( \Re \varphi'(z) \). On the vertical line segment \( z = -1 + \log |ax| + \log |p| + 2k\pi i + vi \) \( (-\pi < v < \pi) \) we have \( |z| \approx \frac{2}{|p|} \), provided \( |p| \) is large enough. Hence

\[
(5,8) \quad \left| \frac{\varphi'}{z} \right| \approx \frac{2}{|p|} |ax|, \quad (|\overline{\Gamma}_m(z)| - 2k\pi) \approx \pi,
\]

\[
1 \approx \Re \varphi'(z) \approx -1 + \log |ax| + \log |p|,
\]

and this completes the proof of the theorem.

Since \( p = 2k\pi i + \log ax \), we have \( |p| \approx |p| \), if at least one of \( |k|, |x| \) is a large positive number. This means that, if \( x \) lies far to the right inside the sector \( S(c, \delta) \), say, \( |x| > x_0 \), each horizontal strip \( \Gamma_k \) \( (|k| \approx 0) \) contains just one saddle point, and this is given by the series \((5,2)\). We will localise these saddle points more precisely. In order to do this we start from \((5,2)\) and take a few terms only \((|x| \approx x_0, |k| \approx 0)\)

\[
(5,9) \quad \varphi_k = p + \log p + \frac{\log p}{p} + O\left( \frac{\log p}{p^2} \right).
\]

Now write

\[
\arg x = \theta, \quad (2k\pi + 0)/\log |ax| = q.
\]

After some simple calculations we find

\[
(5,10) \quad \overline{\Gamma}_m(k) = \frac{2k\pi + \theta + \arctan q + \frac{\arctan q - q \log (1 + q) - q \log |ax|}{(1 + q^2) \log |ax|}}{1 + \frac{\arctan q}{q}}
\]

with an error term which, in absolute value, is less than a constant multiplied by the square of
\((\sqrt{1+q^2 \log |x|})\) \log(\sqrt{1+q^2 \log |x|}).\) Considering the above expression for large and small values of \(|q|\) respectively, it follows that \(|x| \rightarrow \infty\):

\[\theta \sim n \quad (k > 0),\]

\[= \pi/2 < \theta \sim n \quad (k < 0).\]

Next we deal with the case where \(x\) is real and positive but not necessarily large. If \(x \gg 1\) and \(|k|\) is large enough, say, \(|k| \gg k_0\), we have \(|p| = 2kni + \log ax| \gg p_0\). For those values of \(k\) each half-strip \(R_k\) contains just one root of the equation \(\varphi'(x) = 0\), and this root is given by (3.2). If \(a < x < b\) \((a < a < b)\), we may write

\[3.12 \quad \mathcal{I}_k = 2kn + \log 2knax + \frac{\log |2kn|}{2kn} + O(\frac{1}{k}) \quad (|k| \rightarrow \infty);\]

the constant implied in the \(O\)-symbol does not depend on \(x\). Hence, if also \(a < x < b\),

\[3.13 \quad \mathcal{I}_n (\mathcal{L}_k) = 2kn + \frac{a}{2} \text{sgn } [k] - \frac{\log |2kn|}{2kn} + O(\frac{1}{k}) \quad (|k| \rightarrow \infty),\]

where \(\text{sgn } [k]\) denotes the sign of the integer \(k\).

The saddle points \(L_k\) are not the only roots of the equation \(\varphi'(x) = 0\). From the graphs of the functions \(e^x\) and \(\arctan x\) \((x \text{ real})\) we see that \(\varphi'(z) = 0\) has a solution lying close to the origin, provided \(x\) is a large positive number. Denoting this solution by \(z\), we have \(z = (ax)^{-1} + (ax)^{-2} + O(x^{-3}).\)

### 3.5 The behaviour of \(\varphi_n(x)\) for large values of \(|x|\)

In this section we shall apply saddle point analysis to the integral (2.3) in order to obtain information concerning the behaviour of \(\varphi_n(x)\) for \(x\) far to the right in the sector \(S(c,\beta)\):
$\varepsilon = |x| < \infty$, $|\arg x| \leq \pi/2 - \delta$ ($\varepsilon > 0$, $0 < \delta < \pi/2$). All formulae labelled "$x \to \infty$" will hold uniformly with respect to both $n$ ($n = 0, \pm 1, \pm 2, \ldots$) and $\arg x$ as $x \to \infty$ in $S(\varepsilon, \delta)$.

Formula (2.3) reads

$$3.14 \quad F_n(x) = \frac{1}{2\pi i} \int_{W + 2\pi n i} \exp \{ \varphi(z) \} \frac{dz}{z},$$

where, referring to our previous notation,

$$3.15 \quad \varphi(z) = \frac{1}{a} \log z - xz + \frac{1}{a} \int_0^z \frac{e^{s-1}}{s} ds,$$

and $W$ is described in sec. 2.1.

In the previous section we pointed out that every horizontal strip $T_k$ ($|k| = 0$) contains just one saddle point of the function $\varphi(z)$, provided that $|x|$ is large enough. As the path of integration $W + 2\pi n i$ can be kept entirely inside the strip $T_n$, we may expect to be able to restrict ourselves to the saddle point $\xi_n$. We therefore have to look for a path that crosses $\xi_n$, while $\xi_n$ itself is the 'highest point' on it. Clearly, that path will depend on $n$, $\arg x$ and $|x|$, and the major trouble is caused by the dependence on $\arg x$. This difficulty may be overcome by application of conformal mapping. Substituting $z = \xi_n + t$, the saddle point is shifted to the origin and we obtain

$$3.16 \quad F_n(x) = \frac{1}{2\pi i} \int_{W + 2\pi n i - \xi_n} \exp \{ \varphi(\xi_n + t) \} \frac{dt}{\xi_n + t},$$

where $W + 2\pi n i - \xi_n$ is defined as the contour described by $t + 2\pi n i - \xi_n$, if $t$ describes $W$. From lemma 2.1 and formula (3.11) we infer that we may replace this contour by the path $W - i \arg x$, which does not depend on $n$. For convenience we put $\arg x = 0$.

Then

$$3.17 \quad F_n(x) = \frac{1}{2\pi i} \int_{W - i \infty} \exp \{ \varphi(\xi_n + t) \} \frac{dt}{\xi_n + t}.$$
The integrand is an analytic function in the complex $t$-plane cut along the line joining the points $-\xi_n$ and $-\xi_n - \infty$. We notice that this half-line lies far to the left in the half-plane $R_n(t) < 0$. We are free to modify the path such that it passes through the saddle point $t = 0$. In order to know how it must cross the point $t = 0$ we must have a clear idea about the behaviour of the integrand in the neighbourhood of the origin. By Taylor series expansion we have, for all $|t| < |\xi_n|$, \[ \varphi(\xi_n + t) = \varphi(\xi_n) + \varphi'(\xi_n) \frac{t^2}{2!} + \varphi''(\xi_n) \frac{t^3}{3!} + \cdots. \] 

The derivatives of $\varphi(z)$ at the point $\xi_n$ can be evaluated by differentiation of the right-hand side of the expression (3,15).

Using $\varphi'(\xi_n) = 0$, we find \[ (3,19) \quad \varphi(\xi_n) = a \varphi(\xi_n), \quad \varphi'(\xi_n) = x \left( 1 - \frac{1}{\xi_n} \right), \]

and, if $k = 3$, $k$ fixed, \[ (3,20) \quad \varphi^{(k)}(\xi_n) = x \left( 1 + O\left( \frac{1}{\xi_n^k} \right) \right) \quad (x = \infty). \]

The axis of the saddle point (for this concept see [9], p. 84) is the straight line through the origin defined by \[ \varphi''(\xi_n) t^2 \quad \text{real and } < 0. \]

The argument of the axis is $\frac{n}{2} - \frac{1}{2} \arg(\varphi''(\xi_n))$ and this tends to $\frac{n}{2} - \frac{1}{2}$ if $x = \infty$ (the argument is uniquely determined apart from an additional multiple of $\pi$). Along this straight line the value of $\exp\{\varphi''(\xi_n) \frac{t^2}{2}\}$ is in all points exponentially small, apart from a small segment around $t = 0$. Now it is very important to know whether in the Taylor expansion (3,18) the sum of the terms \[ \varphi^{(1)}(\xi_n) \frac{t^3}{3!} + \varphi^{(4)}(\xi_n) \frac{t^4}{4!} + \cdots. \]
is small compared to the term $\varphi''(\xi_n) \frac{k^2}{2}$, if $t$ lies close to the origin. If it is small, the second-order term determines the contribution of the saddle point.

In fact, it can be proved that for any integer $N \geq 2$ and for each point $t$ inside the circle with centre 0 and with radius $R$ ($R$ is a fixed, but arbitrarily large, positive number)

$$\varphi(\xi_n + t) = \varphi(\xi_n) + \sum_{k=2}^{N} \varphi^{(k)}(\xi_n) \frac{k^k}{k!} t^k + O(x t^{N+1}) \quad (x \to \infty).$$

The constant in the $O$-symbol depends on both $N$ and $R$. In order to verify this expression we start from formula (5.15). Putting $z = \xi_n + t$, we obtain

$$\varphi(z) = \varphi(\xi_n) - xt + x \xi_n \int_{0}^{t} \frac{e^s}{s^2 + \xi_n} ds,$$

where the integral is taken along a straight line.

The function $(a + \xi_n)^{-1}$ can be written as $(N \geq 2, |s| \leq R)$

$$(a + \xi_n)^{-1} = \sum_{k=0}^{N-1} (-1)^k a^k \xi_n^{-k-1} + (-1)^N a^N \xi_n^{-N} (a + \xi_n)^{-1}.$$ 

Then we easily find $(N \geq 2, |t| \leq R)$

$$\varphi(z) = \varphi(\xi_n) - xt + x \sum_{k=0}^{N-1} (-1)^k \xi_n^{-k-1} \int_{0}^{t} s^k e^s ds +$$

$$+ O(x \xi_n^{-N} t^{N+1}) \quad (x \to \infty).$$

Expanding the functions on the right into powers of $t$ and neglecting terms $t^{N+1}, t^{N+2}, ...$, the error we make is $O(x t^{N+1})$ and this completes the proof.

We now choose the path of integration such that it crosses the saddle point while the angle it makes in $t = 0$ with the axis of the saddle point, is less than $\pi/4$. Then it is easy to evaluate the contribution from a small segment of the path that includes
the origin as an interior point. It is, however, in no way clear that the contribution of the remaining part of the path will be exponentially small compared to that of the saddle point. It is not immediately obvious how the contour has to be chosen outside a small neighborhood of the origin. A more detailed investigation of the integrand will solve this problem.

Taking \( N = 2 \) in (3.23) we obtain

\[
(3.24) \quad \varphi(\xi_n + t) = \varphi(\xi_n) + X(\xi_n^\dagger - 1 - t) - \frac{X}{\xi_n^2} (\xi_n^t - \xi_n^1) + O\left(\frac{X^3}{\xi_n^2}\right).
\]

Then, using (3.19), we infer \( (x \to \infty) \)

\[
(3.25) \quad \varphi(\xi_n + t) = \varphi(\xi_n) + \varphi''(\xi_n)(\xi_n^t - 1 - t) + O\left(\frac{X^3}{\xi_n^2}\right),
\]

uniformly with respect to \( t \) then \( |t| \leq R \), where \( R \) is a fixed, but arbitrarily large, positive number.

This shows that the function \( \exp\{\varphi''(\xi_n)(\xi_n^t - 1 - t)\} \) is the dominating factor in the integrand along a large part of the path of integration. The function \( \varphi(t) = \xi_n^t - 1 - t \) will now be studied closely by means of conformal mapping. As the path of integration can be kept entirely within the open horizontal strip \( |\xi_n(t)| < 2\pi \), it is sufficient to consider the function inside this strip. First we observe that \( \varphi(t) \) has just one zero in the strip, the zero \( t = 0 \) being of the second order. In order to prove this we take a long rectangle with vertices at the points \( \pm M \pm 2\pi i \), where \( M \) is a large positive number, and we determine the path described by \( \varphi(t) \) if \( t \) runs along the boundary of the rectangle starting at \( t = M \) on the real axis and proceeding in the positive direction. We take the argument of \( \varphi(t) \) at the starting-point \( t = M \) equal to zero. Writing \( t = u + iv \), one has \( \varphi(t) = e^{i\varphi - 1 - t} \).

If \( t \) runs from \( M \) to \( M + 2\pi i \), then arg \( \varphi(t) \) increases steadily from \( 0 \) to almost \( 2\pi \), whereas the modulus of \( \varphi(t) \) remains large. If \( t \) runs through the upper horizontal side of the rectangle, then \( \varphi(t) \) runs through a straight line parallel to the real axis up to 43
\(-2\pi i\) (which is attained if \(t = 2\pi i\)) and back along the same line to a point far to the right. If \(t\) proceeds from \(-\pi + 2\pi i\) to \(-\pi\), then \(\arg\ \Psi(t)\) increases to \(2\pi\) while the modulus of \(\Psi(t)\) remains large. If \(t\) runs along the lower part of the rectangle, then \(\Psi(t)\) proceeds through the complex conjugate of the path described just now, and \(\arg\ \Psi(t)\) increases to \(4\pi\) (which is attained at the end-point \(t = \pi\)). Hence,

\[
(3.26) \quad \frac{1}{2\pi i} \int \frac{\Psi'(t)}{\Psi(t)} \, dt = \frac{1}{2\pi} \text{ multiplied by the increment of }
\arg\ \Psi(t) = \gamma,
\]

where the path of integration is the boundary of the rectangle taken in the positive direction. This completes the proof.

We now consider the function \(\pi(t)\) defined for all \(t\) inside the strip \(|\Im(t)| < 2\pi\) by

\[
(3.27) \quad \pi(t) = \left\{ \Psi(t) \right\}^{1/2} = \sqrt{2\Psi(1)} \exp \left\{ \frac{1}{2} \int_{1}^{t} \frac{\Psi'(s)}{\Psi(s)} \, ds \right\},
\]

the path of integration lying entirely inside the open strip, and \(\sqrt{2\Psi(t)}\) being the ordinary positive square root of \(2\Psi(t)\).

Clearly, \(\pi(t)\) is an analytic function in the open strip, vanishing nowhere except at the point \(t = 0\), which is a simple zero of the function. Moreover, \(\pi(t)\) is continuous in the closed strip, and it provides a conformal mapping of the open strip on to the whole complex \(\pi\)-plane cut along two hyperbolic arcs defined by

\[
(3.28) \quad \Im(\pi) \cdot \Re(\pi) = \pm 2\pi, \quad \Re(\pi) \leq -|\Im(\pi)|.
\]

These hyperbolic arcs are the images of the horizontal lines \(t = \pm 2\pi i + u\) \((-\infty < u < \infty\). The same conformal mapping occurs in an asymptotic problem discussed by N.G. de Brujin (see [19], § 6.9). We shall follow that discussion closely. We want to introduce \(\pi\) as a new integration variable in the integral (3.17).

Therefore we have to investigate \(\frac{\pi'}{\pi}\) which clearly is an analytic function throughout the whole cut plane. As \(\Psi(t) > 0\) when
\( t \gg 0, \) we infer from (3.27) that \( w(t) = \left\{ 2(e^t - 1 - t) \right\}^{1/3}, \) where the ordinary positive square root is used \( (t \gg 0). \) Hence,

\[
(3.29) \quad w(t) = t + \frac{t^2}{6} + \ldots \quad (0 < t < 2\pi),
\]

and this relation holds by analytic continuation for all \( t \) with \( |t| < 2\pi. \) From this expansion we deduce

\[
(3.30) \quad t(w) = w - \frac{w^2}{6} + \ldots \quad (|w| < 2\sqrt{\pi}),
\]

and

\[
(3.31) \quad \frac{dt}{dw} = 1 - \frac{w}{3} + \ldots.
\]

As \( w^2 = 2(e^t - 1 - t), \) we have \( \frac{dt}{dw} = \frac{w(e^t - 1)}{e^t} \). Now \( e^t - 1 \) is, as far as our strip is concerned, close to zero only if \( t \) is close to either 0 or \( \pm 2\pi \) (the corresponding points in the \( w \)-plane being \( w = 0 \) and \( w = 2\sqrt{\pi} e^{\pm \pi i} \)). Therefore, we have

\[
(3.32) \quad \frac{dt}{dw} = 0 \quad (|w| \geq 2\sqrt{\pi}, \arg w \leq \frac{\pi}{2} - \frac{\delta}{2}).
\]

Analyzing the image of \( W - i\delta \) under the conformal mapping, we find that it is a curve starting at \( e^{-\delta \pi/2 - i} \) and tending to \( e^{\pi \delta/2} \) without avoiding the hyperbolic arcs. We have, however, considerable freedom in modifying this path. Now we choose \( W - i\delta \) such that it is mapped on to the straight line \( L(\theta) \) passing through \( w = 0 \) and with arguments \( \pm \pi/2 - \theta/2. \) Along \( W - i\delta \) the function \( e^t - 1 - t \) has the constant argument \( \pi - \theta, \) so that the modulus of \( \exp\left(\pi \left(\frac{\theta}{\pi} \right) (e^t - 1 - t)\right) \) has its maximum at the saddle point \( t = 0. \)

So this path seems to be satisfactory from the point of view of the saddle point method but, unfortunately, it is not quite sure that the function \( \exp\left(\pi \left(\frac{\theta}{\pi} \right) (e^t - 1 - t)\right) \) is the dominating factor in the integrand throughout the whole path. Moreover, we have to show that this path does not intersect the half-line in the \( t \)-plane joining the points \(-\xi_0\) and \(-\xi_1\), on which the integrand is 45
discontinuous. For this reason we shall investigate the curve \( W^{-10} \) more closely.

**Lemma 5.1**

If \( w \) runs through the line \( L(0) \) in the complex \( w \)-plane, then \( t(w) \) proceeds along the curve \( W^{10} \) in such a way that

\[
(3.33) \quad R_e(t) > \log \left( |w^2| \frac{\sin \delta}{4} \right) \quad (|w^2| > 4/\sin \delta),
\]

\[
(3.34) \quad \sin \left( \frac{v(t)}{ \sin \delta} + \theta \right) = o(1) \quad (|w| \to \infty),
\]

where the constant implied in the \( o \)-symbol does not depend on \( \theta \) (\( |\theta| < \pi/2 - \delta \)).

**Proof**

Putting \( t = u + iv \) (\( u, v \) real) in \( w^2 = 2(e^t - 1 - t) \), we find, if \( w \) lies on the line \( L(0) \),

\[
(3.35) \quad |w^2| \cos \theta = 2(1 + u - e^u \cos v),
\]

\[
|w^2| \sin \theta = 2(e^u \sin v - v).
\]

Since \( \cos \theta > \sin \delta \), we infer from (3.35) that \( u \) is positive when \( |w^2| > 4/\sin \delta \). With \( 2e^u > 1 + u - e^u \cos v \) \((u > 0)\) it follows that \( u > \log \left( |w^2| \frac{\sin \delta}{4} \right) \) if \( |w^2| > 4/\sin \delta \). Moreover, (3.35) yields

\[
(3.36) \quad \tan \theta = \frac{\sin v - y e^{-u}}{-\cos v + (1 + u)e^{-u}} \quad (t \neq 0).
\]

Hence \( (|w| \to \infty) \)

\[
\sin (v + \theta) - (1 + u)e^{-u} \sin \theta + v e^{-u} \cos \theta = o(1),
\]

and this completes the proof.

We have now reached the stage where the saddle point method will
actually be carried out. We divide the line $L(\theta)$ into three parts $L_k(\theta)$ ($k = 0, \pm 1$), where

1. $L_{-1}(\theta)$ is the half-line $w = \left| w \right| e^{-(\pi \theta) i/2}$
   \[ \left( \infty > \left| w \right| > A \right) , \]

2. $L_0(\theta)$ is the line segment $w = u e^{(\pi \theta) i/2}$
   \[ (-A < u < A) , \]

3. $L_1(\theta)$ is the half-line $w = \left| w \right| e^{(\pi \theta) i/2}$
   \[ (A < \left| w \right| < \infty) , \]

$A$ is a fixed positive number with $\log(A^2 \sin \delta) > \frac{2n}{\sin \delta}$. The parts of $W = \theta$, corresponding under the conformal mapping to these three line segments, will be denoted by $c_{-1}(\theta), c_0(\theta)$ and $c_1(\theta)$ respectively. Next we take $A$ large enough in order to guarantee that on both $c_{-1}(\theta)$ and $c_1(\theta)$ the absolute value of $\sin(I_n(\theta) + \theta)$ is less than $10^{-1}$.

We note that the sector $\left| w \right| < A, \pi/4 + \delta/2 < \arg w < 2\pi - \delta/2$, which is the set of all points $w$ lying on the line segments $L_0(\theta)$, is the image of a bounded domain in the open strip $\{ z \in \mathbb{C} \mid | z | < 2 \pi \}$.

This means that there exists a positive number $R$ such that the circle with centre $0$ and radius $R$ contains all points $t$ lying on the curves $c_0(\theta)$, where $\theta$ varies from $-\pi/2 + \delta$ to $\pi/2 - \delta$. We first evaluate the contribution from the integral along $c_1(\theta)$. To this end we replace $c_1(\theta)$ by another path. On this path it runs from the starting-point of $c_1(\theta)$ through a vertical line, over a small distance only, to the line parallel to the real axis and passing through the point $(\pi - \theta)i$, and from there along this horizontal line to $(\pi - \theta)i + \infty$. On the small vertical line segment the real part of $\omega^\theta (\zeta_n) (e^{i(\pi - 1 - \theta)}$ has a negative upper bound $-2 |x|$, and therefore the contribution of this part becomes (see (3.17), (3.25))

$$(3.37) \quad \zeta_n^{-1} \exp\{\sigma(\zeta_n)\} \cdot O(e^{-|x|}) \quad (x \to \infty) .$$
On the horizontal part of the path we put \( t = u + (\pi - \theta)i \). Since 
\[
\log(A^2 \sin \delta) > \frac{2\pi}{d} \sin \delta,
\]
we have (see (3,33)) \( u > 2\pi / \sin \delta \). From (3,22) we deduce
\[
(3.38) \quad \varphi(t_n + t) = \varphi(t_n) - x(u + (\pi - \theta)i) + x t_n \int_0^{(\pi - \theta)i} \frac{e^{s^2}}{s + t_n} ds - \\
+ |x| t_n \int_0^{u} \frac{e^{s^2}}{s + t_n + (\pi - \theta)i},
\]
where both integrals on the right are taken along straight lines.

One easily verifies that
\[
(3.39) \quad x t_n \int_0^{(\pi - \theta)i} \frac{e^{s^2}}{s + t_n} ds = -|x|(1 + e^{\theta i} + o(\log^{-1} x)) \quad (x \to \infty),
\]
Furthermore, we notice that the real part of \( t_n (t_n + s + \pi - \theta i)^{-1} \)
is positive when \( s \gg 0 \). Hence, if also \( t = u + (\pi - \theta)i \)
\((u > 2\pi / \sin \delta)\),
\[
(3.40) \quad \exp\{\varphi(t_n + t)\} = \exp\{\varphi(t_n)\} \times \\
\times 0\{\exp(\frac{1}{x} u \sin \delta + (2a - 1)|x|)\}.
\]

Integrating with respect to \( u \) from \( 2\pi / \sin \delta \) to \( \infty \), we see that the
contribution of the horizontal part of the path is also given by
the expression (3,37).

Dealing similarly with the integral along \( c_{-1}(t) \), we find a
contribution of the same order.

Along \( c_0(t) \) we introduce \( \varpi \) as a new integration variable
instead of \( t \). Using (3,17), (3,25), (3,30) and (3,37), we obtain
\[
(3.41) \quad F_n(x) = \xi_n - \exp\{\varphi(t_n)\} \times \left[ \frac{1}{\pi^2} \int_{c_0(t)} \exp\{\varphi(t)\} d\varpi + O\left(\frac{\varpi}{\xi_n}\right)\right] \\
\times \left(1 + \frac{t(x)}{\xi_n}\right)^{-1} dt \cdot e^{-|x|}.
\]
It is, however, preferable to have a fixed integration path, not depending on \( \theta \). Actually, we can achieve this by replacing \( L_\theta(\theta) \) by the line segment joining the points \( \pm 1 \) on the imaginary axis. In virtue of Cauchy’s theorem we must take into account the contribution of the arcs of the circle, with centre 0 and radius \( A \), connecting the points \( \pm 1 \) and \( -A \) with \( A e^{i \pi/2} \) and \( -A e^{i \pi/2} \), respectively. Along these arcs the integrand is \( O(1/|x|^2) \). The new path will be denoted by \( L \). Along \( L \) we have

\[
(3.42) \quad |exp\{v^e(x_n^{(e)} \frac{w}{2} + 0 \left( \frac{w}{x_n^{(e)}} \right)^3 \}| \approx exp\{-|x| \cdot |w|^2 \frac{\sin \beta}{4} \}.
\]

Now the method of Laplace can be applied. The values of \( \frac{dt}{dw} \) and \( \left( 1 + \frac{t(w)}{x_n^{(e)}} \right)^{-1} \) at \( w = 0 \) equal 1. Furthermore, \( \frac{dt}{dw} \) is an analytic function of \( w \) along \( L \), and \( \frac{dt}{dw} = O(w) \) if \( |w| \gg \sqrt{x_n^{(e)}} \). Therefore the integral along \( L \) can be compared with the formula

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp\{v^e(x_n^{(e)} \frac{w}{2}) \} dw = \frac{1}{\sqrt{2\pi}} \exp\{v^e(x_n^{(e)})\}^{-\frac{1}{2}},
\]

where \( (v^e(x_n^{(e)}))^{-\frac{1}{2}} \) is to be interpreted as \( \exp\{-\frac{1}{2} \log v^e(x_n^{(e)})\} \), with the principal value of the logarithm.

Finally, we obtain

\[
(3.43) \quad P_n(x) = \frac{1}{\sqrt{2\pi}} x_n^{(e)} (v^e(x_n^{(e)}))^{-\frac{1}{2}} \exp\{v(x_n^{(e)})\} \cdot (1 + o(1)) \quad (x \to \infty).\tag{3.44}
\]

It is possible, although rather complicated, to replace the \( o(1) \) term by an asymptotic expansion. But since this is not needed for our purposes we shall give only one more term here. In order to do this, it is easier to evaluate the contribution of the middle point in the original \( t \)-plane. Using (3.19), (3.20), (3.21), and integrating along a small segment of the imaginary axis, including the origin as an interior point, we find

\[
(3.44) \quad P_n(x) = \frac{1}{\sqrt{2\pi}} x_n^{(e)} (v^e(x_n^{(e)}))^{-\frac{1}{2}} \exp\{v(x_n^{(e)})\} \cdot (1 - \frac{1}{12} x + O\left(\frac{1}{x \log x}\right)) \quad (x \to \infty),
\]
In conclusion, we will determine the asymptotic behaviour of the function $\tilde{S}(x)$. Again the result will be uniform with respect to $\arg x$ as $x$ tends to infinity in the sector $S(c, b)$. We have

\begin{equation}
\tilde{S}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \exp\{\varphi(z)\} \frac{dz}{z} + \frac{1}{2\pi i} \int_{\mathcal{C}_2} \exp\{\varphi(z)\} \frac{dz}{z},
\end{equation}

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are contours, both starting from the origin and tending to $\pi + \infty$ and $-\pi + \infty$ respectively; $\varphi(z)$ is defined by (3.15). Now let $\mathcal{C}_1$ be the curve consisting of the following parts

(i) the interval $0 \leq z \leq 1$,
(ii) the line segment $z = 1 + u e^{\pi/2}$ ($0 \leq u \leq \pi/\sin \frac{b}{2}$),
(iii) the half-line $z = i u + 1 + \pi/\tan \frac{b}{2} \leq u < \infty$.

Let $\mathcal{C}_2$ be the complex conjugate of $\mathcal{C}_1$. On the line segment (ii) and the half-line (iii) we have $e^{-x^2} = O(e^{-x|x|\sin \frac{b}{2}})$ and the same thing holds for the contour $\mathcal{C}_2$. Then it can easily be seen that

\begin{equation}
\tilde{S}(x) = \frac{1}{\pi i} \int_0^1 z^{1/\alpha - 1} \exp\left[-zx + \frac{1}{\alpha} \int_0^z \frac{e^{s} - 1}{s} \, ds\right] + O(e^{-x|x|\sin \frac{b}{2}}) \quad (x \to \infty).
\end{equation}

Since $\exp\left(\frac{1}{\alpha} \int_0^z \frac{e^{s} - 1}{s} \, ds\right)$ is an analytic function, it can be expanded into a power series

\begin{equation}
\exp\left(\frac{1}{\alpha} \int_0^z \frac{e^{s} - 1}{s} \, ds\right) = \sum_{k=0}^{\infty} c_k x^k,
\end{equation}

converging absolutely and uniformly within the range of integration. This implies that for each integer $N \geq 0$ ($x \to \infty$)

\begin{equation}
\tilde{S}(x) = \frac{1}{\pi i} \sum_{k=0}^{N} c_k \int_0^1 e^{-zx} z^{k+1/\alpha - 1} \, dz + O(|x|^{-N-1/\alpha}).
\end{equation}
Finally, using the relation
\[ \int_0^\infty e^{-xz} z^{k+1/\alpha - 1} \, dz = x^{-1/\alpha} z^k \Gamma(k + 1/\alpha) \quad (R_\alpha(x) > 0), \]
we obtain an asymptotic development in terms of powers of $x^{-1}$
\[ (3.49) \quad \bar{F}(x) = \frac{x^{-1/\alpha}}{\pi} \sum_{k=0}^N c_k \Gamma(k + 1/\alpha) x^{-k} + O(|x|^{-N-1/\alpha}) \quad (x \to \infty); \]
we notice that $c_0 = 1$ and $x^{-1/\alpha} = |x|^{-1/\alpha} e^{-\theta_1/x}$.

3.4 The asymptotic behaviour of $H_n(x)$

In this section we shall apply saddle point analysis in order to investigate the behaviour of the functions $H_n(x)$ for $x$ far to the right inside the sector $S(\epsilon, b)$. Formulas labelled "$x \to \infty$" will hold uniformly in both $n$ ($n = 0, \pm 1, \ldots$) and $\arg x$ as $x \to \infty$ in $S(\epsilon, b)$.

It is convenient to deal with the cases $n \neq 0$ and $n = 0$ separately.

(i) $n \neq 0$.

We start from formula (2.25). Integration by parts yields
\[ (3.50) \quad H_n(x) = -x \int_{-\infty + 2n \pi i}^{\infty + 2n \pi i} \exp \{-q(z)\} \, dz, \]
where, with the notation used in sec. 3.3,
\[ (3.51) \quad q(z) = \frac{1}{\alpha} \log z - xz + \frac{1}{\alpha} \int_0^z \frac{s^{\beta - 1}}{s} \, ds. \]

For the integral (3.50) we have exactly the same saddle point $z_n$ as in the previous section. Substituting $z = z_n + t$, the saddle point is shifted to the origin and we obtain
\[(3.51) \quad H_n(x) = - x \int_{\infty + 2n \pi i - \xi_n}^{\infty + 2n \pi i - \xi_n} \exp \{-\varphi(\xi_n + t)\} \, dt.\]

Next we infer from lemma 2.1 and formula (3.11) that the path of integration can be replaced by a curve consisting of the following parts:

(i) the half-line \( t = u \) (\( u \geq 0 \Rightarrow -A \)),

(ii) the vertical line segment connecting the points \( t = A \) and \( t = A - \theta i \) (again \( \theta = \arg x \)),

(iii) the half-line \( t = u - \theta i \) (\( A < u < \infty \)),

where \( A = \frac{b}{\sin \delta} \log \left( \frac{b}{\sin \delta} \right) \).

This curve turns out to be satisfactory from the point of view of the saddle point method. The main contribution to the asymptotic behaviour will come from a small interval along the real axis, which contains the origin as an interior point.

First we turn our attention to the vertical line segment. Formula (3.25) gives

\[(3.52) \quad \exp \{-\varphi(\xi_n + t)\} = \exp \{-\varphi(\xi_n)\} \varphi'((e^{t - 1 - t}) +
\quad + O\left(\frac{x^{\frac{1}{3}}}{\xi_n}\right) \quad (x \to \infty),\]

uniformly with respect to \( t \) when \( |t| \leq 2A \).

We notice that \( A \) is large enough in order to guarantee that on (ii) the modulus of \((1 + t)e^{-t}\) is less than \( \frac{\sin b}{2} \), so that the contribution furnished by the integral along this part of the curve becomes

\[(3.53) \quad - x \exp \{-\varphi(\xi_n)\} \cdot O(e^{-|x|}) \quad (x \to \infty),\]

We will now estimate the integral along the half-line \( t = u - \theta i \) (\( u \geq A \)). We infer from (3.22) that
\( (3.54) \quad \exp \{-\phi(t_n + t)\} = \exp \{-\phi(t_n) \cdot x(u - 01) + \)

\[\begin{align*}
&\quad - x t_n \int_0^u \frac{s^2}{s + t_n} ds - |x t_n \int_0^u \frac{s e^s ds}{s + t_n + 01},
\end{align*}\]

where both integrals are taken along straight lines. It is not difficult to verify that the real part of \( t_n (s + t_n - 01)^{-1} \) exceeds \( (s + 2)^{-1} \), provided \( s > 0 \). Then the inequality \( \int_0^u e^s (s + 2)^{-1} ds > 4u \)

\( (u \gg A) \) yields

\( (3.55) \quad |\exp \{-|x| t_n \int_0^u \frac{s^2}{s + t_n - 01} ds\}| \leq \exp \{-4|x| u\} \quad (u \gg A) \).

This relation still holds when \( n \) is equal to zero.

Further, we have \( (x \to \infty) \)

\( (3.56) \quad - x t_n \int_0^u \frac{s^2}{s + t_n} ds = - |x| \left\{ t - 8e + 0(\log^{-1} x) \right\}. \)

Combining the various results we find that, if \( t = u - 01 \) \( (u \gg A) \),

\( (3.57) \quad \exp \{-\phi(t_n + t)\} = \exp \{-\phi(t_n) \cdot 0(e^{-1}|x| u) \} \quad (x \to \infty). \)

Integrating with respect to \( u \) from \( A \) to \( \infty \), we obtain a contribution of the same order as before.

We now proceed to the half-line \( t = -u \) \( (\infty \gg u \gg A) \), and first deal with that part where \( u \gg A \). Again using \( (3.22) \) we have

\( (3.58) \quad \exp \{-\phi(t_n + t)\} = \exp \{-\phi(t_n) \cdot w u + x t_n \int_0^u \frac{s e^{-u}}{s + t_n} ds\}, \)
Since \( e^{-\gamma \vert -a + \xi_n \vert} \) is a decreasing function of \( \gamma \) (\( \gamma > 0 \)), we may write \( (u \geq A, 0 \leq a < 1) \)

\[
(3.59) \quad \int_0^u \frac{e^{-s}}{\vert -s + \xi_n \vert} \, ds = \int_0^a \frac{e^{-\gamma \vert -s + \xi_n \vert}}{\vert -s + \xi_n \vert} \, ds + \int_a^u \frac{e^{-\gamma \vert -s + \xi_n \vert}}{\vert -s + \xi_n \vert} \, ds < \\
< \frac{e^{-\gamma \vert \xi_n \vert}}{\vert -a + \xi_n \vert} + \frac{u e^{-\gamma \vert \xi_n \vert}}{\vert -a + \xi_n \vert}.
\]

If \( a = \log(8/\sin \delta) \), the right-hand member of this inequality is less than \( (u \sin \delta)/(3 \vert \xi_n \vert) \). Hence \( (x \to \infty) \)

\[
(3.60) \quad \exp \{-\varphi(\xi_n + t)\} = \exp \{-\varphi(\xi_n)\} \cdot o(x^{1/2 - 1/3(\sin \delta)}).
\]

and this gives rise to a contribution of the same order as in \((3.53)\). We have thus shown that

\[
(3.61) \quad \Pi(x) = -x \int_{-A}^A \exp \{-\varphi(\xi_n + t)\} \, dt + \\
- x \exp \{-\varphi(\xi_n)\} \cdot o(x^{-1/2}) \quad (x \to \infty),
\]

From this point onwards the process runs as in the previous section. Our final result is

\[
(3.52) \quad \Pi(x) = -x \sqrt{2\pi} (\varphi''(\xi_n))^{-\frac{1}{2}} \exp \{-\varphi(\xi_n)\} \cdot \\
\cdot (1 + \frac{1}{12x} + o(\frac{1}{x^{1/3} \log x})) \quad (x \to \infty),
\]

where \( (\varphi''(\xi_n))^{-\frac{1}{2}} \) is to be interpreted as \( \exp(-\frac{1}{2} \log \varphi''(\xi_n)) \), with the principal value of the logarithm.

(ii) \( n = 0 \).
It is not difficult to show that the above formula remains true when \( n \) equals zero. We start from (2.11), (2.12) and (2.20). Partial integration yields

\[
(3.65) \quad \Pi_0(x) = -\frac{x}{2} \int_{P_1}^{P_2} \exp \{-\psi(z)\} \, dz - \frac{x}{2} \int_{P_2}^{P_1} \exp \{\psi(z)\} \, dz.
\]

\( P_1 \) is a curve lying entirely in the upper half-plane \( L_z(z) > 0 \) and leading from \( z = -\infty \) to \( z = \infty \), and \( P_2 \) is the complex conjugate of \( P_1 \).

The main contribution to the asymptotic behaviour will come from a small neighbourhood of the saddle point \( \xi_0 \) that lies far away from the negative real axis where the integrand is discontinuous. Now one can easily verify that

\[
(3.64) \quad \Pi_0(x) = -x \int_{1\,\text{Re}\,L_z(\xi_0)}^{\infty} \exp \{-\psi(z)\} \, dz + O(e^{|x|^2}) \quad (x \to \infty),
\]

where the path of integration lies entirely in the half-plane \( R_{e}(z) > 0 \). Substituting \( z = \xi_0 + t \) it follows that

\[
(3.65) \quad \Pi_0(x) = -x \int_{1-P_e(\xi_0)}^{\infty - \xi_0} \exp \{-\psi(\xi_0 + t)\} \, dt + O(e^{|x|^2}) \quad (x \to \infty),
\]

The integral on the right of (3.65) can be treated in the same way as before and its behaviour for large values of \( |x| \) is given by (3.62). In the next section we shall see that the contribution of the saddle point is exponentially large compared with the \( 0 \) term in (3.65) and this completes the proof.

In conclusion of this section we will determine the asymptotic behaviour of the function \( \Pi(x) \). Again the result will be uniform with respect to \( \arg x \) as \( x \) tends to infinity in the sector \( \mathcal{R}(\xi, \phi) \).
Starting from (2.11), (2.12) and (2.20) we find by means of partial integration

\[
\widehat{\mathcal{H}}(x) = -\frac{x}{2} \int_{0}^{\infty} \exp \left\{ \frac{x}{\alpha} \left( s - \frac{1}{\alpha} \right) \right\} s^{-1/\alpha} \text{d}s,
\]

where the contour \( \mathcal{P} \) comes from \(-\infty\) in the upper half-plane, encircles the origin once in the negative direction, and returns to \(-\infty\) in the lower half-plane.

By Taylor series expansion we have, for all values of \( x \),

\[
\exp \left\{ -\frac{1}{\alpha} \int_{0}^{s} \frac{\text{d}s'}{s} \right\} = \sum_{k=0}^{\infty} \frac{s}{k!} z^k.
\]

Let \( N \) be any integer \( \geq 1/\alpha - 1 \). Then we write

\[
\widehat{\mathcal{H}}(x) = -\frac{x}{2} \sum_{k=0}^{N} \frac{s}{k!} \int_{\mathcal{P}} \exp \left\{ \frac{x}{\alpha} s \right\} s^{-1/\alpha} \text{d}s + \frac{x}{2} \int_{\mathcal{P}} \exp \left\{ \frac{x}{\alpha} s \right\} \sum_{k=N+1}^{\infty} \frac{1}{k!} s^k z^{-1/\alpha} \text{d}s.
\]

It is a well-known result in the theory of the Gamma functions that

\[
\int_{\mathcal{P}} \exp \left\{ \frac{x}{\alpha} s \right\} s^{-1/\alpha} \text{d}s = -\frac{2\pi i}{\Gamma(1/\alpha - k)} x^{-k-1/\alpha} \quad (\text{Re}(x) > 0),
\]

where \( x^{1/\alpha} = |x^{1/\alpha}| e^{\theta i/\alpha} \).

Next we observe that the infinite sum in braces is bounded in the neighbourhood of the origin. Therefore, the path of integration \( \mathcal{P} \) in the last integral on the right of (3.68) may be replaced by a path following the negative real axis from \(-\infty\) to the origin, taking the value of \( x^{-1/\alpha} \) that corresponds to its value in the upper half-plane, and back from the origin to \(-\infty\) with the value taken from the lower half-plane. Hence
\( (3.70) \quad \tilde{F}(x) = \pi x^{1/\alpha} \sum_{k=0}^{N} \frac{d_k}{\Gamma(1/\alpha - k)} x^{-k} + \)

\[ + ix \sin \frac{n}{\alpha} \int_0^\infty e^{-xz} \left\{ \sum_{N=1}^{\infty} (-1)^k d_k z^{k-1/\alpha} \right\} dz. \]

From (3.67), replacing \( z \) by \(-z\), we infer that

\[ (3.71) \quad \exp \left\{ -\frac{1}{\alpha} \int_0^z \frac{s^{\omega} - 1}{s} ds \right\} = \sum_{k=0}^{\infty} (-1)^k d_k z^k. \]

It can easily be seen that \( \sum_{k=0}^{\infty} (-1)^k d_k z^k = O(z^{1/\alpha}) \) \((z \to 1)\), and this implies

\[ (3.72) \quad \sum_{k=N+1}^{\infty} (-1)^k d_k z^{k-1/\alpha} = O(z^{N+1-1/\alpha}) \quad (z \to 1). \]

Obviously, this relation also holds when \( 0 \leq z \leq \alpha. \) So, finally, we obtain

\[ (3.73) \quad \tilde{F}(x) = \pi x^{1/\alpha} \sum_{k=0}^{N} \frac{d_k}{\Gamma(1/\alpha - k)} x^{-k} + O(x^{1/\alpha - N+1}) \quad (x \to \infty). \]

If \( \alpha^{-1} \) is an integer, the integral in (3.70) vanishes, and then we infer that \( \tilde{F}(x) \) is a polynomial in \( x \) of degree \( \alpha^{-1} \). This result can, of course, be obtained immediately by applying the theorem of residues to the integral (3.66).

3.5 Some results

Here follow some immediate applications of the results of secs. 3.3 and 3.4. Most of them will be used in the following chapters. Again, all formulas hold uniformly in both \( x \) and \( \arg x \) as \( x \) tends to infinity in the sector \( S(c, \delta) \). From (5.15) it follows that
\[ (3.74) \quad \exp\{w(\xi_n)\} = \exp\left\{ \frac{1}{\alpha} \int_0^{1/s} \frac{d s - x \xi_n}{s} + \frac{1}{\alpha} \int_0^{\xi_n} \frac{d s}{s} \right\}, \]

where the path of integration \((1, \xi_n)\) is taken along a straight line. Repeated integration by parts yields \((x \to \infty)\)

\[ (3.75) \quad \exp\{w(\xi_n)\} = \exp\left\{ -x(\xi_n - 1) - \frac{1}{\xi_n} - \frac{2}{x^2} + o\left(\frac{1}{\log x}\right) \right\}. \]

From this relation a clear survey of the asymptotic behaviour of the special solutions can be obtained. For the moment, however, we only give a simple result, sufficient, nevertheless, for our present purposes.

First, we infer from \((3.2)\) and \((3.75)\) that

\[ (3.76) \quad \exp\{w(\xi_n)\} = \exp\left\{ -x(2n\alpha + \log x) + \log(2n\alpha + \log x) + \right. \]

\[ \left. - 1 + o\left(\frac{\log \log x}{\log x}\right) \right\}. \]

Then \((3.19), (3.44), (3.62)\) and \((3.76)\) show that \((x \to \infty)\)

\[ (3.77) \quad P_n(x) \sim \xi_n^{-1} \exp\left\{ -x(2n\alpha + \log x) + \log(2n\alpha + \log x) + \right. \]

\[ \left. - 1 + o\left(\frac{\log \log x}{\log x}\right) \right\}, \]

and

\[ (3.78) \quad H_n(x) = \exp\{x(2n\alpha + \log x) + \log(2n\alpha + \log x) + \]

\[ - 1 + o\left(\frac{\log \log x}{\log x}\right) \}. \]

Obviously, if \(n\) is fixed, \(P_n(x)\) tends to zero as \(x\) approaches infinity inside the sector \(S(\epsilon, b)\). But, in general, the convergence is not uniform with respect to \(n\) except when \(x\) runs to infinity through positive real values only. An extensive discussion of this
case is to be found in chapter 7.
We need a formula for \( H_n(x+1)/H_n(x) \) as \( x \to \infty \) in \( S(\xi,\delta) \). Let \( \tilde{t}_n \) be the saddle point of the function

\[
(3.79) \quad \tilde{\varphi}(z) = \frac{1}{\alpha} \log z - (x+1)z + \frac{1}{\alpha} \int_0^z \frac{e^{\frac{s}{\alpha}} - 1}{s} \, ds
\]

inside the half-strip \( \tilde{H}_n(z) \geq 1, \left| \varphi_n(z) - 2n \pi \right| < n \). We can easily verify by (3.2) that

\[
(3.80) \quad \tilde{t}_n = \frac{1}{x} + O\left( \frac{1}{x \log x} \right) \quad (x = \infty).
\]

Since \( \tilde{\varphi}(z) = \varphi(z) - z \), we infer from (3.22) and (3.80) that \( x = \infty \)

\[
(3.81) \quad \exp\{\tilde{\varphi}(t_n)\} = \exp\{\varphi(t_n) - t_n - x t_n \int_0^{t_n} \frac{e^{s} - 1}{s} \, ds + O\left( \frac{1}{x} \right)\}
\]

where \( t = \tilde{t}_n - t_n \).

Hence

\[
(3.82) \quad \exp\{\tilde{\varphi}(t_n)\} = \exp\{\varphi(t_n) - t_n + O\left( \frac{1}{x} \right)\} \quad (x = \infty).
\]

Finally, we obtain by means of (3.62) and (3.82)

\[
(3.83) \quad H_n(x+1) = \alpha x t_n H_n(x) \{1 + O\left( \frac{1}{x} \right)\} \quad (x = \infty).
\]

3.6 The biorthogonality

Let \( f(x) \) be any of the functions \( \tilde{F}(x), F_n(x) \) \( (n = 0, \pm 1, \ldots) \),

and let \( h(x) \) be any of the functions \( \tilde{H}(x), H_n(x) \). Then the inner product

\[
(3.84) \quad \{f, h\} = f(x)h(x) + \frac{1}{\alpha} \int \frac{f(t)h(t + 1)}{t + 1} \, dt
\]
is defined and constant in the whole complex $x$-plane cut along
the negative real axis from $-1$ to $-\infty$. Because of the disconti-
uity of the integrand for $t < -2$, the path of integration is re-
quired to avoid this half-line.

We shall now evaluate these inner products by means of the asymp-
totic relations derived in the previous sections.

**Theorem 3.2**

The sets $\{F_0, F_1, F_2, \ldots\}$ and $\{H_0, H_1, H_2, \ldots\}$ form a
biorthogonal system, i.e.

$$\langle n, m \rangle = \delta_{nm},$$

where $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$.

$$\langle n, \overline{m} \rangle = 1, \quad \langle n, m \rangle = 0, \quad \langle n, \overline{n} \rangle = 0.$$

We first prove (3.05). If $n$ and $m$ are fixed, we have, by (3.77)
and (3.78),

$$P_n(x)H_n(x) = \exp\{-2n\log(n-m) + \frac{\log x}{\log n}\},$$

uniformly with respect to $\arg x$ as $x \to 0$ in $S(\pi, \delta)$.

Then with (3.83) it follows that

$$\langle n, \overline{m} \rangle = 0 \left[\exp\{-2n\log(n-m) + \frac{\log x}{\log n}\}\right] \ (x \to \infty).$$

If $n \neq m$, the leading term in the exponent is $-2n\log(n-m)$. Its
real part tends to $-\infty$ as $x$ runs to infinity along the half-line
through the origin with argument $\pi/4 \cdot \text{sgn}(n-m)$. As the expres-
sion for the inner product does not actually depend on $x$, we in-
fer that $\langle n, \overline{m} \rangle = 0$ (n $\neq$ m). Furthermore, we easily deduce from
(5.44), (5.62) and (5.83) that $\langle n, \overline{n} \rangle = 1 + o(1)$, and so

$$\langle n, \overline{n} \rangle = 1. \quad \text{The first two relations of (3.86) can be proved in a}$$
similar way by means of (3.49), (3.73) and (3.77). In order to verify the last relation of (3.66) we consider the expression for the inner product, with \( f = \tilde{P}_n \), \( h = \tilde{H}_n \) for purely imaginary values of \( x \). Putting \( x = iy \) in (2.23), we find, if \( n \neq 0 \), \( \tilde{H}_n (x) = O(e^{-i \pi y}) \). Further, we have (see (2.4), (2.5) and (2.7)) \( \tilde{F}(x) = O(e^{\pi |x|}) \)
(\( \tilde{H}_n (x) \neq 0 \)). Hence

\[(3.99) \quad \{ \tilde{F}, \tilde{H}_n \} = O(1) \quad (n \neq 0, \text{ fixed}, x = iy, ny \to \infty), \]

and so \( \{ \tilde{F}, \tilde{H}_n \} = 0 \quad (n \neq 0) \).

It remains to be shown that \( \{ \tilde{F}, \tilde{H}_0 \} = 0 \). Proceeding in the way described above, one can easily verify that the inner product \( \{ \tilde{F}, \tilde{Y}_- \} \) is equal to zero; \( \tilde{F}_+ (x) \) and \( \tilde{Y}_- (x) \) are defined in sec. 2.2. Then we infer from (2.6), (2.7), (2.20) and (2.21) that

\[(3.90) \quad \{ \tilde{F}, \tilde{H}_0 \} = -\{ \tilde{F}_0, \tilde{H}_0 \} + \{ \tilde{F}_0, \tilde{Y}_- \} + \{ \tilde{F}_+, \tilde{H}_0 \} + 2\{ \tilde{F}_+, \tilde{Y}_- \} = 0. \]

**Conclusion**

From the above theorem it follows at once that the functions \( \tilde{Y}(x) \), \( \tilde{F}_n (x) \) (\( n = 0, 1, 2, ... \)) are linearly independent, and the same thing is true for the functions \( \tilde{H}(x), \tilde{H}_n (x) \) (\( n = 0, 1, 2, ... \)).
CHAPTER IV

THE SERIES $\Sigma f_n(x)H_n(y)$

4.1 Introduction

In this chapter the convergence of the series $\Sigma f_n(x)H_n(y)$ will be investigated when both $x$ and $y$ are real and greater than $-1$. The behaviour of the terms for large values of $|n|$ will be determined by the saddle point method. For convenience, we first state the final result.

Theorem 4.1

The series $\Sigma f_n(x)H_n(y)$ converges absolutely and uniformly in both $x$ and $y$ in any compact set of the real $x-y$-plane inside the region $x > y > -1$. The convergence is still uniform, but not absolute, in any compact set where $y-1 < x < y$, $x > -1$, $y > -1$. If $x=y$ ($y > -1$) the series does not converge in the ordinary sense, although it converges in the sense that $\Sigma_{-N}^{N} f_n(x)H_n(y)$ tends to a limit as $N$ tends to infinity.

4.2 The behaviour of $f_n(x)$ and $H_n(x)$ for large values of $|n|$

For the time being we confine ourselves to positive values of $x$, and we shall derive asymptotic formulae that hold uniformly with respect to $x$ when $x$ ranges over a finite closed interval lying entirely to the right of the origin. Throughout this section relations labelled $"|n| \to \infty"$ will be uniform in any such interval.

First we solve the asymptotic problem concerning $f_n(x)$. We have

$$ f_n(x) = \frac{1}{2\pi i} \int_{W + 2\pi i} \exp(z) \frac{\partial \phi(z)}{z} \, dz $$
where, with our previous notation,

\[(4.2) \quad \psi(z) = \frac{1}{z} \log \frac{z}{x} - \frac{1}{z} \int_0^\infty \frac{e^{s-1}}{s} \, ds.
\]

In sec. 3.2 we pointed out that, if \(0 < a \ll x \ll b\), every half-strip \(|z_n| - 2n\pi \ll n\), \(\phi_n(a) \gg 1\) contains just one saddle point of the function \(\psi(a)\), provided \(|n|\) is large enough. Since the path of integration \(W = 2n\pi i\) can be kept inside this half-strip, we may expect to deal with the problem in a satisfactory way, if we use the saddle point \(z_n\) only. By the substitution \(z = z_n + t\), the saddle point is shifted to the origin and we obtain

\[(4.3) \quad F_n(x) = \frac{1}{2\pi i} \int_{\mathbb{C}} \exp(\psi(z_n + t)) \frac{dt}{z_n + t}.
\]

It is quite easy now to replace this path by another that depends neither on \(n\) nor on \(x\). It is an immediate consequence of lemma 27 and formula (3.13) that the path of integration can be deformed continuously into the curve \(W\) without affecting the value of the integral. Hence, if also \(|n|\) is large enough,

\[(4.4) \quad F_n(x) = \frac{1}{2\pi i} \int_{\mathbb{C}} \exp(\psi(z_n + t)) \frac{dt}{z_n + t}.
\]

In contrast with the asymptotic problems discussed before, the contribution will here come from a large neighbourhood of the saddle point \(t = 0\). This is caused by the fact that the integrand behaves more or less smoothly within a large region including the origin. In order to verify this we take as a starting-point formula (3.22), from which we can easily deduce that for any integer \(N \gg 1\) and for each point \(t\) inside the infinite strip \(|z_n(t)| \ll n\)

\[(4.5) \quad \exp(\psi(z_n + t)) = \exp(\psi(z_n)) \cdot xt + \sum_{k=0}^{N-1} (-1)^k \int_0^t e^{s-k} ds + (-1)^N \left[ x + \int_0^t e^{s+k} ds \right].
\]

\[\int_0^\infty e^{-\frac{s^2}{2}} \left( e^{-i\varphi} \cos \theta - i e^{-i\varphi} \sin \theta \right) \, ds.
\]

\[\int_0^\infty e^{-\frac{s^2}{2}} \cos \theta \, ds.
\]
We notice that the integrals occurring in the exponent are taken along straight lines. Taking \( N = 1 \), we infer from (4.5) that

\[
(4.6) \quad \exp \{ \varphi(t_n + t) \} = \exp \{ \varphi(t_n) + x(e^{t} - 1 - t) + O(e^{-t}) \},
\]

uniformly with respect to \( t \) throughout the half-strip \( |t_n(t)| \leq n, Re(t) \leq 0 \).
Moreover, we have \( (|n| \to \infty) \)

\[
(4.7) \quad \exp \{ \varphi(t_n + t) \} = \exp \{ \varphi(t_n) + x(e^{t} - 1 - t) + O(e^{-t}) \},
\]

when \( |t_n(t)| \leq n, Re(t) \geq 0 \).

This clearly proves the assertion we made. So, fortunately, there is not much trouble in finding a suitable path of integration. The most natural thing we can do is to integrate along the curve consisting of the line segments \((-ni) \to (ni) \to (ni, ni + \infty) \) \to \((-ni) \).

Since \( e^{t}e^{-t} \) is bounded for \( |t| \leq \log|t_n| \) we divide this path into the following parts -

(i) the part consisting of all points \( t \) with \( |t| \leq \log|t_n| \),
(ii) the half-line \( t = ni + u \quad (A \leq u < \infty) \),
(iii) the half-line \( t = -ni + u \quad (\infty > u \geq A) \),

where \( A \) is related to \( \varphi \) by \( A^2 + u^2 = (\log|t_n|)^2 \).

First we estimate the value of the integral along the half-line (ii). If \( t = ni + u \quad (u \geq A) \), the function \( \exp \{ \varphi(t_n + t) \} \) equals (see (5.22))

\[
(4.8) \quad \exp \{ \varphi(t_n) - nin - xu + x_n \int_0^u \frac{e^s}{s + x_n} ds - x_n \int_0^u \frac{e^s}{s + x_n + ni} ds \}.
\]

Since \( x \) is bounded, the first integral is bounded too. Next we observe that the real part of \( x_n(s + x_n + ni)^{-1} \) exceeds \( s^{-1} \), provided that \( s \gg 0 \). Then the inequality \( \int_0^u e^{s} (s + 2)^{-1} ds > e^{u/2} \)

\( (u \geq A) \) yields \( (|n| \to \infty) \)
\[ (4.9) \quad \exp\{\psi(t_n + t)\} = \exp\{\psi(t_n)\} \cdot 0[\exp(-xu - xz^{1/2})]. \]

Integrating with respect to \( u \) from \( A \) to \( \infty \), we find that the contribution to the integral (4.4) arising from the half-line (ii) is given by

\[ (4.10) \quad \xi_n^{-1} \exp\{\psi(t_n)\} \cdot O\left( \frac{1}{n} \right), \quad (|n| \to \infty). \]

Dealing similarly with the integral along the other half-line, we obtain a contribution of the same order. With (4.7) it then follows \((|n| \to \infty)\)

\[ (4.11) \quad P_n(x) = \xi_n^{-1} \exp\{\psi(t_n)\} \cdot \left[ 1 + O\left( \frac{1}{n} \right) \right], \]

where \( D \) denotes the curve defined in (i).

Now we need some relations which can easily be verified

\[ (4.12) \quad \int_D \exp\{x(e^t - 1 - t)\} \, dt = O(1) \quad (|n| \to \infty), \]

and

\[ (4.13) \quad \int_D \exp\{x(e^t - 1 - t)\} \, dt = O\left( \frac{1}{n} \right), \quad (|n| \to \infty), \]

where the path of integration consists of the two half-lines (ii) and (iii).

Combining the various results, we finally obtain

\[ (4.14) \quad P_n(x) = \xi_n^{-1} \exp\{\psi(t_n)\} \cdot \left[ 1 + O\left( \frac{1}{n} \right) \right] \]

\[ + O\left( \frac{1}{n} \right), \quad (|n| \to \infty). \]

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It may be remarked that the O-term can be replaced by an asymptotic series of the form \( \sum_{k=0}^{\infty} a_k x^{-k} \), the coefficients \( a_k \) depending on \( x \) only. In order to find this series it is necessary to use the equality (4.5) for \( N > 1 \). Further, we observe that

\[
(4.15) \quad \frac{1}{2\pi i} \int_{\mathcal{W}} \exp\{x(e^t - 1 - t)\} dt = \frac{1}{\Gamma(x+1)} \exp(x \log x - x).
\]

The argument is as follows. Consider the conformal mapping \( z = \log(-z) \), where the logarithm has its principal value. The path of integration \( \mathcal{W} \) is, under this conformal mapping, the image of a curve in the \( z \)-plane consisting of the real axis from \( \infty \) to \( 1 \), the circle \( |z| = 1 \) described in the positive direction, and the real axis from \( 1 \) to \( \infty \) again. Introducing \( z \) as a new integration variable we see that the integral (4.15) takes the form

\[
\frac{e^{-x}}{2\pi i} \int_{\mathcal{W}} e^{-z^2} (-z)^{-1} dz. \]

Then the result follows by means of the theory of the Gamma functions.

The problem about \( \Pi_n(x) \) can be dealt with in a similar way. Substituting \( z = R_n + t \) in (3.56) one obtains \((|n| \text{ large})\)

\[
(4.16) \quad \Pi_n(x) = -x \int_{-\infty}^{\infty} \exp\{-x(R_n + t)\} dt.
\]

The contribution coming from the positive real axis can be evaluated by means of (4.7), the details of the process being almost the same as those occurring in the previous case. The integral along the negative real axis can be tackled by formula (4.6). The final result is

\[
(4.17) \quad \Pi_n(x) = -x \exp\{-x(R_n)\} \ast \int_{-\infty}^{\infty} \exp\{-x(e^t - 1 - t)\} dt + O\left(\frac{1}{R_n}\right) \quad (|n| \to \infty).
\]
Again, the $O$-term can be replaced by an asymptotic series in terms of powers of $\xi_n^{-1}$. We note that

$$
\int_{-\infty}^{\infty} \exp\{-x(e^{s} - 1 - s)\} \, dt = \Gamma(x) \exp(x - x \log x),
$$

Now we need an estimate for $\exp\{\psi(\xi_n)\}$. First we derive from (4.2) (see also section 1.5)

$$
(4.18) \quad \exp\{\psi(\xi_n)\} = \exp\{-\gamma/\xi + \pi i/\xi \text{ sgn}[n]\} - x \xi_n +
$$

$$
+ x \xi_n \int_{0}^{\infty} \frac{e^{-s}}{s - \xi_n^{-1}} \, ds,
$$

where $\gamma$ denotes Euler's constant.

Integrating by parts we have

$$
x \xi_n \int_{0}^{\infty} \frac{e^{-s}}{s - \xi_n^{-1}} \, ds = x + O(\frac{1}{\xi_n}) \quad (|n| \to \infty) .
$$

Finally, by (3.12) ($|n| \to \infty$)

$$
(4.19) \quad \exp\{\psi(\xi_n)\} = \exp\{-\gamma/\xi + \pi i/\xi \text{ sgn}[n]\} - 2n \pi i x +
$$

$$
- x \log 2n \pi i \xi + x + O(\frac{1}{n}) .
$$

It is obvious from the results stated above that ($|n| \to \infty$)

$$
(4.20) \quad \Gamma_n(x) = \frac{2^{-x} e^{-\gamma/x}}{\Gamma(x + 1)} (2n \pi i)^{x-1} \exp\{\frac{\pi i}{x} \text{ sgn}[n]\} - 2n \pi i x ,
$$

$$
\cdot (1 + O(\frac{1}{n})) ,
$$
and

\begin{equation}
H_n(x) = -\alpha^x e^{-\alpha x} (x+1)^{(2n \pi i)^x} \exp\{- \frac{ni}{\alpha} \text{sgn}[n] + 2n \pi i x\} \\
\cdot (1 + O\left(\frac{\log n}{n}\right)).
\end{equation}

Hence

\begin{equation}
F_n(x)H_n(y) = \psi(x,y)(2n \pi i)^{(X-Y)x} \exp\{- 2n \pi i (x-y)\} \\
\cdot (1 + O\left(\frac{\log n}{n}\right))
\end{equation}

uniformly with respect to both \(x\) and \(y\) in any compact set of the real \(x-y\)-plane contained in the quadrant \(x > 0, y > 0\). The function \(\psi(x,y)\) is defined by

\begin{equation}
\psi(x,y) = -\alpha^{y-x} \Gamma(y+1) \Gamma^{-1}(x+1).
\end{equation}

It should be noted that this function is continuous in both \(x\) and \(y\) when \(x > -1, y > -1\).

The result stated in (4.22) can be improved if we work with slightly different expressions for \(F_n(x)\) and \(H_n(x)\) instead of those used before. Since \(F_n(x)\) is a solution of (1.1) one has

\begin{equation}
F_n(x) = \alpha(x+1)F_n'(x+1), \text{ and so}
\end{equation}

\begin{equation}
F_n(x) = \frac{\alpha(x+1)}{2 \pi i} \int \exp\{- (x+1)z + \frac{1}{\alpha} \int_0^{\frac{y+s-1}{s}} s^{\alpha-1} \, ds\} z^{1/\alpha} \, ds,
\end{equation}

For \(F_n(x)\) we use formula (2.23). Proceeding as before, we can determine the asymptotic behaviour of these functions when \((x+1)\) is real and positive. Then, after some trivial calculations, it turns out that the relation (4.22) also holds uniformly with respect to \(x\).
and \( y \) in any compact region in the \( x-y \)-plane contained in the quadrant \( x > -1, y > -1 \). After that, the proof of theorem 4.1 follows at once by comparing \( \sum_{n=0}^{\infty} F_n(x) \mathbb{R}_n(y) \) with \( \sum_{n=1}^{\infty} n^{-2} \text{C} \text{e}^{-nt} \) (\( t = x-y \)). We notice that the series converges in the ordinary sense except for \( x = y \).
CHAPTER V

SERIES EXPANSION OF THE GREEN FUNCTION

5.1 Introduction

In this chapter we shall show that the Green function \( G_\alpha(y,x) \) can be expressed in terms of the functions of the biorthogonal system.

Theorem 5.1

Let \( y > 0 \), then we have for all \( x > y - 1 \) (except for \( x = y \))

\[
G_\alpha(y,x) = F(x)\tilde{H}(y) + \sum_{-\infty}^{\infty} F_n(x)\tilde{H}_n(y).
\]

The proof may be obtained from the integral representation of the Green function derived in sec. 1.5. But since we have different formulae according as \( y \) is positive or zero, we must split the proof into two parts. We first need some preliminary results.

5.2 Some preliminary results

We consider the function \( \Psi(x,y,n; t) \) defined by

\[
\Psi(x,y,n; t) = t^{1/\alpha-1} \exp \left\{ -xt + \frac{1}{\alpha} \int_0^t \frac{e^{a(s-1)} ds}{s} \right\} \cdot \int_{t}^{\infty+2\pi i} \exp \left\{ \int_0^y \frac{e^{a(s-1)} ds}{s} \right\} v^{-1/\alpha} dv.
\]

\( x \) and \( y \) are fixed positive numbers (\( x > y \)), and \( n \) is a fixed integer. The multi-valued functions \( t^{1/\alpha-1} \) and \( v^{-1/\alpha} \) are taken to be real on the positive real axis, and the path of integration \( t, \infty+2\pi i \) does not intersect the half-line \( v < 0 \).
With this definition, \( \Psi(x, y, n; t) \) is analytic in the whole \( t \)-plane cut along the negative real axis. We shall investigate the behaviour of \( \Psi \) far to the right inside the infinite strip \( |L_n(t) - 2\pi n| < \pi \).

**Lemma 5.1**

If \( x, y, n \) are given, \( 0 < y < x \), then for \( |L_n(t) - 2\pi n| < \pi \):

\[
\Psi(x, y, n; t) = O(e^{-\psi x t}) \quad (\Re t = \infty).
\]

**Proof**

Let \( t \) be a point inside the half-strip \( |L_n(t) - 2\pi n| < \pi \), \( \Re t > 1 \). For this choice of \( t \) the path of integration \( t, \Re t + 2\pi n \) may be replaced by a curve starting at \( v = t \) and tending to \( v = \infty + 2\pi n \), where \( \varphi = \frac{\pi}{2} (2\pi n - L_n(t)) \). Introducing a new integration variable \( u \) instead of \( v \) by the substitution \( v = u + t \), one obtains

\[
e^{(x-y)t} \Psi(x, y, n; t) = \frac{1}{t} \int_0^\infty \exp \left\{ yu - \frac{1}{\alpha} \int_0^u \frac{s+tt}{s+t} \, ds \right\} \, du,
\]

where the path of integration lies entirely in the half-plane \( \Re u > \Re \alpha \), and the integral occurring in the exponent is taken along a straight line. At this point we apply a conformal mapping. Consider the function \( u = \log(x + t) \), the logarithm having its principal value. The image of the half-line \( L \), defined by \( w = z, z > 0 \) \( (0 < x < \infty) \), is a curve in the \( u \)-plane leading from the origin to \( \infty + i\varphi \). Along this curve we have \( \Re \alpha \) \( > \frac{\varphi}{2} \) \( > 0 \). Introducing \( w \) as a new integration variable, we find

\[
e^{(x-y)t} \Psi(x, y, n; t) = \frac{1}{\pi} \int_0^{\infty} \left( \frac{\varphi}{2} \right)^{\psi - 1} \exp \left\{ \frac{\psi}{\alpha} \int_0^\psi \frac{ds}{s + \log(s + \psi)} \right\} \, dw.
\]
again the integral occurring in the exponent is taken along a straight line. It can now easily be verified that on \( L \)

\[
\exp \left\{ -\frac{t}{a} \int_{\gamma} \frac{ds}{t + \log (s + 1)} \right\} \leq \exp \left\{ -\frac{|w|}{a(1 + \log(|w| + 1))} \right\},
\]

provided that the real part of \( t \) is large enough (it suffices to take \( R_e(t) = 5(|n| + 1)n \)). This means that the integrand in (5.5) has as an upper bound an integrable function not depending on \( t \) if \( R_e(t) \) is large enough. This completes the proof.

As an immediate consequence of this lemma we have

**Lemma 5.2**

If \( x, y, n \) are given, \( 0 < y < x \), then

\[
\int_{\gamma} W(x, y, n; t) dt = 0, \quad W + 2\pi i
\]

where \( \gamma \) is a curve lying inside the strip \( |\gamma_n(t)| < n \) and leading from \( -\pi i \) to \( +\pi i \). Because of the discontinuity of the integrand for \( t \approx 0 \) the path of integration in (5.7) must not intersect the negative real axis.

We next turn our attention to the special case \( n = 0 \) and we want some information about the behaviour of \( W(x, y, 0; t) \) in a neighbourhood of the origin. Needless to say, the function is not defined if \( t \approx 0 \). We shall prove the following lemma.

**Lemma 5.3**

Let \( U \) be the set consisting of all points \( t \) inside and on the unit circle with the exception of the points \( -1 \leq t < 0 \). Then we have, if \( t \in U \),
\( f(x, y, 0; t) = \begin{cases} o(1) & (0 < \alpha < 1) \\ o(\log |t|) & (\alpha = 1) \\ o(|t|^{-1/\alpha}) & (\alpha > 1). \end{cases} \)

Proof

For convenience we write

\[ f(x, y, 0; t) = t^{1/\alpha - 1} \exp \left\{ -xt + \frac{1}{\alpha} \int_0^t \frac{e^{s-1}}{s} ds \right\} \cdot A(t) \]

with

\[ A(t) = \int_{-\infty}^\infty \exp \left\{ yv - \frac{1}{\alpha} \int_0^v \frac{e^{s-1}}{s} ds \right\} v^{-1/\alpha} dv. \]

Replace in (5.10) \( \int_t^\infty \) by \( \int_t^1 + \int_1^\infty \). The second term does not depend on \( t \). The integrand in the first term can be expanded into powers of \( v \)

\[ \exp \left\{ yv - \frac{1}{\alpha} \int_0^v \frac{e^{s-1}}{s} ds \right\} v^{-1/\alpha} = \sum_{k=0}^{[1/\alpha]} c_k v^{k-1/\alpha} + o(1) \]

for all admissible \( v \) inside the unit circle (i.e. \( v \neq 0 \), \(-\pi < \text{arg } v < \pi \)); \([1/\alpha]\) denotes the integral part of \( 1/\alpha \). Hence, if \( t \in U \),

\[ A(t) = \begin{cases} o(|t|^{1-1/\alpha}) & (0 < \alpha < 1) \\ o(\log |t|) & (\alpha = 1) \\ o(1) & (\alpha > 1). \end{cases} \]

The result now easily follows from (5.9) and (5.12).
Remark

If \( n = 0 \), the relation (5.7) still holds when the curve \( W \) passes through the origin. This is an immediate consequence of lemma 5.3.

4.3 The case \( y > 0 \)

For the time being we take \( x > y \). In order to verify (4.1) we start from the integral representation of the Green function derived in sec. 1.5,

\[
\Omega_a(y,x) = \lim_{y \to +\infty} \int_{s=1}^{y} I(t)dt,
\]

where \( \beta \) is an arbitrary positive number, and

\[
I(t) = \frac{\sqrt{\pi}}{2\pi} \exp\left\{-xt + \frac{1}{a} \int_{0}^{t} \frac{e^{-\frac{1}{s}}}{s} ds \right\} (-t)^{1/a-1}.
\]

\[
\cdot \int_{-\infty}^{t} \exp\left\{yv - \frac{1}{a} \int_{0}^{v} \frac{e^{-\frac{1}{s}}}{s} ds \right\} (-v)^{-1/a} dv.
\]

The multi-valued functions \((-t)^{1/a-1}\) and \((-v)^{-1/a}\) are defined by their principal values, and the path of integration \((-\infty, t)\) does not meet the positive real axis. In this way \(I(t)\) is analytic in the complex \( t \)-plane cut along the half-lines \( t \geq 0 \).

The points \( t = (2n+1)i, \, n = 0, \pm 1, \ldots \), split the imaginary axis into an infinite number of intervals of equal length \( 2i \).

Through each of these points we draw the half-line \( t = (2n+1)i + z \) \((z \geq \beta)\). Let \( W \) be the contour consisting of the line segments \((-ni + \infty, -ni + \beta), \, (-ni - \beta, ni - \beta), \, (ni - \beta, ni + \infty)\). Then for each integer \( N \geq 0 \) we have, formally,

\[
\int_{-\beta-(2N+1)i}^{\beta+(2N+1)i} I(t)dt = \sum_{n=-N}^{N} \int_{-\beta-(2N+1)i}^{\beta+(2N+1)i} I(t)dt + \int_{\beta-(2N+1)i}^{\beta+(2N+1)i} I(t)dt.
\]

The two sides are equal, provided that all integrals on the right exist. In order to prove this we first deal with \( \int_{W+2ni} \).

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compare the function \((-t)^{1/\alpha-1}\) with the function \(t^{1/\alpha-1}\). The principal values are related by

\[
(5.16) \quad (-t)^{1/\alpha-1} = -e^{\pi i/\alpha} t^{1/\alpha-1} \quad (I_1(t) > 0),
\]

\[
(5.17) \quad (-t)^{1/\alpha-1} = e^{\pi i/\alpha} t^{1/\alpha-1} \quad (I_1(t) < 0).
\]

A similar relation holds for \((-v)^{1/\alpha}\) and \(v^{1/\alpha}\).

Now we take \(n \neq 0\). Replace in (5.14) the integral \(\int_{-\infty}^{\infty} \) by \(\int_{-2\pi i}^{2\pi i} \) and solve it. Then from (3.50), (5.2), (5.14) and (5.16) it follows that, if \(t\) lies on the path \(W + 2\pi i\),

\[
(5.17) \quad I(t) = H_n(y) - \frac{1}{2\pi i} \exp\left\{-xt + \frac{t}{\alpha} \int_{0}^{t} \frac{e^{s-1}}{s} ds\right\} t^{1/\alpha-1} +
\]

\[
+ \frac{x}{2\pi i} \Psi(x,y,n; t).
\]

Finally, by lemma 5.2 and formula (2.3)

\[
(5.18) \quad \int_{W + 2\pi i} I(t) dt = F_n(x) H_n(y) \quad (n \neq 0).
\]

If \(n = 0\), the integrand \(I(t)\) is not analytic everywhere inside the contour \(W\). Instead of (5.17) we now have (see (3.63), (3.66))

\[
(5.19) \quad I(t) = (H_0(y) + \tilde{H}(y)) - \frac{1}{2\pi i} \exp\left\{-xt + \frac{t}{\alpha} \int_{0}^{t} \frac{e^{s-1}}{s} ds\right\} t^{1/\alpha-1} +
\]

\[
+ \frac{x}{2\pi i} \Psi(x,y,0; t) \quad (I_1(t) > 0),
\]
and

$$I(t) = (H_0(y) - \overline{H}(y)) \cdot \frac{1}{2\pi i} \exp \left( -\frac{t}{\alpha} \right) \int_0^t \frac{\frac{1}{\alpha} \int_0^s -1 \, ds}{s} t^{1/\alpha - 1} +$$

$$+ \frac{y}{2\pi i} \mathcal{W}(x, y, 0; t) \quad (t \neq 0).$$

We now try to pull the path $\mathcal{W}$ as far as possible to the right. But we cannot, of course, pull it over the branchpoint $t = 0$. We shall therefore replace $\mathcal{W}$ by a path consisting of the line segments $(-\infty, -ni], (-ni, ni], (ci, ni], (ni, ni + \infty)$ and the semi-circle with radius $\varepsilon$ around the origin, described in the negative direction, in order to circumvent the singularity at $t = 0$. But, fortunately, the semi-circle can be removed by making its radius tend to zero, as can easily be seen by lemma 5.2 and formulae (5.19), (5.20). So, finally, we obtain by (2.4), (2.5), (2.6), (2.7) and lemma 5.2

$$\int_0^t I(t) \, dt = \mathcal{F}(x)H_0(y) + \mathcal{F}(x)\overline{H}(y).$$

We still have to investigate the last two terms occurring on the right-hand side of (5.15). Obviously, if $I_m(t) \neq 0$, we have

$$I(t) = -\frac{y}{2\pi i} \exp \left( -\frac{t}{\alpha} \right) \int_0^t \frac{\frac{1}{\alpha} \int_0^s -1 \, ds}{s} t^{1/\alpha - 1} \cdot \int_{-\infty}^t \exp \left( -\frac{y}{\alpha} \int_0^s -1 \, ds \right) v^{1/\alpha} \, dv,$$

where the path of integration $(-\infty, t)$ is taken along a straight line. With the substitution $v = u + t$ the above expression becomes

$$I(t) = -\frac{y}{2\pi i} \exp \left( -\frac{t}{\alpha} \right) \int_0^\infty \exp \left( -\frac{y}{\alpha} \int_0^{u-t} -1 \, ds \right) du.$$
If \( s \gg 0 \) and \( \mathbf{I}_n(t) = (2n+1)\pi \), the real part of \( (s + \pi t)^{-1}e^{-\pi t} \) is less than \( \exp\left\{-|s - R(t)|\right\} \). This implies that \( t\mathbf{I}(t)\exp((x-y)t) \) is uniformly bounded with respect to both \( s \) and \( n \) on the horizontal lines \( t = (2n+1)\pi i + z \) \( (-\infty < z < \infty ; n = 0, \pm 1, \pm 2, \ldots ) \). Hence \( (N \gg 0) \)

\[
\int_{-\beta}^{\beta} I(t)dt = \int_{-\beta}^{\beta} I(2n+1)\pi i + z) = 0(\frac{1}{2n+1}).
\]

Consequently, the relation (5.15) holds for any integer \( N \gg 0 \).

Moreover, we have shown \( (N \gg 0) \)

\[
\int_{-\beta-(2n+1)\pi i}^{-\beta+(2n+1)\pi i} I(t)dt = \mathcal{F}(x)\mathcal{H}(y) + \sum_{n=-N}^{N} F_n(x)H_n(y) + 0(\frac{1}{2n+1}).
\]

Finally, keeping \( x \) and \( y \) fixed, \( x > y > 0 \), we make \( N \to \infty \), and that completes the proof of (5.1) in the special case \( x > y > 0 \).

Now suppose that \( y < x < y + 1 \). Term-by-term differentiation in (5.1) gives

\[
\frac{d}{dx} G_a(y, x) = \mathcal{H}(y) \frac{d}{dx} \mathcal{F}(x) + \sum_{-\infty}^{\infty} F_n(x)H_n(y) \frac{d}{dx} F_n(x).
\]

This process will be legitimate if the above series converges uniformly with respect to \( x \) over any closed sub-interval. Since \( \mathcal{F}(x) \)

and \( \mathcal{F}_n(x) \) satisfy (1.1) for all values of \( x \), the right-hand member of (5.26) equals

\[
- \frac{1}{\alpha x} \left\{ \mathcal{F}(x-1)\mathcal{H}(y) + \sum_{-\infty}^{\infty} F_n(x-1)H_n(y) \right\},
\]

and the uniform convergence follows at once from theorem 4.1. Furthermore, we have \( \frac{d}{dx} G_a(y, x) = - \frac{1}{\alpha x} g_a(y, x-1) \) \( (y < x < y + 1) \).

Finally, replacing \( x-1 \) by \( x \), we obtain

\[
g_a(y, x) = \mathcal{F}(x)\mathcal{H}(y) + \sum_{-\infty}^{\infty} F_n(x)H_n(y) \quad (y-1 < x < y). 77
\]
5.4 The case $y = 0$

Taking $x > -1$ ($x \neq 0$) and keeping $x$ fixed we choose $y > 0$ such that either $y - 1 < x < 0$ or $y < x$. Under these conditions we may write

$$\mathcal{C}_a(y, x) = F(x) \mathcal{H}(y) + \sum_{n=1}^{\infty} P_n(x) H_n(y).$$

Theorem 4.1 shows that this series defines a continuous function of $y$ in some neighbourhood of the origin, whence

$$\lim_{y \to 0} \mathcal{C}_a(y, x) = F(x) \mathcal{H}(0) + \sum_{n=1}^{\infty} P_n(x) H_n(0).$$

As $\mathcal{C}_a(y, x)$ is continuous with respect to $y$ for $y > 0$ (except when $y = x$) it follows that

$$\mathcal{C}_a(0, x) = F(x) \mathcal{H}(0) + \sum_{n=1}^{\infty} P_n(x) H_n(0) \quad (x > -1, x \neq 0).$$

This completes the proof of theorem 5.1.
CHAPTER VI
SERIES EXPANSION OF ARBITRARY SOLUTIONS

6.1 Introduction

In this chapter we shall derive a representation of an arbitrary solution of the equation (1.1) in the form of an infinite series. We shall prove the following theorem.

Theorem 6.1

Let \( f(x) \) be a solution of the linear differential-difference equation (1.1) for \( x > y \geq 0 \), in the sense of definition 1.1. Then we have

\[
(6.1) \quad f(x) = \{f_{\infty}\} f(x) + \sum_{n=0}^{\infty} \{f_{n}\} f_{n}(x) \quad (x > y).
\]

The series converges absolutely and uniformly with respect to \( x \) over any compact interval lying entirely to the right of \( y+1 \). For \( x > y \) the convergence is still uniform in any such compact interval but need no longer be absolute.

Moreover, if there exists another series expansion in terms of the same special functions, representing \( f(x) \) for \( x > y \), then the corresponding coefficients in the two series are equal, i.e. the series are identical.

It should be remarked that the series (6.1) converges in the ordinary sense. For convenience, we deal with the cases \( x > y + 1 \) and \( y < x \leq y + 1 \) separately.

6.2 The case \( x > y + 1 \)

We take as a starting-point formula (1.12) which expresses \( f(x) \) in terms of its initial values
\[ f(x) = f(y) \delta_\alpha(y, x) - \frac{1}{\alpha} \int_{y-1}^{y} f(t) \delta_\alpha(t+1, x) \, dt \quad (x > y). \]

Since \( x > t + 1 \) we have by theorem 5.1 \( (y - 1 < t < y) \)
\[ \delta_\alpha(t+1, x) = \tilde{F}(x) H(t+1) + \sum_{-\infty}^{\infty} F_n(x) H_n(t+1). \]

Term-by-term integration in (6.2) gives
\[ f(x) = \langle f(t) \delta_\alpha(y) - \frac{1}{\alpha} \int_{y-1}^{y} f(t) \delta_\alpha(t+1) dt \rangle \tilde{F}(x) + \]
\[ + \sum_{-\infty}^{\infty} \langle f(t) \delta_\alpha(y) - \frac{1}{\alpha} \int_{y-1}^{y} f(t) \delta_\alpha(t+1) dt \rangle F_n(x). \]

The expressions in parentheses are equal to the inner products \( \langle f, \tilde{F} \rangle \) and \( \langle f, F_n \rangle \) respectively, whence
\[ f(x) = \langle f, \tilde{F} \rangle \tilde{F}(x) + \sum_{-\infty}^{\infty} \langle f, F_n \rangle F_n(x) \quad (x > y + 1). \]

Next we observe that \((t+1)^{-\alpha} f(t)\) is bounded in \( y - 1 < t < y \) (see definition 1.1), and therefore the absolute value of \( \langle f, F_n \rangle \) is less than a constant multiplied by \( \max_{y - 1 < t < y} |H_n(t+1)| \). Formula (4.22) now shows that
\[ \langle f, F_n \rangle F_n(x) = O(|n|^{-1+\epsilon}) \quad (|n| \to \infty) \]
uniformly with respect to \( x \) when \( y + \epsilon < x < y + 1 + \epsilon \) (where \( \epsilon \) is an arbitrary positive number with \( \epsilon > a > 0 \)).

The uniqueness of the development will be shown at the end of the next section.

6.3 The case \( y < x \leq y + 1 \)

We first prove the following lemma.
Lemma 6.1

Let \( g(t) \) be a bounded integrable function over \([-1, 1]\), and let \( \xi \) be a positive number with \( 0 < \xi < 1 \). Then

\[
(6.7) \quad \int_{-\delta}^{\delta} g(t) \sum_{n=1}^{\infty} \frac{\sin nt}{n^{1+\xi}} \, dt = o(1) \quad (N = \infty)
\]

uniformly with respect to both \( \delta_1 \) and \( \delta_2 \) if \( 0 < \delta_1 < \delta_2 < \xi \).

We denote the infinite series, occurring in the integrand of (6.7), by \( S_N(t) \). Obviously, \( S_N(t) \) is a continuous function of \( t \) when \( t > -1 \) (\( t \neq 0 \)), but it is not defined for \( t = 0 \). In order to establish the existence of the integral we have to investigate the behaviour of this function in a neighbourhood of the origin. We first choose \( N > 1/\xi \). Then, keeping \( N \) fixed, we split the interval \((-\delta, \delta)\) into the parts \((-\delta, -\frac{1}{N})\), \((-\frac{1}{N}, \frac{1}{N})\), \((\frac{1}{N}, \delta)\), \((\delta, 1)\).

Summation by parts yields

\[
(6.8) \quad \left| S_N(t) \right| \leq \frac{1}{k^{1+\xi}} \sum_{n=1}^{\infty} \frac{1}{\sin \pi nt} \quad (0 < |t| < 1, k > 1, \xi > k).
\]

For \( 0 < |t| < \frac{1}{N} \), we write \( S_N(t) = S + \sum_{n=1}^{N-1} \), where \( \beta \) denotes the integral part of \( |t|^{-1} \). Comparing the first term with \( \int_{N-1}^{\infty} u^{-1-t} \, du \)
and applying the above inequality to the second term, we obtain

\[
(6.9) \quad |S_N(t)| \leq \frac{\beta^{-t}}{t} + \frac{(N-1)^{-t}}{t} + \frac{|t|^{-1+t}}{\sin \pi nt} + 1 + \frac{1}{t} \left( e^{-t \log \beta} - e^{-t \log (N-1)} \right).
\]

Expanding the right-hand side into powers of \( t \) we easily find

\[
(6.10) \quad |S_N(t)| \leq \log |Nt| + 4 \quad (0 < |t| < 1/N),
\]

and this establishes the existence of the integral.
Notice further the following inequality

\begin{equation}
(6.11) \quad |S_N(t)| \leq \frac{1}{\mu^3 \delta} \cdot \frac{b}{|t| \sin \alpha} \quad (1/\mu \leq |t| \leq \delta).
\end{equation}

Since the series occurring in (6.7) converges uniformly with respect to \( t \) when \( \delta \leq |t| \leq 1 \), we have \( S_N(t) = o(1) \) \((N \rightarrow \infty, \delta \leq |t| \leq 1)\). Now (6.7) can easily be verified by means of the relations stated above. Lemma 6.1 of course still holds if we replace \( e^{-\alpha n t} \) by \( e^{-\alpha n t^2} \).

Let \( \varepsilon \) be a small positive number, \( \varepsilon < 1 \). Now we choose \( x \) such that \( y + \varepsilon \leq x \leq y + 1 \). For those values of \( x \) we have by theorem 5.1

\begin{equation}
(6.12) \quad C_\varepsilon(t + 1, x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( f_n H_n(t+1) + \sum_{n=1}^{N} P_n(x) H_n(t+1) \right)
\end{equation}

\( (y - 1 \leq t \leq y, \ t \neq x - 1) \).

The series does not converge in the ordinary sense if \( t = x - 1 \). So it is not quite sure that term-by-term integration in (6.2) is legitimate. Instead of (6.4) we now have

\begin{equation}
(6.13) \quad f(x) = \sum_{n=1}^{N} \left( f_n H_n(x) + R_n(x) \right) \quad (N \gg 1),
\end{equation}

where

\begin{equation}
(6.14) \quad R_n(x) = f(y) \int_{|y| \geq N} P_n(x) H_n(y) - \frac{1}{a} \int_{y-1}^{y+1} P_n(x) H_n(t+1) dt.
\end{equation}

A special investigation is required in order to see whether this function tends to zero uniformly with respect to \( x \) as \( N \) approaches infinity. From (4.22) it follows that \( (n \rightarrow \infty) \)

\begin{equation}
(6.15) \quad P_n(x) H_n(t+1) = C(x, t) \left\{ \frac{a^{2n+i(x-t)}}{x-t} + O\left(\frac{\log N}{n^{1+i}}\right) \right\},
\end{equation}

where
uniformly with respect to both $x$ and $t$ when $y + c \leq x \leq y + 1$, 
$y - 1 \leq t \leq y$, where

\[(6.16) \quad C(x, t) = -a^{t-x} \cdot n^{\frac{t-x}{2}} \cdot \Gamma(t+2) \cdot \Gamma(x+1).\]

Hence,

\[(6.17) \quad \sum_{N} P_n(x) R_n(t+1) = C(x, t) S_n(x-t-1) + o(1) \quad (N \to \infty).\]

Next we consider the integral

\[(6.18) \quad \int_{y-1}^{y} \sum_{N} P_n(x) R_n(t+1) dt = \int_{x-y}^{x-y} \frac{f(x-t)}{x-t} C(x, x-t-1) S_n(t) dt + o(1).\]

We lose nothing by assuming $f(t) = 0$ when $t < y - 1$. In this way the function $(x-t)^{-1} f(x-t) C(x, x-t-1)$ is defined for 
$y + c \leq x \leq y + 1$, $y - 1 \leq t \leq y$ and it is uniformly bounded there.

Applying lemma 6.1 with $b_1 = y+1-x$, $b_2 = x-y$ and $\delta = 1 - \epsilon$, we obtain

\[(6.19) \quad \int_{y-1}^{y} \sum_{N} P_n(x) R_n(t+1) dt = o(1) \quad (N \to \infty),\]

uniformly with respect to $x$ when $y + \epsilon \leq x \leq y + 1$.

The sum $\sum_{N} P_n(x) R_n(t+1)$ can be dealt with in the same way. Finally, we observe that

\[(6.20) \quad \sum_{|n| \gg N} P_n(x) R_n(y) = o\left( \sum_{|n| \gg N} |n|^{-1-\epsilon} \right) = o(1) \quad (N \to \infty).\]

Again the constant implied in the $o$-symbol is independent of $x$. So we have proved that $R_n(x)$ tends to zero uniformly in $x$. 83
\( y + \epsilon \leq x \leq y + 1 \), as \( N \) tends to infinity, and this completes the proof of the first part of theorem 6.1.

In order to show the uniqueness of the development in terms of the special solutions, it suffices to establish that the sum of the series \( \sum_{n=1}^{\infty} a_n \mathcal{P}_n(x) \) vanishes identically if, and only if, all the coefficients are equal to zero. Now let us assume that

\[
(6.21) \quad c_n P(x) + \sum_{n=1}^{\infty} c_n \frac{P_n(x)}{n} = 0
\]

for all \( x > y \geq 0 \), while at least one of the coefficients, say, \( a_k \), does not vanish. From the convergence of the series at \( y + 1 \) we infer that \( c_n P_n(y + 1) = 0 \). Next we deduce from \((6.20)\) that

\[
(6.22) \quad a_n P_n(x) = 0 (|n|^{-x+1}) \quad (|n| \to \infty),
\]

uniformly with respect to \( x \), when \( x \) ranges over a compact interval entirely to the right of the point \( x = y \). Now \((1.7)\) yields

\[
(6.23) \quad \left\{ c_n P(x) + \sum_{n=1}^{\infty} a_n P_n(x), H_k(x) \right\} = \sum_{n=1}^{\infty} a_n \left\{ P_n, H_k \right\} = 0.
\]

As the sets of special solutions form a biorthogonal system we find \( a_k = 0 \). Thus we have arrived at a contradiction and so have proved the uniqueness of the development.
CHAPTER VII

THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS

7.1 Introduction

In this chapter we will investigate the asymptotic behaviour of arbitrary solutions for large positive values of the independent variable. It is obvious from theorem 6.1 that the nature of the solutions, for large $x$, is closely related to that of the special solutions. It is our purpose to explore this connection in greater detail. Further, the stability of the solutions will be discussed.

7.2 Some asymptotic relations

We start from formula (5.44), which describes the behaviour of the functions $F_n(x)$ far to the right inside the sector $S(\varepsilon, \delta)$.

\[
\begin{align*}
(7.1) & \quad F_n(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2} \left( \varphi'(\xi_n) \right)^2} \exp\{\varepsilon_n \} \cdot \\
& \quad \cdot \left( 1 - \frac{3}{12\varepsilon} + O\left( \frac{1}{x \log x} \right) \right) \quad (|x| \to \infty)
\end{align*}
\]

uniformly with respect to both $n$ and $\arg x$ if $|\arg x| \leq \pi/2 = 5$, $0 < \delta < \pi/2$; $n = 0, \pm 1, \pm 2, \ldots$.

Henceforth, we assume $x$ to be real and positive. Formulae labelled "$(x \to \infty)$" will hold uniformly with respect to $n$. Relation (5.75) reads $(x \to \infty)$

\[
(7.2) \quad \exp\{\varepsilon_n \} = \exp\left\{ -x(\xi_n - 1) + \frac{2}{x} \xi_n^2 + O\left( \frac{1}{\log^3 x} \right) \right\}
\]
One has by theorem 3.1 \((x \to \infty)\)

\[
(7.3) \quad \xi_n = p + \log p + \frac{\log p}{p} - \frac{1}{2} \left( \frac{\log p}{p} \right)^2 + \frac{\log p}{p^2} + O\left( \frac{\log \log x}{\log x} \right),
\]

where \(p = 2\pi n + \log \infty\).

Inserting this in \((7.2)\) one obtains

\[
(7.4) \quad \exp\{\psi_n\} = \exp\{-x(P_n(x) + O\left( \frac{\log \log x}{\log x} \right))\} \quad (x \to \infty),
\]

with

\[
(7.5) \quad P_n(x) = p + \log p - 1 + \frac{\log p}{p} - \frac{1}{p} - \frac{1}{2} \left( \frac{\log p}{p} \right)^2 +
\]

\[+ 2 \frac{\log p}{p^2} - \frac{2}{p^2}.
\]

Write \(2\pi n/\log \infty = q, u = \alpha x\). From \((7.3)\) we deduce by some trivial calculations that the real part of \(P_n(x)\) equals

\[
(7.6) \quad \log u + \log \log u + \frac{1}{2} \log(1 + q^2) = 1 + \frac{1}{(1 + q^2) \log u}
\]

\[+ \{ \log \log u + \frac{1}{2} \log(1 + q^2) + q \arctan q = 1 \} +
\]

\[+ \frac{(1 - q^2)}{(1 + q^2)^2 \log^2 u} \{ 2 \log \log u - 2 \log(1 + q^2) +
\]

\[+ \frac{2}{3} (\arctan q)^2 - \frac{2}{3} (\log \log u)^2 - \frac{2}{3} \log(1 + q^2) \log \log u +
\]

\[- \frac{1}{6} (\log(1 + q^2))^2 \} + \frac{2q \arctan q}{(1 + q^2)^2 \log^2 u} \{ 2 - \log \log u +
\]

\[- \frac{1}{2} \log(1 + q^2) \}.
\]

In particular if \(n\) is fixed, say \(n = m\), one obtains \((x \to \infty)\)
\[ (7.7) \quad F_{e}(p_{m}(x)) = \log u + \loglog u - 1 + \frac{\loglog u}{\log u} - \frac{1}{\log u} + \]
\[ - \frac{1}{2} \left( \frac{\loglog u}{\log u} \right)^2 + 2 \frac{\loglog u}{\log^2 u} + \frac{2\pi^2 n^2 - 2}{\log^2 u} \]

with an error term which, in absolute value, is less than a constant multiplied by \((\log^{-1} x \cdot \loglog x)^3\). Next, keeping \(m\) fixed, we infer from (7.6) and (7.7)

\[ (7.8) \quad R_{e}\{p_{n}(x) - p_{m}(x)\} = \frac{1}{2} \log(1 + q^2) - \frac{q^2 \loglog u}{(1 + q^2) \log u} \]
\[ + \left(1 + O\left(\frac{1}{\loglog x}\right)\right) - \frac{2\pi^2 n^2}{\log^2 u} + O\left(\frac{\loglog x}{\log x}\right)^3, \]

uniformly with respect to \(n\) when \(x\) tends to infinity.

It is obvious that, for \(x\) large,

\[ (7.9) \quad \frac{1}{2} \log(1 + q^2) - \frac{q^2 \loglog u}{(1 + q^2) \log u} + \]
\[ + O\left(\frac{1}{\loglog x}\right) - \frac{2\pi^2 n^2}{\log^2 u} \]
\[ \geq \frac{1}{2} \log(1 + q^2) - \frac{2\pi^2 n^2}{(1 + q^2) \log u} \]

The right-hand member of this inequality is an increasing function of \(q^2\). It follows at once from (7.8) and (7.9) that, for \(x\) large, the real part of \(p_{n}(x) - p_{m}(x)\) exceeds \(3/\log^2 x\), provided that \(|n| > |m|\) (and consequently \(q^2 \approx (4\pi^2 n^2 + 4\pi^2)/\log^2 x\)).

Finally, by (7.1) and (7.4)

\[ (7.10) \quad |p_{n}(x)| \leq |p_{m}(x)| \exp\left(-\frac{2\pi}{\log x}\right) \quad (x \text{ large, } m \text{ fixed}, \quad |n| > |x|). \]

Evidently, this inequality holds uniformly with respect to \(n\).

If we compare \(p_{n}(x)\) to \(p_{-n}(x)\), we easily find by means of the integral representations of these functions that, for real values of
\( x, x_n(x) \) and \( P_n(x) \) take conjugate values, whereas \( P_0(x) \) is real. For \( P_0(x) \) we notice the following asymptotic formula \((x \to \infty)\)

\[
(7.11) \quad P_0(x) = \exp\{-x(\log \pi + \log x - 1 + \frac{\log \log x}{\log x})\}.
\]

An uniform estimate of the same ratio \( P_n(x)/P_n(x) \), involving both \( n \) and \( x \), can be obtained from \((7.1), (7.4), (7.8) \) and \((7.9)\)

\[
(7.12) \quad |P_n(x)| \leq |P_n(x)| e^{-\frac{N}{2}} \quad (x \text{ large, } n \text{ fixed, } |n| \gg x).
\]

### 7.3 Some theorems

**Theorem 7.1**

Let \( f(x) \) be a solution of \((1.1)\) for \( x = y \geq 0 \). Then we have for any integer \( n > 0 \) \((x = \infty)\)

\[
(7.13) \quad f(x) = \left( f_n, H_n^{\text{reg}}(x) + \sum_{|n| \leq m} \{ f_n, H_n \} f_n(x) + O(f_{n+1}(x)) \right).
\]

The constant implied in the O-symbol depends on \( m \) and on the initial condition imposed on the solution \( f(x) \), i.e., on the values of \( f(x) \) attained for \( y - 1 < x < y \).

**Proof**

We have, by virtue of theorem 6.1, for \( x > y \),

\[
(7.14) \quad f(x) = \left( f_n, H_n^{\text{reg}}(x) + \sum_{|n| \leq m} \{ f_n, H_n \} f_n(x) + \sum_{|n| > m+1} \{ f_n, H_n \} f_n(x) \right).
\]

For the moment we consider positive values of \( n \) only and we write \((x \text{ large})\)

\[
(7.15) \quad \sum_{n=m+1} \{ f_n, H_n \} f_n(x) = \sum_{n=m} \{ f_n, H_n \} f_n(x) + \sum_{n>x} \{ f_n, H_n \} f_n(x).
\]
From (1.7) and (4.21) we infer that \( \{ f, \eta \} = 0 (|n|^{-1/2}) (|n| > 0) \). The first sum on the right of (7.15) contains a finite number of terms only and so, by (7.10), it is in absolute value less than a constant multiplied by \( |P_{m+1}(x)| \). Comparing the second sum with 
(see (7.12)) the integral \( \int_1^\infty |P_{m+1}(x)|x^{-2} \, dx \), we obtain

\[
\sum_{n=m+1}^\infty \{ f, \eta \} P_n(x) = 0 (P_{m+1}(x)) \quad (x \to \infty).
\]

The sum \( \sum_{n=1}^{m} \{ f, \eta \} P_n(x) \) can be dealt with in the same way. Its contribution to (7.14) is of the same order (notice the equality \( |P_{m+1}(x)| = \xi^{(m+1)}(x) \)). This completes the proof.

We now state some theorems which can be verified by means of theorem 7.1 and the asymptotic relations (3.49), (7.10) and (7.11).

**Theorem 7.2**

Any solution of the linear functional equation (1.1) tends to zero as \( x \) approaches infinity.

**Theorem 7.3**

Let \( f(x) \) be a solution of (1.1) for \( x > y > 0 \), then the ratio \( f(x)/\xi^{(y)}(x) \) tends to \( \{ f, \eta \} \) as \( x \) tends to infinity. If \( \{ f, \eta \} \neq 0 \), \( f(x) \) possesses the following asymptotic series expansion

\[
(7.16) \quad f(x) \sim \{ f, \eta \} \frac{x^{-1/2}}{\xi^{(y)}(x)} \sum_{k=0}^{\infty} a_k \Gamma(k + \frac{1}{2})x^{-k} \quad (x \to \infty),
\]

where the coefficients \( a_k \) are given by the development

\[
(7.17) \quad \exp \left( \frac{1}{\alpha} \int_0^{s} \frac{e^t - 1}{t} \, dt \right) = \sum_{k=0}^{\infty} \alpha_k s^k.
\]
Theorem 7.4

If \( f(x) \) is a solution of (1.1) for \( x > y > 0 \) with the property that the inner product \( \{ f, \eta \} \) is equal to zero, then the quotient \( f(x)/\mathcal{F}_0(x) \) tends to \( \{ f, \eta_0 \} \) as \( x \) approaches infinity.

It is a trivial but important remark that a solution which is positive (negative) for all \( x > a \) decreases (increases) steadily for \( x > a + 1 \). Thus it is obvious from the previous theorems that any real-valued solution \( f(x) \) will be a monotonic function from a certain point onwards, provided that at least one of the inner products \( \{ f, \eta \} \), \( \{ f, \eta_0 \} \) does not vanish. If, on the contrary, both \( \{ f, \eta \} \) and \( \{ f, \eta_0 \} \) are equal to zero, then \( f(x) \) shows a type of asymptotical periodicity.

Theorem 7.5

Let \( f(x) \) be a real- or complex-valued function satisfying (1.1) for \( x > y > 0 \), and assume that \( \{ f, \eta \} = 0 \). Let \( k \) be a positive integer, \( k > 1 \), such that the expression (1.7) for the inner product equals zero when \( h(x) = \eta_1(x) \) \((n = k - 1)\), while at least one of \( \{ f, \eta_1 \} \), \( \{ f, \eta_2 \} \) do not vanish. Then we can find complex constants \( c_1 \) and \( c_2 \), \( c_1 \neq 0 \), such that \( (x \to \infty) \)

\[
(x, 18) \quad f(x)/\mathcal{F}_k(x) = c_1 \sin(a_2 + \arg(\mathcal{F}_k(x))) + \mathcal{O}(e^{-\frac{x}{\log x}}).
\]

Moreover, we have \( (x \to \infty) \)

\[
(7.19) \quad \arg(\mathcal{F}_k(x+a)) = \arg(\mathcal{F}_k(x)) = 2k\pi a + \mathcal{O}\left(\frac{\log \log x}{\log x}\right),
\]

uniformly with respect to \( a \), when \( 0 < a < 1 \).

Proof

We start from (7.15) with \( m = k \). With (7.10) it follows that \( (x \to \infty) \)

\[
(7.20) \quad f(x) = \{ f, \eta \} \mathcal{F}_k(x) + \{ f, \eta_0 \} \mathcal{F}_k(x) + \mathcal{O}(\mathcal{F}_k(x) e^{-\frac{x}{\log x}}).
\]
Since both the real part and the imaginary part of $H_k(x)$ are solutions of the adjoint equation, we have 
$$
\{f, R_k(x)\} = \{f, R_k(x)\} + \{f, \Im R_k(x)\}.
$$
If we compare $H_k(x)$ with $H_{-k}(x)$, we easily find by means of the integral representations of these functions that $H_k(x)$ and $H_{-k}(x)$ take conjugate values, provided $x$ is real.

Hence,

$$
(7.21) \quad f(x) = 2\{f, R_k(x)\} \cdot e^{\frac{-x}{2\alpha}} + 2\{f, \Im R_k(x)\} \cdot \Im R_k(x) + O\left(e^{\frac{-x}{2\alpha}}\right),
$$

and so

$$
(7.22) \quad \frac{f(x)}{|R_k(x)|} = 2\{f, R_k(x)\} \cos(\arg(R_k(x))) + \sin(\arg(R_k(x))) + O\left(e^{\frac{-x}{2\alpha}}\right).
$$

Obviously, we can find complex numbers $a_1$ and $a_2$, $a_1 \neq 0$, such that the expression on the right of (7.22) is equal to

$$
\begin{align*}
&c_1 \sin(c_2 + \arg(R_k(x))) + O\left(e^{\frac{-x}{2\alpha}}\right),
\end{align*}
$$

This clearly completes the proof of the first part of the theorem. In order to verify (7.19) we need a formula for $P_k(x+a)/P_k(x)$, when $x$ is large and $0 < a < 1$. Let $\xi_k$ be the saddle point of the function $q(x) = \frac{1}{2} \log(1 + \frac{1}{x})$ inside the half-strip

$$
P_k(x) \approx 1, \quad |\xi_k - 2kn| \leq a
$$

Then by means of the method used in Sec. 3.3, where we investigated the ratio $P_k(x+1)/P_k(x)$, it can be shown that

$$
(7.23) \quad P_k(x+a) = P_k(x)e^{-ak}(1 + O\left(\frac{1}{x}\right)) \quad (x \to \infty)
$$

uniformly with respect to $a$, $0 < a < 1$. 91.
Finally, we obtain the required result by (3.2).

If \( f(x) \) is a real-valued function which satisfies the conditions enumerated in theorem 7.5, then both \( c_1 \) and \( c_2 \) are real and \( f(x) \) has at least one zero in each interval \( (x, x+1) \), provided \( x \) is large.

We now apply these results to the solution \( G_q(y, x) \).

Theorem 5.1 states \( G_q(y, x) = \Phi(x, y) + \sum_{n=-\infty}^{\infty} \frac{P_n(x)}{P_n(y)} \).

(7.24) \( G_q(y, x) = \Phi(x, y) + \sum_{n=-\infty}^{\infty} \frac{P_n(x)}{P_n(y)} \).

It clearly depends on the real zeros of \( \Phi(y) \) whether the Green function does or does not vanish exponentially. For convenience, we deal with the cases \( \alpha > 1, \alpha = 1 \) and \( \alpha < 1 \) separately.

(i) \( \alpha > 1 \).

By (2.11), (2.12) and (2.20) we have

(7.25) \( \Phi(y) = -\frac{1}{2\pi} \int_{F} z^{-1/\alpha} \exp\left\{ (y+1)z - \frac{1}{\alpha} \int_{0}^{s} \frac{s-1}{s} ds \right\} dz, \)

where the contour \( F \) comes from \(-\infty\) in the upper half-plane, encircles the origin once in the negative direction, and returns to \(-\infty\) in the lower half-plane.

Integrating by parts, we obtain

(7.26) \( \Phi(y) = -\frac{1}{\alpha} \int_{P} \exp\left\{ yz - \frac{1}{\alpha} \int_{0}^{s} \frac{s-1}{s} ds \right\} z^{-1/\alpha} dz \quad (y > 0). \)

Since \( \alpha > 1 \) the contour \( P \) may be replaced by a path following the negative real axis from \(-\infty\) to the origin, taking the value of \( z^{-1/\alpha} \) which corresponds to its value in the upper half-plane, and back from 0 to \(-\infty\) with the value taken from the lower half-plane. Hence \( (y > 0) \)

(7.27) \( \Phi(y) = iy \sin \frac{\pi}{\alpha} \int_{0}^{\infty} \exp\left\{ -yz - \frac{1}{\alpha} \int_{0}^{s} \frac{s-1}{s} ds \right\} z^{-1/\alpha} dz \).
The integral does not converge if \( y = 0 \). In order to evaluate \( \tilde{H}(0) \) we use (7.25). Putting \( y = 0 \) in the integrand, we obtain

\[
(7.26) \quad \tilde{H}(0) = - \frac{1}{2\alpha y} \exp\{1 - \frac{1}{\alpha} \int_0^y \frac{x^{1-1/\alpha} dx}{x} \}.
\]

As the principal values of \((-z)^{-1/\alpha}\) and \(z^{-1/\alpha}\) are related by

\[
(-z)^{-1/\alpha} = e^{\pi i/\alpha} z^{-1/\alpha} \quad (\Re(z) > 0),
\]

\[
(-z)^{-1/\alpha} = e^{-\pi i/\alpha} z^{-1/\alpha} \quad (\Re(z) < 0),
\]

we have

\[
(7.29) \quad \tilde{H}(0) = i \sin \frac{\pi}{\alpha} \lim_{z \to \infty} z^{-1/\alpha} \exp\{1 - \frac{1}{\alpha} \int_0^z \frac{x^{s-1} ds}{x} \}.
\]

Then (1.22) and (1.24) yield the result

\[
(7.30) \quad \tilde{H}(0) = i \sin \frac{\pi}{\alpha} \gamma/\alpha,
\]

where \( \gamma \) is Euler's constant.

In the same way it can be proved that

\[
(7.31) \quad H_n(0) = - e^{\pi i/\alpha} + \gamma/\alpha \quad (n < 0),
\]

\[
H_0(0) = - \cos \frac{\pi}{\alpha} \gamma/\alpha, \quad H_n(0) = - e^{-\pi i/\alpha} + \gamma/\alpha \quad (n > 0).
\]

It should be noted that (7.30) and (7.31) still hold if \( \alpha \ll 1 \). We have thus shown that \( \tilde{H}(y) \neq 0 \quad (y = 0, \alpha > 1) \). Finally, by theorem 7.3,

\[
(7.32) \quad a_{\alpha}(y, x) = \tilde{H}(y) \frac{1}{m} x^{-1/\alpha} \alpha \sum_{k=0}^\infty a_k R(k + \frac{1}{\alpha}) x^{-k} \quad (x \to \infty).
\]
This implies that $G_{\alpha}(y,x)$ is a monotonic function from a certain point onwards. It is even possible to show that the Green function decreases for all $x > y + 1$. To this end we need the following relations which can easily be verified ($x > y$)

\[(7.33) \quad G_{\alpha}(y,x) = \frac{x}{x-1} \int_{x-1}^{x} G_{\alpha}(y,t) \, dt + \frac{\alpha-1}{\alpha x} \int_{y}^{x} G_{\alpha}(y,t-1) \, dt\]

and

\[(7.34) \quad G_{\alpha}(y,x+1) = G_{\alpha}(y,x) - \frac{1}{\alpha} \int_{x-1}^{x} \frac{G_{\alpha}(y,t)}{t+1} \, dt .\]

In order to prove that $G_{\alpha}(y,x)$ is a decreasing function it suffices to show that it is positive for all $x > y$.

Obviously, $G_{\alpha}(y,x) = 1$ for $y < x < y + 1$, and integrating (1.1) we find $G_{\alpha}(y,x) = -\frac{1}{\alpha} \log \frac{x}{y-1}$ ($y + 1 < x < y + 2$). Hence $G_{\alpha}(y,x) > 0$ for $y < x < y + 2$. It follows from (7.33) and (7.34) that

\[G_{\alpha}(y,x+1) = \frac{x}{x-1} \int_{x-1}^{x} G_{\alpha}(y,t) \left[ 1 - \frac{1}{x} \right] \, dt\]

\[(y < x < y + 2),\]

and so $G_{\alpha}(y,x) > 0$ for $y < x < y + 3$. Repeating this process we infer, by means of induction, that $G_{\alpha}(y,x)$ is positive whenever $x > y$.

It should be remarked that the proof also holds if $\alpha = 1$.

\[(11) \quad \alpha = 1 .\]

Applying the theorem of residues to the integral (7.25), we have $H(y) = \pi iy$ ($y > 0$). Next, by theorem 7.3, theorem 7.4 and formula (7.31)

\[(7.35) \quad G_{\alpha}(0,x) = F_{\alpha}(x) \left\{ e^{y} + 0 \left( e^{-\frac{1}{2} \log^{2} x} \right) \right\} \quad (x \to \infty),\]
(7.36) \[ q_1(y,x) \sim \sum_{k=0}^{\infty} a_k x^{-k} \quad (y > 0, x \to \infty) , \]

(iii) \[ a < 1. \]

If \( y = 0 \), we deduce from (7.30), (7.31) and the previous theorems that \( (x \to \infty) \)

(7.37) \[ q_0(x) = \frac{x^{y/a}}{\pi} \sin \frac{\pi}{a} \lesssim \sum_{k=0}^{\infty} a_k x^{-k} \quad (1/a \text{ is not an integer}), \]

\[ a_q(x) = f_0(x) \{ -1 \}^{1/a+1} y/a + O\left( e^{-\log^2 x} \right) \]

(1/a is an integer).

It is in no way simple to localise the zeros of \( \tilde{H}(y) \) which lie on the positive real axis. We confine ourselves to the remark that there is an infinite set of pairs \((y, a)\) with the property that \( \tilde{H}(y) = 0 \). So we have, for instance, the pairs \((2, \frac{3}{2}), (3 \pm \frac{1}{\sqrt{5}}, \frac{1}{2})\) as can easily be verified by the theorem of residues.

7.4 The stability of the solutions

Theorem 7.2 shows that any solution tends to zero as \( x \) approaches infinity. It is even possible to prove that every solution which is initially small for \( y = 1 \leq x < y \) \( (y > 0) \) remains small for all larger values of \( x \). It is evident that this somewhat vague assertion must be given a precise mathematical formulation. To this end we first introduce the following definition.

Definition 7.1

The zero solution of the equation (1.1) - i.e., the solution which is zero for all real values of \( x \) - is said to be stable, if, given any two positive numbers \( y \) and \( s \), there exists a number \( \delta > 0 \) such...
that every function \( f(x) \) which is a solution of (1.1) for \( x \geq y \) and which satisfies \( |f(x)| \leq \delta (y-1 \leq x \leq y) \), will also satisfy \( |f(x)| \leq \varepsilon \) for all \( x \geq y \).

Theorem 7.6

The zero solution of the linear functional equation (1.1) is stable.

Proof

Suppose that \( f(x) \) is a solution if (1.1) for \( x \geq y > 0 \) and let \( M \) be the supremum of the set \( \{|f(x)|, y-1 \leq x \leq y\} \). Then it can easily be shown that

\[
(7.39) \quad |f(x)| \leq M \left( \frac{x}{y} \right)^{1/\alpha} \quad (x \geq y).
\]

In order to verify this inequality we consider the function \( F(x) \), which is continuous for \( x \geq y-1 \) and satisfies the equation

\[
\alpha x F'(x) - F(x-1) = 0 \quad \text{for} \quad x \geq y
\]

under the initial condition

\[
F(x) = M (y-1 \leq x \leq y).
\]

Obviously, \( F(x) \) is an increasing function for \( x \geq y \), whence \( \alpha x F'(x) \leq F(x) (x > y) \). Integrating this inequality we obtain

\[
F(x) \leq M \left( \frac{x}{y} \right)^{1/\alpha} \quad (x \geq y).
\]

Since \( |f(x)| \leq F(x) \), the result follows immediately.

Next we prove that

\[
(7.40) \quad |(f, H_n^\alpha)| \leq c y |n|^{1+\gamma} \quad (|n| > 0),
\]

where \( c \) is a constant not depending on \( n \) and \( N \).

This inner product is, in absolute value, less than

\[
N \left( 1 + \frac{1}{\alpha y} \right) \cdot \max_{y-1 \leq t \leq y} |H_n(t+1)|.
\]

From (4.21) we infer that

\[
H_n(t+1) = O(|n|^{1+\gamma}) \quad (|n| \to \infty),
\]

uniformly with respect to \( t \) when \( y-1 \leq t \leq y \).
Then the result follows from the fact that $u(x, t+1)$ is uniformly bounded with respect to both $t$ and $x$ for $y-1 < t < y$, $0 < |n| < N$ ($N$ is a fixed but arbitrarily large positive number).

By the method used previously in order to establish theorem 7.1, it can be shown that

$$|f(x)| \leq M(c_1 |\tilde{F}(x)| + c_2 \tilde{F}(x)) \quad (x \geq x_0).$$

Here $x_0$ is a sufficiently large positive number, and the constants $c_1$ and $c_2$ are independent of the initial values imposed on $f(x)$. The right-hand side of the above inequality tends to zero as $x \to \infty$. This implies that, given a positive number $\varepsilon$, however small, we can find a large number $x_1 > y_0$, not depending on $x$, such that

$$|f(x)| \leq \varepsilon M \quad \text{for all} \quad x \geq x_1.$$ 

Now define $\delta = \min((y-x_1)^{1/2} \cdot \varepsilon, 1)$. Then $|f(x)| \leq \varepsilon$ for $x \geq y$, provided that $|f(x)| \leq \delta$ for $y-1 < x < y$. This completes the proof of theorem 7.6.
CHAPTER VIII

THE CASE $a<0$

§ 3.1 Introduction

Throughout this chapter the linear functional equation will be investigated for negative values of the parameter $a$. The procedures to be used are exactly the same as those employed previously, while the theory turns out to be almost identical to that developed in the previous chapters. For this reason the working out of most details will be omitted.

It is perhaps appropriate to make, already here, a remark about the essential difference between the cases $a>0$ and $a<0$. If $a>0$, any solution tends to zero as $x \to \infty$, whereas this property fails to exist in the case $a<0$. It is trivially disposed of that in the latter case every solution which is initially positive remains positive for all larger values of $x$ and increases steadily to positive infinity as $x \to \infty$.

It should be noted that already in § 2.3 we introduced a set of analytic solutions $f_0(x), f_{2n+1}(x), n = 0, \pm 1, \pm 2, \ldots$. These solutions constitute a fundamental system of the equation (1.1), i.e. they are linearly independent and any other solution, in the sense of definition 1.1, can be expressed in terms of those by means of finite or infinite linear combinations. From this representation the asymptotic properties of arbitrary solutions can be derived.

§ 3.2 The behaviour of the special solutions for $|x| \to \infty$

Let $S(c, \delta)$ be the sector $c \leq |x| < \infty$, $|\arg x| < \pi/2 - \delta$, where $c$ and $\delta$ are small positive numbers ($\delta < \pi/2$).

We shall study the behaviour of $f_0(x), h_0(x), f_{2n+1}(x)$ and $h_{2n+1}(x)$ as $x \to \infty$ in $S(c, \delta)$. The results will be uniform with respect to
both \( n (n = 0, \pm 1, \pm 2, \ldots) \) and \( a \), we start from formula (4.25) which reads

\[
T_{2n+1}(x) = \frac{1}{n^k} \int_{W(n)} \frac{\exp(\varphi(z))}{z} \, dz,
\]

where, with our previous notation,

\[
\varphi(z) = \frac{1}{a} \log z - xz + \frac{1}{a} \int_0^{\frac{\pi}{a}} \frac{e^{-s} - 1}{s} \, ds,
\]

and \( \varphi \) is described in sec. 2.1.

The saddle points of the function \( \varphi(z) \) are the roots of the equation \( e^z = axz \). Omitting the proof, we state the following theorem about the distribution of the saddle points over the half-plane \( \Re(z) > 1 \).

**Theorem 3.1**

Let \( S_k, k = 0, \pm 1, \pm 2, \ldots \), be the half-strip

\[
2kn < \Im(z) < 2(k+1)n, \quad \Re(z) > 1.
\]

There exists a large positive number \( r_0 \), depending on \( a, \varepsilon \) and \( \delta \) only, such that the equation \( e^z - axz = 0 \) has for each \( z \in S(r_0) \) one and only one solution in every horizontal strip \( S_k \) with

\[
|2(k+1)n + \log(-ax)| > r_0.\]

The root in the \( k \)-th strip, denoted by \( \eta_k \), is the sum of an absolutely convergent double power series

\[
u_k = r + \log r + \mu \sum_{\lambda=0}^{\infty} \sum_{m=0}^{\infty} c_{\lambda m} \lambda^m,
\]

where

\[
\begin{align*}
r &= (2k+1)n + \log(-ax), \quad \lambda = 1/r, \quad \mu = \log r/r,
\end{align*}
\]

all the logarithms having their principal values. The \( c_{\lambda m} \) are the same constants as those occurring in theorem 3.1.
Since \( x = (2k + 1) \pi + \log(-e^x) \), we have \( |x| > \pi \), if at least one of \( |k|, |x| \) is large. This means that, if \( x \) lies far to the right inside the sector \( S(\alpha, \beta) \), say, \( |x| > \xi \), each horizontal strip \( \eta_n, |k| > 0 \), contains just one saddle point, and this saddle point is given by the series (8.4). By the method used in sec. 3.2 it can easily be shown that, if \( |x| \) is large enough,

\[
\begin{align*}
0 &< \eta_n, n = (2k + 1) \pi - \arg x < \frac{\pi}{2} & \quad (k > 0), \\
&< \frac{\pi}{2} < \eta_n, n = (2k + 1) \pi - \arg x < 0 & \quad (k < 0).
\end{align*}
\]

We now return to the integral (3.1). By the substitution \( \tau = \eta_n \), the saddle point \( \eta_n \) is shifted to the origin and we obtain, using lemma 2.1 and formula (8.5),

\[
\begin{align*}
F_{2n+1}(x) & = \frac{1}{2\pi i} \int_{\gamma \to x} \exp\left\{\varphi(\eta_n) + t\right\} \frac{dt}{\eta_n + t},
\end{align*}
\]

The integrand is an analytic function in the complex \( t \)-plane cut along the line joining the points \( -\eta_n \) and \( -\eta_n - \infty \). This half-line lies far to the left in the half-plane \( \Re(t) < 0 \). The axis of the saddle point is the straight line through the origin with arguments \( \frac{\pi}{2} - \frac{1}{2} \arg \varphi(\eta_n) \). Since \( \varphi(\eta_n) = x(1 - \eta_n) \), we have \( \frac{\pi}{2} - \frac{1}{2} \arg \varphi(\eta_n) = \pm \frac{\pi}{2} - \frac{1}{2} \arg x \cdot (1 - \eta_n) \).

From this point onwards the process runs exactly as in the case \( \alpha > 0 \) (sec. 3.2). Omitting the details we state the final result

\[
\begin{align*}
F_{2n+1}(x) = & \frac{1}{\sqrt{2\pi}} \eta_n^{-1}(\varphi(\eta_n))^{-\frac{1}{2}} \exp\{\varphi(\eta_n)\} \cdot (1 - \frac{1}{2\pi} + O(\frac{1}{x \log x})),
\end{align*}
\]

uniformly with respect to both \( n \) and \( \arg x \) if \( x \to \infty \) in \( S(\alpha, \beta) \). We notice that \( (\varphi(\eta_n))^{-\frac{1}{2}} \) is to be interpreted as \( 100 \exp\{-\frac{1}{2} \log \varphi(\eta_n)\} \), with the principal value of the logarithm.
For the integral (2.36) we use the same saddle point. Integrating by parts and putting $z = \eta_n + t$, we find

$$(a) \quad h_{2n+1}(x) = -x \int_{-\infty - i \arg x}^{\infty - i \arg x} \exp\{-t^2\} dt.$$  

Again following usual techniques (see sec. 3.4), we obtain

$$(b) \quad h_{2n+1}(x) = -x(2\pi (\eta_n^{(2)}))^{-1/2} \exp\{-\eta_n\} \cdot$$

$$\cdot \left(1 + \frac{1}{12x} + O\left(\frac{1}{x \log x}\right)\right),$$

uniformly with respect to $n$ and $\arg x$ if $x \to \infty$ in $\mathbb{C}_5$.  

In order to obtain a clear survey of the asymptotic behaviour of the special solutions we need a uniform estimate of $\exp\{\eta_n\}$. Instead of (3.75) we now have

$$(c) \quad \exp\{\eta_n\} = \exp\{-x(\eta_n - \frac{1}{\eta_n} - \frac{2}{\eta_n^2} + O\left(\frac{1}{\log x}\right))\}.$$  

With (3.3) it follows that $(|x| \to \infty)$

$$(d) \quad \exp\{\eta_n\} = \exp\{-x(\log x + 1 + O\left(\frac{\log \log x}{\log x}\right))\},$$

where $x = (2n + 1)i + \log(-\omega x)$.

Needless to say, the above relations hold uniformly with respect to both $n$ and $\arg x$, $|\arg x| < x/2 - 5$.  

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Then (8.1), (8.7), (8.9) and (8.11) show that \(|x| \to \infty\)

(8.12) \( \mathcal{I}_{2n+1}(x) = \frac{1}{\pi} \exp \{-x(r\log x + r - 7 + 0(\frac{\log \log x}{\log x}))\} \),

(8.13) \( h_{2n+1}(x) = \exp \{x(r\log x + r - 7 + 0(\frac{\log \log x}{\log x}))\} \).

Obviously, if \(n\) is fixed, \( \mathcal{I}_{2n+1}(x) \) tends to zero as \(x\) approaches infinity inside the sector \(S(\varepsilon, \delta)\). But, in general, the convergence is not uniform with respect to \(n\) except when \(x\) runs to infinity through positive real values only.

Finally, we notice the relation (see also (3.6))

(8.14) \( h_{2n+3}(x+1) = x^n h_{2n+1}(x) \{1 + O(\frac{1}{x})\} \quad (|x| \to \infty) \).

The constant implied in the \(O\)-symbol depends neither on \(n\) nor on \(\arg x, x \in S(\varepsilon, \delta)\).

We now proceed to the discussion of the asymptotic behaviour of the functions \(\mathcal{I}_0(x)\) and \(h_0(x)\). Again the results will be uniform with respect to \(\arg x\) as \(x\) tends to infinity in the sector \(S(\varepsilon, \delta)\). Formula (2.36) reads

(8.15) \( \mathcal{I}_0(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \{-x + \frac{1}{2\pi i} \int_{0}^{2\pi} \exp \{z(1 - 1)\} \} (-z)^{1/\alpha - 1} dz \),

where the contour comes from positive infinity in the lower half-plane, encircles the origin once in the negative direction, and returns to positive infinity in the upper half-plane. The multi-valued function \((-z)^{1/\alpha - 1}\) is made definite by taking \((-z)^{1/\alpha - 1}\) to be real on the negative real axis. Its value on the upper side of the cut is equal to \(-z)^{1/\alpha - 1} e^{\pi i / \alpha}\), and its value on the lower side is given by \(-z)^{1/\alpha - 1} e^{-\pi i / \alpha}\). By Taylor series expansion we have, for all values of \(\pi\),
\( (8.16) \quad \exp \left\{ \frac{1}{\alpha} \int_0^2 \frac{x^{\frac{\alpha - 1}{2}}}{s} ds \right\} = \sum_{k=0}^{\infty} \frac{\frac{2\pi i}{\alpha}}{-1} d_k z^k. \)

Let \( N \) be any integer \( \geq \frac{1}{\alpha} \). Then we write

\( (8.17) \quad f_0(x) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{2\pi i}{\alpha} d_k \int_\Gamma e^{-\xi z} (-\xi)^{k+1/a - 1} d\xi + \)

\[ + \frac{1}{\pi} \sin \frac{\pi}{\alpha} \int_0^\infty e^{-\xi z} \left\{ \sum_{k=0}^{\infty} (-1)^k d_k (-\xi)^{k+1/a - 1} \right\} d\xi. \]

The infinite sum in braces is bounded in the neighbourhood of the origin. Therefore we can, in the usual way, transform the path in the last integral on the right of (8.17) into a path leading from \( +\infty \) to \( 0 \) and back from \( 0 \) to \( +\infty \), taking different branches of the multi-valued integrand both times. It is a well-known result in the theory of Gamma functions that

\[ \int_\Gamma e^{-\xi z} (-\xi)^{k+1/a - 1} d\xi = \frac{2\pi i}{\Gamma(1 - \frac{k}{\alpha})} x^{-1/\alpha - k} \quad (\Re(x) > 0), \]

where \( x^{-1/\alpha} = |x|^{-1/\alpha} e^{-i \arg x/\alpha} \).

Hence

\( (8.18) \quad f_0(x) = x^{-1/\alpha} \sum_{k=0}^{\infty} \frac{2\pi i}{\alpha} d_k \frac{\xi^{k+1/\alpha - 1}}{\Gamma(1 - \frac{k}{\alpha})} x^{-k} + \)

\[ + \frac{1}{\pi} \sin \frac{\pi}{\alpha} \int_0^\infty e^{-\xi z} \left\{ \sum_{k=0}^{\infty} (-1)^k d_k (-\xi)^{k+1/a - 1} \right\} d\xi. \]

If \( \alpha^{-1} \) is an integer, the integral vanishes, and then we infer that \( f_0(x) \) is a polynomial in \( x \) of degree \( -\alpha^{-1} \). This result can, of course, be obtained immediately by applying the theorem of residues to the integral (8.15).

It is easily seen by (8.16) that

\[ \sum_{k=0}^{\infty} d_k z^k = O(z^{-1/\alpha}) \quad (z > 1). \]

This implies

\( (8.19) \quad \sum_{k=N+1}^{\infty} d_k x^{k+1/a - 1} = O(z^{N+1/\alpha}) \quad (z > 1), \)
Obviously, this relation also holds when $0 < x < 1$. So, finally, we obtain

$$f_0(x) = x^{-1/(a-1)} \sum_{k=0}^{N} \frac{\Gamma(k+1)}{\Gamma(1 - \frac{k}{a-1})} x^{-k} + O(x^{-1/(a-1)})$$

uniformly in $\arg x$ as $x \to \infty$ in $S(\varepsilon, \delta)$.

The constant implied in the $O$-symbol of course depends on $N$.

For $h_0(x)$ we have the asymptotic series expansion

$$h_0(x) \sim x^{1/2} \sum_{k=0}^{\infty} (-1)^k c_k \Gamma(\frac{1}{2} + k - \frac{1}{a}) x^{-k}$$

($x \in S(\varepsilon, \delta)$, $x \to \infty$),

where the coefficients $c_k$ are defined by the development

$$\exp\left( -\frac{1}{\alpha} \int_0^x \frac{s-1}{s} ds \right) = \sum_{k=0}^{\infty} c_k x^{-k}.$$

3.3 The biorthogonality

Let $f(x)$ be any of the functions $f_0(x)$, $f_{2n+1}(x)$, $n = 0, 1, \ldots$, and let $h(x)$ be any of the functions $h_0(x)$, $h_{2n+1}(x)$. Then the inner product $\{ f, h \}$ defined by

$$\{ f, h \} = f(x)h(x) - \frac{1}{\alpha} \int_{x-1}^{x} \frac{f(t)h(t+1)}{t+1} dt$$

is a constant throughout the half-plane $R_0(x) > -1$.

It should be remarked that the path of integration in (3.23) lies entirely in the half-plane $R_0(t) > -2$.

These inner products can be evaluated by means of the asymptotic relations derived in the previous section (see also sec. 3.6).

**Theorem 6.2**

The sets $\{ f_0, f_1, f_{2n+1}, \ldots \}$ and $\{ h_0, h_1, h_{2n+1}, \ldots \}$ form a biorthogonal system, i.e.
(8.24) \[ \{ c_{2n+1} h_{2n+1} \} = b_{nm} \]

where \( b_{nm} = 1 \) if \( n = m \), \( b_{nm} = 0 \) if \( n \neq m \).

(8.25) \[ \{ f_0 h_0 \} = 1, \{ f_{2n+1} h_0 \} = 0, \{ f_0 h_{2n+1} \} = 0 \]*

From the theorem it follows at once that the functions \( f_0(x) \),
\( f_{2n+1}(x), n = 0, \pm 1, \ldots \), are linearly independent. Obviously, the
same thing is true for the functions \( h_0(x), h_{2n+1}(x), n = 0, \pm 1, \ldots \).

8.4 The series \( \Sigma f_{2n+1}(x) h_{2n+1}(y) \)

We shall investigate the convergence of the series
\( \Sigma_{-\infty}^{\infty} f_{2n+1}(x) h_{2n+1}(y) \) when both \( x \) and \( y \) are real and greater than

\(-1\).

Theorem 8.3
The series \( \Sigma_{-\infty}^{\infty} f_{2n+1}(x) h_{2n+1}(y) \) converges absolutely and uniform-
ly in both \( x \) and \( y \) in any compact set of the real \( x \)-\( y \)-plane inside
the region \( x > y > -1 \). The convergence is still uniform, but not
absolute, in any compact set in the strip \( y - 1 < x < y (x > -1, \)
\( y > -1) \). If \( x = y \) (\( y > -1) \), the series does not converge in the
ordinary sense, although it converges in the sense that
\( \Sigma_{-\infty}^{\infty} f_{2n+1}(x) h_{2n+1}(x) \) tends to a limit as \( N \) tends to infinity.

The behaviour of the terms for large values of \( |n| \) can be deter-
mined by means of the saddle point method applied to the integrals
(2.35) and (2.36). Proceeding in the way described in sec 4.2 we
easily find that \( |n| \to \infty \)

(8.26) \[ f_{2n+1}(x) = \frac{\alpha}{\Gamma(x+1)} \left[ \exp\left( -\frac{x}{\alpha} + \frac{\pi i}{\alpha} \text{sgn}(n) \right)
+ (1 + x) \log(2n+1) \pi i - (2n+1) \pi i \right] \cdot (1 + O(\frac{\log n}{n})) \]
and

\( h_{2n+1}(x) = -|a|^x \Gamma(x+1) \exp \left\{ \gamma - \frac{n}{a} \text{sgn}[n] \right\} + \)

\[ + x \log(2n+1) + (2n+1) \text{sgn}[x] \cdot (1 + O\left( \frac{\log n}{n} \right)) \]

uniformly with respect to \( x \) in any interval \( a < x < b \) (\( a > -1, \)
\( a < b \)); \( \gamma \) is Euler's constant and \( \text{sgn}[n] \) denotes the sign of the integer \( n \). Hence

\( f_{2n+1}(x)h_{2n+1}(y) = \Psi(x,y) \exp \left\{ -(x+1-y) \log(2n+1) + \right\} \)

\[- (2n+1) \text{sgn}[x-y] \cdot (1 + O\left( \frac{\log n}{n} \right)) \]

uniformly with respect to both \( x \) and \( y \) in any compact set of the
real \( x-y \)-plane contained in the region \( x > -1, y > -1 \). The function \( \Psi(x,y) \) is defined by

\( \Psi(x,y) = -|a|^{x-y} \Gamma(y+1)/\Gamma(x+1) \).

After that the proof of the theorem follows at once by comparing

\( \lim_{x \to \infty} f_{2n+1}(x)h_{2n+1}(y) \) with the series \( \sum_{n=0}^{\infty} (2n+1)^{-n} - t^{-(2n+1)} \text{sgn}[x-y] \)

(with \( t = x-y \)). We notice that the series converges in the ordinary sense except for \( x = y \).

6.5 Series expansions of arbitrary solutions

Series expansions of arbitrary solutions in terms of the functions of the biorthogonal system can be found following the same procedure as used in the case \( \alpha > 0 \). Omitting the proof we state

\[ \text{the following theorem.} \]
Theorem 8.4

Let \( f(x) \) be a solution of the linear differential-difference equation \((1.1)\) for \( x > y > 0 \), in the sense of definition 1.1. Then we have

\[
(8.30) \quad f(x) = \left( f_0 h_0 \right) e_0(x) + \sum_{-\infty}^{\infty} \left( f_0 h_{2n+1} \right) e_{2n+1}(x) \quad (x > y).
\]

The series converges absolutely and uniformly with respect to \( x \) over any compact interval lying entirely to the right of \( y + 1 \). For \( x > y \) the convergence is still uniform in any such compact interval but need no longer be absolute.

Moreover, if there exists another series expansion in terms of the same special functions, representing \( f(x) \) for \( x > y \), then the corresponding coefficients in the two series are equal, i.e., the series are identical.

It should be remarked that the series in \((8.30)\) converges in the ordinary sense.

If \( f(x) = g(x,y) \), formula \((8.30)\) becomes

\[
(8.31) \quad g(x,y) = \left( f_0 h_0 \right) e_0(y) + \sum_{-\infty}^{\infty} f_0 h_{2n+1}(y) e_{2n+1}(x) \quad (x > y)
\]

This equality still holds if \( y - 1 < x < y \) (see also theorem 5.1).

From \((2.39)\) we infer that \( h_0(y) > 0 \) \((y > 0)\).

8.6 The asymptotic behaviour of the solutions

In this section we shall investigate the asymptotic nature of the solutions for large positive values of the independent variable. A good asymptotic approximation can be derived from \((8.30)\) by means of the uniform estimates (see also sect. 7.2)

\[
(8.32) \quad |f_{2n+1}(x)| \ll |f_{2n+1}(x)| \exp \left( - \frac{2n}{\log^2 x} \right)
\]

\((x \text{ large, } n \text{ fixed, } (2n + 1)^2 > (2m + 1)^2)\).
\[(8,33) \quad |f_{2n+1}(x)| \leq \left| f_{2n+1}(x) \right| \cdot |x+1|^{\frac{x}{2}}\]

\[(x \text{ large, } n \text{ fixed}, \ (2n+1)^2 \approx x^2).\]

If we compare \( f_{2n+1}(x) \) with \( f_{-(2n+1)}(x) \), we easily find by means of the integral representations of these functions that, for real values of \( x \), \( f_{2n+1}(x) \) and \( f_{-(2n+1)}(x) \) take conjugate values. For \( f_1(x) \) we notice the following asymptotic formula

\[(8,34) \quad f_1(x) = \exp \left\{ -x \log(-ax) + \log \log(-ax) + a - 1 + x + O(\frac{\log \log x}{\log x}) \right\} \quad (x \to \infty).\]

Instead of theorem 7.1 we now have

**Theorem 8.5**

If \( f(x) \) is a solution of \((1,1)\) for \( x \approx y \approx 0 \), then we have for any integer \( m \geq 0 \)

\[(8,35) \quad f(x) = \{f, h_0\}_0(f_0(x)) + \sum_{(2n+1)^2 \approx (2m+1)^2} \{f, h_{2n+1}\}_0 f_{2n+1}(x) + O(f_{2m+1}(x)) \quad (x \to \infty).\]

The constant implied in the \( O \)-symbol depends on \( m \) and on the initial condition imposed on the solution \( f(x) \), i.e. on the values of \( f(x) \) attained for \( y - 1 \leq x \leq y \).

We now state some theorems which can easily be verified by means of theorem 8.5 and the asymptotic relations \((8,20), (8,32)\) and \((8,34)\).

**Theorem 8.6**

A function \( f(x) \) which is a solution of \((1,1)\) for \( x \approx y \approx 0 \) will approach zero as \( x \) tends to infinity if, and only if, \( \{f, h_0\} = 0 \).
Theorem 8.7

Let \( f(x) \) be a solution of (1.1) for \( x \gg y \gg 0 \), and suppose that the inner product \( \{ f, h_0 \} \) does not vanish. Then \( |f(x)| \) tends to infinity as \( x \to \infty \). Further, \( f(x) \) possesses the following asymptotic series expansion

\[
(8.36) \quad f(x) \sim \{ f, h_0 \} x^{-1/2} \sum_{k=0}^\infty \frac{d_k}{\Gamma(1 - 1/2 - k)} x^{-k} \quad (x \to \infty),
\]

where the coefficients \( d_k \) are given by the development

\[
(8.37) \quad \exp\left(\frac{1}{a} \int_0^y s^{-1} ds\right) = \sum_{k=0}^\infty d_k x^k.
\]

Obviously, any solution \( f(x) \) which is positive when \( x \) ranges over some interval of length one remains positive for all larger values of \( x \) and increases steadily. Then it follows from the previous theorems that \( \{ f, h_0 \} \neq 0 \) and that \( f(x) \to \infty \) as \( x \to \infty \). Clearly, the Green function has this property as \( G_\alpha(x,y) = 1 \) for \( y \leq x \leq y+1 \). The converse of this result also holds. Any real-valued solution \( f(x) \) with \( \{ f, h_0 \} \neq 0 \) will be a monotonic function from a certain point onwards. If, on the contrary, a real-valued solution \( f(x) \), \( x \gg y \gg 0 \), tends to zero as \( x \to \infty \), then it has at least one zero in each interval \( y+k \leq x \leq y+k+1 \) (\( k=0,1,2, \ldots \)). In this case \( f(x) \) shows a type of asymptotical periodicity. Analogous to theorem 7.5 we have

Theorem 8.8

Let \( f(x) \) be a real- or complex-valued function satisfying (1.1) for \( x \gg y \gg 0 \), and assume that \( \{ f, h_0 \} = 0 \). Let \( k \) be a non-negative integer such that the expression (1.7) for the inner product equals zero when \( h(x) = h_{2n+1}(x) \) (\( |2n+1| < 2k+1 \)), while at least one of \( \{ f, h_{2k+1} \} \) and \( \{ f, h_{(2k+1)} \} \) do not vanish. Then there exist complex numbers \( \alpha_1 \) and \( \alpha_2, \gamma_1 \neq 0 \), such that \( (x \to \infty) \).
(9.59) \[ f'(x)/f_{2k+1}(x) = c_1 \sin(c_2 + \arg(f_{2k+1}(x))) + O\left(\frac{\log^2 x}{x}\right). \]

Moreover, \( (x \to \infty) \)

(9.59) \[ \arg(f_{2k+1}(x+a)) = \arg(f_{2k+1}(x)) + (2k+1)a + O\left(\frac{\log \log x}{\log x}\right), \]

uniformly with respect to \( a \) when \( 0 \leq a \leq 1. \)
CHAPTER IX

A PARKING PROBLEM

During the last few years several mathematicians (A. Rényi [14], A. Dvoretsky and H. Robbins [12] and others, have studied the following random process in which cars of length $1$ are parked in a street $[0,x]$ of length $x > 1$. The first car is parked so that the position of its center is a random variable which is uniformly distributed on $[\frac{x}{2}, x-\frac{1}{2}]$. If enough space is available to park a second car, this is parked so that its center is a random variable which is uniformly distributed over the set of all points in $[\frac{x}{2}, x-\frac{1}{2}]$ whose distance from the first car is $> \frac{1}{2}$. We continue in this random way until each remaining segment is of length less than one. Let $N_x$ denote the number of cars so placed in the street, and define $N_x = 0$ if $0 < x < 1$. The expectation $\mu(x) = E(N_x)$ is a continuous function for $x > 1$ and satisfies the integral equation

\begin{equation}
\mu(x + 1) = \frac{2}{x} \int_0^x \mu(t)dt + 1 \quad (x > 0)
\end{equation}

(together with the initial conditions)

\begin{equation}
\mu(x) = 0 \quad (0 < x < 1), \quad \mu(1) = 1.
\end{equation}

It has to be understood that $\mu(x)$ is continuous to the right at $x = 1$.

In 1958 A. Rényi proved that for large values of $x$

\begin{equation}
\mu(x) = \lambda(x + 1) - 1 + o(x^{-x}) \quad (x \to \infty),
\end{equation}
where the constant $\lambda$ is given by

\[ \lambda = \int_0^\infty \exp \left\{ 2 \int_0^v \frac{e^{-s} - 1}{s} \, ds \right\} dv \approx 0.74759. \]

In 1964, Dvoretzky and Robbins showed that this result can be strengthened to

\[ \mu(x) = \lambda(x+1) - 1 + O\left( \left( \frac{2m}{x} \right)^{x-3/2} \right) \quad (x \to \infty). \]

Again (9.5) can be slightly improved if we use the results derived in the previous chapters.

Since $\mu(x) = 0$ ($0 \leq x < 1$), with the continuity to the right at $x=1$, we have $\mu(x) = 1$ ($1 \leq x \leq 2$).

Obviously, $\mu(x)$ is a differentiable function for $x > 2$, whence

\[ (x\mu(x+1))' = 2\mu(x) + 1 \quad (x > 1). \]

Defining $f(x) = (x+1)[\mu(x+2) + 1]$, we see that $f(x)$ is a solution of the linear differential-difference equation

\[ \frac{\Pi}{2} f'(x) + f(x-1) = 0 \quad (x > 0) \]

under the initial conditions

\[ f(x) = 2(x+1) \quad (-1 \leq x \leq 0). \]

This equation is a special case of (1.1) with $\alpha = -\frac{1}{2}$. Then it follows from theorem 8.5 that

\[ f(x) = \{f,h_0\}f_0(x) + 0(f_1(x)) \quad (x \to \infty), \]

with

\[ \{f,h_0\} = f(x)h_0(x) + 2 \int_{x-1}^x \frac{f(t)}{t+1} h_0(t+1) \, dt \quad (x > 0). \]
As the expression for the inner product does not actually depend on $x$ for $x > 0$, we obtain, putting $x = 0$ in (9.10),

$$\begin{align*}
(f, h_0) &= 2h_0(0) + 4 \int_0^1 h_0(t) dt.
\end{align*}$$

(9.11)

The function $h_0(t)$ is an analytic solution of the adjoint equation throughout the half-plane $\Re(t) > -1$ so that

$$h_0(t) = \frac{1}{\pi} \tan^{-1}(t-1) \quad (t > 0).$$

(9.12)

Next, by Taylor series expansion, we infer that $h_0(t-1) = o(t)$ ($t \downarrow 0$). Inserting (9.12) in (9.11) and integrating by parts, we find

$$\begin{align*}
(f, h_0) &= 2 \int_0^1 h_0(t-1) dt.
\end{align*}$$

(9.13)

Notice that (see (2.39))

$$\begin{align*}
h_0(t-1) &= 2 \int_0^\infty v \exp\{-tv + 2 \int_0^v \frac{s^{s-1}}{s} ds\} dv \quad (t > 0).
\end{align*}$$

(9.14)

Hence

$$\begin{align*}
(f, h_0) &= 4 \int_0^1 dt \int_0^\infty v \exp\{-tv + 2 \int_0^v \frac{s^{s-1}}{s} ds\} dv.
\end{align*}$$

(9.15)

Inverting the order of integration one obtains

$$\begin{align*}
(f, h_0) &= 4 \int_0^\infty (1 - e^{-tv}) \exp\{2 \int_0^v \frac{s^{s-1}}{s} ds\} dv.
\end{align*}$$

(9.16)

Finally, partial integration yields

$$\begin{align*}
(f, h_0) &= 2 \int_0^\infty \exp\{2 \int_0^v \frac{s^{s-1}}{s} ds\} dv = 2\lambda.
\end{align*}$$

(9.17)
In sec. 8.2 we already observed that, if \( a = -\frac{1}{3} \), \( f_0(x) \) is a polynomial in \( x \) of degree 2. In fact, applying the theorem of residues to the integral (8.15), we have

\[
(9.18) \quad f_0(x) = \frac{1}{3} (x+1)(x+3),
\]

and so

\[
(9.19) \quad f(x) = \lambda(x+1)(x+3) + O(f'(x)) \quad (x \to \infty).
\]

Then (8.3), (8.4), (8.7), (8.10) and the equality

\[
\mu(x) = (x-1)^{-1} f(x-2) - 1
\]

yield the final result

\[
(9.20) \quad \mu(x) = \lambda(x+1) - 1 + O\left(\frac{2\log x}{x}\right) \quad (x \to \infty).
\]
REFERENCES


SAMENVATTING

In dit proefschrift wordt de lineaire differentiaal-differentie vergelijking

\[ axf'(x) + f(x - 1) = 0 \]

bestudeerd voor positieve waarden van \( x \). De in (1) optredende parameter \( a \) wordt reëel verondersteld. In het bijzonder wordt nagegaan wat het gedrag is van de oplossingen voor \( x \to \infty \).

De door ons gevolgde methode vertoont grote overeenkomst met die gebruikt door N.C. de Bruijn in zijn artikel [8] over de vergelijking \( F'(x) = e^{\alpha x} F(x - 1) \). Bekende begrippen uit de theorie van de lineaire differentiaal vergelijkingen, zoals Greensche functie, geadjungeerde vergelijking, biorthogonaal stelsel, Laplace transformatie en zeldzame methode, kunnen ook hier met succes worden toegespit.

In dit proefschrift zal een eindige verzameling van lineaire onafhankelijke oplossingen van (1) worden geconstrueerd. Deze oplossingen vormen een fundamentaal stelsel van de vergelijking, dwz. iedere andere oplossing kan op eindigdige wijze in deze worden uitgedrukt met behulp van een eindige of een oneindige lineaire combinatie. Uit deze voorstelling kan het asymptotisch gedrag van een willekeurige oplossing worden afgeleid, uitgedrukt in zijn beginwaarden.

Hoofdstuk I bevat enige inleidende onderzoeken. Het beginnende probleem wordt geformuleerd en de begrippen geadjungeerde vergelijking, inwendig product en Greensche functie worden ingevoerd. In hoofdstuk II worden met behulp van de Laplace-transformatie oneindige verzamelingen analytische oplossingen van respectievelijk (1) en de geadjungeerde vergelijking geconstrueerd. Het is nood-
zakelijk hierbij onderscheid te maken tussen de gevallen $\alpha > 0$ en $\alpha < 0$.

De volgende vijf hoofdstukken zijn gewijd aan een uitvoerige bespreking van de vergelijking in het geval dat $\alpha$ een vast positief getal is.

In hoofdstuk III wordt bewezen dat de verzamelingen analytische oplossingen uit hoofdstuk II een biorthogonaal stelsel vormen. In het bewijs wordt gebruik gemaakt van de asymptotische eigenschappen van deze functies voor grote waarden van $|x|$.

In hoofdstuk IV wordt aangetoond dat de gewenste functie kan worden uitgedrukt in de functies van het biorthogonaal stelsel in de vorm van een oneindige reeks, waarvan de convergentie in onderzoek in hoofdstuk IV.

Dit resultaat kan op een eenvoudige wijze worden uitgebreid tot willekeurige oplossingen (hoofdstuk VI).

De bespreking van de asymptotische eigenschappen van de oplossingen is het onderwerp van het volgende hoofdstuk.

Het geval $\alpha < 0$ wordt behandeld in hoofdstuk VIII.

Dit proefschrift besluit met een toepassing van de theorie op een zekere "parkeer probleem".
CURRICULUM VITAE

STELLINGEN

I

Zij $P(z)$ een polynoom van de graad vier met complexe coefficients. Zij $K(c)$ het convex omhulsel van de wortels van de vergelijking $P(z) = c$, waarin $c$ een willekeurig complex getal is. Zij $K$ het convex omhulsel van de nulpunten van het afgeleide polynoom $P'(z)$.

Dan geldt dat de doorsnede van alle $K(c)$ gelijk is aan $K$, dan en slechts dan, als de nulpunten van $P'(z)$ collineair zijn.

II

Zij $y > 0$, $0 < \alpha < 1$. Voor iedere waarde van $y$ en $x$ is een functie $G_{\alpha}(y,x)$ gedefinieerd voor $x \approx y$ door de eigenschappen

1. $G_{\alpha}(y,x)$ is continu voor $x \approx y$.

2. $G_{\alpha}(y,x)$ voldoet voor $x > y + 1$ aan de differentiaal-differentiaal vergelijking $\alpha f'(x) + f(x+1) = 0$ onder de beginvoorwaarde $G_{\alpha}(y,y) = 1$ $(y < x \leq y + 1)$.

Dan geldt

$$\int_{y}^{\infty} G_{\alpha}(y,x) dx = \frac{y - x}{1 - \alpha}.$$
De functie $G(y,x)$ is in het gehele $x=y$ vlak gedefinieerd en bezit de volgende eigenschappen

1° $G(y,x) = 0$ (x < y); $G(y,y) = 1$.

2° $G(y,x)$ is, bij vaste waarde van $y$, continu in $x$ voor $x > y$.

3° $G(y,x)$ voldoet, als functie van $x$, voor $x > y$ ($x \neq y + 1$) aan de vergelijking $f'(x) + p(x)f(x) + q(x)f(x-1) = 0$.

De functies $p(x)$ en $q(x)$ zijn continu voor $-\infty < x < \infty$.

Dan geldt dat $G(y,x)$, als functie van $y$, voor $y < x$ ($y \neq x - 1$) een oplossing is van de vergelijking $f'(y) + p(y)f(y) - q(y+1)f(y+1) = 0$.

IV

Van een divergente reeks $\Sigma_{n=1}^{\infty} d_n$ zijn de termen positief en begrensd. Zij $S_n = \Sigma_{k=1}^{n} d_k$ ($n > 1$). Dan geldt dat de reeks $\Sigma_{n=1}^{\infty} d_n S_n^{1-\alpha}$ convergeren voor iedere positieve waarde van $\alpha$, terwijl bovendien

$$\Sigma_{n=1}^{\infty} \frac{d_n}{S_n^{1+\alpha}} = \frac{1}{\alpha} + \log \alpha \ + \ \Sigma_{n=2}^{\infty} \frac{d_n}{S_n} + \log(1 - \frac{d_n}{S_n}) = o(\alpha) \quad (\alpha > 0).$$

V

Van een reeks worden de partiele sommen op de gebruikelijke wijze aangegeven met $S_n$. Er bestaan divergente reeksen $\Sigma_{n=1}^{\infty} d_n$, waarvoor $\Sigma_{n=1}^{\infty} \frac{d_n}{S_n}$ convergent is.

VI

Het bewijs van A. Proda dat de constante van Euler irrationaal is, is fout.

VII

In het hieronder vermelde werk van Whittaker - Watson is paragraaf 7.82 getiteld "Expansions in series of inverse factorials". De in deze paragraaf gegeven afleiding van een formule is fout.

Whittaker, E.T. and G.N. Watson: A course of modern analysis,
Cambridge University Press, fourth ed. (1962), 142-144.

VIII

De complexe algebra over een Boole algebra beschreven door A. Rudini kan met een andere notatie eenvoudiger worden behandeld.

Angelo Rudini: Teoria degli elementi complessi nelle
Algebra di Boole,

IX

De definitie die A. Rudini geeft van het begrip inclusie is niet gelukkig.

Angelo Rudini: Teoria degli elementi complessi nelle
Algebra di Boole,

X

De wijze waarop N.J. Trappeniers e.a. de viscositeit als functie van dichtheid en temperatuur beschrijven is fysisch onbevredigend.

N.J. Trappeniers, A. Botzen, R.R. van den Berg and J.
von Cothen: The viscosity of Neon between 25°C and 75°C
at pressures up to 1800 atmospheres. Corresponding
states for the viscosity of the noble gases up to high
densities.
Physica 30 (1964), 985 - 996.
Het verdient aanbeveling bij de opleiding tot de propaedeutische examens aan de Technische Hogescholen een fundamentele opbouw van de Wiskunde te geven zonder echter de examensissen te verzwaren.

Het zou aanbeveling verdienen dat zij die Wiskunde studeren aan een van de instellingen van Hoger Onderwijs in ons land, worden onderricht in de geschiedenis van de Wiskunde.

Eindhoven, 22 november 1966

J. J. A. Beenakker