POINT-FREE CARRIER SPACE
TOPOLOGY FOR
COMMUTATIVE BANACH ALGEBRAS

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE
TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE
HOGESCHOOL TE EINDHOVEN OP GEZAG VAN DE
RECTOR MAGNIFICUS DR. K. POSTHUMUS, HOOGLEERAAR
IN DE AFDeling der Scheikundige Technologie,
VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEIGEN
OP DINSdag 7 NOVEMBER 1967 TE 16.00 UUR

DOOR

Willem van der Meiden
GEBoren TE SChIEDAM

TECHNISCHE HOGESCHOOL EINDHOVEN
DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR
PROF. DR. N.G. DE BRUIN
Contents

0. Introduction; preliminaries  

1. General properties of commutative Banach algebras  
   with identity element  
   18

2. Properties of spectra  
   27

3. The *-operation and the lattice of *-invariant  
   sets of closed ideals  
   35

4. A point-free topology: the lattice of clusters  
   43

5. On compactness  
   53

6. Maximal ideals  
   67

7. Examples  
   74

References  
   77

Samenvatting  
   81

Curriculum vitae  
   83
Chapter 0

Introduction; preliminaries

0.1. It seems that the axiom of choice has become a standard tool in certain branches of mathematics, including functional analysis. Nevertheless, in many cases proofs can also be given without the axiom, although it must be admitted that these proofs are usually less elegant and more cumbersome.

Since the time Cohen obtained his results (cf. e.g. [7]) on the axiom of choice, it is no longer a matter of taste to use it or not; results obtained by the axiom are definitely weaker than the corresponding ones obtained without it.

We remind the reader of Wiener's theorem on absolutely convergent Fourier series: if \( f(t) = \sum_{v = -\infty}^{\infty} y_v e^{i\lambda t}, \sum_{v = -\infty}^{\infty} |y_v| < \infty \) and \( \forall t f(t) \neq 0, \) then \( f(t)^{-1} \) has an absolutely convergent Fourier series \( \sum_{v = -\infty}^{\infty} y_v^* e^{i\lambda t}. \)

Originally published in 1932 (cf. [25], p. 14 or [27], p. 91) and provided with another proof in 1938 by Beurling ([3]; see also [22], pp. 422-426), in both cases without the aid of the axiom, it became a standard example of Banach algebra methods since 1941, when Gelfand published his famous papers; ([10], [11]; cf. also [12], [13]).

Wiener's theorem gained a short proof (in [11]), in contrast with the lengthy ones of Wiener and Beurling; this shortness, however,
was effectuated by the uncontrollable machinery of the axiom of choice.

De Bruijn, during a seminar on functional analysis held at Eindhove Technical University over the years 1961-1964, suggested an approach to the theory of commutative Banach algebras avoiding the axiom, leading to a theory entirely parallel to Gelfand's, such that it is possible at every stage to reach the corresponding stage in Gelfand's theory by a simple application of the axiom of choice. Needless to say, he was not the only one to be unsatisfied with the state of affairs; Cohen published as early as 1961 constructive proofs of theorems related to Wiener's (cf. [5]); Bishop reported on constructive analysis in La Jolla and Moscow ([1], [2]); Carathéodory, de Wilde and Schmet proved for separable algebras the existence of maximal ideals, containing some given proper ideal ([9]).

De Bruijn's suggestion was inspired by an old idea outlined already in Menger's "Dimensionstheorie" in 1928, and more extensively published in 1940 (cf. [20] and the literature cited there); this promising "topology without points", which did not find applications until now, appears to provide an appropriate setting for discussion of Gelfand's theory.

The first part of the problem is to construct a lattice of sets of ideals of the Banach algebra, featuring the main properties of the lattice of all closed sets in a topological space. Since maximal ideals correspond to the points of a topological space, if one wishes to avoid discussing maximal ideals (whose existence
depends on the axiom of choice), one has to avoid discussing the
topology as based upon the underlying point set.

Next, de Bruijn intended to describe elements of the algebra as
mappings from the point-free topology into (the lattice of closed
sets of) the complex number field, in such a way that they should
degenerate into continuous functions when restricted (in one way
or another) to the minimal elements of the point-free topology.

Since 1963 we have co-operated on the subject; this thesis is mainly
an account of results concerning the first part of the problem.

Not only the principal idea, which can be found in sections 3 and
4, but also several details are due to de Bruijn; this particularly applies to the complicated proofs in section 5.

The axiom of choice is not employed in sections 1, 2, 3 and 4; in
section 5, we use the restricted (i.e. the countable) version of
the axiom; section 6 gives connections with the space of maximal
ideals, and there of course, several conclusions depend on the
unrestricted axiom.

A very important theorem in the sequel is 2.8; it provides the
Banach algebra with ideals acting, in a way, as maximal ideals;
this theorem is also the key to a new proof of Wiener's theorem
to be published elsewhere ([5]).

0.2. Although the reader is supposed to be more or less familiar
with the concepts of "Banach algebra" and "lattice" we list here
the axioms for these structures to facilitate references; further
information on Banach algebras can be found in [12] and [21], on lattices in [4] and [23].

A non-empty set \( \mathcal{H} \) is called a Banach algebra over the complex number field \( \mathbb{C} \) if mappings

\[ + : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}, \text{ called addition} \]

\[ * : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}, \text{ called multiplication} \]

\[ \cdot : \mathbb{C} \times \mathcal{H} \rightarrow \mathcal{H}, \text{ called multiplication by scalars} \]

and

\[ \| \| : \mathcal{H} \rightarrow \mathbb{R} \] (\( \mathbb{R} \) denoting the field of real numbers),

called norm

are defined in such a way that

i: \( (\mathcal{H}, *, *) \) is a ring (its zero-element denoted by \( 0 \));

iii: \( (\mathcal{H}, C, +, *) \) is a linear space over \( \mathbb{C} \);

iii: \( \forall x \in \mathcal{H} \quad \forall y \in \mathcal{H} \quad \forall \lambda \in \mathbb{C} \quad \lambda \ast (x \ast y) = (\lambda \ast x) \ast y = x \ast (\lambda \ast y) \);

iv: \( \forall x \in \mathcal{H} \quad \|x\| \geq 0; \quad \forall x \in \mathcal{H} \quad [\|x\| = 0 \Rightarrow x = 0]; \)

\[ \forall \lambda \in \mathbb{C} \quad \forall x \in \mathcal{H} \quad \|\lambda \ast x\| = |\lambda| \ast \|x\|; \]

\[ \forall x \in \mathcal{H} \quad \forall y \in \mathcal{H} \quad [\|x + y\| \leq \|x\| + \|y\| \& \|x \ast y\| \leq \|x\| \ast \|y\|]; \]

v: The metric topology induced by the norm \( \| \| \) is complete.

A non-empty set \( L \) is called a lattice if mappings

\[ \wedge : L \times L \rightarrow L, \text{ called meet (or simply cap)} \]

and
\[ V : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \text{ called join (or sup)} \]
are defined in such a way that

\[ L_1 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \forall c \in \mathcal{L} \ (a \land b) \land c = a \land (b \land c); \]

\[ L_2 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \forall c \in \mathcal{L} \ (a \lor b) \lor c = a \lor (b \lor c); \]

\[ L_3 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \ a \land b = b \land a; \]

\[ L_4 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \ a \lor b = b \lor a; \]

\[ L_5 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \ a \land (a \lor b) = a; \]

\[ L_6 : \forall a \in \mathcal{L} \forall b \in \mathcal{L} \ a \lor (a \land b) = a. \]

In a lattice \( \mathcal{L} \) a partial order is defined by

\[ a \leq b : \iff a \land b = a \]

or equivalently

\[ a \leq b : \iff a \lor b = b. \]

In view of this order we may define:

If \( D \subseteq \mathcal{L} \) and if the greatest lower bound \( \inf D \) exists, then

\[ \land a := \inf D; \]

\[ a \in D \]

analogously \( \lor a := \sup D \) if \( \sup D \) exists;

\[ a \in D \]

we occasionally write \( \land D \) and \( \lor D \) respectively.

A lattice \( \mathcal{L} \) is called \( \land \)-complete if \( \land D \) exists for every subset \( D \) of \( \mathcal{L} \); a \( \land \)-complete lattice is called \( \lor \)-distributive if

\[ \forall a \in \mathcal{L} \forall D \subseteq \mathcal{P}(\mathcal{L}) \ a \lor (\land D) = \land (a \lor d), \mathcal{P}(\mathcal{L}) \text{ denoting the set of } \]

subsets of \( \mathcal{L} \).
$V$-completeness and $\Lambda$-distributiveness are defined analogously.

A lattice $L$ is called distributive if

$$\forall a \in L \forall b \in L \forall c \in L \ (a \land (b \lor c)) = (a \land b) \lor (a \land c).$$

or equivalently, if

$$\forall a \in L \forall b \in L \forall c \in L \ (a \lor (b \land c)) = (a \lor b) \land (a \lor c).$$

A lattice is called complete if it is $\Lambda$-complete as well as $V$-complete.

0.3. A lattice $L$ is called a point-free topology if it has the following properties:

PT$_1$: $L$ contains elements $v$ and $w$, $v < w$ and $\forall a \in L \ v < a < w$;

PT$_2$: $L$ is $\Lambda$-complete;

PT$_3$: $L$ is $V$-distributive;

PT$_4$: $\forall a \in L \forall b \in L \ [b \not\equiv a \equiv \exists c \in L \ (v \not\equiv c \equiv b \land a \land c = v)].$

By virtue of well-known theorems of lattice theory, a point-free topology is distributive and $V$-complete as well.

An element $a$ of $L$ is called minimal if $a \not\equiv v$ and if

$$\forall a \in L \ [v \not\equiv a \equiv m = a = m].$$

Denote the set of minimal elements of $L$ by $M$;

define $F_a := \{m \in M | m \leq a\}$ for every $a \in L$.

The point-free topology $L$ is called atomic if $\forall a \in L \setminus \{v\} F_a \not\equiv \emptyset$.

If the point-free topology $L$ is atomic, then obviously
\( F_v = \emptyset, \ F_w = \emptyset, \ a \triangleleft b \Rightarrow F_a \subset F_b, \ F_a \land b = F_a \cap F_b, \)
\( F_a \lor b = F_a \cup F_b, \ a \in M \Rightarrow F_a = \{a\}; \)
moreover, if \( D \in P(L) \setminus \{\emptyset\}, \) then \( F_{\bigwedge D} = \bigcap_{d \in D} F_d. \)

If \( a \neq b \) then, without loss of generality assuming \( a \triangleleft b, \)
\( \exists c \in L \forall \emptyset \neq c \triangleleft b \land a \land c = v, \) by \( PT_a; \) hence \( F_c \neq \emptyset, \)
implying \( \exists m \in M \ [m \triangleleft c \triangleleft b \land a \land m = v], \) whence \( m \in F_b \setminus F_a \) and \( F_a \neq F_b. \)

All these facts prove that \( \{F_a\}_{a \in L} \) is a topology (in terms of closed sets) for \( L. \) It can be considered as a lattice with set-intersection and set-union as operations. As a lattice it is lattice-isomorphic with \( L. \) It satisfies, moreover, the Fréchet separation condition (i.e., sets consisting of a single point are closed).

We call the topology \( \{F_a\}_{a \in L} \) the companion topology of the point-free topology \( L. \)

The point-free topology \( L \) is called regular if it satisfies

\( PT_5: \) If \( a \in L, \ b \in L, \) and \( b \nmid a, \) there exist elements \( c, d, e \in L \)
with the following properties:

\( i: \) \( v \neq c \triangleleft b; \)
\( ii: \) \( c \land d = v; \)
\( iii: \) \( a \land e = v; \)
\( iv: \) \( a \triangleleft d; \)
\( v: \) \( c \triangleleft e; \)
\( vi: \) \( d \lor e = w. \)
PT is clearly implied by PT; if L is atomic and satisfies PT, then M is a regular topological space in the usual sense.

The point-free topology L is called compact if every subset D with "finite L-intersection property" has AD ≠ ν.

Compactness of L implies compactness of the companion topology.

It is also possible to define the Hausdorff property in terms of the point-free topology (cf. [9]). We do not discuss this here, since the point-free topology to be constructed in this dissertation possesses the stronger property of regularity.

Unfortunately, we did not succeed in proving the normality of our point-free topology. The usual argument leading to normality involves compactness, but the kind of compactness we could obtain (section 5) seems too weak for that purpose. On the other hand, the description of normality for a point-free topology is very close to, and even easier than, that of regularity. So at first sight there seems to be no reason to expect the same kind of difficulties one meets when trying to prove compactness, which is logically of a much more intricate nature.

0.4. The text, sections 1 - 7, consists of theorems, definitions (preceded by the symbol ::), and further statements (corollaries, remarks, etc.; preceded by the symbol :). Every statement is preceded in the usual way by a pair of positive integers to facilitate references.

The * for multiplication we drop almost immediately hereafter; both kinds of multiplication in the Banach algebra are only quite
incidentally denoted by $\cdot$, usually by nothing. Indeed, we use standard set-theoretical, algebraic and topological notations without further comment; for a list of symbols we refer to the end of this section.

In order to reduce the frequency of words like "hence", "consequently", we occasionally replace them by $\therefore$.

The symbols $\forall$ and $\exists$ are used frequently, but somewhat unsystematically; logicians will find several pages where they may insert or omit a few of them.

From ALGOL we borrowed the symbol $:= $ for definitions; from Gillman and Jerison ([14], p. 1) we adopt $\varphi^*$ to describe the inverse image produced by a function $\varphi$:

If $\varphi : A \to B$ is a function, then for every $y \in B$

$$\varphi^*(y) := \{ x \in A \mid \varphi(x) = y \} ;$$

denoting by $P(B)$ the set of all subsets of $B$ then for every $S \in P(B)$

$$\varphi^*(S) := \{ x \in A \mid \varphi(x) \in S \} .$$

If proofs of standard results are omitted (particularly in sections 1 and 2) the reader may find them in [12], [21], [24] or [25] or in modern textbooks on functional analysis; in most cases we do not give explicit references.

0.5. In this dissertation, the fields of complex numbers and real numbers are denoted by $\mathbb{C}$ and $\mathbb{R}$ respectively; complex numbers
are denoted by $\alpha, \beta, \gamma, \ldots$; subsets of $\mathbb{N}$ are denoted by $\mathcal{Y}, \mathcal{Q}, \ldots$.

$\mathbb{N}$ denotes the set of positive integers; integers are denoted by $i, j, k, l, m, n$.

$I, K, A, M, N$ denote index sets, their elements are denoted by $i, j, k, \lambda, \mu, \nu$.

$\mathcal{A}$ denotes a Banach algebra, its elements are denoted by $x, y, z, \ldots$.

$\mathcal{N}$ denotes the set of proper closed ideals in $\mathcal{A}$; ideals are denoted by $\mathcal{A}, \mathcal{B}, \mathcal{I}, \ldots$; subsets of $\mathcal{N}$ are denoted by $\mathcal{K}, \mathcal{L}, \mathcal{L}, \ldots$.

$\mathcal{P}(\mathcal{A})$ is the class of subsets of a set.

$\mathcal{F}(\mathcal{A})$ is the class of finite subsets of a set.

The following symbols have a specified meaning; cf. the cited article.

$A, B, C$ elements of $\mathcal{A}$, subsets of $\mathcal{F}$

$a, b, c$ elements of $\mathcal{F}$

$D := \mathbb{R}^N_{\mathbb{N}}$ \footnote{15.10}

$e$ identity of $\mathcal{A}$

$F := \mathcal{P}(\mathcal{A})$

$\omega$ zero-element of $\mathcal{A}$

$Q := \mathcal{P}(\mathcal{P}(\mathcal{A}))$

$R, R_{\neq}$ sets of regular elements in $\mathcal{A}$ or $\mathcal{F}_{\mathbb{N}}(\mathcal{A})$ respectively; \footnote{1.3, 1.19}
\( S, S' \) \text{ sets of singular elements in } \mathcal{K} \text{ or } \mathcal{K}(\mathcal{K}) \text{ respectively; 1.3, 1.19}

\( \mathcal{Z} \) \text{ set of topological divisors of zero in } \mathcal{K} ; \text{ 1.17}

\( \mathcal{P}, \Delta, E \) \text{ subsets of } \mathcal{P}(\mathcal{K}); \text{ 3.2, 3.3, 4.10}

\( \nu(, ), \nu( ) \) \text{ spectral radius; 1.21}

\( \sigma(, ), \sigma( ) \) \text{ spectrum; 1.23}

\( \varphi, \varphi(, ) \) \text{ special subsets of } \mathcal{C}_m; \text{ 4.1}

\( \varphi_n \) \text{ homomorphism corresponding with an ideal } \mathcal{N}; \text{ 1.11}

\( \mathcal{L}(, ), \mathcal{L}_a, \mathcal{L}_h \) \text{ elements of } \mathcal{L} \text{ clusters; 4.5, 4.7, 4.10, 4.14}

\( \mathcal{M} \) \text{ set of maximal ideals in } \mathcal{K} ; \text{ 6.1}

\( \mathcal{M}( ) \) \text{ bonnet; 2.1}

\( \mathcal{N}, \mathcal{N}_m \) \text{ radical of the algebra } \mathcal{K} \text{ or } \mathcal{K}(\mathcal{K}) \text{ respectively; 1.25}

\( \mathcal{R} \) \text{ set of strong ideals in } \mathcal{K} ; \text{ 1.30}

\( \mathcal{R}_a \) \text{ subset of } \mathcal{N} \text{ with the property } \mathcal{R}_a^* = \mathcal{L}_a ; \text{ 4.10}

\( \mathcal{R}_a = \mathcal{K}_a \) \text{ 5.10}

\( \mathcal{I} \) \text{ the ideal } \{a\} \text{ in } \mathcal{K}

\text{Intersections and unions of sets of a class } \mathcal{A} \text{ will be denoted sometimes as } \cap \mathcal{A} \text{ and } \cup \mathcal{A} \text{ .} 17
General properties of commutative Banach algebras with identity element

1.1. A Banach algebra \( \mathcal{A} \) is said to be a \( \mathcal{B}_1 \)-algebra if the ring \((\mathcal{A}, +, \cdot)\) is commutative and contains an identity element \( e \) with the property \( \|e\| = 1 \).

1.2. We start with a \( \mathcal{B}_1 \)-algebra \( \mathcal{A} \); its identity element \( e \) is unique.

1.3. \( R := \{ x \in \mathcal{A} \mid \exists y \in \mathcal{A} \quad xy = e \} \).
    \( S := \mathcal{A} \setminus R. \)

Elements of \( R \) are called regular, those of \( S \) are called singular.

1.4. If \( x \in R \) then the element \( y \) for which \( xy = e \) is unique; it is called the inverse of \( x \), belongs to \( R \) and is denoted by \( x^{-1} \).

1.5. \( \|e - x\| < 1 \Rightarrow \{ x \in R \& x^{-1} = e + \sum_{n=1}^{\infty} (e - x)^n \}. \)

1.6. \( R \) is open and \( S \) is closed.

Proof: \( \forall x \in R, \forall y \in \mathcal{A} \quad \|x^{-1}y - e\| \leq \|x^{-1}\| \|y - x\| \).

Hence, if \( \|y - x\| < \|x^{-1}\|^{-1}, \) then \( x^{-1}y \in R \) by 1.5, implying \( y \in R \).

1.7. A non-empty subset \( \mathcal{A} \) of \( \mathcal{A} \) is called an ideal if \( \mathcal{A} + \mathcal{A} \subset \mathcal{A} \) and \( \mathcal{A} \cdot \mathcal{A} \subset \mathcal{A} \); the ideal \( \mathcal{A} \) is called a proper ideal if \( \mathcal{A} \neq \mathcal{A}. \)
1.8. If $\mathfrak{a}$ is an ideal then $\mathfrak{a}m \subset \mathfrak{a}$.

1.9. If $\mathfrak{a}$ is a proper ideal then the closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is a proper ideal.

1.10. If $\mathfrak{a}$ is an ideal and $\mathfrak{a} = \overline{\mathfrak{a}}$ then $\mathfrak{a}$ is called a closed ideal; moreover, if $\mathfrak{a}$ is a proper ideal, then $\mathfrak{a}$ is called a proper closed ideal. The set of proper closed ideals of $\mathfrak{A}$ is denoted by $\mathfrak{M}$; the null ideal $\{0\}$ belongs to $\mathfrak{M}$ and is denoted by $\mathfrak{P}$.

1.11. For every $\mathfrak{m} \in \mathfrak{M}$ we denote by $q_{\mathfrak{m}}$ the canonical homomorphism $\mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{m}$; if we take

$$h_{\mathfrak{m}, \mathfrak{p}}(x) = \inf \{ \|y\| \mid y \in q_{\mathfrak{m}, \mathfrak{p}}(\mathfrak{x}) \}$$

then it turns out that $\mathfrak{A}/\mathfrak{m}$ together with this norm $\|\|$ is a $\mathfrak{B}_1$-algebra with identity $e_{\mathfrak{m}}(\mathfrak{x})$; in particular $h_{\mathfrak{m}, \mathfrak{m}}(\mathfrak{x}) = \inf \{ \|y\| \mid y \in \mathfrak{m} \} = 1$.

For a proof see [21], p. 44.

We prefer to write $q_{\mathfrak{m}}(\mathfrak{A})$ for $\mathfrak{A}/\mathfrak{m}$.

1.12. If $\mathfrak{m} \in \mathfrak{M}$ and $\mathfrak{m} \in \mathfrak{M}$ then,

i: $q_{\mathfrak{m}}(\mathfrak{m})$ is an ideal in $q_{\mathfrak{m}}(\mathfrak{A})$,

ii: $q_{\mathfrak{m}}q_{\mathfrak{n}}(\mathfrak{m}\mathfrak{n}) = \mathfrak{m}\mathfrak{n} + \mathfrak{m}\mathfrak{n}$.

Proof:

i: Trivially $q_{\mathfrak{m}}(\mathfrak{m}) + q_{\mathfrak{n}}(\mathfrak{m}\mathfrak{n}) \subset q_{\mathfrak{n}}(\mathfrak{A})$, $q_{\mathfrak{m}}(\mathfrak{m}) \subset q_{\mathfrak{m}}(\mathfrak{m})$ hence $q_{\mathfrak{m}}(\mathfrak{m}\mathfrak{n})$ is an ideal in $q_{\mathfrak{m}}(\mathfrak{A})$.

ii: For every $x \in \mathfrak{A}$ the following statements are easily seen to be equivalent:

$x \in q_{\mathfrak{m}}q_{\mathfrak{n}}(\mathfrak{m}\mathfrak{n})$,

$q_{\mathfrak{m}}(x) \in q_{\mathfrak{n}}(\mathfrak{m}\mathfrak{n})$

$3 y \in \mathfrak{m} \ q_{\mathfrak{m}}(x) = q_{\mathfrak{m}}(y)$

19
\[ y \in m \implies x = y \in n \]
\[ x \in m + n. \]

1.13. If \( m \in \mathcal{M} \), \( n \in \mathcal{M} \) and \( m \subseteq n \) then

i. \( m = \varphi_{mm}(n) \).

ii. \( \varphi_{mm}(n) \) is a proper closed ideal in \( \varphi_{mm}(\mathcal{M}) \).

Proof:

i. \( m \subseteq m = m = m + n \); the result follows from 1.12.i.

ii. \( \varphi_{mm}(n) \) is an ideal in \( \varphi_{mm}(\mathcal{M}) \) by 1.12.i.

If \( \varphi_{mm}(n) \subseteq \varphi_{mm}(n) \) then \( m \in \varphi_{mm}(\mathcal{M}) \); hence by 1.12.i \( m \subseteq m \), contrary to the assumption \( m \in \mathcal{M} \); thus \( \varphi_{mm}(m) \) is a proper ideal.

If \( \varphi_{mm}(x) \subseteq \varphi_{mm}(m) \) and \( \epsilon > 0 \) then \( \exists y \in m \colon \| \varphi_{mm}(x) - \varphi_{mm}(y) \| < \epsilon \); \( \epsilon \| \varphi_{mm}(x-y) \| < \epsilon \); by the definition of \( \| \cdot \| \) in \( \varphi_{mm}(\mathcal{M}) \) it follows that \( \exists z \in \varphi_{mm}(x-y) \colon \| m \| < \epsilon \).

Now \( x - y + z \in m \) and \( y \in m \) imply \( x - z \in m + m = m \); writing \( x - z = w \), we obtain

\[ \forall x \in \varphi_{mm}(n) \colon \exists w \in m \colon \| x - w \| < \epsilon \text{ or} \]
\[ \varphi_{mm}(n) \subseteq \varphi_{mm}(m) \text{ and } m \in \mathcal{M} \text{ whence } \varphi_{mm}(m) \subseteq \varphi_{mm}(n) \]

which proves that \( \varphi_{mm}(m) \) is closed.

1.14. If \( m \in \mathcal{M} \), \( n \in \mathcal{M} \) and \( m \subseteq n \) then \( \varphi_{mm}(\mathcal{M}) \) and \( \varphi_{mm}(\mathcal{M}) / \varphi_{mm}(m) \) are isomorphic and isometric.

Proof: 1.11 and 1.13 imply that \( \varphi_{mm}(\mathcal{M}) \), \( \varphi_{mm}(\mathcal{M}) \) and \( \varphi_{mm}(\mathcal{M}) / \varphi_{mm}(m) \) are \( B \) - algebras.

The algebraic part of the statement is a well-known algebraic property.
Let \( \psi \) be the homomorphism \( \varphi_{\mathcal{M}}(\mathcal{K}) \to \varphi_{\mathcal{M}}(\mathcal{K}) / \varphi_{\mathcal{M}}(\mathcal{M}) \), then the isomorphism between \( \varphi_{\mathcal{M}}(\mathcal{K}) \) and \( \varphi_{\mathcal{M}}(\mathcal{K}) / \varphi_{\mathcal{M}}(\mathcal{M}) \) implies that

\[
\forall x \in \mathcal{K} \quad \varphi_{\mathcal{M}}(\psi \circ \varphi_{\mathcal{M}})(x) = (\psi \circ \varphi_{\mathcal{M}})^{-1}(\psi \circ \varphi_{\mathcal{M}})(x);
\]

\[
\therefore \quad \| (\psi \circ \varphi_{\mathcal{M}})(x) \| = \inf \{ \| \psi_{\mathcal{M}}(y) \| \| \psi_{\mathcal{M}}(y) \| \psi_{\mathcal{M}}(x) \| = \psi_{\mathcal{M}}(y) \} \}
\]

with

\[
\| \psi_{\mathcal{M}}(y) \| = \inf \{ \| z \| \mid z \in \varphi_{\mathcal{M}}^{-1}(\varphi_{\mathcal{M}}(y)) \} = \inf \{ \| z \| \mid \varphi_{\mathcal{M}}(z) = \varphi_{\mathcal{M}}(y) \}
\]

hence

\[
\| (\psi \circ \varphi_{\mathcal{M}})(x) \| = \inf \{ \| z \| \mid \varphi_{\mathcal{M}}(z) \in \psi_{\mathcal{M}}(\varphi_{\mathcal{M}}(x)) \} = \inf \{ \| z \| \mid \varphi_{\mathcal{M}}(z) = \varphi_{\mathcal{M}}(x) \} = \inf \{ \| z \| \mid z \in \varphi_{\mathcal{M}}^{-1}(\varphi_{\mathcal{M}}(x)) \} = \inf \{ \| z \| \mid z \in \varphi_{\mathcal{M}}^{-1}(\varphi_{\mathcal{M}}(x)) \} = \| \varphi_{\mathcal{M}}(x) \| .
\]

1.15. \( \varphi_{\mathcal{M}} \), as a linear operator \( \mathcal{M} \to \varphi_{\mathcal{M}}(\mathcal{M}) \), has a norm \( \| \varphi_{\mathcal{M}} \| \)

defined by \( \| \varphi_{\mathcal{M}} \| = \sup \{ \| \varphi_{\mathcal{M}}(x) \| \mid \| x \| = 1 \} \).

We have \( \| \varphi_{\mathcal{M}}(x) \| = \| x \| \) for all \( x \in \mathcal{M} \), and \( \| \varphi_{\mathcal{M}}(x) \| = 1 = \| x \| \).

Hence \( \| \varphi_{\mathcal{M}} \| = 1 \).

1.16. An element \( x \in \mathcal{M} \) is called a topological divisor of zero if \( \mathcal{M} \) contains a sequence \( \{ x_n \} \) with the properties:

i: \( \forall n \in \mathbb{N} \quad \| x_n \| = 1 \), and

ii: \( \lim_{n \to \infty} x_n = x \).

The set of topological divisors of zero is denoted by \( \mathcal{Z} \).

1.17. \( \mathcal{Z} \in \mathcal{S} \).

1.18. i: If \( x \in \mathcal{S} \) then \( \overline{x} \mathcal{M} \in \mathcal{N} \).

ii: If \( x \in \mathcal{S} \setminus \mathcal{Z} \) then \( \overline{x} \mathcal{M} \in \mathcal{N} \).
Proof:
i: If \( x \in S \) then \( x \mathcal{A} \) is a proper ideal; hence by 1.9 \( x \mathcal{A} \in \mathcal{K} \).

ii: If \( x \in S \setminus \emptyset \) and \( y \in x \mathcal{A} \) then there exists a sequence 
\( \{y_n\}_{n \in \mathbb{N}} \) in \( \mathcal{A} \) with \( \lim_{n \to \infty} xy_n = y \).

Suppose that \( \{y_n\}_{n \in \mathbb{N}} \) is not bounded; then it contains a subsequence 
\( \{y_{n_k}\}_{k \in \mathbb{N}} \) with \( \|y_{n_k}\| \not\to \infty \);

since \( \|y_{n_k}\|^{-1} \cdot xy_{n_k} \leq \|y_{n_k}\|^{-1} (xy_{n_k} - y) + \|y_{n_k}\|^{-1} \cdot y \)
this implies \( \lim_{k \to \infty} \|y_{n_k}\|^{-1} \cdot y_{n_k} = e \);

by \( \|y_{n_k}\|^{-1} \cdot y_{n_k} = e \) we infer that \( x \in S \), contrary to the assumptions.

Consequently \( \{y_n\}_{n \in \mathbb{N}} \) is bounded, and so it contains a convergent subsequence \( \{y_{n_k}\}_{k} \).

Denote the limit of \( \{y_{n_k}\}_{k} \) by \( y' \), then
\( \|x(y_{n_k} - y')\| = \|x\| \cdot \|y_{n_k} - y'\| \) implies that
\( y = \lim_{k \to \infty} xy_{n_k} = xy' \), hence \( y \in x \mathcal{A} \);

this proves that \( x \mathcal{A} = x \mathcal{A} \) and \( x \mathcal{A} \in \mathcal{K} \).

Remark. The employment of the Bolzano–Weierstrass property in the foregoing argument does not involve the axiom of choice.

:1.19. If \( m \in \mathcal{K} \) then 
\( R_m := \{q_{m}(x) \cdot q_{m}(\mathcal{A}) \mid \exists y \in \mathcal{A} \ q_{m}(xy) = q_{m}(e) \} \)

and 
\( S_m := q_{m}(\mathcal{A}) \setminus \{e\} \).
In other words, $R_{ss}$ is the set of regular elements of $\varphi_{ss}(\mathcal{K})$ and $S_{ss}$ the set of singular elements of $\varphi_{ss}(\mathcal{K})$.

The set of proper closed ideals in $\varphi_{ss}(\mathcal{K})$ will be denoted by $\mathcal{K}_{ss}$.

1.20. If $m \in \mathcal{K}$, $n \in \mathcal{K}$ and $m \geq n$ then

$$\varphi_{mn}(S_{mm}) \subset \varphi_{mn}(S_{nn}) \quad \text{and} \quad \varphi_{mn}(R_{mm}) \supset \varphi_{mn}(R_{nn}).$$

Proof: The statements

$$x \in \varphi_{mn}(R_{mm})$$

exist equivalent, since $m \in \mathcal{K}$, the latter of them implies.

$$\exists y \in \mathcal{K} \quad \varphi_{mn}(xy - e) = \varphi_{mn}(e)$$

$$\exists y \in \mathcal{K} \quad xy - e \in \mathcal{K}$$

and this, in turn, is equivalent to

$$x \in \varphi_{mn}(R_{mm});$$

this completes the proof.

1.21. The limit $\nu(m,x) := \lim_{k \to \infty} \frac{1}{k} \|\varphi_{mn}(x^k)\|_k$ exists for every $x \in \mathcal{K}$ and every $m \in \mathcal{K}$, and has the following properties:

i: $\nu(m,x) = \inf\{\|\varphi_{mn}(x^k)\|_k \mid k \in \mathbb{N}\}$,

ii: $0 \leq \nu(m,x) \leq \|\varphi_{mn}(x)\|$,

iii: $\nu(m,ax) = |a| \cdot \nu(m,x)$,

iv: $\nu(m,x^k) = \nu(m,x^k) \quad (k \in \mathbb{N})$,

v: $\nu(m,x + y) \leq \nu(m,x) + \nu(m,y)$,

vi: $\nu(m,xy) \leq \nu(m,x) \cdot \nu(m,y)$.
Proof: [21], pp. 10, 11 is applicable to \( \psi_m(\mathcal{G}) \) without any difficulty.

1.22. If \( \nu(\sigma, e \cdot x) < 1 \) then \( \psi_m(x) \in \mathbb{R} \) and

\[
\psi_m(x)^{-1} = \psi_m\left[ e + \sum_{k=1}^{\infty} (e \cdot x)^k \right].
\]

Proof: Apply the proof in [21], p. 12 to \( \psi_m(\mathcal{G}) \).

:1.23. \( \sigma(\sigma, x) := \{ \gamma \in \mathcal{G} : \psi_m(x \cdot y) \in \mathbb{N} \} \).

\( \sigma(\sigma, x) \) is called the spectrum of \( x \) modulo \( \sigma \).

1.24. \( \sigma(\sigma, x) \) is a non-vacuous closed set in \( \mathcal{G} \) for every \( m \in \mathcal{G} \) and every \( x \in \mathcal{G} \); moreover, \( \max\left\{ |\gamma| \mid \gamma \in \sigma(\sigma, x) \right\} = \nu(\sigma, x) \).

Proof: Apply [21], pp. 26-30 to \( \psi_m(\mathcal{G}) \); this proof is independent of the axiom of choice.

:1.25. \( \mathcal{W}_m := \{ \psi_m(x) \mid \nu(\sigma, x) = 0 \} \).

\( \mathcal{W}_m \) is called the radical of \( \psi_m(\mathcal{G}) \).

The elements of \( \mathcal{W}_m \) are called topologically nilpotent in \( \psi_m(\mathcal{G}) \).

1.26. \( \mathcal{W}_m \subseteq \mathcal{W}_m \).

Proof: \( \mathcal{W}_m \) is an ideal in \( \psi_m(\mathcal{G}) \) by 1.21.v and 1.21.vi; since \( \nu(e \cdot e) = 1 \) is \( \psi_m(e) \notin \mathcal{W}_m \); hence \( \mathcal{W}_m \) is a proper ideal.

1.21.v and 1.21.vii imply \( |\nu(\sigma, x) - \nu(\sigma, y)| \leq \nu(\sigma, x - y) \leq \|\psi_m(x - y)\| = \|\psi_m(x) - \psi_m(y)\| \);

\( \therefore \) \( \nu(\sigma, x) \) is a continuous function; \( \mathcal{W}_m \) as the zero-set of a continuous function, is a closed set in \( \psi_m(\mathcal{G}) \).

\textit{An analogous result can be found in [19], p. 52.}
1.27. $\mathcal{H}_n = \{ q_{xy}(x) | \forall y \in \mathbb{R}, q_{xy}(xy - e) \in \mathbb{R} \}.$

Proof: $q_{xy}(x) \in \mathcal{H}_n$ is by definition equivalent to $v_{\gamma,x} = 0,$ implying $\forall y \in \mathbb{R}, v_{\gamma,xy} = 0$ by 1.21, vi; hence
$
\forall y \in \mathbb{R}, q_{xy}(xy - e) \in \mathbb{R}.$ by 1.22; and conversely the latter formula implies $\forall \lambda \in \mathbb{C}, q_{xy}(\lambda x - e) \in \mathbb{R}$ or, equivalently,
$
\forall \lambda \neq 0, q_{xy}(\lambda x - e) \in \mathbb{R}.$ whence, by 1.24, $\sigma_{\gamma,x} = \{ 0 \}$ and therefore $v_{\gamma,x} = 0,$ $\mathcal{H}_n(x) \in \mathcal{H}_n.$

1.28. Since $\mathbb{R}$ can be identified with $\mathbb{R}(\mathbb{R})$, the expressions $v(\gamma,x), \sigma(\gamma,x)$ and $\mathcal{H}_n$ will be denoted by $v(x), \sigma(x)$ and $\mathcal{H}$ respectively.

1.29. $q_{\gamma,n} \in \mathcal{H}_n$ for every $\gamma \in \mathcal{H}.$

Proof: If $x \in \mathcal{H}$ then $\forall y \in \mathbb{R}, xy - e \in \mathbb{R};$ by 1.20 this implies $\forall \gamma \in \mathcal{H} \forall y \in \mathbb{R}, q_{\gamma}(xy - e) \in \mathbb{R}$ and by 1.27 $\forall \gamma \in \mathcal{H}, q_{\gamma,x}(x) \in \mathcal{H}_n.$

1.30. If $\gamma \in \mathcal{H}$ and $\mathcal{H}_n = \{ q_{\gamma,x}(x) \}$ then $\gamma$ is called a strong ideal in $\mathcal{H}$; the set of strong ideals in $\mathcal{H}$ is denoted by $\mathcal{J};$ analogously $\mathcal{J}_n$ is defined as the set of strong ideals in $\mathcal{H}_n(\mathbb{R})$ for every $\gamma \in \mathcal{H}. \mathcal{K}$. An algebra with $\mathcal{H} \in \mathcal{J}$ is usually called semi-simple.

1.31. If $\gamma \in \mathcal{H}$ then $\gamma \in \mathcal{H}.$

Proof: $\gamma = q_{\gamma,x}(\gamma) = q_{\gamma,x}(\mathcal{H}_n) = q_{\gamma,x}(\mathcal{H})$ by 1.30 and 1.29, and $q_{\gamma,x}(\mathcal{H}_n) = \gamma + \mathcal{H}$ by 1.12; hence $\gamma \in \mathcal{H}.$
1.32. \( M \in \mathcal{C} \), and analogously \( \mathcal{N}_M \in \mathcal{Y}_M \).

Proof: \( M \in \mathcal{C} \) if and only if \( \mathcal{N}_M = \{ \psi_M(\sigma) \} \) or \( \mathcal{N}_M = \psi_M(M) \).

Now, \( \psi_M(M) \in \mathcal{N}_M \) by 1.26 and 1.29.
Conversely, if \( \psi_M(x) \in \mathcal{N}_M \) then by 1.27
\[
\forall y \in \mathcal{C} \quad \psi_M(xy - e) \in \mathcal{N}_M \quad \text{or}
\]
\[
\forall y \in \mathcal{C} \quad \exists z \in \mathcal{C} \quad \psi_M[(xy - e)z - e] = \psi_M(\sigma).
\]

If we take \( u := (xy - e)z - e \) then \( u \in \mathcal{N} \); hence by 1.27
\[
u + u \in \mathcal{N}
\]

from which we conclude
\[
\forall y \in \mathcal{C} \quad \exists z \in \mathcal{C} \quad (xy - e)z \in \mathcal{N}
\]

implying \( \forall y \in \mathcal{C} \quad xy - e \in \mathcal{N} \)

whence \( x \in \mathcal{N} \) by 1.27 and
\[
\psi_M(x) \in \psi_M(M).
\]

1.33. \( M \in \mathcal{C} \iff \psi^-_M(M) = M \).

Proof: \( M \in \mathcal{C} \) means by definition \( \mathcal{N}_M = \{ \psi_M(\sigma) \} \) and \( \psi^-_M(M) = M \)
is equivalent to \( \mathcal{N}_M = \psi_M(M) = \{ \psi_M(\sigma) \} \); hence the result.
Properties of spectra

2.1. If \( m \in \mathcal{M} \) then \( \mathcal{M}(m) = \{ m' \in \mathcal{M} \mid m' \supset m \} \).

\( \mathcal{M}(m) \) is called the homet of \( m \).

2.2. If \( m \in \mathcal{M} \) and \( m' \in \mathcal{M}(m) \) then \( \forall x \in \mathfrak{X}, \sigma(m', x) \subseteq \sigma(m, x) \).

Proof: If \( \lambda \in \sigma(m', x) \) then \( \varphi_{m', m}(x - \lambda e) \in S_{m'} \) and by 1.20
\( x - \lambda e \in \varphi_{m, m}(S_{m'}) \subseteq \varphi_{m, m}(S_{m'}) \) hence \( \varphi_{m, m}(x - \lambda e) \in S_{m'} \) and \( \lambda \in \sigma(m, x) \).

An analogous result can be found in [16], p. 698.

2.3. If \( m \in \mathcal{M} \) and \( m' \in \mathcal{M} \) then \( \forall x \in \mathfrak{X}, \sigma(m \cap m', x) = \sigma(m', x) \cup \sigma(m, x) \).

Proof: \( m \cap m' \in \mathcal{M} \); hence
\( \forall x \in \mathfrak{X}, \sigma(m \cap m', x) \supset \sigma(m, x) \cup \sigma(m', x) \) by 2.2.

Suppose \( \lambda \not\in \sigma(m, x) \cup \sigma(m', x) \), then
\( \varphi_{m', m}(x - \lambda e) \in R_{m'} \) and \( \varphi_{m, m}(x - \lambda e) \in R_{m} \), or
\( \exists y \in \mathfrak{Y}, \varphi_{m', m}[(x - \lambda e)y - e] = \varphi_{m, m}(\phi) \) and \( \exists z \in \mathfrak{Z}, \varphi_{m, m}[(x - \lambda e)z - e] = \varphi_{m, m}(\phi) \), whence
\( (x - \lambda e)y - e \in m' \) and \( (x - \lambda e)z - e \in m \); from \( (x - \lambda e)[(y + z) - (x - \lambda e)yz] - e = - [(x - \lambda e)y - e] \cdot [(x - \lambda e)z - e] \in m' \cap m \)
we infer that
\[
\varphi_{\mathcal{M}}(\{(x-\lambda e)(y+z) - (x-\lambda e)yz\}) = \varphi_{\mathcal{M}}(e)
\]
which by 1.19 means \( \varphi_{\mathcal{M}}(x-\lambda e) \in \mathcal{R}_{\mathcal{M}} \)
and consequently \( \lambda \not\in \mathcal{O}(\mathcal{M}, x) \).
This proves the desired result.

2.4. If \( \mathcal{M} \in \mathcal{H} \) then \( \mathcal{M} \in \mathcal{H}(\mathcal{M}) \) if and only if \( \forall x \in \mathcal{M} \sigma(x) = \sigma(\mathcal{M}, x) \).

Proof:

i: Sufficiency. \( \forall x \in \mathcal{M} \sigma(x) = \sigma(\mathcal{M}, x) \) implies

\[
\forall x \forall \lambda \left[ x - \lambda e \in S \Rightarrow \varphi_{\mathcal{M}}(x-\lambda e) \in \mathcal{S}_{\mathcal{M}} \right],
\]

\[
\forall x \in \mathcal{M} \forall \lambda \left[ x - \lambda e \in S \Rightarrow \varphi_{\mathcal{M}}(x-\lambda e) \in \mathcal{S}_{\mathcal{M}} \right],
\]

\[
\forall x \in \mathcal{M} \forall \lambda \left[ x - \lambda e \in S \Rightarrow \lambda \cdot \varphi_{\mathcal{M}}(e) \in \mathcal{S}_{\mathcal{M}} \right],
\]

\[
\forall x \in \mathcal{M} \forall \lambda \left[ x - \lambda e \in S \Rightarrow \lambda = 0 \right],
\]

\[
\forall x \in \mathcal{M} \sigma(x) = \{0\} \text{ and }
\]

\[
\forall x \in \mathcal{M} \nu(x) = 0 ;
\]

hence \( \mathcal{M} \in \mathcal{H} \).

ii: Necessity. Assume \( \mathcal{M} \in \mathcal{H} \). Since \( \varphi_{\mathcal{M}}(x) \in \mathcal{R}_{\mathcal{M}} \) implies

\[
\exists x \in \mathcal{M} \varphi_{\mathcal{M}}(xy - e) = \varphi_{\mathcal{M}}(x) \text{ and hence } xy - e \in \mathcal{M} \in \mathcal{H}, \text{ we have }
\]

\[
\sigma(xy - e) = \{0\}, xy \in \mathcal{R} \text{ and consequently } x \in \mathcal{R}.
\]

Therefore \( \forall x \left[ \varphi_{\mathcal{M}}(x) \in \mathcal{R}_{\mathcal{M}} \Rightarrow x \in \mathcal{R} \right],
\]

\[
\forall x \left[ x \in S \Rightarrow \varphi_{\mathcal{M}}(x) \in \mathcal{S}_{\mathcal{M}} \right],
\]

\[
\forall x \forall \lambda \left[ x - \lambda e \in S \Rightarrow \varphi_{\mathcal{M}}(x - \lambda e) \in \mathcal{S}_{\mathcal{M}} \right] \text{ and }
\]

\[
\forall x \sigma(x) = \sigma(\mathcal{M}, x);
\]

hence by 2.2 \( \forall x \sigma(x) = \sigma(\mathcal{M}, x) \).
2.5. As a corollary of 2.4 we infer that, if \( m \in \mathcal{H} \) and 
\( m \in \mathcal{H}(\mathcal{H}) \), then \( m \in \mathcal{H}(\mathcal{H}) \) if and only if \( \forall \chi \sigma(m, x) = \sigma(m, x) \).

Proof: If \( \chi \) is the isomorphism \( \varphi_{\mu}(\mathcal{H}) = \varphi_{\mu}(\mathcal{H}) / \varphi_{\mu}(\mathcal{H}) \) described in 1.14 and if \( \phi \), as in 1.14, is the homomorphism
\( \varphi_{\mu}(\mathcal{H}) = \varphi_{\mu}(\mathcal{H}) / \varphi_{\mu}(\mathcal{H}), \) then obviously \( \chi \cdot \varphi_{\mu} = \phi \cdot \varphi_{\mu} \).

Moreover
\[ \sigma(\varphi_{\mu}(\mathcal{H}), \varphi_{\mu}(\mathcal{H})) = \sigma(\chi \cdot \varphi_{\mu}(\mathcal{H}), \varphi_{\mu}(\mathcal{H})) \]

by the definition of the spectrum modulo an element of \( \mathcal{H} \).

It is now obvious that the conditions
\[ \forall \chi \sigma(m, x) = \sigma(m, x) \]

and
\[ \forall \chi \sigma(\phi \cdot \varphi_{\mu}(\mathcal{H}), \phi \cdot \varphi_{\mu}(\mathcal{H})) = \sigma(\varphi_{\mu}(\mathcal{H}), \varphi_{\mu}(\mathcal{H})) \]

are equivalent, the latter can also be read
\[ \forall \chi \sigma(\varphi_{\mu}(\mathcal{H}), \varphi_{\mu}(\mathcal{H})) = \sigma(\varphi_{\mu}(\mathcal{H}), \varphi_{\mu}(\mathcal{H})) \]

which happens to be the condition of 2.4 applied to the algebra \( \varphi_{\mu}(\mathcal{H}) \) instead of \( \mathcal{H} \) and the ideal \( \varphi_{\mu}(\mathcal{H}) \) instead of \( \mathcal{H} \); hence

the condition is by 2.4 equivalent to
\[ \mathcal{H}_{\varphi_{\mu}} = \mathcal{H}_{\varphi_{\mu}} \]

or \( \mathcal{H}_{\varphi_{\mu}}(\mathcal{H}) = \mathcal{H} \).

2.6. If \( n \in \mathcal{H} \) and \( \mu \in \mathcal{H} \) then \( \forall \chi \sigma(m, x) = \sigma(m, x) \) is equivalent to \( \overline{m + \mu} \subset \varphi_{\mu}(\mathcal{H})(\mathcal{H}_{\varphi_{\mu}}(\mathcal{H})) \).

Proof: If \( \sigma(m, x) = \sigma(m, x) \), then they are both equal to \( \sigma(m + \mu, x) \) by 2.3; hence
\[ \forall x \, o(m, x) = o(m', x) \]

implies

\[ m \in \varphi_{m_0 \cap m_0} \left( \mathcal{H}^{m_0 \cap m_0} \right) \]

and

\[ m \in \varphi_{m_0 \cap m_0} \left( \mathcal{H}^{m_0 \cap m_0} \right) \]

by 2.5.

Since \( \varphi_{m_0 \cap m_0} \in \mathcal{H}^{m_0 \cap m_0} \) and \( \varphi_{m_0 \cap m_0} \) is continuous,

\[ m \in \varphi_{m_0 \cap m_0} \left( \mathcal{H}^{m_0 \cap m_0} \right) \]

Conversely \( m \in \varphi_{m_0 \cap m_0} \left( \mathcal{H}^{m_0 \cap m_0} \right) \) implies by 2.5

\[ \forall x \, o(m, x) = o(m \cap m', x) \]

and analogously \( \forall x \, o(m', x) = o(m \cap m', x) \).

2.7. If \( m \in \mathcal{H} \) then \( \mathcal{H} \in \mathcal{H}(m) \) implies \( \mathcal{H} \in \varphi \left( \mathcal{H}^{m_0 \cap m_0} \right) \).

Proof: If \( \mathcal{H} \in \mathcal{H}(m) \) then by 2.4

\[ \forall x \, o(m, x) = o(m', x) \]

This implies

\[ \forall x \, o(m, x) = o(m', x) \]

and

\[ \mathcal{H} \in \varphi \left( \mathcal{H}^{m_0 \cap m_0} \right) \]

2.8. If \( x \in \mathcal{H}, m \in \mathcal{H} \) and \( \lambda \in o(m, x) \), and if we define

\[ m \in \varphi \left( (x - \lambda) \mathcal{H} \right) \]

then \( m \) has the following properties:

i: \( m \in \mathcal{H}(m) \),

ii: \( x - \lambda \epsilon \in m \),

iii: \( o(m, x) = \{ \lambda \} \).

Proof:

i: By the definition of \( o(m, x) \) we have \( \varphi_{m_0} \left( (x - \lambda) \mathcal{H} \right) \in \mathcal{H}_{m_0} \); therefore, \( \varphi_{m_0} \left( (x - \lambda) \mathcal{H} \right) \) is a proper ideal in \( \varphi_{m_0} \left( \mathcal{H} \right) \) and consequently
\( \mathfrak{s}_{\mathfrak{mp}} [(x - \lambda e)\mathfrak{K}] \) is a proper ideal in \( \mathfrak{K} \), whence \( m \in \mathfrak{M} \) by 1.9; \( \textit{obviously} \ m \circ m \); hence \( m \in \mathfrak{M}(m) \).

\[ \text{i}: \quad x - \lambda e = (x - \lambda e)e \in \mathfrak{s}_{\mathfrak{mp}} \mathfrak{m} [(x - \lambda e)\mathfrak{K}] \subset m. \]

\[ \text{iii}: \quad x - \lambda e \in m \text{ implies } \mathfrak{s}_{\mathfrak{mp}} (x - \lambda e) = \mathfrak{s}_{\mathfrak{mp}} (x - e) \in S_{mp}; \text{ hence } \lambda \in \sigma (m, x). \]

If \( \mu \in \sigma (m, x) \), then \( \mathfrak{s}_{\mathfrak{pp}} (x - \mu e) \in S_{mp} \), since

\[ (\lambda - \mu) \mathfrak{s}_{\mathfrak{mp}} (x) = \mathfrak{s}_{\mathfrak{mp}} (x - \mu e) - \mathfrak{s}_{\mathfrak{mp}} (x - \lambda e) = \mathfrak{s}_{\mathfrak{mp}} (x - \mu e) = \mathfrak{s}_{\mathfrak{mp}} (x - \mu e) \]

we have \( (\lambda - \mu) \mathfrak{s}_{\mathfrak{mp}} (x) \in S_{mp} \), whence \( \lambda = \mu \).

2.9. Obvious consequences of 2.8 are

\[ \text{i}: \quad \text{if } x \in \mathfrak{K}, \{ \lambda_{i} \}_{i=1}^{n} \subset \sigma (x) \text{ and } m = \bigcap_{i=1}^{n} (x - \lambda_{i} e)\mathfrak{K} \text{ then } \sigma (m, x) = \{ \lambda_{i} \}_{i=1}^{n} . \]

\[ \text{ii}: \quad \text{if } x \in \mathfrak{K} \text{ and } \mathfrak{q}(x) = \bigcap_{\lambda \in \mathfrak{C} m} (x - \lambda e)\mathfrak{K} \text{ then } \sigma (x) = \sigma (\mathfrak{q}(x), x) . \]

\[ \text{iii}: \quad \cap_{x \in \mathfrak{K}} \mathfrak{q}(x) = \emptyset . \]

\[ \text{iv}: \quad \text{if } y \in \mathfrak{C} m \text{ then } \mathfrak{q}(y) = \overline{y} \mathfrak{K} . \]

Proof:

\[ \text{i}: \quad \text{If } m_{i} = (x - \lambda_{i} e)\mathfrak{K} (i = 1, \ldots, n) \text{ then by 2.8, taking } m = \mathfrak{C} m, \]

\[ \sigma (m_{i}, x) = \{ \lambda_{i} \}_{i=1}^{n} \text{ (i = 1, \ldots, n) and by an corollary of 2.3} \]

\[ \sigma (m, x) = \sigma (\bigcap_{i=1}^{n} m_{i}, x) = \sigma (m_{1}, x) = \{ \lambda_{i} \}_{i=1}^{n} , \ldots, n^* . \]

\[ \text{ii}: \quad \sigma (x) \text{ is clearly a proper closed ideal in } \mathfrak{K}, \text{ hence by } 2.2 \]

\[ \sigma (\mathfrak{q}(x), x) \subset \sigma (x) . \]

31
If \( \lambda \in \sigma(x) \) then \( \mathcal{M} = \frac{(x-\lambda e)\mathcal{X}}{\lambda} \) has the properties of 2.8 (with \( \mathcal{M} = \mathcal{X} \)), moreover \( \mathcal{M} \supset \sigma_{\mathcal{X}}(x) \); hence, again by 2.2 \( \lambda \in \sigma(\sigma(x),x) \); this entails that \( \sigma(x) \subseteq \sigma(\sigma(x),x) \).

iii: If \( y \in \bigcap_{x \in \mathcal{X}} \sigma(x) \), then

\[ y \in \overline{\mathcal{X}} \quad \forall x \in \mathcal{X} \] \[ \lambda \in \mathcal{O} \mathcal{X} \] \[ y \in (x-\lambda e)\overline{\mathcal{X}} \],

whence \( y \in \overline{\sigma \mathcal{X}} = \mathcal{Y} \).

iv: If \( y \in \mathcal{Y} \) then \( \sigma(y) = \{0\} \); hence, if \( \lambda \neq 0 \), then \( y - \lambda e \in \mathcal{Y} \) and \( (y-\lambda e)\mathcal{X} = \mathcal{Y} \).

Now \( \sigma(y) = \frac{(y-0e)\mathcal{X}}{\lambda} = \frac{y}{\lambda} \).

**2.10.** If \( x \in \mathcal{X}, y \in \mathcal{Y} \) and \( \mathcal{M} \in \mathcal{N} \) then

i: \( \sigma(\mathcal{M},x+y) \subseteq \sigma(\mathcal{M},x) + \sigma(\mathcal{M},y) \);

ii: \( \sigma(\mathcal{M},xy) \subseteq \sigma(\mathcal{M},x) \cdot \sigma(\mathcal{M},y) \).

**Proof:**

i: If \( \nu \in \sigma(\mathcal{M},x+y) \) then \( \mathcal{M} : = \frac{\mathcal{M}}{\mathcal{M}} \subseteq \frac{(x+y-\nu e)\mathcal{X}}{\nu} \) has properties which, according to 2.3 and 2.2, guarantee

\( \sigma_{\mathcal{M}}(x+y-\nu e) = \sigma_{\mathcal{M}}(\sigma) \), \( \sigma(\mathcal{M},x+y) = \{\nu\} \) and \( \sigma(\mathcal{M},x) \subseteq \sigma(\mathcal{M},x) \); since \( \sigma(\mathcal{M},x) \) is non-vacuous by 1.24, we can take \( \lambda \in \sigma(\mathcal{M},x) \).

\[ \mathcal{M} = \mathcal{M} \frac{(y-\lambda e)\mathcal{X}}{\lambda} = \frac{(y-\lambda e)\mathcal{X}}{\lambda} = \frac{(y-\lambda e)\mathcal{X}}{\lambda} \in \mathcal{M}, \]

hence \( \nu = \lambda \in \sigma(\mathcal{M},y) \) and

\[ \nu = \lambda + \frac{\sigma(\mathcal{M},x) + \sigma(\mathcal{M},y) \in \sigma(\mathcal{M},x) + \sigma(\mathcal{M},y)}{\sigma(\mathcal{M},x) + \sigma(\mathcal{M},y)} \]

which shows that \( \sigma(\mathcal{M},x+y) \subseteq \sigma(\mathcal{M},x) + \sigma(\mathcal{M},y) \).
11. Analogously, assume \( v \in \sigma(m, x) \); then take

\[ m = \overline{\varphi_{\mathcal{H}}[\lambda(y - \varphi_{\mathcal{H}})]} \text{ and } \lambda \in \sigma(m, x). \]

Since \( xy - ve = (x - \lambda x)y + (\lambda y - ve) \) and \( \varphi_{\mathcal{H}}[(x - \lambda x)y] \in S_{\mathcal{H}} \), we obtain \( \varphi_{\mathcal{H}}[(\lambda y - ve)] \in S_{\mathcal{H}} \).

If \( \lambda = 0 \) this implies \( v = 0 \) and \( v = \lambda \mu \) for every \( \mu \in \sigma \); if \( \lambda \neq 0 \) then \( \forall \lambda^{-1} \in \sigma(m, y) \). In both cases

\[ v \in \sigma(m, x), \sigma(m, y) \subset \sigma(m, x), \sigma(m, y) \text{ for all } v \in \sigma(m, y). \]

2.11. If \( a \in \sigma_{\mathcal{H}}, x \in \mathcal{H} \) and \( m \in \mathcal{H} \) then \( \sigma(a, ax) = a \cdot \sigma(m, x) \).

Proof: Trivial from 2.13.

2.12. Theorems 2.8 and 2.10 can be used for proving Wiener's theorem on absolutely convergent Fourier series by Banach algebra methods, without using the axiom of choice; see [5].

2.13. Another consequence of 2.8 is the following theorem:

If \( x \in \mathcal{H}, m \in \mathcal{H} \) and \( x \in \mathcal{H} \) \( m \in \mathcal{H} \) \( \exists \mathcal{H} \in \mathcal{H} \) \( \varphi_{\mathcal{H}}(x) \in R_{\mathcal{H}} \), then

\[ \varphi_{\mathcal{H}}(x) \in R_{\mathcal{H}}. \]

Proof: If \( \varphi_{\mathcal{H}}(x) \notin R_{\mathcal{H}} \) or, equivalently, \( \varphi_{\mathcal{H}}(x) \in S_{\mathcal{H}} \), then

\[ m = \overline{\varphi_{\mathcal{H}}[\lambda(x \mathcal{H})]} \text{ has the properties of 2.8, particularly } x \in m \text{ and } m \in \mathcal{H} \text{; hence by assumption } \exists \mathcal{H} \in \mathcal{H} \text{ such that } \varphi_{\mathcal{H}}(x) \in R_{\mathcal{H}}, \text{ contradicting the fact that, since } x \in m \text{, } \varphi_{\mathcal{H}}(x) = \varphi_{\mathcal{H}}(x). \]

This proves that \( \varphi_{\mathcal{H}}(x) \in R_{\mathcal{H}} \).

2.14. If \( m \in \mathcal{H} \) then \( \overline{\varphi_{\mathcal{H}}(S_{\mathcal{H}})} = \bigcup_{m \in \mathcal{H}} \overline{\varphi_{\mathcal{H}}(S_{\mathcal{H}})}. \)

Proof: If \( x \in \overline{\varphi_{\mathcal{H}}(S_{\mathcal{H}})} \) then \( \varphi_{\mathcal{H}}(x) \in S_{\mathcal{H}}, \text{ implying } 0 \in \sigma(m, x); \]}
according to 2.8 there exists an $m \in \mathcal{N}(m)$ with $\sigma(m, x) = \{0\}$; hence $\nu(m, x) = 0$, $\varphi_{m}(x) \in \Sigma_{\nu}$, and $x \in \varphi_{m}(\Sigma_{\nu})$.

Conversely, if $\varphi_{m}(x) \in \Sigma_{\nu}$, for any $m \in \mathcal{N}(m)$ then $\nu(m, x) = 0$; this implies $\sigma(m, x) = \{0\}$, $0 \in \sigma(m, x)$ by 2.2 and $\varphi_{m}(x) \in \Sigma_{\nu}$.  

Chapter 3

The *-operation and the lattice of *-invariant sets of closed ideals

3.1. Recall that $\mathcal{K}(\mathcal{M}) := \{ m \in \mathcal{K} | m \geq n \}$. Recall that if $S$ is a set, $P(S)$ denotes the class of subsets of $S$.

3.2. $\mathcal{E} := \{ \mathcal{A} \in P(\mathcal{K}) | \forall n \in \mathcal{A} \mathcal{K}(n) \subseteq \mathcal{A} \}$. $\mathcal{E}$ is clearly a subset of $P(\mathcal{K})$.

3.3. We define the mapping $* : P(\mathcal{K}) \rightarrow P(\mathcal{K})$ by:
   if $\mathcal{A} \in P(\mathcal{K})$ then $\mathcal{A}^* := \{ m \in \mathcal{K} | \forall n \in P(m) \exists k \in \mathcal{K}(m) k \in \mathcal{A} \}$.

By $\mathcal{\Delta}$ we denote the subset of $P(\mathcal{K})$ which contains the *-invariant subsets of $\mathcal{K}$:

$\mathcal{\Delta} := \{ \mathcal{A} \in P(\mathcal{K}) | \mathcal{A} = \mathcal{A}^* \}$.

3.4. i: $\emptyset^* = \emptyset$, hence $\emptyset \in \mathcal{\Delta}$;
   ii: $\mathcal{K}^* = \mathcal{K}$, hence $\mathcal{K} \in \mathcal{\Delta}$;
   iii: $\mathcal{A} \in \mathcal{\Delta}$ implies $\mathcal{A}^* \in \mathcal{\Delta}$.

Proof:
   i: Since $\forall x \in \mathcal{K} k \not\in \emptyset$, we have $\forall m \in P(m) \forall k \in \mathcal{K}(m) k \not\in \emptyset$ and consequently $\forall m \not\in \emptyset$.
ii: Since $\forall m \in \mathcal{M}(m) \; \forall k \in \mathcal{M}(k) \; \exists k \in \mathcal{N}$, we have $\forall m \in \mathcal{N}$, whence $\mathcal{N}^* = \mathcal{N}$.

iii: If $\alpha \in \mathcal{E}^*$ and $\alpha \in \mathcal{N}^*$ then $\forall m \in \mathcal{M}(m) \; \exists k \in \mathcal{N}(k) \; k \in \mathcal{E}$, hence $\alpha \in \mathcal{E}^*$.

3.5. $\forall \alpha \in \mathcal{P}(\mathcal{N}) \; \alpha^* \in \mathcal{E}$.

Proof: If $\alpha \in \mathcal{N}^*$ then $\forall m \in \mathcal{M}(m)$ we have to prove $\alpha \in \mathcal{N}^*$ or equivalently $\forall k \in \mathcal{M}(m) \; \exists k \in \mathcal{N}(k) \; k \in \mathcal{E}$.

But this is trivial since $\alpha \in \mathcal{M}(m)$ and $k \in \mathcal{M}(m)$ imply $k \in \mathcal{N}(m)$, and $m \in \mathcal{N}^*$.

3.6. $\Delta \in \mathcal{E}$.

Proof: Consequence of 3.5 and 3.5.

3.7. $\alpha \in \mathcal{E} \Rightarrow \alpha \in \mathcal{N}^*$.

Proof: If $\alpha \in \mathcal{E}$ and $\alpha \in \mathcal{N}$ then $\forall k \in \mathcal{M}(m)$ and $\forall k \in \mathcal{M}(m)$ $\forall k \in \mathcal{M}(m)$ $\exists k \in \mathcal{N}(k)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)$ $\exists k \in \mathcal{M}(m)
3.10. If \( \{ \alpha_\lambda \} \in P(\Lambda) \) then

i: \( \forall \alpha_\lambda \in \Lambda \) and \( (\bigcup \alpha_\lambda^* ) \ast \bigcup \alpha_\lambda^* \)

ii: \( \forall \alpha_\lambda \in \Lambda \) and \( (\bigcap \alpha_\lambda^* ) \ast \bigcap \alpha_\lambda^* \)

iii: If, moreover, \( \Lambda \) is finite, then \( (\bigcap \alpha_\lambda^* ) \ast = \bigcap \alpha_\lambda^* \).

Proof: i and ii are trivial consequences of 3.2 and 3.4.iii.

iii: Since \( \Lambda \) is finite we have that \( \Lambda = \{ \lambda_1, \ldots, \lambda_l \} \).

If \( m \in \bigcap \alpha_\lambda^* \) then \( \forall j = 1, \ldots, l \), \( \forall m \in M(m) \) \( \exists j \in M(m) \) \( k \in \alpha_\lambda^* \).

Take \( m = m_1 \in M(m) \cap \alpha_\lambda^* \), then \( m_1 \in M(m) \); hence there exists an \( m_2 \in M(m_1) \cap \alpha_\lambda^* \) and since \( \alpha_\lambda \in \Lambda \) we have also \( m_2 \in \alpha_\lambda^* \).

Proceeding in this way we construct a sequence

\[ m_1, m_2, m_3, \ldots, m_l \]

and we see at once that \( m_k \in M(m) \) and \( m_k \in \bigcap \alpha_\lambda^* \); hence

\( m \in \left( \bigcap \alpha_\lambda^* \right)^* \).

This proves the theorem.

3.11. If \( \alpha \in P(\Lambda) \) and \( \beta \in \Sigma \) then \( \alpha^* \cap \beta \in (\alpha \cap \beta)^* \).

Proof: The assumptions together with \( m \in \alpha^* \cap \beta \) and \( m \in M(m) \) imply \( m \in \alpha^* \), \( \exists k \in M(m) \) \( k \in \alpha^* \), \( m \in \beta \) and \( \exists k \in M(m) \) \( k \in \beta \), hence \( m \in M(m) \) \( k \in \alpha \cap \beta \); thus by definition \( m \in (\alpha \cap \beta)^* \).

3.12. If \( \{ \alpha_\lambda \} \in P(\Delta) \) then \( (\bigcap \alpha_\lambda^* ) \ast = \bigcap \alpha_\lambda^* \), whence \( \bigcap \alpha_\lambda \in \Delta. \)

Proof: By 3.10.ii \( (\bigcap \alpha_\lambda^* ) \ast \bigcap \alpha_\lambda^* \) whence by assumption
(0 CL)∗ ∈ CL; the converse inclusion is true by 3.10.ii and 3.7.

3.13. If α ∈ τ(κ), β ∈ δ and (α ∪ β)∗ = α then β ⊆ α.

Proof: α ∉ α implies by assumption α ∉ (α ∪ β)∗ and by definition this entails ∃ m ∈ τ(κ) such that (α ∪ β)∗ = ∅; particularly m ∉ α ∪ β and m ∉ β; since β ∈ δ we conclude m ∉ β.

3.14. The *-operation is not a closure operator in τ in the usual topological sense, since both α ∉ α∗ and ((α ∪ β)∗ = α∗ ∪ β∗) do not always hold in τ(κ); the first of these conditions by 3.7 holds in δ, the latter does not hold always even in δ; for proofs see section 7.

3.15. If {αμ}μ ∈ M ∈ τ(κ) and β ∈ δ then

i: ∀ λ ∈ M αλ > ∩ μ ∈ M αμ,

ii: [∃ λ ∈ M αλ ⊆ β] ⇒ ∩ μ ∈ M αμ ⊆ β,

iii: ∀ λ ∈ M αλ ∈ (U μ ∈ M αμ)∗,

iv: [∀ λ ∈ M αλ ⊆ β] = (U μ ∈ M αμ)∗ ⊆ β.

Proof: i and ii are trivial, iii is a consequence of 3.4.iii and the assumption β ∈ δ; iv follows from the fact that

[U λ ∈ M αλ ⊆ β] = U λ ∈ M αλ ⊆ β, hence, by 3.4.iii and the assumption β ∈ δ, (U μ ∈ M αμ)∗ ⊆ β.
3.16. If \( \{ \mathcal{O}_\mu \}_{\mu \in M} \subseteq \mathcal{P}(\Delta) \) then

i. \( \bigwedge_{\mu \in M} \mathcal{O}_\mu = \bigcap_{\mu \in M} \mathcal{O}_\mu \)

ii. \( \bigvee_{\mu \in M} \mathcal{O}_\mu = \left( \bigcup_{\mu \in M} \mathcal{O}_\mu \right)^* \).

If \( M = \emptyset \) then \( \bigcap_{\mu \in M} \mathcal{O}_\mu = \emptyset \) and \( \bigcup_{\mu \in \emptyset} \mathcal{O}_\mu = \emptyset \); hence \( \bigwedge_{\mu \in \emptyset} \mathcal{O}_\mu = \emptyset \) and \( \bigvee_{\mu \in \emptyset} \mathcal{O}_\mu = \emptyset \); if \( M \) consists of only one index then we have \( A \mathcal{O} = \mathcal{O} \); if \( M \) consists of a finite number \( J \) of indices then we occasionally write

\( \mathcal{O}_1 \wedge \mathcal{O}_2 \cdots \mathcal{O}_j \wedge \mathcal{O}_k \) for \( \mathcal{O}_1 \wedge \mathcal{O}_3 \) and \( \mathcal{O}_1 \wedge \mathcal{O}_2 \cdots \mathcal{O}_j \wedge \mathcal{O}_k \) for \( \mathcal{O}_j \wedge \mathcal{O}_k \).

3.17 (\( \wedge, \vee \)) is a complete, \( \wedge \)-distributive, \( \vee \)-distributive lattice.

Proof:

i: \( \wedge \) and \( \vee \) have the required property of \( \mathcal{O} \in \Delta \) and \( \mathcal{O}' \in \Delta \) implying \( \mathcal{O} \wedge \mathcal{O}' \in \Delta \) and \( \mathcal{O} \vee \mathcal{O}' \in \Delta \); this is shown for \( \wedge \) in 3.12, for \( \vee \) it follows from the definition 3.16,ii and 3.8.

ii: Properties \( L_3 \) and \( L_4 \) of section 0.2 on commutativity and \( L_1 \) on associativity of \( \wedge \) follow trivially from the definitions; if \( \forall \mu \in M \forall \nu \in M \mathcal{O}_\mu \in \Delta \) then

\( (\bigvee_{\nu} \mathcal{O}_\mu^*) = \bigvee_{\nu} \mathcal{O}_\mu^* \) for every \( \mu \in M \), by 3.10.1,

hence \( (\bigvee_{\nu} \mathcal{O}_\mu^*) \mu, \nu = \bigvee_{\nu} (\mathcal{O}_\mu^*) \mu, \nu \mu, \nu \);

since \( [(\bigvee_{\nu} \mathcal{O}_\mu^*)]_{\mu, \nu, \mu, \nu} \) by 3.4,iii

\[39\]
we have \( V(V \alpha_{\mu}) = V \alpha_{\mu} \) by 3.16.ii.

we now prove the converse inclusion:

Putting \( \mathcal{L}_\mu = V \alpha_{\mu} \), we have

\( \alpha_{\mu} \in \mathcal{L}_\mu \) for all \( \mu, \nu \) by 3.4.iii,

hence \( (U \alpha_{\mu})^\ast \in \mathcal{L}_\mu \) for all \( \mu \)

and \( [U(U \alpha_{\mu})^\ast]^\ast \in \mathcal{L}_\mu \) by 3.15.iv.

This means \( V(V \alpha_{\mu}) \in \mathcal{L}_\mu \).

The results combine to the extended law of associativity for \( V \):

\[
V(V \alpha_{\mu}) = V \alpha_{\mu} = V(V \alpha_{\mu}),
\]

implying that \( \alpha V (\mathcal{L} V \mathcal{L}) = (\alpha V \mathcal{L}) V \mathcal{L} \); this establishes property \( L_4 \) of section 0.2.

Before we are allowed to conclude that \( (\Delta, \Lambda, V) \) is a lattice we should have verified \( L_2 \) and \( L_3 \) that for every \( \mathcal{C} \in \Delta \) and for every \( \mathcal{L} \in \Delta \), \( \mathcal{C} \Lambda (\mathcal{L} V \mathcal{L}) = \mathcal{C} V (\mathcal{C} \Lambda \mathcal{L}) \); this, however, is a consequence of the laws of distributivity, see vi below.

iii: Since \( \alpha \Lambda \mathcal{L} = \alpha \cap \mathcal{L} \) we have \( \alpha \in \mathcal{L} \iff \alpha \Lambda \mathcal{L} = \alpha \), implying that the lattice-order of \( (\Delta, \Lambda, V) \) coincides with set-inclusion in \( P(\mathcal{H}) \), which together with 3.12 and 3.15.i and ii implies that \( \Delta \) is \( \Lambda \)-complete. The \( V \)-completeness of \( \Delta \) follows analogously from 3.3 and 3.15.iii and iv.

iv: Let \( \{ \alpha_{\mu} \}_{\mu} \in \mathcal{M} \subseteq P(\Delta) \) and \( \mathcal{L} \in \Delta \); then by 3.11
\[ L \cap (v \circ \mu) = L \cap (u \circ \mu)^* \subset [L \cap (u \circ \mu)]^* = [u(L \cap \circ \mu) ]^* \]

hence \[ L \cap (v \circ \mu) \subset V(L \cap \circ \mu) \].

Conversely, if \[ m \in V(L \cap \circ \mu) \] then

\[ \forall \mu \in \mathcal{M}(m) \exists \nu \in \mathcal{M}(m) E \in L \cap (u \circ \mu) = L \cap (u \circ \mu) \]

implying \[ m \in L^* \subset L \] and \[ m \in (u \circ \mu)^* = v \circ \mu \]

hence \[ m \in L \cap (v \circ \mu) \].

This proves that \((A, \Lambda, V)\) is \(\Lambda\)-distributive.

\(v:\) Again we take \( \{ \alpha_{\mu} \}_{\mu} \in \mathcal{P}(\Delta) \) and \( L \in \Delta \); now

\[ L \cap (\Lambda \circ \mu) = [L \cup (n \circ \mu)]^* = [n(L \cup \circ \mu)]^* \subset \]

\[ n(L \cup \circ \mu)^* \] by 3.10.11 since \( L \cup \circ \mu \in E \)

hence \[ L \cap (\Lambda \circ \mu) \subset \Lambda(L \cap \circ \mu) \).

If, conversely, \( m \notin L \cap (\Lambda \circ \mu) = [L \cup (n \circ \mu)]^* \) then

\[ \exists \mu \in \mathcal{M}(m) \forall \nu \in \mathcal{M}(m_{\nu}) E \in L^* \cup (n \circ \mu) \] or equivalently

\[ \exists \mu \in \mathcal{M}(m) \mathcal{M}(m_{\nu}) \cap [L \cup (n \circ \mu)] = \emptyset \]

implying \( \mathcal{M}(m_{\nu}) \cap L^* = \emptyset \) and \( \mathcal{M}(m_{\nu}) \cap (n \circ \mu) = \emptyset \).

In particular \( \exists \nu \in \mathcal{M} \forall m \in \mathcal{M}(m) \cap \circ \nu = \emptyset \) and consequently

\[ \exists m_2 \in \mathcal{M}(m_1) \mathcal{M}(m_2) \cap \circ \nu = \emptyset \]

implying, since \( \mathcal{M}(m_2) \subset \mathcal{M}(m_1) \)
\[ \forall (m_2) \cap (\mu_\Lambda \cup L_\mu) = \emptyset ; \]

since \( m_2 \in \mathcal{N}(m) \) we can conclude

\[ m \notin (\mu_\Lambda \cup L_\mu)^* = \mu_\Lambda \cup L_\mu \]

whence \( m \notin (\mu_\Lambda \cup L_\mu) = \Lambda(\mu_\Lambda \cup L_\mu) \).

This shows that \( L_\mu \cup (\Lambda \mu_\Lambda) = \Lambda(L_\mu \cup \mu_\Lambda) \)

which connotes that \( (\Lambda, \Lambda, \mu, V) \) is \( V \)-distributive.

vi: If \( \mu \in \Lambda, L_\mu \in \Delta \) then

\[ \mu \Lambda(\mu_\Lambda \cup L_\mu) = (\mu \Lambda \mu_\Lambda) \cup (\mu \Lambda L_\mu) = [\mu_\Lambda \cup (\mu \Lambda \mu_\Lambda)]^* = \mu_\Lambda \cup L_\mu \]

and

\[ \mu \Lambda(\mu_\Lambda \cup L_\mu) = [\mu_\Lambda \cup (\mu \Lambda \mu_\Lambda)]^* = \mu \Lambda \cup L_\mu \]

(cf., final part of ii in this proof) are evident.

This concludes the proof of the theorem.

**3.18.** i: \( (\Lambda \mu_\Lambda) \cup (\Lambda L_\mu) = \Lambda (\mu_\Lambda \cup L_\mu) \),

ii: \( (\mu \Lambda_\Lambda) \cup (\mu L_\mu) = \mu \Lambda(\mu_\Lambda \cup L_\mu) \).

Proof:

i: \( (\Lambda \mu_\Lambda) \cup (\Lambda L_\mu) = \Lambda[\mu_\Lambda \cup (\Lambda L_\mu)] = \Lambda[\Lambda \mu_\Lambda \cup L_\mu] = \Lambda (\mu_\Lambda \cup L_\mu) \).

ii: Analogously.
A point-free topology: the lattice of clusters

4.1. If $\alpha \in \mathcal{C}_m$ and $\varepsilon > 0$ then $\Phi(\alpha, \varepsilon) := \{ \gamma \in \mathcal{C}_m \mid |\gamma - \alpha| \geq \varepsilon \}$ ;
$\Phi := \Phi(0, 1)$.

4.2. $\{\Phi(\alpha, \varepsilon) \mid \alpha \in \mathcal{C}_m, \varepsilon > 0\}$ is a base for the closed sets in $\mathcal{C}_m$.

4.3. By $\mathcal{P}(\mathcal{K})$ or $\mathcal{P}$ we denote the class of finite subsets of (the
$\mathcal{B}_\mathcal{K}$-algebra) $\mathcal{K}$; by $\mathcal{Q}$ we denote the class $\mathcal{P}(\mathcal{P}(\mathcal{K}))$ of subsets of $\mathcal{P}$.

4.4. Recall that if $x \in \mathcal{K}$ then $\{x\} \in \mathcal{Q}$; and if $a \in \mathcal{P}$ then
$\{a\} \in \mathcal{Q}$; particularly $\{\{x\}\} \in \mathcal{Q}$ for every $x \in \mathcal{K}$. The mapping
$\omega: \mathcal{K} \to \mathcal{Q}$, defined by $\omega(x) := \{\{x\}\}$, is evidently an injection.

4.5. If $A \in \omega(\mathcal{K})$, $\alpha \in \mathcal{C}_m$ and $\varepsilon > 0$ then
$\mathcal{L}(A, a, \varepsilon) := \{\nu \in \mathcal{K} \mid c(\nu) \in \omega(\mathcal{K}) \} \in \Phi(\alpha, \varepsilon) \}$.

If we substitute $x$ for $\omega(A)$ in this expression we get
$\mathcal{L}(\{\{x\}\}, a, \varepsilon) = \{\nu \in \mathcal{K} \mid c(\nu) \in \omega(\mathcal{K}) \} \times \{\{x\}\} \times \mathcal{P}(\mathcal{P}(\mathcal{K}))$ for every $x \in \mathcal{K}$.

4.6. $\mathcal{L}(\{\{x\}\}, a, \varepsilon) = \mathcal{L}(\{\{x\}\}, a, \varepsilon) \times \{0, 1\}$.

Proof: The following statements are equivalent:

$\lambda \in \sigma(\nu, x), \varphi(\nu, x - \lambda e) \in S_{\nu}; \varphi(\nu, x - \lambda e, \frac{\lambda - a}{\varepsilon}) \in \sigma(\nu);$

$\frac{\lambda - a}{\varepsilon} \in \sigma(\nu, x - \lambda e)$.

Hence the result.
4.7. For every \( x \in \mathcal{X} \) we denote \( \mathcal{L}(\{x\}, 0, 1) \) by \( \mathcal{L}_x \).

4.8. If \( m \in \mathcal{L}_x \) and \( n \in \mathcal{L}_x \) then \( mn \in \mathcal{L}_x \).

Proof: Corollary of 2.3.

4.9. If \( x \in \mathcal{X} \) then \( \mathcal{L}_x \in \Delta \).

Proof: We observe that \( \mathcal{L}_x \in \mathcal{E} \) since \( m \in \mathcal{L}_x \) as a corollary of 2.2 implies \( \mathcal{M}(m) \subset \mathcal{L}_x \); consequently \( \mathcal{L}_x \in \mathcal{L}_x^* \) by 3.7.

Now suppose \( m \in \mathcal{L}_x^* \), then \( \forall m \in \mathcal{M}(m) \exists \lambda \in \mathcal{M}(m) \in \mathcal{L}_x \) or equivalently, \( \forall m \in \mathcal{M}(m) \exists \lambda \in \mathcal{M}(m) \in \mathcal{L}_x \).

\[ \forall m \in \mathcal{M}(m) \exists \lambda \in \mathcal{M}(m) \forall \lambda \in \mathcal{C}(m) \left[ |\lambda| < 1 \Rightarrow \mathcal{L}(x - \lambda e) \in \mathcal{R}_x \right] \]

or by 2.13 \( \forall \lambda \in \mathcal{C}(m) \left[ |\lambda| < 1 \Rightarrow \mathcal{L}(x - \lambda e) \in \mathcal{R}_x \right] \)

or equivalently \( \mathcal{L}_x \in \mathcal{L}_x \).

Consequently \( \mathcal{L}_x \in \Delta \).

4.10. If \( a \in \mathcal{P} \) then \( \mathcal{L}_a := \bigcup_{x \in a} \mathcal{L}_x \),

\( \mathcal{L}_a := \mathcal{L}_a^* = \bigvee_{x \in a} \mathcal{L}_x \)

and if \( A \subseteq \mathcal{P} \) then \( \mathcal{L}_A := \bigcap_{a \in A} \mathcal{L}_a = \bigwedge_{a \in A} \mathcal{L}_a \).

\( \Gamma := \{ \mathcal{L}_A \mid A \subseteq \mathcal{P} \} \).

The sets \( \mathcal{L}_A \) are called clusters.

4.11. i: \( \mathcal{P} \subseteq \mathcal{P} \), \( \forall x \in \mathcal{X} \mathcal{L}_x \in \mathcal{P} \), \( \forall a \in \mathcal{P} \mathcal{L}_a \in \mathcal{P} \).

ii: \( \mathcal{P} \subseteq \Delta \).
iii: $\Gamma$ is closed in $\Delta$ with respect to $A$: if $\{A_{\mu}\}_{\mu \in M} \in \mathcal{P}(Q)$ then
\[ A \subseteq A_{\mu} \subseteq \Gamma ; \]

iv: $\Gamma$ is finitely closed in $\Delta$ with respect to $V$: if $\{A_{j}\}_{j=1}^{k} \subseteq \mathcal{P}(Q)$ then $V \subseteq \bigcup_{j=1}^{k} A_{j} \subseteq \Gamma$.

Proof:
i: If $x \in \mathcal{X}$ and $a \in \Gamma$ then $\{x\}, \{a\} \subseteq Q$ and $\{a\} \subseteq Q$;
since $\mathcal{L}_{x} = \mathcal{L}_{\{x\}} = \mathcal{L}_{\{x\}}$ and $\mathcal{L}_{a} = \mathcal{L}_{\{a\}}$ we have $\mathcal{L}_{a} \subseteq \Gamma$ and $\mathcal{L}_{a} \subseteq \Gamma$.

Now, denoting as before the zero and identity of $\mathcal{H}$ by $\sigma$ and $e$, we have
\[ \mathcal{L}_{\sigma} = \{ m \in \mathcal{H} \mid c(m, e) = 1 \} = \emptyset \quad \text{and} \quad \mathcal{L}_{e} = \{ m \in \mathcal{H} \mid c(m, e) = 1 \} = \mathcal{H}. \]

ii: Since $\mathcal{L}_{x} \subseteq \Delta$ for every $x \in \mathcal{X}$, also $\mathcal{L}_{a} \subseteq \Delta$ for every $a \in A$ as a consequence of the fact that $\Delta$ is a lattice; and from the completeness of $\Delta$ it follows that $\bigwedge_{a \in A} \mathcal{L}_{a} \subseteq \Delta$ for every $A \subseteq Q$.

iii: If $\{A_{\mu}\}_{\mu \in M} \subseteq \mathcal{P}(Q)$ then
\[ \bigwedge_{\mu \in M} \mathcal{L}_{A_{\mu}} = \bigcap_{\mu \in M} \mathcal{L}_{A_{\mu}} = \bigcap_{\mu \in M} \mathcal{L}_{A_{\mu}} = \mathcal{L}_{\bigcup_{\mu \in M} A_{\mu}}, \quad \text{since} \quad \mathcal{L}_{A_{\mu}} \subseteq \Gamma, \quad \mathcal{L}_{\bigcup_{\mu \in M} A_{\mu}} \subseteq \Gamma. \]

iv: Let $A \subseteq Q$ and $B \subseteq Q$; suppose first that $A = \{a\}$, $B = \{b\}$ with $a \in A$ and $b \in B$.
Then $\mathcal{L}_{A} \vee \mathcal{L}_{B} = \mathcal{L}_{A} \vee \mathcal{L}_{B} = \bigvee_{x \in a} \mathcal{L}_{x} \bigvee_{y \in b} \mathcal{L}_{y} = \bigvee_{x \in a \cup b} \mathcal{L}_{x}$ by the associativity of $\vee$;
hence \( \mathcal{L}_a \lor \mathcal{L}_b = \mathcal{L}_{a \cup b} \).

In the general case we have

\[
\mathcal{L}_A \lor \mathcal{L}_B = \left( \bigwedge_{a \in A} \mathcal{L}_a \right) \lor \left( \bigwedge_{b \in B} \mathcal{L}_b \right), \quad \text{and by } \lambda \text{ we obtain}
\]

\[
\mathcal{L}_A \lor \mathcal{L}_B = \bigwedge_{a \in A} \left( \mathcal{L}_a \lor \mathcal{L}_B \right) = \bigwedge_{(a, b) \in A \times B} \mathcal{L}_{a \cup b}.
\]

Now, if \( A \subseteq \mathfrak{Q} \) and \( B \subseteq \mathfrak{Q} \), then \( \{a \cup b\}_{(a, b) \in A \times B} \subseteq \mathfrak{Q} \);

hence \( \mathcal{L}_A \lor \mathcal{L}_B \in \Gamma \).

From this result it follows by induction that, if

\( \{ A_j \}_{j=1, \ldots, \ell} \subseteq \mathfrak{Q} \), then

\[
\bigwedge_{j=1}^{\ell} \mathcal{L}_{A_j} = \bigwedge_{(a_1, \ldots, a_\ell) \in A_1 \times \ldots \times A_\ell} \mathcal{L}_{a_1 \cup \ldots \cup a_\ell},
\]

and this again is an element of \( \Gamma \).

\( \text{4.12.} \) From \( 4.11 \) it will be clear that \( \Gamma \) is a sublattice of \( \Delta \);

since \( \Gamma \) is \( \Lambda \)-complete and bounded, it is complete by a well-known theorem of lattice theory ([23], p. 63). Since \( \Delta \) is \( \vee \)-distributive

and \( \Gamma \) is \( \Lambda \)-closed in \( \Delta \) we see that \( \Gamma \) is \( \vee \)-distributive. Thus \( \Gamma \), as

a lattice, has the properties \( \text{PT}_1 \), \( \text{PT}_2 \) and \( \text{PT}_3 \) of point-free topologies as stated in section \( 0.3 \). We now proceed to show that \( \Gamma \)

fulfills the conditions \( \text{PT}_4 \) and \( \text{PT}_5 \) as well. Though \( \text{PT}_5 \) implies \( \text{PT}_4 \),

the case of \( \text{PT}_4 \) will be treated separately, mainly because part of

its proof serves as a lemma for the proof of \( \text{PT}_5 \) (\( 4.18 \) and \( 4.19 \)).

4.17 stays a little apart as a minor result.

\( 4.13. \) If \( x \in \mathfrak{X} \) and \( \mathfrak{Y} \) is a closed set in \( \mathfrak{X}_m \), then
\{x \in \mathcal{X} | \sigma(x, \lambda) \in \mathcal{Y}\} \in \mathcal{F}.

Proof: By 4.2 \mathcal{Y} is the intersection of a certain family
\{\Phi(\lambda, \lambda)\}_{(\lambda, \lambda)}; \text{ now}

\{x \in \mathcal{X} | \sigma(x, \lambda) \in \mathcal{Y}\} = \bigcap_{(\lambda, \lambda)} \{x \in \mathcal{X} | \sigma(x, \lambda) \in \Phi(\lambda, \lambda)\} =

\bigcap_{(\lambda, \lambda)} \{x \in \mathcal{X} | \sigma(x, \lambda) \in \Phi(\lambda, \lambda)\} = \bigcap_{(\lambda, \lambda)} \mathcal{L}(x, \lambda) \sigma(\lambda, \lambda) \text{ by 4.6.}

This proves the theorem.

4.14. If \nu \in \mathcal{H}, then \mathcal{L}(\nu) = \{x \in \mathcal{X} | \forall x \in \mathcal{X} [\sigma(x, \lambda) \in \mathcal{Y}]\}.

4.15. i: \mathcal{L}(\nu) \in \mathcal{F} for every \nu \in \mathcal{H}.

ii: \mathcal{H}(\nu) \subset \mathcal{L}(\nu) for every \nu \in \mathcal{H}.

iii: If \nu \in \mathcal{K} and \nu \in \mathcal{L}_x, then \mathcal{L}(\nu) \subset \mathcal{L}_x.

Proof:

i: \mathcal{L}(\nu) = \bigcap_{x \in \mathcal{X}} \{x \in \mathcal{X} | \sigma(x, \nu) \in \mathcal{Y}\}; \text{ since } \sigma(x, \nu) \text{ is closed in } \mathcal{X} \text{ for every } \nu \text{ and } x, \{x \in \mathcal{X} | \sigma(x, \nu) \in \mathcal{Y}\} \in \mathcal{F} \text{ by 4.13; hence the result.}

ii: Trivial consequence of 2.2 and 4.14.

iii: Trivial consequence of 4.6 (and 4.7) and 4.14.

4.16. If \nu \in \mathcal{S} then \nu \mathcal{E} \in \mathcal{H} by 1.15; we call \mathcal{L}(\nu \mathcal{E}) the zero-cluster of \nu.

4.17. If \{x_j\}_{j=1}^k \subset \mathcal{K} and \bigwedge_{j=1}^k \mathcal{L}(\nu_j \mathcal{E}) = \emptyset then \exists e \in \sum_{j=1}^k \nu_j \mathcal{E}. 47
Proof: \( \Sigma y_j \mathcal{K} \) is an ideal in \( \mathcal{K} \); if we assume
\[ a \notin \bigoplus_{j \in \mathbb{N}} y_j \mathcal{K}, \]
then \( a \notin \bigoplus_{j=1}^\infty y_j \mathcal{K} \) by 1.9;

hence \( \Sigma_j y_j \mathcal{K} \in \mathcal{N}(y_1 \mathcal{K}) \) for every \( i = 1, \ldots, \lambda \);

we infer that \( \Sigma_j y_j \mathcal{K} \in \mathcal{L}(y_1 \mathcal{K}) \) (i = 1, \ldots, \lambda) by 4.14, ii, whence
\[ \lambda \in \mathcal{L}(y_1 \mathcal{K}) \neq \emptyset. \]

This proves the theorem.

4.18 If \( \mathcal{L}_A \in \Gamma, \mathcal{L}_B \in \Gamma \) and \( \mathcal{L}_B \notin \mathcal{L}_A \) then

\[ 3 \mathcal{L}_C \in \Gamma \left( \emptyset \neq \mathcal{L}_C \subset \mathcal{L}_B \land \mathcal{L}_C \land \mathcal{L}_A = \emptyset \right). \]

Proof: The assumption \( \mathcal{L}_B \notin \mathcal{L}_A \) implies that \( \mathcal{L}_B \neq \emptyset \); since
\[ \mathcal{L}_A = \bigwedge_{a \in A} \mathcal{L}_a, \quad \exists a \in A \quad \mathcal{L}_B \notin \mathcal{L}_a. \]

Thus \( 3 \mathcal{L}_C \in \mathcal{N}(\mathcal{L}_B \setminus \mathcal{L}_a). \)

Writing \( \mathcal{L}_a = \bigvee_{j=1}^\lambda \mathcal{L}_{y_j} \) we have \( \mathcal{L} \notin \left( \bigcup_{j=1}^\lambda \mathcal{L}_{y_j}{'} \right) \); hence
\[ 3 \mathcal{L}_C \in \mathcal{N}(\mathcal{L}) \forall \mathcal{L} \in \mathcal{N}(\mathcal{L}_a), \quad \mathcal{L} \notin \bigcup_{j=1}^\lambda \mathcal{L}_{y_j} \]
or
\[ 3 \mathcal{L}_C \in \mathcal{N}(\mathcal{L}) \forall \mathcal{L} \in \mathcal{N}(\mathcal{L}_a), \quad \mathcal{L} \notin \bigcup_{j=1}^\lambda \mathcal{L}_{y_j} \quad \forall j = 1, \ldots, \lambda. \]

According to this statement we take \( \mathcal{L}_1 \in \mathcal{N}(\mathcal{L}) \) and \( \xi_1 \in \sigma(\mathcal{L}_1, y_1) \) with \( |\xi_1| < 1 \); next, according to 2.8, we take \( \mathcal{L}_2 \in \mathcal{N}(\mathcal{L}_1) \) to the effect that \( \sigma(\mathcal{L}_2, y_2) = \{\xi_1\} \).

Now we can select \( \xi_2 \in \sigma(\mathcal{L}_2, y_2) \) with \( |\xi_2| < 1 \) and construct
$m_3 \in \mathcal{M}(m_2)$ such that $o(m_3, y_2) = \{x_2\}$; proceeding in this way we get the sequences

$m_1 \in m_2 \ldots \in m_{k+1} = m$

and $x_1, x_2, \ldots, x_k$

with the properties $\forall j=1,\ldots,k \quad |x_j| < 1$

and $\forall j=1,\ldots,k \quad \forall i=1,\ldots,j \quad o(m_{j+1}, y_i) = \{x_i\}$,

whence $m \in \mathcal{M}(m)$ and $\forall j=1,\ldots,k \quad o(m, y_j) = \{x_j\} \wedge |x_j| < 1$.

We now define $\mathcal{L}_C := \mathcal{L}(m) \cap \mathcal{L}_B$ and will prove that $\mathcal{L}_C$ has the desired properties:

Since $m \in \mathcal{L}_B$ and $m \in \mathcal{M}(m)$ also $m \in \mathcal{L}_B$; $m \in \mathcal{L}(m)$ by 4.15 ii;

hence $m \in \mathcal{L}_C$ and $\mathcal{L}_C \neq \emptyset$; that $\mathcal{L}_C \subseteq \mathcal{L}_B$ is obvious.

Finally, $\mathcal{L}_C \setminus \mathcal{L}_A \subseteq \mathcal{L}_C \setminus \mathcal{L}_A = \mathcal{L}_C \setminus \bigcup_{j=1}^{k} \mathcal{L}_{y_j} = \bigcup_{j=1}^{k} \mathcal{L}_C \setminus \mathcal{L}_{y_j}$;

since $q \in \mathcal{L}_C$ implies $q \in \mathcal{L}(m)$ and consequently

$\forall j=1,\ldots,k \quad o(q, y_j) = \{x_j\}$ and $\forall q \notin \mathcal{L}_{y_j}$ we have $\forall j \quad \mathcal{L}_C \setminus \mathcal{L}_{y_j} = \emptyset$;

this proves that $\mathcal{L}_C \setminus \mathcal{L}_A = \emptyset$.

4.19. If $\mathcal{L}_A \in \Gamma$, $\mathcal{L}_B \in \Gamma$ and $\mathcal{L}_B \notin \mathcal{L}_A$, then there exist $\mathcal{L}_C$, $\mathcal{L}_D$ and $\mathcal{L}_E$ in $\Gamma$ with the following properties:

i: $\emptyset \neq \mathcal{L}_C \subseteq \mathcal{L}_D$

ii: $\mathcal{L}_C \setminus \mathcal{L}_D = \emptyset$

iii: $\mathcal{L}_A \setminus \mathcal{L}_E = \emptyset$
iv: \( \mathcal{L}_A \supset \mathcal{L}_B \)

v: \( \mathcal{L}_C \supset \mathcal{L}_B \)

vi: \( \mathcal{L}_D \lor \mathcal{L}_E = \mathcal{H} \).

Proof: Taking \( a = \{ y_j \}_{j=1}^\ell, m, n, m \text{ and } \{ y_j \}_{j=1}^\ell \text{ as in the proof of } 4.18 \) we now proceed as follows:

Let \( \delta \) be a positive number with \( \delta < 1 - \max\{ ||L_j|| \mid j = 1, \ldots, \ell \} \);

then \( \Phi_1 := \{ \lambda \in \mathcal{C} \mathcal{M} \mid |\lambda| < 1 - \frac{\delta}{2} \} \)

\( \Phi_2 := \{ \lambda \in \mathcal{C} \mathcal{M} \mid |\lambda| \geq 1 - \delta \} \)

and for every \( x \in \mathcal{A} \) and \( i = 1, 2 \)

\( \mathcal{L}_{i,x} := \{ \nu \in \mathcal{H} \mid o(\nu, x) \subset \Phi_i \} \).

It will be clear that \( \mathcal{L}_{2,x} \supset \mathcal{L}_x \) for every \( x \in \mathcal{A} \), since \( \Phi_2 \supset \Phi_1 \);

and also that \( \mathcal{L}_{1,x} \in \Gamma \) and \( \mathcal{L}_{2,x} \in \Gamma \) since \( \Phi_1 \) and \( \Phi_2 \) are closed in \( \mathcal{C} \mathcal{M} \).

Thus \( \mathcal{L}_A \subset \mathcal{L}_A = \bigvee_{j=1}^\ell \mathcal{L}_{y_j} \supset \bigvee_{j=1}^\ell \mathcal{L}_{2,y_j} \).

We now define

\( \mathcal{L}_C := \mathcal{L}(m) \cap \mathcal{L}_B \) as in 4.18; hence \( \mathcal{L}_C \subset \mathcal{L}_B \) and \( \mathcal{L}_C \neq \emptyset \).

\( \mathcal{L}_D := \bigvee_{j=1}^\ell \mathcal{L}_{2,y_j} \); hence \( \mathcal{L}_A \subset \mathcal{L}_D \).

\( \mathcal{L}_E := \bigwedge_{j=1}^\ell \mathcal{L}_{1,y_j} \).

Properties i and iv are established by now; we have to prove the remaining ones: ii, iii, v and vi.
\[ \forall \sigma \in \Phi(A) \exists \kappa \in \Phi(A) \exists j \in \{1, \ldots, l\} \sigma(\kappa, y_j) \in \Phi \]

and \( n \in \mathcal{L}(m) \) implies
\[ \forall j = 1, \ldots, l \sigma(n, y_j) = \{ \xi_j \} \] and consequently
\[ \forall \sigma \in \Phi(n) \forall j = 1, \ldots, l \sigma(\rho, y_j) = \{ \xi_j \} \]

Since \( \forall j \xi_j \notin \Phi \), we can conclude that \( \mathcal{L}(m) \cap \mathcal{L}_D = \emptyset \).

Hence \( \mathcal{L}_0 \cap \mathcal{L}_D = \emptyset \).

iii: \( \mathcal{L}_A \cap \mathcal{L}_B = \left( \bigvee_{j=1}^{b} \mathcal{L}_{y_j} \right) \cap \left( \bigwedge_{i=1}^{l} \mathcal{L}_{1, y_i} \right) = \)

\[ = \bigvee_{j=1}^{b} \left( \mathcal{L}_{y_j} \cap \left( \bigwedge_{i=1}^{l} \mathcal{L}_{1, y_i} \right) \right) \]

by the \( \vee \)-distributivity of \( \Gamma \), and
\[ \mathcal{L}_A \cap \mathcal{L}_B \subseteq \bigvee_{j=1}^{b} \left[ \mathcal{L}_{y_j} \cap \mathcal{L}_{1, y_j} \right] \]

Since \( \mathcal{L}_{y_j} \cap \mathcal{L}_{1, y_j} = \{ m \in \mathcal{L}(m) \mid \sigma(m, y_j) \in \Phi \} \) and \( \Phi \cap \Phi = \emptyset \), we conclude that \( \mathcal{L}_A \cap \mathcal{L}_B = \emptyset \); but then also
\( \mathcal{L}_A \cap \mathcal{L}_B = \emptyset \), since \( \mathcal{L}_A \subseteq \mathcal{L}_B \).

iv: If \( m \in \mathcal{L}(m) \) then \( \forall j \sigma(m, y_j) \subseteq \sigma(m, y_j) \), whence
\[ \forall j \sigma(m, y_j) = \{ \xi_j \} \]; since \( \forall j \xi_j \notin \Phi \) we have
\[ \forall j \exists \mathcal{L}_1 y_j \] and consequently \( m \in \mathcal{L}_1 y_j = \mathcal{L}_E \).

This proves \( \mathcal{L}(m) \subseteq \mathcal{L}_E \) and, by the definition of \( \mathcal{L}_0 \), \( \mathcal{L}_0 \subseteq \mathcal{L}_E \).
vi: \mathcal{L}_D \lor \mathcal{L}_N = \left( \bigvee_{i=1}^{L} \mathcal{L}_{2, y_i} \right) \lor \left( \bigwedge_{j=1}^{L} \mathcal{L}_{1, y_j} \right) =

\bigwedge_{j=1}^{L} \left[ \mathcal{L}_{1, y_j} \lor \left( \bigvee_{i=1}^{L} \mathcal{L}_{2, y_i} \right) \right] =

\bigvee_{j=1}^{L} \left[ \mathcal{L}_{1, y_j} \land \mathcal{L}_{2, y_j} \right].

Putting for simplicity a make \mathcal{A} = \mathcal{L}_{1, y_j} \cup \mathcal{L}_{2, y_j} \text{ then}

\mathcal{A} = \{ \omega \in \mathcal{K} | \sigma(\omega, y_j) \in \phi_1 \text{ or } \sigma(\omega, y_j) \in \phi_2 \} \text{ and }

\mathcal{A}^* = \{ \omega \in \mathcal{K} | \exists k \in \mathcal{K}(q) \exists \mu \in \mathcal{K}(\mu) \sigma(\omega, y_j) \in \phi_1 \text{ or } \sigma(\omega, y_j) \in \phi_2 \};

since \phi_1 \cup \phi_2 = \phi_3 \text{ and for every } k \not\in \mathcal{K} \text{ there exists an } \omega \in \mathcal{K}(k) \text{ with the spectrum } \sigma(\omega, y_j) \text{ consisting of one single point, we get } \mathcal{A}^* = \mathcal{K} \text{; consequently } \mathcal{L}_D \lor \mathcal{L}_N = \mathcal{K}.

This concludes the proof of the theorem.
Chapter 5

On compactness

5.1. A topological space \( X \) is said to be compact if every collection \( \{ F_\alpha \}_{\alpha \in \Delta} \) of closed sets in \( X \) with the finite intersection property has a non-void intersection; accordingly we define:

A point-free topology \( L \) is said to be compact if every subset \( D \) of \( L \) has the property

\[
\left( \forall \quad \forall a \not\in v \right) \Rightarrow \left( \exists \quad a \not\in v \right).
\]

\( A \in F(D) \quad a \not\in A \quad a \not\in D \)

It is easily seen that compactness of \( L \) implies compactness of its companion topology (compare section 0.3) in \( L \).

The companion topology of the lattice \( \Gamma \) of clusters is, as will be shown in section 6 below, the Gelfand topology in the space \( \Omega \) of maximal ideals in the \( E_1 \)-algebra \( \mathcal{A} \). Since in this case compactness is a consequence of Tychonoff's theorem, and hence of the axiom of choice, it will be clear that compactness properties for \( \Gamma \) will be hard to derive.

This section features results yielding certain compactness properties, but they depend on the restricted axiom of choice. They are preceded by a few introductory remarks on topological spaces.

5.2. A topological space \( X \) is called first countable if the neighbourhood system of every point of \( X \) has a countable base. In
a first countable topological space $X$ a point $x$ is an accumulation point of the subset $Y$ if and only if there is a sequence in $Y \setminus \{x\}$ which converges to $x$.

5.3. If $\{X_v\}_{v \in N}$ is a family of spaces, then we denote by $X = \bigotimes_{v \in N} X_v$ the cartesian product of these spaces; if $N$ is not finite then $X = \bigotimes_{v \in N} X_v \neq \emptyset$ is equivalent to the restricted or unrestricted axiom of choice.

By $p_\mu (\mu \in N)$ we denote the projection $X = \bigotimes_{v \in N} X_v \to X_\mu$, defined by $p_\mu (\langle x_v \rangle_{v \in N}) := x_\mu$.

If every $X_v$ is a topological space, the "natural" topology for $X = \bigotimes_{v \in N} X_v$ is the weak topology for the family of functions $\{p_v\}_{v \in N}$ which we shall refer to as the "product topology"; it has as a subbase for the open sets $\pi_\mu^{-1}(U_\mu)$, $U_\mu$ open in $X_\mu$, $\forall \mu \in N$; the mappings $p_\mu$ are continuous and open.

If $N$ is countable then $X = \bigotimes_{v \in N} X_v$ is first countable if and only if $X_v$ is first countable for every $v \in N$.

5.4. If the topological spaces $X_j$ ($j \in \bar{N}_t$) are first countable and compact, then $X = \bigotimes_{j \in \bar{N}_t} X_j$ is compact.

Proof: If $X = \bigotimes_{j \in \bar{N}_t} X_j = \emptyset$ then its compactness is trivial.

If $\{\sigma_\beta\}_{\beta \in \mathcal{B}}$ is an open covering of $X = \bigotimes_{j \in \bar{N}_t} X_j$, it might happen that $\exists \beta_0 \in \mathcal{B}$ such that $\sigma_{\beta_0} \cap \bigotimes_{j \in \bar{N}_t} X_j$; in this case $\{\sigma_\beta\}_{\beta \neq \beta_0}$ is a finite subcovering of $\{\sigma_\beta\}_{\beta \neq \beta_0}$.

Suppose $X = \bigotimes_{j} X_j \neq \emptyset$ and $\forall \beta \in \mathcal{B}$, $\sigma_\beta \neq X_j$; consider the refinement $X = \bigotimes_{j} X_j \neq \emptyset$ and $\forall \beta \in \mathcal{B}$, consisting of the basic sets $W_\alpha$ which con-
stitute the $\mathfrak{C}_j$'s; hence $\{\mathfrak{C}_j\}_{j \in \mathbb{N}}$ is an open covering of $X_j \times X_j$ and $\gamma_\alpha \in (\mathfrak{C}_j) \neq X_j \times X_j$.

We denote the family of open sets in $X_j$ by $\mathfrak{C}_j$ and define $\gamma(\alpha) := \max \{ j \in \mathbb{N} \mid \exists \mathfrak{C}_j \subseteq \mathfrak{C}_j \setminus \{X_j \times X_j\} : \mathfrak{C}_j \subseteq \mathfrak{C}_j\}$. $\gamma_\alpha$, as a basic set, is a finite intersection of sets $\mathfrak{C}_j(U)$ with $U \subseteq \mathfrak{C}_j \setminus \{X_j \times X_j\}$; and since $\mathfrak{C}_j \neq X_j \times X_j$, this finite number is not zero; hence $\gamma(\alpha) \in \mathbb{N}$.

We will now prove $\exists n_0 \in \mathbb{N}$ $U_\gamma(\alpha) \subseteq n_0$ $\mathfrak{C}_j \times X_j$.

To this end we assume the opposite

$\forall n \in \mathbb{N}$ $\exists n_0 \in \mathbb{N}$ $U_\gamma(\alpha) < n_0$ $\mathfrak{C}_j \times X_j$.

As $X_j$ is first-countable and compact, $\{ y_n \}_{n \in \mathbb{N}}$ contains a monotonic subsequence $\{ y^*_n \}_{n \in \mathbb{N}}$ such that $\{ p_1 (y^*_n) \}_{n \in \mathbb{N}}$ converges to a point $\eta_1 \in X_1$; as $X_2$ is first-countable and compact, $\{ y^*_{n,n} \}$ contains a monotonic subsequence $\{ y^*_{n_{n},n} \}_{n_{n} \in \mathbb{N}}$ such that $\{ p_2 (y^*_{n,n}) \}_{n_{n} \in \mathbb{N}}$ converges to a point $\eta_2 \in X_2$; of course, $\{ p_1 (y^*_{n,n,n}) \}_{n_{n} \in \mathbb{N}}$ converges to $\eta_1$.

Proceeding in this way we construct the principle of induction sequences $\{ y_n^{(j)} \}_{n \in \mathbb{N}}$, $j \in \mathbb{N}$

with the properties

$\{ y_n^{(j+1)} \}_{n \in \mathbb{N}}$ is a monotonic subsequence of $\{ y_n^{(j)} \}_{n \in \mathbb{N}}$, $j \in \mathbb{N}$

and $\{ p_2 (y_n^{(j)} ) \}_{n \in \mathbb{N}}$ converges to a point $\eta_{j,i} \in X_j$; $j \in \mathbb{N}$, $i < j$.

Now consider the sequence $\{ y_n^{(j)} \}_{n \in \mathbb{N}}$.

For every $j \in \mathbb{N}$ $\{ p_j (y_{n,j}) \}_{n \in \mathbb{N}}$ is a subsequence of $\{ p_j (y_{n,j}^{(j)}) \}$, 55.
from which we infer that for every \( j \in \mathbb{N} \) \( \{ p_j(y^{(n)}) \} \) converges to \( \eta_j \).

Take \( y := \{ \eta_j \}_j \in \mathbb{N} \), then \( y \in \bigcap_j X_j \); consequently there exists an \( \alpha_0 \in A \) such that \( y \in \bigcap_j X_j \), hence \( \bigvee_j \eta_j \in p_j(\mathcal{W}_{\alpha_0}) \) and, since \( p_j \) is open, every \( p_j(\mathcal{W}_{\alpha_0}) \) is a neighbourhood of \( \eta_j \).

But \( n = \gamma(\alpha_0) \) implies by the definition of \( \gamma \),

\[
\gamma_n \not\subseteq \mathcal{W}_{\alpha_0}
\]

and consequently \( y^{(n)}_n \not\subseteq \mathcal{W}_{\alpha_0} \).

Since \( \mathcal{W}_{\alpha_0} \not\subseteq X_j \), there are finitely many indices, say \( j_1, \ldots, j_k \),

such that \( \mathcal{W}_{\alpha_0} = \bigcap_{i=1}^k p_{j_i}(U_i) \) with \( U_i \) open in \( X_{j_i} \) and \( U_i \not\subseteq X_{j_i} \); for those \( j_1, \ldots, j_k \), \( U_i = p_{j_i}(\mathcal{W}_{\alpha_0}) \), \( i = 1, \ldots, k \), and for at least one of these, say \( j_0 \), there are countably many elements of \( \{ y^{(n)} \} \)

such that \( p_{j_0}(y^{(n)}) \not\subseteq U_{j_0} = p_{j_0}(\mathcal{W}_{\alpha_0}) \). This contradicts the earlier observation that \( p_{j_0}(\mathcal{W}_{\alpha_0}) \) is a neighbourhood of the limit \( \eta_{j_0} \)

of the convergent sequence \( \{ p_{j_0}(y^{(n)}) \} \).

We conclude that

\[
\exists_{\alpha_0} \in \mathbb{N} \quad U_{\alpha_0} = X_j \quad \neg X_j
\]

as stated before.

Hence \( \{ \mathcal{W}_{\alpha} \}_{\gamma(\alpha)} \subseteq \mathcal{N} \) is a subcovering of \( \{ \mathcal{W}_{\alpha} \}_{\alpha \in A} \).

If \( p \) is the projection of \( X_j \) onto \( Y := \bigcap_{j=1}^n X_j \) then

\[
56 \quad \{ p(\mathcal{W}_{\alpha}) \}_{\gamma(\alpha)} \subseteq \mathcal{N}
\]
is an open covering of \( Y \) and since \( Y \) as a finite product of compact spaces is compact, there exist \( a_1, \ldots, a_m \)

with \( V_{i-1}, \ldots, m, \gamma(a_i) \subseteq \mu_0 \) and

\[
\bigcup_{i=1}^{m} p(p(\tilde{a_i})) = Y
\]

hence \( \bigcup_{i=1}^{m} p^{-1}(p(\tilde{a_i})) = \chi_j x_j \)

and since \( \gamma(a_i) \subseteq \mu_0 \) by definition implies \( p^{-1}(p(\tilde{a_i})) = \tilde{a_i} \),

\[
\bigcup_{i=1}^{m} \tilde{a_i} = \chi_j x_j
\]

If we now replace the \( \tilde{a_i} \)'s by the corresponding \( \tilde{\beta_i} \)'s from the original covering, we get

\[
\bigcup_{i=1}^{m} \tilde{\beta_i} = \chi_j x_j
\]

5.5. If \( \{ \alpha_i \}_{i \in I} \) is a collection of (not necessarily closed) ideals in the \( B_1 \)-algebra \( \mathcal{K} \), then

\[
\Sigma_{i \in I} \alpha_i := \{ f \in X | x_h | k \in F(I) \land \forall x \in X, x_h \in \alpha_k \}
\]

\( F(I) \) denoting the class of finite subsets of \( I \).

5.6. \( \Sigma_{i \in I} \alpha_i \) is an ideal in \( \mathcal{K} \), and consequently

\[
\Sigma_{i \in I} \alpha_i \in \mathcal{K} \cup \{ \mathcal{K} \}
\]

5.7. If \( \{ \mathcal{M}_i \}_{i \in N \in I} \in P(\mathcal{K}) \) and \( \forall i \in N \in I, \mathcal{M}_i \in \mathcal{M}(\mathcal{M}_i) \), then

\[
\Sigma_{i=1}^{\infty} \mathcal{M}_i = \bigcup_{i=1}^{\infty} \mathcal{M}_i \quad \text{and} \quad \Sigma_{i=1}^{\infty} \mathcal{M}_i \in \mathcal{K}
\]
Proof: The first part of the statement is trivial; since
\[ \forall i \in \mathbb{N} \setminus \mathcal{N}_1 \text{ and } e \notin \bigcup_{i=1}^{\infty} \mathcal{N}_i \; \text{; hence, in connection with the}
\] first part of the statement, 5.6 and 1.9
\[ \sum_{i=1}^{\infty} e_i \in \mathcal{N} \, . \]

5.8. If \( \forall i \in \mathbb{N} \) \( x_i \in \mathcal{K} \) and \( \sum_{i} \in \mathbb{N} \) \( x_i \mathcal{K} = \mathcal{K} \), then there is a
finite subset \( \{i_1, \ldots, i_k\} \) in \( \mathbb{N} \) with the property \( \sum_{j=1}^{k} x_{i_j} \mathcal{K} = \mathcal{K} \).

Proof: \( \sum_{i} \in \mathbb{N} \) \( x_i \mathcal{K} = \mathcal{K} \) implies \( e \in \sum_{i} \in \mathbb{N} \) \( x_i \mathcal{K} \); hence, by de-

5.9. If \( \forall i \in \mathbb{N} \) \( x_i \in \mathcal{K} \) and \( \sum_{i} \in \mathbb{N} \) \( x_i \mathcal{K} = \mathcal{K} \), then there exist

i: a number \( m \in \mathbb{N} \),
ii: a finite subset \( \{i_1, \ldots, i_m\} \) in \( \mathbb{N} \),
iii: positive numbers \( \delta_1, \ldots, \delta_m \)

with the property:

If \( \{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{C} \) and \( \forall j=1, \ldots, m \), \( |\lambda_j| < \delta_j \), then
\[ \sum_{j=1}^{m} (x_{i_j} - \lambda_j e) \mathcal{K} = \mathcal{K} \, . \]

Proof: The assumption yields by 5.8 the existence of a finite set
\( \{i_1, \ldots, i_m\} \) with \( \sum_{j=1}^{m} x_{i_j} \mathcal{K} = \mathcal{K} \); hence \( e = \sum_{j=1}^{m} x_{i_j} y_j \)
with \( \{y_1, \ldots, y_m\} \subset \mathcal{K} \). Let \( \delta_1, \ldots, \delta_m \) be chosen in such a way that
\[ \sum_{j=1}^{m} \delta_j \|y_j\| < 1 \, . \]
Then \( \forall j = 1, \ldots, m \ | \lambda_j | < \delta_j \) implies \( \| \sum_{j=1}^{m} \lambda_j y_j \| < 1 \), whence by 1.5
\[
\alpha = \sum_{j=1}^{m} \lambda_j y_j \in \mathcal{K} ;
\]
since \( e = \sum_{j} \lambda_j y_j = \sum_{j} (x_j - \lambda_j e) y_j \in \mathcal{K} (x_j - \lambda_j e) \mathcal{K} \), the ideal \( \sum_{j} (x_j - \lambda_j e) \mathcal{K} \) contains a regular element; therefore it equals \( \mathcal{K} \).

5.10. By \( \mathcal{N} \) we denote the set of sequences \( \{ x_n \}_{n \in \mathbb{N}} \) with \( \forall n \ x_n \in \mathcal{K} \); elements of \( \mathcal{N} \) will be denoted by symbols like \( x \).

By \( \mathcal{D} \) we denote the set of sequences \( \{ \xi_n \}_{n \in \mathbb{N}} \) with \( \forall n \xi_n \in \mathbb{C}^m \); elements of \( \mathcal{D} \) will be denoted by symbols like \( \xi \).

We now define for every \( \mathcal{K} \)
\[
T(\mathcal{K}) := \{ \xi \in \mathcal{D} | \sum_{i=1}^{\infty} (x_i - \xi_i e) \mathcal{K} \neq \mathcal{K} \} .
\]

5.11. If \( \xi \in T(\mathcal{K}) \) then \( \forall n \in \mathbb{N} \ \left[ \xi_n \in \sigma(x_n) \ \& \ |\xi_n| \ll \nu(x_n) \right] .
\]

Proof: \( \xi_n \notin \sigma(x_n) \) would imply that \( x_n - \xi_n e \in \mathcal{K} \) and \( (x_n - \xi_n e) \mathcal{K} = \mathcal{K} \); hence the result.

5.12. If \( \xi \in T(\mathcal{K}) \) then \( \forall n \in \mathbb{N} \ \sum_{i=1}^{\infty} (x_i + \eta_i e) \mathcal{K} \notin T(\mathcal{K} + \mathcal{K}e) \)
\( (\xi + \eta \mathcal{K} \ e \ \text{being defined in the obvious "pointwise" way}).

Proof: \( x_n - \xi_n e = (x_n + \eta_n e) - (\xi_n + \eta_n e) \); hence the result.

5.13. If \( \mathcal{K} \in \mathcal{N} \) then \( T(\mathcal{K}) \), as a subset of \( \mathcal{D} \), is closed in the product topology of \( \mathcal{D} \).
Proof:

i: Suppose that the zero sequence $\mathcal{O}$ satisfies $\mathcal{O} \not\in T(\mathcal{X})$, then by definition $\Sigma_{i=1}^{\infty} x_i \mathcal{H} = \mathcal{H}$; we apply 5.9 and get a subset 
\[ \{x_{i_1}, \ldots, x_{i_k}\} \] and positive numbers $\varepsilon_1, \ldots, \varepsilon_k$.

Now, if $j \in \{i_1, \ldots, i_k\}$, take $Q_j := \{\gamma \in C \mid |\gamma| < \varepsilon_j\}$
and otherwise $Q_j := C$;

then, with $Q := \bigvee_{j=1}^{\infty} Q_j$,

$Q$ is a neighbourhood of $\mathcal{O}$ in $D$ and
\[ \forall \xi \in Q \bigvee_{i=1}^{\infty} (x_1 - \varepsilon_i e) \mathcal{H} \geq \sum_{j=1}^{l} (x_j - \varepsilon_j e) \mathcal{H} = \mathcal{H} ; \]

hence, by definition, $Q \cap T(\mathcal{X}) = \emptyset$.

ii: If $\mathcal{O} \not\in T(\mathcal{X})$ then $\mathcal{O} \not\in T(\mathcal{X} - \mathcal{O})$ by virtue of 5.12; hence $\mathcal{O} \cap T(\mathcal{X} - \mathcal{O}) = \emptyset$ and $(\mathcal{O} + \mathcal{O}) \cap T(\mathcal{X}) = \emptyset$.

Since $(\mathcal{O} + \mathcal{O})$ is a neighbourhood of $\mathcal{O}$, we see that $D \setminus T(\mathcal{X})$ is open and consequently $T(\mathcal{X})$ is closed in $D$.

:5.14. If $x \in \mathcal{H}$ then $G(x) := \{\gamma \in C \mid |\gamma| < v(x)\}$;
if $x \in N$ then $G(x) := \bigvee_{j=1}^{\infty} G(x_j)$.

5.15. If $x \in N$ then $G(x)$, as a subset of $D$, is compact in the product topology of $D$.

Proof: $G(x)$, as a bounded and closed subset of $C$, is compact for every $x \in \mathcal{H}$; $C$, and therefore also $G(x)$, are first countable; hence $G(x)$ is compact according to 5.4.
5.16. If $x \in \mathcal{H}$ then $T(x)$ is compact in $B$.

Proof: If $x \in \mathcal{H}$ and $y \in T(x)$ then $y \in G(x)$ by 5.11; hence $T(x)$ as a closed subset of the compact set $G(x)$ is compact.

5.17. If $x \in \mathcal{K}$ then

$$\mathcal{L}_x = \left\{ y \in \mathcal{K} \mid \forall \lambda \in C_N \quad |\lambda| < 1 \Rightarrow \varphi_{\eta}(x - \lambda e) = \mathcal{K} \right\}.$$ 

Proof: The following statements are successively equivalent:

1. $x \in \mathcal{L}_x$
2. $o(x, x_1) \subset \mathcal{S}$
3. $\forall \lambda \in C_N \quad |\lambda| < 1 \Rightarrow \varphi_{\eta}(x - \lambda e) \in \mathcal{R}_0$
4. $\forall \lambda \in C_N \quad |\lambda| < 1 \Rightarrow \varphi_{\eta}(x - \lambda e) = \varphi_{\eta}(\mathcal{K})$
5. $\forall \lambda \in C_N \quad |\lambda| < 1 \Rightarrow \varphi_{\eta}(x - \lambda e) = \mathcal{K}$

hence the result.

5.18. If $x \in \mathcal{H}$ and $\sum_{i=1}^{\infty} x_i \mathcal{H} \neq \mathcal{K}$, then $m^* = \sum_{i=1}^{\infty} x_i \mathcal{H}$ has the property $\forall \eta \in \mathcal{H}, o(m^*, x_1) = \{0\}$.

Proof: $\forall \eta \in \mathcal{H}, 2\eta > x_1 \mathcal{H}$ and consequently $m^* \subset x_1 \mathcal{H}$; since $m^* \in \mathcal{H}$ by 1.9 we can apply 2.2 with $\eta = x_1 \mathcal{H}$ to show $\forall \eta \in \mathcal{H}, o(m^*, x_1) = \{0\}$.

Since $x_1 \in S$, $0 \in o(x_1)$; and now $o(m^*, x_1) = \{0\}$ by 2.3;

hence the result.
5.19. If \( \bar{a} \in \mathcal{X} \) and \( \mu \in \mathcal{K} \) then there exist

i: \( \nu \in \mathcal{K} \), and

ii: for every \( i \in \mathbb{N} \) a \( x_i \in \sigma(\nu, x_1) \)

with the property

\[ \forall i \in \mathbb{N} \left[ x_i - x_1 \in \nu \land \sigma(\nu, x_1) = \{x_i\} \right]. \]

Proof: We take \( \nu_1 := \mu \); let \( x_1 \in \sigma(\nu, x_1) \), then

\[ \nu_2 := \frac{\Phi_{\nu_1} \Phi_{\nu_1}}{\Phi_{\nu_2} \Phi_{\nu_2}} ((x_1 - x_2) \mathcal{R}) \]

has the properties of 2.9:

\[ \nu_2 \in \mathcal{K}(x_1), x_1 - x_2 \in \nu_2 \quad \text{and} \quad \sigma(\nu_2, x_1) = \{x_i\}. \]

Let \( x_2 \in \sigma(\nu_2, x_2) \), then \( \nu_3 := \frac{\Phi_{\nu_2} \Phi_{\nu_2}}{\Phi_{\nu_3} \Phi_{\nu_3}} ((x_2 - x_3) \mathcal{R}) \) and again

\[ \nu_3 \in \mathcal{K}(x_2), x_2 - x_3 \in \nu_3 \quad \text{and} \quad \sigma(\nu_3, x_2) = \{x_i\}. \]

Proceeding in this way we construct sequences \( \{x_i\}_{i \in \mathbb{N}} \) and

\( \{\nu_i\}_{i \in \mathbb{N}} \) with properties

\[ \forall i \in \mathbb{N} \quad x_i - x_1 \in \nu_i \]

\[ \forall i \in \mathbb{N} \quad \nu_{i+1} \in \mathcal{K}(x_i) \quad \text{and} \]

\[ \forall i > 2, j \leq i \quad \sigma(\nu_i, x_j) = \{x_j\}. \]

Take \( \nu := \bigcup_{i \in \mathbb{N}} \nu_i \); then \( \nu \in \mathcal{K} \) by 5.7 and

\[ \forall i \in \mathbb{N} \quad x_i - x_1 \in \nu \]

Since \( \forall i \mu \in \mathcal{K}(x_i) \), also \( \forall i \sigma(\mu, x_i) = \{x_i\} \).

5.20. We now choose an \( \bar{a} \in \mathcal{X} \) and define

\[ \mathcal{F}_j := \{ \gamma \in \mathcal{G}_m \mid 1 < |\gamma| < \nu(x_j) \} \]
\[ \mathbb{H}_j := (X_{i=1}^j \overline{a}_i) \times (X_{i=j+1}^{\infty} G(x_i)) , \]
\[ \mathbb{I}_j := T(x) \cap \mathbb{H}_j . \]

5.21. If \( x \in \mathbb{H}_j \) and \( \{ \mathcal{L}_j \}_{j \in \mathbb{N}_t} \) has the property
\[ \forall k \in \mathbb{N}_t \cap \bigcap_{i=1}^k \mathcal{L}_i \neq \emptyset, \text{ then } \bigcap_{j=1}^\infty \mathcal{L}_j \neq \emptyset . \]

Proof: If \( \bigcap_{i=1}^k \mathcal{L}_i \neq \emptyset \), then there exists a proper closed ideal \( \mathcal{M}_k \) with \( \forall_{i=1, \ldots, k} \mathcal{M}_k \in \mathcal{L}_i \); this implies
\[ \forall_{i=1, \ldots, k} \mathcal{M}_k \in \mathcal{L}_i \cap \mathcal{X}_1 . \]

Applying 5.19 to \( \mathcal{M}_k \) we find a proper closed ideal \( \mathcal{M}_k \) and an element \( \mathbb{I}_k \) of \( \mathcal{D} \) with the property
\[ \forall j \in \mathbb{N}_t \cap \mathcal{L}_i = \mathbb{I}_k \in \mathcal{M}_k \text{ implying } \sum_{j=1}^{\infty} (x_j - \mathcal{L}_j) \mathcal{C} \subset \mathcal{M}_k , \]
from which we infer \( \mathbb{I}_k \in T(x) \) by definition 5.10.

Another consequence of 5.19 is that \( \mathbb{I}_k \) can be chosen such that
\[ \mathcal{L}_k \cap \mathbb{X}_j = \mathbb{I}_k \cap \mathcal{X}_j , \]
but this set is a subset of the set \( \mathcal{H}_k \) defined in 5.20; we conclude that \( \mathbb{I}_k \in T(x) \cap \mathcal{H}_k = \mathcal{I}_k \).

Since \( \mathcal{H}_k \) and \( T(x) \) are compact by 5.4 and 5.16 respectively, \( \mathcal{I}_k \) is compact.

The sequence \( \{ \mathcal{I}_k \}_{k \in \mathbb{N}_t} \) is a descending sequence of non-vacuous compact sets in the Hausdorff-space \( \mathcal{D} \); hence
\[ \forall k \in \mathbb{N}_t \mathcal{I}_k \neq \emptyset \text{, let } \mathcal{L} = \bigcap_{k \in \mathbb{N}_t} \mathcal{I}_k \cap \bigcap_{k \in \mathbb{N}_t} \mathcal{I}_k = \{ x_j \}_{j \in \mathbb{N}_t} . \]

then \( \mathcal{L} \in T(x) \) and \( \forall_k \in \mathbb{N}_t \mathcal{L} \in \mathcal{H}_k \) being equivalent to 63
\[ \sum_{j \in N_t} (x_j - \xi_j e) \mathcal{A} \neq \mathcal{A} \quad \text{and} \quad \forall j \in N_t \quad 1 \leq |x_j| < v(x_j). \]

If \( \forall \varepsilon > 0 \exists \eta > 0 \) such that \( x_j \neq \xi_j e \), then by 5.18

\[ \forall j \in N_t \quad \sigma(\varepsilon, x_j) = \{ \xi_j \} \in \mathcal{F} \quad \text{and consequently} \]

\[ \forall j \in N_t \quad m \in \mathcal{L}_{x_j}, \quad \text{or} \quad m \in \bigcap_{j \in N_t} \mathcal{L}_{x_j}. \]

This completes the proof.

**5.22.** If \( \{ a_j \}_{j \in N_t} \) is a sequence in \( P \) with the property

\[ \forall j \in N_t \quad \bigwedge_{i=1}^{j} a_i \neq \emptyset, \quad \text{then} \quad \bigwedge_{j \in N_t} a_j \neq \emptyset. \]

Proof: Recall that \( \gamma_{a} := \bigcup_{x \in a} \gamma_{x} \) and \( \mathcal{L}_{a} = \gamma_{a}^\ast. \)

We first prove \( \forall j \in N_t \quad \bigwedge_{i=1}^{j} \gamma_{a_i} \neq \emptyset. \)

Take \( j \in N_t \) and \( m \in \bigcap_{i=1}^{j} \mathcal{L}_{a_i}. \)

Since \( m \in \mathcal{L}_{a_1} = \gamma_{a_1}^\ast \) we can choose an \( m_1 \in \mathcal{K}(m) \cap \gamma_{a_1} \) by 3.3; notice that \( \mathcal{K}(m_1) \subset \gamma_{a_1} \) by 3.10.1 and 3.2.

Since \( m \in \mathcal{L}_{a_2} \) and \( \mathcal{L}_{a_2} \in \Delta \subset \mathcal{E} \), \( m_1 \in \mathcal{L}_{a_2} \) and we can choose an \( m_2 \in \mathcal{K}(m_1) \cap \gamma_{a_2} \); now, \( m_2 \in \mathcal{K}(m_2) \cap \gamma_{a_1} \cap \gamma_{a_2} \) by the preceding remark. Continuation of this process shows that \( \mathcal{K}(m) \cap \bigwedge_{i=1}^{j} \gamma_{a_i} \neq \emptyset \) and in particular that \( \bigwedge_{i=1}^{j} \gamma_{a_i} \neq \emptyset. \)

as a consequence also \( \forall j \in N_t \quad a_j \neq \emptyset. \)

64 Next, we consider the cartesian product \( \prod_{j=1}^{\infty} a_j \); providing every
a_j with the discrete topology, we infer that \( \prod_{j=1}^{\infty} a_j \) is compact in its product topology by 5.4.

An element \( \alpha \) of \( \prod_{j=1}^{\infty} a_j \) can be considered a mapping on \( N_t \), assigning an element \( \alpha(j) \in a_j \) to every \( j \in N_t \).

Accordingly,

\[ V := \{ \alpha \in \prod_{j=1}^{\infty} a_j \mid \bigwedge_{j=1}^{\infty} \alpha(j) \neq \emptyset \} \]

and \( V_j := \{ \alpha \in \prod_{i=1}^{\infty} a_i \mid \bigwedge_{i=1}^{j} \alpha(i) \neq \emptyset \}, \ j \in N_t \).

\( \forall j \in N_t \) \( V_j \neq \emptyset \) as a consequence of the fact that \( \bigwedge_{i=1}^{\infty} a_i \neq \emptyset \), the latter set being a union of sets like \( \bigwedge_{i=1}^{j} a(i) \).

Since in the definition of \( V_j \) no conditions are imposed on \( \alpha(i) \) for \( i > j \), \( V_j \) is a closed set;

\( V_1 \supset V_2 \supset V_3 \ldots \) is obvious; consequently, \( \{ V_j \}_{j=1}^{\infty} \) having the finite intersection property and \( \prod_{j=1}^{\infty} a_j \) being compact, \( \bigcap_{j=1}^{\infty} V_j \neq \emptyset \).

But \( V = \bigcap_{j=1}^{\infty} V_j \) by 5.21; hence \( V \neq \emptyset \).

If \( \beta \in V \) then \( V \in N_t \), \( \bigwedge_{j=1}^{\infty} \alpha(j) c \gamma_{a_j} c \bigwedge_{j=1}^{\infty} a_j \) by 5.7; hence

\( \bigwedge_{j=1}^{\infty} \beta(j) \neq \emptyset \) implies \( \bigwedge_{j=1}^{\infty} \alpha(j) \neq \emptyset \). This completes the proof.

5.23. The selection of a sequence \( \{ \beta(j) \}_{j=1}^{\infty} \) from the sets \( a_j \), \( j \in N_t \), as in 5.22, is related to a theorem of König, known as the "Unendlichkeitslemma" (cf. [18], pp. 51 - 52); and, in fact, it is possible to give a proof of 5.22 based on König's lemma; but this
Theorem still depends on the restricted axiom of choice as well as 5.3, 5.21 and 5.22 do.

5.24. \( \{ \mathcal{L}_a \}_{a \in \mathcal{F}} \) can be considered as a base for \( \Gamma \) in the following sense: Every element \( \mathcal{L}_a \in \Gamma \) can be written as \( \bigwedge_{a \in \mathcal{A}} \mathcal{L}_a \)
this is simply the definition of \( \mathcal{L}_a \).

5.22 can be reformulated in terms of this base: \( \Gamma \), as a point-free topology, has a countably compact base.

To inform the reader about the position we hold at this moment, we mention two well-known theorems of general topology (cf. [17], p. 139):

If a topological space has a compact base, it is compact.

If a topological space has a compact subbase, it is compact. This result depends again on the axiom of choice.
Maximal ideals

6.1. If \( \mathfrak{m} \) is a maximal ideal in the \( \mathfrak{B} \)-algebra \( \mathfrak{A} \), then \( \mathfrak{m} \) is closed as a consequence of 1.9; we denote the set of maximal ideals in \( \mathfrak{A} \) by \( \mathfrak{M} \) and have \( \mathfrak{M} \in \mathfrak{A} \). In this chapter we suppose \( \mathfrak{M} \neq \emptyset \).

If \( x \in \mathfrak{A} \), \( \mathfrak{m} \in \mathfrak{M} \) and \( \lambda \in \sigma(\mathfrak{m}, x) \), then as a consequence of 2.8 we have

\[
\mathfrak{m} = \mathfrak{m} \cdot [(x - \lambda e) \mathfrak{A}]
\]

and \( \sigma(\mathfrak{m}, x) = \{\lambda\} \).

\( \sigma(\mathfrak{m}, x) \) consists of one complex number for every \( x \in \mathfrak{A} \), and this number is, as usual, denoted by \( \xi(\mathfrak{m}) \); \( \sigma(\mathfrak{m}, x) = \{\xi(\mathfrak{m})\} \).

By 2.10 and 2.11 we have

\[
\xi + \eta(\mathfrak{m}) = \xi(\mathfrak{m}) + \eta(\mathfrak{m})
\]

\[
\xi(\mathfrak{m}) = \xi(\mathfrak{m}) + \eta(\mathfrak{m})
\]

and

\[
\alpha \xi(\mathfrak{m}) = \alpha \xi(\mathfrak{m})
\]

whence \( x \rightarrow \xi(\mathfrak{m}) \) is a multiplicative linear functional of \( \mathfrak{A} \) onto \( \mathbb{C} \) for every \( \mathfrak{m} \in \mathfrak{M} \).

Since \( \|\eta_{\mathfrak{m}}(x - \xi(\mathfrak{m})e)\| = \|\eta_{\mathfrak{m}}(e)\| = 0 \) we have that

\[
\|\eta_{\mathfrak{m}}(x) - \xi(\mathfrak{m})\eta_{\mathfrak{m}}(e)\| = 0
\]
or \( \| \varphi_{\text{reg}}(x) \| = |\hat{x}(\text{reg})| \cdot \| \varphi_{\text{reg}}(e) \| = |\hat{x}(\text{reg})| \)
and moreover: the kernel of the homomorphism \( x \to \hat{x}(\text{reg}) \) is \( \text{reg} \).
This means that \( \varphi_{\text{reg}}(x) \to \hat{x}(\text{reg}) \) is an isometric isomorphism from \( \varphi_{\text{reg}}(\mathcal{A}) \) onto \( \mathcal{M} \) for every \( \text{reg} \in \pi \).

6.2. \( \mathcal{M} \in \mathcal{T}^* \).
(Recall that \( \mathcal{T}^* \) is the set of strong ideals in \( \mathcal{A} \), compare 1.30).
Proof: Since for every \( \text{reg} \in \pi \) \( \varphi_{\text{reg}}(\mathcal{A}) \) can be identified with \( \mathcal{M} \),
we have for every \( x \in \mathcal{A} \)
\[
\| \varphi_{\text{reg}}(x) \| \| = |\hat{x}(\text{reg})| \| = |\hat{x}(\text{reg})| \quad \quad \Rightarrow \quad \quad \hat{\mathcal{M}} = \{ \varphi_{\text{reg}}(x) | \hat{x}(\text{reg}) = 0 \}
\]
\( = \{ \varphi_{\text{reg}}(\sigma) \} \); this proves the theorem.

6.3. i: \( \mathcal{N} \subseteq \pi \).
ii: If \( S = \cup \mathcal{M} \) then \( \mathcal{N} = \cap \mathcal{M} \).
Proof:

i: Trivial consequence of 1.31 and 6.2.
ii: If \( x \not\in \mathcal{N} \) then by 1.27 (with \( m = \mathcal{N} \))
\[
3 \gamma \in \mathcal{M} \quad xy - e \in S;
\]
hence by assumption
\[
3 \mu \in \mathcal{N} \quad xy - e \in \text{reg}
\]
implying \( xy \not\in \text{reg} \) and \( x \not\in \text{reg} \).

6.4. If \( \text{reg} \in \mathcal{M} \) and if we denote the set of maximal ideals in
\( S_{\text{reg}}(\mathcal{A}) \) by \( \mathcal{M}_{\text{reg}} \) then (again assuming \( S_{\text{reg}} = \cup \mathcal{M}_{\text{reg}} \)) we have
\( \mathcal{X}_m = \mathcal{N}_m \); observe that \( S_m = \mathcal{N}_m \) implies that \( \mathcal{N}_m \notin \mathcal{X}_m \).

It is easy to check that \( \mathcal{X}_m = \mathcal{N}_m \mathcal{X}_m ) \cap \mathcal{N}_m \), hence

\[
\mathcal{X}_m = \mathcal{N}_m (\mathcal{N}_m \mathcal{X}_m) \cap \mathcal{N}_m .
\]

We now have by 1.33 that for every \( m \in \mathcal{X} \)

\[
m = \mathcal{N}_m (\mathcal{N}_m \mathcal{X}_m) \cap \mathcal{N}_m .
\]

This reveals that, with the definition of kernel as in [21] p.115, \( \mathcal{X} \) is the set of kernels of \( \mathcal{X}_m \).

6.5. If \( \{ A_{\lambda} \}_{\lambda \in \Lambda} \in P(Q) \) then

\[
i i: \bigcap_{\lambda \in \Lambda} (\mathcal{L}_{\lambda} \cap \mathcal{N}_m) = \bigg( \bigcap_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) \cap \mathcal{N}_m
\]

\[
i i: \bigcup_{\lambda \in \Lambda} (\mathcal{L}_{\lambda} \cap \mathcal{N}_m) = \bigg( \bigvee_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) \cap \mathcal{N}_m .
\]

Proof: The first part of the proof is trivial; in the second we have

\[
\bigg( \bigvee_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) \cap \mathcal{N}_m = \bigg( \bigcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) * (\bigcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda}) \cap \mathcal{N}_m
\]

by 3.7; and if, conversely, \( m \in \bigg( \bigvee_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) \cap \mathcal{N}_m \) then

\[
\forall \lambda \in \Lambda \impliedby \bigg( \bigcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) * (\bigcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda}) \cap \mathcal{N}_m
\]

which by definition 3.3, observing that

\[
m \in \mathcal{N}_m \] implies \( \mathcal{N}_m (m) = \{ m \} \), entails \( m \in \bigg( \bigcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda} \bigg) \); hence the result.

6.6. For every \( x \in \mathcal{X} \) we consider \( \mathcal{X} \) as a function from \( \mathcal{Y} \) into \( \mathcal{X}_m \). The weakest topology in \( \mathcal{Y} \) with the property that \( \mathcal{X} \) is continuous for every \( x \in \mathcal{X} \) is usually called the Gelfand topology.

We define \( \mathcal{Y} | \mathcal{N}_m = \{ \mathcal{L}_{\lambda} \cap \mathcal{N}_m \}_{\lambda \in \Lambda} \).
6.7. $\mathcal{F}_\mathcal{K}$ is the class of Gelfand closed sets in $\mathcal{K}$.

Proof: The class of closed sets has a subbase
$$\{x^\rightarrow(\phi) \mid x \in \mathcal{K}, \phi \text{ closed in } \mathcal{C}_\mathcal{K}\}.$$ 

Now
$$\mathcal{A}(\phi) = \{\mu \in \mathcal{K} \mid \check{x}(\mu) \subseteq \phi\} = \mathcal{K} \cap \{\mu \in \mathcal{K} \mid \sigma(\mu, x) \subseteq \phi\}$$

is an element of $\mathcal{F}_\mathcal{K}$ by 4.13.

As a consequence of this and 6.5 the Gelfand closed sets are contained in $\mathcal{F}_\mathcal{K}$.

The converse is also true: if $A \in \mathcal{Q}$ then
$$\mathcal{L}_A \cap \mathcal{K} = \left\{ \begin{array}{l}
\Lambda \in A \\
\bigcup \mathcal{L}_x \in \mathcal{A}
\end{array} \right\} \cap \mathcal{K} \text{ and again by 5.5}
$$

$$\mathcal{L}_A \cap \mathcal{K} = \bigcap_{x \in A} \left( \bigcup_{x \in A} \mathcal{L}_x \cap \mathcal{K} \right) = \bigcap_{x \in A} \left( \bigcup_{x \in A} \mathcal{L}_x \cap \mathcal{K} \right) = \bigcap_{x \in A} \left( \bigcup_{x \in A} x^\rightarrow(\phi) \right)$$

which, since $x \in \mathcal{F}(\mathcal{K})$, is a Gelfand closed set.

6.8. If $\forall \mu \in \mathcal{K} \check{x}(\mu) \subseteq \mathcal{K} \neq \emptyset$ then
$$\forall A \in \mathcal{Q} \forall B \in \mathcal{Q} \left[ \mathcal{L}_A \cap \mathcal{K} = \mathcal{L}_B \cap \mathcal{K} = [\mathcal{L}_A = \mathcal{L}_B] \right].$$

Proof: Suppose $\forall x \in \mathcal{C}_\mathcal{A}$; since $\mathcal{L}_A = \mathcal{L}_A^*$

$$\exists \mu \in \mathcal{K}(\sigma) \cap \mathcal{A} \neq \emptyset$$

implying by virtue of Zorn's lemma.
\[3 \in \mathcal{M}(\mu) \cap \mathcal{M} \setminus \mathcal{L} \setminus \mathcal{M} \quad \text{and} \quad \mu \notin \mathcal{L} \setminus \mathcal{M}
\]

whence \(\mu \notin \mathcal{L} \setminus \mathcal{M} \) and \(\mu \notin \mathcal{L} \setminus \mathcal{M} \).

Since \(\bar{\mathcal{L}} \in \Gamma \subseteq \mathbb{E}\), we see that

\[\mu \notin \mathcal{L} \setminus \mathcal{M} \]

hence \(\mathcal{L} \subseteq \mathcal{L} \setminus \mathcal{M} \) and analogously \(\mathcal{L} \supset \mathcal{L} \setminus \mathcal{M} \),

which proves the theorem.

### 6.9. If \(\mu \in \mathcal{M}, \mathcal{L} \subseteq \Gamma \) and \(\mu \in \mathcal{L}\), then \(\mathcal{L}(\mu) \subseteq \mathcal{L}\).

**Proof:** There is no loss of generality if we suppose \(\exists \sigma_i \in \mathcal{L} \neg \mathcal{L} = \mathcal{L} \setminus \mathcal{L} \mathcal{a}\); \(\mathcal{L} = \mathcal{L} \mathcal{a} \) is equivalent to \(\forall \sigma \in \mathcal{M}(\mu) \exists \sigma \in \mathcal{M}(\nu) \in \mathcal{L} \mathcal{a}\); since \(\mathcal{M}(\mu) = \{\nu\}, \nu \in \mathcal{L} \mathcal{a}\); implying \(\exists \sigma_0 \in \mathcal{L} \in \mathcal{L} \mathcal{a}\); hence \(\mathcal{L}(\mu) \subseteq \mathcal{L} \mathcal{a}\) by 4.15.iii and \(\mathcal{L}(\mu) \subseteq \mathcal{L} \mathcal{a} \mathcal{a}\).

### 6.10. If \(\mu \in \mathcal{M}\) then \(\mathcal{L}(\mu) \cap \mathcal{M} = \{\mu\}\).

**Proof:** \(\mu \in \mathcal{L}(\mu) \cap \mathcal{M}\) is obvious.

If \(\sigma \in \mathcal{M}\), then either \(\sigma = \mu\) or \(\exists x \in \mathcal{X} \setminus \mathcal{M}\); the latter case implies \(\mathcal{L}(\mu) = 0, \mathcal{L}(\nu) \neq 0\); hence \(\sigma(\sigma, x) \neq \sigma(\mu, x)\) and \(\sigma \notin \mathcal{L}(\mu)\). This completes the proof.

### 6.11. If \(\mu \in \mathcal{M}\), then by Zorn's lemma \(\mathcal{L}(\mu)\) is a minimal element of \(\Gamma\).

**Proof:** If \(\emptyset \neq \mathcal{L} \subseteq \mathcal{L}(\mu)\) then \(\exists \mathcal{M} \subseteq \mathcal{L} \setminus \mathcal{M}\); by Zorn's lemma \(\exists \sigma \in \mathcal{M}(\mu) \subseteq \mathcal{M}\), implying \(\sigma \in \mathcal{L}\) since \(\mathcal{L} \in \Gamma \subseteq \mathcal{E}\); hence
\[ \mathcal{C}(\sigma_1) \subseteq \mathcal{C} \text{ by 6.9 and } \mathcal{C}(\sigma_1) \subseteq \mathcal{L}(\sigma_2) \text{ by assumption; by virtue of 6.10 } \mathcal{L}(\sigma_1) = \mathcal{L}(\sigma_2) \text{, and consequently } \mathcal{L} = \mathcal{L}(\sigma_2); \text{ this proves that } \mathcal{L}(\sigma_2) \text{ is minimal.} \]

6.12. If \( \mathcal{L} \) is a minimal element of \( \Gamma \) then by Zorn's lemma 

\[ \exists m \in \mathcal{M} \mathcal{L} = \mathcal{L}(\sigma) \text{.} \]

Proof: \( \mathcal{L} \neq \emptyset \) by assumption, whence \( \exists m \in \mathcal{M} m \in \mathcal{L} \).

By Zorn's lemma \( \exists m \in \mathcal{M} \forall \mathcal{N}(\sigma), \) implying \( m \in \mathcal{L} \) and, by 6.9, \( \mathcal{L}(\sigma) \subseteq \mathcal{L} \); consequently \( \mathcal{L}(\sigma) = \mathcal{L} \) by assumption.

6.13. The Gelfand topology \( \Gamma | \mathcal{M} \) and the companion topology of \( \Gamma \) are homeomorphic by virtue of the axiom of choice.

Proof: The mapping \( \Theta : \mathcal{M} \to \Gamma \) defined by 

\[ \Theta(\mathcal{M}) = \mathcal{L}(\sigma) \text{ is injective by 6.10; } \Theta(\mathcal{M}) \text{ is the set of minimal elements of } \Gamma \text{ by 6.11 and 6.12.} \]

Since \( m \in \mathcal{L} \cap \mathcal{M} \) if and only if \( m \in \mathcal{M} \) and \( \mathcal{L}(\sigma) \subseteq \mathcal{L} \) we have 

\[ \Theta(\mathcal{L} \cap \mathcal{M}) = \{ \mathcal{L}(\sigma) \mid \mathcal{L}(\sigma) \subseteq \mathcal{L} \} \text{.} \]

The left-hand side of this equality is the \( \Theta \)-image of a Gelfand closed set in \( \mathcal{M} \) by virtue of 6.7, the right-hand side is a closed set in the companion topology; this explains that \( \Theta \) is a homeomorphism.

6.14. Assuming Zorn's lemma we have: If \( \exists j_{m,1}^x \) \( y_j \not\in \mathcal{A} \) for a set of elements \( y_1, \ldots, y_x \) of \( \mathcal{M} \), then \( \exists m \in \mathcal{M} \forall j_{m,1}, \ldots, j_{m,x} \mathcal{A} = 0 \).

Proof: If \( \exists j_{m,1} \) \( y_j \not\in \mathcal{A} \) then \( m \not\in j_{m,1} \mathcal{A} \), \( y_j \not\in \mathcal{A} \), \( y_j \in \mathcal{M} \) and,
by 4.17, \( A \bigcup_{j=1}^k \mathcal{L}(y_j, \mathcal{A}) \neq \emptyset \); hence \( \exists m \in \mathcal{A} \forall j=1, \ldots, k. m \in \mathcal{L}(y_j, \mathcal{A}) \).

Applying Zorn's lemma we get

\( \exists m \in \mathcal{M} \cap \mathcal{M}(m) \forall j=1, \ldots, k. m \in \mathcal{L}(y_j, \mathcal{A}) \)

equivalent to

\( \exists m \in \mathcal{M} \cap \mathcal{M}(m) \forall j=1, \ldots, k. \forall x \in \mathcal{A}. k(m) \in o(y_j, x) \).

Particularly

\( \forall j=1, \ldots, k. \forall x \in \mathcal{A}. k(m) \in o(y_j, x) = \{0\} . \)
Chapter 7

Examples

Let \( T \) be a set consisting of three elements \( a, b, c \); \( T \) is topologized with the discrete topology, and \( C(T) \) denotes the ring of continuous functions on \( T \), which turns into a \( \mathbb{B}_1 \) algebra if
\[
\| f \| := \max \{|f(a)|, |f(b)|, |f(c)|\}
\]
for every \( f \in C(T) \).

The set \( \mathcal{N} \) of proper closed ideals consists of
\[
\mathcal{N} := \{ f \in C(T)| f(a) = 0 \},
\]
\[
\mathcal{E} := \{ f \in C(T)| f(b) = 0 \},
\]
\[
\mathcal{L} := \{ f \in C(T)| f(c) = 0 \},
\]
\[
\mathcal{L}' := \mathcal{E} \cap \mathcal{L}, \quad \mathcal{L}'' := \mathcal{E} \cap \mathcal{L}' \quad \text{and} \quad \mathcal{L}''' := \mathcal{E} \cap \mathcal{L}''.
\]

If \( \mathcal{O} := \{ \mathcal{O}, \mathcal{O}' \} \) then \( \mathcal{O}^* = \{ \mathcal{O} \} \); this shows that 3.7 does not always hold in \( F(\mathcal{N}) \).

If \( \mathcal{O} := \{ \mathcal{O}, \mathcal{O}', \mathcal{O}'' \} \) then \( \mathcal{O}^* = \{ \mathcal{O}, \mathcal{O}', \mathcal{O}'' \} \),
showing that the converse of 3.7 is untrue.

If \( \mathcal{O} := \{ \mathcal{O} \} \) and \( \mathcal{E} := \{ \mathcal{E} \} \) then \( (\mathcal{O} \cup \mathcal{E})^* = \{ \mathcal{E} \} \) and
\( \mathcal{O}^* \cup \mathcal{E}^* = \{ \mathcal{O}, \mathcal{E} \} \); this explains why equality in 3.10.1 does not always hold, not even if \( \{ \mathcal{O}_\lambda \}_{\lambda} \in \mathcal{P}(\mathcal{A}) \).

\( x := (0,2,2) \) and \( y := (2,0,0) \) are elements of \( C(T) \):
\[
74 \quad \mathcal{L}_x = \{ \mathcal{O}', \mathcal{E}, \mathcal{L} \}, \quad \mathcal{L}_y = \{ \mathcal{O} \} \quad \text{and} \quad (\mathcal{L}_x \cup \mathcal{L}_y)^* = \mathcal{N};
\]
since $\mathcal{L}_x = \mathcal{L}_x^*$ and $\mathcal{L}_y = \mathcal{L}_y^*$, $\mathcal{L}_x^* \cup \mathcal{L}_y^* \neq (\mathcal{L}_x \cup \mathcal{L}_y)^*$; hence, in 3.10.4 equality even does not always hold if 
$\{\Theta_{\lambda} \}_{\lambda} \in \mathcal{P}(\mathcal{R})$. 
References


5. Bruin, N.G. de, W. van der Maidan; Notes on Gelfand's theory; to be published in Ind. Math.


7. ---; Set theory and the continuum hypothesis; Benjamin New York 1966.


11. --- ; Uber absolut konvergente trigonometrische
    Reihen und Integrale; Mat. Sb. 2(51) (1941) p. 51 - 66.

    Algebren; W&K Deutscher Verlag der Wissenschaften,
    Berlin 1964.

    Einführung der Topologie in die Menge der maximalen
    Ideale eines normierten Ringes; Mat. Sb. 9(51) (1941)

14. Gillman, L, M. Jerison; Rings of continuous functions; Van

15. Fraenkel, A.A, Y. Bar-Hillel; Foundations of set theory;

16. Hille, E, R.S. Phillips; Functional analysis and semi-groups;
    A.M.S. Coll. Publ. 1957.


18. König, D; Theorie der endlichen und unendlichen Graphen;

19. Loomis, L.H; An introduction to abstract harmonic analysis;

20. Menger, K; Topology without points; The Rice Institute
    Pamphlet 47 (1940) # 1, p. 80 - 107.

21. Rickart, C.E; General theory of Banach algebras; Van Nostr-
22. Riesz, F., Sz.-Nagy; Leçons d'Analyse fonctionnelle; Akadémiai Kiadó, Budapest 1952.

23. Szasz, G.; Einführung in die Verbandstheorie; Akadémiai Kiadó, Budapest 1962.


25. Waerden, B.L. van der; Algebra; Springer Verlag, Berlin 1966.


27. ---; The Fourier integral and certain of its applications; Dover Publ., New York 1959.
**Samenvatting**

In dit proefschrift wordt beschreven hoe men voor een kommutatieve Banach algebra met eenheids-element in de verzameling der echte gesloten idealen een klasse van deelverzamelingen kan aangeven die de eigenschappen heeft van een topologie; bij de beschrijving van deze klasse spelen de maximale idealen (waarvan de existentie op het gebruik van het keuze axioma berust) geen rol; dit wil zeggen dat een topologie is beschreven zonder dat van een puntverzameling wordt gebruik gemaakt en die daarom een "puntvrije topologie" kan worden genoemd.

Deze puntvrije topologie impliceert bij toepassing van het keuze-axioma achteraf een topologie voor de ruimte der maximale idealen zoals die door Gelfand in 1941 is beschreven.

De in dit proefschrift gebruikte methode is te danken aan de Brujin; het beginsel van een puntvrije topologie is reeds in 1928 door Menger beschreven, maar vond tot nog toe weinig weerklink.
Curriculum vitae

De schrijver van dit proefschrift, geboren op 23 november 1928 te Schiedam, bezocht aldaar de lagere school en behaalde aan de Christelijke hogere burgerschool te Vlaardingen in 1946 het B-diploma.


Van 1954 tot 1956 werd hij opgeleid tot en te werk gesteld als luchtvaartmeteoroog bij de Marine Luchtvaart Dienst.

In september 1956 werd hij tot wiskunde leraar aan de Christelijke h.b.s. "Charlois" te Rotterdam benoemd; vanaf augustus 1959 is hij als wetenschappelijk medewerker verbonden aan de onderafdeling wiskunde van de Technische Hogeschool te Eindhoven.
Stellingen

behorende bij het proefschrift van W. van der Meiden

1. Het vaak gemaakte onderscheid tussen "zuivere wiskunde" en "toegepaste wiskunde" is verouderd en als middel om onderdelen van de wiskunde te karakteriseren onbruikbaar.

2. In de ontwikkeling van sommige delen van de wiskunde zijn drie fasen te onderscheiden.
Ten eerste worden problemen opgelost die hetzij elementair wiskundig formuleerbaar of uit een andere discipline afkomstig zijn; dit is de heuristische fase.
Ten tweede worden bereikte resultaten geordend en in hun onderlinge samenhang (vaak vanuit een axiomatisch gezichtspunt) beschreven; dit is de beschrijvende fase.
Ten derde worden problemen geformuleerd die wijzen naar aanleiding van of in verband met de verkregen axiomatisering; dit is de abstracte fase.

3. Bij het onderwijs aan hen die wiskunde als hulpwetenschap willen gaan hanteren dient op de heuristische aspecten der wiskunde de nadruk te worden gelegd; het onderwijs aan aanstaande wiskundige ingenieurs behoort zo weinig mogelijk abstract te zijn.

4. Het onderwijs in beschrijvende meetkunde wordt door de belanghebbenden vaak, en niet ten onrechte, verdedigd met een beroep op de bevordering van het ruïnestelijk inzicht; bij de algebraïsering van het meetkundeonderwijs wordt deze functie op onvoldoende wijze overgenomen.
5. Het invoeren van zogenaamde "objectieve studietoetsen" (in plaats van proefweken en tentamens van klassieke makelij) wordt vaak verdedigd met een beroep op de grotere rechtvaardigheid van de op grond daarvan te nemen beslissingen. De woorden objectief en rechtvaardig bezitten dan evenwel slechts propagandistische betekenis.

6. De cirkels door de voetpunten van de loodlijnen, die uit een vast punt van een kegelsnede worden neergelaten op de zijden van driehoeken waarvan de hoekpunten worden bepaald door een derdegraads involutie op de kegelsnede, vormen een bundel.

7. De begrenste lineaire operatoren $B_n$ die op $C([0,1])$ volgens een van Bernstein afkomstig procedé worden gedefinieerd door

$$Vf(x) := \sum_{v=0}^{n} f\left(\frac{v}{n}\right) \cdot \left(\frac{x}{n}\right)^{v}(1-x)^{n-v}$$

hebben als spectrum \{1, \frac{n-1}{n}, \frac{(n-1)(n-2)}{n^2}, \ldots, \frac{(n-1)!}{n^{n-1}}\}; (n \in \mathbb{N})$.


$$\int_0^1 F(y)|x-y|^{-\alpha} \, dy = f(x), \quad 0 < x < 1$$

en

$$\int_0^1 F(y)|x-y|^{-\alpha} \text{sgn}(x-y) \, dy = f(x), \quad 0 < x < 1,$$

waarin $0 < \alpha < 1$ is, kunnen op meer elementaire wijze worden afgeleid met behulp van een door Peters (Comm. Pure Appl. Math. 16 (1963), 57 - 61) aangegeven methode; de alzins verkregen oplossingen van de eerste en de tweede integraalvergelijking blijken bovendien geldig te zijn voor $-1 < \alpha < 1, \alpha \neq 0$, respectievelijk $0 < \alpha < 2$. 
9. Men kan op het in het proefschrift in hoofdstuk 4 beschreven lattice $\mathcal{P}$ van clusters afbeeldingen in de klasse $\mathcal{P}(G)$ definiëren door

$$\mathcal{X}(\mathcal{L}) := \{ \lambda \in \mathcal{P}(G) | \exists \mu \in \mathcal{L} \text{ en } \lambda \in \mathcal{O}(\mu, x) \}$$

voor alle $x \in \mathcal{X}$ en voor alle $\mathcal{L} \in \mathcal{P}$.

Deze afbeeldingen hebben de volgende eigenschappen:

a: $\mathcal{X}(\mathcal{L}) = \bigcup_{\mu \in \mathcal{L}} \mathcal{O}(\mu, x)$

b: $\mathcal{X}(\vee_{\alpha} \mathcal{L}_{\alpha}) = \bigcup_{\alpha} \mathcal{X}(\mathcal{L}_{\alpha})$

c: $\mathcal{X}(\mathcal{O} \land \mathcal{L}) \subseteq \mathcal{X}(\mathcal{O}) \cap \mathcal{X}(\mathcal{L})$

d: $\mathcal{X}(\mathcal{F}) \subseteq \mathcal{X}(\mathcal{X}) \land \mathcal{X}(\mathcal{L})$

e: $\mathcal{X}(\mathcal{L}) \subseteq \mathcal{X}(\mathcal{X}) \cdot \mathcal{X}(\mathcal{L})$.

Als $\mathcal{X}$ op de gebruikelijke wijze wordt gedefinieerd (vergelijk 6.1 in het proefschrift) kan met behulp van het keuze-axioma worden bewezen:

f: $\tilde{\mathcal{X}}(\mathcal{L}) = \mathcal{X}(\mathcal{L} \cap \mathcal{M})$ en

g: $\bigvee_{\mu \in \mathcal{M}} \tilde{\mathcal{X}}(\mathcal{L} \cap \mathcal{M}) = \{ \mathcal{X}(\mathcal{M}) \}.$

De afbeeldingen $\mathcal{X}$ gaan dus bij restrictie tot $\mathcal{M}$ over in de continue functies $\tilde{\mathcal{X}}$ uit de theorie van Gelfand.

10. Uitdrukkingen zoals "this result is best-possible" en "this condition cannot be weakened" geven in wiskundige teksten vaak aanleiding tot verwarring. Veelal zijn dergelijke mededelingen niet voldoende gespecificeerd om betekenis te hebben.

11. Zolang de equivalentie van het (tot een aftelbare klasse) beperkte keuze-axioma en het onbeperkte keuze-axioma niet bewezen is dienen bij de toepassing van het keuze-axioma in bewijzen deze gevallen duidelijk onderscheiden te worden.
12. Het kerkelijk ambt van predikant heeft de neiging te ontsierden tot het beroep van (geestelijk) verzorgingstechnicus; het was na te bevelen dominoes te ontslaan van hun administratieve en organisatorische taken en deze over te dragen aan daartoe opleide, benoemde onderlingen.