SEMIGROUPS, INVARINACE AND TIME-INVARIANT LINEAR SYSTEMS

PROEFSCHRIFT

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CONTENTS

0 Introduction i

1 On locally convex topological vector spaces 1
   1.1 Seminorms and locally convex topologies 2
   1.2 Pre-F-spaces and F-spaces 8
   1.3 Projective and inductive limits of F-spaces 10

2 One-parameter (semi-)groups on sequentially complete topological vector spaces 19
   2.0 Scope of the main results 20
   2.1 Semigroups: General Theory 24
       2.1.1 Semigroups and flows 25
       2.1.2 The translation semigroup on \( C(\mathbb{R}^n, V) \) and its relation to arbitrary semigroups on \( V \) 28
       2.1.3 Integration and differentiation in \( C(\mathbb{R}^n, V) \) 31
       2.1.4 Polynomials in the Differentiation Operator 34
       2.1.5 Translation invariant operators 37
       2.1.6 \( \alpha \)-semigroups; invariance 41
   2.2 \( \alpha \)-groups, in summary 44
   2.3 \( \alpha \)-semigroups and \( \alpha \)-groups on strict LF-spaces 47

3 Convolution algebras and their ideals; Closed translation-invariant subspaces 55
   3.0 Convolution on \( \mathcal{D}_+(\mathbb{R}) \) and \( \mathcal{D}_-(\mathbb{R}) \) 56
   3.1 Convolution on \( \mathcal{D}(\mathbb{R}) \) and \( \mathcal{E}(\mathbb{R}) \) 60
       3.1.1 The spaces \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \) 60
       3.1.2 A convolution structure for \( \mathcal{E}(\mathbb{R}) \) 62
       3.1.3 Closed, translation-invariant subspaces of \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \) 67
   3.2 Convolution structures for \( \mathcal{D}_-(\mathbb{R}) \) and \( \mathcal{D}_-(\mathbb{R}) \) 71
       3.2.1 Convolution on \( \mathcal{D}_-(\mathbb{R}) \) 71
INTRODUCTION

The title of this thesis "Semigroups, Invariance and Time-invariant Linear Systems" refers to the key concepts of the study presented here.

For a proper understanding of the motives and ideas leading to the analysis described in the next chapters, we focus on the third key item in the title. Many people working in the field of mathematical system theory regard systems as mathematical models of real-world phenomena, formulated in terms of sets of equations. Well-known and often used in control theory is the classical state space model. In this model, the evolutions of states $x$ enforced by inputs $u$ and causing outputs $y$ of an input/output system $\Sigma$, is described by

$$
\begin{align*}
\dot{x}(t) &= A(x(t)) + B(u(t)) & t > 0, & x(0) = x_0, \\
y(t) &= C(x(t))
\end{align*}
$$

(0.1)

Here at each time-instance $t$, $x(t)$ is the state of $\Sigma$ at time $t$, $u(t)$ the input and $y(t)$ the output at time $t$. Furthermore, $A$, $B$ and $C$ are (linear) operators (often matrices) of appropriate dimensions. So heuristically, the input/output behaviour of the system $\Sigma$, described by the state space model (0.1), is

$$
y(t) = C e^{A t} x_0 + \int_0^t C e^{(t-r)A} B(u(r)) \, dr \quad (t > 0).
$$

(0.2)

In spite of being a natural starting point in control theory, this classical approach to systems in terms of input/output representations is not always possible. Given the external variables, it is not always possible to classify them as inputs and outputs. This was recognized by Willems (see [Will]) and therefore, he introduced the so-called "behavioural approach". In the behavioural approach to dynamical systems, one starts from the behaviour of a system, i.e. the collection of all possible time-evolutions of the external (measurable) variables, and not of one of its representations such as the state space model (0.1), where an internal state variable has to be introduced. For convenience and further reference, we recall the definition of dynamical system in the behavioural approach (cf. [Will])
Definition 0.1 A dynamical system $\Sigma$ is a triple $(T, \mathcal{W}, \mathcal{B})$, where $T$ is the time-axis, where $\mathcal{W}$, an abstract set, is the signal alphabet, and where $\mathcal{B}$, a subset of $\mathcal{W}^T$, is the behaviour of $\Sigma$.

Naturally, input/output systems $\Sigma$ fit in the behavioural set-up. To this extent, let $\Sigma$ be an input/output system. Let $\mathcal{W}_U$ be the set in which the inputs $u(t)$ to $\Sigma$ take their values at each time, and let $\mathcal{W}_Y$ be the set in which the outputs $y(t)$ take their values at each time. Let $U$ be the set of all possible input-evolutions and let $Y$ be the set of all possible output-evolutions. Then, the input/output system $\Sigma$ can be regarded as a mapping, say $f_\Sigma$, from $U$ into $Y$. So, the behavioural description of $\Sigma$ is the triple $(T, \mathcal{W}_U \times \mathcal{W}_Y, \mathcal{B})$, where the behaviour $\mathcal{B}$ is a subset of $U \times Y$ ($\subseteq \mathcal{W}_U^{T} \times \mathcal{W}_Y^{T}$), satisfying

$$(u, y) \in \mathcal{B} \iff f_\Sigma(u) = y.$$ 

From the behavioural point of view, representation forms for mathematical systems $\Sigma = (T, \mathcal{W}, \mathcal{B})$ are expressed in terms of intrinsic properties of their behaviours $\mathcal{B}$. For instance, a system $\Sigma = (T, \mathcal{W}, \mathcal{B})$ is linear if and only if its behaviour $\mathcal{B}$ is a linear set.

For discrete-time systems $\Sigma = (T, \mathcal{W}, \mathcal{B})$, where $T = \mathbb{Z}$ or $T = \mathbb{N}$, a complete axiomatic set-up has been developed for the behavioural approach using algebraic methods. In his thesis [Soe], Soethoudt studied the behavioural set-up for continuous-time systems $\Sigma = (T, \mathcal{W}, \mathcal{B})$, where $T = \mathbb{R}$. His study focussed on so-called AR-systems $\Sigma = (\mathbb{R}, \mathbb{R}^n, \mathcal{B})$, i.e. systems which can be represented by a collection of ordinary differential equations. Hence for each AR-system $\Sigma = (\mathbb{R}, \mathbb{R}^n, \mathcal{B})$, there exists a polynomial matrix $P$ such that

$$y \in \mathcal{B} \iff P \frac{d}{dt} y = 0. \quad (0.3)$$

Soethoudt encountered a typical problem for continuous-time systems, namely, the problem that in most mathematical models concerned with a continuous-time evolution, restrictions are imposed on the type of evolution. For instance, considering only $L^1(\mathbb{R})$-type of signals, one assumes some kind of energy restriction on the evolutions. This yields the paradox that each restriction or extension of the class of signals leads to a different type of system. In [Soe], systems are studied in which the external variables are continuous functions of time and where, as a consequence, the behaviours are subspaces of $C(\mathbb{R}, \mathbb{R}^n)$, the space of all continuous functions from the time-axis $\mathbb{R}$ into the signal alphabet $\mathbb{R}^n$. In [Soe], a solution is presented to the following problem: "What are necessary and sufficient conditions on behaviours $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^n)$ such that $\mathcal{B}$ can be described in terms of an AR-relation (cf. (0.3))?"

Since AR-systems are linear and time-invariant, they correspond to linear subsets of $C(\mathbb{R}, \mathbb{R}^n)$ satisfying

$$y \in \mathcal{B} \iff \forall_{s \in \mathbb{R}} \sigma_s y \in \mathcal{B},$$

where $(\sigma_t)_{t \in \mathbb{R}}$ denotes the translation group on $C(\mathbb{R}, \mathbb{R}^n)$, i.e.

$$(\sigma_t y)(s) := y(t + s) \quad (y \in C(\mathbb{R}, \mathbb{R}^n), s, t \in \mathbb{R}).$$
Hence, a behaviour of an AR-system is a \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \(C(\mathbb{R}, \mathbb{R}^n)\).

It is recognized in [Soe] that an assumption has to be made with considerable consequences; the natural topological structure of \(C(\mathbb{R}, \mathbb{R}^n)\) has to be involved in the definition of a continuous-time behaviour. This natural topology is the topology of uniform convergence on compact subsets of \(\mathbb{R}\). Equipped with this topology, \(C(\mathbb{R}, \mathbb{R}^n)\) is a Fréchet space.

There may be criticism on the use of topological methods in system theory. One may argue that in dealing with real-world problems, the use of topology leads to topological difficulties which may seem rather artificial. Despite this argument, the author believes that, in dealing with infinite-dimensional structures, it is inevitable and useful to invoke topological concepts. In all kinds of mathematical theories, one can see that solutions of mathematical problems are given not in purely algebraic terms (sums), but by a sequence of more and more accurate approximations (series). So, a topology or, at least, a concept of convergence is needed for a proper interpretation of these solutions. Moreover, when dealing with differential equations, the often used concept of weak-solution has indeed a topological nature, although it is not always recognized as such.

After this rather philosophical intermezzo, let us proceed with the mathematics. For the natural (Fréchet) topology of \(C(\mathbb{R}, \mathbb{R}^n)\), behaviours of AR-systems are closed \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspaces of \(C(\mathbb{R}, \mathbb{R}^n)\). The topology of \(C(\mathbb{R}, \mathbb{R}^n)\) has the additional property that the translation group \((\sigma_t)_{t \in \mathbb{R}}\) is a strongly continuous one-parameter group (briefly \(c_0\)-group), i.e. each of the operators \(\sigma_t\) is linear and continuous, satisfying \(\sigma_t \sigma_s = \sigma_{t+s}\), \(\sigma_0\) equals the identity mapping, and

\[
\lim_{t \to 0} \sigma_t y = y \quad (y \in C(\mathbb{R}, \mathbb{R}^n)).
\]

This explains our interest in one-parameter \(c_0\)-groups on Fréchet spaces and our interest in closed subspaces of a Fréchet space which are invariant under a \(c_0\)-group. With the intention to replace the Fréchet space \(C(\mathbb{R}, \mathbb{R}^n)\) in [Soe] by other Fréchet spaces, we introduce in Chapter 4 the class of translatable Fréchet spaces of \(\mathcal{L}(\mathbb{R})\)-type.

The second motivation for studying the functional analytic concepts, as presented in this thesis, is Yamamoto’s work ([Y1, Y2, Y3, Y4, Y5]) on the subject of realization theory for continuous-time input/output systems. Realization deals with the question whether an input/output system can be represented by a state space model. The realization problem for discrete-time systems has been solved completely by Kalman [Kal]. As for the discrete-time case, Yamamoto focus on systems with input and output signals with support bounded in the past, in fact, input/output systems with signals from \(L^2_{\text{loc (+)}}(\mathbb{R})\). The space \(L^2_{\text{loc (+)}}(\mathbb{R})\) consists of all locally square Lebesgue integrable functions \(y\) on \(\mathbb{R}\), for which there exists \(T \in \mathbb{R}\) such that \(\text{supp}(y) \subseteq [T, \infty)\). Considering input/output systems \(\Sigma\) which are linear and time-invariant, their input/output mappings \(f_\Sigma\) are linear mappings on \(L^2_{\text{loc (+)}}(\mathbb{R})\), satisfying

\[
\sigma_t \circ f_\Sigma = f_\Sigma \circ \sigma_t \quad (t \in \mathbb{R}).
\]

By demanding \(f_\Sigma\) to be continuous, Yamamoto also introduced the natural topological structure of \(L^2_{\text{loc (+)}}(\mathbb{R})\). The natural topological structure of \(L^2_{\text{loc (+)}}(\mathbb{R})\) is more involved.
than the one of $C(R, R^*)$; it is a strict LF-topology and not a Fréchet topology (see Chapter 1). Correspondingly, when equipped with this strict LF-topology, $L^2_{loc,+}(R)$ is a strict LF-space. With respect to the strict LF-topology, the translation group is a $\mathbb{R}$-group on $L^2_{loc,+}(R)$. In [Y1], the realization space, i.e. the collection of all possible state evolutions in the state space model, turns out to be a closed subspace of the Fréchet space $L^2_{loc}(R^+)$ which is invariant under the translation $\mathbb{R}$-semigroup $(\sigma_t)_{t \geq 0}$ on $L^2_{loc}(R^+)$. This explains our interest in one-parameter $\mathbb{R}$-groups on strict LF-spaces, our interest in continuous linear operators on strict LF-spaces which are invariant with respect to a $\mathbb{R}$-group and our interest in $\mathbb{R}$-semigroups on Fréchet spaces. With the intention to replace the strict LF-space $L^2_{loc,+}(R)$ in Yamamoto's work by other strict LF-spaces, in Chapter 4, we introduce the class of translatable strict LF-spaces of $D_+^*(R)$-type.

Summarizing, exploring the subjects of $\mathbb{R}$-group and $\mathbb{R}$-semigroup, of closed subspaces and continuous linear operators invariant under a $\mathbb{R}$-group, in the framework of Fréchet spaces and strict LF-spaces, has been the leading motive for writing this thesis. However, presenting an extensive study of these functional analytic subjects is not the only goal. We shall focus especially on those strict LF-spaces $V$ which are related to Yamamoto's signal space $L^2_{loc,+}(R)$, in the sense that translation invariant operators on $V$ have a similar form as translation invariant operators on $L^2_{loc,+}(R)$. To this extent, in Chapter 4, we introduce the class of translatable strict LF-spaces of $D_+^*(R)$-type, i.e. translation invariant subspaces $V$ of $D_+^*(R)$ with a strict LF-topology such that the translation group is a $\mathbb{R}$-group on $V$ and the $\mathbb{R}$-domain of the translation group on $V$ is $D_+^*(R)$.

Yamamoto restricted the attention to input/output systems $\Sigma$, whose input/output mapping $f_\Sigma$ is a convolution operator. In particular, these convolution operators are of the form

$$f_\Sigma(y) = \mu * y \quad (y \in L^2_{loc,+}(R)), \quad (0.4)$$

where $\mu$ is a Radon measure with support in $[0, \infty)$ and where $*$ denotes convolution in $D'_+(R)$. This measure $\mu$ is the impulse response of the system $\Sigma$. The convolution mapping (0.4) extends to a much larger class of function spaces. In particular, for each translatable strict LF-space $V$ of $D_+^*(R)$-type and each Radon measure with support in $[0, \infty)$, the mapping

$$y \in V \mapsto \mu * y, \quad (0.5)$$

is a continuous linear operator from $V$ into $V$, which is invariant with respect to the translation group on $V$. So, (0.5) can be viewed as an input/output mapping of an input/output system with signals in $V$. Thus, we obtain classes, say $C_\mu$, of input/output systems having the same form being generated by the same impulse response $\mu$, but each based on another signal space $V$.

Now, assuming that a state space realization for an input/output system with signals in $L^2_{loc,+}(R)$ is available by Yamamoto's set-up, the natural question arises whether a state space realization exists also for the other systems in the corresponding class $C_\mu$, i.e. replacing $L^2_{loc,+}(R)$ by another translatable strict LF-space of $D_+^*(R)$-type. And
INTRODUCTION

if so, the subsequent question arises whether these state space realizations are of the same form. We shall solve a part of this realization problem for all translatable strict LF-spaces of $D_+(\mathbb{R})$-type. In fact, we shall solve the so-called factorization problem. In case the factorization problem yields a finite-dimensional state space for one element of $C_p$, then the realization problem is solved completely for all elements of $C_p$.

Concluding, this thesis is about three main subjects, two with a functional analytic nature, and one with a system theoretical nature. As far as the functional analysis is concerned, first, we study $c_0$-semigroups and $C_0$-groups on sequentially complete locally convex topological vector spaces, such as Fréchet spaces and strict LF-spaces. Secondly, we introduce classes of Fréchet spaces and strict LF-spaces, so-called translatable spaces, on which the translation group is defined and deduce characterization results on translation invariant subspaces and operators. As far as the system theory is concerned, we consider the factorization problem, being a part of the realization problem, for translatable strict LF-spaces of $D_+(\mathbb{R})$-type.

We end this introduction with a description of the chapters of this monograph. Since a rather detailed introduction is attached to each chapter, each description of the chapters is kept short. At the beginning of the Chapters 2, 3 and 4, there is a short summary of the material needed in the subsequent chapters.

Chapter 1 is preliminary. Amongst others, we introduce the concepts of Fréchet space (briefly F-space), of strict inductive limit of Fréchet spaces (briefly strict LF-space), of weak topology and of locally equicontinuous set of operators. We give a brief introduction to the theory of locally convex topological vector spaces and focus on the case that the topology of a locally convex topological vector space is metrizable. Classical functional analytic results such as the Closed Graph Theorem and the Open Mapping Theorem are mentioned.

In Chapter 2, we focus on one-parameter $c_0$-groups and $C_0$-semigroups on sequentially complete locally convex topological vector spaces. This is not a new subject to literature. But, the existing theory on this topic is developed merely for the purpose of finding alternatives for the celebrated Hille-Yosida Theorem (see [Yoe], p.246). In particular, one wants to obtain necessary and sufficient conditions for operators such that these operators generate a $C_0$-semigroup (or $c_0$-group). The reader is referred to [Bab], [Dem], [Li], [Koz], [Ou] and [Wae].

Since we want to investigate invariance aspects of operators and of subspaces with respect to $c_0$-(semi-)groups, our intentions are different. It has turned out that there is need for new theory dealing with this type of invariance. This has resulted in the theory for $c_0$-semigroups as presented in Chapter 2. Meanwhile, we present a functional analytic calculus for the infinitesimal generator of a $c_0$-semigroup and prove that polynomials in this operator are closed when equipped with their natural domain. In a separate section, we study $c_0$-groups and $C_0$-semigroups on F-spaces and strict LF-spaces. The $C_0$-domain of the infinitesimal generator of a $c_0$-semigroup on an F-space is an F-space again. However, for a strict LF-space the problem how to find a suitable strict LF-topology for the $C_0$-domain is more involved. We suggest a natural solution to this problem.
In Chapter 3, we discuss basic types of convolution: distributional convolutions on the Schwartz spaces $\mathcal{D}'(\mathbb{R})$, $\mathcal{D}''(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$ are introduced. Our introduction differs from the classical one, as presented by Schwartz (see [Schw2]). Indeed, showing that these distribution spaces are in one-one correspondence with the continuous linear translation-invariant operators on $\mathcal{D}'(\mathbb{R})$, $\mathcal{D}''(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ respectively, product structures on $\mathcal{D}'(\mathbb{R})$, $\mathcal{D}''(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ are introduced. Similarly, the spaces of Radon measures $\mathcal{M}_+(\mathbb{R})$, $\mathcal{M}_-(\mathbb{R})$ and $\mathcal{M}_c(\mathbb{R})$ are treated and put in one-one correspondence with continuous linear translation-invariant operators on the spaces of continuous functions $C_-(\mathbb{R})$, $C_+(\mathbb{R})$ and $C(\mathbb{R})$.

In Chapter 4, we introduce three types of translatable spaces: translatable strict LF-spaces of $\mathcal{D}_c(\mathbb{R})$-type and of $\mathcal{D}(\mathbb{R})$-type, and translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type. For these classes of spaces, we characterize translation-invariant closed subspaces and translation-invariant (closed) linear operators. Additionally, for the class of translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type, we have found Kahane-Schwartz-type characterizations of translation-invariant closed subspaces in terms of exponential-polynomials.

Chapter 5 deals with the factorization problem for continuous-time systems. Imposing mild conditions on translatable strict LF-spaces $V$ of $\mathcal{D}_c(\mathbb{R})$-type, we prove that the factorization problem can be solved in Yamamoto’s sense replacing $J^2_{\text{max}}(\mathbb{R})$ by $V$. Adopting Yamamoto’s concept of pseudo-rational approximately coprime systems, we present a condition, which yields that the state space resulting from a factorization is the closed linear span of exponential polynomials for any $V$.

For convenience, we summarise in Appendix A all function spaces, distribution spaces and measures spaces that are used, without further introduction, in this thesis.
ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

In this thesis, topological concepts like Fréchet spaces (F-spaces) and strict inductive limits of Fréchet spaces (strict LF-spaces) are frequently used. In fact, we are interested especially in closed subspaces of, and closed operators on these types of topological vector spaces.

In this chapter, a brief introduction into the concept of locally convex topological vector spaces is given (§1.1). In section §1.2 and §1.3, we focus on two classes of locally convex topological vector spaces concepts. The first class consists of pre-Fréchet spaces and their complete versions, the so-called Fréchet spaces. The second class consists of strict inductive limits of pre-Fréchet spaces and their complete versions, strict inductive limits of Fréchet spaces. For each of these two classes, we investigate various topological concepts, such as continuous or closed linear operators, graph topologies, completion, relative, and quotient topology. Furthermore, we mention the existence of classical functional analytic theorems such as the Closed Graph Theorem and the Open Mapping Theorem for these classes of topological vector spaces.

This chapter is not intended to be complete. For non-specialists in this area of mathematics, it gives a short introduction into the field of Fréchet spaces and strict LF-spaces, which occur frequently in this thesis. For the specialists in this area, the interesting points are the emphasize laid on the families of seminorms generating such topologies (also in the case of strict LF-spaces!), the discussion of continuous linear operators on non-complete spaces such as pre-F-spaces and strict pre-LF-spaces and, finally, the fact that the completion of a strict inductive limit of pre-F-spaces is indeed a strict LF-space. However, for a more transparent introduction of these kind of topological vector spaces, we refer to the monograph of Conway [Con]. Other standard references are the monographs of Köthe [Köhl], Schaefer [Sch], Robertson and Robertson [R-R] and Treves [Tre].
1.1 SEMINORMS AND LOCALLY CONVEX TOPOLOGIES

In this section, we discuss the concept of *locally convex topological vector space*. Emphasize is laid especially on the point of view that a locally convex topology is generated by a family of seminorms. Furthermore, topological manipulations as completion, relative topology, quotient topology and graph topology pass in review.

A topological vector space endowed with a linear topology $\mathcal{T}$ is called *locally convex* if $\mathcal{T}$ is Hausdorff and for every null-neighbourhood $W \in \mathcal{T}$, there exists $U \in \mathcal{T}$ with $U \subseteq W$ such that

\[
\forall_{\lambda \in [0,1]} \left\{ \lambda U \subseteq U \right\} \quad (U \text{ is balanced}),
\forall_{\lambda > 0} \left\{ \lambda x + (1 - \lambda) y \in U \right\} \quad (U \text{ is convex}),
\forall_{\varepsilon > 0} \exists_{\lambda > 0} \left\{ \lambda^{-1} x \in U \right\} \quad (U \text{ is absorbing}).
\]

All locally convex topological vector spaces over the real or complex field arise in the following way. Let $V$ be a vector space, and let $\Pi$ be a family of seminorms on $V$. Let $\mathcal{T}_\Pi$ be the topology on $V$ in which the convex sets

\[ x_0 + U_{p,\varepsilon} := x_0 + \{ x \in V \mid p(x) < \varepsilon \} \quad (\varepsilon > 0, p \in \Pi, x_0 \in V), \tag{1.1} \]

form a subbasis. Thus a subset $U$ of $V$ is open if and only if for every $x_0 \in U$ there are $p_1, \ldots, p_n \in \Pi$ and $\varepsilon > 0$ such that $x_0 + \bigcap_{i=1}^n U_{p_i,\varepsilon} \subseteq U$. We say that the family of seminorms $\Pi$ *generates* the topology $\mathcal{T}_\Pi$ on $V$. The topology $\mathcal{T}_\Pi$ is linear on $V$, i.e. the algebraic operations addition and scalar multiplication are continuous with respect to $\mathcal{T}_\Pi$. Therefore, $(V, \mathcal{T}_\Pi)$ is a topological vector space. The topology is Hausdorff if and only if the family of seminorms $\Pi$ is *separating* on $V$, i.e. $x = 0 \Rightarrow \forall_{p \in \Pi} p(x) = 0$.

Summarizing:

**Theorem 1.1** Let $V$ be a vector space and let $\Pi$ be a separating set of seminorms on $V$. Let $\mathcal{T}_\Pi$ be the linear topology on $V$ generated by $\Pi$ (cf. (1.1)). Then, $(V, \mathcal{T}_\Pi)$ is an Hausdorff, locally convex topological vector space. Conversely, every locally convex topology is generated by a set of seminorms $\Pi$.

**Proof.**
The first assertion is readily checked. The proof of the converse statement is based on so-called *Minkowski-functions* or *gauges* (see [Con]).

We have to be cautious in the relation between a locally convex topology and a generating family of seminorms. Scaling and combining the generating seminorms generates the same topology, so the relation is not one-to-one.

**Definition 1.2** A set of seminorms $\Pi$ on a vector space $V$ is called *directed*, if for each $p, q \in \Pi$, there is an $r \in \Pi$ and a $C > 0$ such that $p \leq C \cdot r$ and $q \leq C \cdot r$.

We restrict ourselves in this thesis to locally convex topological vector spaces with a directed set of generating seminorms.
1.1. Seminorms and Locally Convex Topologies

Assumption 1.3 Every locally convex topological vector space \((V, T)\) in this thesis has a directed set of seminorms \(\Pi\) generating its topology \(T\).

The notion of convergence of a net in a locally convex topological vector space \((V, T_0)\) can be expressed in terms of the family \(\Pi\) of generating seminorms. The net \((x_\lambda)_{\lambda \in D}\) in \(V\) converges to \(x \in V\) if and only if \(\lim_{\mu} p(x_\lambda - x) = 0\) for all \(p \in \Pi\). A net \((y_\lambda)_{\lambda \in D}\) is called a Cauchy net if \(\lim_{\mu} p(y_\lambda - y_\mu) = 0\) for all \(p \in \Pi\). A locally convex topological vector space \(V\) is called complete if every Cauchy net is convergent within \(V\). Similarly, the concepts of convergence of a sequence, Cauchy sequence can be introduced, replacing the net \((x_\lambda)_{\lambda \in D}\) by the sequence \((x_n)_{n \in \mathbb{N}}\). A locally convex topological vector space \(V\) is called sequentially complete if every Cauchy sequence is convergent within \(V\).

As for normed spaces, the notion of continuity of linear mappings on locally convex topological vector spaces can be translated into terms of each family of seminorms generating the locally convex topology. We quote the following lemma.

Lemma 1.4 Let \((V, T_0)\) and \((W, T_1)\) be locally convex topological vector spaces whose topologies are generated by the directed families of seminorms \(\Pi\) and \(\Gamma\), respectively. Then

i. a seminorm \(p\) is continuous on \((V, T_0)\) if and only if \(q \in \Pi\) and \(C > 0\) exist such that \(p(x) \leq C \cdot q(x)\) for all \(x \in V\).

ii. A linear mapping \(L: (V, T_0) \rightarrow (W, T_1)\) is continuous if and only if for each seminorm \(q \in \Gamma\) the seminorm \(x \mapsto q(Lx)\) is continuous on \((V, T_0)\).

iii. A set of linear mappings \(\{L_i: (V, T_0) \rightarrow (W, T_1) \mid i \in I\}\) is equicontinuous if and only if for each seminorm \(q \in \Gamma\) the seminorm \(x \mapsto \sup_{i \in I} q(L_i x)\) is continuous on \((V, T_0)\).

Given a locally convex topological vector space \((V, T_0)\), its topological dual, denoted by \((V, T_0)^\prime\), consists of all continuous linear functionals on \((V, T_0)\). Applying Lemma 1.4 yields that a linear functional \(F: V \rightarrow \mathbb{C}\) is continuous on \((V, T_0)\) if and only if \(C > 0\) and \(q \in \Pi\) exist such that for all \(x \in V\),

\[ |F(x)| \leq C \cdot q(x). \tag{1.2} \]

Next, we mention some consequences of the classical Hahn-Banach Theorem for locally convex topological vector spaces.

Lemma 1.5 Let \((V, T_0)\) be a non-trivial locally convex topological vector space and let \(M\) be a subspace of \(V\). Then the following statements hold.

- \((V, T_0)^\prime\) is non-trivial.
- Let \(F\) a linear functional on \(M\) which is continuous with respect to the relative \(V\)-topology (see 1.20)). Then \(F_{\text{ext}} \in (V, T_0)^\prime\) exists such that \(F_{\text{ext}}|_M = F\).
Let $x \in V$ be such that $x \notin \overline{M}$, i.e., the closure of $M$. Then $F \in (V, T_v)$ exists such that $F|_M = 0$ and $F(x) = 1$.

If $M$ is a subspace of $(V, T_v)$, its polar $M^\circ$ equals the collection

$$M^\circ = \{ F \in (V, T_v) : F(x) = 0 \text{ for all } x \in M \}.$$  \hspace{1cm} \text{(1.3)}

The polar $M^\circ$ of $M$ is a subspace of $(V, T_v)^\prime$. Similarly, we define for a subspace $N$ of $(V, T_v)^\prime$ the subspace $N^\circ$ of $V$ by

$$N^\circ = \{ x \in V | F(x) = 0 \text{ for all } F \in N \}.$$  \hspace{1cm} \text{(1.4)}

Notice that we did not equip $(V, T_v)^\prime$ with a topology, so $N^\circ$ is, strictly spoken, not the polar of $N$. In fact, $N^\circ$ is the polar of $N$ only if $(V, T_v)$ is reflexive, i.e., $(V, T_v)^\prime$ is a topological vector space satisfying $V = ((V, T_v)^\prime)^\prime$. Nevertheless, we call $N^\circ$ the polar of $N$.

Taking the polar of a polar of a subspace $M$ of $V$, an application of Hahn-Banach’s Theorem yields

$$(M^\circ)^\circ = \overline{M},$$  \hspace{1cm} \text{(1.5)}

where the closure is in $(V, T_v)$-sense. Hence the bipolar of a closed subspace $M$ of $(V, T_v)$ is $M$ itself.

A concept closely related to the concept of (topological) dual is the purely algebraic concept of dual system.

**Definition 1.6** Let $V$ and $W$ be vector spaces over the same field (real or complex). Then $V$ and $W$ are said to be in duality by means of a bilinear form $\langle \ldots \rangle$ if

i. $\langle x_0, y \rangle = 0$ for all $y \in W$ implies $x_0 = 0$,

ii. $\langle x, y_0 \rangle = 0$ for all $x \in V$ implies $y_0 = 0$.

We call the triple $(V, W, \langle \ldots \rangle)$ a dual system. It is customary to write $\langle V, W \rangle$ instead of $(V, W, \langle \ldots \rangle)$.

Obviously, if $V$ and $W$ are in duality, then also $W$ and $V$ are in duality. Furthermore, every locally convex topological vector space $(V, T_v)$ and its topological dual form a canonical dual system defining the bilinear form $\langle \ldots \rangle$ on $(V, T_v) \times (V, T_v)^\prime$ by

$$\langle x, F \rangle := F(x) \quad (x \in V, F \in (V, T_v)^\prime).$$

An example of two vector spaces that are in duality, but not each others topological duals are the locally convex topological vector spaces $D(\mathbb{R})$ and $D_+(\mathbb{R})$. In fact, defining the bilinear form $\langle \ldots \rangle$ on $D(\mathbb{R}) \times D_+(\mathbb{R})$ by

$$\langle \phi, \psi \rangle := \int_{\mathbb{R}} \phi(t) \psi(t) \, dt,$$

$\langle D(\mathbb{R}), D_+(\mathbb{R}) \rangle$ forms a dual system. Although $D(\mathbb{R})$ and $D_+(\mathbb{R})$ are no topological duals of one another, the mapping $\langle \ldots \rangle$ is continuous in both arguments, so that $D_+(\mathbb{R}) \subseteq D^\prime(\mathbb{R})$ and $D(\mathbb{R}) \subseteq D_+^\prime(\mathbb{R})$.

Although the concept of dual system is purely algebraic, it induces the so-called weak topology.
1.1. SEMINORMS AND LOCALLY CONVEX TOPOLOGIES

Definition 1.7 Let $V$ and $W$ be vector spaces over the same field ($\mathbb{R}$ or $\mathbb{C}$) and let $\langle V, W \rangle$ be a dual system. Then the weak topology on $V$ induced by $W$, denoted as $\sigma(V, W)$, is the locally convex topology generated by the seminorms $\{p_y \mid y \in W\}$ with for each $y \in W$

$$p_y(x) := |\langle x, y \rangle| \quad (x \in V).$$

Corollary 1.8 The topology $\sigma(V, W)$ is the coarsest locally convex topology on $V$ such that the linear functional

$$f \in V \mapsto \langle x, f \rangle,$$

is continuous for each $y \in W$.

Recalling that any locally convex topological vector space $(V, T_V)$ is involved in the duality $\langle V, (V, T_V)^* \rangle$, we observe that the weak topology $\sigma(V, (V, T_V)^*)$ on $V$ is coarser than $T_V$. It is customary to call the weak topology $\sigma((V, T_V)^*, V)$ on $(V, T_V)^*$ the weak-star topology.

Lemma 1.9 Let $(V, T_V)$ be a locally convex topological vector space, then $V$ is the dual of the vector space $(V, T_V)^*$ endowed with the weak-star topology $\sigma((V, T_V)^*, V)$.

Proof.
See, for instance [Con], p.125.

Besides the weak-star topology on the dual of a locally convex topological vector space, often the so-called strong topology is used.

Definition 1.10 Let $(V, T_V)$ be a locally convex topological vector space. A subset $U$ of $V$ is called $T_V$-bounded if for each continuous seminorm $p$ on $(V, T_V)$

$$\sup_{x \in U} p(x) < \infty.$$

Definition 1.11 Let $(V, T_V)$ be a locally convex topological vector space with dual $(V, T_V)^*$. A net $(F)_{\alpha \in N}$ is said to converge strongly to zero if it converges uniformly to zero on every bounded subset in $(V, T_V)$.

Obviously, the weak-star topology on the dual $(V, T_V)^*$ is coarser than the strong topology on $(V, T_V)^*$. Sometimes, convergence of sequences coincide for both topologies (cf. [Tre], p.358).

Lemma 1.12 For the dual spaces $\mathcal{D}'(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$ of the spaces $\mathcal{E}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ (see Appendix A), a sequence $(G_n)_{n \in \mathbb{N}}$ converges weak-star-sense if and only if $(G_n)_{n \in P}$ converges strongly.

An interesting result for a polar set is the following.
Lemma 1.13 Let $M$ be a closed subspace of the locally convex topological vector space $(V, T_0)$. Then $M^*$ is closed in $(V, T_0)'$ endowed with the weak-star topology.

An important example of a weak (star) topology is given in the case of the distribution space $\mathcal{D}'(\mathbb{R})$.

Example 1.14 Let $\mathcal{D}'(\mathbb{R})$ be the set of all continuous, linear functionals on $\mathcal{D}(\mathbb{R})$. The (bilinear) mapping $\langle \cdot, \cdot \rangle$ on $\mathcal{D}'(\mathbb{R}) \times \mathcal{D}(\mathbb{R})$

$\langle L, x \rangle := L(x)$ \quad ($L \in \mathcal{D}'(\mathbb{R})$, $x \in \mathcal{D}(\mathbb{R})$),

induces a dual system $\langle \mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R}) \rangle$. So, the weak-star topology $\sigma(\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ on $\mathcal{D}'(\mathbb{R})$ is the locally convex topology on $\mathcal{D}'(\mathbb{R})$ generated by the seminorms $p_x \mid x \in \mathcal{D}(\mathbb{R}) \rangle$ with for each $x \in \mathcal{D}(\mathbb{R})$:

$p_x(L) := |\langle L, x \rangle| = |L(x)|$. \quad ($L \in \mathcal{D}'(\mathbb{R})$).

The space $\mathcal{D}(\mathbb{R})$ is the dual of the topological vector space $\mathcal{D}'(\mathbb{R})$ with weak-star topology.

Next, we consider closed linear operators.

Definition 1.15 Let $V$ and $W$ be locally convex topological vector spaces. Then the mapping $L : V \to W$ with domain $\text{Dom}(L)$ is closed if one of the following equivalent statements holds true.

i. The set $\{(x, Lx) \mid x \in \text{Dom}(L)\}$ is closed in $V \times W$ with product topology.

ii. Let $\{x_\nu\}_{\nu \in I}$ be a net in $V$. Then $x_\nu \to x$ and $Lx_\nu \to y \Rightarrow x \in \text{Dom}(L)$ and $Lx = y$.

If the set $\{(x, Lx) \mid x \in \text{Dom}(L)\}$ is sequentially closed in $V \times W$ or equivalently (ii) holds true only for sequences, then $L$ is called sequentially closed.

In case of a closed linear mapping on a locally convex topological vector space its domain can be topologized as a locally convex topological vector space, also.

Definition 1.16 Let $(V, T_0)$ be a locally convex topological vector space whose topology is generated by the family of seminorms $\Pi$. Let $L$ be a closed linear operator on $V$ with domain $\text{Dom}(L)$. Then the seminorms $\{p + p \circ L \mid p \in \Pi\}$ are separating and generate a locally convex topology on $\text{Dom}(L)$. This topology is called the graph topology of $L$.

Proposition 1.17 Let $(V, T_0)$ and $L$ be as in Definition 1.16 (sequentially) closed linear operator on $V$ with domain $\text{Dom}(L)$. If $(V, T_0)$ is (sequentially) complete, then $\text{Dom}(L)$ equipped with the graph topology is (sequentially) complete.

An important notion, frequently used in this thesis, is the notion of continuously embedded.
1.1. SEMINORMS AND LOCALLY CONVEX TOPOLOGIES

**Definition 1.18** Let \((V,T)\) and \((W,\Theta)\) be locally convex topological vector spaces. By writing

\[ (V,T) \hookrightarrow (W,\Theta) \]

we mean that there is a continuous linear injection from \((V,T)\) into \((W,\Theta)\). We say \((V,T)\) is continuously embedded in \((W,\Theta)\).

In case \(V = W\), we call the topology \(T\) on \(V\) finer (or stronger) than \(\Theta\), or equivalently \(\Theta\) coarser (or weaker) than \(T\) on \(V\).

Next, we discuss the concept of completion. From [Kōd1], §18.4, we quote the following result.

**Theorem 1.19** For every locally convex topological vector space \((V,T_{\alpha})\), there exists a, up-to topological isomorphism unique, smallest complete locally convex topological vector space \((\overline{V},\overline{T}_{\alpha})\) in which \((V,T_{\alpha})\) is embedded. If the topology \(T_{\alpha}\) is generated by the collection seminorms \(\{p_{v} | v \in D\}\), then \(\overline{T}_{\alpha}\) is generated by the collection \(\{\overline{p}_{v} | v \in D\}\), where \(\overline{p}_{v}\) is the uniquely determined continuous extension from \(p_{v}\) to \(\overline{V}\).

The locally convex topological vector space \((\overline{V},\overline{T}_{\alpha})\) is called the completion of \((V,T_{\alpha})\).

We end this section discussing the subjects of relative topologies and quotient topologies.

Let \(M\) be a closed subspace of some complete locally convex topological vector space \(V\). Then, the relative topology for \(M\) is the coarsest locally convex topology on \(M\) such that the canonical inclusion of \(M\) into \(V\) is continuous.

**Definition 1.20** Let \(M\) be a closed subspace of some locally convex topological vector space \((V,T_{\alpha})\) with topology generated by the seminorms \(\Pi\). Then the relative (or induced) topology on \(M\) is the locally convex topology generated by the seminorms \(\{p_{M} | p \in \Pi\}\).

For a closed subspace \(M\) of some complete locally convex topological vector space \(V\), we consider the quotient space \(V/M\) consisting of all cosets

\[ x + M \quad (x \in V) \]

The quotient topology for \(V/M\) is the finest locally convex topology on \(V/M\) such that the quotient mapping \(\phi : V \rightarrow V/M\), defined as

\[ \phi(x) := x + M \quad (x \in M) \]

is continuous.

**Definition 1.21** Let \(M\) be a closed subspace of some locally convex topological vector space \((V,T_{\alpha})\) with topology generated by the directed family of seminorms \(\Pi\). The quotient topology of \(V/M\) is the locally convex topology generated by the seminorms \(\{\widetilde{p} | p \in \Pi\}\), defined as

\[ \widetilde{p}(x + M) := \inf_{y \in M} p(x + y) \]

where \(x \in V\) and \(p \in \Pi\).
Remark 1.21.1 The directness of the generating family of seminorms II is essential in Definition 1.21. If II is not directed, then the topology generated by the above seminorms is not necessarily the quotient topology in the classical sense. We refer to [Köt1] and Bourbaki [Bou2].

1.2 PRE-F-SPACES AND F-SPACES

Some locally convex topologies are generated by an uncountable number of seminorms. However, many of the locally convex topological vector spaces used in this thesis have a topology generated by a countable number of seminorms. We call such a topological vector space a pre-Frèchet space, and if the space is also complete, we call it a Frèchet space. So, in the case of (pre-)Frèchet spaces the 0-subbasis can be taken countable. Pre-Frèchet spaces are considerably more structured than arbitrary locally convex topological vector spaces. Amongst others, they are metrizable. The converse is true also; every metrizable locally convex topological vector space is a pre-Frèchet space. Furthermore, classical results in functional analysis such as the Closed Graph Theorem and Open Mapping Theorem have their versions in the context of Frèchet spaces.

Definition 1.22 Let \((V, T_0)\) be a locally convex topological vector space. If \(T_0\) is generated by a countable number of seminorms we call \((V, T_0)\) a pre-Frèchet space, or briefly pre-F-space. If, in addition, \((V, T_0)\) is (sequentially) complete then we call \((V, T_0)\) a Frèchet space, or briefly F-space.

Remark 1.22.1 Some authors distinguish Frèchet spaces from F-spaces. From Rudin [Rud2] p.8, we recall that an F-space is a complete metrizable topological vector space and a Frèchet space is a locally convex F-space. Since we consider locally convex topological vector spaces only, the use of F-space instead of the more correct terminology Frèchet space leads to no confusion.

Each normed space is an example of a pre-F-space, where its topology is generated by one (semi-)norm. Examples of F-spaces are all Banach spaces, and the spaces \(C(\mathbb{R})\), \(B_1(\mathbb{R})\) and \(L_\infty(\mathbb{R})\) (see Appendix A).

Remark 1.22.2 In order to distinguish the class of pre-F-spaces from the more general class of locally convex topological vector spaces, we use, in this thesis, the symbol \(F\) to designate a pre-F-space, and we use the symbol \(V\) for an arbitrary locally convex topological vector space.

Remark 1.22.3 For each pre-Frèchet space \(F\), we can always choose an ordered countable generating set of seminorms \((p_n)_{n \in \mathbb{N}}\), i.e. \(p_n(x) \leq p_{n+1}(x)\) for all \(x \in F, n \in \mathbb{N}\). To this extent, if \((q_n)_{n \in \mathbb{N}}\) generates the topology \(T_0\) on \(F\), then the family of ordered seminorms \((p_n := \sum_{i=1}^{n} q_i)_{n \in \mathbb{N}}\) generates the same topology. From now on, unless explicitly mentioned otherwise, we assume generating sets of seminorms for a pre-Frèchet space to be ordered.
1.2. Pre-F-spaces and F-spaces

Pre-F-spaces are always metrizable. In fact, if \((p_n)_{n \in \mathbb{N}}\) is a generating family of semi-norms for the pre-Frèchet space \((F, T_0)\), then, for instance, the mapping \(d : F \times F \to [0, \infty)\) defined by

\[
d(x, y) := \sum_{n=1}^{\infty} \frac{2^{-n} p_n(x - y)}{1 + p_n(x - y)}
\]

defines a metric on \(F\) generating the topology \(T_0\).

We return to the concepts introduced in the previous section.

Lemma 1.23 The completion of a pre-F-space is an F-space.

For the concepts of relative topology and quotient topology for (pre-) F-spaces we have the following permanence properties.

Proposition 1.24 Let \(F\) be a pre-F-space and let \(M\) be a closed subspace of \(F\). Then the following statements hold true.

- The space \(M\) equipped with relative topology is a pre-F-space. If \(F\) is an F-space, then \(M\) with relative topology is an F-space.
- The space \(F/M\) equipped with quotient topology is a pre-F-space. If \(F\) is an F-space, then \(F/M\) is an F-space.

For the graph topology in F-spaces we obtain the following result.

Proposition 1.25 Let \(L\) be a closed operator on the F-space \(F\) with domain \(\operatorname{Dom}(L)\), then \(\operatorname{Dom}(L)\) endowed with graph-topology is an F-space.

A classical result on F-spaces is the Open Mapping Theorem.

Theorem 1.26 (Open Mapping Theorem) Let \(L\) be a continuous linear surjection from an F-space \(F\) onto an F-space \(G\). Then \(L\) is open, i.e. for each open subset \(U\) of \(F\), its image \(L(U)\) is open.

Corollary 1.27 Every continuous linear bijection from an F-space \(F\) onto an F-space \(G\) is an homeomorphism.

Next, we consider continuous linear mappings on pre-F-spaces. Since every pre-F-space is metrizable, sequential continuity equals continuity. We quote the following classical result from [Tre], Proposition 8.5.

Proposition 1.28 Let \(F\) be a pre-F-space and \(V\) a locally convex topological vector space. Then a linear mapping \(L\) from \(F\) into \(V\) is continuous if and only if it is sequentially continuous.

For F-spaces, Proposition 1.28 can be extended with the so-called Closed Graph Theorem.
Theorem 1.29 (Closed Graph Theorem). Let $F$ and $G$ be $F$-spaces and let $L$ be an everywhere defined linear mapping from $F$ into $G$. Then the following statements are equivalent.

i. The mapping $L$ is continuous.

ii. The mapping $L$ is sequentially continuous.

iii. The mapping $L$ is closed.

iv. The graph of $L$ is sequentially closed in $F \times G$ with respect to the product topology.

Proof.
The equivalence of (i) and (ii) is an immediate consequence of Proposition 1.28. The equivalence of (i) and (iii) can be read as the Closed Graph Theorem for $F$-spaces (see Yosida [Yos], p.79). The equivalence of (iii) and (iv) is due to the equivalence of the notion closed subspace and sequentially closed subspace in case of metrizable topological vector spaces (see [Tre], Chapter 8).

1.3 PROJECTIVE AND INDUCTIVE LIMITS OF $F$-SPACES

In this section we focus on (countable) projective, and (countable) strict inductive limits of (pre-) Fréchet spaces. A (countable) projective limit of (pre-) $F$-spaces is again a (pre-) $F$-space. A strict inductive limit of (pre-) $F$-spaces is generally not metrizable, however it will be shown that its (locally convex) topology is connected closely to a countable collection of seminorms. Most results we mentioned for (pre-) $F$-spaces have their equivalents in the case of a strict inductive limit of (pre-) $F$-spaces.

First, we discuss the concept of a (countable) projective limit of (pre-) $F$-spaces. Suppose the pre-$F$-spaces $(F_i, T_i), i \in \mathbb{N}$, satisfy

$$
\ldots \hookrightarrow (F_3, T_3) \hookrightarrow (F_2, T_2) \hookrightarrow (F_1, T_1).
$$

(1.6)

Then, the sequence $\left( (F_n, T_n) \right)_{n \in \mathbb{N}}$ is called a (countable) left-sided chain of pre-$F$-spaces. Similarly, right-sided and two-sided chains of pre-$F$-spaces are introduced.

Left-sided chains of locally convex topological vector spaces $\left( (F_i, T_i) \right)_{i \in \mathbb{N}}$ invoke the natural question whether there exists a "largest" topological vector space $(F, T)$, which is continuously embedded in every $(F_i, T_i)$. If $\tilde{F}_i$ denotes the embedding of $F_i$ in $F_1$, then the problem is equivalent to searching for a suitable topology for the intersection of the vector spaces $\tilde{F}_i$ ($i \in \mathbb{N}$). Therefore, we may consider only the case that $F_{i+1} \subseteq F_i$ for all $i \in \mathbb{N}$.

Definition 1.30 Let $\left( (F_n, T_n) \right)_{n \in \mathbb{N}}$ be a left-sided chain of (pre-) $F$-spaces with the property that $F_{n+1} \subseteq F_n$ for each $n \in \mathbb{N}$. Then the projective limit of $\left( (F_n, T_n) \right)_{n \in \mathbb{N}}$, denoted by $\text{proj}_n F_n$ is the vector space $P = \cap_{n \in \mathbb{N}} F_n$ endowed with the so-called projective limit topology $T_{\text{proj}}$, i.e. the coarsest locally convex topology for $P$ such that each inclusion of $P$ into $F_n$ is continuous.
1.3. PROJECTIVE AND INDUCTIVE LIMITS OF F-SPACES

Proposition 1.31 Let \( \text{proj}_n \, F_n \) a projective limit of pre-\( F \)-spaces, where \( F_{n+1} \subseteq F_n \) for each \( n \in N \). Let each topology \( T_n \) be generated by the family of seminorms \( \Pi^{(n)} = \{ p_{m}^{(n)} \mid m \in N \} \) on \( F_n \), \( n \in N \). Then the projective limit topology \( T_{\text{proj}} \) of \( \text{proj}_n \, F_n \) is generated by the separating set of seminorms \( \bigcup_{n \in N} \Pi^{(n)} = \{ p_{m}^{(n)} \mid n, m \in N \} \), where each seminorm is restricted to \( F \).

Corollary 1.32 The projective limit of a (countable!) left-sided chain of (pre-) \( F \)-spaces is a (pre-) \( F \)-space.

An example of a projective limit of \( F \)-spaces is the \( F \)-space \( E(R) \). In particular, we have \( E(R) = \text{proj}_n \, C^n(R) \).

Dual to the concept of projective limit of a left-sided chain is a (countable) inductive limit of a right-sided chain.

Definition 1.33 Let \( (F_n)_{n \in N} \) be a family of (pre-) Fréchet spaces with associated topologies \( T_n \). Then this collection is called a (countable) strict inductive system of (pre-) Fréchet spaces if

- \( F_n \) is a closed subspace of \( F_{n+1} \), \( n \in N \),
- the topology \( T_n \) for \( F_n \) equals the relative topology \( T_{n+1} \mid_{F_n} \).

The vector space \( \bigcup_{n=1}^{\infty} F_n \) is called the strict inductive limit of the system \( (F_n)_{n \in N} \).

The sequence \( (F_n)_{n \in N} \) is called a defining sequence for the strict inductive limit.

This leaves us the problem how to find a "natural" locally convex topology for a strict inductive limit, the so-called strict inductive limit topology. In literature often the more general case of an inductive limit and inductive limit topology is treated. In case of inductive limit the conditions of Definition 1.33 are weakened, requiring that \( F_n \) is a (proper) subspace of \( F_{n+1} \) and requiring that the topology \( T_n \) is finer than relative topology \( T_{n+1} \mid F_n \). Although the strict inductive limit topology as presented below is nothing else but an inductive limit topology (see for example [Con], II.4.5), the given definition is not in the usual form. In fact, the definition of a strict inductive limit topology as presented below is a consequence of the usual definition. We restrict our attention to strict inductive systems of (pre-) \( F \)-spaces.

Definition 1.34 Let \( (F_n)_{n \in N} \) be a strict inductive system of (pre-) Fréchet spaces. Then the strict inductive limit topology for \( \bigcup_{n=1}^{\infty} F_n \) is the Hausdorff locally convex topology generated by the collection of all seminorms \( p \) on \( \bigcup_{n=1}^{\infty} F_n \) with the property that \( p \mid_{F_n} \) is continuous on \( F_n \) for every \( n \in N \). The vector space \( \bigcup_{n=1}^{\infty} F_n \) endowed with its strict inductive limit topology is denoted by \( \text{ind} \, F_n \). We call a (countable) strict inductive limit \( \text{ind} \, F_n \) of (pre-) Fréchet spaces a strict (pre-) \( LF \)-space.

Remark 1.34.1 Notice that the case that all \( F_n \) are equal may occur, contrary to most literature, where often it is assumed that each \( F_n \) is a proper closed subspace of \( F_{n+1} \). Hence, in this set-up a (pre-) \( F \)-space is a strict (pre-) \( LF \)-space also. So all properties we derive for strict \( LF \)-spaces are true in particular for \( F \)-spaces.
Remark 1.34.2 The Hausdorff property of strict (pre-) LF-spaces is one of the great advantages of considering strict LF-spaces instead of the more general concept of (pre-) LF-spaces. Furthermore, we mention that every (countable) strict LF-space is complete (see [Ködl] §19.5). A strict (pre-) LF-space is metrizable iff it is a (pre-) F-space. In fact, if a strict LF-space $\text{ind } F_n$ is metrizable, then in $\mathcal{N}$ exists such that $F_k = F_n$ for all $k \geq m$ (see for example Floret [Flo], p.209).

Examples of strict LF-spaces are $D(R), D_+(R), L^{1}_{\text{loc}}(R)$ and $L^{1}_{\text{comp}}(R)$ (see Appendix A).

Although in general, a strict pre-LF-space is generated by an uncountable family of seminorms, its topology is directly related to a countable family of seminorms. To illustrate this, consider a strict pre-LF-space $\text{ind } F_n$, where for each $n \in \mathcal{N}$ the separating set of seminorms $\Pi_n = \{p_{n,k} \mid k \in \mathcal{N}\}$ generates the Fréchet-topology of $F_n$. So, a seminorm $p$ is continuous on $\text{ind } F_n$ if and only if

$$\forall n \in \mathcal{N} \exists (n) \in \mathcal{N} \exists (n) \in \mathcal{N} \forall e \in F_n \{ p(x) \leq C(n) \cdot p_{n,(n)}(x) \}. $$

Instead of the countable collection $\{\Pi_n \mid n \in \mathcal{N}\}$ of families of generating seminorms, we can construct a single countable, separating family of seminorms on $\text{ind } F_n$ strongly related to (but not generating) the strict inductive limit topology. To do so, we apply the following Lemma (cf. [Con], Corollary 5.15).

Lemma 1.35 Let $(F_n)_{n \in \mathcal{N}}$ be a strict inductive system of (pre-) F-spaces and let $n_0 \in \mathcal{N}$ be fixed. Then for every continuous seminorm $p$ on $F_{n_0}$, there exists a continuous seminorm $\tilde{p}$ on $\text{ind } F_n$ such that $\tilde{p}|_{F_{n_0}} = p$.

Now extend for every $k, n \in \mathcal{N}$, $p_{n,k}$ to a continuous seminorm $\tilde{p}_{n,k}$ on $\text{ind } F_n$. Define for every $k \in \mathcal{N}$ the seminorm $p_k$ by

$$p_k := \max\{\tilde{p}_{1,k}, \ldots, \tilde{p}_{k,k}\},$$

then the family $\tilde{\Pi} := \{p_k \mid k \in \mathcal{N}\}$ induces the topology of $\text{ind } F_n$, in the sense that $\tilde{\Pi}|_{F_k} := \{p_k \mid p_k \in \mathcal{N}\}$ generates the topology of $F_n$ for each $n \in \mathcal{N}$. We have obtained the following result:

Proposition 1.36 Let $\text{ind } F_n$ be a strict (pre-) LF-space. Then there exists a countable separating family of (ordered) continuous seminorms $\tilde{\Pi}$ on $\text{ind } F_n$ such that for each $n \in \mathcal{N}$ the seminorms $\{p_n \mid p \in \tilde{\Pi}\}$ generate the topology of $F_n$.

Again we emphasize that, in general, it is not true that $\tilde{\Pi}$ generates the inductive limit topology of $\text{ind } F_n$. Of course, $\tilde{\Pi}$ does generate a pre-Fréchet topology on $\bigcup_{n=1}^{\infty} F_n$. The reader should be warned, the constructed pre-Fréchet topology $\tilde{\Pi}$ is far from unique. To illustrate this we give the following example.
1.3. Projective and inductive limits of $F$-spaces

Example 1.37 Consider the vector space of real valued sequences $\mathbb{R}^N$. Let $\psi$ denote the subset of all finite sequences, i.e.

$$\psi := \{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^N \mid \exists k \geq N \text{ s.t. } x_k = 0 \}.$$

The space $\psi$ admits a strict inductive structure if we write it as follows

$$\psi = \bigcup_{m \in \mathbb{N}} \psi_m, \quad \text{where } \psi_m := \{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^N \mid \forall k \geq m \text{ s.t. } x_k = 0 \}.$$

Each $\psi_m$ is a finite dimensional subspace of $\psi$. Let $\| \cdot \|$ be an arbitrary norm on $\psi$. Endow each $\psi_m$ with the restricted norm, denoted by $\| \cdot \|_m$. The sequence $((\psi_m, \| \cdot \|_m))_{m \in \mathbb{N}}$ is a strict inductive system of Banach spaces. Since all norms on a finite dimensional space are equivalent, the strict LB-space $\text{ind} (\psi_m, \| \cdot \|_m)$ does not depend on the choice of the norm $\| \cdot \|$. Obviously, we have that

$$\text{ind} (\psi_m, \| \cdot \|_m) \rightarrow (\psi, \| \cdot \|).$$

However, the normed space $(\psi, \| \cdot \|)$ depends heavily on the choice of the norm $\| \cdot \|$. For example, taking

$$\| (x_n)_{n \in \mathbb{N}} \|_1 := \sum_{i=1}^{\infty} |x_i| \quad \text{and} \quad \| (x_n)_{n \in \mathbb{N}} \|_2 := \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}},$$

it is easily seen that

$$(\psi, \| \cdot \|_1) \rightarrow (\psi, \| \cdot \|_2),$$

but the converse is not true. In fact, the completion of $(\psi, \| \cdot \|_1)$ is the Banach space $l_1$ and the completion of $(\psi, \| \cdot \|_2)$ is the Banach space $l_2$.

Proposition 1.38 Let $(F_n)_{n \in \mathbb{N}}$ be a strict inductive system of (pre-) $F$-spaces with associated topologies $T_n$. Then a pre-Fréchet topology $T$ exists for $(\bigcup_{n=1}^{\infty} F_n, T_n)$ such that

$$\text{ind} F_n \rightarrow (\bigcup_{n=1}^{\infty} F_n, T),$$

and $T |_{F_n} = T_n$ for each $n \in \mathbb{N}$.

Since the completion of a pre-$F$-space is an $F$-space (Lemma 1.23), Corollary 1.38 yields the following result:

Proposition 1.39 Let $(F_n)_{n \in \mathbb{N}}$ be a strict inductive system of (pre-) $F$-spaces with associated topologies $T_n$. Then an $F$-space $F$ exists with the following properties:

- $\bigcup_{n=1}^{\infty} F_n$ is dense in $F$,
- If $T$ denotes the topology of $F$, then $T |_{F_n} = T_n$. 
• \( F_\alpha \) is closed in \( F \) if and only if \( F_\alpha \) is an \( F \)-space.

We use Proposition 1.39 to investigate the completion of a strict pre-LF-space.

**Theorem 1.40** Let \((F_\alpha)_{\alpha \in \mathbb{N}}\) be a strict inductive system of pre-\( F \)-spaces. Then a strict inductive system of \( F \)-spaces \((\overline{F}_\alpha)_{\alpha \in \mathbb{N}}\) exists such that
\[
F_\alpha \hookrightarrow \overline{F}_\alpha \quad \text{and} \quad \overline{F}_\alpha \cap \text{ind}_\alpha F_\alpha = F_\alpha
\]
for each \( \alpha \in \mathbb{N} \). In particular, we have
\[
\text{ind}_\alpha F_\alpha \hookrightarrow \text{ind}_\alpha \overline{F}_\alpha,
\]
where the embedding is dense. The strict LF-space \( \text{ind}_\alpha \overline{F}_\alpha \) is a completion of the strict pre-LF-space \( \text{ind}_\alpha F_\alpha \).

**Proof:**
Let \((F_\alpha)_{\alpha \in \mathbb{N}}\) be a strict inductive system of pre-\( F \)-spaces with associated topologies \( T_\alpha \). Then, by Proposition 1.39, there is an \( F \)-space \( F \) with Fréchet topology \( T \) such that
\[
F_\alpha \hookrightarrow \text{ind}_\alpha F_\alpha \hookrightarrow F,
\]
and \( T|F_\alpha = T_\alpha \). Now, let \( \overline{F}_\alpha \) be the closure of \( F_\alpha \) within \( F \). With induced \( F \)-topology (= \( T|F_\alpha \)) each \( \overline{F}_\alpha \) is an \( F \)-space, satisfying
\[
(T|F_{\alpha+1})|\overline{F}_\alpha = T|\overline{F}_\alpha . \tag{1.7}
\]
Since \( \overline{F}_\alpha \) is closed in \( F \), equality (1.7) implies that \( \overline{F}_\alpha \) is closed in \( \overline{F}_{\alpha+1} \). So \((\overline{F}_\alpha)_{\alpha \in \mathbb{N}}\)

is a strict inductive system of \( F \)-spaces which satisfies the inclusions of the statement.

It remains to prove that \( F_\alpha \cap \text{ind}_\alpha F_\alpha = F_\alpha \). Therefore, let \( x \in F_\alpha \cap \text{ind}_\alpha F_\alpha \). Then \( x \in F_{\alpha+k} \) for some \( k \in \mathbb{N} \). Moreover, there is a sequence \((x_\beta)_{\beta \in \mathbb{N}}\) in \( F_\alpha \)(\( \subseteq F_{\alpha+k} \)), such that \( x_\beta \to x \) in \( \overline{F}_\alpha \). So, \( x_\beta \rightarrow x \) in \( \overline{F}_{\alpha+k} \), and since \((x_\beta)_{\beta \in \mathbb{N}}\) is in \( \overline{F}_{\alpha+k} \) this yields that \( x_\beta \rightarrow x \in \overline{F}_{\alpha+k} \). Since \( F_\alpha \) is closed in \( \overline{F}_{\alpha+k} \), we conclude that \( x \in F_\alpha \). Hence \( \overline{F}_\alpha \cap \text{ind}_\alpha F_\alpha \subseteq F_\alpha \). Conversely, it is obvious that \( \overline{F}_\alpha \cap \text{ind}_\alpha F_\alpha \supseteq F_\alpha \), which proves the statement.

For strict LF-spaces, the performance of relative topologies is not that good.

**Proposition 1.41** Let \( \text{ind}_\alpha F_\alpha \) be a strict LF-space. Let \( M \) be a closed subspace of \( \text{ind}_\alpha F_\alpha \). Then the following statements hold true.

• The family \((F_\alpha \cap M)_{\alpha \in \mathbb{N}}\) is a strict inductive system of \( F \)-spaces.
• \( \text{ind}_\alpha (F_\alpha \cap M) \hookrightarrow M \), where \( M \) is equipped with the relative \( \text{ind}_\alpha \) topology.
1.3. Projective and inductive limits of F-spaces

Proof.
The proof of the first statement is straightforward and therefore omitted.
Now, let \( p \) be a continuous seminorm on \( M \) equipped with relative \( \text{ind} F_n \)-topology. Then there is a continuous seminorm \( q \) on \( \text{ind} F_n \) such that \( p \leq q|_M \). Hence, for all \( n \in N \) we have that
\[
p(n) \leq q(M)M_{\infty}.
\]
Since \( q \) is continuous on every F-space \( M \cap F_n \), this yields that \( p \) is continuous on \( \text{ind} (F_n \cap M) \). So the identity mapping from \( \text{ind} (F_n \cap M) \) onto \( M \) is continuous, which proves the assertion.

\[ \blacksquare \]

Remark 1.4.1 Closed subspaces \( M \) of a strict LF-space \( \text{ind} F_n \) satisfying
\[
M = \text{ind} (F_n \cap M),
\]
where \( M \) and each \( M \cap F_n \) are equipped with induced topology, are called limit-subspaces. In [P1], Pták gave a sufficient condition on a closed subspace of a strict LF-space to be a limit subspace and in [P2], Pták gave an example of a strict LB-space with a closed subspace, which is not a limit subspace. For more details we refer also to [F].

Next, we discuss the performance of quotient topologies. Although a quotient topology of a complete locally convex topology is not necessarily complete, many classes of locally convex topologies are closed under the action of making a quotient topology. For strict LF-spaces we quote the following result from Saxon and Narayanaswami [S-N], Theorem 2.

Proposition 1.42 Let \( V \) be a strict LF-space. Let \( M \) be a closed subspace of \( V \). Then \( V/M \), endowed with quotient topology, is a (not necessarily strict) LF-space.

Taking a closer look at strict LF-spaces, we find the following result.

Proposition 1.43 Let \( \text{ind} F_n \) be a strict LF-space. Let \( M \) be a closed subspace of \( \text{ind} F_n \). Then the following assertions hold true.

i. \( F_k/(M \cap F_k) \hookrightarrow F_n/(M \cap F_n) \hookrightarrow (\text{ind} F_n)/M \) for each \( k \leq n \),

ii. the family \( (F_k/(M \cap F_k))_{k \in N} \) is a strict inductive system of F-spaces,

iii. \( \text{ind} F_k/(M \cap F_k) \equiv (\text{ind} F_n)/M, \)

where \( F_k/(M \cap F_k) \) and \( \text{ind} F_n/M \) are equipped with respective quotient topologies.
Proof.

(i) Define for each \( k, n \in \mathbb{N}, k \geq n \) the mapping \( id_{k,n} : F_k/(M \cap F_k) \to F_n/(M \cap F_n) \) by

\[
id_{k,n}(x + M \cap F_k) := x + M \cap F_n \quad (x \in F_k).
\]

Then each \( id_{k,n} \) is a well-defined linear injection. To show that \( id_{k,n} \) is continuous, let \( \Pi \) be a (in general non-countable) family of seminorms generating the topology of \( \text{ind } F_n \). Let \( k, n \in \mathbb{N} \) be fixed, \( k \leq n \). Recall from Proposition 1.24 that the topology of \( F_k \) is generated by the seminorms \( \bar{\beta}_k \) := \{ \beta_p | p \in \Pi \}. Define for each \( p \in \Pi \) the seminorm \( \beta_k \) on \( F_k/(M \cap F_k) \) by

\[
\beta_k(x + M \cap F_k) := \inf_{m \in M \cap F_n} p(x + m) \quad (x \in F_k).
\]

By definition the family of seminorms \( \bar{\beta}_k := \{ \beta_k | p \in \Pi \} \) generates the quotient topology of \( F_k/(M \cap F_k) \). Similarly, we have that the family of seminorms \( \bar{\beta}_n := \{ \beta_n | p \in \Pi \} \) generates the quotient topology of \( F_n/(M \cap F_n) \). For each \( x \in F_k \) and each \( p \in \Pi \) we have that

\[
\beta_k(id_{k,n}(x + M \cap F_k)) = \inf_{m \in M \cap F_n} p(x + m) \leq \inf_{m \in M \cap F_n} p(x + m) = \beta_k(x + M \cap F_k),
\]

proving the first part of the assertion. To prove the second part, define for each \( k \in \mathbb{N} \) the mapping \( id_k : F_k/(M \cap F_k) \to (\text{ind } F_n)/M \) by

\[
id_k(x + M \cap F_k) := x + M \quad (x \in F_k).
\]

Then \( id_k \) is a well-defined linear injection. To prove that \( id_k \) is continuous observe that the family of seminorms \( \bar{\beta}_n := \{ \beta | p \in \Pi \} \), defined for each \( p \in \Pi \) by

\[
\beta(x + M) := \inf_{m \in M} p(x + m) \quad (x \in \text{ind } F_n),
\]

generates the quotient topology of \( \text{ind } F_n)/M \). So, for each \( \beta \in \bar{\beta}_n, x \in F_k \) we have that

\[
\beta(id_k(x + M \cap F_k)) \leq \inf_{m \in M \cap F_n} p(x + m) = \beta_k(x + M \cap F_k),
\]

so \( id_k \) is continuous.

(ii) Here we have to be careful since the \( F \)-spaces (Proposition 1.42) \( F_k/(M \cap F_k) \), \( k \in \mathbb{N} \), are not subspaces nor super-spaces of one another. Therefore a more general definition of inductive system has to be used. We follow the definition used by Floret [Flo], §1, which states that \( (F_k/(M \cap F_k))_{k \in \mathbb{N}} \) is an inductive system if continuous linear mappings \( \pi_{k,n} : F_k/(M \cap F_k) \to F_n/(M \cap F_n) \) exist for \( k \leq n \) such that

\[
\pi_{k,k} = \text{id} \quad \text{and} \quad \pi_{n,l} \circ \pi_{k,n} = \pi_{k,l} \quad \text{for all } k \leq n \leq l,
\]

where \( \text{id} \) denotes the identity mapping. Observing that the mappings \( id_{k,n} \) satisfy these conditions, the assertion follows.
1.3. Projective and inductive limits of $F$-spaces

(iii). Again we have to be careful in this slightly different concept of inductive system. Notice that
\[ \left( \text{ind}_n F_n \right) / M = \bigcup_k \text{id}_k \left( F_k / (M \cap F_k) \right), \]
and recall from (i) each of the mappings $\text{id}_k$ is continuous from $F_k / (M \cap F_k)$ into $\left( \text{ind}_n F_n \right) / M$. Furthermore, we have $\text{id}_k \circ \text{id}_{k \to n} = \text{id}_k$ for each $k \leq n$. Then, by definition (see [Fol]), the inductive limit of the inductive system $\left( F_k / (M \cap F_k) \right)_{k \in \mathbb{N}}$ is the collection $\bigcup_{k \in \mathbb{N}} F_k / M$ endowed with finest locally convex Hausdorff topology, such that each of the mappings $\text{id}_k$ is continuous, we have that
\[ \text{ind}_k \left( F_k / (M \cap F_k) \right) \leftarrow \left( \text{ind}_n F_n \right) / M. \]
However, since $\left( \text{ind}_n F_n \right) / M$ is an LF-space already, the Open Mapping Theorem 1.46 yields that
\[ \text{ind}_k \left( F_k / (M \cap F_k) \right) \equiv \left( \text{ind}_n F_n \right) / M. \]

Finally, we consider continuous linear mappings on strict LF-spaces. Since a strict (pre-) LF-space has in general a non-countable generating family of seminorms the characterization of continuous linear mappings in terms of this family (Lemma 1.4) is not satisfactory. In fact, we can do better. Particularly, the following extension of Theorem 1.29 is due to Grothendieck [Gro].

**Theorem 1.44** Let $\text{ind}_n F_n$ and $\text{ind}_m G_m$ be two strict LF-spaces and let $L$ be an everywhere defined linear mapping from $\text{ind}_n F_n$ into $\text{ind}_m G_m$. Then the following statements are equivalent.

i. The mapping $L$ is continuous.

ii. The mapping $L$ is sequentially continuous.

iii. For each $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $L(F_n) \subseteq G_m$ and $L |_{F_n} : F_n \to G_m$ is continuous.

iv. The mapping $L$ is closed.

v. The graph of $L$ is sequentially closed in $\text{ind}_n F_n \times \text{ind}_m G_m$ with respect to the product topology.

**Proof.**
For the equivalence of (i), (ii) and (iii) we refer to [Con], §IV.4. To prove the factorization result (iii) a Baire-Category argument has to be used. The equivalence of (i) and (iv) can be read as the Closed Graph Theorem for strict LF-spaces (see [Köt2], §34.8). The equivalence of (iv) and (v) is due to the equivalence of the notion closed subspace and sequentially closed subspace [see [Köt1], §6.19].

An immediate consequence of the latter remark is the following.
Corollary 1.45 Let $V$ and $W$ be two strict LF-spaces. Then the linear mapping $L$ from $V$ into $W$ is closed if and only if $L$ is sequentially closed.

There is an Open Mapping Theorem for strict LF-spaces too, see Dieudonné [Die], p.71.

Theorem 1.46 (Open Mapping Theorem) If $L$ is a continuous linear surjection from a strict LF-space $V$ onto a strict LF-space $W$, then $L$ is open.

Köthe extended the Open Mapping Theorem to the case of (not necessarily strict) LF-spaces (see [Kö61], p.43). Theorem 1.44 can be extended to some extent to strict pre-LF-spaces.

Theorem 1.47 Let $\text{ind} F_n$ and $\text{ind} G_m$ be two strict pre-LF-spaces. Let $L$ be a linear mapping from $\text{ind} F_n$ into $\text{ind} G_m$. Then the following statements are equivalent.

i. The mapping $L$ is continuous.

ii. $L$ extends continuously to a continuous linear operator $\overline{L}$ from $\text{ind} \overline{F}_n$ into $\text{ind} \overline{G}_m$, the respective completions of $\text{ind} F_n$ and $\text{ind} G_m$.

iii. For all $n \in N$ there is an $m \in N$ such that

$$L(F_n) \subseteq G_m \quad \text{and} \quad L|_{F_n} : F_n \rightarrow G_m \text{ is continuous.} \quad (1.8)$$

Proof

We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

First, let $L$ be a continuous linear mapping from $\text{ind} F_n$ into $\text{ind} G_m$. Since $\text{ind} F_n$ and $\text{ind} G_m$ are sequentially dense in $\overline{F}_n$ and $\overline{G}_m$, respectively, and since the latter spaces are complete, the mapping $L$ extends to a sequentially continuous linear mapping $\overline{L}$ from $\text{ind} \overline{F}_n$ into $\text{ind} \overline{G}_m$. In fact, $\overline{L}$ is continuous, because of Theorem 1.44. So $(i) \Rightarrow (ii)$.

To prove $(ii) \Rightarrow (iii)$, suppose $L$ has a continuous linear extension $\overline{L}$ from $\text{ind} \overline{F}_n$ into $\text{ind} \overline{G}_m$. Now, let $n \in N$ be fixed, then by Theorem 1.44, there is an $m \in N$ such that $\overline{L}(F_n) \subseteq \overline{G}_m$. In particular, applying Theorem 1.40 yields

$$L(F_n) \subseteq G_m \cap \text{ind} G_m \quad (1.49) \subseteq G_m,$$

and $L|_{F_n} : F_n \rightarrow G_m$ is continuous.

To prove $(iii) \Rightarrow (i)$, let $q$ be a continuous seminorm on $\text{ind} G_m$. Then, for each $m \in N$, $q|_{G_m}$ is a continuous seminorm on $G_m$. Applying (1.8) yields for each $n \in N$ that $(q \circ L)|_{F_n}$ is a continuous seminorm on $F_n$. Hence, $q \circ L$ is a continuous seminorm on $\text{ind} F_n$. We conclude that $L$ is continuous linear mapping from $\text{ind} F_n$ into $\text{ind} G_m$. \qed
ONE-PARAMETER
(SEMI-)GROUPS ON
SEQUENTIALIY COMPLETE
TOPOLOGICAL VECTOR
SPACES

In this chapter, we focus on one-parameter groups and semigroups of continuous linear mappings on sequentially complete locally convex topological vector spaces such as F-spaces and strict LF-spaces.

In section §2.0, we summarize the most important terminology and results deduced in this chapter. These results will be used in the remaining part of this thesis. The reason for separating these statements and definitions from the theory in which they arise, is that the theory presented in this chapter is a separate subject. For a good understanding of this thesis the reader has to take notice of section §2.0 only. However, we believe that the theory developed in this chapter presents a fruitful approach to general \(c_0\)-(semi)-groups.

In section §2.1, we investigate the theory of \(c_0\)-semi-groups \((\alpha_t)_{t \in \mathbb{R}}\) on locally convex topological vector spaces by exploring the mapping \(t \in \mathbb{R}^+ \mapsto \alpha_t x\) for all \(x \in V\).

In section §2.1.1, we establish the link between local equicontinuity of \((\alpha_t)_{t \in \mathbb{R}}\) and the continuity of this mapping. In case of a locally equicontinuous \(c_0\)-semigroup on a non-complete locally convex topological vector space \((V, T_0)\) the semigroup extends to a locally equicontinuous \(c_0\)-semigroup on the completion of \((V, T_0)\). In section §2.1.2, we restrict ourselves to locally equicontinuous \(c_0\)-semigroups on sequentially complete locally convex topological vector spaces \((V, T_0)\). In that case, the mapping \(t \in \mathbb{R}^+ \mapsto \alpha_t x\) defines for all \(x \in V\) a continuous function from \(\mathbb{R}^+\) into \(V\), a so-called \(C(\mathbb{R}^+, V)\)-function. We consider the translation \(c_0\)-semigroup on \(C(\mathbb{R}^+, V)\) and its relation to the \(c_0\)-semigroup \((\alpha_t)_{t \in \mathbb{R}}\) in §2.1.3. We characterize the infinitesimal generator of the translation \(c_0\)-semigroup on \(C(\mathbb{R}^+, V)\) by introducing the concepts of integration and differentiation on \(C(\mathbb{R}^+, V)\). In §2.1.4, we show that any polynomial in the differentiation operator is a closed linear operator on \(C(\mathbb{R}^+, V)\). As a consequence of this result any polynomial in the infinitesimal generator of \((\alpha_t)_{t \in \mathbb{R}}\) is closed also.

In §2.1.5, we introduce a class of convolution operators. These operators are used to show that the \(c_0\)-domain of the infinitesimal generator of any \(c_0\)-semigroup is dense.

Subsection §2.1.6 presents some results on operators and subspaces invariant under the action of the elements of a semigroup.
In section §2.2, we reformulate the theory developed in section §2.1 for \( c_0 \)-groups. It turns out that the main results and concepts from §2.1 have analogues in the context of \( c_0 \)-groups.

Finally, in section §2.3, we investigate the theory developed both for \( c_0 \)-semigroups and for \( c_0 \)-groups on strict LF-spaces and F-spaces. An important result, due to Dixmier and Malliavin (D-M), on the \( c_0 \)-domain of the infinitesimal generator of a \( c_0 \)-group on strict LF-spaces is given.

### 2.0 SCOPE OF THE MAIN RESULTS

**Definition 2.1** Let \( (\alpha_t)_{t \in \mathbb{R}} \) be a family of continuous linear mappings on a locally convex topological vector space \((V, T_V)\) having the following properties

i. \( \alpha_0 \) is the identity mapping on \((V, T_V)\),

ii. \( \alpha_t \alpha_s = \alpha_{t+s} \) for all \( s, t \in \mathbb{R} \),

iii. \( \lim_{t \to 0} \alpha_t x = x \) for all \( x \in V \).

Then \( (\alpha_t)_{t \in \mathbb{R}} \) is called a **strongly continuous group** of continuous linear mappings, briefly a \( c_0 \)-group, on \((V, T_V)\).

Dealing with families of continuous linear operators, the concept of equicontinuous subsets becomes interesting.

**Definition 2.2** A one-parameter group \( (\alpha_t)_{t \in \mathbb{R}} \) on \((V, T_V)\) is said to be **locally equicontinuous** if for all compact \( K \subseteq \mathbb{R} \) the collection

\[ \{ \alpha_t : t \in K \} \]

is equicontinuous, i.e. for each \( p \in \Pi \), where \( \Pi \) is a family of seminorms generating the topology \( T_V \), the seminorm

\[ x \in V \mapsto \sup_{t \in K} p(\alpha_t x) \]

is continuous on \((V, T_V)\).

In §2.2, we show that in a barrelled space any \( c_0 \)-semigroup is locally equicontinuous. For a definition of barrelled space, see Treves [T], §33.

**Theorem 2.3** Let \( (\alpha_t)_{t \in \mathbb{R}} \) be a \( c_0 \)-group on the barrelled locally convex topological vector space \((V, T_V)\). Then \( (\alpha_t)_{t \in \mathbb{R}} \) is locally equicontinuous.

For non-complete locally convex topological vector spaces \( V \) the following result is deduced (see Theorem 2.28).

**Theorem 2.4** Let \( (\alpha_t)_{t \in \mathbb{R}} \) be a locally equicontinuous \( c_0 \)-group on the locally convex topological vector space \( V \). Then \( (\alpha_t)_{t \in \mathbb{R}} \) extends continuously to a locally equicontinuous \( c_0 \)-group \( (\tilde{\alpha}_t)_{t \in \mathbb{R}} \) on the completion \( \tilde{V} \) of \( V \).
2.0. **Scope of the main results**

An important concept associated to a $c_0$-group is its *infinitesimal generator*.

**Definition 2.5** Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on a locally convex topological vector space $(V, T_0)$. The *infinitesimal generator* $\delta_\alpha$ of $(\alpha_t)_{t \in \mathbb{R}}$ is defined as

$$
\delta_\alpha x := \lim_{t \to 0} \frac{\alpha_t x - x}{t} \quad (x \in \text{Dom}(\delta_\alpha)),
$$

where $\text{Dom}(\delta_\alpha) := \{x \in V \mid \lim_{t \to 0} \frac{1}{t}(\alpha_t x - x) \text{ exists}\}$.

**Remark 2.5.1** An important aspect of the infinitesimal generator $\delta_\alpha$ of a $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ on $V$ is in the following observation. Suppose $x_0 \in \text{Dom}(\delta_\alpha)$. Then a solution of the evolution equation

$$
\begin{align*}
\left\{ x(t) &= \delta_\alpha(x(t)) + f(t) \\
x(0) &= x_0
\end{align*}
$$

where $f : \mathbb{R} \to V$ is continuous, is

$$
x(t) = \alpha_t x_0 + \int_0^t \alpha_{t-s} f(s) \, ds \quad (t \in \mathbb{R}).
$$

In advance, it is not clear that the solution (2.1) is unique. However, if $(\alpha_t)_{t \in \mathbb{R}}$ is *jointly equicontinuous*, i.e. the mapping

$$(t, x) \mapsto \alpha_t x,$$

is continuous on $\mathbb{R} \times V$, then uniqueness is guaranteed. A family of operators $(\alpha_t)_{t \in \mathbb{R}}$ is jointly equicontinuous if and only if it is a locally equicontinuous $c_0$-group. In case of a barrelled space, each $c_0$-group is jointly equicontinuous (see Theorem 2.3).

Naturally, we can apply the operator $\delta_\alpha$ more than once. However, since the operator $\delta_\alpha$ need not be defined everywhere, we have to be cautious. In fact, the operator $\delta_\alpha^k$ is defined recursively, i.e. for each $k \in \mathbb{N}$

$$
\delta_\alpha^k x = \delta_\alpha(\delta_\alpha^{k-1} x) \quad (x \in \text{Dom}(\delta_\alpha)),
$$

where

$$
\text{Dom}(\delta_\alpha^k) = \{ x \in \text{Dom}(\delta_\alpha^{k-1}) \mid \delta_\alpha^{k-1} x \in \text{Dom}(\delta_\alpha) \}
$$

is the domain of the operator $\delta_\alpha^k$ on $V$. Furthermore, the subspace

$$
\text{Dom}^{\omega}(\delta_\alpha) = \bigcap_{k \in \mathbb{N}} \text{Dom}(\delta_\alpha^k).
$$

is called the $c_\omega$-domain of the operator $\delta_\alpha$.

For the remainder of this section we restrict ourselves to the case of a *sequentially complete* locally convex topological vector space. In §2.2, the following result is proved (Theorem 2.57).
Theorem 2.6 Let $V$ be a sequentially complete locally convex topological vector space. Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on $V$ with infinitesimal generator $\delta_o$. And let $p \in \mathbb{C}[x]$ be a polynomial with $k = \deg(p)$. Then the linear operator $p(\partial_o)$ with domain $\text{Dom}(\partial_o)$, is a closed linear mapping in $V$.

An important class of operators closely connected to a locally equicontinuous $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ arises by introducing the space $\mathcal{M}(\mathbb{R})$ of Radon measures, i.e. the dual of the strict LF-space $C_c(\mathbb{R})$ (cf. Bourbaki [Bou1], p.47). For each compact $K \subset \mathbb{R}$, the restriction of $\mu \in \mathcal{M}(\mathbb{R})$ to $C_c(K)$, i.e. the collection of all $\varphi \in C_c(\mathbb{R})$ with support in $K$, has bounded variation, denoted by $\text{var}_K(\mu)$. So,

$$\text{var}_K(\mu) := \int_K |\mu(dt)|.$$

By $\mathcal{M}_c(\mathbb{R})$, we denote the subspace of $\mathcal{M}(\mathbb{R})$ consisting of all $\mu \in \mathcal{M}(\mathbb{R})$ with compact support. The space $\mathcal{M}_c(\mathbb{R})$ equals the dual of $C_c(\mathbb{R})$ (cf. Gaal [Gaa], p.135). For $\mu \in \mathcal{M}_c(\mathbb{R})$, we denote by $\text{var}(\mu)$ the total variation of $\mu$, i.e.

$$\text{var}(\mu) := \sup_{K \subseteq \mathbb{R}, \text{compact}} \text{var}_K(\mu).$$

Since the mapping $t \mapsto \alpha_t x$ is continuous for each $x \in V$ and since $V$ is locally convex and sequentially complete, for each $\mu \in \mathcal{M}_c(\mathbb{R})$, the integral

$$\alpha[\mu] x := \int_{\mathbb{R}} \alpha_t x \mu(dt),$$

is properly defined. In fact, each $\alpha[\mu]$ is a continuous linear operator on $(V, T_\mathbb{R})$. In particular, for each $p \in \Pi$ there are $q \in \Pi$ and $C > 0$ such that

$$p(\alpha[\mu] x) \leq \text{var}(\mu) \cdot \max_{t \in \text{supp}(\mathcal{M}_c(\mathbb{R}))} p(\alpha_t x) \leq C \cdot q(x),$$

where the latter inequality is due to the local equicontinuity of $(\alpha_t)_{t \in \mathbb{R}}$.

Notice that for each $t \in \mathbb{R}$, the Dirac measure at $t$, $\delta_t \in \mathcal{M}_c(\mathbb{R})$, defined by

$$\delta_t(x) := x(t) \quad (x \in C_c^\infty(\mathbb{R})).$$

satisfies $\alpha[\delta_t] = \alpha_t$.

We list some properties of the operators $\alpha[\mu]$, $\mu \in \mathcal{M}_c(\mathbb{R})$.

Lemma 2.7 (cf. [Bou1], p.72) For each $\mu \in \mathcal{M}_c(\mathbb{R})$ there exists $(\mu_t)_{t \in \mathbb{R}}$ in $\text{span}(\delta_t \mid t \in \mathbb{R})$ such that for every $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ on $(V, T_\mathbb{R})$,

$$\lim_{k \to \infty} \alpha[\mu_k] x = \alpha[\mu] x$$

for all $x \in V$. In other words, the linear span, $\text{span}(\delta_t \mid t \in \mathbb{R})$, is strongly sequentially dense in the collection $\{\alpha[\mu] \mid \mu \in \mathcal{M}_c(\mathbb{R})\}$.

Lemma 2.8 For all $\mu_1, \mu_2 \in \mathcal{M}_c(\mathbb{R})$, we have $\alpha[\mu_1] \alpha[\mu_2] = \alpha[\mu_2] \alpha[\mu_1]$ in $\mathcal{M}_c(\mathbb{R})$. In particular, we have $\alpha[\mu_1] \alpha_t = \alpha_t \alpha[\mu_1]$ for all $t \in \mathbb{R}$. 

2.0. Scope of the main results

The space $D(R)$ can be regarded as a subspace of $M_c(R)$ in the following way: Let $\phi \in D(R)$. Then $\mu_\phi \in M_c(R)$ is defined by

$$
\mu_\phi(x) := \int R x(t)\phi(t)\, dt \quad (x \in C_c(R)).
$$

In the sequel, we shall write $\phi$ instead of $\mu_\phi$ and correspondingly $\alpha[\phi]$ instead of $\alpha[\mu_\phi]$. For $\phi \in D(R)$ and for each $k \in \mathbb{N}$, we have

$$
\delta^n_k \alpha[\phi] = (-1)^k \alpha[\phi^{(k)}].
$$

In particular for each $x \in V$, we have $\alpha[\phi]x \in \text{Dom}^{\omega}(\delta_n)$.

Lemma 2.9 (approximate identity) There exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ in $D(R)$ such that for all $x \in V$, $\alpha[\phi_n]x \to x$ as $n \to \infty$.

As an immediate consequence of the latter result we have the following theorem.

Theorem 2.10 Let $V$ be a sequentially complete locally convex topological vector space. Let $(\alpha_t)_{t \in R}$ be a locally equicontinuous $c_0$-group on $V$ with infinitesimal generator $\delta_0$. Then $\text{Dom}^{\omega}(\delta_0)$ is dense in $V$.

So, the operators from Theorem 2.6 are densely defined. In §2.3, we present the following important result for strict LF-spaces.

Theorem 2.11 Let $(\alpha_t)_{t \in R}$ be a $c_0$-group on the strict LF-space $V$, with infinitesimal generator $\delta_0$. Then

$$
\text{span}\{ \alpha[\phi]x \mid \phi \in D(R), x \in V \} = \text{Dom}^{\omega}(\delta_0).
$$

In fact, for each $x \in \text{Dom}^{\omega}(\delta_0)$ there are $y \in \text{Dom}^{\omega}(\delta_0)$ and $\phi_1, \phi_2 \in D(R)$ such that $x = \alpha[\phi_1]x + \alpha[\phi_2]y$.

Remark 2.11.1 The proof of Theorem 2.11 is based upon a result due to Dixmier and Malliavin [D-M]. Although Dixmier and Malliavin proved a similar result in the more general context of Lie-group representation theory, they proved their result for F-spaces only.

Remark 2.11.2 The vectors $\alpha[\phi]x$, $\phi \in D(R)$, $x \in V$, are called Gårding vectors (see Warner [War]).

Throughout this thesis, invariance under a $c_0$-group or $c_0$-semigroup plays an important role.

Definition 2.12 Let $(\alpha_t)_{t \in R}$ be a locally equicontinuous $c_0$-group on the sequentially complete locally convex topological vector space $V$. Then the subspace $M$ of $V$ is called $(\alpha_t)_{t \in R}$-invariant if

$$
\alpha_t(M) \subseteq M
$$

for all $t \in R$. 

We mention the following results from §2.2.

Lemma 2.13 Let \( M \) be a sequentially closed, \((\alpha_t)_{t \geq 0}\)-invariant subspace of \( V \). Then \( \alpha[\mu](M) \subseteq M \) for each \( \mu \in \mathcal{M}_c(\mathbb{R}) \).

Theorem 2.14 Let \( M \) be a sequentially closed, \((\alpha_t)_{t \geq 0}\)-invariant subspace of \( V \). Then \( \text{Dom}^\infty(\delta_0) \cap M \) is sequentially dense in \( M \).

For linear operators, we introduce \((\alpha_t)_{t \geq 0}\)-invariance likewise.

Definition 2.15 The operator \( L \) with domain \( \text{Dom}(L) \) in \( V \) is called \((\alpha_t)_{t \geq 0}\)-invariant if \( \text{Dom}(L) \) is \((\alpha_t)_{t \geq 0}\)-invariant and

\[
L_\alpha x = \alpha_t L x
\]

for all \( x \in \text{Dom}(L) \) and all \( t \in \mathbb{R} \).

For (sequentially) closed linear \((\alpha_t)_{t \geq 0}\)-invariant operators we have the following result.

Theorem 2.16 Let \( L \) be a (sequentially) closed linear \((\alpha_t)_{t \geq 0}\)-invariant operator on \( V \) with domain \( \text{Dom}(L) \). Then the following assertions hold.

- For all \( \mu \in \mathcal{M}_c(\mathbb{R}) \) and for all \( x \in \text{Dom}(L) \), we have \( \alpha[\mu] x \in \text{Dom}(L) \) and \( L \alpha[\mu] x = \mu L x \).

Next, assume \( \text{Dom}^\infty(\delta_0) \subseteq \text{Dom}(L) \). Then

- \( L(\text{Dom}^\infty(\delta_0)) \subseteq \text{Dom}^\infty(\delta_0) \), and
- \( \text{graph}(L) \) is the closure in \( V \times V \) (with product topology) of
  \[
  \text{graph}(L|_{\text{Dom}^\infty(\delta_0)}) = \{ (x, Lx) \mid x \in \text{Dom}^\infty(\delta_0) \}.
  \]

i.e. \( \text{Dom}^\infty(\delta_0) \) is a core for \( L \).

2.1 SEMIGROUPS: GENERAL THEORY

Contrary to the previous section, where we considered \( c_0 \)-groups of continuous linear operators on locally convex topological vector spaces, we focus in this section on one-parameter \( c_0 \)-semigroups. In fact, we need both concepts; \( c_0 \)-groups in all following chapters and \( c_0 \)-semigroups in Chapter 5. The reason for presenting the theory for \( c_0 \)-semigroups here, is that \( c_0 \)-groups can be regarded as a special subclass of \( c_0 \)-semigroups. Indeed, if \( (\alpha_t)_{t \geq 0} \) is a \( c_0 \)-group on \( V \), then it is fully characterized by the \( c_0 \)-semigroup \( (\alpha_t)_{t \geq 0} \) with the property that every operator \( \alpha_t \) has a continuous inverse.

Besides that \( R \) must be replaced by \( R^+ \) in their definition, the most essential difference between \( c_0 \)-groups \( (\alpha_t)_{t \geq 0} \) and \( c_0 \)-semigroups \( (\alpha_t)_{t \geq 0} \) is that for \( c_0 \)-groups strong
2.1. SEMIGROUPS: GENERAL THEORY

continuity means that for all $x$, $\lim_{t \to 0} \alpha_t x = x$, where as for $c_0$-semigroups strong continuity means that for all $x$, $\lim_{t \to 0} \alpha_t x = x$. In literature, one often encounters another strong continuity concept for semigroups that is more closely connected to the group-strong continuity, namely for all $x$ and all $s > 0$

$$\lim_{t \to 0} \alpha_t x = x \text{ and } \lim_{t - \delta \to 0} \alpha_{t+s} x = \alpha_s x.$$  

Using the latter concept of strong continuity, one can copy the theory on semigroups presented in this section largely from the theory of one-parameter semigroups. Only the theory presented in the first subsection (§2.1.1) becomes (a bit) less involved.

2.1.1 Semigroups and flows

Definition 2.17 Let $(\alpha_t)_{t \geq 0}$ be a family of continuous linear mappings on a locally convex topological vector space $(V, T_{\Pi})$ having the following properties

i. $\alpha_0$ is the identity mapping on $(V, T_{\Pi})$,

ii. $\alpha_t \alpha_s = \alpha_{t+s}$ for all $s, t \geq 0$,

iii. $\lim_{t \to 0} \alpha_t x = x$ for all $x \in V$.

Then $(\alpha_t)_{t \geq 0}$ is called a strongly continuous semigroup of continuous linear mappings, briefly a $c_0$-semigroup, on $(V, T_{\Pi})$.

Let $(V, T_{\Pi})$ be a locally convex topological vector space, where the collection of seminorms $\Pi$ generates the topology of $V$. Let $(\alpha_t)_{t \geq 0}$ be a $c_0$-semigroup on $(V, T_{\Pi})$. Notice that strong continuity of $(\alpha_t)_{t \geq 0}$ means that for every $x \in V$, the mapping $t \in \mathbb{R}^+ \mapsto \alpha_t x$ is continuous from the right as a function from $\mathbb{R}^+$ into $V$. The following question arises:

What are (necessary and sufficient) conditions on $(\alpha_t)_{t \geq 0}$ (and $V$) for the mapping $t \in \mathbb{R}^+ \mapsto \alpha_t x$ to be a continuous function for each $x \in V$?

Let us introduce some terminology first. Let $C(\mathbb{R}^+, V)$ denote the vector space of all continuous functions from $\mathbb{R}^+$ into $V$. Hence, $f \in C(\mathbb{R}^+, V)$ if

$$\forall_{t \geq 0} \forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{x \in V} \left[ \left| t - s \right| < \delta \Rightarrow | f(t) - f(s) | < \epsilon \right].$$

By the triangle inequality for seminorms, the mapping $t \in \mathbb{R}^+ \mapsto p(f(t))$ is continuous for all $p \in \Pi$ and $f \in C(\mathbb{R}^+, V)$. So, we can define the following seminorms on $C(\mathbb{R}^+, V)$

$$p_K(f) := \max_{i \in K} p(f(t)) \quad (f \in C(\mathbb{R}^+, V)), \quad (2.6)$$
where $K \subseteq \mathbb{R}^+$ is a compact subset and $p \in \Pi$. Consequently, we equip $C(\mathbb{R}^+, V)$ with the locally convex topology $T_m$ generated by the seminorm collection

$$\{ p_K \mid p \in \Pi, K \subseteq \mathbb{R}^+ \text{ compact} \}.$$ 

So a net in $C(\mathbb{R}^+, V)$ is convergent if it converges uniformly on each compact subset $K$ of $\mathbb{R}^+$. The topology $T_m$ is called the compact-open topology.

Define the flow operator $\mathcal{F}_\alpha : V \to C(\mathbb{R}^+, V)$ with domain

$$\text{Dom}(\mathcal{F}_\alpha) := \{ x \in V \mid \alpha t \in C(\mathbb{R}^+, V) \}$$

by

$$\mathcal{F}_\alpha x(t) := \alpha t x \quad (t \in \mathbb{R}^+, x \in \text{Dom}(\mathcal{F}_\alpha)).$$

The following lemma shows that the answer to our question is connected with the concept of local equicontinuity.

**Lemma 2.18** Let $(\alpha_t)_{t \geq 0}$ be a $\alpha$-semigroup on the locally convex topological vector space $(V, T_1)$. Then, $(\alpha_t)_{t \geq 0}$ is locally equicontinuous if and only if $\text{Dom}(\mathcal{F}_\alpha) = V$ and $\mathcal{F}_\alpha : V \to C(\mathbb{R}^+, V)$ is continuous.

**Proof.**

First, suppose $(\alpha_t)_{t \geq 0}$ is locally equicontinuous. Let $t_0 > 0$, $x \in V$. Then, the mapping $t \mapsto \alpha_t x$ is right-continuous at $t_0$. Now for $0 \leq t \leq t_0$ and $p \in \Pi$ we have

$$p(\alpha_t x - \alpha_{t_0} x) = p(\alpha_t (x - \alpha_{t_0} x)).$$

Since $(\alpha_t)_{t \geq 0}$ is locally equicontinuous, $q \in \Pi$ and $C > 0$ exist such that

$$p(\alpha_t x - \alpha_{t_0} x) \leq C \cdot q(x - \alpha_{t_0} x),$$

for all $t \in [0, t_0]$. So, the mapping $t \mapsto \alpha_t x$ is left-continuous at any $t = t_0$. Hence, $x \in \text{Dom}(\mathcal{F}_\alpha).$ In particular, $\text{Dom}(\mathcal{F}_\alpha) = V$ Furthermore, knowing that $\text{Dom}(\mathcal{F}_\alpha) = V$, we have $\mathcal{F}_\alpha$ is continuous if and only if $(\alpha_t)_{t \geq 0}$ is locally equicontinuous. Indeed, if one of these assertions is true, then for all $p \in \Pi$ and all compact $K \subseteq \mathbb{R}^+$, there exist for all $p \in \Pi$ and all compact $K \subseteq \mathbb{R}^+$, a constant $C > 0$ and some $q \in \Pi$, such that for all $x \in V$

$$p_K(\mathcal{F}_\alpha x) = \max_{t \in K} p(\alpha_t x) \leq C \cdot q(x).$$

Lemma 2.18 proves that local equicontinuity of $(\alpha_t)_{t \geq 0}$ is sufficient for the image of $\mathcal{F}_\alpha$ to be in $C(\mathbb{R}^+, V)$. It is also necessary if we impose the additional condition that $V$ is barrelled. We recall the following result from [Tre], Theorem 33.1.

**Proposition 2.19** Let $V$ and $W$ be locally convex topological vector spaces, where $V$ is barrelled. Then, a subset $\mathcal{H}$ of continuous linear operators from $V$ into $W$ is equicontinuous if and only if $\mathcal{H}$ is bounded for the topology of pointwise convergence.
2.1. Semigroups: General Theory

Now, suppose $V$ is barrelled and let $W = V$. Let $(\alpha_t)_{t \geq 0}$ be a $C_0$-semigroup on $V$ such that $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$. Let $K \subseteq \mathbb{R}^+$ be compact. Then for all $p \in \Pi$ and $x \in V$

$$\sup_{t \in K} p(\alpha_t x) < \infty,$$

because $t \mapsto p(\alpha_t x)$ is continuous on $\mathbb{R}^+$. So, the set $K = \{\alpha_t \mid t \in K\}$ is bounded for the topology of pointwise convergence, whence equicontinuous according to the above proposition. Thus we derived the following result (cf. Kömura [Kom], Proposition 1.1).

**Lemma 2.20** Suppose $V$ is a barrelled locally convex topological vector space. Let $(\alpha_t)_{t \geq 0}$ be a $C_0$-semigroup on $V$ such that the flow $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$. Then, $F_\alpha$ is continuous from $V$ into $C(\mathbb{R}^+, V)$.

Combining Lemma 2.18 and 2.20 yields the following result.

**Theorem 2.21** Let $(\alpha_t)_{t \geq 0}$ be a $C_0$-semigroup on the barrelled locally convex topological vector space $(V, T_0)$. Then $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$ if and only if $(\alpha_t)_{t \geq 0}$ is locally equicontinuous. Moreover, if $(\alpha_t)_{t \geq 0}$ is locally equicontinuous, then the flow operator $F_\alpha$ maps $V$ continuously into $C(\mathbb{R}^+, V)$.

Examples of barrelled locally convex topological vector spaces are F-spaces and strict LF-spaces ([Tre], §33). Normed spaces, thus pre-F-spaces and strict pre-LF-spaces, are not necessarily barrelled (see Floret and Wiorko [Fi-Wi], §10.1.3). Conversely, not every barrelled space is complete (see Khalilulla [Khal], p.38).

**Remark 2.2.1.1** The result of Theorem 2.21 is not so surprising. Let us approach the problem from another angle. From literature (see [Köti2], §34), we recall that for barrelled spaces often a version of Closed Graph Theorem exists. Now, let $(\alpha_t)_{t \geq 0}$ be a $C_0$-semigroup on the locally convex topological vector space $V$. It is easily shown that the operator $F_\alpha : \text{Dom}(F_\alpha) \to C(\mathbb{R}^+, V)$ is linear and closed. Now, suppose we can apply a Closed Graph Theorem to the operator $F_\alpha$. Then, the condition $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$ is equivalent to the operator $F_\alpha$ being everywhere defined, thus continuous. So, in this particular case, Lemma 2.18 yields the condition $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$ to be equivalent to $(\alpha_t)_{t \geq 0}$ being locally equicontinuous (cf. [Kom], Proposition 1.1).

**Corollary 2.22** Let $(V, T_0)$ be an F-space or a strict LF-space and let $(\alpha_t)_{t \geq 0}$ be a $C_0$-semigroup on $(V, T_0)$. Then $F_\alpha x \in C(\mathbb{R}^+, V)$ for all $x \in V$ if and only if $(\alpha_t)_{t \geq 0}$ is locally equicontinuous. Moreover, if $(\alpha_t)_{t \geq 0}$ is locally equicontinuous, the flow operator $F_\alpha : V \to C(\mathbb{R}^+, V)$ is continuous.

If $(V, T_0)$ is (sequentially) complete, we have the following result. The proof is straightforward and therefore omitted.

**Lemma 2.23** Let $V$ be a (sequentially) complete locally convex topological vector space. Then the locally convex topological vector space $C(\mathbb{R}^+, V)$ is (sequentially) complete.
Next, consider a $c_0$-semigroup $(\alpha_t)_{t \geq 0}$ on a non-complete locally convex topological vector space $V$. Let $\overline{V}$ be the completion of $V$ (Theorem 1.19). We may assume that $V \subseteq \overline{V}$ without loss of generality. So for each $t \geq 0$, $\alpha_t$ is continuous as a mapping from $V$ into $\overline{V}$. From [Tre], Theorem 5.2, we recall the following result.

**Proposition 2.24** Let $V, W$ be locally convex topological vector spaces and suppose $W$ is complete. Let $\overline{V}$ be the completion of $V$. Then to every continuous linear mapping $L : V \rightarrow W$ there is a unique continuous linear mapping $\overline{L} : \overline{V} \rightarrow W$ extending $L$.

Applying Proposition 2.24 to the mappings $\alpha_t$ yields extensions $\alpha_\overline{t}$ of $\alpha_t$ to $\overline{V}$. Obviously, the collection $\{\alpha_\overline{t} \mid t \geq 0\}$ is a semigroup of continuous linear operators, but at this point it is not clear whether $(\alpha_\overline{t})_{t \geq 0}$ is strongly continuous.

**Theorem 2.25** Let $(\alpha_t)_{t \geq 0}$ be a locally equicontinuous $c_0$-semigroup on the locally convex topological vector space $V$. Then $(\alpha_\overline{t})_{t \geq 0}$ extends continuously to a locally equicontinuous $c_0$-semigroup $(\alpha_\overline{t})_{t \geq 0}$ on the completion $\overline{V}$ of $V$.

**Proof.**

We assume without loss of generality that $V \subseteq \overline{V}$. Let $(\overline{\alpha}_t)_{t \geq 0}$ be the semigroup of extensions of $(\alpha_t)_{t \geq 0}$. By Lemma 2.18, the flow operator $\mathcal{F}_\alpha : V \rightarrow C(\mathbb{R}^+, V)$ is everywhere defined and continuous. So, the flow operator $\overline{\mathcal{F}}_\alpha$ is everywhere defined and continuous as a mapping from $V$ into $C(\mathbb{R}^+, \overline{V})$. Since $C(\mathbb{R}^+, \overline{V})$ is complete (Lemma 2.23), applying Proposition 2.24 to $\overline{\mathcal{F}}_\alpha$ yields that there is a continuous linear mapping $\mathcal{F}_{\overline{\alpha}}$ from $\overline{V}$ into $C(\mathbb{R}^+, V)$ extending $\mathcal{F}_\alpha$. Obviously, we have for all $t \geq 0$, $x \in V$

$$(\mathcal{F}_{\overline{\alpha}}x)(t) = \overline{\alpha}_t x,$$

so $\mathcal{F}_{\overline{\alpha}} = \overline{\mathcal{F}}_\alpha$. Since $\mathcal{F}_{\overline{\alpha}}$ is everywhere defined, $(\overline{\alpha}_t)_{t \geq 0}$ is a $c_0$-semigroup. Moreover, since $\mathcal{F}_{\overline{\alpha}}$ is continuous also, Lemma 2.18 yields $(\overline{\alpha}_t)_{t \geq 0}$ to be locally equicontinuous. \(\blacksquare\)

### 2.1.2 The translation semigroup on $C(\mathbb{R}^+, V)$ and its relation to arbitrary semigroups on $V$

In the sequel we consider locally equicontinuous $c_0$-semigroups $(\alpha_t)_{t \geq 0}$ on a sequentially complete locally convex topological vector spaces $(V, T_\alpha)$. Hence, flow operators $\mathcal{F}_\alpha$ are everywhere defined and continuous.

So far, we have not exploited the special structure of a flow operator $\mathcal{F}_\alpha$. Let $t, s \geq 0$, $x \in V$. Then

$$(\mathcal{F}_\alpha x)(t+s) = \alpha_{t+s} x = (\mathcal{F}_\alpha(\alpha_t x))(s).$$

So, if we introduce the translation semigroup $(\sigma_t)_{t \geq 0}$ on $C(\mathbb{R}^+, V)$,

$$(\sigma_t f)(s) := f(t+s), \quad (f \in C(\mathbb{R}^+, V), s \geq 0), \quad (2.9)$$
2.1. SEMIGROUPS: GENERAL THEORY

where \( t \geq 0 \), then the above relation can be written as

\[
\sigma_t \circ T_n = T_n \circ \alpha_t, \quad (t \geq 0).
\]

(2.10)

The importance of equation (2.10) is that it relates all locally equicontinuous \( e \)-semigroups \((\alpha_t)_{t \geq 0}\) on \( V \) by means of its flow to the one translation semigroup on \( C(\mathbb{R}^+, V) \). In fact, by deriving properties of the translation semigroup \((\sigma_t)_{t \geq 0}\) on \( C(\mathbb{R}^+, V) \), we obtain general properties for \( e \)-semigroups on \( V \) exploiting (2.10).

Let us study the translation semigroup \((\sigma_t)_{t \geq 0}\) on \( C(\mathbb{R}^+, V) \). For all \( p \in \Pi, K \subseteq \mathbb{R}^+ \) compact, we have

\[
p_K(\sigma_t f) = p_{K_t}(f) \quad (f \in C(\mathbb{R}^+, V)),
\]

where \( K_t = K + \{t\} \) compact. Hence, each \( \alpha_t, t \geq 0 \), is a continuous linear operator on \( C(\mathbb{R}^+, V) \). To show that \((\sigma_t)_{t \geq 0}\) is strongly continuous we need the following lemma.

**Lemma 2.26** For \( K \subseteq \mathbb{R}^+ \) compact and each \( f \in C(\mathbb{R}^+, V) \), the restriction \( f|_K \) is uniformly continuous from \( K \) into \( V \), i.e.

\[
\forall \epsilon > 0 \exists \delta > 0 \forall t, s \in K \quad | t - s | < \delta \implies p(f(t) - f(s)) < \epsilon.
\]

**Proof.**

The proof of this statement is completely analogous to the classical situation with \( V = \mathbb{C} \).

\[ \blacksquare \]

**Proposition 2.27** The translation semigroup \((\sigma_t)_{t \geq 0}\) on \( C(\mathbb{R}^+, V) \) is strongly continuous and locally equicontinuous.

**Proof.**

Let \( \delta_0 \in \mathbb{R}^+, p \in \Pi, K \subseteq \mathbb{R}^+ \) compact. Then for all \( f \in C(\mathbb{R}^+, V) \)

\[
\lim_{t \to \delta_0} p_K(\sigma_t f - \sigma_{\delta_0} f) = \lim_{t \to \delta_0} \max_{t \in K} p(f(t + s) - f(\delta_0 + s)) = 0,
\]

due to the uniform continuity of \( f \) on compact subsets of \( \mathbb{R}^+ \) (Lemma 2.26). It follows that \( \mathcal{F} f \in C(\mathbb{R}^+, C(\mathbb{R}^+, V)) \) for all \( f \in C(\mathbb{R}^+, V) \), so \((\sigma_t)_{t \geq 0}\) is strongly continuous. To show that \((\sigma_t)_{t \geq 0}\) is locally equicontinuous, it is sufficient to show that \( \mathcal{F} \) is continuous (Lemma 2.18). Therefore, let \( K_1, K_2 \subseteq \mathbb{R}^+ \) compact and let \( p \in \Pi \), then \( K_1 + K_2 \) compact and for all \( f \in C(\mathbb{R}^+, V) \)

\[
\max_{t \in K_1} p_{K_2}((\mathcal{F} f)(t)) = \max_{t \in K_1} p_{K_2}(\sigma_t f) = \max_{t \in K_1} \max_{s \in K_2} p(f(t + s))
\leq \max_{t \in K_1 + K_2} p(f(t)) = p_{K_1 + K_2}(f),
\]

proving the assertion.

\[ \blacksquare \]

Now, let \( \delta_0 \) denote the infinitesimal generator of \((\sigma_t)_{t \geq 0}\) with domain \( \text{Dom}(\delta_0) \). Let \( \delta_0 \) denote the infinitesimal generator of the locally equicontinuous \( e \)-semigroup \((\alpha_t)_{t \geq 0}\)
on $V$. Then, we define the operators $\delta_n^k$ and $\delta_n^k$ recursively. In particular, for the operators $\delta_n^k$ we have

$$
\delta_n^k x = \delta_n(\delta_n^{k-1} x) \quad (x \in \text{Dom}(\delta_n^k)).
$$

where

$$
\text{Dom}(\delta_n^k) = \{ x \in \text{Dom}(\delta_n^{k-1}) \mid \delta_n^{k-1} x \in \text{Dom}(\delta_n) \}
$$

is the domain of the operator $\delta_n^k$ on $V$.

We apply equation (2.10) to connect $\delta_n$ with $\delta_n^k$.

**Lemma 2.28** For each $k \in \mathbb{N}$ the following two assertions are equivalent.

- $x \in \text{Dom}(\delta_n^k)$.
- $\mathcal{F}_n x \in \text{Dom}(\delta_n^k)$.

For $x \in \text{Dom}(\delta_n^k)$ we have $\mathcal{F}_n \delta_n^k x = \delta_n \mathcal{F}_n x$.

**Proof.**

We prove this lemma by induction to $k$.

For $k = 1$, let $x \in \text{Dom}(\delta_n)$. Then by the continuity of $\mathcal{F}_n$

$$
\lim_{t \to 0} \frac{\sigma_t x - \sigma_0 x}{t} \overset{(2.30)}{=} \lim_{t \to 0} \mathcal{F}_n \left( \frac{\alpha_t x - \alpha_0 x}{t} \right) = \mathcal{F}_n \delta_n x.
$$

So, $\mathcal{F}_n \delta_n x \in \text{Dom}(\delta_n)$ with $\delta_n \mathcal{F}_n x = \mathcal{F}_n \delta_n x$.

Conversely, suppose $\mathcal{F}_n x \in \text{Dom}(\delta_n)$. Then

$$
\lim_{t \to 0} \frac{\alpha_t x - x}{t} = \lim_{t \to 0} \left( \mathcal{F}_n \left( \frac{\alpha_t x - x}{t} \right) \right)(0) = (\delta_n \mathcal{F}_n x)(0),
$$

proving the assertion for $k = 1$.

Now, let $k \geq 2$, and suppose the assertion holds for all $l = 1, \ldots, k - 1$. Then,

- $x \in \text{Dom}(\delta_n^k)$ if and only if $x \in \text{Dom}(\delta_n^{k-1})$ and $\delta_n^{k-1} x \in \text{Dom}(\delta_n)

\quad \overset{\text{induction}}{\Rightarrow} \quad \mathcal{F}_n x \in \text{Dom}(\delta_n^{k-1})$ and $\delta_n^{k-1} \mathcal{F}_n x = \mathcal{F}_n \delta_n^{k-1} x \in \text{Dom}(\delta_n)$

- $x \in \text{Dom}(\delta_n^k)$ if and only if $\mathcal{F}_n \delta_n^k x = \mathcal{F}_n \delta_n(\delta_n^{k-1} x) = \delta_n \mathcal{F}_n (\delta_n^{k-1} x) = \delta_n \delta_n^{k-1} \mathcal{F}_n x = \delta_n^k \mathcal{F}_n x$.

\[ \blacksquare \]

Obviously, Lemma 2.28 extends to the $c_{\alpha}$-domains of $(\alpha_t)_{t > 0}$ and $(\alpha_t)_{t \geq 0}$.

**Corollary 2.29** The following two assertions are equivalent.

- $x \in \text{Dom}^{c_{\alpha}}(\delta_n) = \bigcap_{t > 0} \text{Dom}(\delta_n^t)$.  

2.1. Semigroups: General Theory

- $F_\alpha x \in \text{Dom}^\alpha(\delta_\alpha) = \bigcap_k \text{Dom}(\delta_\alpha^k)$.

For each $k \in \mathbb{N}$ and each $x \in \text{Dom}(\delta_\alpha^k)$ we have $F_\alpha \delta_\alpha^k x = \delta_\alpha^k F_\alpha x$.

We conclude from Lemma 2.28 that the intertwining relation (2.10) extends to the respective infinitesimal generators. However, since the infinitesimal generator $\delta_\alpha$ of the translation semigroup on $C(\mathbb{R}^+, V)$ is unknown up to this point in the discussion, Lemma 2.28 seems rather academic. In the next subsection we characterize the operator $\delta_\alpha$. Moreover, we show in subsection 2.1.4 that every polynomial $p(\delta_\alpha)$ in the infinitesimal generator $\delta_\alpha$ with domain $\text{Dom}(\delta_\alpha^k)$, with $k$ the degree$(p)$, is a closed linear operator on $C(\mathbb{R}^+, V)$ (see Theorem 2.35). Taking Theorem 2.35 for granted for the moment, we can apply Lemma 2.28 to obtain the following analogue of Theorem 2.35 for all locally equicontinuous $\alpha$-semigroups on the sequentially complete topological vector space $(V, T_0)$. This theorem is one of the main results of this chapter.

**Theorem 2.30** Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then the linear operator $p(\delta_\alpha)$ with domain $\text{Dom}(\delta_\alpha^k)$, $k = \text{degree}(p)$, is a closed linear mapping in $V$.

**Proof.**

By definition $x \in \text{Dom}(\delta_\alpha^k)$ implies that $x \in \text{Dom}(\delta_\alpha^l)$ ($l = 0, 1, \ldots, k$), and so $p(\delta_\alpha)$ is well defined. Let $(x_n)$ be a net in $\text{Dom}(\delta_\alpha^k)$ for which there are $x, y \in V$ such that

$x_n \to x$ and $p(\delta_\alpha)x_n \to y$

in $V$ sense. Then the continuity of $F_\alpha$ ensures that

$F_\alpha x_n \to F_\alpha x$ and $F_\alpha p(\delta_\alpha)x_n \to F_\alpha y$\n
in $C(\mathbb{R}^+, V)$ sense. Notice that for each $n, F_\alpha x_n \in \text{Dom}(\delta_\alpha)$ and $F_\alpha p(\delta_\alpha)x_n = p(\delta_\alpha)F_\alpha x_n$ (Lemma 2.28). So, the hypothesized closedness of the operator $p(\delta_\alpha)$ with domain $\text{Dom}(\delta_\alpha^k)$ in $C(\mathbb{R}^+, V)$ (see Theorem 2.35) yields that $F_\alpha x \in \text{Dom}(\delta_\alpha^k)$ and $p(\delta_\alpha)F_\alpha x = F_\alpha y$. Consequently, Lemma 2.28 yields that $x \in \text{Dom}(\delta_\alpha^k)$ and $y = p(\delta_\alpha)F_\alpha x(0) = p(\delta_\alpha)x$.

\[ \square \]

**2.1.3 Integration and differentiation in $C(\mathbb{R}^+, V)$**

In this subsection, we investigate the infinitesimal generator of the translation $\alpha$-semigroup $(\alpha t)\log$ on $C(\mathbb{R}^+, V)$ as promised. It will come as no surprise that this operator is a differentiation operator on $C(\mathbb{R}^+, V)$. Before we can formalize differentiation on $C(\mathbb{R}^+, V)$, we introduce a concept of integration. In fact, we introduce integration on $C(\mathbb{R}^+, V)$. This is where sequential completeness of $V$ is needed.

The space $\mathcal{M}_c(\mathbb{R}^+)$ is the subspace of $\mathcal{M}_c(\mathbb{R})$ consisting of all $\mu \in \mathcal{M}_c(\mathbb{R})$ with support in $[0, \infty)$. The space $\mathcal{M}_c(\mathbb{R}^+)$ can also be seen as the dual of the $F$-space $C(\mathbb{R}^+)$ consisting of all continuous functions on $[0, \infty)$.

Since $V$ is locally convex and sequentially complete, we can introduce the integral operator $I[\mu] : C(\mathbb{R}^+, V) \to V$ for each $\mu \in \mathcal{M}_c(\mathbb{R}^+)$ by

\[ I[\mu]f = \int_{\mathbb{R}^+} f(t) \mu(\text{d}t). \quad (2.11) \]
It follows that for each $\mu \in \mathcal{M}_{c}(\mathbb{R}^{+})$ and for each continuous seminorm $p$ on $V$

$$p(l_{\mu}[f]) \leq \text{var}(\mu) \cdot p_{\mu_{\mathcal{M}_{c}}}(f) \quad (f \in C(\mathbb{R}^{+}, V)).$$

So, $l_{\mu} : C(\mathbb{R}^{+}, V) \to V$ is continuous. By taking a suitable $\mu_{a,b} \in \mathcal{M}_{c}(\mathbb{R}^{+})$, we can obtain

$$l_{\mu_{a,b}}[f] = \int_{\mathbb{R}^{+}} f(t) \mu_{a,b}(dt) = \int_{\mu_{a,b}} f(t) dt =: \int_{a}^{b} f(t) \, dt,$$

for all $f \in C(\mathbb{R}^{+}, V)$, where the latter integral can be introduced also in the Riemann-Stieltjes sense (see [vEdR1]).

We proceed by introducing the integral operator $J$ from $C(\mathbb{R}^{+}, V)$ into $C(\mathbb{R}^{+}, V)$. Let $Jf : \mathbb{R}^{+} \to V$ for each $f \in C(\mathbb{R}^{+}, V)$ be defined by

$$(Jf)(t) = \int_{0}^{t} f(r) \, dr, \quad (t \in \mathbb{R}^{+}).$$

Notice that $Jf(0) = 0$. Since for all $a > 0$, $s, t \in [0,a]$, $p \in \Pi$

$$p(Jf(t) - Jf(s)) \leq |t - s| \cdot p_{\mu_{0,a}}(f),$$

each $Jf \in C(\mathbb{R}^{+}, V)$, and the operator $J$ is continuous.

Next, we show that $J$ is injective. To this extent, suppose $Jf = 0$. By the construction of the integral, we have for each continuous linear functional $F \in V'$ and each $t \in \mathbb{R}^{+}$

$$F(Jf(t)) = \int_{0}^{t} F(f(r)) \, dr.$$ 

Since $F(Jf(t)) = 0$ we get from ordinary calculus $F(f(t)) = 0$ for all $F \in V'$ and all $t \in \mathbb{R}^{+}$. We conclude that $f = 0$.

Next, we introduce the concept of differentiation on $C(\mathbb{R}^{+}, V)$. By $C^{1}(\mathbb{R}^{+}, V)$ we denote the subspace of $C(\mathbb{R}^{+}, V)$ consisting of all $f \in C(\mathbb{R}^{+}, V)$ for which there exist $x_{0} \in V$ and $g \in C(\mathbb{R}^{+}, V)$ such that

$$f(t) = x_{0} + Jg(t) \quad (t \geq 0).$$

If $f$ can be represented in this way, then this representation is unique. Indeed, suppose $x_{0} + Jg(t) = 0$ for all $t \geq 0$. Since $Jg(0) = 0$ we have $x_{0} = 0$, and hence with $g = 0$ ($J$ is injective).

**Definition 2.31** The operator $D$ in $C(\mathbb{R}^{+}, V)$ with domain Dom($D$) $= C^{1}(\mathbb{R}^{+}, V)$ is defined by

$$Df = g \iff f = \phi_{0}f(0) + Jg,$$

where $\phi_{0}(t) = 1$ for all $t \geq 0$. Here $\phi_{0}f(0)$ denotes the $C(\mathbb{R}^{+}, V)$-function $t \in \mathbb{R}^{+} \mapsto f(0)$. The operator $D$ is called the **differentiation operator** in $C(\mathbb{R}^{+}, V)$.
2.1. SEMIGROUPS: GENERAL THEORY

From the definition of $D$ and $J$ we see that $J$ maps $C(\mathbb{R}^+, V)$ into $\text{Dom}(D)$ and $DJ = I$ (the identity operator).

Now, we can characterize the infinitesimal generator of the translation semigroup $(\sigma_t)_{t \geq 0}$ on $C(\mathbb{R}^+, V)$.

**Theorem 2.32** The infinitesimal generator of the translation semigroup $(\sigma_t)_{t \geq 0}$ on $C(\mathbb{R}^+, V)$ is the differentiation operator $D$ with domain $C^1(\mathbb{R}^+, V)$.

**Proof**

Let $\delta_t$ denote the infinitesimal generator of $(\sigma_t)_{t \geq 0}$. Let $f \in C^1(\mathbb{R}^+, V)$ with $Df = g$.

Put differently, we have for all $s \geq 0$

$$ f(s) = f(0) + \int_0^s g(\tau) \, d\tau. $$

Therefore, for all $t \in [0,1]$ and all $s \geq 0$ we have

$$\frac{\sigma_t f(s) - f(s)}{t} - g(s) = \frac{1}{t} \int_s^{s+t} (g(\tau) - g(s)) \, d\tau. \quad (2.13)$$

Now, let $p \in J$ and $K \subseteq \mathbb{R}^+$ be an arbitrary compact subset. Then

$$ f_p \left( \frac{\sigma_t f - f}{t} - g \right) \leq \max_{\gamma \in K} \max_{s \geq 0} p(g(\tau + s) - g(s)) $$

$$ = \max_{\gamma \in [0,1]} \max_{s \in K} p(g(\tau + s) - g(s)). $$

Since $g$ is uniformly continuous on $[0,1] + K$, the right-hand side term tends to zero for $t \to 0$, so $f \in \text{Dom}(\delta_t)$ with $\delta_t f = g = Df$.

Conversely, let $f \in \text{Dom}(\delta_t)$ with $\delta_t f = g$. Since for all $s \geq 0$

$$ f(s) = \lim_{t \to 0} \frac{1}{t} \int_s^{s+t} f(\tau) \, d\tau $$

in $V$, we have

$$ f(s) - f(0) = \lim_{t \to 0} \frac{1}{t} \left( \int_0^{s+t} f(\tau) \, d\tau - \int_0^s f(\tau) \, d\tau \right) $$

$$ = \lim_{t \to 0} \frac{1}{t} \int_0^s (\sigma_t f - f)(\tau) \, d\tau = \int_0^s g(\tau) \, d\tau. $$

Hence, $f \in C^1(\mathbb{R}^+, V)$ with $Df = g$.

**Remark 2.32.1** The differentiation operator $D$ is the ordinary differentiation operator on $C(\mathbb{R}^+, V)$, as is to be expected. Since point-evaluation on $C(\mathbb{R}^+, V)$ is continuous, we have

$$ (Df)(t) = \left( \lim_{h \to 0} \frac{\sigma_h f - f}{h} \right)(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}, $$

where $t \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^+, V)$. 

2.1.4 Polynomials in the Differentiation Operator

In the previous subsection, we considered integration and differentiation. In this subsection, we investigate how to apply these actions more than once. We follow a similar setup as in the previous subsection. Emphasis is put on the operator $J^k$. In particular, we introduce the spaces of all $k$-times differentiable $C(\mathbb{R}^+,V)$-functions as $k$-times integrated $C(\mathbb{R}^+,V)$-functions (see (2.15)).

Since the integration operator $J$ is injective and continuous, the operator $J^k$ is injective and continuous for each $k \in \mathbb{N}$. Moreover, because for each $F \in V'$ and each $f \in C(\mathbb{R}^+,V)$

$$F(J^k f(t)) = \int_0^t \int_0^t \ldots \int_0^t F(f(\tau_j)) \, d\tau_1 \ldots d\tau_k = \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} F(f(\tau)) \, d\tau,$$

we see that

$$J^k f(t) = \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} f(\tau) \, d\tau. \quad (2.14)$$

For arbitrary $k \in \mathbb{N}$, we denote by $C^k(\mathbb{R}^+,V)$ the subspace of $C(\mathbb{R}^+,V)$ consisting of all $f \in C(\mathbb{R}^+,V)$ for which there exists a $V$-valued polynomial $q$ of degree less or equal to $k-1$, i.e.

$$q(t) = x_0 + tx_1 + \ldots + t^{k-1}x_{k-1} \quad (t \geq 0),$$

with $x_0, x_1, \ldots, x_{k-1} \in V$, and for which there exists a $g \in C(\mathbb{R}^+,V)$ such that

$$f = q + J^k g.$$

Analogous to the case $k = 1$, this representation, when it exists, is unique. Indeed, suppose $q + J^k g = 0$, then for all $t \geq 0$ and $F \in V'$ we have

$$0 = F(q(t)) + F((J^k q)(t)) \quad \text{(2.14)} \quad = \sum_{j=0}^{k-1} F(x_j) t^j + \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} F(g(\tau)) \, d\tau.$$

Differentiating the above expression $k$ times yields that $F(q(t)) = 0$. So $g = 0$ and thence $q = 0$.

Let us return to differentiation. The operators $D^k$ with domain Dom$(D^k)$ are defined recursively, i.e.

$$\text{Dom}(D^k) = \{ f \in \text{Dom}(D^{k-1}) \mid D^{k-1} f \in \text{Dom}(D) \} \quad \text{and} \quad D^k f = D(D^{k-1} f).$$

Since Dom$(D) = C^1(\mathbb{R}^+,V)$, an induction argument yields for all $k \in \mathbb{N}$

$$\text{Dom}(D^k) = C^k(\mathbb{R}^+,V) \quad \text{and} \quad D^k f = g :\iff f = \sum_{j=0}^{k-1} \phi_j(D^j f)(0) + J^k g, \quad (2.15)$$

where $\phi_j(t) = \frac{t^j}{j!}$ (cf. the Riemann remainder formula). Obviously, for all $f \in C(\mathbb{R}^+,V)$ we have $D^k J^k f = f$.

An immediate observation from equation (2.15) is the following.
2.1. SEMIGROUPS: GENERAL THEORY

Lemma 2.33 For each \( k \in \mathbb{N} \), we have
\[
\{ f \in C^k(\mathbb{R}^+, V) \mid D^k f = 0 \} = \text{span}\{ \phi_j x \mid x \in V, j = 0, \ldots, k-1 \}.
\]
Put differently, \( \ker(D^k) = F_{k-1}(V) \), the space of all \( V \)-valued polynomials with degree at most \( k-1 \).

We intend to prove that for each polynomial \( p \) the differential operator \( p(D) \) with domain \( C^k(\mathbb{R}^+, V) \), \( k = \text{degree}(p) \), is closed as a linear mapping in \( C(\mathbb{R}^+, V) \). The closedness of \( p(D) \) is used to prove closedness of \( p(\delta_u) \), where \( \delta_u \) denotes the infinitesimal generator of a locally equicontinuous semigroup \( (\alpha_t)_{t \geq 0} \).

Lemma 2.34 Let \( k \in \mathbb{N} \) be fixed. Then
\[
\ker(D^k) = \text{span}\{ \phi_j x \mid x \in V, j = 0, \ldots, k-1 \} = F_{k-1}(V)
\]
is closed in \( C(\mathbb{R}^+, V) \).

Proof.

We need the following result.

Claim 2.34.1 The matrix \( (\phi_j(i+1))_{j=0}^{k-1} \) is invertible.

Proof Claim 2.34.1.

We calculate the determinant of the matrix \( (\phi_j(i+1))_{j=0}^{k-1} \).
\[
\det((\phi_j(i+1))_{j=0}^{k-1}) = \prod_{i=1}^{k-1} \frac{1}{i!} \cdot \det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & k \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2^{k-1} & \cdots & k^{k-1}
\end{pmatrix}.
\]

In the latter matrix we recognize a Vandermonde matrix, with well known determinant.
\[
\det((\phi_j(i+1))_{j=0}^{k-1}) = \prod_{i=1}^{k-1} \frac{1}{i!} \prod_{j=2}^{k} (j-i) = \prod_{i=1}^{k-1} \frac{1}{i!} \cdot \prod_{j=2}^{k} (j-1)! = 1.
\]

Now, let \( (f_n) \) be a net in \( \text{span}\{ \phi_j x \mid x \in V, j = 0, \ldots, k-1 \} \) converging to \( f \in C(\mathbb{R}^+, V) \). Then, there are \( x_{n,t} \in V \), \( t = 0, \ldots, k-1 \), such that
\[
f_n(t) = \sum_{i=0}^{k-1} \phi_i(t)x_{n,i} \quad (t \in \mathbb{R}^+).
\]

Let \( (\alpha_i)_{j=0}^{k-1} \) denote the inverse of \( (\phi_j(i+1))_{j=0}^{k-1} \). Since
\[
\sum_{i=0}^{k-1} \phi_i(i+1)x_{i,j} = f_n(i+1) \rightarrow f(i+1) \quad (i = 0, 1, \ldots, k-1),
\]
we have for all \( j = 0, 1, \ldots, k - 1 \)

\[
x_{i,j} = \sum_{t=0}^{k-1} \sum_{l=0}^{k-1} a_{j,i} f(i+1) x_{i,l} \rightarrow \sum_{i=0}^{k-1} a_{j,i} f(i+1) =: x_j \in V.
\]

Consequently,

\[
f = \lim_{\nu} \sum_{i=0}^{k-1} f_i x_{i,j} = \sum_{i=0}^{k-1} f_i x_i \in \text{span}\{ f_j x | x \in V, j = 0, \ldots, k - 1 \}.
\]

We come to one of the main results of this chapter.

**Theorem 2.35** Let \( p \in \mathbb{C}[x] \) be a polynomial of degree \( k \). Then, the differential operator \( p(D) \), with domain \( C^k(\mathbb{R}^+, V) \), is a closed linear mapping in \( C(\mathbb{R}^+, V) \), i.e. the graph of \( p(D) \) is closed in the product vector space \( C(\mathbb{R}^+, V) \times C(\mathbb{R}^+, V) \) with respect to the product topology.

**Proof.**

First, we prove the statement for the monomials \( D^k \). Let \( k \in \mathbb{N} \) be fixed. Let \( \{ f_0 \} \) be a set in \( C^k(\mathbb{R}^+, V) \) for which there are \( f \) and \( g \) in \( C(\mathbb{R}^+, V) \) such that

\[
f_0 \rightarrow f \quad \text{and} \quad D^k f_0 \rightarrow g
\]

in \( C(\mathbb{R}^+, V) \)-sense. Then \( f_0 - J^k D^k f_0 \in \ker(D^k) \) and so, since \( \ker(D^k) \) is closed by Lemma 2.34, we have \( f - J^k g \in \ker(D^k) \). It follows that

\[
f = (f - J^k g) + J^k g \in \ker(D^k) + C^k(\mathbb{R}^+, V) = C^k(\mathbb{R}^+, V),
\]

and

\[
D^k f = D^k(f - J^k g) + D^k J^k g = g,
\]

which proves the assertion for monomials.

Now, let \( p \in \mathbb{C}[x] \) be a polynomial of degree \( k \). Since \( D^k f = f \) for all \( f \in C^k(\mathbb{R}^+, V) \), there is a polynomial \( q \) of degree \( k \), such that

\[
p(D)f = D^k q(J)f \quad (f \in C^k(\mathbb{R}^+, V)).
\]

Since \( q(J)(C^k(\mathbb{R}^+, V)) \subseteq C^k(\mathbb{R}^+, V) \) and since \( q(J) \) is continuous, the assertion follows from the closedness of \( D^k \).

We recall from \( \S 2.1.2 \) the following consequence of Theorem 2.35. For every sequentially complete locally convex topological vector space \( (V, T_0) \) and every locally equicontinuous \( \alpha \)-group \( (\alpha_1)_{\alpha \in \mathbb{W}} \) on \( V \), we have that each polynomial \( p(\delta_0) \) in the infinitesimal generator \( \delta_0 \) of \( (\alpha_1) \) with domain \( \text{Dom}(\delta_0) \), \( k \) denoting the degree of \( p \), is closed on \( V \).
2.1.5 Translation invariant operators

For every sequentially complete locally convex topological vector space \((V, T_0)\), we have by Lemma 2.23 that \(C(\mathbb{R}^+, V)\) is also sequentially complete. Hence we can apply the integration theory as developed in §2.1.3 also to the locally convex topological vector space \(C(\mathbb{R}^+, V)\). Since for all \(f \in C(\mathbb{R}^+, V)\) the function \(t \mapsto \sigma_t f\) belongs to \(C(\mathbb{R}^+, C(\mathbb{R}^+, V))\), we have its integral \(I[\mu] f\) for each \(\mu \in \mathcal{M}_c(\mathbb{R}^+)\), and define

\[
\sigma[\mu] f := I[\mu] f = \int_{\mathbb{R}^+} \sigma_t f \, \mu(\mathrm{d}t).
\] (2.16)

So \(\sigma[\mu]\) is a linear operator from \(C(\mathbb{R}^+, V)\) into \(C(\mathbb{R}^+, V)\), which is continuous because \(I[\mu]\) and \(F_\mu\) are continuous. By definition \(\mu \mapsto \sigma[\mu]\) is a linear mapping. Further, it can be checked that

\[
\sigma[\mu_1] \ast \sigma[\mu_2] = \sigma[\mu_1 \ast \mu_2],
\]

for all \(\mu_1, \mu_2 \in \mathcal{M}_c(\mathbb{R}^+)\), where \(\ast\) is the classical convolution in \(\mathcal{M}_c(\mathbb{R}^+)\), i.e.

\[
(\mu_1 \ast \mu_2)(x) := \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} x(t + s) \mu_1(\mathrm{d}t) \mu_2(\mathrm{d}s) \quad (x \in C(\mathbb{R}^+)).
\] (2.17)

Notice that for each \(t \geq 0\), \(\sigma_t \epsilon_t = \epsilon_t\), where \(\epsilon_t \in \mathcal{M}_c(\mathbb{R}^+)\) is the Dirac measure at \(t\) (see (2.5)).

**Lemma 2.36** The linear span, \(\text{span}\{\sigma_t \mid t \in \mathbb{R}^+\}\), is strongly sequentially dense in the collection \(\{\sigma[\mu] \mid \mu \in \mathcal{M}_c(\mathbb{R}^+)\}\), i.e. for each \(\mu \in \mathcal{M}_c(\mathbb{R}^+)\) there exists a sequence \(\mu_k_{k \in \mathbb{N}}\) in \(\text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\}\) such that

\[
\lim_{k \to \infty} \sigma[\mu_k] f = \sigma[\mu] f
\]

for all \(f \in C(\mathbb{R}^+, V)\).

**Proof.**

Let \(\mu \in \mathcal{M}_c(\mathbb{R}^+)\) with \(\text{supp}(\mu) \subseteq [0, T)\), \(T > 0\). Let for each \(k \in \mathbb{N}\) and \(l = 0, 1, \ldots, k\), \(t_{k,l} := \frac{l}{k} T\). Define \(\mu_k \in \text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\}\) by

\[
\mu_k := \sum_{i=1}^{k} \mu([t_{k,i-1}, t_{k,i}]) \epsilon_{t_{k,i-1}}.
\]

Then

\[
(\sigma[\mu] - \sigma[\mu_k]) f(t) = \sum_{i=1}^{k} \int_{t_{k,i-1}}^{t_{k,i}} \left( f(t + \tau) - f(t + t_{k,i-1}) \right) \mu(\mathrm{d}\tau).
\]

So, for all \(p \in \Pi\) and \(K \subseteq \mathbb{R}^+\) compact

\[
p(\sigma[\mu] f - \sigma[\mu_k] f) \leq \max_{\epsilon \in (0, \frac{1}{K})} \max_{\mu_k \in \text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\}} p(f(t + \tau) - f(t + t_{k,i-1})).
\] (2.18)

Since \(f\) is uniformly continuous on compact subsets of \(\mathbb{R}^+\) the right-hand side of (2.18) tends to zero as \(k \to \infty\) which proves the assertion.

An immediate consequence of Lemma 2.36 is that the operators \(\sigma[\mu]\) commute mutually. So each \(\sigma[\mu]\) is a translation-invariant operator on \(C(\mathbb{R}^+, V)\).
Lemma 2.37 Let \( \mu_1, \mu_2 \in \mathcal{M}_c(\mathbb{R}^+), \) then

\[
\sigma[\mu_1] \circ \sigma[\mu_2] = \sigma[\mu_2] \circ \sigma[\mu_1].
\]

In particular, for all \( t \geq 0 \) we have \( \sigma[\mu_1] = \sigma[\mu_1] \sigma_t. \)

The space \( \mathcal{D}(\mathbb{R}^+) \) can be regarded as a subspace of \( \mathcal{M}_c(\mathbb{R}^+) \) in the following way: Let \( \phi \in \mathcal{D}(\mathbb{R}^+). \) Then \( \mu_\phi \in \mathcal{M}_c(\mathbb{R}^+) \) is defined by

\[
\mu_\phi(x) := \int_{\mathbb{R}^+} \phi(t)x(t)dt \quad (x \in C(\mathbb{R}^+)).
\]

Again, we shall write \( \phi \) instead of \( \mu_\phi \) and correspondingly \( \sigma[\phi] \) instead of \( \sigma[\mu_\phi]. \)

An important subspace of \( \sigma[\mu] \)'s arises taking the Radon measure \( \mu \in \mathcal{D}(\mathbb{R}^+). \) For \( \phi \in \mathcal{D}(\mathbb{R}^+), f \in C(\mathbb{R}^+, V), \) we have

\[
(\sigma[\phi]f)(t) = \int_{\mathbb{R}^+} \phi(t) f(t + \tau) d\tau = \int_0^\infty \phi(t) f(t) d\tau.
\]

Notice that the latter term resembles the classical convolution product of \( \phi \) and \( f \) up to a reflection. Moreover, we see that Range(\( \sigma[\phi] \)) \( \subseteq \bigcap_{j \in \mathbb{N}} C^j(\mathbb{R}^+, V) =: C^\infty(\mathbb{R}^+, V) \) with

\[
D^k \circ \sigma[\phi] = (-1)^k \circ \sigma[\phi^{(k)}].
\]

A sequence \( (\phi_n)_{n \in \mathbb{N}} \) in \( \mathcal{D}(\mathbb{R}^+) \) is said to be an approximate identity of reguletors whenever for all \( f \in C(\mathbb{R}^+, V) \)

\[
\lim_{n \to \infty} \sigma[\phi_n]f = f.
\]

Let \( \phi \in \mathcal{D}(\mathbb{R}^+) \) with \( \phi(t) \geq 0 \) and \( \int_{\mathbb{R}^+} \phi(t)dt = 1. \) Define

\[
\phi_n(t) := n\phi(nt) \quad (t \in \mathbb{R}, n \in \mathbb{N}). \tag{2.19}
\]

Then for \( t \geq 0 \)

\[
(\sigma[\phi_n]f - f)(t) = \int_{\mathbb{R}^+} \left( n\phi(n\tau) f(t + \tau) - \phi(t) f(t) \right) d\tau
= \int_{\mathbb{R}^+} \phi(t) \left( f(t + \frac{\tau}{n}) - f(t) \right) d\tau.
\]

So the uniform continuity of \( f \) on compact subsets of \( \mathbb{R}^+ \) ensures the right-hand side to tend to zero as \( n \to \infty. \) Hence, there exists an approximate identity in \( \mathcal{D}(\mathbb{R}^+). \) This is no surprising, since the existence of an approximate identity has been proved for \( V = \mathbb{C} \) classically (see [Schw2]).

Lemma 2.38 The subspace \( C^\infty(\mathbb{R}^+, V) \) is sequentially dense in \( C(\mathbb{R}^+, V). \)
As we did in §2.1.2, we use the intertwining property (2.10) to deduce general results on arbitrary locally equicontinuous \( c_0 \)-semigroups \( (\alpha_t)_{t \geq 0} \) on sequentially complete locally convex topological vector spaces \( (V, \mathcal{T}_V) \) from results on the one translation semigroup on \( C(\mathbb{R}^+, V) \). Let \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \) and a \( c_0 \)-semigroup \( (\alpha_t)_{t \geq 0} \) on \( V \) be given. Define the linear operator \( \alpha[\mu] \) on \( V \) by

\[
\alpha[\mu]x := \left( \sigma[\mu]\mathcal{F}_\alpha x \right)(0) \quad (x \in V).
\] (2.20)

We can also introduce the operators \( \alpha[\mu] \) by means of integration on \( V \). In fact, we can write \( \alpha[\mu]x \) as

\[
\alpha[\mu]x = \left( \int_{\mathbb{R}^+} \sigma[\mu] \mathcal{F}_\alpha x \mu(\mathrm{d}t) \right)(0) = \int_{\mathbb{R}^+} \alpha_t x \mu(\mathrm{d}t).
\]

Since point-evaluation on \( C(\mathbb{R}^+, V) \) is continuous as a mapping from \( C(\mathbb{R}^+, V) \) into \( V \), \( \alpha[\mu] \) is also continuous. Notice that by (2.10), we have for all \( t \geq 0 \)

\[
\alpha[\epsilon] = \left( \sigma[\epsilon] \mathcal{F}_\alpha x \right)(0) = \alpha_t.
\]

Lemma 2.36 has the following analogue for the operators \( \alpha[\mu] \).

**Lemma 2.39** For each \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \) there exists \( (\mu_k)_{k \in \mathbb{N}} \) in \( \text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\} \) such that for every locally equicontinuous \( c_0 \)-semigroup \( (\alpha_t)_{t \geq 0} \) on \( (V, \mathcal{T}_V) \) we have

\[
\lim_{k \to \infty} \alpha[\mu_k]x = \alpha[\mu]x
\]

for all \( x \in V \). In other words, the linear span, \( \text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\} \), is strongly sequentially dense in the collection \( \{\alpha[\mu] \mid \mu \in \mathcal{M}_c(\mathbb{R}^+)\} \).

**Proof.**

Let \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \). Choose the sequence \( (\mu_k)_{k \in \mathbb{N}} \) in \( \text{span}\{\epsilon_t \mid t \in \mathbb{R}^+\} \) as in Lemma 2.36. Let \( (\alpha_t)_{t \geq 0} \) be a locally equicontinuous \( c_0 \)-semigroup on \( V \). Then for all \( x \in V \)

\[
\lim_{k \to \infty} \alpha[\mu_k]x = \lim_{k \to \infty} \left( \sigma[\mu_k] \mathcal{F}_\alpha x \right)(0) = \left( \sigma[\mu] \mathcal{F}_\alpha x \right)(0) = \alpha[\mu]x.
\]

Using Lemma 2.39 we derive the following intertwining results.

**Lemma 2.40** For all \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \), we have \( \sigma[\mu] \mathcal{F}_\alpha = \mathcal{F}_\alpha \sigma[\mu] \).

**Proof.**

Let \( (\mu_k)_{k \in \mathbb{N}} \) as in Lemma 2.39. Then for all \( x \in V \)

\[
\sigma[\mu] \mathcal{F}_\alpha x = \lim_{k \to \infty} \sigma[\mu_k] \mathcal{F}_\alpha x \quad \overset{(1.10)}{=} \quad \lim_{k \to \infty} \mathcal{F}_\alpha \sigma[\mu_k]x = \mathcal{F}_\alpha \sigma[\mu]x.
\]

Recalling the definition of \( \mu_1 \ast \mu_2 \) from (2.17), we have the following consequence of Lemma 2.40.
Lemma 2.41 For all μ₁, μ₂ ∈ M_{c}(\mathbb{R}^+), we have α[μ₁]α[μ₂] = α[μ₁ * μ₂] = α[μ₂]α[μ₁].

In case φ ∈ D(\mathbb{R}^+), we have F_{ α}φ(α)x = φ(α)f_{ α}x ∈ C_{ c}^{\infty}(\mathbb{R}^+, V). So by Corollary 2.29, α[φ] ∈ Dom_{ c}^{\infty}(δ_α). In fact, δ^{k}_{α}φ[φ] = (-1)^{k}φ[φ^{(k)}] for each k ∈ N.

Lemma 2.42 Let (φ_n) ∈ D(\mathbb{R}^+) be an approximate identity. Then, for all x ∈ V, α[φ_n]x ∈ Dom_{ c}^{\infty}(δ_α) → x as n → ∞. As a consequence, the subspace Dom_{ c}^{\infty}(δ_α) is sequentially dense in V.

We conclude that the closed linear operators p(δ_α) on V from Theorem 2.30 are densely defined.

In Chapter 1, we introduced the graph topology for a closed linear operator. In fact, by Proposition 1.17, each subspace Dom(δ_α^k) of (V, T_B) equipped with the graph topology, i.e., the topology generated by the seminorms \{p o δ_α^k | p ∈ Π\}, is sequentially complete. The following result shows that the family (Dom(δ_α^k))_{k∈N} forms a left-sided chain.

Proposition 2.43 Let (V, T_B) be a sequentially complete locally convex topological vector space. Let (α_δ)_{δ>0} be a locally equicontinuous α-semigroup on V with infinitesimal generator ε_α. Then each Dom(δ_α^k) equipped with graph topology satisfies

\[ Dom(δ_α^{k+1}) \hookrightarrow Dom(δ_α^{k}) \twoheadrightarrow V. \]

As a consequence, the graph topology of each Dom(δ_α^k) is generated by the seminorms \{p o δ_α^k | p ∈ Π, j = 0, 1, \ldots, k\}.

Proof. Let k ∈ N be fixed. By definition, we have Dom(δ_α^{k+1}) ⊆ Dom(δ_α^k) ⊆ V. Now, suppose the net (x_ν) converges to x in Dom(δ_α^{k+1}), i.e., x_ν → x and δ_α^{k+1}x_ν → δ_α^{k+1}x in V-sense. Recall from equation (2.15) that for each t ≥ 0 and each ν we have

\[ (F_{α}x_ν)(t) = \sum_{j=0}^{k} \frac{t^j}{j!} (D^j F_{α}x_ν)(0) + \int_{0}^{t} \frac{(t - τ)^k}{k!} (D^{k+1} F_{α}x_ν)(τ) dτ, \]

or equivalently, applying Lemma 2.28

\[ α_δ x_ν = \sum_{j=0}^{k} \frac{t^j}{j!} δ_α^j x_ν + \int_{0}^{t} \frac{(t - τ)^k}{k!} α_δ δ_α^{k+1} x_ν dτ. \]

Letting ν → ∞ we find for all t ≥ 0

\[ \sum_{j=0}^{k} \frac{t^j}{j!} δ_α^j x_ν \rightarrow \sum_{j=0}^{k} \frac{t^j}{j!} δ_α^j x, \]

so applying Claim 2.34.1, we find for each j = 0, \ldots, k

\[ δ_α^j x_ν \rightarrow δ_α^j x. \]
2.1. SEMIGROUPS: GENERAL THEORY

In particular, \( x_n \to x \) in \( \text{Dom}(\delta_n) \), proving the first assertion. The latter assertion is an immediate consequence of the above argument also.

By Proposition 2.43 it seems natural to endow \( \text{Dom}^\infty(\delta_n) = \cap_k \text{Dom}(\delta_n^k) \) with the projective limit topology of the left-sided chain \( (\text{Dom}(\delta_n^k))_{k \in \mathbb{N}} \). For this topology we have the following result.

Proposition 2.44 Let \( (V, T_1), \ (\alpha_1)_{t \geq 0} \) and \( \delta_n \) be as in Proposition 2.43. Then the following assertions are true.

i. The projective limit topology \( T_{\text{graph}} \) of the left-sided chain \( (\text{Dom}(\delta_n^k))_{k \in \mathbb{N}} \) for \( \text{Dom}^\infty(\delta_n) \) is brought about by the seminorms \( \{ p \circ \delta_n^k | p \in \Pi, k \in \mathbb{N}_0 \} \).

ii. \( (\text{Dom}^\infty(\delta_n), T_{\text{graph}}) \) is sequentially complete.

iii. The operators \( (\alpha_1 | \text{Dom}^\infty(\delta_n))_{t \geq 0} \) form a locally equicontinuous \( c_0 \)-semigroup on \( (\text{Dom}^\infty(\delta_n), T_{\text{graph}}) \).

iv. For each \( \mu \in M_c(\mathbb{R}^+), \) the linear mapping \( \alpha[\mu] : \text{Dom}^\infty(\delta_n) \to \text{Dom}^\infty(\delta_n) \) is continuous with respect to \( T_{\text{graph}} \).

Proof.

i. See [F-W], p.34.

ii. Since each \( \text{Dom}(\delta_n^k) \) is sequentially complete, \( \text{proj} \text{Dom}(\delta_n^k) \) is sequentially complete (see [Köt], §19.1).

iii. Since \( (\alpha_1)_{t \geq 0} \) is a \( c_0 \)-semigroup on each \( \text{Dom}(\delta_n^k) \), \( (\alpha_1)_{t \geq 0} \) is a \( c_0 \)-semigroup on \( (\text{Dom}(\delta_n), T_{\text{graph}}) \). Since \( (\alpha_1)_{t \in \mathbb{R}} \) is locally equicontinuous on \( (V, T_1) \), it is locally equicontinuous on \( (\text{Dom}(\delta_n), T_{\text{graph}}) \) by i.

iv. Since the operators \( \delta_n^k \) are closed and since \( \text{span}(\alpha_1 | t \in \mathbb{R}^+) \) is strongly sequentially dense in the collection \( \{ \alpha[\mu] | \mu \in M_c(\mathbb{R}^+) \} \) by Lemma 2.39, we have that each \( \alpha[\mu] \) commutes on \( \text{Dom}^\infty(\delta_n) \) with each of the operators \( \delta_n^k \), \( k \in \mathbb{N} \). So, the assertion follows by i and the continuity of \( \alpha[\mu] \) on \( (V, T_1) \).

\[ \blacksquare \]

2.1.6 \( c_0 \)-semigroups; invariance

We conclude this section with the special topic of \( (\alpha_1)_{t \geq 0} \)-invariant closed subspaces of \( V \) and \( (\alpha_1)_{t \geq 0} \)-invariant closed linear operators on \( V \). Since the setting is that of an arbitrary locally equicontinuous \( c_0 \)-semigroup \( (\alpha_1)_{t \geq 0} \) on an arbitrary sequentially complete locally convex topological vector spaces \( V \), no characterization result can be expected.

Definition 2.45 Let \( (\alpha_1)_{t \geq 0} \) be a locally equicontinuous \( c_0 \)-semigroup on the sequentially complete locally convex topological vector space \( V \). Then the subspace \( M \) of \( V \) is called \( (\alpha_1)_{t \geq 0} \)-invariant if

\[ \alpha_t(M) \subseteq M \]

for all \( t \geq 0 \).
Let \( M, V \) and \((\alpha_t)_{t\geq0}\) as in the above definition. Let for each \( t \geq 0 \), \( \beta_t \) denote the restriction of \( \alpha_t \) to \( \bar{M} \), and equip \( M \) with induced \( V \)-topology. Then \((\beta_t)_{t\geq0}\) is a locally equicontinuous \( c_0 \)-semigroup on \( M \). Let \( \delta_0 \) denote the infinitesimal generator of \((\beta_t)_{t\geq0}\), then \( \text{Dom}(\delta_0) \subseteq \text{Dom}(\delta_0) \cap M \) with 
\[
\delta_0 x = \delta_0 x \quad (x \in \text{Dom}(\delta_0)).
\]

If, furthermore, \( M \) is sequentially closed in \( V \), then for all \( x \in \text{Dom}(\delta_0) \cap M \) we have
\[
\delta_0 x = V \lim_{t \to 0} \frac{\alpha_t x - x}{t} = M \lim_{t \to 0} \frac{\beta_t x - x}{t},
\]
so that \( x \in \text{Dom}(\delta_0) \) with \( \delta_0 x = \delta_0 x \). So, \( \text{Dom}(\delta_0) = \text{Dom}(\delta_0) \cap M \) and \( \delta_0 \) = \( \delta_0 \)|\( M \). In particular, we have \( \text{Dom}^\infty(\delta_0) = \text{Dom}^\infty(\delta_0) \cap M \). Applying Lemma 2.42 we obtain the following result.

**Theorem 2.46** Let \( M \) be a sequentially closed, \((\alpha_t)_{t\geq0}\)-invariant subspace of \( V \). Then \( \text{Dom}^\infty(\delta_0) \cap M \) is sequentially dense in \( M \).

Every sequentially closed subspace \( M \) of \( V \) with induced topology is sequentially complete. Hence, we can introduce the operators \( \beta[\mu] \) on \( V \) for each \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \) (see §2.1.5). In fact, choosing for fixed \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \) the sequence \((\alpha_t)_{t\in\mathbb{N}}\) in \( \text{span}\{\varepsilon_t \mid t \in \mathbb{R}^+\} \) as in Lemma 2.39 we find for all \( x \in M \)
\[
\beta[\mu] x = M \lim_{k \to \infty} \beta[\mu_k] x = V \lim_{k \to \infty} \alpha[\mu_k] x = \alpha[\mu] x.
\]

**Lemma 2.47** Let \( M \) be a sequentially closed, \((\alpha_t)_{t\geq0}\)-invariant subspace of \( V \). Then \( \alpha[\mu](M) \subseteq M \) for each \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \).

A useful result on closed \((\alpha_t)_{t\geq0}\)-invariant subspaces of \((\text{Dom}^\infty(\delta_0), \mathcal{T}_{\text{graph}})\) (see Proposition 2.44) is the following.

**Lemma 2.48** Let \( M \) be a (sequentially) closed subspace of \((\text{Dom}^\infty(\delta_0), \mathcal{T}_{\text{graph}})\) and suppose \( M \) is \((\alpha_t)_{t\geq0}\)-invariant. Let \( \overline{M} \) be its (sequential) closure in \((V, \mathcal{T}_{\overline{M}})\). Then \( M = \overline{M} \cap \text{Dom}^\infty(\delta_0) \).

**Proof.**
It is clear that \( M \subseteq \overline{M} \cap \text{Dom}^\infty(\delta_0) \). Take \( x \in \overline{M} \cap \text{Dom}^\infty(\delta_0) \) and let \( \phi \in \mathcal{D}(\mathbb{R}^+) \). Then there is a net \((\beta_t)x \mid t \in \mathbb{N}\) in \( M \) converging to \( x \) in \( V \)-sense. Since \( \delta_0^\infty \alpha[\phi] = (-1)^k \alpha[\phi_k] \) for each \( k \in \mathbb{N} \), we have that the net \((\alpha[\phi]x_n)\mid n \in \mathbb{N}\) converges to \( \alpha[\phi]x \) in \((\text{Dom}^\infty(\delta_0), \mathcal{T}_{\text{graph}})\). Further, for all \( \mu \in I \), \( \alpha[\mu]x_n \in M \) and therefore \( \alpha[\phi]x \in M \). Letting \((\phi_n)_{n\in\mathbb{N}}\) be an approximate identity in \( \mathcal{D}(\mathbb{R}^+) \), we get \( \alpha[\phi_n]x \to x \) in \((\text{Dom}^\infty(\delta_0), \mathcal{T}_{\text{graph}})\), so that \( x \in M \).

**Corollary 2.49** Let \( M \) be a \((\alpha_t)_{t\geq0}\)-invariant subspace of \( \text{Dom}^\infty(\delta_0) \). Then
\[
\overline{M'} \cap \text{Dom}^\infty(\delta_0) = \overline{M'},
\]
where \( \overline{M'} \) is the closure of \( M \) in \((\text{Dom}^\infty(\delta_0), \mathcal{T}_{\text{graph}})\) and where \( \overline{M'} \) is the closure of \( M \) in \( V \).
Corollary 2.50 Let \( M \) be a \((\alpha_t)_{t \geq 0}\)-invariant subspace of \( \text{Dom}^{\omega}(\delta_a) \) such that \( M \) is (sequentially) dense in \( V \). Then \( M \) is (sequentially) dense in \( \text{Dom}^{\omega}(\delta_a) \) with respect to \( T_{\text{graph}} \).

For linear operators, we introduce \((\alpha_t)_{t \geq 0}\)-invariance likewise.

Definition 2.51 The operator \( L \) with domain \( \text{Dom}(L) \) in \( V \) is called \((\alpha_t)_{t \geq 0}\)-invariant if \( \text{Dom}(L) \) is \((\alpha_t)_{t \geq 0}\)-invariant and

\[
L_{\alpha_t} x = \alpha_t L x
\]

for all \( x \in \text{Dom}(L) \) and all \( t \geq 0 \).

For (sequentially) closed linear \((\alpha_t)_{t \geq 0}\)-invariant operators we have the following result.

Theorem 2.52 Let \( L \) be a (sequentially) closed linear \((\alpha_t)_{t \geq 0}\)-invariant operator on \( V \) with domain \( \text{Dom}(L) \). Then the following assertions hold.

- For all \( \mu \in \mathcal{M}_c(\mathbb{R}^1) \) and for all \( x \in \text{Dom}(L) \) we have \( \alpha[\mu]x \in \text{Dom}(L) \) and \( L\alpha[\mu] = \alpha[\mu]L \).

Assume \( \text{Dom}^{\omega}(\delta_a) \subseteq \text{Dom}(L) \). Then

- \( L(\text{Dom}^{\omega}(\delta_a)) \subseteq \text{Dom}^{\omega}(\delta_a) \), and

- \( \text{graph}(L) \) is the closure in \( V \times V \) (with product topology) of

\[
\text{graph}(L|_{\text{Dom}^{\omega}(\delta_a)}) = \{ (x, Lx) \mid x \in \text{Dom}^{\omega}(\delta_a) \}.
\]

Proof.

(i). \( \text{Graph}(L) = \{ (x, Lx) \mid x \in \text{Dom}(L) \} \) is a closed \((\alpha_t)_{t \geq 0}\) \times \((\alpha_t)_{t \geq 0}\)-invariant subspace of \( V \times V \). So the result follows from Lemma 2.47.

(ii). Let \( (\phi_n)_{n \in \mathbb{N}} \) be an approximate identity in \( \mathcal{D}(\mathbb{R}^1) \). Let \( x \in \text{Dom}^{\omega}(\delta_a) \) and \( k \in \mathbb{N} \). Then we have

\[
\delta_k \alpha[\phi_n] L x = (-1)^k L \alpha[\phi_n] \delta_k x = \alpha[\phi_n] L \delta_k x.
\]

Since \( \delta_k \) is closed we obtain \( Lx \in \text{Dom}(\delta_k) \) and \( \delta_k Lx = L \delta_k x. \) Since \( k \) was arbitrary the assertion follows.

(iii). Let \( x \in \text{Dom}(L) \). Let \( (\phi_n)_{n \in \mathbb{N}} \) be an approximate identity in \( \mathcal{D}(\mathbb{R}^1) \). Then

\[
(x, Lx) = \lim_{n \to \infty} (\alpha[\phi_n] x, \alpha[\phi_n] Lx) = \lim_{n \to \infty} (\alpha[\phi_n] x, L \alpha[\phi_n] x).
\]

\[ \blacksquare \]
2.2 $C_0$-GROUPS, IN SUMMARY

In this section we sketch how the general theory developed for $c_0$-semigroups on locally convex topological vector spaces can be applied in the development of a general theory for $c_0$-groups. We mention two approaches. The first approach is to develop a theory by analyzing the flow mapping $t \in \mathbb{R} \mapsto \alpha_t x$ for each $x \in V$. In fact, a theory almost similar to the one presented in §2.1 can be obtained. We will not follow that approach here. The interested reader is referred to Van Eijndhoven [vEi72].

The approach we present here arises from regarding $c_0$-groups as a special subclass of $c_0$-semigroups. In particular, suppose $(\alpha_t)_{t \in \mathbb{R}}$ is a $c_0$-group on a locally convex topological vector space $V$. Then the family $(\alpha_t)_{t \in \mathbb{R}}$ forms a $c_0$-semigroup on $V$. Obviously, it has some additional properties:

i. each operator $\alpha_t$ has a continuous inverse, and

ii. $\text{Dom}(\mathcal{F}_t) = V$, i.e., $t \in \mathbb{R}^+ \mapsto \alpha_t x \in C(\mathbb{R}^+; V)$ for each $x \in V$.

Notice that property (ii) is due to the concept of strong continuity for $c_0$-groups that differs slightly from the concept of strong continuity for semigroups (see the introduction of §2.1). Conversely, if $(\alpha_t)_{t \in \mathbb{R}}$ is a $c_0$-semigroup satisfying the conditions (i) and (ii), then it can be extended to a one-parameter $c_0$-group of continuous, linear operators $(\alpha_t)_{t \in \mathbb{R}}$, where $\alpha_t = \alpha_t^{-1}$ for all $t \geq 0$. Therefore, we consider each $c_0$-group to be a $c_0$-semigroup satisfying (i) and (ii) additionally.

In the sequel, we show that the main results from §2.1 have analogues in the case of $c_0$-groups. To do so, it is essential that the concepts of local equicontinuity and infinitesimal generator for $c_0$-groups and $c_0$-semigroups correspond. For the concept of local equicontinuity we have the following result:

**Lemma 2.53** Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on the locally convex topological vector space $(V, T_V)$. Let $(\beta_t)_{t \geq 0}$ denote its corresponding $c_0$-semigroup, i.e., $\beta_t = \alpha_t$ for all $t \geq 0$. Then the following two assertions are equivalent.

i. $(\alpha_t)_{t \in \mathbb{R}}$ is locally equicontinuous.

ii. $(\beta_t)_{t \geq 0}$ is locally equicontinuous.

**Proof.**

The implication $(i) \Rightarrow (ii)$ is obvious.

For $(ii) \Rightarrow (i)$, let $p \in \Pi$ and let $K$ be a compact subset of $\mathbb{R}$. Then $t_0 \geq 0$ exists such that $t_0 + K \subseteq \mathbb{R}^+$. Since the seminorm $x \mapsto (p \circ \alpha_{-t_0})(x)$ is continuous on $V$, the local equicontinuity of $(\beta_t)_{t \geq 0}$ guarantees that the seminorm

$$\max_{t \in \mathbb{R}} p(\alpha_t x) = \max_{t \in \mathbb{R} \cap K} (p \circ \alpha_{-t_0})(\beta_t x),$$

is continuous on $V$. Hence, $(\alpha_t)_{t \in \mathbb{R}}$ is locally equicontinuous.

By Lemma 2.53 we conclude that every result from section §2.1 concerning local equicontinuity has an analogue for $c_0$-groups. So, we have the following analogues for Theorem 2.21 and Theorem 2.22.
2.2. c₀-Gruppens, in Summary

Theorem 2.54 Let \((V, T_0)\) be barrelled (for example, \(V\) is an \(F\)-space or a strict \(L\)-space) and let \((\alpha_t)_{t \in \mathbb{R}}\) be a \(c_0\)-group on \((V, T_0)\). Then \((\alpha_t)_{t \in \mathbb{R}}\) is locally equicontinuous.

Proof.
Let \((\beta_t)_{t \geq 0}\) be as in Lemma 2.53. Since \(\text{Dom}(\mathcal{F}_g) = C(\mathbb{R}^+, V)\), Theorem 2.21 yields that \((\beta_t)_{t \geq 0}\) is locally equicontinuous. By Lemma 2.53 the statement follows.

Theorem 2.55 Let \((\alpha_t)_{t \in \mathbb{R}}\) be a locally equicontinuous \(c_0\)-group on \((V, T_0)\). Then \((\alpha_t)_{t \in \mathbb{R}}\) extends continuously to a locally equicontinuous \(c_0\)-group \((\bar{\alpha}_t)_{t \in \mathbb{R}}\) on the completion \(\overline{V}\) of \((V, T_0)\).

Proof.
Applying Theorem 2.25 the restricted \(c_0\)-semigroup \((\beta_t)_{t \geq 0}\) extends to a locally equicontinuous \(c_0\)-semigroup \((\bar{\beta}_t)_{t \geq 0}\) on the completion \(\overline{V}\) of \(V\). Since each \(\beta_t\) is continuously invertible, \(\bar{\beta}_t\) is continuously invertible. Furthermore, it is easily shown that \(\text{Dom}(\mathcal{F}_g) = C(\mathbb{R}^+, \overline{V})\), proving the assertion.

The infinitesimal generator for a \(c_0\)-group and its corresponding \(c_0\)-semigroup are the same.

Lemma 2.56 Let \((\alpha_t)_{t \in \mathbb{R}}, (\beta_t)_{t \geq 0}\) and \(V\) be as in Lemma 2.55 with corresponding infinitesimal generators \(\delta_a\) and \(\delta_\beta\) respectively. Then \(\delta_a = \delta_\beta\) in the sense that \(\text{Dom}(\delta_a) = \text{Dom}(\delta_\beta)\) and \(\delta_a x = \delta_\beta x\) for all \(x \in \text{Dom}(\delta_a)\).

Proof.
Obviously, we have \(\text{Dom}(\delta_a) \subseteq \text{Dom}(\delta_\beta)\) and \(\delta_a x = \delta_\beta x\) for all \(x \in \text{Dom}(\delta_a)\). So, let \(x \in \text{Dom}(\delta_\beta)\). Then for \(t \geq 0\) and \(p \in \Pi\) we have

\[
p \left( \frac{\alpha_t x - x}{t} - \delta_\beta x \right) \leq p \left( \frac{\beta_t x - x}{t} - \delta_\beta x \right) + p \left( \alpha_t \delta_\beta x - \delta_\beta x \right).
\]

(2.21)

Since \(x \in \text{Dom}(\delta_\beta)\) and since \((\alpha_t)_{t \in \mathbb{R}}\) is locally equicontinuous, the first part of the right-hand side of (2.21) tends to zero as \(t \to 0\). Furthermore, the strong continuity of \((\alpha_t)_{t \in \mathbb{R}}\) guarantees the second part of the right-hand side of (2.21) to tend to zero as \(t \to 0\). We conclude that \(x \in \text{Dom}(\delta_a)\), with \(\delta_a x = \delta_\beta x\).

As an immediate consequence of Lemma 2.56, we have the following analogue of Theorem 2.30.

Theorem 2.57 Let \((\alpha_t)_{t \in \mathbb{R}}\) be a \(c_0\)-group on the sequentially complete topological vector space \(V\) with infinitesimal generator \(\delta_a\). Let \(p\) be a polynomial with degree \(k\). Then the linear operator \(p(\delta_a)\) with domain \(\text{Dom}(\delta_a)\) is a closed linear mapping in \(V\).
Similar to the case of \( \omega \)-semigroups on sequentially complete topological vector spaces \( V \), we introduce “convolution” operators corresponding to a \( \alpha \)-group \( (\alpha_t)_{t \in \mathbb{R}} \). Therefore, let \( \mu \in \mathcal{M}_c(\mathbb{R}) \) (for the definition see §2.0), with \( \text{supp} (\mu) \subseteq [-T, \infty) \). Define \( \mu_T \in \mathcal{M}_c(\mathbb{R}^+ \) by

\[
\mu_T := \mu \ast e_T
\]

Then the operator \( \alpha[\mu] \) on \( V \) is defined by

\[
\alpha[\mu] := \alpha_{-T} \alpha[\mu_T].
\]

(2.22)

It is not hard to show that the definition of \( \alpha[\mu] \) is independent of the particular choice of \( T \geq 0 \). We observe that every \( \alpha[\mu], \mu \in \mathcal{M}_c(\mathbb{R}) \), is a continuous linear operator on \( V \). Moreover, \( \alpha[\mu] x = \int_{\mathbb{R}} \alpha_T x \mu(\cdot t) \) for all \( x \in V, \mu \in \mathcal{M}_c(\mathbb{R}) \), since for \( T \geq 0 \) large enough;

\[
\alpha[\mu] x = \alpha_T \left( \int_{\mathbb{R}^+} \alpha_T x \mu_t (dt) \right) = \alpha_T \left( \int_{[-T, \infty)} \alpha_{-T} x \mu (dt) \right) = \int_{\mathbb{R}} \alpha_T x \mu (dt).
\]

By definition (2.22), we have the following analogue of Lemma 2.39 for \( \mathcal{M}_c(\mathbb{R}) \)-elements (see [Bou2], p.71).

**Lemma 2.58** For each \( \mu \in \mathcal{M}_c(\mathbb{R}) \) there exists \( (\mu_k)_{k \in \mathbb{N}} \) in \( \text{span} \{ \alpha_t \mid t \in \mathbb{R} \} \) such that

\[
\lim_{k \to \infty} \alpha[\mu_k] x = \alpha[\mu] x
\]

for all \( x \in V \). In other words, the linear span \( \text{span} \{ \alpha_t \mid t \in \mathbb{R} \} \), is strongly sequentially dense in the collection \( \{ \alpha[\mu] \mid \mu \in \mathcal{M}_c(\mathbb{R}) \} \).

The operators \( \{ \alpha[\mu] \mid \mu \in \mathcal{M}_c(\mathbb{R}) \} \) form a strong continuous representation of the convolution algebra \( (\mathcal{M}_c(\mathbb{R}), +, \ast) \).

**Lemma 2.59** Let \( \mu_1, \mu_2 \in \mathcal{M}_c(\mathbb{R}) \). Then

\[
\alpha[\mu_1] \alpha[\mu_2] = \alpha[\mu_1 \ast \mu_2] = \alpha[\mu_2] \alpha[\mu_1].
\]

where \( \ast \) is the convolution product on \( \mathcal{M}_c(\mathbb{R}) \).

In particular, for all \( t \in \mathbb{R} \) and \( \mu \in \mathcal{M}_c(\mathbb{R}) \) we have \( \alpha_T \alpha[\mu] = \alpha[\mu_T] \alpha[T] \).

Also, the existence of an approximate identity in \( \mathcal{D}(\mathbb{R}) \) is guaranteed by Lemma 2.42.

**Lemma 2.60** There exists a sequence \( (\delta_k)_{k \in \mathbb{N}} \) in \( \mathcal{D}(\mathbb{R}) \) such that for all \( x \in V, \alpha[\delta_k] x \in \text{Dom}^\infty(\delta_k) \to x \) as \( k \to \infty \). As a consequence the subspace \( \text{Dom}^\infty(\delta_k) \) is sequentially dense in \( V \).

We end this section by a brief discussion of invariance under a \( \omega \)-group \( (\alpha_t)_{t \in \mathbb{R}} \).

**Theorem 2.61** Let \( M \) be a sequentially closed, \( (\alpha_t)_{t \in \mathbb{R}} \)-invariant subspace of \( V \). Then \( \text{Dom}^\infty(\delta_k) \cap M \) is sequentially dense in \( M \).
2.3. $c_0$-SEMINIGROPS AND $c_0$-GROUPS ON STRICT LF-SPACES

Proof.
Any sequentially closed, $(\alpha_t)_{t \in \mathbb{R}}$-invariant subspace $M$ of $V$ is a sequentially closed, $(\beta_t)_{t \geq 0}$-invariant subspace of $V$, where $(\beta_t)_{t \geq 0}$ denotes the $c_0$-semigroup corresponding to $(\alpha_t)_{t \in \mathbb{R}}$.

Lemma 2.62 Let $M$ be a sequentially closed, $(\alpha_t)_{t \in \mathbb{R}}$-invariant subspace of $V$. Then $\alpha_\mu(M) \subseteq M$ for each $\mu \in \mathcal{M}_c(\mathbf{R})$.

Proof.
Since $M$ is a sequentially closed, $(\beta_t)_{t \geq 0}$-invariant subspace of $V$, we have

$$\alpha_\mu(M) = \alpha_{-T}(\alpha_\mu(M)) \subseteq \alpha_{-T}(M) = M,$$

where $T \geq 0$ is chosen as in relation (2.22).

Finally, there is the following analogue of Theorem 2.52.

Theorem 2.63 Let $L$ be a (sequentially) closed linear $(\alpha_t)_{t \in \mathbb{R}}$-invariant operator on $V$ with domain $\text{Dom}(L)$. Then the following assertions hold.

- $\alpha_\mu x \in \text{Dom}(L)$ and $L \alpha_\mu x = \alpha_\mu L x$ for each $\mu \in \mathcal{M}_c(\mathbf{R})$ and for each $x \in \text{Dom}(L)$, we have $\alpha_\mu x \in \text{Dom}(L)$ and $L \alpha_\mu x = \alpha_\mu L x$.

- Next, assume $\text{Dom}^{\infty}(\alpha_t) \subseteq \text{Dom}(L)$, then

  - $L(\text{Dom}^{\infty}(\alpha_t)) \subseteq \text{Dom}^{\infty}(\alpha_t)$, and
  - graph($L$) is the closure in $V \times V$ (with product topology) of

    $$\text{graph}(L|_{\text{Dom}^{\infty}(\alpha_t)}) = \{(x, Lx) \mid x \in \text{Dom}^{\infty}(\alpha_t)\}.$$

2.3 $C_0$-SEMINIGROPS AND $C_0$-GROUPS ON STRICT LF-SPACES

For $F$-spaces $F$, the space $C(\mathbf{R}^+, F)$ is an $F$-space.

Lemma 2.64 Let $F$ be an $F$-space. Then $C(\mathbf{R}^+, F)$, equipped with compact-open topology, is an $F$-space.

Proof.
Suppose that the collection of seminorms $\Pi = \{p_n \mid n \in \mathbf{N}\}$ generates the $F$-topology of $F$. Then the countable collection of seminorms $\{q_n \mid n \in \mathbf{N}\}$, where

$$q_n(f) := \max_{t \in [0, n]} p_n(f(t)) \quad (f \in C(\mathbf{R}^+, F)),$$

generates the topology of $C(\mathbf{R}^+, F)$, thus $C(\mathbf{R}^+, F)$ is an $F$-space.
In this section, we focus on the special case that $V$ is a strict (pre-)LF-space.

In case $V$ is a strict inductive limit of $F$-spaces, the (sequentially) complete locally convex topology $T_0$ is more complicated, in general $T_0$ is not a strict inductive limit topology. For more details we refer to [Ve2] p. 25-27.

The structure of strict LF-spaces leads to the following properties of locally equicontinuous $C_0$-semigroups.

**Theorem 2.65** Let $V = \text{ind} F_n$ be a strict LF-space. Let $(\alpha_t)_{t \geq 0}$ be a locally equicontinuous $C_0$-semigroup on $V^n$. Let $\delta_n$ be the infinitesimal generator of $(\alpha_t)_{t \geq 0}$. Then the following assertions hold true.

i. $\forall K \subseteq \mathbb{R}^+ \exists n \in \mathbb{N} \forall \xi \in K \left[ \alpha_n(F_n) \subseteq F_n \right].$

ii. $\forall K \subseteq \mathbb{R}^+ \exists n \in \mathbb{N} \forall \mu \in L^1(\mathbb{R}^+, \mathbb{R}) \supseteq K \left[ \alpha_n(F_n) \subseteq F_n \right].$

iii. $\forall K \subseteq \mathbb{R}^+ \exists n \in \mathbb{N} \forall \mu \in L^1(\mathbb{R}^+, \mathbb{R}) \supseteq K \left[ \alpha_n(F_n) \subseteq F_n \right].$

iv. $\forall K \subseteq \mathbb{R}^+ \exists n \in \mathbb{N} \forall \xi \in K \left[ \delta_n(x) \in F_n \right].$

v. $\forall K \subseteq \mathbb{R}^+ \exists n \in \mathbb{N} \forall \xi \in K \left[ \delta_n(x) \in F_n \right].$

**Proof.**

(i). Let $K \subseteq \mathbb{R}^+$ be compact and let $n \in \mathbb{N}$. Since for each $x \in F_n$, $\mathcal{F}_n x$ is a continuous function from $\mathbb{R}^+$ into $V$, the set $(\mathcal{F}_n x)(K) = \{\alpha_t x : t \in K\}$ is a compact subset of $V$. Hence, $m \in \mathbb{N}$ exists such that $(\mathcal{F}_n x)(K) \subseteq F_m$ (see [Con], Corollary 5.17). Now, write

$$F_n = \bigcup_{m \in \mathbb{N}} \{ x \in F_n \mid \alpha_n x \in F_m \ \text{for all} \ t \in K \} =: \bigcup_{m \in \mathbb{N}} A_m.$$  

Since each $A_m$ is a closed subspace of the $F$-space $F_n$ and each $F$-space is of second category ([Tre] 10), $m_0$ exists such that $F_n = A_{m_0}$, proving the assertion.

(ii). Recalling the proof of Lemma 2.36 and recalling Lemma 2.39, there is for each $\mu \in L^1(\mathbb{R}^+, \mathbb{R})$, with $\text{supp}(\mu) \subseteq K$, a sequence $(\mu_t)_{t \geq 0}$ in $L^1(\mathbb{R}^+, \mathbb{R})$ with $\text{supp}(\mu_t) \subseteq K$ for each $t \in K$, such that $\alpha_n(x) \rightarrow \alpha_n(x)$ for all $x \in F_n$. So, the assertion follows directly from (i).

(iii). Since $(\alpha_n)_{t \geq 0}$ is locally equicontinuous, the assertion follows directly from (i).

(iv). Let $K \subseteq \mathbb{R}^+$ and $n \in \mathbb{N}$ be fixed. Then by (ii) there is an $m \in \mathbb{N}$ such that $\alpha_n(F_n) \subseteq F_m$ for all $\mu \in L^1(\mathbb{R}^+, \mathbb{R})$ with $\text{supp}(\mu) \subseteq [0, 1]$. In particular, let $(\phi_t)_{t \geq 0}$ be an approximate identity in $\mathcal{D}(\mathbb{R}^+)$ with $\text{supp}(\phi_t) \subseteq [0, 1]$. Let $x \in \text{dom}(\phi_t) \cap F_n$. Then for each $j \in \{0, 1, \ldots, k\}$ we have

$$\alpha_n[\phi_t] \delta_n^j x = \delta_n[\phi_t] \alpha_n x = (-1)^j \alpha_n(\phi_t^j) \alpha_n x,$$

where the operators $\alpha_n[\phi_t]$ and $\delta_n^j$ can be interchanged since $\delta_n^j$ is closed. The above sequence is in the $F$-space $F_m$, hence we have

$$\delta_n^k x = \lim_{k \to \infty} (-1)^j \cdot \alpha_n(\phi_t^j) \alpha_n x \in F_m.$$
which proves the assertion.
(v). Follows from the proof of (iv) directly.

Remark 2.65.1 There is an analogue of Theorem 2.65 for $c_0$-groups. In fact, since every strict LF-space is barrelled, every $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ on a strict LF-space is locally equicontinuous. So, admitting compact subsets from $\mathbb{R}$ instead of $\mathbb{R}^+$, the statement can be copied directly.

There is an analogue of Theorem 2.65 for locally equicontinuous $c_0$-semigroups (and $c_0$-groups) for strict pre-LF-spaces also. Recall from Theorem 1.40 that the completion of a strict pre-LF-space $V$ is a strict LF-space $\overline{V}$ and that every locally equicontinuous $c_0$-semigroup on $V$ extends to a locally equicontinuous $c_0$-semigroup (Theorem 2.25) on $\overline{V}$. So, applying Theorem 2.65 to (the extensions of) locally equicontinuous $c_0$-semigroups on strict pre-LF-spaces gives the following result.

Lemma 2.66 Let $V = \text{ind}\ F_n$ be a strict pre-LF-space. Let $(\alpha_t)_{t \geq 0}$ be a locally equicontinuous $c_0$-semigroup on $V$. Then the following assertions hold true.

i. $\forall_{\text{compact } K \subseteq \mathbb{R}^+} \forall_{n \in \mathbb{N}} \exists_{m \in \mathbb{N}} \forall_{t \in \mathbb{R}^+} [\alpha_t(F_n) \subseteq F_m]$.

ii. $\forall_{\text{compact } K \subseteq \mathbb{R}^+} \forall_{n \in \mathbb{N}} \exists_{m \in \mathbb{N}} [(\alpha_t(F_n))_{t \in K} \text{ is an equicontinuous subset of } L(F_n, F_m)]$.

Proof.

i. Let $\overline{V} = \text{ind}\overline{F_n}$ be the completion of $V$ (cf. Theorem 1.40). Let $(\beta_t)_{t \geq 0}$ be the extension of $(\alpha_t)_{t \geq 0}$ to $\overline{V}$. Let $K \subseteq \mathbb{R}^+$ be compact and let $n \in \mathbb{N}$. Then, there exists $m \in \mathbb{N}$, such that for all $t \in K$

$$\beta_t(F_n) \subseteq F_m.$$ 

by Theorem 2.65.i. In particular, we have for each $t \in K$

$$\alpha_t(F_n) = \beta_t(F_n) \cap V \subseteq \overline{F_n} \cap V = F_m,$$

by Theorem 1.40.

The assertion ii is an immediate consequence of i and Theorem 2.65.iii.

Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on the sequentially complete locally convex topological vector space $V$. Recall from §2.2 that for every $\mu \in \mathcal{M}^0(\mathbb{R})$ and for every $x \in V$ we have $\alpha(\mu x) \in \text{Dom}^\infty(\delta_0)$. In case $V$ is a strict LF-space we have the following remarkable result.

Theorem 2.67 Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on the strict LF-space $V$, with infinitesimal generator $\delta_0$. Then

$$\text{span}\{ \alpha(\phi) x \mid \phi \in \mathcal{D}(\mathbb{R}), x \in V \} = \text{Dom}^\infty(\delta_0),$$

In fact, for each $x \in \text{Dom}^\infty(\delta_0)$ there are $y \in \text{Dom}^\infty(\delta_0)$ and $\phi_1, \phi_2 \in \mathcal{D}(\mathbb{R})$ such that $x = \alpha(\phi_1) x + \alpha(\phi_2) y$. 


For the proof of Theorem 2.67, we refer to [vEi2], Theorem 3.11. It is based on a result of Dixmier and Malliavin [D-M] in Lie-group representation theory. Their result deals with F-spaces only. In [vEi2], the result of Dixmier and Malliavin is extended in the special case of a $c_0$-group on a strict LF-space.

We end this chapter with a brief discussion of the problem how to impose a suitable topology on the $c_0$-domain of the infinitesimal generator $\delta_n$ of a $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ (or semigroup $(\alpha_t)_{t \geq 0}$) on F-spaces and strict LF-spaces. Recall from Proposition 2.44 the introduction of the topology $T_{\text{graph}}$ for the $c_0$-domain. In case of a $c_0$-group on an F-space, we have the following result.

**Proposition 2.68** Let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on the F-space $F$ with infinitesimal generator $\delta_n$. Then $(\text{Dom}^{\infty}(\delta_n), T_{\text{graph}}) = \text{proj}_k \text{Dom}(\delta_n^k)$ is an F-space.

**Proof.**

By Proposition 1.25 each subspace $\text{Dom}(\delta_n^k)$ of $F$, equipped with the graph topology of the operator $\delta_n^k$, is an F-space. Hence, by Proposition 2.43 the family $(\text{Dom}(\delta_n^k))_{k \in \mathbb{N}}$ forms a countable left-sided chain of F-spaces. Hence, $\text{proj}_k \text{Dom}(\delta_n^k)$ is an F-space by Corollary 1.32.

The topological structure of $(\text{Dom}^{\infty}(\delta_n), T_{\text{graph}})$ in case of an F-space is satisfactory. Concepts as continuity and convergence are well described (see Chapter 1). In the case of a strict LF-space, the choice of a suitable topology for $\text{Dom}^{\infty}(\delta_n)$ is more complicated. In fact, it will turn out that there are three natural possibilities.

Let $V = \text{ind}_n F_n$ be a strict LF-space and let $(\alpha_t)_{t \in \mathbb{R}}$ be a $c_0$-group on $V$ with infinitesimal generator $\delta_n$. The first natural attempt for a topology for $\text{Dom}^{\infty}(\delta_n)$ arises applying Proposition 2.44 again. In particular, equip each $\text{Dom}(\delta_n^k)$ with its graph topology and equip $\text{Dom}^{\infty}(\delta_n)$ with the projective limit topology arising from the chain $(\text{Dom}(\delta_n^k))_{k \in \mathbb{N}}$, i.e.

\[
(\text{Dom}^{\infty}(\delta_n), T_{\text{graph}}) = \text{proj}_k \text{Dom}(\delta_n^k).
\]  

The topology $T_{\text{graph}}$ is not very satisfactory for us. In the sequel, it will be shown that $T_{\text{graph}}$ is not a strict inductive limit topology. In fact, we do not even know whether $\text{Dom}(\delta_n^k)$ is a strict LF-space. So, in describing convergence of sequences and continuity of operators in $(\text{Dom}^{\infty}(\delta_n), T_{\text{graph}})$, we lose all additional results for strict LF-spaces as presented in Chapter 1, such as the Grothendieck factorization result, the Open Mapping Theorem and the Closed Graph Theorem. We have to deal with an uncountable collection of seminorms. And what is even worse, in most concrete examples, characterizations of these seminorms are unknown.

In the second attempt for a suitable topology for $\text{Dom}^{\infty}(\delta_n)$, we solve this lack of a nice topological structure problem by the use of Theorem 2.65. In fact, define $\text{Dom}(\delta_n^k) := V$. Then by Theorem 2.65, we may decompose each $\text{Dom}(\delta_n^k)$ as follows: Define for each $n, k \in \mathbb{N}$

\[
F_{k,n} := \{ x \in \text{Dom}(\delta_n^k) \mid \delta_n^k x \in F_n \ (i = 0, 1, \ldots, k) \}.
\]  

\[
F_k := \{ x \in \text{Dom}(\delta_n^k) \mid \delta_n^k x \in F_n \ (i = 0, 1, \ldots, k) \}.
\]
Then

\[ \text{Dom}(\delta_n^k) = \bigcup_{n=1}^{\infty} F_{k,n}, \]

so we write \( \text{Dom}^{\infty}(\delta_n) \) as

\[ \text{Dom}^{\infty}(\delta_n) = \bigcap_{k=0}^{\infty} \text{Dom}(\delta_n^k) = \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{\infty} F_{k,n}, \]  \hspace{1cm} (2.25)

Obviously, we have for each \( k \in \mathbb{N}_0, n \in \mathbb{N} \)

\[ F_{k+1,n} \subseteq F_{k,n} \subseteq F_n \quad \text{and} \quad F_{k,n} \subseteq F_{k,n+1} \subseteq \text{Dom}(\delta_n^k). \]

Notice that each \( F_{k,n} \) is a closed subspace of \( \text{Dom}(\delta_n^k) \) endowed with graph topology. So, equip each \( F_{k,n} \) with restricted graph topology \( T_{k,n} \). Since each \( \text{Dom}(\delta_n^k) \) is complete, it follows that \((F_{k,n}, T_{k,n})\) is complete. Furthermore, if \( \{p_m \mid m \in \mathbb{N}\} \) is a collection of seminorms generating the Fréchet-topology of \( F_n \), then the inclusion \( F_{k,n} \subseteq F_n \) yields that the seminorms \( \{p_m \circ \delta_n^k \mid m \in \mathbb{N}, i = 0, 1, \ldots, k\} \) generate the topology \( T_{k,n} \). Thus \( T_{k,n} \) is metrizable. We conclude that each \((F_{k,n}, T_{k,n})\) is an F-space. By the seminorm description of the topologies \( T_{k,n} \) it is seen directly that

\[ (F_{k+1,n}, T_{k+1,n}) \hookrightarrow (F_{k,n}, T_{k,n}) \hookrightarrow F_n, \]  \hspace{1cm} (2.26)

and

\[ (F_{k,n}, T_{k,n}) \hookrightarrow (F_{k,n+1}, T_{k,n+1}) \hookrightarrow \text{Dom}(\delta_n^k). \]  \hspace{1cm} (2.27)

Each family \( \{(F_{k,n}, T_{k,n})\}_{k \in \mathbb{N}} \) is a strict inductive system of F-spaces. Hence, for each \( k \in \mathbb{N} \) we have

\[ \text{ind}_n (F_{k,n}, T_{k,n}) \hookrightarrow \text{Dom}(\delta_n^k). \]

Equipping \( \text{Dom}^{\infty}(\delta_n) \) with the projective limit topology \( T_{proj} \) of the family of strict LF-spaces \( \text{ind}_n (F_{k,n}), k \in \mathbb{N} \), we obtain the second suitable topology for \( \text{Dom}^{\infty}(\delta_n) \);

\[ (\text{Dom}^{\infty}(\delta_n), T_{proj}) = \text{proj ind}_n (F_{k,n}, T_{k,n}). \]  \hspace{1cm} (2.28)

The projective limit \((\text{Dom}^{\infty}(\delta_n), T_{proj})\) is a complete locally convex topological vector space. Unfortunately, the topology \( T_{proj} \) has in general not the desirable strict LF-structure.

From now on, we write \( F_{k,n} \) instead of \((F_{k,n}, T_{k,n})\).

The third possibility to topologize \( \text{Dom}^{\infty}(\delta_n) \), we obtain by applying Theorem 2.65.v. In fact, by Theorem 2.65.v we can decompose \( \text{Dom}^{\infty}(\delta_n) \) as follows

\[ \text{Dom}^{\infty}(\delta_n) = \bigcup_{n=1}^{\infty} \bigcap_{k=0}^{\infty} F_{k,n}. \]  \hspace{1cm} (2.29)
By the inclusion (2.26) the family \( (F_{\infty,n})_{n \in \mathbb{N}} \) is a left-sided chain of F-spaces. So, equip \( F_{\infty,n} := \bigcap_{k \in \mathbb{N}} F_{k,n} \) with the projective limit topology of the F-spaces \( F_{k,n} \). Then \( F_{\infty,n} \) is an F-space also (Corollary 1.32). In fact, its Fréchet topology is brought about by the seminorms \( \{(p \circ \delta_{n}^{i})|_{F_{\infty,n}} \mid p \in \Pi, i = 0, 1, \ldots\} \). Moreover, by the inclusion (2.27) we have for each \( n \in \mathbb{N} \)

\[
F_{\infty,n} \hookrightarrow F_{\infty,n+1}.
\]

In fact, the family of F-spaces \( (F_{\infty,n})_{n \in \mathbb{N}} \) is a strict inductive system. The associated strict inductive limit topology for \( \text{Dom}^{\omega}(\delta_{n}) \) we denote by \( T_{\text{ind}} \), i.e.

\[
(\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) = \text{ind proj } F_{k,n}.
\]

We emphasize that \( (\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) \) is indeed a strict LF-space. So, in dealing with convergence of sequences in \( (\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) \) and in dealing with continuity of linear operators in \( (\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) \), we can apply the results of Chapter 1. In fact, it is precisely the topology \( T_{\text{ind}} \) for \( \text{Dom}^{\omega}(\delta_{n}) \) we will use in the sequel.

Summarizing, we have introduced the following topologies for \( \text{Dom}^{\omega}(\delta_{n}) \).

**Definition 2.69** Let \( (\alpha_{n})_{n \in \mathbb{N}} \) be a \( c_{0} \)-group on the strict LF-space \( \text{ind } F_{n} \). Let \( \delta_{n} \) denote the infinitesimal generator of \( (\alpha_{n})_{n \in \mathbb{N}} \). Then

\[
(\text{Dom}^{\omega}(\delta_{n}), T_{\text{proj}}) := \text{proj Dom}(\delta_{n}^{n}),
\]

\[
(\text{Dom}^{\omega}(\delta_{n}), T_{\text{graph}}) := \text{proj ind } F_{k,n},
\]

\[
(\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) := \text{proj proj } F_{k,n},
\]

where for each \( n \in \mathbb{N}, k \in \mathbb{N} \) the F-space \( F_{k,n} \) is defined as

\[
F_{k,n} := \{ x \in \text{Dom}(\delta_{n}^{k}) \mid \delta_{n}^{k} x \in F_{n} (i = 0, 1, \ldots, k) \},
\]

equipped with restricted graph topology of \( \text{Dom}(\delta_{n}^{k}) \).

By definition the topology \( T_{\text{ind}} \) is finer than the topology \( T_{\text{proj}} \) on \( \text{Dom}^{\omega}(\delta_{n}) \), and the topology \( T_{\text{proj}} \) is finer than the topology \( T_{\text{ind}} \) on \( \text{Dom}^{\omega}(\delta_{n}) \).

**Proposition 2.70** Let \( \delta_{n} \) be the infinitesimal generator of the \( c_{0} \)-group \( (\alpha_{n})_{n \in \mathbb{N}} \) on the strict LF-space \( \text{ind } F_{n} \). Then

\[
(\text{Dom}^{\omega}(\delta_{n}), T_{\text{ind}}) \hookrightarrow (\text{Dom}^{\omega}(\delta_{n}), T_{\text{proj}}) \hookrightarrow (\text{Dom}^{\omega}(\delta_{n}), T_{\text{graph}}) \hookrightarrow \text{ind } F_{n}.
\]

Although the topologies \( T_{\text{graph}}, T_{\text{proj}} \) and \( T_{\text{ind}} \) are different in general, sequential convergence and therefore sequential continuity is the same for each of them.

**Proposition 2.71** Let \( \delta_{n} \) be the infinitesimal generator of the \( c_{0} \)-group \( (\alpha_{n})_{n \in \mathbb{N}} \) on the strict LF-space \( \text{ind } F_{n} \). Then the following statements are equivalent.
2.3. \( c_0 \)-semigroups and \( c_0 \)-groups on strict LF-spaces

- The sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\) in \((\text{Dom}^{\omega}_\omega(\delta_n), T_{\text{graph}})\).
- The sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\) in \((\text{Dom}^{\omega}_\omega(\delta_0), T_{\text{proj}})\).
- The sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\) in \((\text{Dom}^{\omega}_\omega(\delta_n), T_{\text{ind}})\).

**Proof.**

We need to prove that a convergent sequence in \((\text{Dom}^{\omega}(\delta_n), T_{\text{graph}})\) is convergent in \((\text{Dom}^{\omega}(\delta_0), T_{\text{ind}})\) only. Therefore, suppose that \(x_n \to x\) in \((\text{Dom}^{\omega}_\omega(\delta_n), T_{\text{graph}})\) = \(\text{proj} \text{Dom}^{\omega}_\omega(\delta_n)\). Then \(x_n \to x\) in \(\text{Dom}^{\omega}_\omega(\delta_n)\) for each \(k \in \mathbb{N}\). Particularly, \(x_n \to x\) in \(\text{ind} \mathcal{F}_n\), so \(m_0 \in \mathbb{N}\) exists such that \(x_n \to x\) in \(\mathcal{F}_m\). Applying Theorem 2.65.v, there exists \(m \in \mathbb{N}\) such that \(x_n \to x\) in \(\mathcal{F}_{k,n}\) for each \(k \in \mathbb{N}\). Hence, \(x_n \to x\) in \(\text{proj} \mathcal{F}_{k,n}\).

We conclude that \(x_n \to x\) in \(\text{ind} \mathcal{F}_{k,n}\), or put differently, in \(T_{\text{ind}}\)-sense.

\[ \blacksquare \]

By Theorem 1.44 we recall that sequential continuity of an operator \(L\) on a strict LF-space is equivalent to continuity of \(L\). So, applying Proposition 2.71 we obtain the following results.

**Proposition 2.72** Let \(\delta_n\) be the infinitesimal generator of the \(c_0\)-group \((\alpha_t)_{t \in \mathbb{R}}\) on the strict LF-space \(\text{ind} \mathcal{F}_n\). Let \(L\) be a linear operator on \(\text{Dom}^{\omega}(\delta_n)\). Then the following assertions are equivalent.

- \(L\) is continuous with respect to \(T_{\text{ind}}\).
- \(L\) is sequentially continuous with respect to \(T_{\text{proj}}\).
- \(L\) is sequentially continuous with respect to \(T_{\text{graph}}\).

**Proposition 2.73** Let \(\delta_n\) be the infinitesimal generator of the \(c_0\)-group \((\alpha_t)_{t \in \mathbb{R}}\) on the strict LF-space \(\text{ind} \mathcal{F}_n\). Let \(V\) be a strict LF-space and suppose \(L\) is a linear operator from \(\text{Dom}^{\omega}(\delta_n)\) into \(V\). Then the following two assertions are equivalent.

- \(L\) is continuous as a mapping from \((\text{Dom}^{\omega}(\delta_n), T_{\text{proj}})\) into \(V\).
- \(L\) is sequentially continuous as a mapping from \((\text{Dom}^{\omega}(\delta_n), T_{\text{graph}})\) into \(V\).

Finally, we give an example where the topologies \(T_{\text{proj}}\) and \(T_{\text{ind}}\) differ.

**Example 2.74** Consider the strict LF-space \(C_{\text{w}}(\mathbb{R}) = \text{ind} C_{\text{w},n}(\mathbb{R})\) with translation group \((\alpha_t)_{t \in \mathbb{R}}\). The infinitesimal generator of \((\alpha_t)_{t \in \mathbb{R}}\) equals the differentiation operator \(\frac{d}{dt}\) with domain \(C^1_{\text{w}}(\mathbb{R})\), i.e. all continuously differentiable complex valued functions on \(\mathbb{R}\) with support bounded on the left. Then the F-space \(C_{\text{w},n}(\mathbb{R})\), as in (2.25), is defined for each \(k, n \in \mathbb{N}\) by

\[ C_{\text{w},n}(\mathbb{R}) := \{ x \in C^1_{\text{w}}(\mathbb{R}) \mid x^{(i)} \in C_{\text{w},n}(\mathbb{R}) \quad (i = 0, 1, \ldots, k) \} =: C_{\text{w},n}^k(\mathbb{R}), \]

endowed with the topology of uniform convergence up to the \(k\)-th derivative on every compact subset of \(\mathbb{R}\). Besides, \(\text{Dom}^{\omega}(\delta_n) = D_{\omega}(\mathbb{R})\) (see §3.0). In fact, we have

\[ (\text{Dom}^{\omega}(\delta_n), T_{\text{proj}}) = \text{proj} \text{ind} C_{\text{w},n}^k(\mathbb{R}) =: \text{proj} C_{\text{w}}^k(\mathbb{R}), \]
so \((\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{proj}}) = C_{\omega}(\mathcal{R}) \cap \mathcal{E}(\mathcal{R})\) with intersection topology. Moreover, we have
\[
(\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{ind}}) = \operatorname{ind proj} C^{\omega}_{n_{\omega}}(\mathcal{R}) =: \operatorname{ind} D_{\omega}(\mathcal{R}),
\]
where \(D_{\omega}(\mathcal{R})\) consists of all \(C^{\omega}\)-functions on \(\mathcal{R}\) with support in \([-n, \infty)\) endowed with the topology of uniform convergence on compact subsets of \(\mathcal{R}\) in every derivative. Since the topological duals of \((\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{proj}})\) and \((\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{ind}})\) are different, the topologies \(T_{\text{proj}}\) and \(T_{\text{ind}}\) differ. For example, define the functional \(L\) on \(D_{\omega}(\mathcal{R})\) by
\[
Lx := \sum_{i=1}^{\infty} x^{(i)}(-i) \quad (x \in C^{\omega}_{\omega}(\mathcal{R})).
\]
Then \(L\) is continuous on \((\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{ind}})\), but not on \((\text{Dom}^{\omega}(\delta_{\alpha}), T_{\text{proj}})\).

Finally, it is not true that the topologies \(T_{\text{proj}}\) and \(T_{\text{graph}}\) differ in this particular example. In fact, it is not hard to show that for each \(k \in \mathbb{N}\), the space \(C^{\omega}_{k}(\mathcal{R})\) equipped with graph topology is homeomorphic with \(C_{k}(\mathcal{R})\) (apply the homeomorphism \(\frac{\partial}{\partial x}\)). Hence, \(C^{\omega}_{k}(\mathcal{R})\) equipped with graph topology is a strict LF-space, so that the Open Mapping Theorem for strict LF-spaces (Theorem 1.46) yields that \(T_{\text{graph}}\) and \(T_{\text{proj}}\) equal for \(\text{Dom}^{\omega}(\delta_{\alpha})\).

The above example of a linear functional on \(\text{Dom}^{\omega}(\delta_{\alpha})\) that is continuous with respect to \(T_{\text{ind}}\), but not with respect to \(T_{\text{proj}}\), is an example from a class of linear functionals on \(\text{Dom}^{\omega}(\delta_{\alpha})\) with this property. In particular, let \(\text{ind} F_{\alpha}\) be a proper strict LF-space, proper in the sense that it is not an F-space. Let \((\alpha_{n})_{n \in \mathbb{N}}\) be a \(c_{0}\)-group on \(\text{ind} F_{\alpha}\) with infinitesimal generator \(\delta_{\alpha}\). Then by the Hahn-Banach Theorem, there is a sequence of continuous linear functionals on \(\text{ind} F_{\alpha}\), say \((G_{n})_{n \in \mathbb{N}}\), with for each \(n \in \mathbb{N}\)
\[
G_{n} \neq 0 \quad \text{and} \quad G_{n}|_{F_{m}} = 0 \quad \text{for} \ m \leq n - 1.
\]

Now, define the linear functional \(H\) on \(\text{Dom}^{\omega}(\delta_{\alpha})\) by
\[
H := \sum_{n=1}^{\infty} G_{n} \circ \delta_{\alpha}^{n}.
\]

Then \(H\) is continuous with respect to \(T_{\text{ind}}\), but \(H\) is not continuous with respect to \(T_{\text{proj}}\). For the latter \(H\) would be extendable to \(\text{ind} F_{\alpha}\) for some \(n \in \mathbb{N}\), which is obviously not the case.
CONVOLUTION ALGEBRAS
AND THEIR IDEALS; CLOSED
TRANSLATION-IN Variant
SUBSPACES

This chapter presents three basic types of convolution algebras of distributions; in
§3.1 convolution algebras consisting of distributions with compact support (\(\mathcal{D}(\mathbb{R})\)) and
\(\mathcal{E}'(\mathbb{R}))\), in §3.2 and §3.3 convolution algebras consisting of distributions with
half-infinite support (\(\mathcal{D}_+(\mathbb{R})\) and \(\mathcal{D}_-(\mathbb{R})\), \(\mathcal{D}_\mathcal{F}(\mathbb{R})\) and \(\mathcal{D}_\mathcal{F}'(\mathbb{R})\)), and in §3.3 a convolution
algebra consisting of distributions defined on a half-line (\(\mathcal{E}'(\mathbb{R}^+)\)). Although each
of these algebras is well known from literature (see Schwartz [Schw2]), the set-up
presented here differs from the classical set-up, as will be explained now.
Indeed, let \(V\) be one of the spaces \(\mathcal{E}(\mathbb{R})\), \(\mathcal{D}_+(\mathbb{R})\), \(\mathcal{D}_-(\mathbb{R})\), and let \((\sigma_t)_{t \in \mathbb{R}}\) be the
translation group on \(V\). Then \((\sigma_t)_{t \in \mathbb{R}}\) is a \(\sigma_t\)-group on \(V\). Furthermore, define for
each \(F \in V'\) (\(\subseteq \mathcal{D}'(\mathbb{R})\)) the continuous, linear, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operator \(\sigma[F]\) on \(V\) by
\[
(\sigma[F]x)(t) = F(\sigma t x) \quad (x \in V, t \in \mathbb{R}).
\]

It turns out that \(V'\) is in one-one correspondence with the collection of all continuous,
linear \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on \(V\) via this definition. So, for each \(F_1, F_2 \in V'\)
there exists a unique \(F \in V\), such that
\[
\sigma[F_1] \circ \sigma[F_2] = \sigma[F].
\]
Defining \(F_1 \ast F_2 := F\), a product on \(V'\) is introduced herewith. Since the algebra
of all continuous, linear \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on \(V\) is commutative, the algebra
\((V', +, \ast)\) is commutative over \(\mathbb{C}\). The introduced product equals the classical
convolution product on \(V'\). Therefore, \(\mathcal{E}'(\mathbb{R})\), \(\mathcal{D}_+(\mathbb{R})\) and \(\mathcal{D}_\mathcal{F}(\mathbb{R})\) are commutative
convolution algebras (cf. [Schw2]). The commutative convolution product on \(\mathcal{E}'(\mathbb{R}^+)\)
is introduced similarly, considering the translation \(\sigma_t\)-semigroup on \(\mathcal{E}'(\mathbb{R}^+)\).

Within the perspective of the relation between the convolution product and translation-invariance, we consider for each of the spaces \(\mathcal{E}(\mathbb{R})\), \(\mathcal{D}_+(\mathbb{R})\), \(\mathcal{D}_-(\mathbb{R})\) and \(\mathcal{E}(\mathbb{R}^+)\), closed,
translation-invariant subspaces. It is shown that each closed, translation-invariant subspace
corresponds to an ideal in the corresponding convolution algebra.
The set-up of this chapter is as follows; In §3.0, we investigate the convolution algebras
\( \mathcal{D}_+(\mathbb{R}) \) and the algebra \( \mathcal{D}_-(\mathbb{R}) \). These algebras arise as necessary tools in the factorization theory presented in Chapter 5. The results in §3.0 are presented without proof, being straightforward analogues of the results proved in §3.2. In section 3.1, we give a brief introduction into the classical theory of Schwartz-distributions. We present the convolution algebra \( \mathcal{E}'(\mathbb{R}) \) and its subalgebra \( \mathcal{D}(\mathbb{R}) \) and the Fourier-transformation on both. In §3.2, we investigate the convolution algebra \( \mathcal{D}_+(\mathbb{R}) \) and the algebra \( \mathcal{D}_-(\mathbb{R}) \), following a similar set-up as in §3.1. However, there exists no Fourier transformation on these spaces. In §3.3, we focus on the convolution algebra \( \mathcal{E}'(\mathbb{R}^n) \). This algebra is used in spectral considerations in Chapter 5.

### 3.0 CONVOLUTION ON \( \mathcal{D}_+(\mathbb{R}) \) AND \( \mathcal{D}_-(\mathbb{R}) \)

Let \( \mathcal{E}(\mathbb{R}) \) be the Fréchet space consisting of all arbitrarily many times differentiable functions on \( \mathbb{R} \) equipped with compact-open topology (see §3.1.1). Let \( \mathcal{D}_+(\mathbb{R}) \) be the subspace of the Fréchet space \( \mathcal{E}(\mathbb{R}) \) defined by

\[
\mathcal{D}_+(\mathbb{R}) = \bigcup_n \{ f \in \mathcal{E}(\mathbb{R}) \mid f(t) = 0 \text{ for all } t \leq -n \} =: \bigcup_n \mathcal{D}_{+,n}(\mathbb{R}).
\]

So \( f \in \mathcal{E}(\mathbb{R}) \) belongs to \( \mathcal{D}_+(\mathbb{R}) \) if \( f \) has support bounded on the left, i.e., supp(\( f \)) \( \subseteq [-n, \infty) \) for some \( n \in \mathbb{N} \). Each \( \mathcal{D}_{+,n}(\mathbb{R}) \) being a closed subspace of \( \mathcal{E}(\mathbb{R}) \), the family \( \{ \mathcal{D}_{+,n}(\mathbb{R}) \}_{n \in \mathbb{N}} \) is a strict inductive system. Correspondingly, \( \mathcal{D}_+(\mathbb{R}) \) is equipped with the related strict LF-topology (cf. Definition 1.33).

\[
\mathcal{D}_+(\mathbb{R}) = \bigcap_n \mathcal{D}_{+,n}(\mathbb{R}).
\]  

(3.1)

We have the following (dense) inclusions

\[
\mathcal{D}(\mathbb{R}) \hookrightarrow \mathcal{D}_+(\mathbb{R}) \hookrightarrow \mathcal{E}(\mathbb{R}).
\]

(3.2)

So, the topological dual of \( \mathcal{D}_+(\mathbb{R}) \) is a subspace of \( \mathcal{D}'(\mathbb{R}) \): DEFINITION 3.1 The subspace \( \mathcal{D}'_+(\mathbb{R}) \) of \( \mathcal{D}'(\mathbb{R}) \) consists of all \( F \in \mathcal{D}'(\mathbb{R}) \) with distributional support \( \text{supp}(F) \subseteq (-\infty, T] \) for some \( T \in \mathbb{R} \). The elements of \( \mathcal{D}'_+(\mathbb{R}) \) are called distributions with support bounded on the right.

Indeed, we can extend each distribution \( F \in \mathcal{D}'(\mathbb{R}) \) continuously to \( \mathcal{D}_+(\mathbb{R}) \) if and only if \( F \) has support bounded on the right (see [Schw2], p.172). Hence, \( \mathcal{D}'_+(\mathbb{R}) = (\mathcal{D}_+(\mathbb{R}))' \). Dual to the inclusion (3.2) we have

\[
\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{D}'_+(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}).
\]

(3.3)

if we identify \( \mathcal{E}'(\mathbb{R}) \) with the subspace of \( \mathcal{D}'(\mathbb{R}) \) consisting of all distributions with compact support (see §3.1.1).

Similar to \( \mathcal{D}_+(\mathbb{R}) \), the subspace \( \mathcal{D}_-(\mathbb{R}) \) of \( \mathcal{E}(\mathbb{R}) \) is defined,

\[
\mathcal{D}_-(\mathbb{R}) = \bigcup_n \{ \phi \in \mathcal{E}(\mathbb{R}) \mid \text{supp}(\phi) \subseteq [-\infty, n] \} =: \bigcup_n \mathcal{D}_{-,n}(\mathbb{R}).
\]
3.0. Convolutions on $\mathcal{D}_+(\mathbb{R})$ and $\mathcal{D}'_-(\mathbb{R})$

So, $f \in \mathcal{C}(\mathbb{R})$ belongs to $\mathcal{D}_-(\mathbb{R})$ if $f$ has support bounded on the right.

The vector space $\mathcal{D}_-(\mathbb{R})$ can be embedded into a subspace of $\mathcal{D}'_-(\mathbb{R})$: Let $\psi \in \mathcal{D}_-(\mathbb{R})$ and $\phi \in \mathcal{D}_+(\mathbb{R})$. Define the pairing $\langle \phi, \psi \rangle$ by

$$ \langle \phi, \psi \rangle := \int_{\mathbb{R}} \phi(t) \psi(t) \, dt. $$

The mapping $\phi \mapsto \langle \phi, \psi \rangle$ is a continuous linear functional on $\mathcal{D}_+(\mathbb{R})$ for each fixed $\psi \in \mathcal{D}_-(\mathbb{R})$. Hence, with this observation

$$ \mathcal{D}_-(\mathbb{R}) \subseteq \mathcal{D}'_-(\mathbb{R}) = \mathcal{D}_+(\mathbb{R})'. $$

In addition to being a topological vector space, $\mathcal{D}_+(\mathbb{R})$ is a ring. Indeed, for each $\psi, \phi \in \mathcal{D}_+(\mathbb{R})$ the convolution product $\psi * \phi$ in $\mathcal{D}_+(\mathbb{R})$ is defined as

$$ (\psi * \phi)(t) := \int_{\mathbb{R}} \psi(t-\zeta) \phi(\zeta) \, d\zeta \quad (t \in \mathbb{R}), \quad (3.4) $$

and satisfies $(\psi * \phi)' = \psi' * \phi = \psi * \phi'$ and $\text{supp}(\psi * \phi) \subseteq \text{supp}(\psi) + \text{supp}(\phi)$.

**Theorem 3.2** (Cf. [Schw2], p.117) The vector space $\mathcal{D}_+(\mathbb{R})$ is a commutative algebra over $\mathbb{R}$ with respect to convolution. It has no zero-divisors.

Next, we relate the convolution product on $\mathcal{D}_+(\mathbb{R})$ to the convolution product on $\mathcal{D}'_-(\mathbb{R})$ (see [Schw2]). To do so, we use the concept of translation group and the fact that $\mathcal{D}_+(\mathbb{R}) \to \mathcal{D}'_-(\mathbb{R})$. Define for each $t \in \mathbb{R}$ the translation operator $\sigma_t$ on $\mathcal{D}_+(\mathbb{R})$ by

$$ (\sigma_t \phi)(s) := \phi(t+s) \quad (s \in \mathbb{R}, \phi \in \mathcal{D}_+(\mathbb{R})). \quad (3.5) $$

**Lemma 3.3** The family $(\sigma_t)_{t \in \mathbb{R}}$ forms a locally equicontinuous $\sigma$-group of continuous linear operators on $\mathcal{D}_+(\mathbb{R})$. The infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ is the everywhere defined, continuous, linear differentiation operator $\frac{d}{dt}$ on $\mathcal{D}_+(\mathbb{R})$.

Rearrange the convolution product of $\psi, \phi \in \mathcal{D}_+(\mathbb{R})$ as follows

$$ (\psi * \phi)(t) = \langle \sigma_t \psi, \phi \rangle \quad (t \in \mathbb{R}), \quad (3.6) $$

where the reflection operator is defined as $\bar{\phi}(t) = \phi(-t)$. Notice that $\bar{\phi} \in \mathcal{D}_-(\mathbb{R}) \subseteq \mathcal{D}'_-(\mathbb{R})$. So, the mapping $\psi \mapsto \psi * \phi$ defines for each $\phi \in \mathcal{D}_+(\mathbb{R})$ a continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator $\mathcal{D}_+(\mathbb{R})$. Now, replace $\phi$ in (3.6) by a $\mathcal{D}'_-(\mathbb{R})$-distribution $F$.

**Definition 3.4** Let $F \in \mathcal{D}'_-(\mathbb{R})$. Then the convolution operator $\sigma[F]$ on $\mathcal{D}_+(\mathbb{R})$ is defined as

$$ (\sigma[F] \phi)(t) := F(\sigma_t \phi) \quad (t \in \mathbb{R}, \phi \in \mathcal{D}_+(\mathbb{R})). $$
Notice that each $F \in D'_c (\mathbb{R})$ gives rise to a unique convolution operator on $D_c (\mathbb{R})$.
The collection $\{\sigma[F] | F \in D_+(\mathbb{R})\}$ consists of all linear, continuous, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators on $D_c (\mathbb{R})$ (cf. Theorem 3.46).

**Theorem 3.5** A continuous linear mapping $L$ on $D_c (\mathbb{R})$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant if and only if $L$ is of the form $L = \sigma[F]$ for some $F \in D'_c (\mathbb{R})$.

Since for $F_1, F_2 \in D'_c (\mathbb{R})$, the product of $\sigma[F_1] \sigma[F_2]$ is again a continuous linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant mapping on $D_c (\mathbb{R})$, Theorem 3.5 gives rise to a product structure on $D'_c (\mathbb{R})$.

**Definition 3.6** Let $F_1, F_2 \in D'_c (\mathbb{R})$. Then the convolution product $F_1 \ast F_2$ of $F_1$ and $F_2$ is defined by the equation $\sigma[F] \sigma[F_2] = \sigma[F_1 \ast F_2]$.

The convolution product from Definition 3.6 corresponds to the classical convolution product on $D'_c (\mathbb{R})$ (see [Schw2]). So, a different angle in considering convolution products is starting from continuous linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings on $D_c (\mathbb{R})$. To investigate whether or not $D'_c (\mathbb{R})$ has zero-divisors and to investigate commutativity of $D'_c (\mathbb{R})$, we use the following lemma (cf. Lemma 3.48).

**Lemma 3.7** All non-zero convolution operators $\sigma[F]$ are injective on $D_c (\mathbb{R})$.

We obtain the following well-known result ([Schw2], p.172).

**Theorem 3.8** The collection $\{\sigma[F] | F \in D'_c (\mathbb{R})\}$ is a commutative algebra over $\mathbb{C}$ with respect to composition and addition. It has an identity (the identity operator) and no zero-divisors.

Identifying each $F \in D'_c (\mathbb{R})$ with $\sigma[F]$, the vector space $D'_c (\mathbb{R})$ forms a commutative algebra over $\mathbb{C}$ with respect to convolution. $D'_c (\mathbb{R})$ has an identity and no zero-divisors.

The vector space $\mathcal{M}_c (\mathbb{R})$ consists of all Radon measures with half-infinite support bounded on the right, i.e. all $\mu \in \mathcal{M}(\mathbb{R})$ for which supp$(\mu) \subseteq (-\infty, T)$ for some $T > 0$. The space $\mathcal{M}_c (\mathbb{R})$, introduced in §2.0, is a subspace of $\mathcal{M}_c (\mathbb{R})$. The space $\mathcal{M}_c (\mathbb{R})$ can be defined as the dual of the strict LF-space $C_c (\mathbb{R})$ from example 2.74, see for example [EdR4], Theorem 1.12, where instead of $\mathcal{M}_c (\mathbb{R})$ the corresponding set $b_{\text{fin}} (\mathbb{R})$ is used.

The space $\mathcal{M}_c (\mathbb{R})$ can be seen as a subspace of $D'_c (\mathbb{R})$ in the following way.

**Lemma 3.9** For every $\mu \in \mathcal{M}_c (\mathbb{R})$ the mapping

$$\phi \in D_+ (\mathbb{R}) \rightarrow \int_{\mathbb{R}} \phi(r) \mu(dr) =: \langle \phi, \mu \rangle,$$

(3.7) defines a continuous linear functional on $D_c (\mathbb{R})$. 

3.0. Convolution on $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}_c'(\mathbb{R})$

Define for each $\mu \in \mathcal{M}_-(\mathbb{R})$, the convolution mapping $\sigma[\mu]$ by

$$(\sigma[\mu]\phi)(t) := \langle \phi, \mu \rangle, \quad (\phi \in \mathcal{D}_c(\mathbb{R}), \ t \in \mathbb{R}).$$

It is checked straightforwardly, that for $\mu_1, \mu_2 \in \mathcal{M}_-(\mathbb{R})$

$$\sigma[\mu_1]\sigma[\mu_2] = \sigma[\mu_1 \ast \mu_2],$$

where $\ast$ is the classical convolution product in $\mathcal{M}_-(\mathbb{R})$, i.e.

$$(\mu_1 \ast \mu_2)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} x(t + s) \mu_1(dt) \mu_2(ds) \quad (x \in C_c(\mathbb{R})).$$

**Theorem 3.10** The space $(\mathcal{M}_-(\mathbb{R}), +, \ast)$ is a subalgebra of $(\mathcal{D}_c'(\mathbb{R}), +, \ast)$, relating each $\mu \in \mathcal{M}_-(\mathbb{R})$ to the distribution $\phi \in \mathcal{D}_c(\mathbb{R}) \mapsto \left\langle \phi, \mu \right\rangle$. In particular, $(\mathcal{M}_-(\mathbb{R}), +, \ast)$ is a commutative convolution algebra over $\mathbb{C}$. It has an identity and no zero-divisors.

An important subclass of $\mathcal{M}_-(\mathbb{R})$ is established by the Radon measures corresponding to $\mathcal{D}_c(\mathbb{R})$-functions. The operators $\sigma[\phi]$, where $\phi \in \mathcal{D}_c(\mathbb{R})$, act by convolution.

**Theorem 3.11** Let $\phi \in \mathcal{D}_c(\mathbb{R})$. Then

$$\sigma[\phi]\psi = \phi * \psi = \int_{\mathbb{R}} \psi(t + \tau) \phi(\tau) d\tau,$$

for all $\psi \in \mathcal{D}_c(\mathbb{R})$.

From the theory of Chapter 2, we recall that for each $\mu \in \mathcal{M}_c(\mathbb{R})$ the mapping

$$\sigma[\mu] := \int_{\mathbb{R}} \sigma_t \mu(dt)$$

defines strongly a continuous, linear, $(c_t)_{t \in \mathbb{R}}$-invariant mapping on $\mathcal{D}_c(\mathbb{R})$. Furthermore, for each $\mu, \nu \in \mathcal{M}_c(\mathbb{R})$, the convolution product of $\mu$ and $\nu$ satisfies the equality (cf. Lemma 2.59)

$$\sigma[\mu]\sigma[\nu] = \sigma[\mu \ast \nu].$$

Notice that this product on $\mathcal{M}_-(\mathbb{R})$ corresponds to the classical convolution product on $\mathcal{M}_-(\mathbb{R})$ (see [Schw2]). So, $\mathcal{M}_-(\mathbb{R})$ is a subalgebra of $\mathcal{D}_c'(\mathbb{R})$, with identity $\sigma[\delta_0]$, where $\delta_0$ denotes the Dirac measure at 0.

The discussion of closed $(c_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}_c(\mathbb{R})$ is kept short; there are no non-trivial ones (cf. Theorem 3.55 and 3.54).

**Theorem 3.12** Let $M$ be a closed $(c_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{D}_c(\mathbb{R})$. Then $M = \mathcal{D}_c(\mathbb{R})$ or $M = \{0\}$. Equivalently, let $I$ be a weak-star closed ideal in $\mathcal{D}_c(\mathbb{R})$. Then $I = \{0\}$ or $I = \mathcal{D}_c(\mathbb{R})$. 

3.1 CONVOLUTION ON $\mathcal{D}(\mathbb{R})$ AND $\mathcal{E}'(\mathbb{R})$

We study in this section the F-space $\mathcal{E}(\mathbb{R})$ and its topological dual $\mathcal{E}'(\mathbb{R})$ and we study the strict LF-space $\mathcal{D}(\mathbb{R})$ and its topological dual $\mathcal{D}'(\mathbb{R})$ (§3.1.1). A convolution structure on $\mathcal{E}'(\mathbb{R})$, extending the classical convolution product on $\mathcal{D}(\mathbb{R})$, is introduced in relation with continuous, linear, $(\sigma_{t})_{t \in \mathbb{R}_{\geq 0}}$-invariant operators on $\mathcal{E}(\mathbb{R})$ (§3.1.2). By means of this convolution product $\mathcal{E}'(\mathbb{R})$ becomes an algebra. Introducing the classical Fourier transform on $\mathcal{D}(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$, it is proved that the algebras are integral, i.e. there are no divisors of zero. Furthermore, the space $\mathcal{M}_{e}(\mathbb{R})$, introduced in Chapter 2, is shown to be a subalgebra of $\mathcal{E}'(\mathbb{R})$. In §3.1.3 we investigate the relation between closed $(\sigma_{t})_{t \in \mathbb{R}_{\geq 0}}$-invariant subspaces of $\mathcal{E}(\mathbb{R})$ and weak-star closed ideals in $\mathcal{E}'(\mathbb{R})$.

3.1.1 The spaces $\mathcal{E}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$

Let $\mathcal{E}(\mathbb{R})$ denote the space of all arbitrarily many times differentiable functions on $\mathbb{R}$. Equipped with the topology generated (for example) by the family of seminorms $\{q_{n} | n \in \mathbb{N}_{0}\}$,

$$q_{n}(f) := \max_{t \in [-n,n]} |f^{(n)}(t)| \quad (f \in \mathcal{E}(\mathbb{R}), n \in \mathbb{N}_{0}),$$

$\mathcal{E}(\mathbb{R})$ is an F-space.

An important subspace of $\mathcal{E}(\mathbb{R})$ is the space $\mathcal{D}(\mathbb{R})$ consisting of all $f \in \mathcal{E}(\mathbb{R})$ with compact support.

Definition 3.13 Let $f$ be a continuous function on $\mathbb{R}$. Then the support of $f$, supp$(f)$, is the complement of the largest open set $U$ in $\mathbb{R}$ such that $f$ is zero on $U$.

Being closed, the support of a function is compact, if it is bounded.

The classical way to topologize $\mathcal{D}(\mathbb{R})$ is the following. Write $\mathcal{D}(\mathbb{R})$ as

$$\mathcal{D}(\mathbb{R}) = \bigcup_{n} \{ f \in \mathcal{E}(\mathbb{R}) | \text{supp}(f) \subseteq [-n,n] \} =: \bigcup_{n} \mathcal{D}_{n}(\mathbb{R}). \quad (3.8)$$

The family $(\mathcal{D}_{n}(\mathbb{R}))_{n \in \mathbb{N}}$ is a strict inductive system of closed subspaces of $\mathcal{E}(\mathbb{R})$. Therefore, $\mathcal{D}(\mathbb{R})$ is equipped with the related strict LF-topology

$$\mathcal{D}(\mathbb{R}) = \text{ind} \mathcal{D}_{n}(\mathbb{R}) \hookrightarrow \mathcal{E}(\mathbb{R}), \quad (3.9)$$

where the latter inclusion is dense.

The topological duals of $\mathcal{D}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$, denoted by $\mathcal{D}'(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$, therefore satisfy

$$\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}).$$

The dual space $\mathcal{D}'(\mathbb{R})$ is well known from literature ([Schw2]) and called the distribution space of Schwartz. Accordingly, the elements of $\mathcal{D}'(\mathbb{R})$ are called distributions.

The dual space $\mathcal{E}'(\mathbb{R})$ is a well-described subspace of $\mathcal{D}'(\mathbb{R})$; it consists of all distributions with compact support ([Schw2], p.89).
3.1. Convolution on $\mathcal{D}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$

**Definition 3.14** Let $F \in \mathcal{D}'(\mathbb{R})$. The (distributional) support of $F$, $\text{supp}(F)$, is the complement of the largest open set $U$ in $\mathbb{R}$ for which

$$\forall \phi \in \mathcal{D}(\mathbb{R}) : \text{supp}(\phi) \subseteq U \Rightarrow F(\phi) = 0.$$ 

Since the support of a distribution is closed, it is compact if and only if it is bounded.

The spaces $\mathcal{E}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ will be regarded as subspaces of $\mathcal{D}'(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$ in the following way. Define the pairing $<.,.>$ on $\mathcal{E}(\mathbb{R}) \times \mathcal{D}(\mathbb{R})$ by

$$\int_{\mathbb{R}} f(\tau) \phi(\tau) \, d\tau := <f, \phi> \quad (f \in \mathcal{E}(\mathbb{R}), \phi \in \mathcal{D}(\mathbb{R})).$$

We regard $\phi$ as the $\mathcal{E}'(\mathbb{R})$-distribution $f \in \mathcal{E}(\mathbb{R}) \mapsto <f, \phi>$ and $f$ as the $\mathcal{D}'(\mathbb{R})$-distribution $\phi \in \mathcal{D}(\mathbb{R}) \mapsto <f, \phi>$. So,

$$\mathcal{E}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}) \text{ and } \mathcal{D}(\mathbb{R}) \subseteq \mathcal{E}'(\mathbb{R}).$$

Define for each $t \in \mathbb{R}$ the translation operator $\sigma_t$ on $\mathcal{E}(\mathbb{R})$ by

$$(\sigma_t f)(x) := f(x + t) \quad (s \in \mathbb{R}, f \in \mathcal{E}(\mathbb{R})). \quad (3.10)$$

**Lemma 3.15** The family $(\sigma_t)_{t \in \mathbb{R}}$ is a $c_0$-group of continuous linear operators on $\mathcal{E}(\mathbb{R})$. The infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ is the (everywhere defined) differentiation operator $\frac{d}{dt}$ on $\mathcal{E}(\mathbb{R})$.

The translation group $(\sigma_t)_{t \in \mathbb{R}}$ is locally equicontinuous, being a $c_0$-group on an F-space (Theorem 2.54).

Since $\sigma_t(\mathcal{D}(\mathbb{R})) \subseteq \mathcal{D}(\mathbb{R})$ for all $t \in \mathbb{R}$, restricting the translation group $(\sigma_t)_{t \in \mathbb{R}}$ to $\mathcal{D}(\mathbb{R})$ defines the translation group on $\mathcal{D}(\mathbb{R})$, which, for notational convenience, we denote again by $(\sigma_t)_{t \in \mathbb{R}}$.

**Lemma 3.16** The family $(\sigma_t)_{t \in \mathbb{R}}$ forms a locally equicontinuous $c_0$-group of continuous linear operators on $\mathcal{D}(\mathbb{R})$. The infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ is the differentiation operator $\frac{d}{dt}$ on $\mathcal{D}(\mathbb{R})$.

The topological vector space $\mathcal{D}(\mathbb{R})$ has a ring-structure. Indeed, the convolution product $\psi \ast \phi$ in $\mathcal{D}(\mathbb{R})$ is defined as

$$(\psi \ast \phi)(t) := \int_{\mathbb{R}} \psi(t - \tau) \phi(\tau) \, d\tau \quad (t \in \mathbb{R}), \quad (3.11)$$

satisfying $(\psi \ast \phi)' = \psi' \ast \phi = \psi \ast \phi'$ and $\text{supp}(\psi \ast \phi) \subseteq \text{supp}(\psi) + \text{supp}(\phi)$.

Thus, $\mathcal{D}(\mathbb{R})$ is a commutative algebra over $\mathbb{C}$ (see [Schw2]). To deduce more properties of the algebra $(\mathcal{D}(\mathbb{R}), +, \ast)$, we introduce the classical Fourier-transformation on $\mathcal{D}(\mathbb{R})$. 

Definition 3.17 For every \( \phi \in \mathcal{D}(\mathbb{R}) \) the Fourier transform \( \mathcal{F}\phi \) of \( \phi \) is defined by

\[
(\mathcal{F}\phi)(\omega) := \int_{\mathbb{R}} \phi(t)e^{-i\omega t} \, dt = <\epsilon_\omega, \phi> \quad (\omega \in \mathbb{C}),
\]

where \( \epsilon_\omega(t) = e^{-i\omega t} \) for all \( t \in \mathbb{R} \).

It is well known from literature that the Fourier transformation is an algebra homomorphism from \( \mathcal{D}(\mathbb{R}) \) into the commutative algebra \( \mathcal{A}(\mathbb{C}) \) of entire functions. So, since the algebra \( \mathcal{A}(\mathbb{C}) \) has no zero-divisors, \( \mathcal{D}(\mathbb{R}) \) has no zero-divisors.

The following version of the famous Paley-Wiener-Schwartz Theorem presents a complete characterization of the Fourier-image of \( \mathcal{D}(\mathbb{R}) \) in terms of a subalgebra of \( \mathcal{A}(\mathbb{C}) \).

**Theorem 3.18 (Paley-Wiener-Schwartz I)** An entire function \( h \) is the Fourier transform of some \( \phi \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp}(\phi) \subseteq [-a, a] \) if and only if there exists for each \( m \in \mathbb{N}_0 \) a constant \( C(m) > 0 \) such that

\[
|h(\omega)| \leq C(m) \cdot (1 + |\omega|)^{-m} e^{|\text{Im} \omega|} \quad (\omega \in \mathbb{C}).
\]

### 3.1.2 A convolution structure for \( \mathcal{E}'(\mathbb{R}) \)

In this subsection, we extend the convolution structure of \( \mathcal{D}(\mathbb{R}) \) to \( \mathcal{E}'(\mathbb{R}) \), so that \( \mathcal{D}(\mathbb{R}) \) is a subalgebra of \( \mathcal{E}'(\mathbb{R}) \). We show that \( \mathcal{M}_c(\mathbb{R}) \) is a subalgebra of \( \mathcal{E}'(\mathbb{R}) \). Also, we extend the Fourier transformation from \( \mathcal{D}(\mathbb{R}) \) to \( \mathcal{E}'(\mathbb{R}) \).

Let us rewrite the convolution product \( \psi \ast \phi \) in \( \mathcal{D}(\mathbb{R}) \) a slightly. For \( \psi, \phi \in \mathcal{D}(\mathbb{R}) \), we have

\[
(\psi \ast \phi)(t) = \int_{\mathbb{R}} \psi(t + \tau) \phi(\tau) \, d\tau = <\sigma_t \psi, \phi>, \quad (3.12)
\]

where \( \hat{\phi} \) is the reflection of \( \phi \). In particular, we have \( \psi \ast \phi = \sigma_t \psi \). Moreover, for each \( \phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}) \), we have

\[
\sigma_t [\phi_1] \sigma_t [\phi_2] \psi = \hat{\phi}_2 \ast (\hat{\phi}_1 \ast \psi) = \sigma_t [\phi_1 \ast \phi_2] \psi
\]

for all \( \psi \in \mathcal{D}(\mathbb{R}) \). So, if we identify each \( \phi \in \mathcal{D}(\mathbb{R}) \) with its action \( \sigma_t \phi \) as a continuous, linear, \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant operator on \( \mathcal{D}(\mathbb{R}) \), then the convolution product \( \phi \ast \psi \) in \( \mathcal{D}(\mathbb{R}) \) corresponds with the operator product \( \sigma_t [\phi] \sigma_t [\psi] \) on \( \mathcal{D}(\mathbb{R}) \). In fact, in the sequel we show that the convolution of \( \mathcal{E}'(\mathbb{R}) \)-elements corresponds to the product of continuous, linear, \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant operators on \( \mathcal{D}(\mathbb{R}) \) in the same way. To avoid too many technicalities, we focus on continuous, linear, \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant operators on \( \mathcal{E}(\mathbb{R}) \) instead of \( \mathcal{D}(\mathbb{R}) \).

Since \( (\sigma_t)_{t \in \mathbb{R}} \) is a locally equicontinuous \( \epsilon_0 \)-group on \( \mathcal{E}(\mathbb{R}) \), each \( \sigma_t [\phi] \) is a continuous,
linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $\mathcal{E}(\mathbb{R})$ (see the theory of Chapter 2). Thus, the convolution $\ast$ between $\mathcal{E}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ can also be defined by:

$$(f \ast \phi)(t) = \langle \sigma_t f, \phi \rangle = \langle \mathcal{F}(\phi), \mathcal{F}(\sigma_t f) \rangle \quad (f \in \mathcal{E}(\mathbb{R}), \phi \in \mathcal{D}(\mathbb{R})).$$

We lose none of the properties of the mapping $\sigma[\phi]$ when replacing the function $\phi$ by an $\mathcal{E}'(\mathbb{R})$-distribution $F$. In particular, define for each $F \in \mathcal{E}'(\mathbb{R})$ and for each $f \in \mathcal{E}(\mathbb{R})$ the function $g$ by

$$g(t) := F(\sigma_t f) \quad (t \in \mathbb{R}).$$

By the strong continuity of $(\sigma_t)_{t \in \mathbb{R}}$, the function $g$ is continuous on $\mathbb{R}$. In fact, we have that $g \in \mathcal{E}(\mathbb{R})$, with $g'(t) = \mathcal{F}(\sigma_t \mathcal{F}' f)$.

**Definition 3.19** Let $F \in \mathcal{E}'(\mathbb{R})$. Then the convolution operator $\sigma[F]$ on $\mathcal{E}(\mathbb{R})$ is defined as

$$(\sigma[F] f)(t) := F(\sigma_t f) \quad (t \in \mathbb{R}, f \in \mathcal{E}(\mathbb{R})).$$

Translation operators are the simplest examples of convolution operators. Let the delta distribution $\delta_t$ in $t \in \mathbb{R}$ be defined by

$$\delta_t(f) = f(t) \quad (f \in \mathcal{E}(\mathbb{R})), \quad (3.13)$$

then for each $t \in \mathbb{R}$

$$\sigma[\delta_t] = \sigma_t. \quad (3.14)$$

**Theorem 3.20** A mapping $L$ on $\mathcal{E}(\mathbb{R})$ is a convolution operator if and only if $L$ is a continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $\mathcal{E}(\mathbb{R})$.

**Proof.**

"If"-part. The functional $f \mapsto (Lf)(0)$ is continuous on $\mathcal{E}(\mathbb{R})$. Denote this $\mathcal{E}'(\mathbb{R})$-distribution by $F$, then for all $f \in \mathcal{E}(\mathbb{R})$ and all $t \in \mathbb{R}$

$$(Lf)(t) = L(\sigma_t f)(0) = F(\sigma_t f),$$

so $L = \sigma[F]$.

"Only if"-part. Since the operator $\sigma[F]$ is closed, the Closed Graph Theorem for F-spaces (Theorem 1.29) yields that $\sigma[F]$ is continuous.

Notice that the correspondence between $F \in \mathcal{E}'(\mathbb{R})$ and the convolution operator $\sigma[F]$ is one-one. So, since the product of two continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators on $\mathcal{E}(\mathbb{R})$ is again continuous, linear and $(\sigma_t)_{t \in \mathbb{R}}$-invariant on $\mathcal{E}(\mathbb{R})$, Theorem 3.20 gives rise to another way of introducing of the convolution structure on $\mathcal{E}'(\mathbb{R})$ than the classical introduction (cf. [Schw2]).

**Definition 3.21** Let $F_1, F_2 \in \mathcal{E}'(\mathbb{R})$. Then the convolution product $F_1 \ast F_2$ of $F_1$ and $F_2$ is defined by the equation $\sigma[F_1] \sigma[F_2] = \sigma[F_1 \ast F_2]$. 
The following lemma will be used frequently.

**Lemma 3.22** Let $F, G \in \mathcal{E}'(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$. Then $(F \ast G)(f) = F(\sigma[G]f)$.

In [Soe], Corollary 2.4.27, Soethoudt deduced the following useful characterization for $\mathcal{E}'(\mathbb{R})$.

**Proposition 3.23** A linear functional $F$ on $\mathcal{E}(\mathbb{R})$ is continuous if and only if

$$F(f) := \int_{\mathbb{R}} \left( P \left( \frac{d}{dt} \right) f \right)(\tau) \mu(d\tau) \quad (f \in \mathcal{E}(\mathbb{R})). \quad (3.15)$$

for some $\mu \in \mathcal{M}_c(\mathbb{R})$ and some polynomial $P$.

**Remark 3.23.1** In the sequel we use the notation $F = \langle \mu, P \rangle$ for the representation of $F \in \mathcal{E}'(\mathbb{R})$ in terms of (3.15). So, if we define $\mathcal{P}$ to be the collection of all polynomials, then $\mathcal{E}'(\mathbb{R})$ corresponds to $\mathcal{M}_c(\mathbb{R}) \otimes \mathcal{P}$. The reader should be warned that this correspondence is not one-one. For example, let $\phi \in \mathcal{D}(\mathbb{R})$ be differentiable, then for all $f \in \mathcal{E}(\mathbb{R})$

$$\int_{\mathbb{R}} f'(\tau) \phi(\tau) d\tau = -\int_{\mathbb{R}} f(\tau) \phi'(\tau) d\tau.$$

So $(\phi, \mathcal{D})$ and $(\phi', -\mathcal{D})$ correspond to the same $\mathcal{E}'(\mathbb{R})$-functional.

Applying the representation result 3.23 to Theorem 3.20, we find the following representation of convolution operators on $\mathcal{E}(\mathbb{R})$.

**Theorem 3.24** Every continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mapping on $\mathcal{E}(\mathbb{R})$ is of the form

$$L = \sigma[\mu] P \left( \frac{d}{dt} \right) \quad (3.16)$$

for some $\mu \in \mathcal{M}_c(\mathbb{R})$ and $P \in \mathcal{P}$.

Theorem 3.24 relates the theory of Chapter 2 to the theory of convolution operators on $\mathcal{E}(\mathbb{R})$. Recall from Lemma 2.7, that the linear span of all translation operators $\sigma_t$ is strongly dense in the collection of all $\sigma[\mu]$. So, since the infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{E}(\mathbb{R})$ is the differentiation operator, we obtain the following results.

**Lemma 3.25** Let $L$ be a continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mapping on $\mathcal{E}(\mathbb{R})$. Then

$$Lf \in \overline{\text{span}\{\sigma_t f \mid t \in \mathbb{R}\}},$$

for each $f \in \mathcal{E}(\mathbb{R})$.

**Proposition 3.26** For every $F \in \mathcal{E}'(\mathbb{R})$, $F \in \overline{\text{span}\{\sigma_t \mid t \in \mathbb{R}\}}$, where the closure is in weak-star sense.
3.1. CONVOLUTION ON $\mathcal{D}(\mathbb{R})$ AND $\mathcal{E}'(\mathbb{R})$

Proof.
Let $F \in \mathcal{E}'(\mathbb{R})$. Then, Lemma 3.25 yields for each $f \in \mathcal{E}(\mathbb{R})$

$$F(f) = (σ[F](f)(0) \in \text{span}\{σ[t](0) \mid t \in \mathbb{R}\} = \text{span}\{δ, f \mid t \in \mathbb{R}\}.$$ 

Using the characterization of all convolution operators on $\mathcal{E}(\mathbb{R})$ from Theorem 3.20, the convolution product on $\mathcal{E}'(\mathbb{R})$ follows directly. Let $μμ_1, μμ_2 \in \mathcal{M}_x(\mathbb{R})$. Recall from Chapter 2 that $σ[μμ_1]σ[μμ_2] = σ[μμ_1 * μμ_2]$, where $*$ denotes the convolution on $\mathcal{M}_x(\mathbb{R})$.

Then

$$σ[μμ_1, P_1](μμ_2, P_2) = σ[μμ_1, P_1]σ[μμ_2, P_2] = σ[μμ_1, P_1]σ[μμ_2, P_2](\frac{d}{dt}) = σ[μμ_1, P_1](μμ_2, P_2)(P_1, P_2)(\frac{d}{dt}) = σ[μμ_1 * μμ_2, P_1, P_2].$$

We conclude that $(μμ_1, P_1) * (μμ_2, P_2) = (μμ_1 * μμ_2, P_1, P_2)$. Since the operators $σ[μμ_1]$ and $σ[μμ_2]$ commute (see Lemma 2.8), this yields the well known fact that the convolution product on $\mathcal{E}'(\mathbb{R})$ is commutative (see [Schw2], p.172). To show that the convolution algebra $\mathcal{E}'(\mathbb{R})$ has no zero-divisors, one uses Fourier theory. Since $(Fφ)(ω) = <φ, ω >, ω \in \mathbb{C}$, $φ \in \mathcal{D}(\mathbb{R})$, Fourier transformation extends from $\mathcal{D}(\mathbb{R})$ to $\mathcal{E}'(\mathbb{R})$.

Definition 3.27 For every $F \in \mathcal{E}'(\mathbb{R})$ the Fourier transform $\mathcal{F}F$ of $F$ is defined by

$$(\mathcal{F}F)(ω) := F(ω), \quad (ω \in \mathbb{C}),$$

where $e_ω(t) := e^{-iωt}, t \in \mathbb{R}, ω \in \mathbb{C}$.

The Fourier transformation, the convolution product of $\mathcal{E}'(\mathbb{R})$ becomes (function) multiplication in $\mathcal{A}(\mathbb{C})$. Some properties:

Proposition 3.28 Let $F, G \in \mathcal{E}'(\mathbb{R})$, then the following statements hold true.

i. If $F$ is represented by $(μ, P) \in \mathcal{M}_x(\mathbb{R}) \oplus P$ in the sense of Proposition 3.23, then

$$(\mathcal{F}F)(ω) = P(-iω) \cdot \int_\mathbb{R} e_ω(τ) μ(dτ). \quad (3.17)$$

ii. $\mathcal{F}F$ is an entire function.

iii. The Fourier transform $\mathcal{F}$ from $\mathcal{E}'(\mathbb{R})$ into $\mathcal{A}(\mathbb{C})$ is linear and injective.

iv. For every $ω \in \mathbb{C}$, $σ[φ]e_ω = F(e_ω) = (\mathcal{F}F)(ω) \cdot e_ω$, i.e. $e_ω$ is an eigenvector of $σ[φ]$ with eigenvalue $(\mathcal{F}F)(ω)$.

v. $\mathcal{F}(F * G) = \mathcal{F}(F) \mathcal{F}(G) = \mathcal{F}(G * F)$.
Proof.
Assertions i, iv follow after straightforward calculation.
ii. For each $\mu \in \mathcal{M}(\mathbb{R})$, $\omega \in \mathbb{C}$ and for each $n \in \mathbb{N}$ we have
\[
\frac{d^n}{d\omega^n} \int_{\mathbb{R}} e^{-i\omega t} d\mu(t) = (-i)^n \int_{\mathbb{R}} t^n e^{-i\omega t} \mu(dt).
\]
Combining this with (3.17) the statement follows.
iii. Suppose $\mathcal{F}(\mu, P) = 0$. Then by ii, $\int_{\mathbb{R}} t^n \mu(dt) = 0$ for each $n \in \mathbb{N}$. Since $\mathcal{M}(\mathbb{R})$ represents the dual of the F-space $C(\mathbb{R})$ by means of this integral and since the collection of all polynomials is dense in $C(\mathbb{R})$ as a consequence of the Weierstrass Approximation Theorem, the assertion follows.
v. $(\mathcal{F}(F * G))(\omega) = (\sigma[F]G)(\omega) = (\mathcal{F}(G))(\omega)$.
\[\square\]

Remark 3.28.1 Notice that if $(\mu, P) = (\nu, Q) \in \mathcal{E}'(\mathbb{R})$, then for all $\omega \in \mathbb{C}$
\[
P(-i\omega) \cdot \int_{\mathbb{R}} e_{\omega}(t) \mu(dt) = Q(-i\omega) \cdot \int_{\mathbb{R}} e_{\omega}(t) \nu(dt).
\]
So, the representation $\mathcal{M}(\mathbb{R}) \otimes \mathbb{R}$ for $\mathcal{E}'(\mathbb{R})$ is unique up to exchangeable zeros in its Fourier image.

The following version of the Paley-Wiener-Schwartz Theorem ([Schw2], p.271) fully characterises the Fourier transforms of $\mathcal{E}'(\mathbb{R})$-elements.

Theorem 3.29 (Paley-Wiener-Schwartz II) An entire function $\phi \in \mathcal{A}(\mathbb{C})$ is the Fourier transform of some $F \in \mathcal{E}'(\mathbb{R})$ if and only if there are $C, a > 0$, $N \in \mathbb{N}_0$ such that
\[
|\phi(\omega)| \leq C \cdot (1 + |\omega|)^N e^{3|\omega|} \quad (\omega \in \mathbb{C}).
\]

Remark 3.29.1 If the entire function $h$ fulfills the inequality of Theorem 3.29, $h$ is the Fourier transform of a distribution $F$ supported in $[-a, a]$. If $F = (\mu, P)$ this implies that $\text{supp} (\mu) \subseteq [-a, a]$, and degree($P$) $\leq N + 1$ can be taken.

Since the algebra $\mathcal{A}(\mathbb{C})$ has no zero-divisors, Proposition 3.28 yields the following.

Theorem 3.30 The collection $\{\sigma[F] \mid F \in \mathcal{E}'(\mathbb{R})\}$ forms a commutative algebra over $\mathbb{C}$ with respect to composition and addition. It has an identity (the identity operator) and no zero-divisors.
Identifying $F \in \mathcal{E}'(\mathbb{R})$ with $\sigma[F]$, the vector space $\mathcal{E}'(\mathbb{R})$ forms a commutative algebra over $\mathbb{C}$ with respect to convolution. $\mathcal{E}'(\mathbb{R})$ has an identity and no zero-divisors.

As mentioned before, we could have introduced the convolution structure on $\mathcal{E}'(\mathbb{R})$ considering continuous, linear and translation-invariant mappings on $\mathcal{D}(\mathbb{R})$. This assertion is an immediate consequence of the following result.
3.1. CONVOLUTION ON \( \mathcal{D}(\mathbb{R}) \) AND \( \mathcal{E}'(\mathbb{R}) \)

Lemma 3.31  Every continuous linear \((\sigma_l)_{l \in \mathbb{N}}\)-invariant operator on \( \mathcal{D}(\mathbb{R}) \) can be extended uniquely to a continuous linear \((\sigma_l)_{l \in \mathbb{N}}\)-invariant operator on \( \mathcal{E}'(\mathbb{R}) \).

Proof. Suppose \( L \) is a continuous linear translation-invariant operator on \( \mathcal{D}(\mathbb{R}) \). Then \( F \in \mathcal{D}'(\mathbb{R}) \) exists such that

\[
F(\sigma_l f) = (Lf)(t) \quad (t \in \mathbb{R}, f \in \mathcal{D}(\mathbb{R})).
\]  (3.18)

We show that \( F \) has compact support, so that \( F \) can be extended to an element of \( \mathcal{E}'(\mathbb{R}) \). In that case \( L \) can be extended continuously to \( \mathcal{E}'(\mathbb{R}) \) by (3.18).

Since \( L \) is continuous on the strict LF-space \( \mathcal{D}(\mathbb{R}) = \text{ind} \mathcal{D}_m(\mathbb{R}) \) there is an \( m \in \mathbb{N} \), such that

\[
L(\mathcal{D}_m(\mathbb{R})) \subseteq \mathcal{D}_m(\mathbb{R}).
\]  (3.19)

Writing every \( \phi \in \mathcal{D}_k(\mathbb{R}) \) as the sum of translates of \( \mathcal{D}_1(\mathbb{R}) \)-functions, (3.19) yields that

\[
L(\mathcal{D}_k(\mathbb{R})) \subseteq \mathcal{D}_{m+k-1}(\mathbb{R})
\]

for each \( k \in \mathbb{N} \). Next, let \( \phi \in \mathcal{D}(\mathbb{R}) \). Then there are \( t \in \mathbb{R}, l \in \mathbb{N} \) such that \( \text{supp}(\phi) \subseteq [t-l,t+l] \). Since \( \sigma_l \phi \in \mathcal{D}_1(\mathbb{R}) \), we obtain

\[
L\phi = \sigma_{-l} \sigma_l \phi = \sigma_{-l}(L \sigma_l \phi) \in \sigma_{-l}(\mathcal{D}_{m+l-1}(\mathbb{R})).
\]  (3.20)

Hence, for all \( |l| \leq m+l-1 \) we have

\[
F(\phi) = (L\phi)(0) = 0.
\]

We conclude that for all \( \phi \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp}(\phi) \subseteq \mathbb{R} \setminus [-m+1, m-1] \) we have \( F(\phi) = 0 \). So \( F \) has compact support, in particular \( \text{supp}(F) \subseteq [-m+1, m-1] \).

By Proposition 3.23, we obtain the following representation result for continuous linear translation-invariant operators on \( \mathcal{D}(\mathbb{R}) \).

Corollary 3.32  A mapping \( L \) on \( \mathcal{D}(\mathbb{R}) \) is continuous, linear and translation-invariant if and only if \( L \) is of the form \( L = \sigma(\mu)(P(\frac{d}{dx})) \) for some \( \mu \in \mathcal{M}(\mathbb{R}) \) and some polynomial \( P \).

We conclude that the algebra of continuous, linear \((\sigma_l)_{l \in \mathbb{N}}\)-invariant mappings on \( \mathcal{D}(\mathbb{R}) \) and the convolution algebra \( \mathcal{E}'(\mathbb{R}) \) are isomorphic.

3.1.3 Closed, translation-invariant subspaces of \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \)

We end this section with a discussion of closed, \((\sigma_l)_{l \in \mathbb{N}}\)-invariant subspaces of \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \) and their relation with closed ideals in the convolution ring \( \mathcal{E}'(\mathbb{R}) \) and modules over the convolution ring \( \mathcal{E}'(\mathbb{R}) \).

From Schwartz [Schw1], we mention the following characterization result for closed, \((\sigma_l)_{l \in \mathbb{N}}\)-invariant subspaces of \( \mathcal{E}(\mathbb{R}) \).
Theorem 3.33 A closed linear subspace \( M \) of \( E(\mathbb{R}) \) is \((\sigma_t)_{t \in \mathbb{R}}\)-invariant if and only if there is a countable set \( \Sigma \subseteq \mathbb{C} \) and a mapping \( N : \Sigma \to \mathbb{N} \) such that
\[
M = \overline{\text{span}}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 0, 1, \ldots, N(\lambda) - 1\},
\]
where \( e_{\lambda,j}(t) = e^{\lambda t j} \), \( t \in \mathbb{R} \).

Remark 3.33.1 In [Kah] and [Kah2], Kahane has given a simpler proof for a related statement in the context of mean periodic functions, i.e., \( E(\mathbb{R}) \)-functions for which the set of all its translates is not total in \( E(\mathbb{R}) \). In fact in [Kah2] p.29, the above assertion is proved for closed proper subspace \( M \) of \( E(\mathbb{R}) \) which are of the form \( M = \overline{\text{span}}\{\sigma_t f \mid t \in \mathbb{R}\} \) for some \( f \in E(\mathbb{R}) \). The above proposition is then an immediate corollary to the Hahn-Banach Theorem. A posteriori, it follows that each closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace \( M \) of \( E(\mathbb{R}) \) is of the form \( M = \overline{\text{span}}\{\sigma_t f \mid t \in \mathbb{R}\} \).

We present the Schwartz-approach ([Schw1]) to this problem. For a closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace \( M \) of \( E(\mathbb{R}) \), \((M^o)^o\) equals \( M(1.5) \), where \((M^o)^o\) is the bipolar of \( M \). So, \( M \) is characterized in terms of its polar \( M^o \). The subspace \( M^o \) of \( E'(\mathbb{R}) \) has the following properties.

Lemma 3.34 Let \( M \) be a closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( E(\mathbb{R}) \). Let \( M^o \subseteq E'(\mathbb{R}) \) be the polar of \( M \). Then the following assertions are true.

i. For each \( F \in E'(\mathbb{R}) \) we have \( F \in M^o \) if and only if \( \sigma[F](M) = \{0\} \).

ii. For each \( F \in E'(\mathbb{R}) \) we have \( \sigma[F](M) \subseteq M \).

Proof.

i. The sufficiency of the assertion is obvious. Now, let \( F \in M^o \) and \( f \in M \). Since \( \sigma_t f \in M \) for all \( t \in \mathbb{R} \), we have \( (\sigma[F]f)(t) = F(\sigma_t f) = 0 \).

ii. Let \( F \in E'(\mathbb{R}) \) and \( f \in M \). Then, Lemma 3.22 yields for each \( G \in M^o \)
\[
G(\sigma[F]) = (G * F)(f) = F(\sigma[G]f) = 0.
\]
We conclude that \( \sigma[F]f \in M \).

We come to the following characterization of closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspaces of \( E(\mathbb{R}) \).

Theorem 3.35 Let \( M \) be a \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( E(\mathbb{R}) \), then \( M^o \) is a weak-star closed ideal in \( E'(\mathbb{R}) \). Conversely, let \( I \) be an ideal in \( E'(\mathbb{R}) \), then \( I^* = \{f \in E(\mathbb{R}) \mid \forall \phi \in I, \phi(f) = 0\} \) is a closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( E(\mathbb{R}) \).

Proof.
Let \( M \) be a \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( E(\mathbb{R}) \). Then, \( M^o \) is a subalgebra of \( E'(\mathbb{R}) \) by Lemma 3.34.i. Moreover, the polar \( M^o \) is weak-star closed by Lemma 1.13. Now, let \( F \in M^o \) and let \( G \in E'(\mathbb{R}) \). Then for all \( f \in M \)
\[
(F * G)(f) = G(\sigma[F]f) \overset{\text{(3.24)}}{=} 0.
\]
3.1. CONVOLUTION ON $\mathcal{D}(\mathbb{R})$ AND $\mathcal{E}'(\mathbb{R})$

Hence, $F * G \in M^o$ and $M^o$ is an ideal in $\mathcal{E}'(\mathbb{R})$.

Next, let $I$ be an ideal in $\mathcal{E}'(\mathbb{R})$. Then $I^o$ is a closed subspace of $\mathcal{E}'(\mathbb{R})$. Moreover, let $f \in I^o$ and $t \in \mathbb{R}$ be fixed. Since $I$ is an $\mathcal{E}'(\mathbb{R})$-ideal, we have for each $F \in I$, $F * \delta_t \in I$ and

$$F(\sigma_t f) = F(\sigma_t \delta_t f) = (F * \delta_t)(f) = 0,$$

where $\delta_t$ denotes a delta-distribution. We conclude that $\sigma_t f \in I^o$.

Theorem 3.35 relates $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{E}'(\mathbb{R})$ to weak-star closed ideals in $\mathcal{E}'(\mathbb{R})$. The following result can be used to check whether a weak-star closed subspace of $\mathcal{E}'(\mathbb{R})$ is an ideal.

**Lemma 3.36** Let $I$ be a weak-star closed subspace of $\mathcal{E}'(\mathbb{R})$. Then, the following two assertions are equivalent:

1. $I$ is an ideal in $\mathcal{E}'(\mathbb{R})$.
2. For all $t \in \mathbb{R}$ and for all $F \in I$, $\delta_t * F \in I$.

**Proof.**

1. $\Rightarrow$ 2. is true by definition.
2. $\Rightarrow$ 1. is an immediate consequence of Proposition 3.26.

Applying Theorem 3.33 and Lemma 3.34 we can be more precise about the structure of the ideal $M^o$.

A closed linear subspace $M$ of $\mathcal{E}(\mathbb{R})$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant if and only if there is a countable set $\Sigma \subseteq \mathbb{C}$ and mapping $N : \Sigma \to \mathbb{N}$ such that

$$M = \overline{\text{span}}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 0, 1, \ldots, N(\lambda) - 1\},$$

where $e_{\lambda,j}(t) := \lambda e^{-\lambda t}$, $t \in \mathbb{R}$.

**Theorem 3.37** Let $M$ be a closed, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{E}(\mathbb{R})$ with polar $M^o \subseteq \mathcal{E}'(\mathbb{R})$. Then there is a countable set $\Sigma \subseteq \mathbb{C}$ and mapping $N : \Sigma \to \mathbb{N}$ such that the following assertions are equivalent.

- $F \in M^o$.
- $\forall \lambda \in \Sigma, \forall j = 0, 1, \ldots, N(\lambda) - 1 \quad [F(e_{\lambda,j}) = 0].$
- $\forall \lambda \in \Sigma, \forall j = 0, 1, \ldots, N(\lambda) - 1 \quad [\langle F \rangle^{(j)}(\lambda) = 0].$

Next, the closed, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}(\mathbb{R})$ are considered. Since there are no exponential-polynomials $e_{\lambda,j}$ in $\mathcal{D}(\mathbb{R})$, no characterization result like Theorem 3.33 is to be expected. In particular, there are no non-trivial finite dimensional (closed) $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}(\mathbb{R})$. For the polar of a closed $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{D}(\mathbb{R})$ we have the following result.
Lemma 3.38 Let \( M \) be a closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \). Let \( M^* \subseteq \mathcal{D}'(\mathbb{R}) \) be the polar of \( M \). Then the following assertions are true.

i. For each \( F \in \mathcal{E}'(\mathbb{R}) \) we have \( \sigma(F)(M) \subseteq M \).

ii. If \( M \neq \{0\} \), then \( M^* \cap \mathcal{E}'(\mathbb{R}) = \{0\} \).

Proof.
The proof of the assertion i is conform the proof of Lemma 3.34.ii.

ii. Suppose \( M \neq \{0\} \) and let \( F \in M^* \cap \mathcal{E}'(\mathbb{R}) \). Let \( x \in M \setminus \{0\} \) be fixed. Define \( X \in \mathcal{E}'(\mathbb{R}) \) by

\[
X(\phi) = \langle \phi, \tilde{x} \rangle \quad (\phi \in \mathcal{D}(\mathbb{R})).
\]

Then \( X \neq 0 \) and for every \( \phi \in \mathcal{D}(\mathbb{R}) \) we have by straightforward calculation

\[
0 = \sigma(F)x \ast \phi = \sigma(F)x_0 \phi,
\]

so \( F \ast X = 0 \). Since \( X \neq 0 \) we conclude that \( F = 0 \). ■

The polar of a non-trivial closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \), does not contain any compactly supported distribution (Lemma 3.38.ii). From this observation, we obtain the following.

Proposition 3.39 The following assertions hold true.

i. Every non-trivial \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \) is dense in \( \mathcal{E}(\mathbb{R}) \).

ii. There are no mean-periodic functions in \( \mathcal{E}(\mathbb{R}) \) belonging to \( \mathcal{D}(\mathbb{R}) \).

Proof.
i. Let \( M \) be a \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \), with \( M \neq \{0\} \). Let \( N \subseteq \mathcal{E}'(\mathbb{R}) \) be the polar of \( M \) regarded as a subspace of \( \mathcal{E}(\mathbb{R}) \). Since \( M^* = M^\circ \) and since \( M^\circ \cap \mathcal{E}'(\mathbb{R}) = N \), the assertion follows from Lemma 3.38.ii.

ii. For each \( \phi \in \mathcal{D}(\mathbb{R}) \) the span \( \{\sigma_t \phi \mid t \in \mathbb{R}\} \) is a \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \).

Hence by assertion (i), span \( \{\sigma_t \phi \mid t \in \mathbb{R}\} \) equals \( \{0\} \) or \( \mathcal{E}(\mathbb{R}) \). ■

There is an analogue of Theorem 3.35 for the collection of all closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspaces of \( \mathcal{D}(\mathbb{R}) \). However, since the polar of such a subspace is a subspace of \( \mathcal{D}'(\mathbb{R}) \), and since \( \mathcal{D}'(\mathbb{R}) \) is not a convolution ring, we need to be careful. From literature [Schw2], we mention the existence of a convolution of \( \mathcal{D}'(\mathbb{R}) \), and \( \mathcal{E}'(\mathbb{R}) \)-distributions yielding \( \mathcal{D}'(\mathbb{R}) \)-distributions. This product extends the convolution product on \( \mathcal{E}'(\mathbb{R}) \).

So, \( \mathcal{D}'(\mathbb{R}) \) is a module over \( \mathcal{E}'(\mathbb{R}), +, * \). The polars of closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspaces of \( \mathcal{D}(\mathbb{R}) \) correspond with weak-star closed submodules of \( \mathcal{D}'(\mathbb{R}) \) over the convolution ring \( \mathcal{E}'(\mathbb{R}) \).

Theorem 3.40 Let \( M \) be a closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{D}(\mathbb{R}) \). Then \( M^* \) is a weak-star closed submodule of \( \mathcal{D}'(\mathbb{R}) \) over \( \mathcal{E}'(\mathbb{R}), +, * \). Conversely, let \( I \) be a weak-star closed submodule of \( \mathcal{D}'(\mathbb{R}) \) over \( \mathcal{E}'(\mathbb{R}), +, * \). Then the closed subspace \( \{ f \in \mathcal{D}(\mathbb{R}) \mid \forall F \in I F(f) = 0 \} \) of \( \mathcal{D}(\mathbb{R}) \) is \((\sigma_t)_{t \in \mathbb{R}}\)-invariant.
3.2. CONVOLUTION STRUCTURES FOR $\mathcal{D}_-(\mathbb{R})$ AND $\mathcal{D}'_+(\mathbb{R})$

The following example indicates that there are indeed non-trivial closed $(\sigma_i)_{i \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}(\mathbb{R})$.

Example 3.41 Let $\phi_0 \in \mathcal{D}(\mathbb{R})$ be such that
\[
\begin{aligned}
\text{supp}(\phi_0) &\subseteq [-1,1], \\
\phi_0(t-1) &= -\phi_0(t) \quad \text{for } t \in [0,1],
\end{aligned}
\]
Let $F \in \mathcal{D}'(\mathbb{R})$ be defined by
\[
F := \sum_{i \in \mathbb{Z}} \delta_i,
\]
where $\delta_i$ ($i \in \mathbb{Z}$) denotes the delta-distribution in $i$, i.e. $\delta_i \phi = \phi(i)$. Then for all $t \in \mathbb{R}$, we have that
\[
F(\sigma_i \phi_0) = \sum_{i \in \mathbb{Z}} \phi_0(t+i) = \phi_0([-t] + t) + \phi_0([-t] + t - 1) = 0,
\]
where $[t]$ is the largest integer $i$ in $\mathbb{Z}$ such that $i \leq t$. Hence,
\[
F \in \text{span}\{\sigma_i \phi_0 \mid t \in \mathbb{R}\}^\circ = \overline{\text{span}\{\sigma_i \phi_0 \mid t \in \mathbb{R}\}}^\circ.
\]
We conclude that $\overline{\text{span}\{\sigma_i \phi_0 \mid t \in \mathbb{R}\}}$ is a non-trivial closed $(\sigma_i)_{i \in \mathbb{R}}$-invariant subspace of $\mathcal{D}(\mathbb{R})$.

3.2 CONVOLUTION STRUCTURES FOR $\mathcal{D}_-(\mathbb{R})$ AND $\mathcal{D}'_+(\mathbb{R})$

In this section, we consider the subspace $\mathcal{D}_-(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$ and its topological dual $\mathcal{D}'_+(\mathbb{R})$. Similar to the previous section we show that a convolution structure on $\mathcal{D}_+(\mathbb{R})$ can be introduced considering continuous, linear, $(\sigma_i)_{i \in \mathbb{R}}$-invariant operators on $\mathcal{D}_-(\mathbb{R})$. Since these convolution products have no zero-divisors, there are no non-trivial closed $(\sigma_i)_{i \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}_-(\mathbb{R})$.

3.2.1 Convolution on $\mathcal{D}_-(\mathbb{R})$

Let $\mathcal{D}_-(\mathbb{R})$ be the subspace of $\mathcal{E}(\mathbb{R})$ defined by
\[
\mathcal{D}_-(\mathbb{R}) = \bigcup_n \{ \phi \in \mathcal{E}(\mathbb{R}) \mid \text{supp}(\phi) \subseteq (-\infty, n] \} =: \bigcup_n \mathcal{D}_{-n}(\mathbb{R}).
\]
So, $f \in \mathcal{E}(\mathbb{R})$ belongs to $\mathcal{D}_-(\mathbb{R})$ if $f$ has support bounded on the right. The family $(\mathcal{D}_{-n}(\mathbb{R}))_{n \in \mathbb{N}}$ is a strict inductive system of closed subspaces of $\mathcal{E}(\mathbb{R})$. Correspondingly, $\mathcal{D}_-(\mathbb{R})$ is equipped with the related strict LF-topology
\[
\mathcal{D}_-(\mathbb{R}) = \varinjlim \mathcal{D}_{-n}(\mathbb{R}).
\]
(3.21)
and satisfies
\[ \mathcal{D}(\mathbb{R}) \subseteq \mathcal{D}_-(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{R}). \]  
(3.22)

Similarly, the subspace \( \mathcal{D}_+(\mathbb{R}) \) of \( \mathcal{E}(\mathbb{R}) \) is introduced by
\[ \mathcal{D}_+(\mathbb{R}) = \bigcup_n \{ f \in \mathcal{E}(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, \infty) \} = \bigcup_n \mathcal{D}_{+,n}(\mathbb{R}). \]

So, \( f \in \mathcal{E}(\mathbb{R}) \) belongs to \( \mathcal{D}_+(\mathbb{R}) \), if \( f \) has support bounded on the left. Each \( \mathcal{D}_{+,n}(\mathbb{R}) \) being a closed subspace of \( \mathcal{E}(\mathbb{R}) \), the family \( (\mathcal{D}_{+,n}(\mathbb{R}))_{n \in \mathbb{N}} \) is a strict inductive system. Correspondingly, \( \mathcal{D}_+(\mathbb{R}) \) is equipped with the related strict LF-topology
\[ \mathcal{D}_+(\mathbb{R}) = \text{ind} \mathcal{D}_{+,n}(\mathbb{R}). \]

(3.23)

The reflection mapping \( \phi \in \mathcal{D}_-(\mathbb{R}) \mapsto \check{\phi} \) is a homeomorphism from \( \mathcal{D}_-(\mathbb{R}) \) onto \( \mathcal{D}_+(\mathbb{R}) \). Because of this fact, all results mentioned in this section for \( \mathcal{D}_-(\mathbb{R}) \) have their analogues for the space \( \mathcal{D}_+(\mathbb{R}) \). These results have been summarized in §3.0.

Because of (3.22), the topological dual of \( \mathcal{D}_-(\mathbb{R}) \) is a subspace of \( \mathcal{D}'(\mathbb{R}) \).

**Definition 3.42** The subspace \( \mathcal{D}_+(\mathbb{R}) \) of \( \mathcal{D}'(\mathbb{R}) \) consists of all \( F \in \mathcal{D}'(\mathbb{R}) \) such that \( \text{supp}(F) \subseteq [T, \infty) \) for some \( T \in \mathbb{R} \). The elements of \( \mathcal{D}_+(\mathbb{R}) \) are called distributions with support bounded on the left.

Indeed, in case of the topological dual of \( \mathcal{D}_-(\mathbb{R}) \) we can extend the distribution \( F \in \mathcal{D}'(\mathbb{R}) \) to \( \mathcal{D}_-(\mathbb{R}) \) continuously if and only if \( F \) has support bounded on the left (see [Schw2], p.172). Hence, we may regard \( \mathcal{D}_+(\mathbb{R}) \) as the topological dual of \( \mathcal{D}_-(\mathbb{R}) \). Dual to the inclusions (3.22) are
\[ \mathcal{E}'(\mathbb{R}) \subseteq \mathcal{D}_+(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}). \]

(3.24)

Now, define the canonical pairing \( < \cdot, \cdot > \) between \( \mathcal{D}_+(\mathbb{R}) \) and \( \mathcal{D}_-(\mathbb{R}) \) by
\[ < \phi, \psi > := \int_{\mathbb{R}} \phi(t) \psi(t) \, dt \quad (\phi \in \mathcal{D}_+(\mathbb{R}), \psi \in \mathcal{D}_-(\mathbb{R})). \]

Since the mapping \( \psi \mapsto < \phi_0, \psi > \) defines for fixed \( \phi_0 \in \mathcal{D}_+(\mathbb{R}) \) a continuous linear functional on \( \mathcal{D}_-(\mathbb{R}) \), we have
\[ \mathcal{D}_+(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}). \]

Let \( (\sigma_t)_{t \in \mathbb{R}} \) be the translation group on \( \mathcal{E}(\mathbb{R}) \). Since \( \sigma_t(\mathcal{D}_-(\mathbb{R})) \subseteq \mathcal{D}_-(\mathbb{R}) \), we may restrict \( (\sigma_t)_{t \in \mathbb{R}} \) to this strict LF-space.

**Lemma 3.43** The family \( (\sigma_t)_{t \in \mathbb{R}} \) forms a group of continuous linear operators on \( \mathcal{D}_-(\mathbb{R}) \). The infinitesimal generator of \( (\sigma_t)_{t \in \mathbb{R}} \) is the differentiation operator \( \frac{d}{dt} \) on \( \mathcal{D}_-(\mathbb{R}) \).
3.2. Convolution structures for $\mathcal{D}_-(\mathbb{R})$ and $\mathcal{D}_+^*(\mathbb{R})$

The topological vector space $\mathcal{D}_-(\mathbb{R})$ has a ring-structure in addition. Indeed, for each $\psi, \phi \in \mathcal{D}_-(\mathbb{R})$ the convolution product of $\psi \ast \phi$ in $\mathcal{D}_-(\mathbb{R})$ is defined by

$$
(\psi \ast \phi)(t) := \int_\mathbb{R} \psi(t - \tau)\phi(\tau) \, d\tau \quad (t \in \mathbb{R}),
$$

(3.25)

satisfying $(\psi \ast \phi)' = \psi' \ast \phi = \psi \ast \phi'$, and $\text{supp}(\psi \ast \phi) \subseteq \text{supp}(\psi) + \text{supp}(\phi)$.

**Theorem 3.44** The vector space $\mathcal{D}_-(\mathbb{R})$ is a commutative algebra over $\mathbb{C}$ with respect to convolution. It has no zero-divisors.

**Proof.**
The assertion that $\mathcal{D}_-(\mathbb{R})$ is a commutative algebra over $\mathbb{C}$ with respect to convolution can be checked straightforwardly. The assertion that $\mathcal{D}_-(\mathbb{R})$ has no zero-divisors is an application of a result due to Titchmarsh [T14], p.327.

3.2.2 Convolution on $\mathcal{D}_+^*(\mathbb{R})$

Following a scheme similar to §3.1, we extend the convolution product on $\mathcal{D}_-(\mathbb{R})$ to its topological dual $\mathcal{D}_+^*(\mathbb{R})$.

For $\phi \in \mathcal{D}_-(\mathbb{R})$, the mapping $\psi \mapsto \psi \ast \phi$ defines a continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $\mathcal{D}_-(\mathbb{R})$. Rearranging

$$
(\psi \ast \phi)(t) = \langle \sigma_t \psi, \phi \rangle \quad (t \in \mathbb{R}),
$$

we see that we can replace $\phi$ by a $\mathcal{D}_+^*(\mathbb{R})$-distribution $F$.

**Definition 3.45** Let $F \in \mathcal{D}_+^*(\mathbb{R})$. Then, the convolution operator $\sigma[F]$ on $\mathcal{D}_-(\mathbb{R})$ is defined as

$$
(\sigma[F] \psi)(t) := F(\sigma_t \psi) \quad (t \in \mathbb{R}, \phi \in \mathcal{D}_-(\mathbb{R})).
$$

The collection $\{\sigma[F] \mid F \in \mathcal{D}_+^*(\mathbb{R})\}$ consists of all linear, continuous, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators on $\mathcal{D}_-(\mathbb{R})$.

**Theorem 3.46** The continuous linear mapping $L$ on $\mathcal{D}_-(\mathbb{R})$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant if and only if $L$ is of the form $L = \sigma[F]$ for some $F \in \mathcal{D}_+^*(\mathbb{R})$.

**Proof.**
The proof of this assertion is based on the same arguments as the proof of Theorem 3.30.

Since the collection of all continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings on $\mathcal{D}_-(\mathbb{R})$ establish an algebra, Theorem 3.46 gives rise to a product structure on $\mathcal{D}_+^*(\mathbb{R})$. 
Definition 3.47 Let \( F_1, F_2 \in \mathcal{D}'\left( \mathbb{R} \right) \). Then the convolution product \( F_1 \ast F_2 \) is defined by the equation \( \sigma[F_1]\psi[F_2] = \sigma[F_1 \ast F_2] \).

Again, the convolution product from Definition 3.47 corresponds to the classical convolution product on \( \mathcal{D}'\left( \mathbb{R} \right) \). So, a different angle in considering the convolution product on \( \mathcal{D}'\left( \mathbb{R} \right) \) arises starting from continuous linear \((\alpha, \lambda)\)-invariant mappings on \( \mathcal{D}_{-}(\mathbb{R}) \). Contrary to the case of \( \mathcal{C}'(\mathbb{R}) \), there is no Fourier Transformation on \( \mathcal{D}'\left( \mathbb{R} \right) \). Therefore, to investigate whether \( \mathcal{D}'\left( \mathbb{R} \right) \) has zero-divisors, and to investigate commutativity of \( \mathcal{D}'\left( \mathbb{R} \right) \), we need a different approach.

Lemma 3.48 Every non-zero convolution operator \( \sigma[F] \) is injective on \( \mathcal{D}_{-}(\mathbb{R}) \).

Proof. Let \( \psi \in \ker(\sigma[F]) \). Take \( \psi \neq \ker(\sigma[F]) \). Then, we have for all \( t \in \mathbb{R} \)

\[
0 = (\psi \ast \sigma[F] \phi)(t) = \int_{\mathbb{R}} \psi(t - \tau)F(\sigma_{\tau}\phi) \, d\tau = \int_{\mathbb{R}} \sigma(t - \tau)\sigma_{\tau}\phi \, d\tau
\]

\[
= \int_{\mathbb{R}} \phi(t - \tau)\sigma_{\tau}\psi \, d\tau = (\phi \ast \sigma[F] \psi)(t).
\]

Since the algebra \( \mathcal{D}_{-}(\mathbb{R}) \) has no zero-divisors (Theorem 3.44) it follows that \( \phi = 0 \).

We obtain the following well known result (cf. [Schw2], p.173).

Theorem 3.49 The collection \( \{\sigma[F] \mid F \in \mathcal{D}'\left( \mathbb{R} \right)\} \) is a commutative algebra over \( \mathbb{C} \) with respect to composition and addition. It has an identity (the identity operator) and no zero-divisors.

Identifying each \( F \in \mathcal{D}'\left( \mathbb{R} \right) \) with \( \sigma[F] \), the vector space \( \mathcal{D}'\left( \mathbb{R} \right) \) forms a commutative algebra over \( \mathbb{C} \) with respect to convolution. \( \mathcal{D}'\left( \mathbb{R} \right) \) has an identity and no zero-divisors.

Proof. By Lemma 3.48, the algebra \( \{\sigma[F] \mid F \in \mathcal{D}'\left( \mathbb{R} \right)\} \) has no zero-divisors. To show that the convolution product on \( \{\sigma[F] \mid F \in \mathcal{D}'\left( \mathbb{R} \right)\} \) is commutative, we use the commutativity of \( \mathcal{D}_{-}(\mathbb{R}) \) and the techniques from the proof of Lemma 3.48, namely that for each \( \psi, \phi \in \mathcal{D}_{-}(\mathbb{R}) \) and for each \( F \in \mathcal{D}'\left( \mathbb{R} \right) \)

\[
\psi \ast \sigma[F] \phi = \sigma[F] \psi \ast \phi.
\]

Now, let \( F_1, F_2 \in \mathcal{D}'\left( \mathbb{R} \right) \). Then for \( \psi, \phi \in \mathcal{D}_{-}(\mathbb{R}) \),

\[
\psi \ast (\sigma[F_1] \sigma[F_2] \phi) = (\sigma[F_1] \psi) \ast (\sigma[F_2] \phi) = (\sigma[F_1] \psi) \ast (\sigma[F_2] \phi) - (\sigma[F_2] \sigma[F_1] \psi) \ast \phi
\]

\[
= (\sigma[F_2] \sigma[F_1] \psi) \ast \phi - (\sigma[F_2] \sigma[F_1] \psi) \ast \phi = 0.
\]

We conclude that \( \sigma[F_1] \sigma[F_2] = \sigma[F_2] \sigma[F_1] \).

The vector space \( \mathcal{M}_{+}(\mathbb{R}) \) consists of all Radon measures with half-infinite support bounded on the left, i.e. all \( \mu \in \mathcal{M}(\mathbb{R}) \) for which \( \text{supp}(\mu) \subseteq [-T, \infty) \) for some \( T \in \mathbb{R} \). Naturally, the space \( \mathcal{M}_{+}(\mathbb{R}) \) is a subspace of \( \mathcal{M}_{-}(\mathbb{R}) \). The space \( \mathcal{M}_{+}(\mathbb{R}) \) can be defined as the dual of the strict LF-space \( \mathcal{C}_{-}(\mathbb{R}) \). The space \( \mathcal{M}_{+}(\mathbb{R}) \) can be seen as a subspace of \( \mathcal{D}'\left( \mathbb{R} \right) \) in the following way.
3.2. Convolution structures for $\mathcal{D}_+^*(\mathbb{R})$ and $\mathcal{D}_-^*(\mathbb{R})$

Lemma 3.50 For every $\mu \in \mathcal{M}_+^1(\mathbb{R})$ the mapping

$$
\phi \in \mathcal{D}_-^*(\mathbb{R}) \mapsto \int_{\mathbb{R}} \psi(\tau) \mu(\text{d}\tau) \quad =: \langle \phi, \mu \rangle,
$$

(3.26)

defines a continuous linear functional on $\mathcal{D}_-^*(\mathbb{R})$.

Define for each $\mu \in \mathcal{M}_+^1(\mathbb{R})$, the convolution mapping $\sigma[\mu]$ by

$$
(\sigma[\mu]\phi)(t) := \langle \phi, \mu \rangle \quad (\phi \in \mathcal{D}_-^*(\mathbb{R}), \ t \in \mathbb{R}).
$$

It is checked straightforwardly, that for $\mu_1, \mu_2 \in \mathcal{M}_+^1(\mathbb{R})$

$$
\sigma[\mu_1]\sigma[\mu_2] = \sigma[\mu_1 \ast \mu_2],
$$

where $\ast$ is the classical convolution product in $\mathcal{M}_+^1(\mathbb{R})$, i.e.

$$
(\mu_1 \ast \mu_2)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(t + \tau) \mu_1(\text{d}t) \mu_2(\text{d}\tau) \quad (x \in C_c(\mathbb{R})).
$$

Theorem 3.51 The space $(\mathcal{M}_+^1(\mathbb{R}), +, \ast)$ is a subalgebra of $(\mathcal{D}_+^*(\mathbb{R}), +, \ast)$, relating each $\mu \in \mathcal{M}_+^1(\mathbb{R})$ to the distribution $\phi \in \mathcal{D}_-^*(\mathbb{R}) \mapsto \langle \phi, \mu \rangle$. In particular, $(\mathcal{M}_+^1(\mathbb{R}), +, \ast)$ is a commutative convolution algebra over $\mathbb{C}$. It has an identity and no zero-divisors.

An important subclass of $\mathcal{M}_+^1(\mathbb{R})$ is are the measures corresponding to $\mathcal{D}_+^*(\mathbb{R})$-functions. The operators $\sigma[\phi]$, where $\phi \in \mathcal{D}_+^*(\mathbb{R})$, act by convolution.

Theorem 3.52 Let $\phi \in \mathcal{D}_+^*(\mathbb{R})$. Then

$$
(\sigma[\phi]\psi)(t) = \langle \phi, \psi \rangle(t) = \int_{\mathbb{R}} \psi(t + \tau) \phi(\tau) \text{d}\tau,
$$

for all $\psi \in \mathcal{D}_-^*(\mathbb{R})$ and $t \in \mathbb{R}$.

Proof. The proof follows from straightforward calculation.

3.2.3 Closed translation-invariant subspaces of $\mathcal{D}_-^*(\mathbb{R})$

We conclude this section with a brief discussion of closed $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}_-^*(\mathbb{R})$. In fact, there are no non-trivial ones.

Theorem 3.53 Let $M$ be a closed $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{D}_-^*(\mathbb{R})$. Then $M = \mathcal{D}_-^*(\mathbb{R})$ or $M = \{0\}$. 
Proof.
Let $M$ be a closed $(\sigma_i)_{i\in\mathbb{R}}$-invariant subspace of $\mathcal{D}_-(\mathcal{R})$ and suppose $M \neq \mathcal{D}_-(\mathcal{R})$. Then, the Hahn-Banach Theorem ensures that

$$M := \bigcap_{F \in M^*} \ker(\sigma[F]),$$

where $M^*$ is the polar of $M$. Since $M^* \neq \{0\}$, Lemma 3.48 yields that $M = \{0\}$.

Theorem 3.12 is equivalent to the following result on weak-star closed ideals in $\mathcal{D}'_*(\mathcal{R})$.

**Theorem 3.54** Let $I$ be a weak-star closed ideal in $\mathcal{D}'_*(\mathcal{R})$. Then $I = \mathcal{D}'_*(\mathcal{R})$ or $I = \{0\}$.

Proof.
Let $I$ be a weak-star closed ideal in $\mathcal{D}'_*(\mathcal{R})$. Define $M \subseteq \mathcal{D}_-(\mathcal{R})$ by

$$M := \{\phi \in \mathcal{D}_-(\mathcal{R}) \mid \forall \epsilon \in I, F(\phi) = 0\}.$$

Then $M$ is a closed, $(\sigma_i)_{i\in\mathbb{R}}$-invariant subspace of $\mathcal{D}_-(\mathcal{R})$, hence by Theorem 3.12 $M = \{0\}$ or $M = \mathcal{D}_-(\mathcal{R})$. Since $M^* = I$, the assertion follows.

Convolution products are well-described in literature ([Schw2]). For sake of completeness, we give for all treated spaces a scheme of possible convolution products.

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*Figure 3.1* Convolution products for distributions
3.3. THE SPACE $\mathcal{E}(\mathbb{R}^+)$ AND THE
CONVOLUTION PRODUCT ON $\mathcal{E}'(\mathbb{R}^+)$

In this section we consider the vector space $\mathcal{E}(\mathbb{R}^+)$. The space $\mathcal{E}(\mathbb{R}^+)$ will be used in
the factorization theory presented in Chapter 5.

Definition 3.55 The vector space $\mathcal{E}(\mathbb{R}^+)$ consists of all functions $f$ on $\mathbb{R}^+$ which
are arbitrary many times differentiable on $(0, \infty)$, such that for each $k \in \mathbb{N}$ limit
\[ \lim_{t \to 0^+} f^{(k)}(t) \]
exists. We write $f^{(k)}(0) = \lim_{t \to 0^+} f^{(k)}(t)$.

We can approach the vector space $\mathcal{E}(\mathbb{R}^+)$ in various ways. Recall the following result
due to Borel (see Donoghue [Don], §10).

Theorem 3.56 (Borel) Let $(a_n)_{n \in \mathbb{N}^0}$ be a sequence in $\mathbb{C}$. Then there exists $f \in \mathcal{E}(\mathbb{R})$
such that $f^{(k)}(0) = a_k$ for each $k \in \mathbb{N}_0$.

We conclude that for each $f \in \mathcal{E}(\mathbb{R}^+)$ there is a function $f_{\text{ext}} \in \mathcal{E}(\mathbb{R})$ such that
$f_{\text{ext}}|_{[0, \infty)} = f$. In fact, $f$ can be taken within $\mathcal{D}_+(\mathbb{R})$.

Let $\mathcal{D}(\mathbb{R})$ be the closed subspace of $\mathcal{D}_+(\mathbb{R})$ consisting of all $f \in \mathcal{D}_+(\mathbb{R})$ with
$\operatorname{supp}(f) \subseteq (-\infty, 0]$. Let $\phi$ be the quotient mapping from $\mathcal{D}_+(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})/\mathcal{D}(\mathbb{R}^-)$
(cf. Definition 1.21). By the Borel Theorem, the mapping $\pi : \mathcal{D}_+(\mathbb{R})/\mathcal{D}(\mathbb{R}^-) \rightarrow
\mathcal{E}(\mathbb{R}^+)$, defined by

\[ \pi(\phi(f)) := f|_{[0, \infty)} \quad (f \in \mathcal{D}_+(\mathbb{R})) \tag{3.27} \]

is an isomorphism. So, $\mathcal{E}(\mathbb{R}^+)$ and the quotient space $\mathcal{D}_+(\mathbb{R})/\mathcal{D}(\mathbb{R}^-)$ are isomorphic.

We equip $\mathcal{E}(\mathbb{R}^+)$ with the $F$-topology of uniform convergence in every derivative on
compact subsets of $\mathbb{R}^+$, the compact open topology. This topology is generated (for example) by the ordered family of seminorms $\{q_n | n \in \mathbb{N}_0\}$, defined as

\[ q_n(f) := \sum_{k=0}^{n} \max_{|x| \leq d} |f^{(k)}(t)| \quad (f \in \mathcal{E}(\mathbb{R}^+), n \in \mathbb{N}_0), \]

The following result shows that the $F$-topology of $\mathcal{E}(\mathbb{R}^+)$ equals the quotient topology
of the quotient space $\mathcal{D}_+(\mathbb{R})/\mathcal{D}(\mathbb{R}^-)$.
Theorem 3.57 The spaces $\mathcal{E}(\mathbb{R}^+)$ and $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$ are homeomorphic.

Proof. The proof consists of two parts.

First, we consider the topology of the quotient space $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$. Recall from Proposition 1.42, that $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$ is a LP-space (not necessarily strict), satisfying by Proposition 1.43

$$\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-) \equiv \text{ind}_{n} \mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-)).$$

In particular, we have

$$\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-)) \overset{\text{id}}{\hookrightarrow} \mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-),$$

where the embedding $\text{id} : \mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-)) \to \mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$ is given as in the proof of Proposition 1.43:

$$\text{id}(f + \mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-)) := (f + \mathcal{D}(\mathbb{R}^-)) \quad (f \in \mathcal{D}_+).$$

(3.28)

Or briefly, let $\phi_1$ be the quotient mapping of $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$ onto $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$, then

$$\text{id}(\phi_1(f)) = \phi(f) \quad (f \in \mathcal{D}_+).$$

The embedding $\text{id}$ is bijective, as a consequence of the Borel Theorem 3.56, and therefore a homeomorphism, as a consequence of the Open Mapping Theorem 1.46. So, since $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$ is an F-space, so is $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$.

Next, we show that the F-spaces $\mathcal{E}(\mathbb{R}^+)$ and $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$ are homeomorphic, which proves the theorem. Indeed, since both $\pi$ and $\text{id}$ are bijections, the mapping $\pi \circ \text{id}$ is a bijection from $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$ into $\mathcal{E}(\mathbb{R}^+)$. In fact, $\pi \circ \text{id}$ is a homeomorphism. Recall from Definition 1.21 and §3.0 that the topology of $\mathcal{D}_+ / (\mathcal{D}_+ \cap \mathcal{D}(\mathbb{R}^-))$ is generated by the seminorms

$$\phi(f) \mapsto \inf_{g \in \mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)} \sum_{i=0}^{n} \max_{t \in [-1,1]} |f + g(t)|.$$ 

So, for each $n \in \mathbb{N}^+$

$$q_n((\pi \circ \text{id}) \phi_1(f)) = \sum_{i=0}^{n} \max_{t \in [0,1]} |f(t)|$$

$$\leq \inf_{g \in \mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)} \sum_{i=0}^{n} \max_{t \in [-1,1]} |f + g(t)|.$$ 

So, $\pi \circ \text{id}$ is continuous and therefore a homeomorphism.

In the sequel, we characterize the dual of $\mathcal{E}(\mathbb{R}^+)$ in terms of the dual of the quotient space $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$. Topological duals of quotient spaces are well described in literature. In particular, let $M$ be a closed subspace of the topological vector space $V$, and let $\phi : V \to V / M$ be its quotient mapping (see Definition 1.21). Then, the adjoint $\phi^*$ of $\phi$ is an isomorphism from $(V/M)'$ onto the polar $M^*$ ([K64], §22.1). Replacing $V$ by $\mathcal{D}_+ / \mathcal{D}(\mathbb{R}^-)$ and replacing $M$ by $\mathcal{D}(\mathbb{R}^-)$, we obtain the following result.
3.3. The space $E'(R^+)$ and the convolution product on $E'(R^+)$

Theorem 3.58 The dual space $E'(R^+)'$ is isomorphic to $E'(R^+)^* = D'(R^-)$, i.e. the $D'(R)$-distributions with compact support within $[0, \infty)$.

Remark 3.58.1 We did not specify the relation between $E'(R^+)'$ and $E'(R^+)$ in Theorem 3.58, since this relation seems rather technical. The correct assertion should be the following: each $G \in E'(R^+)'$ corresponds to $\phi^*(G \circ \pi) \in E'(R^+)^*$. Conversely, each $F \in E'(R^+)^*$ corresponds uniquely to $(\phi^*)^{-1}(F) \circ \pi^{-1} \in E'(R^+)'$.

We obtain from the Soehnlein result (Proposition 3.23) a more concrete characterization of $E'(R^+)$.

Theorem 3.59 A linear functional $F$ on $E'(R^+)$ is continuous if and only if

$$F(f) := \int_{R^+} \left( P \left( \frac{d}{dr} \right) f \right)(\tau) \, \mu(d\tau) \quad (f \in E'(R^+)) \tag{3.29}$$

for some $\mu \in M_u(R^+)$ and some polynomial $P$.

Similar to the previous sections we introduce a convolution product on $E'(R^+)$ considering the collection of all continuous, linear translation-invariant operators on $E'(R^+)$. However, we have to be careful, because there is a semigroup of translations on $E'(R^+)$ only.

From the theory on quotient spaces, we obtain the following result.

Lemma 3.60 For each continuous, linear mapping $L$ on $D_+(R)$, additionally satisfying $L(D(R^+)) \subseteq D(R^-)$, the quotient mapping $\tilde{L}$ of $L$ on $D_+(R)/D(R^-)$, defined by

$$\tilde{L}(\phi(f)) := \phi(Lf) \quad (f \in D_+(R)),$$

is linear and continuous for the quotient topology, where $\phi$ denotes the quotient mapping from $D_+(R)$ onto $D_+(R)/D(R^-)$.

Since $\sigma_t(D(R^+)) \subseteq D(R^-)$ if and only if $t \geq 0$, there is a semigroup of translations on $D_+(R)/D(R^-)$ only.

Lemma 3.61 Let $\sigma_t$ denote the quotient mapping of $\sigma_t$ on $D_+(R)/D(R^-)$ for each $t \geq 0$. So, $\tilde{\sigma}_t \phi(f) = \phi(\sigma_t f)$ for each $f \in D_+(R)$. Then, $(\tilde{\sigma}_t)_{t \geq 0}$ is a $\sigma_0$-semigroup on $D_+(R)/D(R^-)$ with everywhere defined infinitesimal generator.

Proof. The family $(\tilde{\sigma}_t)_{t \geq 0}$ consists of continuous, linear operators on $D_+(R)/D(R^-)$ by Lemma 3.60. Since for each $s, t \geq 0$ and each $f \in D_+(R)$

$$\tilde{\sigma}_t \tilde{\sigma}_s \phi(f) = \phi(\sigma_t \sigma_s f) = \phi(\sigma_{s+t} f) = \tilde{\sigma}_{s+t} \phi(f).$$

$(\tilde{\sigma}_t)_{t \geq 0}$ is a semigroup of linear operators. Finally, since $\phi$ is continuous, we have for each $f \in D_+(R)$

$$\lim_{t \to 0} \tilde{\sigma}_t \phi(f) = \phi(\lim_{t \to 0} \sigma_t f) = \phi(f).$$
which proves that \((\delta_t)_{t \geq 0}\) is a \(c_0\)-semigroup on \(D_-(\mathbb{R})/D(\mathbb{R}^+)\).

Let \(\delta_t\) be the infinitesimal generator of \((\delta_t)_{t \geq 0}\). By the continuity of \(\phi\), we have for all \(f \in D_+(\mathbb{R})\)

\[
\frac{\partial_t}{t} \phi(f) = \phi \left( \frac{\partial_t - \partial_0}{t} f \right) \to \phi \left( \frac{d}{dt} f \right),
\]
as \(t \downarrow 0\). So, \(\delta_t\) is everywhere defined, with \(\delta_t \circ \phi = \phi \circ \delta_t\).

Define for each \(t \geq 0\) the translation operator \(\sigma_t^+\) on \(E(\mathbb{R}^+)\) by

\[
\sigma_t^+ = \pi \circ \delta_t \circ \pi^{-1}.
\]

Then, we have for each \(t \geq 0\) and each \(f \in E(\mathbb{R}^+)\)

\[
(\sigma_t^+ f)(s) = f(t + s) \quad (s \geq 0).
\]

**Lemma 3.62** The family \((\sigma_t^+)_{t \geq 0}\) forms a \(c_0\)-semigroup of continuous linear operators on \(E(\mathbb{R}^+)\). The infinitesimal generator of \((\sigma_t^+)_{{t \geq 0}}\) is the differentiation operator \(\frac{d}{dt}\) on \(E(\mathbb{R}^+)\).

The \(c_0\)-semigroup \((\sigma_t^+)_{t \geq 0}\) is locally equicontinuous (apply Theorem 2.21) and called the translation semigroup on \(E(\mathbb{R}^+)\).

Again, continuous linear \((\sigma_t^+)_{t \geq 0}\)-invariant operators on \(E(\mathbb{R}^+)\) can be fully characterized.

**Theorem 3.63** The continuous linear mapping \(L\) on \(E(\mathbb{R}^+)\) is \((\sigma_t^+)_{t \geq 0}\)-invariant if and only if \(F \in E(\mathbb{R}^+)\) exist such that \((L f)(t) = F(\sigma_t^+ f)\) for every \(f \in E(\mathbb{R}^+)\) and \(t \in \mathbb{R}\). In this case, \(L\) is called a convolution operator on \(E(\mathbb{R}^+)\) and we denote \(L = \sigma^*[F]\).

As the dual of \(E(\mathbb{R}^+)\) is related to a subalgebra of \(D_-(\mathbb{R}), E'(\mathbb{R}^+)\), by Theorem 3.58, so are convolution operators on \(E(\mathbb{R}^+)\) related to convolution operators on \(D_+(\mathbb{R})\).

Therefore, let \(F \in E'(\mathbb{R}^+)\). Then, the convolution operator \(\sigma[F]\) on \(D_+(\mathbb{R})\) satisfies

\[
s[F](D(\mathbb{R}^+)) \subseteq D(\mathbb{R}^+).
\]

So, let \(\delta[F]\) be the quotient mapping of \(\sigma[F]\) on \(D_+(\mathbb{R})/D(\mathbb{R}^+)\), i.e. \(\delta[F] \circ \phi = \phi \circ \delta[F]\).

Let \(G\) be the \(\mathbb{E}(\mathbb{R}^+)^\ast\)-functional related to \(F\) by Theorem 3.58, i.e. \(\phi^\ast(G \circ \pi) = F\).

Then for each \(f \in D_+(\mathbb{R})\) and \(t \geq 0\)

\[
\left( \sigma^\ast[G] \circ \pi(\phi(f)) \right)(t) = G(\sigma_t^+ \circ \pi(\phi(f))) = \left( \phi^\ast \right)^{-1}(F) \left( \partial_t \phi(f) \right)
\]

\[
= \left( \phi^\ast \right)^{-1}(F) \left( \phi(\sigma_t f) \right) = F(\sigma_t f) = \left[ \sigma[F] f \right](t)
\]

\[
= \phi(\sigma_t f) = (\pi \circ \phi \circ \sigma[F] f)(t) = (\pi \circ \delta[F] \circ \phi)(f)(t).
\]

We obtain the following result.
3.3. The space $\mathcal{E}(\mathbb{R}^+)$ and the convolution product on $\mathcal{E}'(\mathbb{R}^+)$

**Proposition 3.64** For each $F \in \mathcal{E}'(\mathbb{R}^+)$ we have $\pi \circ \sigma[F] = \sigma^+ [G]$ o $\pi$, where $G$ is the $\mathcal{E}(\mathbb{R}^+)$-functional related to $F$ by Theorem 3.58.

Theorem 3.59 yields the following characterization of convolution operators on $\mathcal{E}(\mathbb{R}^+)$.  

**Corollary 3.65** Each continuous, linear, $(\sigma^+)^{\tau\omega}$-invariant mapping $L$ on $\mathcal{E}(\mathbb{R}^+)$ is of the form 

$$ (Lf)(t) = \int_{\mathbb{R}} (P(\frac{d}{dt})f)(\tau + t) \mu(d\tau) = (\sigma^+[p]P(\frac{d}{dt})f)(t), $$

for some $\mu \in \mathcal{M}_e(\mathbb{R}^+), p \in P$.

Theorem 3.63 gives rise to a convolution product on $\mathcal{E}(\mathbb{R}^+)'$.  

**Definition 3.66** Let $F_1, F_2 \in \mathcal{E}(\mathbb{R}^+)'$. Then the convolution product $F_1 \ast F_2$ of $F_1$ and $F_2$ is defined by the equation $\sigma^+ [F_1] \sigma^+ [F_2] = \sigma^+ [F_1 \ast F_2]$.

Proposition 3.64 enables us to use the properties of the subalgebra $\mathcal{E}'(\mathbb{R}^+)$ of $\mathcal{E}(\mathbb{R}_0)$ to investigate the convolution product on $\mathcal{E}(\mathbb{R}^+)$, We obtain the following result. 

**Theorem 3.67** The collection $\{\sigma^+[F] \mid F \in \mathcal{E}(\mathbb{R}^+)'\}$ forms a commutative algebra over $\mathbb{C}$ with respect to composition and addition. It has an identity (the identity operator) and no zero-operators. Equivalently, identifying each $F \in \mathcal{E}(\mathbb{R}^+)'$, a vector space $\mathcal{E}(\mathbb{R}^+)'$ forms a commutative algebra over $\mathbb{C}$ with respect to convolution. $\mathcal{E}(\mathbb{R}^+)'$ has an identity and no zero-operators. 

We proceed this section introducing the subalgebra of $\mathcal{E}'(\mathbb{R}^+)$ consisting of Radon measures. In §2.1.3, we introduced the vector space $\mathcal{M}_e(\mathbb{R}^+)$ consisting of Radon measures $\mu$ on $\mathbb{R}^+$ with compact support in $\mathbb{R}^+$. The space $\mathcal{M}_e(\mathbb{R}^+)$ is a subspace of $\mathcal{E}(\mathbb{R}^+)'$ in the following way.  

**Lemma 3.68** For every $\mu \in \mathcal{M}_e(\mathbb{R}^+)$ the mapping 

$$ f \in \mathcal{E}(\mathbb{R}^+) \rightarrow \int_{\mathbb{R}^+} f(\tau) \mu(d\tau) = :< f, \mu >, $$

(3.32)

defines a continuous linear functional on $\mathcal{E}(\mathbb{R}^+)$. 

Furthermore, we defined in §2.1.5 the convolution mappings $\sigma^+ [\mu], \mu \in \mathcal{M}_e(\mathbb{R}^+)$, by 

$$ (\sigma^+[\mu]f)(t) := \int_{\mathbb{R}^+} \sigma^+_t f \mu(d\tau) = < \sigma^+_t f, \mu > \quad (f \in \mathcal{E}(\mathbb{R}^+), t \geq 0). $$

It is shown that for $\mu_1, \mu_2 \in \mathcal{M}_e(\mathbb{R}^+)$ 

$$ \sigma^+ [\mu_1 \ast \mu_2] = \sigma^+ [\mu_1] \ast \mu_2, $$

where $\ast$ is the convolution product on $\mathcal{M}_e(\mathbb{R}^+)$ (see section 2.1.3).
Theorem 3.69 The space \( \mathcal{M}_c(\mathbb{R}^+) \) is a subalgebra of \( \mathcal{E}'(\mathbb{R}^+) \), relating each \( \mu \in \mathcal{M}_c(\mathbb{R}^+) \) with the distribution \( f \in \mathcal{E}(\mathbb{R}^+) \mapsto \langle f, \mu \rangle \). In particular, \( \mathcal{M}_c(\mathbb{R}^+) \) is a commutative convolution algebra over \( \mathbb{C} \). It has an identity and no zero-divisors.

We end this section with a brief discussion of closed \( (\sigma_t^+)_{t \geq 0} \)-invariant subspaces of \( \mathcal{E}(\mathbb{R}^+) \). Since there is no essential difference with the closed \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant subspaces of \( \mathcal{E}(\mathbb{R}) \), we mention the main result only.

Theorem 3.70 Let \( M \) be a closed, \( (\sigma_t^+)_{t \geq 0} \)-invariant subspace of \( \mathcal{E}(\mathbb{R}^+) \), then \( M^* \) is a weak-star closed ideal in \( \mathcal{E}(\mathbb{R}^+) \). Conversely, let \( I \) be a weak-star closed ideal in \( \mathcal{E}(\mathbb{R}^+) \), then \( \{ f \in \mathcal{E}(\mathbb{R}^+) \mid \forall R, \forall F(f) = 0 \} \) is a closed, \( (\sigma_t^+)_{t \geq 0} \)-invariant subspace of \( \mathcal{E}(\mathbb{R}^+) \).

In §3.1.2 we deduced that a closed \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant subspace \( M \) of \( \mathcal{E}(\mathbb{R}) \) are spanned by the exponential-polynomials contained in \( M \). In Chapter 5, we present conditions on a \( (\sigma_t^+)_{t \geq 0} \)-invariant subspace \( M \) of \( \mathcal{E}(\mathbb{R}^+) \) such that a similar result can be obtained.
TRANSLATION INVARIANT SUBSPACES OF DISTRIBUTIONS

In this chapter, we introduce classes of subspaces of $\mathcal{D}'_{\sigma}(\mathcal{R})$ and $\mathcal{C}'(\mathcal{R})$ which will be used in the factorization theory for input-/output systems to be developed in Chapter 5.

Let us think of a time-invariant, linear system $\Sigma$ as a device, accepting certain signals (inputs) and producing other signals (outputs) according to some specific rules. In this chapter, we study two types of descriptions of time-invariant, linear systems. The first way to describe a system is by means of a mapping from the space of all possible input signals into space of all possible output signals, the so-called input-/output mapping. The second, more general, approach describes a linear system as a subspace of the product space of the input space and output space, hereby disregarding the possible input or output character of certain signals. This approach towards systems is called the behavioral approach (cf. Willems [Wil]). In this chapter, we propose a class of possible signal spaces for each of these approaches.

The search for the first type of spaces, which can serve as signal space in the working mode description of time-invariant single-input-single-output systems (SISO-systems), is inspired by works of Kamen ([Kam1] and [Kam2]) and Yamamoto ([Y1] and [Y2]). Kamen used $\mathcal{D}'_{\sigma}(\mathcal{R})$ as signal space, yielding an almost algebraic theory using the convolutional aspects of $\mathcal{D}'_{\sigma}(\mathcal{R})$, which we described in §3.0. Yamamoto considered $L^1_{\infty,c}(\mathcal{R})$ as signal space, involving topological considerations as well. We construct a class $\mathcal{C}$ of subspaces of $\mathcal{D}'_{\sigma}(\mathcal{R})$, containing Yamamoto's choice $L^1_{\infty,c}(\mathcal{R})$, and also strict LF-spaces, such as $\mathcal{D}'_{\sigma}(\mathcal{R})$, $C'_{\sigma}(\mathcal{R})$ and $L^1_{\infty,c}(\mathcal{R})$.

The basic idea for the construction of this class arises from the time-invariance of the systems under consideration. To formalize this time-invariance of the systems, we use the concept of translation or time-shift. Since we are interested in subspaces of $\mathcal{D}'_{\sigma}(\mathcal{R})$, it suffices to introduce translations on $\mathcal{D}'_{\sigma}(\mathcal{R})$. This is done by duality:

\[ \langle \sigma F | \phi \rangle := F(\sigma^{-1} \phi) \]

where $\mathcal{D}'_{\sigma}(\mathcal{R})$ is the topological dual of $\mathcal{D}_{\sigma}(\mathcal{R})$ and where $\sigma_{-t}$ is the translation over $-t$ in $\mathcal{D}_{\sigma}(\mathcal{R})$. It is readily checked that $(\sigma_t)_{t \in \mathbb{R}}$ forms a one-parameter group of
Chapter 4. Translation invariant subspaces of distributions

(continuous) linear operators on \( \mathcal{D}'(\mathbb{R}) \).

"Distributional" translations formalize the time-invariance of a system in the case that inputs cannot be considered as functions. Now, a system is called time-invariant (or constant) if the following holds: Let \( U, Y \in C \) be the input space and output space of a system \( \Sigma \). Let \( u \in U \) an input to \( \Sigma \) causing the output \( y \in Y \). Then for every \( t \in \mathbb{R} \), \( \sigma_{tU} \) is an acceptable input, i.e., \( \sigma_{tU} \in U \), causing the output \( \sigma_{tY} \in Y \).

So, to be able to introduce translation-, or time-invariance of a system, translations need to be defined on the whole of \( U \) and \( Y \). Put differently,

signal spaces have to be translation-invariant subspaces of \( \mathcal{D}'(\mathbb{R}) \).

We impose an additional condition on the class \( C \), namely, we require the translation group \( \{\sigma_t\}_{t \in \mathbb{R}} \) to be strongly continuous on each of the signal spaces from \( C \). This strong-continuity condition is an additional condition on the topologies of these signal spaces. As a consequence, the theory of Chapter 2 can be applied. Particularly, there is a densely defined differentiation operator on each element of \( C \).

Finally, we require that the elements of \( C \), contain \( \mathcal{D}'(\mathbb{R}) \) densely and continuously.

We show (Theorem 4.3) that the \( c_0 \)-domain of the infinitesimal generator of the translation group on each \( V \in C \) is \( \mathcal{D}'(\mathbb{R}) \), also topologically. Then, as a consequence of Theorem 2.16, \( \mathcal{D}'(\mathbb{R}) \) is a core for every input-/output mapping from \( V \) into \( V \), for arbitrary \( V \in C \). So, the "smooth" part of the input-/output systems with signals in \( V, V \in C \), determines the system entirely. Herewith we classify systems, each with different assumptions on the signals. This diminishes the role of the topologies involved. We show in Chapter 5 that the factorization theories as developed by Kamen and Yamamoto apply to the entire constructed class of signal spaces \( C \). So, to some extent, the factorization theory will be independent of the choice of the signal space taken from \( C \). Since each of these spaces carries a topology, one might say that this new factorization theory is robust towards topology.

The second approach towards systems, the behavioural approach, considers systems in terms of (closed) translation-invariant subspaces of distributions. We follow a similar set-up as for the input-/output systems. In this case, we consider translation-invariant subspaces of \( \mathcal{D}'(\mathbb{R}) \) (!), on which the translation group is strongly continuous, and which contain \( \mathcal{E}(\mathbb{R}) \) densely and continuously. Now, the \( c_0 \)-domain of the infinitesimal generator of the translation group is \( \mathcal{E}(\mathbb{R}) \) for each of these spaces. So by Theorem 2.10, we conclude that in the behavioural approach, a system is determined entirely by its "smooth" part also.

The set-up of this chapter is as follows: In section 4.0, we introduce a class of \( \{\sigma_t\}_{t \in \mathbb{R}} \) invariant strict LF-spaces in \( \mathcal{D}'(\mathbb{R}) \), so-called translatable strict LF-spaces of \( \mathcal{D}'(\mathbb{R}) \)-type. Each subspace, in this class, can take the role as signal space in the next chapter. Results on closed, \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant subspaces and closed, linear, \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant operators on these spaces are derived. In subsection 4.0.4, we prove some results on continuous, linear \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant operators for the translatable strict LF-spaces of \( \mathcal{D}'(\mathbb{R}) \)-type, \( \mathcal{C}_2(\mathbb{R}) \) and \( L_{\infty, \text{loc}}^1(\mathbb{R}) \). In section 4.1, we introduce a class of \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant F-spaces in \( \mathcal{D}'(\mathbb{R}) \), so-called translatable F-spaces of \( \mathcal{E}(\mathbb{R}) \)-type. Results on closed, \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant subspaces and closed, linear, \( \{\sigma_t\}_{t \in \mathbb{R}} \)-invariant operators on these spaces are derived, extending results due to Kabanov [Kah] and Schwarz [Schw1].
4.0. Translation invariant subspaces of $\mathcal{D}_+^*(\mathbb{R})$

(for closed, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces on $\mathcal{E}(\mathbb{R})$) and Soethoudt [Soe] (for closed, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators on $\mathcal{E}(\mathbb{R})$). In section 4.2, we introduce a class of $(\sigma_t)_{t \in \mathbb{R}}$-invariant strict LF-spaces in $\mathcal{E}'(\mathbb{R})$, so-called translatable strict LF-spaces of $\mathcal{D}(\mathbb{R})$-type. Results on closed, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces and closed, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators on these spaces are summarized.

4.0 TRANSLATION INvariant SUBSPACES
OF $\mathcal{D}_+^*(\mathbb{R})$

In this section, we study a class of subspaces of the distribution space $\mathcal{D}_+^*(\mathbb{R})$. Defining the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{D}_+^*(\mathbb{R})$ by duality, i.e.

$$ (\sigma_t F)(\phi) := F(\sigma_t \phi) \quad (F \in \mathcal{D}'_+(\mathbb{R}), \phi \in \mathcal{D}_-(\mathbb{R}), t \in \mathbb{R}), $$

we require these subspaces $V$ to be translation-invariant, or briefly $(\sigma_t)_{t \in \mathbb{R}}$-invariant. So, if $V$ is a translation-invariant subspace of $\mathcal{D}_+^*(\mathbb{R})$, then we may speak without ambiguity of the translation group on $V$. In fact, we consider only topological subspaces of $\mathcal{D}_+^*(\mathbb{R})$, on which the translation group is strongly continuous.

4.0.1 Translatable strict LF-spaces of $\mathcal{D}_+^*(\mathbb{R})$-type

The first simple example of an $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{D}_+^*(\mathbb{R})$, on which the translation group is strongly continuous, is the space $\mathcal{D}_+^*(\mathbb{R})$ from §3.0. For reasons that will become clear in §4.0.2, we search for complete, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of $\mathcal{D}_+^*(\mathbb{R})$ which contain $\mathcal{D}_+(\mathbb{R})$ densely. In fact, they are completions of $\mathcal{D}_+(\mathbb{R})$. Such a topological vector space is the strict LF-space $C_+(\mathbb{R})$.

Example 4.1 (The space $C_+(\mathbb{R})$)

Recall from Lemma 2.64 that the vector space $C(\mathbb{R}, \mathbb{C})$, consisting of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$ and equipped with compact open topology, is an F-space. In the sequel, we abbreviate $C(\mathbb{R}, \mathbb{C})$ to $C(\mathbb{R})$. Let $C_+(\mathbb{R})$ be the subspace of $C(\mathbb{R})$ defined by

$$ C_+(\mathbb{R}) = \bigcup_{n} C_{+,n}(\mathbb{R}), \quad \text{where } C_{+,n}(\mathbb{R}) := \{ f \in C(\mathbb{R}) : \text{supp}(f) \subseteq [-n, n) \}. $$

So, $f \in C(\mathbb{R})$ belongs to $C_+(\mathbb{R})$ if $f$ has support bounded on the left. Each $C_{+,n}(\mathbb{R})$ being a closed subspace of $C(\mathbb{R})$, the family $(C_{+,n}(\mathbb{R}))_{n \in \mathbb{N}}$ is a strict inductive system. Correspondingly, $C_+(\mathbb{R})$ is equipped with the related strict LF-topology

$$ C_+(\mathbb{R}) = \varinjlim_n C_{+,n}(\mathbb{R}). \quad (4.2) $$

Notice the resemblance between the introduction of $C_+(\mathbb{R})$ and the introduction of $\mathcal{D}_+^*(\mathbb{R})$ in §3.0. It will be shown (see example 4.8) that $C_+(\mathbb{R})$ is the completion of $\mathcal{D}_+(\mathbb{R})$ with respect to a suitable strict pre-LF-topology, namely the induced $C_+(\mathbb{R})$-topology on $\mathcal{D}_+(\mathbb{R})$. In fact, each $C_{+,n}(\mathbb{R})$ is a completion of $\mathcal{D}_{+,n}(\mathbb{R})$ for the $C(\mathbb{R})$-topology. The space $C_+(\mathbb{R})$ is a subspace of $\mathcal{D}_+^*(\mathbb{R})$ in canonical way; for each $f \in$
Chapter 4. Translation invariant subspaces of distributions

\( C_c(\mathbb{R}) \), the mapping

\[
\phi \in D'(\mathbb{R}) \mapsto \int_{\mathbb{R}} f(\tau) \phi(\tau) \, d\tau
\]

(4.3)

defines a \( D'(\mathbb{R}) \)-element. We have the following (dense) inclusions

\[
D'_m(\mathbb{R}) \subset C_c(\mathbb{R}) \subset D'_m(\mathbb{R}),
\]

(4.4)

where \( D'_m(\mathbb{R}) \) is equipped with weak-star topology induced by \( D_m(\mathbb{R}) \). So, the topological dual of \( C_c(\mathbb{R}) \) is a subspace of \( D'_m(\mathbb{R}) \).

The strict LF-space \( C'_c(\mathbb{R}) \) is a \( (\mathcal{L}_0)_{\mathbb{R}_+} \)-invariant subspace of \( D'_m(\mathbb{R}) \). The translation operators \( (\sigma_t)_{t \in \mathbb{R}} \) on \( C'_c(\mathbb{R}) \), defined in the standard way by

\[
(\sigma_t f)(s) = f(t + s) \quad (t, s \in \mathbb{R}, f \in C'_c(\mathbb{R})),
\]

form a \( c_0 \)-group of continuous linear operators. Naturally, the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) on \( V \) extends the translation group on \( D'_m(\mathbb{R}) \). It can also be seen as the restriction of the translation group on \( D'_m(\mathbb{R}) \).

A translatable space \( V \) of \( D'_m(\mathbb{R}) \)-type is a translation-invariant, topological vector subspace of \( D'_m(\mathbb{R}) \), such that the natural embedding from \( V \) into \( D'_m(\mathbb{R}) \), equipped with weak-star topology \( \sigma(D'_m(\mathbb{R}), D_m(\mathbb{R})) \) is continuous and for which the restricted translation group is strongly continuous. We focus especially on the case where the topological structure of \( V \) is a strict LF-space.

**Definition 4.2** Let \( V \) be a strict LF-space, satisfying the following properties:

i. \( D'_m(\mathbb{R}) \subseteq V \subseteq D'_m(\mathbb{R}) \).

ii. The inclusions from i. are dense and continuous, where \( D'_m(\mathbb{R}) \) is equipped with weak-star topology \( \sigma(D'_m(\mathbb{R}), D_m(\mathbb{R})) \).

iii. \( \sigma_t(V) \subseteq V \) for all \( t \in \mathbb{R} \).

iv. The (restricted) translation group on \( V \) is a \( c_0 \)-group.

Then \( V \) is called a **translatable strict LF-space of \( D'_m(\mathbb{R}) \)-type**.

**Remark 4.2.1** In the nomenclature of a translatable strict LF-space of \( D'_m(\mathbb{R}) \)-type, "strict LF-space" refers to the topological structure of the vector spaces under consideration, the adjective "translatable" refers to the existence of the translation group on these spaces and the addition "of \( D'_m(\mathbb{R}) \)-type" refers to the \( c_0 \)-domain of the infinitesimal generator of the translation group on these spaces (cf. Theorem 4.3). The strict LF-space \( D'_m(\mathbb{R}) \) is the smallest space with these properties.

**Remark 4.2.2** Notice that, by definition, a translatable space of \( D'_m(\mathbb{R}) \)-type consists of distributions rather than functions. Nevertheless, we regard translatable function spaces, such as \( D'_m(\mathbb{R}) \) and \( C'_c(\mathbb{R}) \), both as distribution spaces and as function spaces.
Remark 4.2.3 In [deR2], the author presented a more general approach to translatable spaces. The essential difference between the translatable spaces as presented in [vELS1] and [deR2], and the translatable spaces from Definition 4.2 is the absence of the inclusion properties 4.2.1 and 4.2.2 in the corresponding definitions. Instead, it is assumed that there exists a continuous linear injection from every translatable space into $\mathcal{D}'_+(\mathcal{R})$, equipped with weak-star topology.

Since the translation group is strongly continuous on each translatable strict LF-space $V$ of $\mathcal{D}'_+(\mathcal{R})$-type, we may apply the theory on $\mathcal{G}$-groups from Chapter 2 to $V$. Being interested in closed subspaces and closed linear operators on these spaces, we focus on the $c_0$-domain of $(\sigma_t)_{t \in \mathcal{R}}$ on $V$. As a consequence of the Dixmier-Malliavin result, Theorem 2.11, the $c_0$-domain is equal to $\mathcal{D}_+(\mathcal{R})$ for any $V$.

**Theorem 4.3** Let $V$ be a translatable strict LF-space of $\mathcal{D}_+(\mathcal{R})$-type. Let $\delta_\varepsilon$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathcal{R}}$ on $V$. Then

$$(\text{Dom}^{\infty}(\delta_\varepsilon), \mathcal{T}_{\text{ind}}) = \mathcal{D}_+(\mathcal{R}),$$

where $\mathcal{T}_{\text{ind}}$ is the strict inductive limit topology from Definition 2.69.

**Proof.**

The proof consists of two parts.

First, we prove that $\text{Dom}^{\infty}(\delta_\varepsilon)$ and $\mathcal{D}_+(\mathcal{R})$ are equal as sets:

Since $\mathcal{D}_+(\mathcal{R}) \subset V$, and since the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathcal{R}}$ on $\mathcal{D}_+(\mathcal{R})$ is everywhere defined, we have $\mathcal{D}_+(\mathcal{R}) \subset \text{Dom}^{\infty}(\delta_\varepsilon)$.

To prove the inclusion $\text{Dom}^{\infty}(\delta_\varepsilon) \subset \mathcal{D}_+(\mathcal{R})$, we apply the Dixmier-Malliavin Theorem 2.11. Recall that

$$\text{Dom}^{\infty}(\delta_\varepsilon) = \text{span}\{ \sigma[\phi]F \mid \phi \in \mathcal{D}(\mathcal{R}), F \in V \}. \quad (4.5)$$

So, by showing that $\sigma[\phi]F \in \mathcal{D}_+(\mathcal{R})$, for every $\phi \in \mathcal{D}(\mathcal{R})$ and for every $F \in V$, we obtain (4.5) that $\text{Dom}^{\infty}(\delta_\varepsilon) \subset \mathcal{D}_+(\mathcal{R})$. To this extent, let $F \in V$ be fixed. Since $\mathcal{D}_+(\mathcal{R})$ is dense in $V$, there is a sequence $(\phi_n)_{n \in \mathcal{N}}$ in $\mathcal{D}_+(\mathcal{R})$ converging to $F$ in $V$-sense, hence in $\mathcal{D}_+(\mathcal{R})$-sense. Now, let $\phi \in \mathcal{D}(\mathcal{R})$. Then, by Theorem 3.11

$$\sigma[\phi]F = \lim_{n \to \infty} \sigma[\phi_n]F \overset{3.11}{=} \lim_{n \to \infty} \phi \ast \phi_n = \phi \ast F,$$

because $\lim_{n \to \infty} \phi_n = F$ in $\mathcal{D}_+(\mathcal{R})$-sense. Since $\phi \ast F \in \mathcal{D}_+(\mathcal{R})$ (see Figure 3.1.1), we have that $\text{Dom}^{\infty}(\delta_\varepsilon) \subset \mathcal{D}_+(\mathcal{R})$ by (4.5), and herewith $\text{Dom}^{\infty}(\delta_\varepsilon) = \mathcal{D}_+(\mathcal{R})$.

Next, we show that the topology $\mathcal{T}_{\text{ind}}$ for $\text{Dom}^{\infty}(\delta_\varepsilon)$ (see Definition 2.69) equals the strict LF-topology of $\mathcal{D}_+(\mathcal{R})$. By the Open Mapping Theorem 1.46 for strict LF-spaces, it is sufficient to prove that the strict LF-topology for $\mathcal{D}_+(\mathcal{R})$ is stronger than the strict LF-topology $\mathcal{T}_{\text{ind}}$. To this extent, let $(\phi_n)_{n \in \mathcal{N}}$ be a sequence converging to $\phi$ in $\mathcal{D}_+(\mathcal{R})$. Since the differentiation operator $\frac{d}{dt}$ on $\mathcal{D}_+(\mathcal{R})$ is continuous, for each $k \in \mathcal{N}$ the sequence $(\phi_n^{(k)})_{n \in \mathcal{N}}$ converges to $\phi^{(k)}$ in $\mathcal{D}_+(\mathcal{R})$, so in $V$-sense. Hence, $(\phi_n)_{n \in \mathcal{N}}$ converges to $\phi$ in $\mathcal{T}_{\text{graph}}$ (see Definition 2.69). Applying Proposition 2.71, we
find that \((\phi_n)_{n \in \mathbb{N}}\) converges to \(\phi\) in \(\mathcal{T}_{\text{ind}}\). Since both \(\mathcal{D}_+(\mathbb{R})\) and \((\text{Dom}^{\infty}(\delta_x), \mathcal{T}_{\text{ind}})\) are strict LF-topologies, this yields that the inclusion of \(\mathcal{D}_+(\mathbb{R})\) into \((\text{Dom}^{\infty}(\delta_x), \mathcal{T}_{\text{ind}})\) is continuous. Or equivalently, the strict LF-topology for \(\mathcal{D}_+(\mathbb{R})\) is stronger than the strict LF-topology \(\mathcal{T}_{\text{ind}}\).

There are no translatable spaces of \(\mathcal{D}_+(\mathbb{R})\)-type with a topological structure, which is simpler than the one of strict LF-spaces, such as Hilbert spaces, Banach spaces or even Fréchet spaces.

**Proposition 4.4** There exists no translatable \(F\)-space of \(\mathcal{D}_+(\mathbb{R})\)-type.

**Proof.**
Let \(V\) be a translatable \(F\)-space. Then \((\text{Dom}^{\infty}(\delta_x), \mathcal{T}_{\text{graph}})\) is an \(F\)-space by Proposition 2.68. So, by an application of the Open Mapping Theorem, \((\text{Dom}^{\infty}(\delta_x), \mathcal{T}_{\text{ind}})\) is an \(F\)-space. Hence \((\text{Dom}^{\infty}(\delta_x), \mathcal{T}_{\text{ind}}) \neq \mathcal{D}_+(\mathbb{R})\). Contradiction!

We present some examples of translatable strict LF-space of \(\mathcal{D}_+(\mathbb{R})\)-type.

**Example 4.5 (The spaces \(L^p_{\text{loc}^+}(\mathbb{R})\)**
For each \(p \geq 1\), the space \(L^p_{\text{loc}^+}(\mathbb{R})\) consists of all locally \(p\)-Lebesgue integrable functions with support bounded on the left. Define for each \(n \in \mathbb{N}, L^p_{\text{loc}^+}(\mathbb{R}) := \{ x \in L^p_{\text{loc}^+}(\mathbb{R}) | \text{supp}(x) \subseteq [-n, \infty) \} \). Then

\[
L^p_{\text{loc}^+}(\mathbb{R}) = \bigcup_n L^p_{\text{loc}^+}(\mathbb{R})
\]

Each \(L^p_{\text{loc}^+}(\mathbb{R})\) being a closed subspace of the \(F\)-space \(L^p_{\text{loc}}(\mathbb{R})\) (see Appendix A), the family \((L^p_{\text{loc}^+}(\mathbb{R}))_{n \in \mathbb{N}}\) is a strict inductive system of \(F\)-spaces. Correspondingly, \(L^p_{\text{loc}^+}(\mathbb{R})\) is equipped with the related strict LF-topology

\[
L^p_{\text{loc}^+}(\mathbb{R}) = \text{ind}_{n} L^p_{\text{loc}^+}(\mathbb{R}) \quad \text{(4.6)}
\]

The strict LF-space \(L^p_{\text{loc}^+}(\mathbb{R})\) is a translation-invariant subspace of \(\mathcal{D}_+(\mathbb{R})\), in fact the translation group on \(L^p_{\text{loc}^+}(\mathbb{R})\) is strongly continuous. So, the space \(L^p_{\text{loc}^+}(\mathbb{R})\) is a strict \(F\)-space of \(\mathcal{D}_+(\mathbb{R})\)-type.

The strict LF-space \(L^p_{\text{loc}^+}(\mathbb{R})\) is not translatable, since the translation group \((\sigma_t)_{t \in \mathbb{R}}\) on \(L^p_{\text{loc}^+}(\mathbb{R})\) is not strongly continuous. For example, for the Heaviside function \(H_0 \in L^p_{\text{loc}^+}(\mathbb{R})\), defined by \(H_0(t) = 0\) for \(t \leq 0\) and \(H_0(t) = 1\) for \(t > 0\), we have

\[
\sigma_{t} H_0 \not\to H_0, \quad \text{in} \ L^p_{\text{loc}^+}(\mathbb{R}).
\]

as \(t\) tends to zero.

**4.0.2 Translatable strict LF-spaces of \(\mathcal{D}_+(\mathbb{R})\)-type as \(\mathcal{D}_+(\mathbb{R})\)-completions**

Since \(\mathcal{D}_+(\mathbb{R})\) is a dense subspace of each translatable space \(V\) of \(\mathcal{D}_+(\mathbb{R})\)-type by definition, we can regard \(V\) as a completion of \(\mathcal{D}_+(\mathbb{R})\) equipped with a topology
weaker than the $\mathcal{D}_s(ℝ)$-topology. In this subsection, we investigate the locally convex topologies on $\mathcal{D}_s(ℝ)$ leading to a translatable space of $\mathcal{D}_s(ℝ)$-type by completion. These topologies show to be strict pre-LF-topologies, satisfying certain conditions on the countable set of seminorms related to the strict pre-LF-topology as in Proposition 1.36.

In Chapter 3, we introduced for each $n ∈ ℕ$ the subspaces $\mathcal{D}_{+,n}(ℝ)$ of $\mathcal{D}_s(ℝ)$ as

$$\mathcal{D}_{+,n}(ℝ) := \{ f ∈ \mathcal{D}_s(ℝ) | \text{ supp}(f) ⊆ [-n, n] \}.$$  \hfill (4.7)

With induced $\mathcal{E}(ℝ)$-topology, the family $(\mathcal{D}_{+,n}(ℝ))_{n ∈ ℕ}$ is a strict inductive system of F-spaces, satisfying

$$\mathcal{D}_s(ℝ) = \text{ind}_n \mathcal{D}_{+,n}(ℝ).$$ \hfill (4.8)

Defining the subspaces $\mathcal{D}'_{+,n}(ℝ)$ of $\mathcal{D}_s(ℝ)$ for each $n ∈ ℕ$ by

$$\mathcal{D}'_{+,n}(ℝ) := \{ f ∈ \mathcal{D}_s(ℝ) | \text{ supp}(f) ⊆ [-n, n] \},$$ \hfill (4.9)

we observe that

$$\mathcal{D}'_{+,n}(ℝ) \cap \mathcal{D}_s(ℝ) = \mathcal{D}_{+,n}(ℝ) \quad \text{and} \quad \mathcal{D}_s(ℝ) ≤ \mathcal{D}'_{+,n}(ℝ).$$

Now, let $(F_n)_{n ∈ ℕ}$ be a strict inductive system of F-spaces satisfying for each $n ∈ ℕ$

$$\mathcal{D}_{+,n}(ℝ) ≤ F_n ≤ \mathcal{D}'_{+,n}(ℝ),$$ \hfill (4.10)

where the inclusions are continuous, and where the inclusions are dense. Then $\text{ind}_n F_n$ satisfies

$$\mathcal{D}_s(ℝ) ≤ \text{ind}_n F_n ≤ \mathcal{D}'_{+,n}(ℝ),$$ \hfill (4.11)

where all inclusions are continuous and dense.

The question arises what are necessary (extra) conditions on the strict inductive system $(F_n)_{n ∈ ℕ}$, satisfying (4.10), to ensure that $\text{ind}_n F_n$ is a translatable space of $\mathcal{D}_s(ℝ)$-type.

Since every F-space $F_n$ is the completion of $\mathcal{D}_{+,n}(ℝ)$ equipped with a pre-F-topology, we can, to answer the question, search for conditions on pre-F-topologies on the spaces $\mathcal{D}_{+,n}(ℝ)$ as well.

Let $\mathcal{T}_n$ be pre-Préchet topologies for $\mathcal{D}_{+,n}(ℝ)$ $n ∈ ℕ$, making $(\mathcal{D}_{+,n}(ℝ))_{n ∈ ℕ}$ a strict inductive system. If we choose the topologies $\mathcal{T}_n$ such that

$$\mathcal{D}_{+,n}(ℝ) ↪ (\mathcal{D}_{+,n}(ℝ), \mathcal{T}_n) ↪ \mathcal{D}'_{+,n}(ℝ),$$

then the completion of $\text{ind}_n (\mathcal{D}_{+,n}(ℝ), \mathcal{T}_n)$ in $\mathcal{D}'_{+,n}(ℝ)$ satisfies (4.11) by Theorem 1.40, and for each $n ∈ ℕ$, the (Préchet-) completion of $(\mathcal{D}_{+,n}(ℝ), \mathcal{T}_n)$ satisfies (4.10). Moreover, if we choose the topologies such that $(\sigma_n)_{n ∈ ℕ}$ is a locally equicontinuous $C₀$-group on $\text{ind}_n (\mathcal{D}_{+,n}(ℝ), \mathcal{T}_n)$, then $(\sigma_n)_{n ∈ ℕ}$ extends continuously to a $C₀$-group on the completion (Theorem 2.4). Naturally, this extended group is the restriction of the $\mathcal{D}_s(ℝ)$-translation group to the completion of $\text{ind}_n (\mathcal{D}_{+,n}(ℝ), \mathcal{T}_n)$. 
Recalling from Proposition 1.36 that the topology of a strict inductive system of pre-F-spaces is described in terms of one countable separating set of seminorms II, we end up with conditions on II. So, in case of a translatable strict LF-space of \( D_\infty (\mathbb{R}) \)-type, the topology related to II, has to be weaker than the \( D_\infty (\mathbb{R}) \)-topology, it has to be stronger than the restricted \( \sigma (D'_\infty (\mathbb{R}), D_\infty (\mathbb{R})) \)-topology and the translation group has to be locally equicontinuous with respect to this topology.

**Theorem 4.6** Let \( II = (p_k)_{k \in \mathbb{N}} \) be a separating family of ordered seminorms, i.e. \( p_k \leq p_{k+1} \) for each \( k \in \mathbb{N} \), on \( D_\infty (\mathbb{R}) \) satisfying the following conditions on each \( D_{+,n}(\mathbb{R}) \):

I. \( \forall \varepsilon \in \mathbb{R}^+ \exists c_0 \in \mathbb{R}^+ \forall \varepsilon \in D_{+,n}(\mathbb{R}) \left[ p_k(\psi) \leq C \cdot \sum_{i=0}^{n} \max_{t \in [-N,N]} |\psi^{(i)}(t)| \right] \).

II. \( \forall \varepsilon \in D_\infty (\mathbb{R}) \exists c_0 \in \mathbb{R}^+ \forall \varepsilon \in D_{+,n}(\mathbb{R}) \left[ \| f \|_\varepsilon \psi(\tau) \nu(\tau) \, d\tau \leq C \cdot p_k(\psi) \right] \).

III. \( \forall \varepsilon \in \mathbb{R}^+ \exists c_0 \in \mathbb{R}^+ \forall \varepsilon \in D_{+,n}(\mathbb{R}) \left[ \sup_{t \in I} \sigma_t(\psi) \leq C \cdot p(\psi) \right] \).

Let, for each \( n \in \mathbb{N} \), \( T_n \) be the pre-F-topology on \( D_{+,n}(\mathbb{R}) \) generated by the seminorms \( (p_{k,n})_{k \in \mathbb{N}} \) and let \( F_n \) be the completion of \( D_{+,n}(\mathbb{R}) \) for \( T_n \) in \( D_\infty (\mathbb{R}) \). Then

i. \((D_{+,n}(\mathbb{R}), T_n)_{n \in \mathbb{N}} \) is a strict inductive system of pre-F-spaces.

ii. \((F_n)_{n \in \mathbb{N}} \) is a strict inductive system of Fréchet spaces.

iii. \( F_n \) is a translatable strict LF-space of \( D_\infty (\mathbb{R}) \)-type.

**Proof.**
The proof uses results from Chapter 1 and 2.

i. Let \( T_n \) be the pre-F-topology on \( D_{+,n}(\mathbb{R}) \) generated by the restricted seminorms \( (p_{k,n})_{k \in \mathbb{N}} \). Then condition I and II yield

\[ D_{+,n}(\mathbb{R}) \hookrightarrow (D_{+,n}(\mathbb{R}), T_n) \hookrightarrow D_\infty (\mathbb{R}). \]

Notice that \( T_{n+1} \mid D_{+,n}(\mathbb{R}) = T_n \) by construction. It remains to prove that \( D_{+,n}(\mathbb{R}) \) is a closed subspace of \((D_{+,n+1}(\mathbb{R}), T_{n+1})\) for each \( n \in \mathbb{N} \). Since \( T_n \) is finer than the restricted \( \sigma (D_\infty (\mathbb{R}), D_\infty (\mathbb{R})) \)-topology, it is sufficient to show that \( D_{+,n+1}(\mathbb{R}) \setminus D_{+,n}(\mathbb{R}) \) is an \( \sigma (D_\infty (\mathbb{R}), D_\infty (\mathbb{R})) \)-open subset of \( D_{+,n+1}(\mathbb{R}) \). To this extent, let \( \psi_0 \in D_{+,n+1}(\mathbb{R}) \setminus D_{+,n}(\mathbb{R}) \). Choose \( \varphi \in D_\infty (\mathbb{R}) \) such that supp(\( \varphi \)) \( \subseteq (-\infty, -n] \) and \( \int_{\mathbb{R}} \psi_0(\tau) \varphi(\tau) \, d\tau = 1 \). Then

\[ \{ \psi \in D_{+,n+1}(\mathbb{R}) \mid \int_{\mathbb{R}} (\psi_0 - \psi)(\tau) \varphi(\tau) \, d\tau < 1 \} \subseteq D_{+,n+1}(\mathbb{R}) \setminus D_{+,n}(\mathbb{R}), \]

so \( D_{+,n+1}(\mathbb{R}) \setminus D_{+,n}(\mathbb{R}) \) is an \( \sigma (D_\infty (\mathbb{R}), D_\infty (\mathbb{R})) \)-open subset of \( D_{+,n+1}(\mathbb{R}) \), and therefore \( T_{n+1} \)-open subset of \( D_{+,n+1}(\mathbb{R}) \). Thus, we proved that \((D_{+,n}(\mathbb{R}), T_n)_{n \in \mathbb{N}} \) is a strict inductive system of pre-F-spaces.

ii. Let \( F_n \) be the completion of \((D_{+,n}(\mathbb{R}), T_n)\) in \( D_\infty (\mathbb{R}) \). Then by Theorem 1.40,
4.0. Translation invariant subspaces of $\mathcal{D}_+(\mathbb{R})$

$(F_n)_{n \in \mathbb{N}}$ is a strict inductive system of Fréchet spaces.

iii. By Theorem 1.40, $\text{ind } F_n$ satisfies the inclusions (4.11). Moreover, since $\mathcal{D}_+(\mathbb{R}) \hookrightarrow \text{ind } \mathcal{D}_{+,n}(\mathbb{R}, \mathcal{T}_n)$, the strong continuity of $(\sigma_n)_{n \in \mathbb{N}}$ on $\mathcal{D}_+(\mathbb{R})$ yields that $(\sigma_n)_{n \in \mathbb{N}}$ is strongly continuous on $\text{ind } \mathcal{D}_{+,n}(\mathbb{R}, \mathcal{T}_n)$. So, since $(\sigma_n)_{n \in \mathbb{N}}$ is locally equicontinuous on $\text{ind } \mathcal{D}_{+,n}(\mathbb{R}, \mathcal{T}_n)$ by condition III, it extends to a $\sigma_n$-group on $\text{ind } F_n$ by Theorem 2.4. We conclude that $\text{ind } F_n$ is a translatable space of $\mathcal{D}_+(\mathbb{R})$-type.

Each strict LF-space constructed according to Theorem 4.6 is a translatable space of $\mathcal{D}_+(\mathbb{R})$-type. The converse is also true: each translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type arises from such a construction. In other words; the conditions I, II and III are necessary and sufficient for a strict LF-space $\text{ind } F_n$ to be of $\mathcal{D}_+(\mathbb{R})$-type.

**Theorem 4.7** Let $V = \text{ind } V_n$ be a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type and let $(p_n)_{n \in \mathbb{N}}$ be a family of seminorms related to $V$'s-topology in the sense of Proposition 1.36. Then the following statements hold true.

i. The seminorms $(p_n)_{n \in \mathbb{N}}$ satisfy the conditions I, II and III from Theorem 4.6 on $\mathcal{D}_+(\mathbb{R})$.

ii. A strict inductive system of $F$-spaces $(F_n)_{n \in \mathbb{N}}$ exists satisfying

- $\mathcal{D}_{+,n}(\mathbb{R}) \subseteq F_n \subseteq \mathcal{D}_+(\mathbb{R})$, where the inclusions are continuous and the first inclusion is dense,

- $V = \text{ind } F_n$.

**Proof.**

Let $V = \text{ind } V_n$ be a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type. Since, by definition, $\mathcal{D}_+(\mathbb{R}) \hookrightarrow V \hookrightarrow \mathcal{D}_+(\mathbb{R})$ and since the translation group is locally equicontinuous on $V$, the conditions I, II and III from Theorem 4.6 on $\mathcal{D}_+(\mathbb{R})$ are satisfied. Equip each $\mathcal{D}_{+,n}(\mathbb{R})$ with the relative $V$-topology, $\mathcal{T}_n$, and let $F_n$ be its completion in $\mathcal{D}_+(\mathbb{R})$. By Theorem 4.6, $(F_n)_{n \in \mathbb{N}}$ is a strict inductive system of $F$-spaces, and $\text{ind } F_n$ is a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type. It remains to prove that $V = \text{ind } F_n$.

Since $V$ is complete, we have $\text{ind } F_n \hookrightarrow V$. So, if $\cup_{n=1}^{\infty} F_n = V$, then the Open Mapping Theorem 1.46 ensures that $\text{ind } F_n = V$. To this extent, let $\phi \in V$ and suppose $\text{supp}(\phi) \subseteq [-m, \infty)$. Let $(\phi_n)_{n \in \mathbb{N}}$ be an approximate identity of regularizers with $\text{supp}(\phi_n) \subseteq [-1, 0]$. Then $\sigma(\phi_n) \phi \in \mathcal{D}_{+,n}(\mathbb{R})$ and $\sigma(\phi_n) \phi \to \phi$ in $V$-sense. Hence, $\phi \in \text{ind } F_n \subseteq \cup_{n=1}^{\infty} F_n$.

The following example illustrates that the translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type $C_+(\mathbb{R})$ can be constructed by the above theorems.

**Example 4.8** Define the seminorms $\Pi := (p_n)_{n \in \mathbb{N}}$ on $C_+(\mathbb{R})$ by

$$ p_n(\phi) := \max_{n \in [-1, 1]} | \phi(t) | \quad (\phi \in C_+(\mathbb{R})). $$


The seminorms \((p_k)_{k \in \mathbb{N}}\) are related to the \(C_c(\mathbb{R})\)-topology in the sense of Proposition 1.36. We will show that the seminorms \(p_k\) restricted to \(D_+(\mathbb{R})\) satisfy the conditions of Theorem 4.6, as is to be expected. Naturally, condition I of Theorem 4.6 is satisfied. Moreover, let \(n \in \mathbb{N}\) and \(\phi \in D_-(\mathbb{R})\) be fixed. Let \(k \geq n\) be such that \(\text{supp}(\phi) \subseteq (-\infty, k]\) and let \(C = \int_{-n}^{k} |\phi(r)| \, dr\). Then for all \(\psi \in D_{+n}(\mathbb{R})\)

\[
|\int_{\mathbb{R}} \psi(r)^k \, dr| \leq \max_{t \in [-n,k]} |\psi(t)| \cdot \int_{-n}^{k} |\phi(r)| \, dr = C \cdot p_k(\psi),
\]

so condition II of Theorem 4.6 is satisfied. Finally, let \(n, k \in \mathbb{N}\) and let \(I \subseteq \mathbb{R}\) compact. Choose \(m \in \mathbb{N}\) such that \(I \subseteq [-m, m]\). Then for all \(\psi \in D_{+n}(\mathbb{R})\)

\[
\sup_{t \in \mathbb{R}} p_k(\sigma_t \psi) \leq \max_{t \in [-m,m]} |\psi(t)| \leq p_{m+n}(\psi),
\]

which yields condition III.

**Example 4.9** Let \(p \geq 1\). Define the seminorms \(\Gamma := \{q_k \mid k \in \mathbb{N}\}\) on \(D_+(\mathbb{R})\) by

\[
q_k(\phi) := \left( \int_{-k}^{k} |\phi(r)|^p \, dr \right)^{1/p} \quad (\phi \in L^p_{loc,-k}(\mathbb{R})).
\]

The seminorms \(\{q_k \mid k \in \mathbb{N}\}\) are related to the restricted \(L^p_{loc,-k}(\mathbb{R})\)-topology in the sense of Proposition 1.36. We show that the seminorms \(\Gamma\) satisfy the conditions of Theorem 4.6. To this extent, fix \(n \in \mathbb{N}\), and let \(k \in \mathbb{N}\). Then, for each \(\psi \in D_{+n}(\mathbb{R})\)

\[
q_k(\psi) = \left( \int_{-k}^{k} |\psi(r)|^p \, dr \right)^{1/p} \leq (2k)^{1/p} \cdot \max_{t \in [-k,k]} |\psi(t)| .
\]

So, the seminorms \(\Gamma\) satisfy condition I of Theorem 4.6. Furthermore, let \(\phi \in D_-(\mathbb{R})\). Let \(k \geq n\) be such that \(\text{supp}(\phi) \subseteq (-\infty, k]\). Then for each \(\psi \in D_{+n}(\mathbb{R})\), we have by the H"{o}lder inequality

\[
|\int_{\mathbb{R}} \psi(r)^k \phi(r) \, dr| \leq \left( \int_{-n}^{k} |\phi(r)|^q \, dr \right)^{1/q} \cdot \left( \int_{-n}^{k} |\psi(r)|^p \, dr \right)^{1/p} \leq \left( \int_{-n}^{k} |\phi(r)|^q \, dr \right)^{1/q} \cdot q_k(\psi),
\]

where \(q \geq 1\) is such that \(\frac{1}{q} + \frac{1}{p} = 1\). Finally, let \(I \subseteq \mathbb{R}\) be compact and let \(k \in \mathbb{N}\). Let \(m \in \mathbb{N}\) be such that \(I \subseteq [-m, m]\). Then for each \(\psi \in D_{+n}(\mathbb{R})\),

\[
\sup_{t \in \mathbb{R}} q_k(\sigma_t \psi) = \sup_{t \in \mathbb{R}} \left( \int_{-m}^{m} |\psi(r+t)|^p \, dr \right)^{1/p} \leq q_{k+m}(\psi),
\]

so the seminorms \(\Gamma\) on \(D_+(\mathbb{R})\) satisfy condition III also. Since \(L^p_{loc,-k}(\mathbb{R})\) is the completion of \(D_+(\mathbb{R})\) with restricted \(L^p_{loc,-k}(\mathbb{R})\)-topology, \(L^p_{loc,-k}(\mathbb{R})\) is a translatable strict LF-space of \(D_+(\mathbb{R})\)-type.
4.0. Translation invariant subspaces of $\mathcal{D}_+(\mathbb{R})$

We obtain the following result from Theorem 4.7.

**Corollary 4.10** Let $V = \text{ind } V_m$ be a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type. Then for each $m \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that

$$\phi \in V_m \Rightarrow \text{supp}(\phi) \subseteq [-n, \infty).$$

Put differently, $V_m$ consists of $\mathcal{D}_+(\mathbb{R})$-distributions with support in $[-n, \infty)$ only.

**Proof.**
Let the F-spaces $F_n$ be as in Theorem 4.7. Since $\mathcal{D}_+(\mathbb{R}) = F_n \hookrightarrow \mathcal{D}_+(\mathbb{R})$, each $F_n$ consists of $V$-distributions with support in $[-n, \infty)$. Since $\text{ind } V_m = \text{ind } F_n$, there exists, by Theorem 1.44, for each $m \in \mathbb{N}$ an $n \in \mathbb{N}$, such that

$$V_m \hookrightarrow F_n,$$

which proves the assertion.

The following result provides us a method to construct new translatable strict LF-spaces of $\mathcal{D}_+(\mathbb{R})$-type.

**Lemma 4.11** Let $V = \text{ind } V_m$ be a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type. Let $\delta_{e}$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $V$. Let for each $m \in \mathbb{N}$, the subspace $V_{1,m}$ be defined by

$$V_{1,m} := \{ x \in \text{Dom}(\delta_{e}) \mid x, \delta_{e}x \in F_n \}.$$

Equip each $V_{1,m}$ with the relative graph topology of $\text{Dom}(\delta_{e})$. Then the following assertions hold true.

i. $(V_{1,m})_{m \in \mathbb{N}}$ is a strict inductive system of F-spaces.

ii. $\text{ind } V_{1,m}$ is a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type.

**Proof.**
For the proof of i, we refer to §2.3, pages 51 and 51.

ii. Let $\Pi$ be a family of seminorms generating the strict inductive limit topology of $V$ in the sense of Proposition 1.36. Then, by §2.3, the strict LF-topology of $\text{ind } V_{1,m}$ is generated by the family of seminorms $\Pi_{1} := \{ p + p \circ \delta_{e} \mid p \in \Pi \}$ in the sense of Proposition 1.36. Since the seminorms $\Pi$ satisfy the conditions I, II and III of Theorem 4.6, the seminorms $\Pi_{1}$ satisfy the conditions I, II and III of Theorem 4.6. Hence by Theorem 4.6, $\text{ind } V_{1,m}$ is a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type.

Applying Lemma 4.11 recursively to $\mathcal{C}_0(\mathbb{R})$ and $L^p_{\text{loc},+}(\mathbb{R})$, $p \geq 1$, we see that each of the strict LF-spaces $\mathcal{C}_0^+(\mathbb{R})$ and the Sobolev spaces $H^p_{\text{loc},+}(\mathbb{R})$ are also examples of translatable strict LF-spaces of $\mathcal{D}_+(\mathbb{R})$-type.
4.0.3 Closed, \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspaces and operators on translatable strict LF-spaces of \(D_+(\mathbb{R})\)-type

In this subsection, we consider closed, translation-invariant subspaces and operators on translatable strict LF-spaces of \(D_+(\mathbb{R})\)-type. These will be shown to be extensions of closed, translation-invariant subspaces and operators on \(D_+(\mathbb{R})\), which are described in §3.8.

The discussion of closed subspaces of translatable strict LF-space of \(D_+(\mathbb{R})\)-type is kept brief; there are only trivial ones.

**Theorem 4.12** Let \(V\) be a translatable strict LF-space of \(D_+(\mathbb{R})\)-type. Let \(M\) be a closed \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspace of \(V\). Then \(M = V\) or \(M = \{0\}\).

**Proof.**

Let \(M\) be a closed \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspace of \(V\), and suppose \(M \neq \{0\}\). Since \(D_+(\mathbb{R}) = \text{Dom}^{\omega}(\delta_0)\), \(M \cap D_+(\mathbb{R})\) is sequentially dense in \(M\) by Theorem 2.14. So, \(M \cap D_+(\mathbb{R}) \neq \{0\}\). Since \(M \cap D_+(\mathbb{R})\) is closed and \((\sigma_1)_{t \in \mathbb{R}}\)-invariant in \(D_+(\mathbb{R})\), \(M \cap D_+(\mathbb{R}) = D_+(\mathbb{R})\) by Theorem 3.12. Hence, \(M = V\).

\[\square\]

**Remark 4.12.1** There is another approach in showing the weaker assertion that there exist no finite dimensional, \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspaces of \(D_+(\mathbb{R})\). In fact, in [deR] it is shown (Theorem 2.14) that every finite dimensional (so closed) \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspace is the linear span of exponential-polynomials, i.e. functions of the form \(p(t)e^{\lambda t}\), where \(p\) a polynomial. Since \(D_+(\mathbb{R})\) contains the trivial Exponential Function 0 only, this yields that the only finite dimensional \((\sigma_1)_{t \in \mathbb{R}}\)-invariant subspace of \(D_+(\mathbb{R})\) is \(\{0\}\).

We can reformulate Theorem 4.12 in terms of mean periodic functions (see Remark 3.33.1). Recall that a function \(\phi\) in a translatable space \(V\) is mean periodic iff the set of all its translates is not total in \(V\).

**Corollary 4.13** Let \(V\) be a translatable strict LF-space of \(D_+(\mathbb{R})\)-type. Then there are no mean periodic functions in \(V\) but the null function 0.

**Proof.**

The subspace \(\text{span}\{\sigma_1 \phi \mid t \in \mathbb{R}\}\) of \(V\) is closed and \((\sigma_1)_{t \in \mathbb{R}}\)-invariant for every \(\phi \in V\).

\[\square\]

Next, we focus on \((\sigma_1)_{t \in \mathbb{R}}\)-invariant operators \(L\) on a translatable strict LF-space \(V\) of \(D_+(\mathbb{R})\)-type, i.e. \(\sigma_1(\text{Dom}(L)) \subseteq \text{Dom}(L)\) and \(\sigma_1 L = L \sigma_1\) for all \(t \in \mathbb{R}\) (cf. Definition 2.15). We obtain the following lemma from Theorem 2.16.

**Lemma 4.14** Let \(V\) be a translatable strict LF-space of \(D_+(\mathbb{R})\)-type. Let \(L_1, L_2\) be closed, linear \((\sigma_1)_{t \in \mathbb{R}}\)-invariant operators on \(V\) with domains \(\text{Dom}(L_1)\) and \(\text{Dom}(L_2)\). If \(D_+(\mathbb{R}) \subseteq \text{Dom}(L_1) \cap \text{Dom}(L_2)\), then

\[L_1 |_{D_+(\mathbb{R})} = L_2 |_{D_+(\mathbb{R})} \iff L_1 = L_2.\]
Proof.
Since $D_+(R) = \text{Dom}^\infty(\delta_2)$ is a core for every closed, linear $(\sigma_t)_{t \in R}$ -invariant operator on $V$ by Theorem 2.16, the assertion follows.
\[ \blacksquare \]

Remark 4.14.1 We emphasize that $L_1 = L_2$ means $\text{Dom}(L_1) = \text{Dom}(L_2)$ and $L_1 = L_2$ on $\text{Dom}(L_1)$.

Proposition 4.15 Let $V$ be a translatable strict LF-space of $D_+(R)$-type. Let $L$, with domain $\text{Dom}(L)$, be a closed linear operator on $V$, satisfying

i. $L$ is $(\sigma_t)_{t \in R}$ -invariant,

ii. $D_+(R) \subseteq \text{Dom}(L)$.

Then $L(D_+(R)) \subseteq D_+(R)$ and the restricted mapping $L|_{D_+(R)}$ is a continuous linear, $(\sigma_t)_{t \in R}$ -invariant operator on $D_+(R)$.

Proof.
Let $\delta_2$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in R}$ on $V$ with $c_\omega$ -domain $\text{Dom}^\infty(\delta_2)$. Then $L(\text{Dom}^\infty(\delta_2)) \subseteq \text{Dom}^\infty(\delta_2)$ by Theorem 2.16. Since $\text{Dom}^\infty(\delta_2) = D_+(R)$ (Theorem 4.3), the first part of the assertion follows. The continuity of the restricted mapping follows from $D_+(R) \hookrightarrow V$ and an application of the Closed Graph Theorem (Theorem 1.44).
\[ \blacksquare \]

We characterized the collection of all continuous, linear, $(\sigma_t)_{t \in R}$ -invariant operators on $D_+(R)$ in Theorem 3.5. So, we obtain a characterization of these closed, linear, $(\sigma_t)_{t \in R}$ -invariant operators on arbitrary translatable strict LF-spaces of $D_+(R)$-type by Lemma 4.14.

Theorem 4.16 Let $V$ be a translatable strict LF-space of $D_+(R)$-type. Let $L$ be a linear mapping on $V$ with domain $\text{Dom}(L)$, where $D_+(R) \subseteq \text{Dom}(L)$. Then the following two assertions are equivalent.

i. $L$ is closed and $(\sigma_t)_{t \in R}$ -invariant.

ii. A distribution $F \in D_+(R)$ exists such that

- $\text{Dom}(L) = \{ \phi \in V \mid F \ast \phi \in V \}$
- $L \phi = F \ast \phi$ for each $\phi \in \text{Dom}(L)$.

Proof.

ii.$\Rightarrow$i. Let $F \in D_+(R)$ and let the linear mapping $L$ with domain $\text{Dom}(L) := \{ \phi \in V \mid F \ast \phi \in V \}$ be defined by

\[ L \phi := F \ast \phi \quad (\phi \in \text{Dom}(L)). \]
Since $V$ is a strict LF-space, Theorem 1.44 yields that $L$ is closed if and only if $L$ is sequentially closed. To this extent, let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Dom}(L)$, such that
\[ \phi_n \to \phi \quad \text{and} \quad L\phi_n \to \chi \quad (n \to \infty), \]
in $V$-sense. Since both limits exist in $\mathcal{D}'_+(\mathbb{R})$-sense, we have
\[ \lim_{n \to \infty} F * \phi_n = \chi, \]
Hence, $F * \phi = \chi \in V$, and therefore $\phi \in \text{Dom}(L)$ and $L\phi = \chi$. We conclude that $L$ is closed. It remains to prove that $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant. To this extent, let $t \in \mathbb{R}$ and let $\phi \in \text{Dom}(L)$. Then
\[ \sigma_t L\phi = \delta_t * (F * \phi) = F * (\delta_t * \phi) = F * \sigma_t \phi. \]
Since $\sigma_t(F * \phi) \in V$, we conclude that $\sigma_t \phi \in \text{Dom}(L)$ and $\sigma_t L\phi = L \sigma_t \phi$, i.e. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant.
To prove the implication $i \Rightarrow ii$, let $L$ be a closed, $(\sigma_t)_{t \in \mathbb{R}}$-invariant, linear mapping on $V$ with domain $\text{Dom}(L) \supseteq \mathcal{D}_+(\mathbb{R})$. Then, there exists $G \in \mathcal{D}'_+(\mathbb{R})$, such that
\[ L\phi = \sigma[G] \phi \quad (\phi \in \mathcal{D}_+(\mathbb{R})) \quad (4.12) \]
by Proposition 4.15 and Theorem 3.5. Define $F \in \mathcal{D}'_+(\mathbb{R})$ by
\[ F(\psi) := G(\check{\psi}) \quad (\psi \in \mathcal{D}_-(\mathbb{R})). \]
Then, for each $\phi \in \mathcal{D}_+(\mathbb{R})$ and each $t \in \mathbb{R}$
\[ (L\phi)(t) = (4.11) \quad G(\sigma_t \phi) =: F(\sigma_t \check{\phi}) = (F * \phi)(t). \]
Hence, $L\phi = F * \phi$ for each $\phi \in \mathcal{D}_+(\mathbb{R})$. Recall from the first part of the proof that $F$ induces a closed, linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant mapping on $V$ also. So, Lemma 4.14 yields that $\text{Dom}(L) = \{ \phi \in V \mid F * \phi \in V \}$ and $L\phi = F * \phi$ for each $\phi \in \text{Dom}(L)$.

Remark 4.16.1 The characterizing distribution $F$ for a $(\sigma_t)_{t \in \mathbb{R}}$-invariant closed linear operator $L$ on a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type is called impulse response (in System Theory) or convolution kernel (in Functional Analysis) of the operator $L$. In fact, $F^\vee = L^0$.

The absence of non-trivial closed, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces of a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type has the following consequence for closed linear operators.

Proposition 4.17 Let $V$ be a translatable strict LF-space of $\mathcal{D}_+(\mathbb{R})$-type. Let $\xi$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $V$. Let $L$ be an $(\sigma_t)_{t \in \mathbb{R}}$-invariant closed linear operator on $V$ with domain $\text{Dom}(L)$ such that $\mathcal{D}_+(\mathbb{R}) \subseteq \text{Dom}(L)$ and assume that $L$ is not a scalar multiple of the identity mapping. Then
- $L$ has a dense range.
4.0. Translation invariant subspaces of $D_\sigma^\star (\mathbb{R})$

- $L$ has no eigenvalues.

**Proof.**

Observe that $\text{Range}(L)$ is an $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant subspace of $V$. Hence, its closure is a closed, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant subspace of $V$. Since $\text{Range}(L) \neq \{0\}$, Theorem 4.12 yields that $\text{Range}(L) = V$, or equivalently $L$ has a dense Range.

The second statement follows from the observation that each $L - \lambda I$ is a closed, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant operator on $V$, so that $\text{Kern}(L - \lambda I)$ is a closed, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant subspace of $V$, and Theorem 4.12.

- **4.0.4 Continuous, linear, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant operators**

Since a continuous operator is closed, each continuous linear $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant operator $L$ on any translatable strict LF-space $V$ of $D_\sigma^\star (\mathbb{R})$-type is of the form $L \phi = F * \phi$, $\phi \in V$, for some $F \in D_{\sigma}^\star (\mathbb{R})$ (cf. Theorem 4.16(ii)). However, this result is not satisfactory, since it does not reveal the form of the convolution kernels $F$. In case of the translatable space $D_{\sigma}^\star (\mathbb{R})$, the algebra $D_{\sigma}^\star (\mathbb{R})$ corresponds to the collection of all continuous, linear, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant mappings on $D_{\sigma}^\star (\mathbb{R})$ (cf. Theorem 5.3), but for other translatable strict LF-spaces of $D_{\sigma}^\star (\mathbb{R})$-type, the class of all continuous, linear, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant mappings corresponds to a proper subalgebra of $D_{\sigma}^\star (\mathbb{R})$.

In case of the translatable space $C_{\sigma}^\star (\mathbb{R})$, the subalgebra $M_{\sigma} (\mathbb{R})$ of $D_{\sigma}^\star (\mathbb{R})$ characterizes all continuous, linear, $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant mappings.

**Proposition 4.18** Let $L$ be a mapping on $C_{\sigma}^\star (\mathbb{R})$. Then the following assertions are equivalent:

i. $L$ is continuous, linear and $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant.

ii. There exists $\mu \in M_{\sigma} (\mathbb{R})$, such that $L \phi = \mu * \phi$ for all $\phi \in C_{\sigma}^\star (\mathbb{R})$.

**Proof.**

i $\Rightarrow$ ii. Let $L$ be a continuous linear $(\sigma_\lambda)_{\lambda \in \mathbb{R}}$-invariant mapping on $C_{\sigma}^\star (\mathbb{R})$. Then $\phi \in C_{\sigma}^\star (\mathbb{R}) \mapsto (L \phi)(0)$ defines a continuous linear functional on $C_{\sigma}^\star (\mathbb{R})$. From [vEdR4], Theorem 1.12, we recall that every continuous linear functional $F$ on $C_{\sigma}^\star (\mathbb{R})$ is of the form

$$F(\phi) = \int_{\mathbb{R}} \phi(\tau) \nu(d\tau) \quad (\phi \in C_{\sigma}^\star (\mathbb{R})), \quad (4.13)$$

for some $\nu \in M_{\sigma} (\mathbb{R})$. Now, let $\nu \in M_{\sigma} (\mathbb{R})$ correspond to the $C_{\sigma}^\star (\mathbb{R})$-functional $\phi \mapsto (L \phi)(0)$. Define $\mu \in M_{\sigma} (\mathbb{R})$ by

$$\mu(x) := \nu(x) \quad (x \in C_{\sigma}(\mathbb{R})).$$

We show that $L \phi = \mu * \phi$ for all $\phi \in D_{\sigma}(\mathbb{R})$, so that the assertion follows from Lemma 4.14. To this extent, let $\phi \in D_{\sigma}(\mathbb{R})$ and let $t \in \mathbb{R}$. Then,

$$(L \phi)(t) = L(\sigma_t \phi)(0) = \int_{\mathbb{R}} \phi(\tau + t) \nu(d\tau) = \int_{\mathbb{R}} \phi(t - \tau) \mu(d\tau) = (\mu * \phi)(t)$$
We conclude that $L\phi = \mu * \phi$.

ii $\Rightarrow$ i. By Theorem 4.16, the everywhere defined mapping $L$ is closed, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant, hence continuous.

**Remark 4.18.1** Let $\mu \in \mathcal{M}_e(\mathbb{R})$. Define $\nu \in \mathcal{M}_e(\mathbb{R})$ by $\nu = \frac{\mu}{\mu}$. Then, it can be shown (see [deR2]) that the mapping $\phi \mapsto \mu * \phi$ on $C_+^\infty(\mathbb{R})$ extends the mapping $\sigma[\nu]$, introduced in §3.9, on $D_+(\mathbb{R})$ continuously.

The following result extends a result due to Yamamoto ([Y2], Theorem 3.11), and is presented in our terminology.

**Theorem 4.19** Let $L$ be a linear operator on $L^p_{loc,+}(\mathbb{R})$, where $p \geq 1$. Then the following two statements are equivalent.

i. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant and continuous, satisfying $L(C_+^\infty(\mathbb{R})) \subseteq C_+^\infty(\mathbb{R})$.

ii. $\mu \in \mathcal{M}_e(\mathbb{R})$ exists such that $L\phi = \mu * \phi$ for all $\phi \in L^p_{loc,+}(\mathbb{R})$.

Moreover, if additionally for all $\phi \in C_+^\infty(\mathbb{R})$

$$\text{supp}(\phi) \subseteq [0, \infty) \implies \text{supp}(L(\phi)) \subseteq [0, \infty),$$

then $\text{supp}(\mu) \subseteq [0, \infty)$.

**Proof.**

i $\Rightarrow$ ii. The restricted mapping $L|_{C_+^\infty(\mathbb{R})}$ is continuous, linear and $(\sigma_t)_{t \in \mathbb{R}}$-invariant on $C_+^\infty(\mathbb{R})$ by an application of the Closed Graph Theorem. So, there exists by Proposition 4.18 an $\mu \in \mathcal{M}_e(\mathbb{R})$, such that

$$L\phi = \mu * \phi \quad (\phi \in C_+^\infty(\mathbb{R})).$$

Since $D_+(\mathbb{R}) \subseteq C_+^\infty(\mathbb{R})$ is a core for the operator $L$ by Lemma 4.14, the equation (4.14) holds true for every $\phi \in L^p_{loc,+}(\mathbb{R})$.

ii $\Rightarrow$ i. It is an immediate consequence of Theorem 4.16.

The proof of the latter assertion is straightforward, hence omitted.

**Remark 4.19.1** Yamamoto proved the above result for the case $p = 2$ only, not using the core property of $D_+(\mathbb{R})$.

Theorem 4.19 can be extended to more general translatable strict LF-spaces of $D_+(\mathbb{R})$-type. Indeed, let $V$ be a translatable strict LF-space, containing $C_+^\infty(\mathbb{R})$, i.e., the strict LF-space (of $D_+(\mathbb{R})$-type) consisting of all $k$-times continuously differentiable functions on $\mathbb{R}$ with support bounded on the left. Then, a linear mapping $L$ on $V$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant, continuous and satisfies $L(C_+^\infty(\mathbb{R})) \subseteq C_+^\infty(\mathbb{R})$ if and only if there exists $\mu \in \mathcal{M}_e(\mathbb{R})$ such that $L\phi = \mu * \phi$ for all $\phi \in V$. We refer to [deR2], §5.

In the special case $p = 1$ the second condition of Theorem 4.19 is redundant.
4.1. Translation Invariant Subspaces of $\mathcal{D}'(\mathbb{R})$

**Theorem 4.20** Let $L$ be a linear operator on $L^1_{loc,+}(\mathbb{R})$. Then the following two statements are equivalent:

i. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant and continuous.

ii. $\mu \in \mathcal{M}_+(\mathbb{R})$ exists such that $L\phi = \mu \ast \phi$ for all $\phi \in L^1_{loc,+}(\mathbb{R})$.

**Proof.**
We refer to [vEdRd]. Theorem 4.10.

4.1 TRANSLATION INvariant SUBSPACES OF $\mathcal{D}'(\mathbb{R})$

In this section, we study a class of subspaces of the distribution space $\mathcal{D}'(\mathbb{R})$. We follow a set-up similar to the previous section.

4.1.1 Translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type

Define the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{D}'(\mathbb{R})$ by duality, i.e.

$$(\sigma_t F)(\phi) = F(\sigma_{-t} \phi) \quad (F \in \mathcal{D}'(\mathbb{R}), \phi \in \mathcal{D}(\mathbb{R}), t \in \mathbb{R}).$$

We search for translation-invariant subspaces $V$ on which the translation group is strongly continuous.

The first simple example of a $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{D}'(\mathbb{R})$, on which the translation group is strongly continuous, is the F-space $\mathcal{E}(\mathbb{R})$ from §3.1. In fact, we search for complete, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces $V$ of $\mathcal{D}'(\mathbb{R})$, which contain $\mathcal{E}(\mathbb{R})$ densely. So, $V$ may be regarded a completion of $\mathcal{E}(\mathbb{R})$. Such a topological vector space is the F-space $C(\mathbb{R})$.

**Example 4.21 (The space $C(\mathbb{R})$)**

Recall from Example 4.1 that the F-space $C(\mathbb{R})$ consists of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$ and is equipped with the compact open topology. It will be shown (see example 4.29) that $C(\mathbb{R})$ is the completion of $\mathcal{E}(\mathbb{R})$ with respect to a suitable strict pre-F topology, namely the induced $C(\mathbb{R})$-topology on $\mathcal{E}(\mathbb{R})$. The space $C(\mathbb{R})$ is a subspace of $\mathcal{D}'(\mathbb{R})$ in the canonical way; for each $f \in C(\mathbb{R})$, the mapping

$$\phi \in \mathcal{D}(\mathbb{R}) \mapsto \int_{\mathbb{R}} f(\tau)\phi(\tau) \, d\tau \quad (4.15)$$

defines a $\mathcal{D}'(\mathbb{R})$-element. We have the following (dense) inclusions

$$\mathcal{E}(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \hookrightarrow \mathcal{D}'(\mathbb{R}), \quad (4.16)$$

where $\mathcal{D}'(\mathbb{R})$ is equipped with weak-star topology induced by $\mathcal{D}(\mathbb{R})$. So, the topological dual of $C(\mathbb{R})$ is a subspace of $\mathcal{E}'(\mathbb{R})$. 
The $F$-space $C(\mathbb{R})$ is an $c_0(\mathbb{R})$-invariant subspace of $\mathcal{D}'(\mathbb{R})$. The translation operators $(\sigma_t)_{t \in \mathbb{R}}$ on $C(\mathbb{R})$, defined in the standard way by

$$(\sigma_t f)(s) = f(t + s) \quad (t, s \in \mathbb{R}, f \in C(\mathbb{R})),$$

form a $c_0$-group of continuous linear operators. Naturally, the translation group $(\gamma_t)_{t \in \mathbb{R}}$ on $V$ extends the translation group on $\mathcal{S}(\mathbb{R})$.

A translatable space $V$ of $\mathcal{E}(\mathbb{R})$-type is a translation-invariant, topological vector subspace of $\mathcal{D}'(\mathbb{R})$, such that the natural embedding from $V$ into $\mathcal{D}'(\mathbb{R})$, equipped with weak-star topology $\sigma(\mathcal{D}'(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$ is continuous and for which the restricted translation group is strongly continuous. We focus especially on the case where the topological structure of $V$ is an $F$-space.

**Definition 4.22** Let $V$ be an $F$-space, satisfying the following properties;

i. $\mathcal{E}(\mathbb{R}) \subseteq V \subseteq \mathcal{D}'(\mathbb{R})$.

ii. The inclusions from i. are dense and continuous, where $\mathcal{D}'(\mathbb{R})$ is equipped with weak-star topology $\sigma(\mathcal{D}'(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$.

iii. $\sigma_t(V) \subseteq V$ for all $t \in \mathbb{R}$.

iv. The (restricted) translation group on $V$ is a $c_0$-group.

Then $V$ is called a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type.

**Remark 4.22.1** In [vEj1], [deR1], and [vEd2], a more general approach to translatable $F$-spaces was presented. The essential difference between the translatable spaces as presented in [vEj1] and [deR2], and the translatable spaces from Definition 4.22 is the absence of the inclusion properties i and ii. Instead, there exists a continuous linear injection from every translatable space into $\mathcal{D}'(\mathbb{R})$ with weak-star topology.

Since the translation group is strongly continuous on each translatable $F$-space $V$ of $\mathcal{E}(\mathbb{R})$-type, we may apply the theory on $c_0$-groups from Chapter 2 to $V$. Being interested in closed subspaces and closed linear operators on these spaces, we focus on the $c_0$-domain of $(\sigma_t)_{t \in \mathbb{R}}$ on $V$. As a consequence of the Dixmier-Malliavin-type result, Theorem 2.11, the $c_0$-domain is equal to $\mathcal{E}(\mathbb{R})$ for any $V$.

**Theorem 4.23** Let $V$ be a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type. Let $\delta_0$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $V$. Then

$$(\text{Dom}^* \mathcal{E}(\mathbb{R}), \mathcal{T}_{\text{prop}}) = \mathcal{E}(\mathbb{R}),$$

where $\mathcal{T}_{\text{prop}}$ is the Fréchet topology from Proposition 2.68.

**Proof.**

The assertion can be proved similar as for Theorem 4.3.

There are no translatable spaces of $\mathcal{E}(\mathbb{R})$-type with a topological structure, which is simpler than the one of $F$-spaces, such as Hilbert spaces or Banach spaces.
4.1. Translation invariant subspaces of $\mathcal{D}'(\mathbb{R})$

Proposition 4.24 There exists no translatable Banach space of $\mathcal{E}(\mathbb{R})$-type.

Proof. Let $V$ be a translatable Banach space, with norm $\| \cdot \|$. By a simple application of the uniform boundedness principle, there are constants $C, a > 0$ such that for all $t \in \mathbb{R}$

$$\|\sigma_t\| \leq C \cdot e^{a|t|}. \tag{4.17}$$

Let $e_b \in \mathcal{E}(\mathbb{R})$ be defined by $e_b(t) := e^{bt}$, $t \in \mathbb{R}$. Then, for all $b \in \mathbb{R}$

$$\|\sigma_b e_b\| = e^{bt} \cdot \|e_b\|,$$

contradicting (4.17) when $b > a$.

We present some examples of translatable strict LF-space of $\mathcal{E}(\mathbb{R})$-type.

Example 4.25 (The spaces $L^p_{\text{loc}}(\mathbb{R})$)
For each $p \geq 1$, the space $L^p_{\text{loc}}(\mathbb{R})$ consists of all locally $p$-Lebesgue integrable functions. The F-space $L^p_{\text{loc}}(\mathbb{R})$ is a translation-invariant subspace of $\mathcal{D}'(\mathbb{R})$. In fact, the translation group on $L^p_{\text{loc}}(\mathbb{R})$ is strongly continuous. So, the space $L^p_{\text{loc}}(\mathbb{R})$ is an F-space of $\mathcal{E}(\mathbb{R})$-type. The strict LF-space $L^\infty_{\text{loc}}(\mathbb{R})$ is not translatable, since the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $L^\infty_{\text{loc}}(\mathbb{R})$ is not strongly continuous. The Heaviside function provides again a counter example (cf. Example 4.5).

The following result provides a method to construct new translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type.

Lemma 4.26 Let $V$ be a translatable F-space of $\mathcal{E}(\mathbb{R})$-type. Let $\delta_v$ be the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $V$. Then Dom($\delta_v$), equipped with graph topology is a translatable F-space of $\mathcal{E}(\mathbb{R})$-type.

Proof. Since $\delta_v$ is closed, Dom($\delta_v$), equipped with graph topology, is an F-space by Proposition 1.25. The F-space Dom($\delta_v$) satisfies

$$\mathcal{E}(\mathbb{R}) \hookrightarrow \text{Dom}(\delta_v) \hookrightarrow V \hookrightarrow \mathcal{D}'(\mathbb{R}),$$

where all inclusions are dense. Observing that the translation group on Dom($\delta_v$) is strongly continuous, the assertion follows.

Examples of translatable spaces that can be constructed applying Lemma 4.26 are the F-spaces $C^k(\mathbb{R})$ and the Sobolev spaces $H^k_{\text{loc}}(\mathbb{R})$, $k \in \mathbb{N}$, $p \geq 1$. 

4.1.2 Translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type as $\mathcal{E}(\mathbb{R})$-completions

Since $\mathcal{E}(\mathbb{R})$ is a dense subspace of each translatable space $V$ of $\mathcal{E}(\mathbb{R})$-type by definition, $V$ is a completion of $\mathcal{E}(\mathbb{R})$ equipped with a pre-$F$-topology weaker than the $\mathcal{E}(\mathbb{R})$-topology.

Let $T$ be a pre-Frèchet topology for $\mathcal{E}(\mathbb{R})$, such that

$$\mathcal{E}(\mathbb{R}) \hookrightarrow (\mathcal{E}(\mathbb{R}), T) \hookrightarrow \mathcal{D}'(\mathbb{R}).$$

Then the completion of $(\mathcal{E}(\mathbb{R}), T)$ in $\mathcal{D}'(\mathbb{R})$, $(\mathcal{E}(\mathbb{R}), T)$, satisfies by Theorem 1.40

$$\mathcal{E}(\mathbb{R}) \hookrightarrow (\mathcal{E}(\mathbb{R}), T) \hookrightarrow \mathcal{D}'(\mathbb{R}).$$

Moreover, if we choose the topology so that $(\sigma_t)_{t \in \mathbb{R}}$ is a locally equicontinuous $\alpha$-group on $(\mathcal{E}(\mathbb{R}), T)$, then $(\alpha_t)_{t \in \mathbb{R}}$ extends continuously to a $\alpha$-group on the completion $(\mathcal{E}(\mathbb{R}), T)$ by Theorem 2.4. Naturally, this extended group is the restriction of the $\mathcal{D}'(\mathbb{R})$-translation group to $(\mathcal{E}(\mathbb{R}), T)$.

Since the topology of a pre-$F$-space is generated by a countable separating set of ordered seminorms $II$, we end up with conditions on $II$. So, in case of a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type, the topology generated by $II$ has to be weaker than the $\mathcal{E}(\mathbb{R})$-topology, the topology generated by $II$ has to be stronger than the restricted $\sigma(\mathcal{D}'(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$-topology and the translation group has to be locally equicontinuous with respect to this topology.

We obtain the following result.

**Theorem 4.27** Let $(p_k)_{k \in \mathbb{N}}$ be a separating family of ordered seminorms on $\mathcal{E}(\mathbb{R})$ satisfying the following conditions

1. $\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall f \in \mathcal{E}(\mathbb{R}) \left[ p_k(f) \leq C \cdot \sum_{i=1}^{n} \max_{t \in (-N, N)} | f^{(i)}(t) | \right],$

2. $\forall f \in \mathcal{D}(\mathbb{R}) \exists n \in \mathbb{N} \forall \varphi \in \mathcal{E}(\mathbb{R}) \left[ \int \varphi(t) f(t) \, dt \leq C \cdot p_k(f) \right],$

3. $\forall \text{cocompact}_{\mathbb{R}} \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall f \in \mathcal{E}(\mathbb{R}) \left[ \sup_{\text{cocompact}_{\mathbb{R}}} | p_k(\sigma f) | \leq C \cdot p_k(f) \right].$

Let $T_{II}$ be the pre-$F$-topology on $\mathcal{E}(\mathbb{R})$ generated by the family of seminorms $(p_k)_{k \in \mathbb{N}}$ and let $F$ be the completion of $\mathcal{E}(\mathbb{R})$ for $T$ in $\mathcal{D}'(\mathbb{R})$. Then $F$ is a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type.

**Remark 4.27.1** The conditions from Theorem 4.27 can be interpreted in the following way. Condition I is equivalent to the assertion that the topology $T_{II}$ is weaker than the $\mathcal{E}(\mathbb{R})$-topology on $\mathcal{E}(\mathbb{R})$. Condition II is equivalent to the assertion that $T_{II}$ is stronger than the $\sigma(\mathcal{D}'(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$-topology restricted to $\mathcal{E}(\mathbb{R})$, and condition III is equivalent to the assertion that the translation group on $\mathcal{E}(\mathbb{R})$ is locally equicontinuous with respect to $T_{II}$.

The converse of Theorem 4.27 is also true.
4.1. Translation invariant subspaces of $\mathcal{D}'(\mathbb{R})$

**Theorem 4.28** Let $V$ be a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type and let $(p_k)_{k \in \mathbb{N}}$ be a separating family of ordered seminorms generating $V$'s-topology. Then, the seminorms $(p_k)_{k \in \mathbb{N}}$ satisfy the conditions I, II and III from Theorem 4.27 on $\mathcal{E}(\mathbb{R})$.

The following example illustrates that the translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type $C(\mathbb{R})$ can be constructed by the above theorems.

**Example 4.29** Define the seminorms $\Pi := \{p_n \mid n \in \mathbb{N}\}$ on $C(\mathbb{R})$ by

$$p_n(f) := \max_{t \in [-n,n]} |f(t)| \quad (f \in C(\mathbb{R})).$$

Thus, $\Pi$ generates the Fréchet topology of $C(\mathbb{R})$. In fact, the seminorms $\Pi$ restricted to $\mathcal{E}(\mathbb{R})$ satisfy the conditions of Theorem 4.6, as is to be expected. Naturally, condition I of Theorem 4.6 is satisfied. Moreover, let $n \in \mathbb{N}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be fixed. Let $k \geq n$ be such that $\text{supp}(\varphi) \subseteq [-k,k]$ and let $C = \int_{-k}^{k} |\varphi(\tau)| \, d\tau$. Then for all $f \in \mathcal{E}(\mathbb{R})$

$$\left| \int_{\mathbb{R}} f(\tau) \varphi(\tau) \, d\tau \right| \leq \max_{t \in [-k,k]} |f(t)| \cdot \int_{-k}^{k} |\varphi(\tau)| \, d\tau = C \cdot p_n(f).$$

So, condition II of Theorem 4.27 is satisfied. Finally, let $k \in \mathbb{N}$ and let $I \subseteq \mathbb{R}$ compact. Choose $m \in \mathbb{N}$ such that $I \subseteq [-m,m]$. Then for all $f \in \mathcal{E}(\mathbb{R})$

$$\sup_{t \in I} p_n(\varphi, f) \leq \max_{t \in [-m,m]} \max_{\tau \in [-k,k]} |f(t + \tau)| \leq p_{n+k}(f),$$

which yields condition III.

**Example 4.30** Let $p \geq 1$. Define the seminorms $\Gamma := \{q_n \mid n \in \mathbb{N}\}$ on $\mathcal{E}(\mathbb{R})$ by

$$q_n(f) := \left( \int_{-n}^{n} |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \quad (f \in \mathcal{E}(\mathbb{R})).$$

The seminorms $\{q_n \mid n \in \mathbb{N}\}$ generate the relative $L^p_{c(\mathbb{R})}$-topology. We show that the seminorms $\Gamma$ satisfy the conditions of Theorem 4.27. To this extent, let $k \in \mathbb{N}$. Then, for each $f \in \mathcal{E}(\mathbb{R})$

$$q_k(f) = \left( \int_{-k}^{k} |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \leq (2k)^{\frac{1}{p}} \cdot \max_{t \in [-k,k]} |f(t)|.$$

So, the seminorms $\Gamma$ satisfy condition I of Theorem 4.6. Furthermore, let $\varphi \in \mathcal{D}(\mathbb{R})$. Let $k \geq n$ be such that $\text{supp}(\varphi) \subseteq [-k,k]$. Then for each $f \in \mathcal{E}(\mathbb{R})$, we have by the Hölder inequality

$$\left| \int_{\mathbb{R}} f(\tau) \varphi(\tau) \, d\tau \right| \leq \left( \int_{-k}^{k} |\varphi(\tau)|^q \, d\tau \right)^{\frac{1}{q}} \cdot \left( \int_{-k}^{k} |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \leq \left( \int_{-k}^{k} |\varphi(\tau)|^q \, d\tau \right)^{\frac{1}{q}} \cdot q_k(f).$$
where $q \geq 1$ is such that $\frac{1}{q} + \frac{1}{p} = 1$. Finally, let $I \subseteq \mathbb{R}$ be compact and let $k \in \mathbb{N}$. Then for each $f \in \mathcal{E}(\mathbb{R})$,

$$\sup_{t \in I} k(t) f(t) = \sup_{t \in I} \left( \int_{-\infty}^{t} |f(t + \tau)|^p \, d\tau \right)^{\frac{1}{p}} \leq k_{t+m}(f),$$

so the seminorms $\Gamma$ on $\mathcal{E}(\mathbb{R})$ satisfy condition III also. Since $L^p_{loc}(\mathbb{R})$ is the completion of $\mathcal{E}(\mathbb{R})$ with restricted $L^p_{loc}(\mathbb{R})$-topology, it is a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type.

It is well known from literature that each Sobolev space $H^{s,p}_{loc}(\mathbb{R})$, $1 \leq p < \infty$, is embedded in $C(\mathbb{R})$ continuously. A similar assertion holds true for each $F$-space of $\mathcal{E}(\mathbb{R})$-type: an $F$-space of $\mathcal{E}(\mathbb{R})$-type consist only of distributions with irregularity up to a fixed order.

**Proposition 4.31** Let $V$ be a translatable $F$-space of $\mathcal{E}(\mathbb{R})$-type. Let $\delta_o$ be the infinitesimal generator of the translation group $\{\sigma_t\}_{t \in \mathbb{R}}$ on $V$. Then $k \in \mathbb{N}_0$ exists such that

$$\text{Dom}(\delta^k_o) \hookrightarrow C(\mathbb{R}),$$

where $\text{Dom}(\delta^k_o)$ is equipped with the graph topology of the operator $\delta^k_o$.

**Proof.**

We prove that for some $k \in \mathbb{N}_0$ the graph-topology $T_k$ of the operator $\delta^k_o$, restricted to $\mathcal{E}(\mathbb{R})$, is finer than the $C(\mathbb{R})$-topology restricted to $\mathcal{E}(\mathbb{R})$. So, let $\Pi := \{p_n\}_{n \in \mathbb{N}}$ be a separating family of ordered seminorms generating the topology of $V$. Let $k \in \mathbb{N}$ be fixed. Recall from Proposition 2.43 that the graph-topology $T_k$ of the operator $\delta^k_o$ is generated by the family of seminorms $\Pi^{(k)} := \{p_n^{(k)} \mid n \in \mathbb{N}\}$, where

$$p_n^{(k)}(f) := \sum_{i=0}^{k} p_n(\delta^i_o f) \quad (f \in \text{Dom}(\delta^k_o), n \in \mathbb{N}).$$

Since $\mathcal{E}(\mathbb{R}) = (\text{Dom}^n(\delta_o), T_{\text{graph}}) = \text{proj} \, \text{Dom}(\delta^k_o)$ by Theorem 4.23, and since the $\text{proj}$ projective limit topology of $\text{proj} \, \text{Dom}(\delta^k_o)$ is generated by the seminorms $\{p_n^{(k)} \mid n \in \mathbb{N}\}$ (see Proposition 1.31), there exists $k \in \mathbb{N}$, $C_1 > 0$ such that for all $f \in \mathcal{E}(\mathbb{R})$

$$\max_{t \in [-1,1]} |f(t)| \leq \sum_{i=0}^{1} \max_{t \in [-1,1]} |f^{(i)}(t)| \leq C_1 \cdot p_0^{(k)}(f). \quad (4.18)$$

So, for each $f \in \mathbb{N}_0$, there are $C_2 > 0$ and $m \in \mathbb{N}$ such that for all $f \in \mathcal{E}(\mathbb{R})$

$$\max_{t \in [-1,1]} |f(t)| = \sup_{t \in [-1,1]} \max_{n \in \mathbb{N}_0} |f^{(n)}(t)| \stackrel{(4.18)}{=} \sup_{t \in [-1,1]} C_1 \cdot \sum_{i=0}^{k} p_i(\sigma_{t} f^{(i)}) \leq C_2 \cdot \sum_{i=0}^{k} p_i(\sigma_{t} f^{(i)}) \leq C_2 \cdot \sum_{i=0}^{m} p_i(f^{(i)}) =: C_2 \cdot p_m(f). \quad (4.19)$$
where we used condition 4.27.iii. We conclude that the graph-topology of $\text{Dom}(\delta'_e)$ restricted to $E(R)$ is finer than the $C(R)$-topology restricted to $E(R)$. So, their completions satisfy $(\text{Dom}(\delta'_e), T_e) \hookrightarrow C(R)$, which proves the assertion.

### 4.1.3 Closed $(\sigma_i)_{i\in \mathbb{R}}$-invariant subspaces and operators on translatable F-spaces of $E(R)$-type

In this subsection, we consider closed, translation-invariant subspaces and operators on translatable F-spaces of $E(R)$-type. Similar to the case of translatable strict LF-spaces of $D_x(R)$-type, both turn out to be extensions of closed, translation-invariant subspaces and operators on $E(R)$, which are described in §3.1.2 and §3.1.3.

First, we extend the Kahane result, Theorem 3.33, to arbitrary translatable F-spaces of $E(R)$-type. To do so, we need the following lemma.

**Lemma 4.32** Let $V$ be a translatable F-space of $E(R)$-type. Then the following assertions hold true.

i. Let $M$ be a closed, $(\sigma_i)_{i\in \mathbb{R}}$-invariant subspace of $V$. Then $M \cap E(R)$ is a closed, $(\sigma_i)_{i\in \mathbb{R}}$-invariant subspace of $E(R)$.

ii. Let $M$ be a closed, $(\sigma_i)_{i\in \mathbb{R}}$-invariant subspace of $E(R)$ and let $\overline{M}$ be its closure in $V$. Then $\overline{M} \cap E(R) = M$.

**Proof.**

i. Since $E(R) \hookrightarrow V$, the subspace $M \cap E(R)$ is closed in $E(R)$. Moreover, since both $M$ and $E(R)$ are $(\sigma_i)_{i\in \mathbb{R}}$-invariant, $M \cap E(R)$ is $(\sigma_i)_{i\in \mathbb{R}}$-invariant.

ii. Since both $\overline{M} \cap E(R)$ and $M$ are closed in $E(R)$, and since $M \subseteq \overline{M} \cap E(R)$, we need to show only that $M$ is dense in $\overline{M} \cap E(R)$. To this extent, let $f \in \overline{M} \cap E(R)$. Then, a sequence $(f_n)_{n\in \mathbb{N}}$ exists in $M$ converging to $f$ in $V$-sense. So, $\sigma[f_n] \to \sigma[f]$ in $V$-sense for each $\phi \in D_x(R)$. Since $\hat{\phi}(0) = 0$, we have $\sigma[f_n] \to \sigma[f]$ in $(\text{Dom}^{\infty}(\delta_x), T_{\text{graph}})$, so in $E(R)$-sense (Theorem 4.23). Since $M$ is closed in $E(R)$, we have $\sigma[f] \in M$. Letting $(\phi_n)_{n\in \mathbb{N}}$ be an approximate identity of regulizers, we see that the sequence $(\sigma[\phi_n] f)_{n\in \mathbb{N}}$ in $M$ converges to $f$. So, $M$ is dense in $\overline{M} \cap E(R)$, proving the assertion.

**Theorem 4.33** Let $V$ be a translatable F-space of $E(R)$-type. Then a closed subspace $M$ of $V$ is $(\sigma_i)_{i\in \mathbb{R}}$-invariant if and only if there is a countable set $\Sigma \subseteq C$ and a mapping $\lambda \in \Sigma \mapsto \eta_\lambda \in R$, such that

$$M = \overline{\text{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 0, 1, \ldots, \eta_\lambda - 1\}},$$

where $e_{\lambda,j}(t) := \nu e^{-i\lambda t}$, $t \in R$. 


Proof.
The sufficiency of the assertion is obvious.
Let \( M \) be a closed \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant subspace of \( V \). Let \( \delta_{s} \) be the infinitesimal generator of the translation group on \( V \). Since \( \mathcal{E}(\mathbb{R}) = \text{Dom}^{\infty}(\delta_{s}) \) by Theorem 4.23, \( M \cap \mathcal{E}(\mathbb{R}) \) is sequentially dense in \( M \) by Theorem 2.14. Since \( M \cap \mathcal{E}(\mathbb{R}) \) is closed and \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant in \( \mathcal{E}(\mathbb{R}) \) by Lemma 4.32, there is, by Theorem 3.33, a countable set \( \Sigma \subseteq \mathbb{C} \) and a mapping \( \lambda : \Sigma \to n_{\lambda} \in \mathbb{N} \) such that
\[
M \cap \mathcal{E}(\mathbb{R}) = \text{span}\{e_{\lambda,j} | \lambda \in \Sigma, j = 0, 1, \ldots, n_{\lambda} - 1\},
\]
where the closure is in \( \mathcal{E}(\mathbb{R})\)-sense. Since \( \text{span}\{e_{\lambda,j} | \lambda \in \Sigma, j = 0, 1, \ldots, n_{\lambda} - 1\} \) is \( \mathcal{E}(\mathbb{R})\)-dense in \( M \cap \mathcal{E}(\mathbb{R}) \), hence \( V\)-dense in \( M \cap \mathcal{E}(\mathbb{R}) \), and since \( M \cap \mathcal{E}(\mathbb{R}) \) is \( V\)-dense in \( M \), the assertion follows.

Schwartz presented in [Schw1], Theorem 13, p.914, a characterization for closed, \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant subspaces of \( C(\mathbb{R}) \). Put in our terminology:

Lemma 4.34 (Schwartz) Let \( M \) be a closed subspace of \( C(\mathbb{R}) \). Then \( M \) is \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant if and only if there are \( \mu_{1}, \mu_{2} \in \mathcal{M}_{c}(\mathbb{R}) \) such that
\[
M = \ker(\sigma[\mu_{1}]) \cap \ker(\sigma[\mu_{2}]).
\]

We extend this characterisation to arbitrary translatable \( F \)-spaces of \( \mathcal{E}(\mathbb{R})\)-type.

Theorem 4.35 Let \( V \) be a translatable \( F \)-space of \( \mathcal{E}(\mathbb{R})\)-type. Then a closed subspace \( M \) of \( V \) is \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant if and only if there are \( \mu_{1}, \mu_{2} \in \mathcal{M}_{c}(\mathbb{R}) \) such that
\[
M = \ker(\sigma[\mu_{1}]) \cap \ker(\sigma[\mu_{2}]).
\]

Proof.
Both \( \sigma[\mu_{1}] \) and \( \sigma[\mu_{2}] \) being continuous, linear, \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant operators of \( V \), the sufficiency of the assertion is obvious.
Conversely, let \( M \) be a closed, \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant subspace of \( V \). By Lemma 4.32, \( M \cap \mathcal{E}(\mathbb{R}) \) is a closed, \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant subspace of \( \mathcal{E}(\mathbb{R}) \). Consider the closure of \( M \cap \mathcal{E}(\mathbb{R}) \) in \( C(\mathbb{R}) \): \( \overline{M \cap \mathcal{E}(\mathbb{R})} \). Since \( \overline{M \cap \mathcal{E}(\mathbb{R})} \) is \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant, there are by Lemma 4.34, \( \mu_{1}, \mu_{2} \in \mathcal{M}_{c}(\mathbb{R}) \) such that
\[
\overline{M \cap \mathcal{E}(\mathbb{R})} = \ker(\sigma[\mu_{1}]) \cap \ker(\sigma[\mu_{2}]).
\]
Now, regarding \( \sigma[\mu_{1}] \) and \( \sigma[\mu_{2}] \) as continuous, linear, \((\sigma_{t})_{t \in \mathbb{R}}\)-invariant operators on \( \mathcal{E}(\mathbb{R}) \), we have
\[
M \cap \mathcal{E}(\mathbb{R}) \subseteq \overline{M \cap \mathcal{E}(\mathbb{R})} \subseteq \mathcal{E}(\mathbb{R}) = \ker(\sigma[\mu_{1}]) \cap \ker(\sigma[\mu_{2}]). \tag{4.20}
\]
Since \( \mathcal{E}(\mathbb{R}) = \text{Dom}^{\infty}(\delta_{s}) \) by Theorem 4.23, \( M \cap \mathcal{E}(\mathbb{R}) \) is sequentially dense in \( M \) by Theorem 2.14. Moreover, the right-hand side of (4.20) is dense in \( \ker(\sigma[\mu_{1}]) \cap \mathcal{E}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{R}) \) and
4.1. Translation invariant subspaces of $D'(R)$

$ker(\sigma[\mu])$, where the operators $\sigma[\mu_1]$ and $\sigma[\mu_2]$ are taken as continuous, linear, $(\sigma_\alpha)_{\alpha \in R}$-invariant operators on $V$, as can be seen using an approximate identity argument. So, we obtain by taking closures

$$M = ker(\sigma[\mu_1]) \cap ker(\sigma[\mu_2]),$$

proving the assertion.

Next, we focus on closed, $(\sigma_\alpha)_{\alpha \in R}$-invariant operators $L$ on a translatable F-space $V$ of $D(R)$-type. There are direct analogues of Lemma 4.14 and Proposition 4.15.

**Lemma 4.36** Let $V$ be a translatable F-space of $E(R)$-type. Let $L_1, L_2$ be closed, linear, $(\sigma_\alpha)_{\alpha \in R}$-invariant operators on $V$ with domains $Dom(L_1)$ and $Dom(L_2)$. If $E(R) \subseteq Dom(L_1) \cap Dom(L_2)$, then

$$L_1|_{E(R)} = L_2|_{E(R)} \iff L_1 = L_2.$$

**Proof.**

Since $E(R) = Dom^\infty(\delta_u)$ is a core for every closed, linear $(\sigma_\alpha)_{\alpha \in R}$-invariant operator on $V$ by Theorem 2.16, the assertion follows.

**Proposition 4.37** Let $V$ be a translatable F-space of $E(R)$-type. Let $L$, with domain $Dom(L)$, be a closed linear operator on $V$, satisfying

i. $L$ is $(\sigma_\alpha)_{\alpha \in R}$-invariant,

ii. $E(R) \subseteq Dom(L)$.

Then $L(E(R)) \subseteq E(R)$ and the restricted mapping $L|_{E(R)}$ is a continuous linear, $(\sigma_\alpha)_{\alpha \in R}$-invariant operator on $E(R)$.

**Proof.**

Let $\delta_u$ be the infinitesimal generator of the translation $\alpha$-group $(\alpha_\alpha)_{\alpha \in R}$ on $V$ with $c_0$-domain $Dom^\infty(\delta_u)$. Then $L(Dom^\infty(\delta_u)) \subseteq Dom^\infty(\delta_u)$ by Theorem 2.16. Since $Dom^\infty(\delta_u) = E(R)$ (Theorem 4.23), the first part of the assertion follows. The continuity of the restricted mapping follows from $E(R) \hookrightarrow V$ and an application of the Closed Graph Theorem for F-spaces (Theorem 1.29).

Having characterized the collection of all continuous, linear, $(\sigma_\alpha)_{\alpha \in R}$-invariant operators on $E(R)$ by the Theorems 3.20 and 3.24, characterizations of closed, linear, $(\sigma_\alpha)_{\alpha \in R}$-invariant operators on arbitrary translatable F-spaces of $E(R)$-type are obtained applying Lemma 4.36.

**Theorem 4.38** Let $V$ be a translatable F-space of $E(R)$-type. Let $L$ be a linear mapping on $V$ with domain $Dom(L)$, where $E(R) \subseteq Dom(L)$. Then the following two assertions are equivalent.
i. $L$ is closed and $(\sigma_t)_{t \in \mathbb{R}}$-invariant.

ii. A distribution $F \in \mathcal{E}'(\mathbb{R})$ exists such that
   - $\text{Dom}(L) = \{ f \in V \mid F * f \in V \}$, and
   - $L f = F * f$ for each $f \in \text{Dom}(L)$.

iii. There are $\mu \in \mathcal{M}_c(\mathbb{R})$ and $P \in \mathcal{P}$ such that
   - $\text{Dom}(L) = \{ f \in V \mid \sigma[\mu] f \in \text{Dom}(\delta_0^k) \}$, where $k$ is the degree of $P$, and
   - $L f = P(\delta_0) \sigma[\mu] f$,

   where $\delta_0$ is the infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ on $V$.

Proof.
The equivalence of i and ii can be proved by a straightforward analogue of the proof of Theorem 4.16, hence it is omitted.

i$\Rightarrow$iii. Let $\mu \in \mathcal{M}_c(\mathbb{R})$ and $P \in \mathcal{P}$ with degree $k$. Since $\sigma[\mu]$ is a continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $V$ and since $P(\delta_0)$ with domain $\text{Dom}(\delta_0^k)$ is a closed, linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $V$ (see Theorem 2.6), the composition $P(\delta_0) \sigma[\mu]$ with domain $\{ f \in V \mid \sigma[\mu] f \in \text{Dom}(P(\delta_0)) \}$ is a closed linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $V$.

i$\Rightarrow$iii. Let $L$ be a closed, linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $V$ with domain $\text{Dom}(L)$, where $\mathcal{E}(\mathbb{R}) \subseteq \text{Dom}(L)$. By Proposition 4.37, the restricted mapping $L|_{\mathcal{E}(\mathbb{R})}$ is a continuous linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator on $\mathcal{E}(\mathbb{R})$. So, there are $\mu \in \mathcal{M}_c(\mathbb{R})$ and $P \in \mathcal{P}$ by Theorem 3.24, such that for all $f \in \mathcal{E}(\mathbb{R})$

$$L f = P(\delta_0) \sigma[\mu] f.$$  

Let $k$ be the degree of $P$. Then, by the proof of i$\Rightarrow$iii, the linear operator $P(\delta_0) \sigma[\mu]$ with domain $\{ f \in V \mid \sigma[\mu] f \in \text{Dom}(\delta_0^k) \}$ is closed and $(\sigma_t)_{t \in \mathbb{R}}$-invariant on $V$. Since these operators equal $L$ on $\mathcal{E}(\mathbb{R})$, the assertion follows from Lemma 4.36.

4.1.4 Continuous linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operators

Since a continuous linear operator is closed, each continuous linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator $L$ on any translatable strict LF-space $V$ of $\mathcal{E}(\mathbb{R})$-type is of the form $L x = F * x$, $x \in V$, for some $F \in \mathcal{E}'(\mathbb{R})$ (cf. Theorem 4.38.ii). For particular choices of translatable F-spaces of $\mathcal{E}(\mathbb{R})$-type, we can be more precise. For example, in case of the translatable space $C(\mathbb{R})$, the subalgebra $\mathcal{M}_c(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$ characterizes all continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings. So, the polynomial $P$ in Theorem 4.38.iii can be chosen with degree zero.

We use the following result on the duals of $C(\mathbb{R})$ and $C^k(\mathbb{R})$, $k \geq 1$.

Lemma 4.39 The following assertions hold true.
4.1. Translation invariant subspaces of $\mathcal{D}'(\mathbb{R})$

i. A linear functional $F$ on $C(\mathbb{R})$ is continuous if and only if $\mu \in \mathcal{M}_c(\mathbb{R})$ exists, such that

$$F(f) = \int_{\mathbb{R}} f(\tau) \mu(d\tau) \quad (f \in C(\mathbb{R})).$$

ii. A linear functional $F$ on $C^k(\mathbb{R})$, $k \geq 1$, is continuous if and only if there are $\nu \in \mathcal{M}_c(\mathbb{R})$ and a polynomial $P$ of degree at most $k$, such that

$$F(f) = \int_{\mathbb{R}} P(\frac{d}{dt})f(\tau) \nu(d\tau) \quad (f \in C^k(\mathbb{R})).$$

Proof.
We refer to [Soe], Theorem 2.1.15 and Corollary 2.4.26.

Proposition 4.40 For a mapping $L$ on $C(\mathbb{R})$, the following assertions are equivalent.

i. $L$ is continuous, linear and $(\sigma_\tau)_{\tau \in \mathbb{R}}$-invariant.

ii. $L = \sigma[\mu]$, for some $\mu \in \mathcal{M}_c(\mathbb{R})$.

iii. There exists $\nu \in \mathcal{M}_c(\mathbb{R})$, such that $L \nu = \nu \ast f$ for all $f \in C(\mathbb{R})$.

Proof.

i $\Rightarrow$ ii. Let $L$ be a continuous linear $(\sigma_\tau)_{\tau \in \mathbb{R}}$-invariant mapping on $C(\mathbb{R})$. Then, $f \in C(\mathbb{R}) \mapsto (Lf)(0)$ defines a continuous linear functional on $C(\mathbb{R})$. So, by Lemma 4.39.1 there is $\mu \in \mathcal{M}_c(\mathbb{R})$, such that

$$(Lf)(0) = \int_{\mathbb{R}} f(\tau) \mu(d\tau) \quad (f \in C(\mathbb{R})). \quad (4.21)$$

Then, for each $f \in C(\mathbb{R})$

$$(Lf)(t) = L(\sigma_t f)(0) = \int_{\mathbb{R}} f(\tau + t) \mu(d\tau) = (\sigma[\mu] f)(t)$$

We conclude that $L = \sigma[\mu]$.

ii $\Rightarrow$ iii. Let $\mu \in \mathcal{M}_c(\mathbb{R})$, such that $L = \sigma[\mu]$. Define $\nu = \hat{\mu} \in \mathcal{M}_c(\mathbb{R})$, i.e.

$$\nu(x) = \mu(\bar{x}) \quad (x \in C_c(\mathbb{R})).$$

Then, for all $f \in C(\mathbb{R})$, $t \in \mathbb{R}$

$$(\sigma[\mu] f)(t) = \int_{\mathbb{R}} f(\tau + t) \mu(d\tau) = \int_{\mathbb{R}} f(t - \tau) \nu(d\tau) = (\nu \ast f)(t).$$

iii $\Rightarrow$ i. By Theorem 4.38, the everywhere defined mapping $L$ is closed, linear, $(\sigma_\tau)_{\tau \in \mathbb{R}}$-invariant, hence continuous.

From Proposition 4.31, we recall that for each translatable F-space of $\mathcal{E}(\mathbb{R})$-type, there is $k \in \mathbb{N}$ such that $\text{Dom}(\sigma_k^2) \hookrightarrow C(\mathbb{R})$. If this subspace $\text{Dom}(\sigma_k^2)$ contains $C^r(\mathbb{R})$, i.e. all $r$-times continuously differentiable functions on $\mathbb{R}$, for some $r \in \mathbb{N}$, then the polynomial $P$ in Theorem 4.38.ii can be taken with degree at most $r$.
Proposition 4.41 Let $V$ be a translatable $F$-space of $E(R)$-type, satisfying
\[ C^n(R) \subset \text{Dom}(d) \subset C(R) \]
for some $n, k \in \mathbb{N}$, where $d$ is the infinitesimal generator of the translation group on $V$. Then, the polynomial $P$ from Theorem 4.38.i i can be chosen with degree less or equal to $n$.

Proof. Let $n, k$ as in the inclusions (4.22). Since $L$ is continuous and $(\sigma_t)_{t \in \mathbb{R}}$-invariant, we have $L(\text{Dom}(d^k)) \subset \text{Dom}(d^k)$, whence $L(C^n(R)) \subset C(R)$. In fact, the restricted mapping $L : C^n(R) \rightarrow C(R)$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant, and continuous by an application of the Closed Graph Theorem. Therefore, the functional $f \in C^n(R) \rightarrow (Lf)(0)$ is continuous on $C^n(R)$. So by Lemma 4.39.ii, there are $\mu \in \mathcal{M}_c(R)$ and a polynomial $P$ of degree at most $n$, such that for all $f \in C^n(R)$
\[ (Lf)(0) = \int_R P(\frac{d}{dt})f(\tau) \mu(d\tau). \]
Hence, for each $f \in E(R) \subset C^n(R)$, $t \in R$
\[ (Lf)(t) = (L\sigma_t f)(0) = \int_R P(\frac{d}{dt})f(\tau + t) \mu(d\tau) = (\sigma_t \mu) P(\frac{d}{dt})f(t). \]
Applying Lemma 4.36 yields the result.

Examples of translatable strict $L^p$-spaces with property (4.22) are the $F$-spaces $L^p_{\text{loc}}(R)$, $p \geq 1$. It is well known from literature that $C^1(R) \subset H^1_{\text{loc}}(R) \subset C(R)$. So, we obtain the following result from Proposition 4.41.

Theorem 4.42 Let $p \geq 1$. Then, every continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator $L$ on $L^p_{\text{loc}}(R)$, is of the form
\[ Lf = C_1 \cdot \sigma[\mu] + C_2 \cdot \delta_x \sigma[\mu], \]
for some $\mu \in \mathcal{M}_c(R)$ and $C_1, C_2 \in \mathbb{C}$. Here $\delta_x$ is the infinitesimal generator of the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $L^p_{\text{loc}}(R)$.

In the special case $p = 1$, $C_2 = 0$ can be chosen.

Theorem 4.43 Let $L$ be a linear operator on $L^1_{\text{loc}}(R)$. Then the following assertions are equivalent.

i. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant and continuous.

ii. $L = \sigma[\mu]$ for some $\mu \in \mathcal{M}_c(R)$.

iii. $\nu \in \mathcal{M}_c(R)$ exists such that $Lf = \nu * f$ for all $f \in L^1_{\text{loc}}(R)$.

Proof. We refer to [vEdR4], Theorem 4.4.
4.2 TRANSLATION INVARIANT SUBSPACES OF $\mathcal{E}'(\mathbb{R})$

In this section, we study a class of subspaces of the distribution space $\mathcal{E}'(\mathbb{R})$. Following a similar set-up as in §4.0, we derive results very similar to the results for translatable strict LF-spaces of $\mathcal{D}_{\alpha}(\mathbb{R})$-type. In fact, most proofs go along the same lines as the proofs of the analogous assertions in §4.0, and therefore are omitted.

4.2.1 Translatable strict LF-spaces of $\mathcal{D}(\mathbb{R})$-type

Define the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{E}'(\mathbb{R})$ by duality, i.e.

$$(\sigma_t F)(f) := F(\sigma_{-t} f) \quad (F \in \mathcal{E}'(\mathbb{R}), f \in \mathcal{E}'(\mathbb{R}), t \in \mathbb{R}).$$

We search for translation-invariant subspaces $V$ of $\mathcal{E}'(\mathbb{R})$ on which the translation group is strongly continuous.

The first simple example of a $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspace of $\mathcal{E}'(\mathbb{R})$, on which the translation group is strongly continuous, is the strict LF-space $\mathcal{D}(\mathbb{R})$ from §3.1. In fact, we search for complete, $(\sigma_t)_{t \in \mathbb{R}}$-invariant subspaces $V$ of $\mathcal{E}'(\mathbb{R})$, which contain $\mathcal{D}(\mathbb{R})$ densely. So, $V$ may be regarded a completion of $\mathcal{D}(\mathbb{R})$. Such a topological vector space is the strict LF-space $C_0(\mathbb{R})$.

**Example 4.44 (The space $C_0(\mathbb{R})$)**

Let $C_0(\mathbb{R})$ be the subspace of $C(\mathbb{R})$ defined by

$$C_0(\mathbb{R}) = \bigcup_{n} C_n(\mathbb{R}), \quad \text{where } C_n(\mathbb{R}) := \{ f \in C(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, n] \}. $$

So, $f \in C(\mathbb{R})$ belongs to $C_0(\mathbb{R})$ if $f$ has bounded support. Each $C_n(\mathbb{R})$ being a closed subspace of $C(\mathbb{R})$, the family $(C_n(\mathbb{R}))_{n \in \mathbb{N}}$ is a strict inductive system. Correspondingly, $C_0(\mathbb{R})$ is equipped with the related strict LF-topology

$$C_0(\mathbb{R}) = \text{ind}_{n} C_n(\mathbb{R}). \quad (4.23)$$

The space $C_0(\mathbb{R})$ is the completion of $\mathcal{D}(\mathbb{R})$ with respect to respect to a suitable strict pre-LF-topology, namely the induced $C_0(\mathbb{R})$-topology on $\mathcal{D}(\mathbb{R})$. In fact, each $C_n(\mathbb{R})$ is a completion of $\mathcal{D}_n(\mathbb{R})$ for the $C_n(\mathbb{R})$-topology. The space $C_0(\mathbb{R})$ is a subspace of $\mathcal{E}'(\mathbb{R})$ in canonical way, for each $\phi \in C_0(\mathbb{R})$, the mapping

$$f \in \mathcal{E}'(\mathbb{R}) \mapsto \int_{\mathbb{R}} f(\tau) \phi(\tau) \, d\tau \quad (4.24)$$

defines a $\mathcal{E}'(\mathbb{R})$-element. We have the following (dense) inclusions

$$\mathcal{D}(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \hookrightarrow \mathcal{E}'(\mathbb{R}), \quad (4.25)$$

where $\mathcal{E}'(\mathbb{R})$ is equipped with weak-star topology induced by $\mathcal{E}(\mathbb{R})$. So, the topological dual of $C_0(\mathbb{R})$ is a subspace of $\mathcal{E}'(\mathbb{R})$. 

The strict LF-space \( C_c(\mathbb{R}) \) is a \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant subspace of \( \mathcal{E}'(\mathbb{R}) \). The translation operators \((\sigma_t)_{t \in \mathbb{R}}\) on \( C_c(\mathbb{R})\), defined in the standard way by

\[
(\sigma_t f)(s) = f(t + s) \quad (t, s \in \mathbb{R}, f \in C_c(\mathbb{R})),
\]

form a \( \mathbb{C}\)-group of continuous linear operators. Naturally, the translation group \((\sigma_t)_{t \in \mathbb{R}}\) on \( V \) extends the translation group on \( \mathcal{D}(\mathbb{R}) \). It can also be seen as the restriction of the translation group on \( \mathcal{E}'(\mathbb{R}) \).

A translatable space \( V \) of \( \mathcal{D}(\mathbb{R}) \)-type is a translation-invariant, topological vector subspace of \( \mathcal{E}'(\mathbb{R}) \), such that the natural embedding from \( V \) into \( \mathcal{E}'(\mathbb{R}) \), equipped with weak-star topology \( \sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R})) \) is continuous and for which the restricted translation group is strongly continuous. We focus especially on the case where the topological structure of \( V \) is a strict LF-space.

**Definition 4.45** Let \( V \) be a strict LF-space, satisfying the following properties:

i. \( \mathcal{D}(\mathbb{R}) \subseteq V \subseteq \mathcal{E}'(\mathbb{R}) \).

ii. The inclusions from i. are dense and continuous, where \( \mathcal{E}'(\mathbb{R}) \) is equipped with weak-star topology \( \sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R})) \).

iii. \( \sigma_t(V) \subseteq V \) for all \( t \in \mathbb{R} \).

iv. The (restricted) translation group on \( V \) is a \( \mathbb{C}\)-group.

Then \( V \) is called a translatable strict LF-space of \( \mathcal{D}(\mathbb{R}) \)-type.

Besides the spaces \( \mathcal{D}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \), the strict LF-spaces \( L^p_{\text{loc}}(\mathbb{R}) \), \( p \geq 1 \), consisting of all locally \( p \)-Lebesgue integrable functions with bounded support, are translatable strict LF-space of \( \mathcal{D}(\mathbb{R}) \)-type. The strict LF-space \( L^p_{\text{loc}}(\mathbb{R}) \) is not translatable, since the translation group \((\sigma_t)_{t \in \mathbb{R}}\) on \( L^p_{\text{loc}}(\mathbb{R}) \) is not strongly continuous.

The translation group being strongly continuous on each translatable strict LF-space \( V \) of \( \mathcal{D}(\mathbb{R}) \)-type, we may apply the theory on \( \mathbb{C}\)-groups from Chapter 2 to \( V \). As a consequence of the Dixmier-Malliavin result, Theorem 2.11, the \( c_\infty \)-domain is equal to \( \mathcal{D}(\mathbb{R}) \) for any \( V \) (cf. Theorem 4.3).

**Theorem 4.46** Let \( V \) be a translatable strict LF-space of \( \mathcal{D}(\mathbb{R}) \)-type. Let \( \delta_v \) be the infinitesimal generator of the translation group \((\sigma_t)_{t \in \mathbb{R}}\) on \( V \). Then

\[
(\text{Dom}^{c_\infty}(\delta_v), T_{\text{ind}}) = \mathcal{D}(\mathbb{R}),
\]

where \( T_{\text{ind}} \) is the strict inductive limit topology from Definition 2.69.

There are no translatable spaces of \( \mathcal{D}(\mathbb{R}) \)-type with a topological structure more simple than the one of strict LF-spaces, such as Hilbert spaces, Banach spaces or even Fréchet spaces (cf. Proposition 4.4).

**Proposition 4.47** There exists no translatable \( F \)-space of \( \mathcal{D}(\mathbb{R}) \)-type.
4.2. Translation invariant subspaces of $E'(R)$

4.2.2 Translatable strict LF-spaces of $D(R)$-type as $D(R)$-completions

Since $D(R)$ is a dense subspace of each translatable space $V$ of $D(R)$-type by definition, $V$ is a completion of $D(R)$ equipped with a strict pre-LF-topology weaker than the $D(R)$-topology. Similar to Theorem 4.6, we obtain the following result.

**Theorem 4.48** Let $(p_k)_{k \in N}$ be a separating family of ordered seminorms on $D(R)$ satisfying the following conditions on each $D_n(R)$

1. \[ \forall k \in N \exists \epsilon > 0 \forall \phi \in D_n(R) \left[ p_k(\psi) \leq C \cdot \max_{i=0}^{N} \left| \psi(i) \right| \right]. \]
2. \[ \forall f \in L_1(R) \exists \epsilon > 0 \forall \phi \in D_n(R) \left[ \left| \int_{R} \psi(r) f(r) \, dr \right| \leq C \cdot p_k(\psi) \right]. \]
3. \[ \forall \sigma \in \text{compact} \subset R \exists \epsilon > 0 \forall \phi \in D_n(R) \left[ \sup_{x \in \epsilon} p_k(\sigma \phi) \leq C \cdot p_k(\psi) \right]. \]

Let $T_n$ be the pre-F-topology on $D_n(R)$ generated by the seminorms $(p_k|_{D_n(R)})_{k \in N}$ and let $F_n$ be the completion of $D_n(R)$ for $T_n$ in $D_n'(R)$. Then

i. $((D_n(R), T_n))_{n \in N}$ is a strict inductive system of pre-F-spaces.
ii. $(F_n)_{n \in N}$ is a strict inductive system of Fréchet spaces,
iii. $F_n$ is a translatable strict LF-space of $D(R)$-type.

Each strict LF-space constructed according to Theorem 4.48 is a translatable space of $D(R)$-type. The converse is also true, each translatable strict LF-space of $D(R)$-type arises from such a construction. In other words, the conditions I, II and III are necessary for a strict LF-space $F_n$ to be of $D(R)$-type.

**Theorem 4.49** Let $V$ be a translatable strict LF-space of $D(R)$-type and let $(p_k)_{k \in N}$ be a family of seminorms related to $V$'s-topology in the sense of Proposition 1.36. Then the following statements hold true.

i. The seminorms $(p_k)_{k \in N}$ satisfy the conditions I, II and III from Theorem 4.6 on $D(R)$.
ii. A strict inductive system of F-spaces $(F_n)_{n \in N}$ exists satisfying
   - $D_n(R) \subseteq F_n \subseteq E'(R)$, where the inclusions are continuous and the first inclusion is dense,
   - $V = \text{ind } F_n$.

We obtain the following result from Theorem 4.49.

**Corollary 4.50** Let $V = \text{ind } V_n$ be a translatable strict LF-space of $D(R)$-type. Then for each $m \in N$, there is $n \in N$ such that

\[ \phi \in V_n \Rightarrow \supp(\phi) \subseteq [-n, n]. \]

Put differently, $V_n$ consists of $E'(R)$-distributions with support in $[-n, n]$ only.
4.2.3 Closed, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on translatable strict LF-spaces of \(\mathcal{D}(\mathbb{R})\)-type

In this subsection, we consider closed, translation-invariant operators on translatable strict LF-spaces of \(\mathcal{D}(\mathbb{R})\)-type. These turn out to be extensions of closed, translation-invariant operators on \(\mathcal{D}(\mathbb{R})\), which are described in §3.1.2. We will not consider closed, translation-invariant subspaces of \(\mathcal{D}(\mathbb{R})\)-type. Although it can be shown that these are closures of closed, translation-invariant subspaces of \(\mathcal{D}(\mathbb{R})\), we have found no explicit characterization result.

Similar to the Lemmas 4.14 and 4.36, we have the following result.

\textbf{Lemma 4.51} Let \(V\) be a translatable strict LF-space of \(\mathcal{D}(\mathbb{R})\)-type. Let \(L_1, L_2\) be closed, linear \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on \(V\) with domains \(\text{Dom}(L_1)\) and \(\text{Dom}(L_2)\).

If \(\mathcal{D}(\mathbb{R}) \subseteq \text{Dom}(L_1) \cap \text{Dom}(L_2)\), then,

\[ L_1 |_{\mathcal{D}(\mathbb{R})} = L_2 |_{\mathcal{D}(\mathbb{R})} \iff \text{Dom}(L_1) = \text{Dom}(L_2) \text{ and } L_1 = L_2 \text{ on } \text{Dom}(L_1). \]

\textbf{Proposition 4.52} Let \(V\) be a translatable strict LF-space of \(\mathcal{D}(\mathbb{R})\)-type. Let \(L\), with domain \(\text{Dom}(L)\), be a closed linear operator on \(V\), satisfying

\begin{enumerate}
  \item \(L\) is \((\sigma_t)_{t \in \mathbb{R}}\)-invariant,
  \item \(\mathcal{D}(\mathbb{R}) \subseteq \text{Dom}(L)\).
\end{enumerate}

Then \(L(\mathcal{D}(\mathbb{R})) \subseteq \mathcal{D}(\mathbb{R})\) and the restricted mapping \(L |_{\mathcal{D}(\mathbb{R})}\) is a continuous linear, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operator on \(\mathcal{D}(\mathbb{R})\).

In Corollary 3.32, the collection of all continuous, linear, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on \(\mathcal{D}(\mathbb{R})\) are characterized. So applying Lemma 4.51 and Proposition 4.52, we obtain a characterization of closed, linear, \((\sigma_t)_{t \in \mathbb{R}}\)-invariant operators on arbitrary translatable strict LF-spaces of \(\mathcal{D}(\mathbb{R})\)-type (cf. Theorem 4.16 and 4.38)

\textbf{Theorem 4.53} Let \(V\) be a translatable strict LF-space of \(\mathcal{D}(\mathbb{R})\)-type. Let \(L\) be a linear mapping on \(V\) with domain \(\text{Dom}(L)\), where \(\mathcal{D}(\mathbb{R}) \subseteq \text{Dom}(L)\). Then the following two assertions are equivalent.

\begin{enumerate}
  \item \(L\) is closed and \((\sigma_t)_{t \in \mathbb{R}}\)-invariant.
  \item A distribution \(F \in \mathcal{E}'(\mathbb{R})\) exists such that
    \begin{itemize}
      \item \(\text{Dom}(L) = \{ \phi \in V \mid F \ast \phi \in V \}\)
      \item \(L\phi = F \ast \phi\) for each \(\phi \in \text{Dom}(L)\).
    \end{itemize}
  \item There are \(\mu \in \mathcal{M}_c(\mathbb{R})\) and \(P \in \mathcal{B}\) such that
    \begin{itemize}
      \item \(\text{Dom}(L) = \{ f \in V \mid \sigma[\mu]f \in \text{Dom}(\delta_k) \}\), where \(k\) is the degree of \(P\), and
      \item \(Lf = P(\delta_k)\sigma[\mu]f\).
    \end{itemize}
\end{enumerate}
4.2. Translation invariant subspaces of $E'(\mathbb{R})$

where $\delta_0$ is the infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ on $V$.

Since a continuous operator is closed, each continuous linear $(\sigma_t)_{t \in \mathbb{R}}$-invariant operator $L$ on any translatable strict LF-space $V$ of $D(\mathbb{R})$-type is of the form $Lx = F * x$, $x \in V$, for some $F \in E'(\mathbb{R})$ (cf. Theorem 4.53.11). However, this result is not satisfactory, since it does not reveal the form of the convolution kernels $F$. In case of the translatable space $D(\mathbb{R})$, the algebra $E'(\mathbb{R})$ corresponds to the collection of all continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings on $D(\mathbb{R})$, but for other translatable strict LF-spaces of $D(\mathbb{R})$-type, the class of all continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings corresponds to a proper subalgebra of $E'(\mathbb{R})$.

In case of the translatable space $C_c(\mathbb{R})$, the subalgebra $M_\epsilon(\mathbb{R})$ of $E'(\mathbb{R})$ characterizes all continuous, linear, $(\sigma_t)_{t \in \mathbb{R}}$-invariant mappings (cf. Proposition 4.40).

**Proposition 4.54** Let $L$ be a mapping on $C_c(\mathbb{R})$. Then the following assertions are equivalent.

i. $L$ is continuous, linear and $(\sigma_t)_{t \in \mathbb{R}}$-invariant.

ii. There exists $\nu \in M_\epsilon(\mathbb{R})$, such that $L \phi = \nu * \phi$ for all $\phi \in C_c(\mathbb{R})$.

iii. $L = \sigma[\mu]$, for some $\mu \in M_\epsilon(\mathbb{R})$.

The following result extends Proposition 4.54 to the strict LF-spaces $L^p_{\text{comp}}(\mathbb{R})$, $p \geq 1$ (cf. Theorem 4.19).

**Theorem 4.55** Let $L$ be a linear operator on $L^p_{\text{comp}}(\mathbb{R})$, where $p \geq 1$. Then the following two statements are equivalent.

i. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant and continuous, satisfying $L(C_c(\mathbb{R})) \subseteq C_c(\mathbb{R})$.

ii. $L = \sigma[\mu]$ for some $\mu \in M_\epsilon(\mathbb{R})$.

In the special case $p = 1$ the second condition of Theorem 4.55 is redundant.

**Theorem 4.56** Let $L$ be a linear operator on $L^1_{\text{comp}}(\mathbb{R})$. Then the following two statements are equivalent.

i. $L$ is $(\sigma_t)_{t \in \mathbb{R}}$-invariant and continuous.

ii. $L = \sigma[\mu]$ for some $\mu \in M_\epsilon(\mathbb{R})$.

**Proof.**

We refer to [vEdR4], Theorem 4.2.

□
5

FACTORIZATIONS OF
INPUT/OUTPUT MAPPINGS

In this chapter, we concentrate on a problem in the field of system theory, the factorization problem for continuous-time systems.

We may consider a system as a device accepting certain signals (inputs) and producing other signals (outputs) according to some specific rules. A continuous-time system Σ is a system with signals defined on the whole real time-axis. So think of the signals (inputs and outputs) as being "functions" on R. If the input signals of a system Σ determine the output signals of Σ uniquely, then a system is called an input/output system (IO-system). So an IO-system can be regarded as a mapping (fΣ) from the set of input signals (U) into the set of output signals (Y). Schematically the system can be presented by the following scheme.

\[ U \xrightarrow{\ f_\Sigma \ } Y \]

The mapping \( f_\Sigma \) is called the input/output mapping of the system Σ.

A system Σ is completely specified by the triple \((U, Y, f_\Sigma)\). The triple \((U, Y, f_\Sigma)\) is called the working mode of the system Σ (cf. Yamamoto [Y2]). The working mode of a system Σ is an external description, it carries all external properties of Σ within.

For example, the system Σ is linear if and only if the signal spaces \( U \) and \( Y \) are vector spaces and if the mapping \( f_\Sigma \) is linear. The system Σ is time-invariant if and only if for each input \( u \in U \) causing the output \( y (= f_\Sigma(u)) \) and for each \( t \in R \), we have that its translate \( u(.,+t) \) is also an input \((\in U)\), causing the output \( y(.,+t) \). In particular, \( U \) and \( Y \) have to be translation invariant vector spaces. A linear system is causal if and only if \( u|_{(-\infty,T]} = 0 \) yields that \( f_\Sigma(u)|_{(-\infty,T]} = 0 \) for any \( T \in R \).

System theory deals, more or less, with the internal behaviour of causal systems. In particular, system theory is interested in entities related to a system, carrying at each moment the information about the past behaviour of the system that is relevant for the future behaviour of the system, the so called state variables of the system. In modelling physical phenomena, state variables often appear. The problem of factor-
Factorizations of input/output mappings

Chapter 5. Factorizations of input/output mappings

ization deals with the choice of state variables, as will be explained now. We start from the working mode \((U, Y, f_E)\) of a causal, time-invariant system \(\Sigma\). Next, we treat the system differently, by considering the pairs \((\omega, \gamma)\), where \(\omega\) is an input from \(U\), with support only before the time instant zero and where \(\gamma\) is the corresponding output only considered on the time-interval \([0, \infty)\). We call this description of a system \(\Sigma\) the static mode of \(\Sigma\). The static mode of a system is completely specified by the triple \((\Omega, \Gamma, f_{\text{stat}})\), where \(\Omega\) consists of all inputs from \(U\) varying only before the time instant zero, where \(\Gamma = U \setminus \{0, \infty\}\) and where \(f_{\text{stat}} : \Omega \rightarrow \Gamma\) defined by

\[
    f_{\text{stat}}(\omega) := f_E(\omega)|_{[0, \infty)} \quad (\omega \in \Omega).
\]

A factorization of \(f_{\text{stat}}\) is a triple \((X, g, h)\), where \(g\) is a mapping from \(\Omega\) into \(X\) and where \(h\) is a mapping from \(\Omega\) into \(\Gamma\) such that \(f_{\text{stat}} = h \circ g\). The problem of factorization is how to construct, to some extent, “a unique” and “minimal” factorization \((X, g, h)\).

In the minimal factorization, the vector \(g(\omega) \in X\), \(\omega \in \Omega\), indicates precisely the information of the input vector \(\omega\) that is relevant to determine the corresponding output \(\gamma\). Hence, this vector \(g(\omega)\) is the most likely candidate for the state of the system at time zero, given the input \(\omega\) before time instant zero. Correspondingly, \(X\) is the state space, i.e. the space where the state vector takes its values.

A search for a factorization of \(f_{\text{stat}}\) is the first natural step in the search for a realization of \(f_{\text{stat}}\), i.e. the description of the system by the set of differential equations

\[
    \begin{align*}
    x(t) &= A(x(t)) + B(u(t)) \quad t > 0, x(0) = x_0, \\
    y(t) &= C(x(t))
    \end{align*}
\]

where at each time \(t_0\), \(x(t_0)\) denotes the state of the system, \(u(t_0)\) the input and \(y(t_0)\) the output. Furthermore, \(A, B\) and \(C\) are linear mappings. We will not tackle the realization problem in this thesis. Nevertheless, it can be solved in a few cases. In particular, we present in §5.2.2 a (finite-dimensional) state space realization for systems with a finite-dimensional factorization. Furthermore, Yamamoto presented in [Y2] and [Y4] state space realizations for systems belonging to the class of systems studied in this chapter.

In this chapter, we restrict the attention to linear time-invariant causal continuous-time systems. In particular, we focus on the case that both the input- and the output signals are scalar valued functions of time (single-input single-output -, or SISO-systems). Furthermore, we assume that the input space and output space are equal. The case of multivariable inputs and outputs, MIMO-systems, can be treated similar to the theory presented in this chapter, up to obvious modifications.

In contrast with most literature ([Y1, Y2, Y4, Y5], [Kam1, Kam2] and [KH]), we do not want to make an a priori choice for the signal spaces. Indeed, we take signal spaces from the class of translatable strict LP-spaces of \(D_0^*(\mathbb{R})\)-type (cf. Definition 4.2). Herewith, we included Yamamoto's choice \(L_0^*_{\text{stat}}(\mathbb{R})\) for the signal spaces (see [Y1, Y2, Y4, Y5]). Moreover, each input/output mapping \(f_E : U \rightarrow Y\) can be extended to an input/output mapping \(f_{\text{ext}}\) from \(D_0^*(\mathbb{R})\) into \(D_0^*(\mathbb{R})\). So, \((U, Y, f_E)\) is a subsystem of \((D_0^*(\mathbb{R}), D_0^*(\mathbb{R}), f_{\text{ext}})\). The latter type of systems were considered by Kamen ([Kam2]).
5.1. DESCRIPTIONS OF INPUT/OUTPUT SYSTEMS

Taking signals from a translatable strict LF-space of \( D_\ell (\mathbb{R}) \)-type, we consider only signals with support bounded on the left. This condition on the support of the signals expresses the philosophy that the feeding of an input to the system has started only a finite time span ago. However, since the time-point zero (the present) may be chosen arbitrarily (time-invariance), this initial point may also be chosen arbitrarily. In fact, this assumption was made by Kalman in his approach to solve the realization problem for discrete-time dynamical systems (see [Kal], Chapter 10).

The set-up of this chapter is as follows. In section 5.1.1, we investigate the working mode of linear time-invariant input/output systems with signals from a translatable strict LF-space. It turns out that each such system is the unique extension of a system with smooth (\( = D_\ell (\mathbb{R}) \) ) signals. Since different choices of signal spaces can lead to the same smooth subsystem, this may lead to a classification of such systems. In §5.1.2, we discuss the static mode of linear time-invariant input/output systems. Also in the static mode description of systems, systems can be classified, when having the same behaviour for smooth inputs. In §5.1.3, we show that under the condition of strict causality, each system has a unique working mode, and a unique static mode description. In §5.2, we introduce the basics of factorization theory. A concept of canonical (\( = \) "unique and minimal") factorization, due to Yamamoto [Y1], is proposed. In §5.2.2, we investigate systems with finite-dimensional factorizations, i.e., factorizations where the state space \( X \) is finite-dimensional. It is shown that, finite-dimensional state spaces are spanned by exponential-polynomials and that systems with finite-dimensional factorizations have an impulse response that is the sum of exponential-polynomials. For systems with finite-dimensional factorizations, a (finite-dimensional) state space realization is given. Finally, in §5.3, we focus on so-called pseudo-rational systems. The canonical state space of a subclass of pseudo-rational systems is characterized in terms of their impulse responses. A necessary and sufficient condition is given for which canonical state spaces are spanned by a countable collection of exponential-polynomials.

5.1 DESCRIPTIONS OF INPUT/OUTPUT SYSTEMS

In this section, we discuss two types of external descriptions of linear time-invariant continuous-time systems, the working mode of systems and the static mode of systems. Each of these modes of a system can be described by a mapping, the so-called input/output mapping. In §5.1.3, we show that for strictly causal systems, these descriptions correspond.

5.1.1 The working mode of a system

Let \( \Sigma \) be a linear, time-invariant, continuous-time SISO-system accepting and producing signals from the translatable strict LF-space \( V \) of \( D_\ell (\mathbb{R}) \)-type. Schematically, the system \( \Sigma \) can be represented by the following diagram.
Since \( \Sigma \) admits an input/output description, the system \( \Sigma \) can be described by a mapping relating each input to its corresponding output, the input/output mapping. We denote the input/output mapping of the system \( \Sigma \) by \( f_\Sigma \). Since the system \( \Sigma \) is linear and time-invariant, the mapping \( f_\Sigma \) is linear on \( V \) satisfying

\[
\sigma_t \circ f_\Sigma = f_\Sigma \circ \sigma_t,
\]

for all \( t \in \mathbb{R} \). Here \( \sigma_t \) denotes the translation operator on \( V \) over \( t \) for each \( t \in \mathbb{R} \).

In the terminology of Chapter 4, the mapping \( f_\Sigma \) is \((\sigma_t)_{t \in \mathbb{R}}\)-invariant. Involving the topology of \( V \) as well, we shall demand the input/output mapping to be well-conditioned by requiring that the input/output mapping \( f_\Sigma \) is continuous with respect to \( V \).

Definition 5.1 The working mode of a SISO-system \( \Sigma \) with signals from translatable strict LF-space \( V \) of \( \mathcal{D}_+(\mathbb{R}) \)-type, is the pair \((V, f_\Sigma)\). Here \( f_\Sigma \) is the input/output mapping of \( \Sigma \). The mapping \( f_\Sigma \) is linear, continuous and \((\sigma_t)_{t \in \mathbb{R}}\)-invariant on \( V \).

In the remainder of this chapter, we write \( \Sigma = (V, f_\Sigma) \), where \( V \) is always a translatable strict LF-space of \( \mathcal{D}_+(\mathbb{R}) \)-type, and where \( f_\Sigma \) is the input/output mapping of \( \Sigma \).

With the assumption that input/output mappings are continuous, we obtain the following characterisation of input/output mappings from Theorem 4.16.

**Proposition 5.2** Let \( V \) be a translatable strict LF-space of \( \mathcal{D}_+(\mathbb{R}) \)-type. Let \( \Sigma = (V, f_\Sigma) \) be a SISO-system. Then, the input/output mapping \( f_\Sigma \) of \( \Sigma \) is of the form

\[
f_\Sigma (x) = G * x \quad (x \in V),
\]

for a unique \( G \in \mathcal{D}_+'(\mathbb{R}) \). Conversely, for \( G \in \mathcal{D}_+'(\mathbb{R}) \) satisfying \( G * x \in V \) for each \( x \in V \), (5.3) defines a continuous linear translation-invariant mapping \( f_G \) on \( V \). Correspondingly, \((V, f_\Sigma)\) is a SISO-system.

The distribution \( G \) introduced in Proposition 5.2 is called the impulse response of \( \Sigma = (V, f_\Sigma) \).

**Remark 5.2.1** Recall from §4.1.3 that for \( V = C^k_+(\mathbb{R}) \), \( k \in \mathbb{N}_0 \), and for \( V = L^1_{\text{loc},+}(\mathbb{R}) \), the impulse response of a SISO-system \((V, f_\Sigma)\) is always a Radon measure with support bounded on the left. Conversely, each \( \mu \in \mathcal{M}_+(\mathbb{R}) \) is the impulse response of a SISO-system \((V, f_\Sigma)\), where \( V = C^k_+(\mathbb{R}) \), \( k \in \mathbb{N}_0 \), or \( V = L^1_{\text{loc},+}(\mathbb{R}) \).

Proposition 5.2 has the following consequences for SISO-systems \( \Sigma = (V, f_\Sigma) \). Let \( G \) be the impulse response of \( \Sigma \). Then, the continuous linear \((\sigma_t)_{t \in \mathbb{R}}\)-invariant mapping \( f_\Sigma : \mathcal{D}_+'(\mathbb{R}) \rightarrow \mathcal{D}_+'(\mathbb{R}) \), defined by

\[
f_\Sigma^G (U) := G * U \quad (U \in \mathcal{D}_+'(\mathbb{R})),
\]

is a SISO-system with impulse response \( G \).
extends the input/output mapping \( f_2 \) to \( D_+(\mathbb{R}) \).

Furthermore, the mapping \( f^{\text{sm}}_2 : D_+(\mathbb{R}) \to D_+(\mathbb{R}) \), defined by

\[
f^{\text{sm}}_2(\psi) := C \ast \psi \quad (\psi \in D_+(\mathbb{R})),
\]

is a continuous linear \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant mapping on \( D_+(\mathbb{R}) \). In particular, \( f^{\text{sm}}_2 \) is the restriction of \( f_2 \) to \( D_+(\mathbb{R}) \).

**Definition 5.3** The SISO-system \( \Sigma_1 = (V_1, f_{21}) \) is called a subsystem of the SISO-system \( \Sigma_2 = (V_2, f_{22}) \) when the following conditions are satisfied:

i. \( V_1 \subseteq V_2 \),

ii. \( f_{22}|_{V_1} \subseteq V_1 \),

iii. \( f_{21}|_{V_1} = f_{22} \).

By Definition 5.3, each SISO-system \( \Sigma = (V, f_2) \) is a subsystem of a SISO-system, \( \Sigma_{\text{ext}} = (D_+(\mathbb{R}), f^{\text{ext}}_2) \), admitting \( D_+(\mathbb{R}) \)-signals. Systems with \( D_+(\mathbb{R}) \)-signals were considered by Kamen in [Kam2]. So, the systems, considered in this chapter, are subsystems of the ones considered by Kamen. Since \( D_+(\mathbb{R}) \) is not a translatable strict LF-space of \( D_+(\mathbb{R}) \)-type, we deal with proper subsystems of Kamen’s systems only. Furthermore, we observe that each SISO-system \( \Sigma = (V, f_2) \) contains a smooth subsystem, \( \Sigma^{\text{sm}} = (D_+(\mathbb{R}), f^{\text{sm}}_2) \). In fact, by Lemma 4.14 and Proposition 4.15, the mapping \( f_2 \) is the unique continuous linear \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant extension to \( V \) of the input/output mapping \( f^{\text{sm}}_2 \) on \( D_+(\mathbb{R}) \).

**Theorem 5.4** Let \( (V, f_2) \) be a SISO-system. Let \( f^{\text{sm}}_2 \) be the restriction of \( f_2 \) to \( D_+(\mathbb{R}) \) as in (5.5). Then, \( \Sigma^{\text{sm}} = (D_+(\mathbb{R}), f^{\text{sm}}_2) \) is a subsystem of \( \Sigma \), and \( \Sigma \) is the unique SISO-system, with signals in \( V \), having \( \Sigma^{\text{sm}} \) as a subsystem.

By Theorem 5.4, we may regard any SISO-system \( \Sigma = (V, f_2) \) as the completion of its smooth subsystem \( \Sigma^{\text{sm}} = (D_+(\mathbb{R}), f^{\text{sm}}_2) \). However, this smooth subsystem \( \Sigma^{\text{sm}} = (D_+(\mathbb{R}), f^{\text{sm}}_2) \) may very well be the smooth subsystem of a number of SISO-systems. Obviously, these systems will have the same impulse response. For example, recall that the mapping \( \sigma_\mu \) is linear, continuous and \( (\sigma_t)_{t \in \mathbb{R}} \)-invariant on every translatable strict LF-space \( V \) of \( D_+(\mathbb{R}) \)-type for each \( \mu \in M_+(\mathbb{R}) \). So, each pair \( (V, \sigma_\mu) \), with \( \mu \in M_+(\mathbb{R}) \) and with \( V \) a translatable strict LF-space of \( D_+(\mathbb{R}) \)-type, defines a SISO-system, all these systems densely containing the same smooth subsystem \( \Sigma^{\text{sm}} = (D_+(\mathbb{R}), \sigma_\mu) \) and having the same impulse response \( \mu \). In fact, one can classify systems having the same smooth subsystem in their working mode description.

We are especially interested in causal and strictly causal IO-systems. In particular, let \( \Sigma = (V, f_2) \) be a SISO-system. Then \( \Sigma \) is called causal, if its input/output mapping \( f_2 \) satisfies for all \( T \in \mathbb{R} \)

\[
u_1|_{[T, \infty)} = u_2|_{[T, \infty)} \Rightarrow f_2(\nu_1)|_{[T, \infty)} = f_2(\nu_2)|_{[T, \infty)}.
\]

The following assertion is readily checked.
Proposition 5.5 A SISO-system \((V, f_2)\) is causal if and only if the support of the impulse response \(G\) of \(\Sigma\) is contained in \([0, \infty)\). Furthermore, a SISO-system \((V, f_2)\) is causal if and only if its smooth subsystem \((\mathcal{D}_+(\mathbb{R}), f_2^{sm})\) is causal.

In §5.1.3, we show that causality of a system is not sufficient to guarantee a proper introduction of the "state" of a system. Therefore, we introduce the stronger concept of "strict causality". We need some terminology first.

Definition 5.6 Let \(U\) be an open set in \(\mathbb{R}\). Let \(G \in \mathcal{D}'(\mathbb{R})\). Then \(G\) is called regular on \(U\), if there exists \(g \in L^1_{loc}(\mathbb{R})\) such that for all \(\psi \in \mathcal{D}(\mathbb{R})\)

\[
\text{supp}(\psi) \subseteq U \implies G(\psi) = \int_{\mathbb{R}} \psi(t)g(t)\,dt.
\]

We employ the following concept of strict causality due to Kamen [Kam2], Definition 2.3.

Definition 5.7 A SISO-system \(\Sigma = (V, f_2)\) is called strictly causal if \(\Sigma = (V, f_2)\) is causal and its impulse response \(G\) is regular in a neighbourhood of 0.

Again, a SISO-system \((V, f_2)\) is strictly causal if and only if its smooth subsystem \((\mathcal{D}_+(\mathbb{R}), f_2^{sm})\) is strictly causal.

5.1.2 Static SISO-systems

In this subsection, we introduce a second type of description for causal SISO-systems, the so-called static mode of a system (cf. [Y2]). The static mode of a system \(\Sigma\) relates an input with compact support, so actually taking place during a finite time span, to its corresponding output only considered after the actual input has ended. Dealing with time-invariant systems, we may take inputs on the negative time-axis \((-\infty, 0]\) (the past) and outputs on the positive time-axis \([0, \infty)\) (the future). We emphasize that such a description makes sense only if the system \(\Sigma\) is causal.

Let \(\Sigma = (V, f_2)\) be a causal SISO-system. The input space \(\Omega\) in the static mode of \(\Sigma\) consists of all input signals in \(V\) with distributional support in \((-\infty, 0]\), i.e.

\[
\Omega := V \cap \mathcal{E}'(\mathbb{R}^-) = \{x \in V \mid \text{supp}(x) \subseteq (-\infty, 0]\}, \tag{5.6}
\]

where \(\mathcal{E}'(\mathbb{R}^-)\) is the subspace of \(\mathcal{D}'(\mathbb{R})\) consisting of all distributions with (compact) support in \((-\infty, 0]\\)

The output space \(\Gamma\) in the static mode is a quotient space, namely

\[
\Gamma := V/\Omega. \tag{5.7}
\]

So, output signals \(x_1, x_2 \in V\) belong to the same equivalence class in \(\Gamma\) if and only if \(\text{supp}(x_1 - x_2) \subseteq (-\infty, 0]\), or equivalently, if \((x_1 - x_2)!_{[0,\infty)} = 0\).

Now, let \(\omega \in \Omega(\subseteq V)\) be an input to the system \(\Sigma = (V, f_2)\). Since the corresponding
output in the working mode of \( \Sigma \) is \( f_2(\omega) \), it is natural to take the cost \( f_2(\omega) + \Omega \) as output in the static mode of the system \( \Sigma \). So, the mapping \( f_{\text{stat}} : \Omega \rightarrow \Gamma \) defined by

\[
f_{\text{stat}}(\omega) := f_2(\omega) + \Omega \quad (\omega \in \Omega),
\]
describes the behaviour of \( \Sigma \) in the static mode. Notice that \( f_{\text{stat}} \) is linear, satisfying for all \( t \geq 0 \)

\[
f_{\text{stat}}(\sigma_t \omega) = \sigma_t(f_2(\omega)) + \Omega \quad (\omega \in \Omega). \tag{5.8}
\]

Introducing translations on \( \Omega \) and \( \Gamma \), equality (5.8) can be read as an intertwining property of \( f_{\text{stat}} \). Since \( \sigma_t(\Omega) \subseteq \Omega \) if and only if \( t \geq 0 \), we deal with semigroups of translations on \( \Omega \) and \( \Gamma \) only.

**Definition 5.8** Let \( V \) be a translatable strict LF-space of \( D_+ (\mathbb{R}) \)-type with translation group \( (\sigma_t)_{t \in \mathbb{R}} \). The translation semigroup \( (\sigma_t^\Gamma)_{t \geq 0} \) on \( \Omega = V \cap E'(\mathbb{R}^+) \) is defined by

\[
\sigma_t^\Gamma x := \sigma_t x \quad (t \geq 0, \ x \in \Omega),
\]
i.e. the translation semigroup \( (\sigma_t)_{t \geq 0} \) restricted to the subspace \( \Omega \). On \( \Gamma = V/\Omega \), the translation semigroup \( (\sigma_t^\Gamma)_{t \geq 0} \) is defined by

\[
\sigma_t^\Gamma(x + \Omega) := \sigma_t x + \Omega \quad (t \geq 0, \ x \in V),
\]
i.e. \( \sigma_t^\Gamma \) is the quotient mapping of \( \sigma_t \) on \( \Gamma \).

So, (5.8) can be read as the intertwining relation

\[
f_{\text{stat}} \circ \sigma_t = \sigma_t^\Gamma \circ f_2 \quad (t \geq 0). \tag{5.9}
\]

Now, we are ready to introduce the concept of static SISO-system.

**Definition 5.9** A static SISO-system \( \Sigma_{\text{stat}} \) is the quadruple \( (V, \Omega, \Gamma, f_{\text{stat}}) \) satisfying the following conditions:

i. \( V \) is a translatable strict LF-space of \( D_+ (\mathbb{R}) \)-type,

ii. \( \Omega = V \cap E'(\mathbb{R}^+) \),

iii. \( \Gamma = V/\Omega \), and

iv. \( f_{\text{stat}} : \Omega \rightarrow \Gamma \) is a linear operator satisfying the intertwining relation (5.9).

The space \( \Omega \) is called the static input space of \( \Sigma_{\text{stat}} \), \( \Gamma \) is called the static output space of \( \Sigma_{\text{stat}} \) and the mapping \( f_{\text{stat}} \) is called the static input/output mapping of \( \Sigma_{\text{stat}} \).

**Remark 5.9.1** The terminology “static system” is a bit unfortunate, since it usually refers to a system without any dynamics. However, since we want to present a theory which captures the one introduced by Yamamoto ([Y1]), we adopt his terminology. The mapping from past inputs into future outputs is also referred to as the “Hankel operator”, so an alternative for the term “static” could be “Hankel”.
**Remark 5.9.2** In concrete situations the static output space $\Gamma$ is taken isomorphic to the complex quotient space $V/\Omega$. For instance, if $V = L^2_{\text{loc},\mu}(\mathcal{R})$, we take $\Gamma = L^2_{\text{loc}}(\mathcal{R})$ (cf. Yamamoto [Y1, Y2] and Corollary 5.13).

**Remark 5.9.3** In §5.1.3, we show that each working mode description $(V, f_\Omega)$ of a SISO-system $\Sigma$ corresponds uniquely to a static SISO-system $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$. This static SISO-system $\Sigma_{\text{stat}}$ is called the static mode description of the system $\Sigma$ (see §5.1.3).

In the Definitions 5.10 and 5.19, we refine the definition of a static SISO-system, involving topological aspects as well. To this extent, we need to introduce topological structures for the signal spaces of a static SISO-system $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$.

First, we consider the topology for $\Omega$. Since $V \hookrightarrow D'_\sigma(\mathcal{R})$ and $\mathcal{E}(-\mathcal{R}^*)$ is closed in $D'_\sigma(\mathcal{R})$, $\Omega = V \cap \mathcal{E}(-\mathcal{R}^*)$ is a closed subspace of $V$. So, from a topologist's point of view, it would seem natural to endow $\Omega$ with the relative $V$-topology. However, the topological structure of this relative $V$-topology does not need to be a strict inductive limit topology. There are strict LF-spaces having closed subspaces which are not a (strict) LF-space, when equipped with the relative topology (see Remark 1.41.1). Therefore, we equip $\Omega$ with a topology possibly finer than the relative $V$-topology. To this end, let $V = \text{ind} \ F_n$. Since $F_n \cap \Omega = F_n \cap \mathcal{E}(\mathcal{R}^*)$ for each $n \in \mathcal{N}$, we have that $\bigcup_n (F_n \cap \mathcal{E}(\mathcal{R}^*)) = \Omega$. Since $\Omega$ is a closed subspace of $V$, Proposition 1.41 yields that $(F_n \cap \mathcal{E}(\mathcal{R}^*))_{n \in \mathcal{N}}$ is a strict inductive system of $\mathcal{F}$-spaces. We equip $\Omega$ with the strict inductive limit topology of $(F_n \cap \mathcal{E}(\mathcal{R}^*))_{n \in \mathcal{N}}$. Thus, of course, we have $\Omega \hookrightarrow D'_\sigma(\mathcal{R})$.

Next, we investigate the topological structure of the output space $\Gamma$. Since $\Gamma = V/\Omega$ is a quotient space, it is natural to equip $\Gamma$ with corresponding quotient topology. Equipped with quotient topology, $\Gamma$ is a strict LF-space. In particular, in Proposition 1.43 it is shown that, if $V = \text{ind} \ F_n$, then $(F_n/(F_n \cap \Omega))_{n \in \mathcal{N}}$ is a strict inductive system of $\mathcal{F}$-spaces with strict inductive limit $\Gamma$. Summarizing, we have obtained the following topologies for the signal spaces in the static mode of a system.

**Assumption 5.10** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a static SISO-system, where $V = \text{ind} \ F_n$. Then, the static input space $\Omega$ is equipped with the strict inductive limit topology arising from the strict inductive system $(F_n \cap \mathcal{E}(\mathcal{R}^*))_{n \in \mathcal{N}}$, i.e.

$$\Omega := \text{ind} \left( \bigcap_n (F_n \cap \mathcal{E}(\mathcal{R}^*)) \right).$$

Moreover, the static output space $\Gamma$ is equipped with the strict inductive limit topology arising from the strict inductive system $(F_n/(F_n \cap \Omega))_{n \in \mathcal{N}}$, i.e.

$$\Gamma := \text{ind} \left( \bigcap_n (F_n/(F_n \cap \Omega)) \right).$$

In §3.3, we discussed the $\mathcal{F}$-space $\mathcal{E}(\mathcal{R}^*)$. By Theorem 3.57, $\mathcal{E}(\mathcal{R}^*)$ equals the quotient space $D'_\sigma(\mathcal{R})/D(\mathcal{R})$. Since $D(\mathcal{R}) = D'_\sigma(\mathcal{R}) \cap \mathcal{E}(\mathcal{R}^*)$, this yields that the static output space $\Gamma$ of each "smooth" static SISO-system $(D'_\sigma(\mathcal{R}), \Omega, \Gamma, f_{\text{stat}})$ equals the
5.1. DESCRIBATIONS OF INPUT/OUTPUT SYSTEMS

$F$-space $\mathcal{E}(R^n)$. In the sequel, we will show that $\Gamma$ can be taken an $F$-space for a class of translatable strict LF-spaces of $\mathcal{D}(R)$-type.

Let $V = \text{ind} \, F_n$ be a translatable strict LF-space of $\mathcal{D}(R)$-type, where the F-spaces $F_n$ are defined as in Theorem 4.6, i.e. for each $n \in \mathbb{N}$

$$F_n = \{ \xi \in V \mid \text{supp}(\xi) \subseteq [-n, \infty) \},$$

(5.10)

which are equipped with relative $V$-topology. The canonical injection id from $F_1/(F_1 \cap \Omega)$ into $\Gamma$ is defined by

$$\text{id}(\xi + (F_1 \cap \Omega)) := \xi + \Omega \quad (\xi \in F_1).$$

(5.11)

By definition of the topology of $\Gamma$, the injection id is continuous. So, if id were a bijection, then the Open Mapping Theorem (1.40) ensures that id is a homeomorphism. Since $F_1/(F_1 \cap \Omega)$ is an F-space (Proposition 1.24), this would yield that $\Gamma$ is an F-space. Hence, we search for conditions on $V$ for id to be surjective.

First, we consider a simpler problem for $\mathcal{D}_1(R)$-distributions: “Does there exist for each $\xi \in \mathcal{D}_1(R)$ a $\eta \in \mathcal{D}_1(R)$ with supp$(\eta) \subseteq [-1, \infty)$ such that supp$(\xi - \eta) \subseteq (-\infty, 0]$?” This question has a positive answer. In particular, let $\psi \in \mathcal{D}_1(R)$ and $F \in \mathcal{D}_1(R)$. Define the distributional product of $\psi$ and $F$ by

$$(\psi \cdot F)(\phi) := F(\psi \cdot \phi) \quad (\phi \in \mathcal{D}_0(R)).$$

(5.12)

Since the mapping $\phi \in \mathcal{D}_0(R) \rightarrow \psi \cdot \phi$ is continuous on $\mathcal{D}_0(R)$, (5.12) defines a new $\mathcal{D}_1(R)$-distribution, denoted by $\psi \cdot F$. The support of the distribution $\psi \cdot F$ is included in supp$(F) \cap$ supp$(\psi) \subseteq$ supp$(\psi)$. Now, choose $\psi_0 \in \mathcal{D}_1(R)$ such that $\psi_0(t) = 0$ for $t \leq -1$ and $\psi_0(t) = 1$ for $t \geq 0$ and let $\psi \in \mathcal{D}_1(R)$, then we can write $\xi$ as follows

$$\xi = \psi_0 \cdot \xi + (1 - \psi_0) \cdot \xi.$$  

(5.13)

Notice that $\text{supp}(\psi_0 \cdot \xi) \subseteq [-1, \infty)$ and $\text{supp}((1 - \psi_0) \cdot \xi) \subseteq (-\infty, 0]$, so we have indeed that every $\mathcal{D}_1(R)$-distribution can be written as the sum of two $\mathcal{D}_1(R)$-distributions with support in $(-\infty, 0]$ and in $[-1, \infty)$ respectively.

Returning to the mapping id from (5.11), suppose that $\psi \cdot \xi \in V$ for each $\psi \in \mathcal{D}_1(R)$, $\xi \in V$. Then $\psi_0 \cdot \xi \in V$ and $(1 - \psi_0) \cdot \xi \in V$ for all $\xi \in V$, where $\psi_0$ as above. Hence, $\xi = \psi_0 \cdot \xi + (1 - \psi_0) \cdot \xi \in \mathcal{D}_1(R) \cap V = \text{ind}(\mathcal{D}_1(R) \cap V) = \text{id}(\mathcal{D}_1(R) \cap V)$. We conclude that the following condition on $V$ is sufficient for id to be surjective.

Condition 5.11

For all $\psi \in \mathcal{D}_1(R)$ and all $\xi \in V$, we have $\psi \cdot \xi \in V$.

Now, suppose $V$ satisfies Condition 5.11. Then, as we have seen, $V = F_1 + (E'(-\infty) \cap V)$. Summarizing, we have obtained the following result.

Proposition 5.12 Let $\Sigma_{stat} = (V, \Omega, \Gamma, f_{\text{feq}})$ be a static SISO-system. If $V$ satisfies Condition 5.11, then $\Gamma$ is an $F$-space. In fact, if $V = \text{ind} \, F_n$, where the F-spaces $F_n$ are as in (5.10), then $\Gamma \equiv F_1/(F_1 \cap E'(-\infty))$. 
All translatable strict LF-spaces of $\mathcal{D}_0(\mathbb{R})$-type mentioned explicitly in this thesis satisfy Condition 5.11. For example, the distributional product of a $\mathcal{D}_0(\mathbb{R})$-function with a $\mathcal{D}_0(\mathbb{R})$-function is again a $\mathcal{D}_0(\mathbb{R})$-function, namely the pointwise product of the two. Without proof, we mention some examples of translatable strict LF-spaces of $\mathcal{D}_0(\mathbb{R})$-type satisfying Condition 5.11.

**Corollary 5.13** Let $\Sigma_{\text{mat}} = (V, \Omega, \Gamma, f_{\text{mat}})$ be a static SISO-system. Then, for each of the following translatable strict LF-spaces $V$ of $\mathcal{D}_0(\mathbb{R})$-type, the static output space $\Gamma$ is an $F$-space; $V = \mathcal{D}_0(\mathbb{R}), C^k_0(\mathbb{R}), L^p_{\text{loc, loc}}(\mathbb{R})$, where $k \in \mathbb{N}_0$ and $p \geq 1$.

Yamamoto's choice for the static output space $\Gamma$ was $L^2_{\text{loc}}(\mathbb{R}^+)$ (see Appendix A). The following result shows that this choice for $\Gamma$ arises when taking $V = L^2_{\text{loc, loc}}(\mathbb{R})$ (up to identification).

**Lemma 5.14** Let the spaces $L^p_{\text{loc}}(\mathbb{R}^+)$ and $C^k(\mathbb{R}^+)$ be equipped with their natural $F$-topologies. Then

- The quotient space $L^p_{\text{loc}}(\mathbb{R}^+)/(L^p_{\text{loc}}(\mathbb{R}^+) \cap E'(\mathbb{R}^+))$, equipped with quotient topology, is homeomorphic to the space $L^p_{\text{loc}}(\mathbb{R}^+)$ for each $p \geq 1$.
- The quotient space $C^k(\mathbb{R}^+)/(C^k(\mathbb{R}) \cap E'(\mathbb{R}^+))$, equipped with quotient topology, is homeomorphic to the space $C^k(\mathbb{R}^+)$ for $k = 0, 1, \ldots, \infty$.

**Proof.**
We refer to [deR2], Lemma 6.11.

**Corollary 5.15** Let $\Sigma_{\text{mat}} = (V, \Omega, \Gamma, f_{\text{mat}})$ be a static SISO-system. If $V$ is one of the following spaces: $\mathcal{D}_0(\mathbb{R}), C^k_0(\mathbb{R}), L^p_{\text{loc, loc}}(\mathbb{R})$, where $k \in \mathbb{N}_0$, $p \geq 1$, then $\Gamma$ can be taken $E(\mathbb{R}^+), C^k(\mathbb{R}^+), L^p_{\text{loc}}(\mathbb{R}^+)$.

Next, we consider the translation semigroups on the signal spaces of static SISO-systems. The following result is based on the assertion that $(\sigma_t)_{t \in \mathbb{R}}$ is locally equicontinuous on each translatable strict LF-space $V$ and Theorem 2.65.

**Lemma 5.16** Let $\Sigma_{\text{mat}} = (V, \Omega, \Gamma, f_{\text{mat}})$ be a static SISO-system. Let $(\sigma^+_t)_{t \geq 0}$ and $(\sigma^-_t)_{t \geq 0}$ be the translation semigroups on $\Omega$ and $\Gamma$. Then $(\sigma^+_t)_{t \geq 0}$ and $(\sigma^-_t)_{t \geq 0}$ form $c_0$-semigroups of continuous linear operators on $\Omega$ and $\Gamma$.

Since $\Omega$ and $\Gamma$ are strict LF-spaces, the $c_0$-semigroups $(\sigma^+_t)_{t \geq 0}$ and $(\sigma^-_t)_{t \geq 0}$ are locally equicontinuous.

We are especially interested in the $c_0$-domain of the infinitesimal generators, $\delta^+_\omega$ and $\delta^-_\omega$, of the $c_0$-semigroups $(\sigma^+_t)_{t \geq 0}$ and $(\sigma^-_t)_{t \geq 0}$. First, we consider $\delta^-_\omega$.

**Lemma 5.17** Let $\Sigma_{\text{mat}} = (V, \Omega, \Gamma, f_{\text{mat}})$ be a static SISO-system. Let $\delta_\omega$ be the infinitesimal generator of $(\sigma_t)_{t \geq 0}$ on $V$ and let $\delta^-_\omega$ be the infinitesimal generator of $(\sigma^-_t)_{t \geq 0}$. Then $\text{Dom}(\delta^-_\omega) = \text{Dom}(\delta_\omega) \cap \Omega$. Moreover, $\delta^-_\omega \omega = \delta_\omega \omega$ for all $\omega \in \text{Dom}(\delta^-_\omega)$. 
5.1. DESCRIBEDS OF input/OUTPUT SYSTEMS

Proof.
The assertion that $\text{Dom}(\delta_x \cap \Omega) \subseteq \text{Dom}(\delta_x)$ and the assertion that $\delta_x \omega = \delta_{\omega}$ for all $\omega \in \text{Dom}(\delta_x) \cap \Omega$ are obvious. Since $\text{Dom}(\delta_x) \subseteq \Omega$, it remains to be proved that $\text{Dom}(\delta_x) \subseteq \text{Dom}(\delta_x)$. To this end, let $\omega \in \text{Dom}(\delta_x)$ and let $y := \delta_x \omega$. Since $\Omega \rightarrow V$ and since $\frac{\omega}{t} \in \omega$ in $\Omega$-sense as $t \downarrow 0$, we have $\frac{\omega}{t} \in \omega$ in $\text{V}$-sense as $t \downarrow 0$. So, $\omega \in \text{Dom}(\delta_x)$ if and only if $\frac{\omega}{t} \in \omega$ as $t \downarrow 0$. To this end, let $p$ be a continuous seminorm on $V$. Then for all $t > 0$,

$$p\left(y - \frac{\omega}{t} \right) \leq p(y - \sigma_t y) + p\left(y - \frac{\omega}{t} \right).$$

(5.14)

By the strong continuity of $(\sigma_t)_{t \in \mathbb{R}}$, the first part of the right-hand side of (5.14) tends to 0 as $t \downarrow 0$. Furthermore, since $(\sigma_t)_{t \in \mathbb{R}}$ is locally equicontinuous on $V$, the latter part of the right-hand side of (5.14) tends to 0 as $t \downarrow 0$. So, $\frac{\omega}{t} \in \omega$ in $V$ as $t \downarrow 0$, i.e. $\omega \in \text{Dom}(\delta_x) \cap \Omega$ and $\delta_x \omega = y$.

Theorem 5.18 Let $(V, \Omega, \Gamma, f_{\text{stat}})$ be a static SISO-system. Let $\delta_x$ be the infinitesimal generator of $(\sigma_t)_{t \in \mathbb{R}}$ on $V$ and let $\delta_x^\ast$ be the infinitesimal generator of $(\sigma_t^\ast)_{t \in \mathbb{R}}$. Then

$$\text{Dom}^{\text{cos}}(\delta_x) = \text{Dom}^{\text{cos}}(\delta_x) \cap \Omega \equiv \mathcal{D}(\mathcal{R})^\ast.$$

Next, we consider the infinitesimal generator $\delta_x^\ast$ of the translation semigroup $(\sigma_t^\ast)_{t \in \mathbb{R}}$ on $\Gamma$. Similarly to Theorem 5.18, we would like that $\text{Dom}^{\text{cos}}(\delta_x^\ast) = \mathcal{E}(\mathcal{R}^\ast)$, i.e. the smooth static output space. Since $\sigma_t^\ast \circ \phi = \phi \circ \sigma_t$ for all $t \geq 0$, where $\phi$ denotes the (continuous) quotient mapping from $V$ into $\Gamma$, we have that $\phi(x) \in \text{Dom}(\delta_x^\ast)$ with $\delta_x^\ast (\phi(x)) = \phi(\delta_x x)$ for each $x \in \text{Dom}(\delta_x)$. Consequently, we have

$$\{x + \Omega \mid x \in \text{Dom}^{\text{cos}}(\delta_x)\} \subseteq \text{Dom}^{\text{cos}}(\delta_x^\ast).$$

Since $\text{Dom}^{\text{cos}}(\delta_x) = \mathcal{D}(\mathcal{R})$ for every translatable strict LF-space $V$, the mapping $J$ from $\mathcal{D}(\mathcal{R}) / \mathcal{D}(\mathcal{R})^\ast$ into $\{x + \Omega \mid x \in \text{Dom}^{\text{cos}}(\delta_x)\}$, defined by

$$J(\psi + \mathcal{D}(\mathcal{R})^\ast) := \psi + \Omega \quad (\psi \in \mathcal{D}(\mathcal{R})),$$

is an isomorphism. So by Theorem 3.57, we have

$$\mathcal{E}(\mathcal{R}^\ast) \equiv \mathcal{D}(\mathcal{R}) / \mathcal{D}(\mathcal{R})^\ast \equiv \{x + \Omega \mid x \in \text{Dom}^{\text{cos}}(\delta_x)\} \subseteq \text{Dom}^{\text{cos}}(\delta_x^\ast) \quad (5.15)$$

Although it is true for the spaces $l_{\text{per}, +}(\mathcal{R})$ and $C_+^\ast(\mathcal{R})$, the author can not prove the converse inclusion $\text{Dom}^{\text{cos}}(\delta_x^\ast) \subseteq \{x + \Omega \mid x \in \text{Dom}^{\text{cos}}(\delta_x)\}$. So, we do not know whether $\mathcal{E}(\mathcal{R}^\ast)$ equals $\text{Dom}^{\text{cos}}(\delta_x^\ast)$.

Now, let $(V, \Omega, \Gamma, f_{\text{stat}})$ be a static SISO-system. Having introduced topologies on the static input space $\Omega$ and on the static output space $\Gamma$, we require the static input/output mapping $f_{\text{stat}}$ to be well-conditioned, by demanding it to be continuous.
Assumption 5.19 Let \((V, \Omega, \Gamma, f_{\text{stat}})\) be a static SISO-system. Then, the static input/output mapping \(f_{\text{stat}} : \Omega \rightarrow \Gamma\) is assumed to satisfy the following conditions

- \(f_{\text{stat}}\) is linear and continuous,
- \(\sigma^+_t \circ f_{\text{stat}} = f_{\text{stat}} \circ \sigma^-_t\) for all \(t \geq 0\).

Static input/output mappings are uniquely determined by their “smooth” signals.

Lemma 5.20 Let \((V, \Omega, \Gamma, f_{\text{stat}})\) be a static SISO-system. Then, the static input/output mapping \(f_{\text{stat}} : \Omega \rightarrow \Gamma\) satisfies

i. \(\text{Dom}^\omega(\delta^+_t) \subseteq \text{Dom}^\omega(\delta^+_t)\).

ii. \(\{\omega, f_{\text{stat}}(\omega)\} \mid \omega \in \text{Dom}^\omega(\delta^+_t)\) is sequentially dense in \(\text{graph}(f_{\text{stat}})\).

Proof.

i. Let \(\omega \in \text{Dom}(\delta^+_t)\). Then

\[
\begin{align*}
\sigma^+_t \omega &= f_{\text{stat}}(\delta^+_t \omega) = \lim_{t \downarrow 0} \frac{\sigma^+_t \omega - \omega}{t} = \frac{\sigma^+_t f_{\text{stat}}(\omega) - f_{\text{stat}}(\omega)}{t}.
\end{align*}
\]

So, \(f_{\text{stat}}(\omega) \in \text{Dom}(\delta^+_t)\) and \(f_{\text{stat}}(\delta^+_t \omega) = \delta^+_t f_{\text{stat}}(\omega)\). Applying this equality inductively, the assertion follows.

ii. The assertion is a direct consequence of an extended version of Theorem 2.52.iii, where instead of a continuous linear \((\alpha_1, \alpha_2)\)-invariant mappings from \(V\) into \(V\), continuous linear mappings \(L\) from \(V\) into \(W\) are considered satisfying the intertwining relation \(\beta_t L = L \beta_t\) with \((\alpha_1, \alpha_2)\) and \((\beta, \beta)\) as \(\omega\)-groups on \(V\) and \(W\) (with similar proof as for Theorem 2.52.iii).

So by Lemma 5.20, every static input/output mapping \(f_{\text{stat}} : \Omega \rightarrow \Gamma\) is uniquely determined by its behaviour on the \(\omega\)-domain of the infinitesimal generators of the translation semigroups on the static signal spaces. Recall from Theorem 5.18 that \(\text{Dom}^\omega(\delta^+_t) \equiv \mathcal{D}(R^+)\), and recall from (5.16) that \(\mathcal{E}(R^+) \subseteq \text{Dom}^\omega(\delta^+_t)\). So, if the latter inclusion were an equality, then each static input/output mapping would be the unique extension of a static input/output mapping \(f_{\text{stat}} : D(R^+) \rightarrow \mathcal{E}(R^+)\). Since \(\mathcal{D}(R^+)\) is the static input space of a system admitting \(\mathcal{D}_+(R)\)-signals and since \(\mathcal{E}(R^+)\) is the static output space of a system admitting \(\mathcal{D}_+(R)\)-signals, this would yield, similar to the working mode of system, that each static system \(\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})\) is the unique extension of its smooth subsystem \(\Sigma^\omega_{\text{stat}} = (\mathcal{D}_+(R), D(R^+), \mathcal{E}(R^+), f_{\text{stat}})\).

In the remainder of this thesis, we restrict ourselves to the class \(C\) of translatable strict LF-spaces of \(\mathcal{D}_+(R)\)-type introduced in the following definition.

Definition 5.21 The class \(C\) consist of all translatable strict LF-spaces \(V\) of \(\mathcal{D}_+(R)\)-type, satisfying the following conditions

i. For all \(\psi \in \mathcal{D}_+(R)\) and all \(x \in V\), we have \(\psi \cdot x \in V\), i.e. Condition 5.11, and
5.1. Descriptions of input/output systems

\[ \text{ii. } \text{Dom}^\omega(\delta_\Omega^*) = \{x + \Omega \mid x \in \text{Dom}^\omega(\delta_\Omega)\} \equiv \mathcal{E}(R^*) \]

where \( \Omega \) is the static input space \( V \cap \mathcal{E}(R^+) \) and where \( \delta_\Omega^* \) is the infinitesimal generator of the \( \sigma_\Omega \)-semigroup \( (\sigma_\Omega^t)_{t \geq 0} \) on the static output space \( \Gamma = V/(V \cap \mathcal{E}(R^*)) \).

**Remark 5.21.1** Both conditions of Definition 5.21 have only consequences for the static output space \( \Gamma \). Condition 5.21.i ensures that \( \Gamma \) is an F-space. Condition 5.21.ii ensures that the \( \sigma_\infty \)-domain of the infinitesimal generator of the translation semigroup \( (\sigma_\Omega^t)_{t \geq 0} \) on \( \Gamma \) equals \( \mathcal{E}(R^*) \).

**Assumption 5.22** In the remainder of this thesis, we will consider only static SISO-systems \( (V, \Omega, \Gamma, f_{\text{stat}}) \) with \( V \in \mathcal{C} \).

Static signal spaces contain the smooth static signal spaces densely.

**Lemma 5.23** Let \( (V, \Omega, \Gamma, f_{\text{stat}}) \) be a static SISO-system. Then

\[ D(R^-) \hookrightarrow \Omega \quad \text{and} \quad \mathcal{E}(R^+) \hookrightarrow \Gamma, \]

where the inclusions are dense.

**Proof.**
The proof of Lemma 5.23 is straightforward. This result is also true for translatable strict LF-spaces which do not belong to the class \( \mathcal{C} \).

For static SISO-systems \( \Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}}) \), the input/output mapping is uniquely determined by the smooth signals in \( \Sigma_{\text{stat}} \).

**Theorem 5.24** Let \( (V, \Omega, \Gamma, f_{\text{stat}}) \) be a static SISO-system. Identify \( \text{Dom}^\omega(\delta_\Omega^-) \) and \( \text{Dom}^\omega(\delta_\Omega^+ \mathcal{E}(R^+)

The restricted input/output mapping \( f_{\text{stat}}^\omega : D(R^-) \to \mathcal{E}(R^+) \) satisfies

i. \( f_{\text{stat}}^\omega \) is linear and continuous.

ii. \( \sigma_\Omega^- \circ f_{\text{stat}}^\omega = f_{\text{stat}}^\omega \circ \sigma_\Omega^- \) for all \( t \geq 0 \).

Put differently, \( f_{\text{stat}} \) is the unique extension of the static input/output mapping \( f_{\text{stat}}^\omega \) of the static SISO-system \( (D_\omega(R), D(R^-), \mathcal{E}(R^+), f_{\text{stat}}^\omega) \).

**Proof.**

i. The assumption that \( f_{\text{stat}}^\omega \) is linear is obvious. The continuity of \( f_{\text{stat}}^\omega \) is a direct consequence of the embeddings \( D(R^-) \hookrightarrow \Omega \) and \( \mathcal{E}(R^+) \hookrightarrow \Gamma \) (cf. Lemma 5.23), the continuity of \( f_{\text{stat}} \) and the Closed Graph Theorem.

ii. This is a consequence of the definition of a static input/output mapping.

The restricted input/output mapping \( f_{\text{stat}}^\omega : D(R^-) \to \mathcal{E}(R^+) \) of Theorem 5.24 is called the smooth static input/output mapping. From Theorem 5.24, we deduce the following characterization of smooth static input/output mappings.
Proposition 5.25 Let $(V, \Omega, \Gamma, f_{\text{stat}})$ be a static SISO-system. Then, the smooth static input/output mapping $f_{\text{stat}}^{\text{re}} : \mathcal{D}(\mathbb{R}^-) \to \mathcal{E}(\mathbb{R}^+)$ has the form
\[ f_{\text{stat}}^{\text{re}}(\psi)(t) = F(\sigma^-_t \psi) \quad (\psi \in \mathcal{D}(\mathbb{R}^-), t \geq 0), \tag{5.16} \]
where $F \in \mathcal{D}'(\mathbb{R})$. The distribution $F$ corresponds to $f_{\text{stat}}^{\text{re}}$ uniquely up to distributions with compact support in $[0, \infty)$. Conversely, every $F \in \mathcal{D}'(\mathbb{R})$ defines a smooth static input/output mapping by (5.16).

Proof.
Define the linear functional $F^-$ on $\mathcal{D}(\mathbb{R}^-)$ by
\[ F^-(\psi) := f_{\text{stat}}^{\text{re}}(\psi)(0) \quad (\psi \in \mathcal{D}(\mathbb{R}^-)). \]

Then, $F^-$ is continuous. So, applying the Hahn-Banach Theorem (Lemma 1.5.ii), there exists $F \in \mathcal{D}'(\mathbb{R})$ such that $F|_{\mathcal{D}(\mathbb{R}^-)} = F^-$. Next, let $F_1, F_2 \in \mathcal{D}'(\mathbb{R})$ be such that $F_1|_{\mathcal{D}(\mathbb{R}^-)} = F_2|_{\mathcal{D}(\mathbb{R}^-)}$. Then, supp$(F_1 - F_2) \subseteq [0, \infty)$.

The proof of the last assertion is straightforward.

Remark 5.25.1 The distribution $F$ in Proposition 5.25 can be taken with support in $(-\infty, 0]$.

Smooth static input/output mappings are convolution operators: Define $G \in \mathcal{D}'(\mathbb{R})$ by $G := \tilde{F}$, i.e. $G(\phi) = F(\dot{\phi})$, where $F$ as in Proposition 5.25. Then $G \in \mathcal{D}'(\mathbb{R})$ and
\[ F(\sigma^-_t \psi) = (G * \psi)(t) \quad (t \geq 0, \psi \in \mathcal{D}(\mathbb{R}^-)). \]

We obtained the following analogue of Proposition 5.25.

Proposition 5.26 Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a static SISO-system. Then its smooth static input/output mapping $f_{\text{stat}}^{\text{re}} : \mathcal{D}(\mathbb{R}^-) \to \mathcal{E}(\mathbb{R}^+)$ has the following form
\[ f_{\text{stat}}^{\text{re}}(\psi) = (G * \psi)|_{[0, \infty]} \quad (\psi \in \mathcal{D}(\mathbb{R}^-)), \tag{5.17} \]
for some $G \in \mathcal{D}'(\mathbb{R})$. The distribution $G$ is unique up to distributions with compact support in $(-\infty, 0]$. Conversely, for every $G \in \mathcal{D}'(\mathbb{R})$ defines by (5.17) a static input/output mapping.

The distribution $G \in \mathcal{D}'(\mathbb{R})$ can be taken with support in $[0, \infty)$.

Since the characteristic distribution $G$ of Proposition 5.26 is not unique, we cannot speak of the "impulse response" $G$ of $\Sigma_{\text{stat}}$. Requiring that supp$(G) \subseteq [0, \infty)$ does not solve this problem, since the delta distribution (and its distributional derivatives) $\delta_0$ can always be added.

Definition 5.27 A smooth static input/output mapping $f_{\text{stat}}^{\text{re}} : \mathcal{D}(\mathbb{R}^-) \to \mathcal{E}(\mathbb{R}^+)$ is called strictly causal if the distribution $F \in \mathcal{D}'(\mathbb{R})$ in Proposition 5.25, or equivalently the distribution $G \in \mathcal{D}'(\mathbb{R})$ in Proposition 5.26, can be chosen regular in a neighbourhood of 0.
5.1. DESCRIPTIONS OF INPUT/OUTPUT SYSTEMS

Again, the characteristic distribution $G \in \mathcal{D}'(R)$ for a strictly causal input/output mapping can be taken regular in a neighbourhood of 0 and with support in $[0, \infty)$. The latter distribution is unique.

Definition 5.28 A static SISO-system $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ is called strictly causal if its smooth static input/output mapping $f_{\text{stat}} : \mathcal{D}(R^+) \to \mathcal{E}(R^+)$ is strictly causal. The impulse response of $\Sigma_{\text{stat}}$ is the (unique) distribution $G \in \mathcal{D}'(R)$, which is regular in a neighbourhood of 0 and with supp$(G) \subseteq [0, \infty)$, related to $f_{\text{stat}}$ in the sense of Proposition 5.25.

By definition, a static SISO-system $\Sigma_{\text{stat}}$ is strictly causal if and only if its smooth subsystem $\Sigma_{\text{stat}}^\gamma$ is strictly causal.

5.1.3 Connection working mode - static mode

In the previous two subsections, we introduced two types of SISO-systems, one indicated by the pair $(V, f_\Sigma)$, the other indicated by the quadruple $(V, \Omega, \Gamma, f_{\text{stat}})$. In this subsection, we show that each strictly causal SISO-system can be described in terms of both types. As a consequence, in addition to the working mode of a SISO-system (cf. Definition 5.1), we can speak of the static mode of a SISO-system. In particular, the static mode of a system $\Sigma$ will be the static SISO-system related to $\Sigma$.

Each causal SISO-system is related to a static SISO-system.

Theorem 5.29 Let $\Sigma = (V, f_\Sigma)$ be a causal SISO-system, where $V \in \mathcal{C}$. Let $\Omega_\Sigma = V \cap \mathcal{E}'(R^+)$ and let $\Gamma_\Sigma = V / \Omega_\Sigma$. Define $f_{\text{stat}} : \Omega_\Sigma \to \Gamma_\Sigma$ by

$$f_{\text{stat}} := \phi \circ f_\Sigma \circ i,$$

where $i$ is the canonical injection from $\Omega_\Sigma$ into $V$ and where $\phi : V \to \Gamma_\Sigma$ denotes quotient mapping from $V$ into $\Gamma_\Sigma$. Then the quadruple $(V, \Omega_\Sigma, \Gamma_\Sigma, f_{\text{stat}})$ is a static SISO-system, $\Sigma_{\text{stat}}$. If $\Sigma$ is strictly causal, then $\Sigma_{\text{stat}}$ is strictly causal. In the latter case, the impulse response of $\Sigma$ equals the impulse response of $\Sigma_{\text{stat}}$.

Proof.

Equip $\Omega_\Sigma$ and $\Gamma_\Sigma$ with the usual strict LF-topologies. Let $(\sigma_t^{-})_{t \geq 0}$ and $(\sigma_t^{+})_{t \geq 0}$ be the translation semigroups on $\Omega_\Sigma$ and $\Gamma_\Sigma$. Then the mapping $f_{\text{stat}} = \phi \circ f_\Sigma \circ i$ is a continuous, linear mapping from $\Omega_\Sigma$ into $\Gamma_\Sigma$, satisfying for all $t \geq 0$

$$f_{\text{stat}} \circ \sigma_t^{-} = \phi \circ f_\Sigma \circ \sigma_t^{-} \circ i = \phi \circ \sigma_t \circ f_\Sigma \circ i = \sigma_t^{+} \circ f_{\text{stat}}.$$

So, $f_{\text{stat}}$ is a static input/output mapping. We conclude that $(V, \Omega_\Sigma, \Gamma_\Sigma, f_{\text{stat}})$ is a static SISO-system.

Next, assume that $\Sigma$ is strictly causal. Then the impulse response $G$ of $\Sigma$ is regular in a neighbourhood of 0 with supp$(G) \subseteq [0, \infty)$. Let $f_{\text{stat}}$ be the smooth static input/output mapping of $\Sigma_{\text{stat}}$. Then for each $\psi \in \mathcal{D}(R^+)$, we have

$$f_{\text{stat}}(\psi) = f_\Sigma(\psi)|_{[0,\infty)} = (G * \psi)|_{[0,\infty)}.$$

We conclude that $G$ is the impulse response of $\Sigma_{\text{stat}}$. So, $\Sigma_{\text{stat}}$ is strictly causal and the impulse response of $\Sigma$ equals the impulse response of $\Sigma_{\text{stat}}$.

We call the static SISO-system $\Sigma_{\text{stat}}$ of Theorem 5.29 the static mode description of $\Sigma$.

The following result shows that there is an one-one relation between smooth strictly causal SISO-systems and smooth strictly causal static SISO-systems.

**Lemma 5.30** Let $\Sigma_{\text{stat}} = (\mathcal{D}_+(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_{\text{stat}})$ be a smooth strictly causal static SISO-system. Then, there exists a unique smooth strictly causal SISO-system $\Sigma = (\mathcal{D}_+(\mathbb{R}), f_\Sigma)$ such that the following diagram

\[
\begin{array}{cccc}
\mathcal{D}_+(\mathbb{R}) & \xrightarrow{f_\Sigma} & \mathcal{D}_+(\mathbb{R}) \\
\downarrow i & & \downarrow \phi \\
\mathcal{D}(\mathbb{R}^-) & \xrightarrow{f_{\text{stat}}} & \mathcal{E}(\mathbb{R}^+) \\
\end{array}
\]

commutes. Here $i : \mathcal{D}(\mathbb{R}^-) \rightarrow \mathcal{D}_+(\mathbb{R})$ is the canonical injection from $\mathcal{D}(\mathbb{R}^-)$ into $\mathcal{D}_+(\mathbb{R})$ and $\phi : \mathcal{D}_+(\mathbb{R}) \rightarrow \mathcal{E}(\mathbb{R}^+)$ denotes quotient mapping from $\mathcal{D}_+(\mathbb{R})$ into $\mathcal{E}(\mathbb{R}^+)$. 

**Proof.**

Let $G$ be the impulse response of $\Sigma_{\text{stat}}$. Define the mapping $f_\Sigma : \mathcal{D}_+(\mathbb{R}) \rightarrow \mathcal{D}_+(\mathbb{R})$ by

\[f_\Sigma(\psi) := G \ast \psi \quad (\psi \in \mathcal{D}_+(\mathbb{R})).\]

Then, $f_\Sigma$ is continuous and $(\sigma_t)_{t \in \mathbb{R}}$-invariant on $\mathcal{D}_+(\mathbb{R})$ (see Proposition 5.2). Correspondingly, the pair $(\mathcal{D}_+(\mathbb{R}), f_\Sigma)$ defines a (strictly causal) SISO-system $\Sigma$ with impulse response $G$. So, the quadruple $(\mathcal{D}_+(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_\Sigma \circ i)$ defines a static SISO-system $\Sigma_{\text{stat}}$, with impulse response $G$ by Theorem 5.29. We conclude that $\Sigma_{\text{stat}} = \Sigma_{\text{stat}}$, so that the diagram (5.1) commutes.

The following (simple) example shows that the condition of strict causality can not be omitted in Lemma 5.30.

**Example 5.31** Let $I$ be the identity mapping on $\mathcal{D}_+(\mathbb{R})$ and let $N$ be the null mapping on $\mathcal{D}_+(\mathbb{R})$. Since $\phi(\mathcal{D}(\mathbb{R}^-)) = \{0\}$, we have $\phi \circ I \circ i = 0$. As a consequence, the causal SISO-systems $\Sigma_I = (\mathcal{D}_+(\mathbb{R}), I)$ and $\Sigma_N = (\mathcal{D}_+(\mathbb{R}), N)$ correspond to the same static SISO-system $(\mathcal{D}_+(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), N_{\text{stat}})$, where $N_{\text{stat}}(\psi) = 0$ for each $\psi \in \mathcal{D}(\mathbb{R}^-)$.
5.1. Descriptions of Input/Output Systems

We conclude that there can be loss of information in the static mode description of a causal SISO-system, namely the information in the point zero of the impulse response of the system. This observation was overlooked by Yamamoto in [Y2].

For general strictly causal static SISO-systems $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ the assertion analogous to Lemma 5.30 is not necessarily true.

**Theorem 5.32** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system. Then, there exists at most one strictly causal SISO-system $\Sigma = (V, f_{\Sigma})$ such that the following diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f_{\Sigma}} & V \\
\downarrow i & & \downarrow \phi \\
\Omega & \xrightarrow{f_{\text{stat}}} & \Gamma \\
\end{array}
$$

commutes. Here $i : \Omega \to V$ denotes the canonical injection from $\Omega$ into $V$ and $\phi : V \to \Gamma$ denotes quotient mapping from $V$ into $\Gamma$.

If the strictly causal SISO-system $\Sigma = (V, f_{\Sigma})$ exists, then the impulse response $G$ of $\Sigma_{\text{stat}}$ satisfies

$$G \ast x \in V \quad (5.18)$$

for each $x \in V$. Conversely, if the impulse response $G$ of $\Sigma_{\text{stat}}$ satisfies the condition (5.18) for each $x \in V$, then the existence of the strictly causal SISO-system $\Sigma = (V, f_{\Sigma})$ is guaranteed.

**Proof.**

Let $\Sigma_1 = (V, f_{\Sigma_1})$ and $\Sigma_2 = (V, f_{\Sigma_2})$ be strictly causal SISO-systems such that the diagram (5.2) commutes. By Theorem 5.29, the impulse responses $G_1$ and $G_2$ of $\Sigma_1$ and $\Sigma_2$ equal the impulse response of $\Sigma_{\text{stat}}$. So, $G_1 = G_2$ and $f_{\text{stat}}(x) = G_1 \ast x = G_2 \ast x = f_{\Sigma_2}(x)$ for each $x \in V$. We conclude that $\Sigma_1 = \Sigma_2$. Moreover, since $f_{\Sigma_2}$ is a mapping from $V$ into $V$ by definition, we observe that $G_1$ satisfies condition (5.18).

Next, suppose the impulse response $G$ of $\Sigma_{\text{stat}}$ satisfies the condition (5.18). Then, the mapping $f_{\Sigma} : V \to V$, defined by

$$f_{\Sigma}(x) := G \ast x \quad (x \in V),$$

is continuous and $(\sigma_t)_{t \in \mathbb{R}}$-invariant on $V$ (see Proposition 5.2). Correspondingly, the pair $(V, f_{\Sigma})$ defines a (strictly causal) SISO-system $\Sigma$ with impulse response $G$. By Theorem 5.29, the quadruple $(V, \Omega, f_{\Sigma}, \phi \circ f_{\Sigma} \circ i)$ is a static SISO-system $\Sigma_{\text{stat},1}$ with impulse response $G$. We conclude that $\Sigma_{\text{stat},1} = \Sigma_{\text{stat}}$, so that the diagram (5.2)
Chapter 5. Factorizations of input/output mappings

commutes.

In [6e2], Proposition 5.4, the author proved that each \( \mu \in \mathcal{M}_4(\mathcal{R}) \) satisfies condition 5.18. As a consequence, we have that for each strictly causal static SISO-systems with impulse response in \( \mathcal{M}_4(\mathcal{R}) \), the existence of a strictly causal SISO-system \( \Sigma \), such that the diagram (5.2) commutes, is guaranteed. This captures a result due to Yamamoto, [Y2], for SISO-systems admitting \( L^1_{loc} (\mathcal{R}) \)-signals (see also Theorem 4.19).

5.2 Factorizations: General Theory

In the previous section, we described SISO-systems in terms of the relation between input signals and output signals. Since these signals can be measured, at least to some extent, they are called external signals. Correspondingly, the working mode description and the static mode description of systems from §5.1.1 and §5.1.2 are called external descriptions of systems.

Often, system theoreticians consider the so-called internal description of a system, i.e., the form in which most systems appear when arising from modelling a (physical) problem. In particular, linear time-invariant systems of the following form are studied

\[
\begin{align*}
\dot{x}(t) &= A(x(t)) + B(u(t)) & t > 0, x(0) = x_0, \\
y(t) &= C(x(t))
\end{align*}
\]  

(5.19)

where at each time \( t_0 \), \( x(t_0) \) denotes the state of the system, \( u(t_0) \) the input and \( y(t_0) \) the output. Furthermore, \( A, B \) and \( C \) are linear mappings. In this setting the state space \( X \), the vector space in which the state at each time takes its values, can be infinite dimensional.

The existence of a "state" is the essence of the internal description of systems. This variable cannot always be measured and it carries internal information of the system. The state of a system contains at any moment the information about the past input/output behaviour of the system that is relevant for the future input/output behaviour. So, intuitively, the state of a system contains all information about the past of the system that is relevant for its future.

Realization theories deal with the connection of internal descriptions and external descriptions of systems. From the system (5.19), we can heuristically deduce the following relation between the input \( u \), the state \( x \) the output \( y \):

\[
\begin{align*}
x(t) &= e^{tA}x_0 + \int_0^t e^{(t-s)A}B(u(s))\,ds & t > 0, x(0) = x_0, \\
y(t) &= C(x(t))
\end{align*}
\]  

(5.20)

The first equation of (5.20) can be interpreted as a state transition equation. By (5.20), the output \( y \) can heuristically be expressed in terms of the input \( u \):

\[
y(t) = Ce^{tA}x_0 + \int_0^t Ce^{(t-s)A}B(u(s))\,ds & (t > 0)
\]  

(5.21)
5.2. Factorizations: general theory

So, given the internal description (5.19) of a system $\Sigma$, an external description (input/output mapping) can be derived from equality (5.21) taking $x_0 = 0$. The problem of realization is the following: Given its external description of a system $\Sigma$, does $\Sigma$ have an internal description of the form (5.19)? And if so, what is a canonical internal description, i.e. $\alpha$, to some extent, unique one?

In the general setting of §5.1, we can not solve the realization problem entirely. The operator $A$ can be found, but we can not always find the operator $B$, which is of great importance for system control purposes. Instead of the realization problem, we consider a related problem, the so-called factorization problem: “Given a strictly causal static input/output mapping, does there exist a space $X$, a mapping $g : \Omega \rightarrow X$ and a mapping $h : X \rightarrow \Gamma$, such that $f = h \circ g$?” And if so, what is a canonical factorization, i.e. $\alpha$, to some extent, unique one?

5.2.1 Factorizations of static input/output mappings

We employ the following concept of factorization due to Yamamoto [Y1], Definition 3.28.

Definition 5.33 Let $\Sigma_{stat} = (V, \Omega, \Gamma, f_{stat})$ be a strictly causal static SISO-system. A factorization of $\Sigma_{stat}$ is a quadruple $(X, (\Phi_t)_{t\geq0}, g, h)$, such that

i. $X$ is a complete locally convex topological vector space,

ii. $g : \Omega \rightarrow X$ and $h : X \rightarrow \Gamma$ are continuous linear mappings,

iii. $(\Phi_t)_{t\geq0}$ is a strongly continuous semigroup on $X$,

iv. $g \circ \sigma_t^{-1} = \Phi_t \circ g$ and $h \circ \Phi_t = \sigma_t \circ h$ for all $t \geq 0$ and

v. $f_{stat} = h \circ g$.

For a factorization $(X, (\Phi_t)_{t\geq0}, g, h)$ of $\Sigma_{stat}$, the vector space $X$ is called state space, vectors $x \in X$ are called state vectors, the $\sigma_t$-semigroup $(\Phi_t)_{t\geq0}$ is called the associated semigroup, the mapping $g : \Omega \rightarrow X$ is called the reachability mapping and the mapping $h : X \rightarrow \Gamma$ is called the observability mapping.

A factorization $(X, (\Phi_t)_{t\geq0}, g, h)$ of a strictly causal static SISO-system $\Sigma$ is called reachable if $g(\Omega) = X$, quasi-reachable if $g(\Omega)$ is dense in $X$, observable if $h$ is injective, and topologically observable if the mapping $h : X \rightarrow h(X)$ is a homeomorphism, where $h(X)$ is equipped with induced $\Gamma$-topology.

The existence of factorizations of a strictly causal static SISO-system is guaranteed.

Lemma 5.34 Let $\Sigma_{stat} = (V, \Omega, \Gamma, f_{stat})$ be a strictly causal static SISO-system. Then, the quadruple Fac = $(\tilde{f}_{stat}(\Omega), (\sigma_t^*)_{t\geq0}, f_{stat}, j)$ is a factorization of $\Sigma_{stat}$. Here $j : \mathfrak{R}(\tilde{f}_{stat}) \rightarrow \Gamma$ is the inclusion and the closure $\tilde{f}_{stat}(\Omega)$ is in $\Gamma$. The factorization Fac is quasi-reachable and topologically observable.
The proof of Lemma 5.34 is a straightforward generalization of the proof given by Yamamoto in [Y1], Theorem 5.2, for the case $V = L^2_{loc}(\mathbb{R})$.

Next, we deal with the problem of uniqueness of factorizations of strictly causal static SISO-systems.

**Definition 5.35** Let $(V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system. Let $\text{Fac}_1 = (X_1, (\Phi_{1t})_{t \geq 0}, g_1, h_1)$ and $\text{Fac}_2 = (X_2, (\Phi_{2t})_{t \geq 0}, g_2, h_2)$ be factorizations of $\Sigma_{\text{stat}}$. A morphism from $\text{Fac}_1$ into $\text{Fac}_2$ is a continuous linear mapping $T : X_1 \to X_2$ such that

1. $T \circ g_1 = g_2$,
2. $h_2 \circ T = h_1$, and
3. $\Phi_{2t} \circ T = T \circ \Phi_{1t}$ for each $t \geq 0$.

We say that $\text{Fac}_1$ is isomorphic to $\text{Fac}_2$ if $T$ is a homeomorphism.

It is readily checked that the identity and the composition of two (homeo-)morphisms are (homeo-)morphisms. Furthermore, if $\text{Fac}_1$ is isomorphic to $\text{Fac}_2$, then $\text{Fac}_2$ is isomorphic to $\text{Fac}_1$. Hence, isomorphisms yield a equivalence relation on the collection of all factorizations of a strictly causal static SISO-system $(V, \Omega, \Gamma, f_{\text{stat}})$. In case $\text{Fac}_1$ is quasi-reachable, then the third condition of Definition 5.35 is superfluous.

In searching for uniqueness conditions (up to isomorphisms) on factorizations, we follow the approach of Yamamoto [Y1].

**Definition 5.36** A factorization $(X, \Phi, g, h)$ of a strictly causal static SISO-system $f_{\text{stat}}$ is canonical if it is quasi-reachable and topologically observable.

Similar to Yamamoto, [Y1], Theorem 5.5, we have the following result.

**Theorem 5.37** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system and let $\text{Fac}_1 = (X_1, (\Phi_{1t})_{t \geq 0}, g_1, h_1)$ and $\text{Fac}_2 = (X_2, (\Phi_{2t})_{t \geq 0}, g_2, h_2)$ be two canonical factorizations of $\Sigma_{\text{stat}}$. Then $\text{Fac}_1$ and $\text{Fac}_2$ are isomorphic.

**Proof.**

Define the morphism $T : X_1 \to X_2$ be $T := h_2^{-1} \circ h_1$. Then, $T$ is a homeomorphism.

So, by Lemma 5.34 the existence of an unique canonical factorization is guaranteed.

Canonical state spaces have the following property.

**Lemma 5.38** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system with canonical factorization $(X, (\Phi_t)_{t \geq 0}, g, h)$. Then $X$ is homeomorphic to a closed, $(\sigma^*_{t})_{t \geq 0}$-invariant subspace of $\Gamma$.

**Proof.**

Since $\sigma^*_{t} \circ f_{\text{stat}} = f_{\text{stat}} \circ \sigma^*_{t}$, we have that $f_{\text{stat}}(\Omega)$ is a $(\sigma^*_{t})_{t \geq 0}$-invariant subspace of $\Gamma$. Hence, $f_{\text{stat}}(\Omega)$ is a closed $(\sigma^*_{t})_{t \geq 0}$-invariant subspace of $\Gamma$. Since every canonical state space $X$ for $\Sigma_{\text{stat}}$ is homeomorphic to $f_{\text{stat}}(\Omega)$ by Theorem 5.37, the assertion follows.

By Corollary 2.50, we have that each closed $(\sigma^*_{t})_{t \geq 0}$-invariant subspace of $\Gamma$ is the $\Gamma$-closure of a closed $(\sigma^*_{t})_{t \geq 0}$-invariant subspace of $\tilde{E}(\mathbb{R}^+)$ ($= \text{Dom}^{\infty}(\delta^+_t)$).
5.2. Factorizations: general theory

Proposition 5.39 Let $\Sigma_{\text{stat}} = (V, Q, \Omega, f_{\text{stat}})$ be a strictly causal static SISO-system with canonical factorization $(X, (\Phi_t)_{t \geq 0}, g, h)$. Then $X$ is homeomorphic to the $C$-closure of a closed, $(\sigma_t^*)_{t \geq 0}$-invariant subspace of $C(R^+)$.

We end this subsection with a nice consequence of the topological observability property of canonical factorizations.

Lemma 5.40 Let $\Sigma_{\text{stat}} = (V, Q, \Omega, f_{\text{stat}})$ be a strictly causal static SISO-system with canonical factorization $(X, (\Phi_t)_{t \geq 0}, g, h)$. Then $X$ is an $F$-space.

Proof. Since $V \subset C$ by Assumption 5.22, we have that $C$ is an $F$-space (see Proposition 5.12). Moreover, since the mapping $h : X \rightarrow h(X)$ is a homeomorphism, where $h(X)$ is equipped with induced $C$-topology, $h(X)$ is a closed subspace of $C$, hence an $F$-space. We conclude that $X$ is an $F$-space.

In [Y1] and [Y2], Yamamoto developed a realization theory for SISO-systems accepting $L^2_{\text{loc}}(R)$-signals based on the factorization theory presented here in the special case of $V = L^2_{\text{loc}}(R)$. By demanding that an input/output mapping is a continuous linear $(\sigma_t)_{t \in R}$-invariant mapping on $L^2_{\text{loc}}(R)$, for which $C_{\text{a}}(R)$ is an invariant subspace, Yamamoto restricted himself to causal SISO-systems with impulse responses in $M_{\text{a}}(R)$ (see Theorem 4.19). Given the canonical factorization $(X, (\sigma_t^*)_{t \geq 0}, f_{\text{stat}}, \cdot)$ of the static SISO-system $\Sigma_{\text{stat}}$ with impulse response $\mu \in M_{\text{a}}(R)$, he defined the state transition mapping $\phi(t, \cdot, \cdot) : X \times L^2([0, t)) \rightarrow X$ for each $t \geq 0$ by

$$\phi(t, x, u) := \Phi_t x + f_{\text{stat}}(\sigma_t u) = \Phi_t x + (\mu * \sigma_t u)|_{[0, t)}.$$

(5.22)

Intuitively, $\phi(t, x, u)$ is the state at time $t$ caused by an initial state $x$ and input $u$ during the time span $[0, t)$. In our general setting, the state transition mapping (5.22) can not always be introduced following Yamamoto's approach straightforwardly. For systems with $L^2_{\text{loc}}(R)$-signals nothing really changes. However, considering a system with $D_{\text{a}}(R)$-signals, we observe that when taking a smooth input $u$ in (5.22), the term $(\mu * \sigma_t u)|_{[0, t)}$ is a piecewise smooth function, so not an element in $C(R^+)$, unless the measure $\mu$ is brought about by a regular distribution. So, the canonical state space $X$ arising from our factorization theory, thus a subspace of $C(R^+)$, may be too small. Restricting to systems with regular impulse response would solve this problem, but has the disadvantage that interesting systems like distributed time-delay systems are excluded. Therefore, we have chosen to disregard the latter possibility to develop a realization theory. Nevertheless, for systems with finite-dimensional canonical state space, studied in the next subsection, a state space realization can be, and is, given indeed.

In [Y2], Yamamoto showed that for systems with impulse response in $H^1_{\text{loc}}(R)$ a state space realization can be found using the state transition mapping $\phi(t, \cdot, \cdot)$. The reader interested in a realization theory for systems with distributional signals and smooth impulse responses, we refer to Kalman and Hautus [KH].
5.2.2 Finite dimensional factorizations

In this subsection, we investigate the class of strictly causal static systems \((V, \Omega, \Gamma, f_{\text{stat}})\) which have a finite-dimensional canonical factorization \((X, (\Phi_i)_{i \geq 0}, g, h)\), i.e. \(\dim(X) < \infty\). It turns out that finite-dimensional state spaces \(X\) are spanned by exponential-polynomials and impulse responses of systems having a finite-dimensional canonical state space are regular distributions, which are the linear combination exponential-polynomials. Furthermore, we deduce for strictly causal static systems \(\Sigma_{\text{stat}}\) with finite-dimensional canonical factorization a state space realization.

**Definition 5.41** Let \(\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})\) be a strictly causal static SISO-system. A factorization \((X, (\Phi_i)_{i \geq 0}, g, h)\) of \(\Sigma_{\text{stat}}\) is called finite-dimensional if \(\dim(X) < \infty\).

Let \(\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})\) be a strictly causal static SISO-system with finite-dimensional factorization \((X, (\Phi_i)_{i \geq 0}, g, h)\). Then \(f_{\text{stat}}(\Omega) = (h \circ g)(\Omega) \subseteq h(X)\), so \(\dim(f_{\text{stat}}(\Omega)) < \infty\). So by Proposition 5.39, \(f_{\text{stat}}(\Omega) = f_{\text{stat}}(\mathcal{D}(\mathbb{R}^+))\). As a consequence, any canonical factorization of \(\Sigma_{\text{stat}}\) is finite-dimensional and any canonical factorization of the smooth subsystem \(\Sigma_{\text{stat}}^\infty = (\mathcal{D}_s(\mathbb{R}), \mathcal{D}(\mathbb{R}^+), \mathcal{E}(\mathbb{R}^+), f_{\text{stat}}^\infty)\) is finite-dimensional. Applying Lemma 5.38, we obtain the following result.

**Lemma 5.42** Let \(\Sigma_{\text{stat}}\) be a strictly causal static SISO-system with finite-dimensional canonical factorization \((X, (\Phi_i)_{i \geq 0}, g, h)\). Then the state space \(X\) is isomorphic to a finite-dimensional \((\mathbb{C}^n)_{i \geq 0}\)-invariant subspace of \(\mathcal{E}(\mathbb{R}^+)\).

Now, let \(M\) be a finite-dimensional \((\mathbb{C}^n)_{i \geq 0}\)-invariant subspace of \(\mathcal{E}(\mathbb{R}^+)\). Since the differentiation operator \(\frac{d}{dt}\) is the (everywhere defined) infinitesimal generator of \((\mathbb{C}^n)_{i \geq 0}\) on \(\mathcal{E}(\mathbb{R}^+)\), we have that \(\frac{d}{dt}(M) \subseteq M\). Let \(\{\lambda_1, \ldots, \lambda_k\}\) be the spectrum of the operator \(\frac{d}{dt}\) restricted to \(M\) with respective multiplicities \(m_1, \ldots, m_k\). Then, the Jordan Decomposition Theorem (see Halmos [Hal], §58) yields that

\[
M = \bigoplus_{j=1}^k \ker \left( \left( \frac{d}{dt} - \lambda_j I \right)^{m_j} \right) = \ker \left( P \left( \frac{d}{dt} \right) \right),
\]

where \(P\) is the annihilating polynomial of \(\frac{d}{dt}|_M\) with zeros \(\{\lambda_1, \ldots, \lambda_k\}\) and multiplicities \(m_1, \ldots, m_k\). By definition, \(P\) has highest order term 1.

Defining \(u_{\lambda,i} \in \mathcal{E}(\mathbb{R}^+)\) for each \(\lambda \in \mathbb{C}, i \in \mathbb{N}_0\) by

\[
u_{\lambda,i}(t) := t^i \cdot e^{\lambda t}, \quad (t \geq 0),
\]

we see that (5.23) yields that

\[
M = \text{span}\{u_{\lambda,j} | j = 1, \ldots, m, i = 0, 1, \ldots, m_j - 1\},
\]

i.e. \(M\) is the linear span of exponential-polynomials.

Returning to finite-dimensional canonical factorizations of a strictly causal static SISO-system \(\Sigma_{\text{stat}}\), we have obtained the following result.
5.2. Factorizations: general theory

Proposition 5.43 Let \( \Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}}) \) be a strictly causal static SISO-system with finite-dimensional canonical factorization \( (X, (\Phi_i)_{i \geq 0}, g, h) \). Then, there are \( m \in \mathbb{N} \) and \( (\lambda_1, r_1), \ldots, (\lambda_m, r_m) \in \mathbb{C} \times \mathbb{N} \) such that

\[
X = \text{span}\{h^{-1}(s_{\lambda_j, j}) \mid j = 1, \ldots, m, i = 0, 1, \ldots, m_j - 1\}.
\]

Next, we focus on the impulse response \( G \) of a strictly causal SISO-system \( \Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}}) \) with a finite-dimensional canonical factorization. By (5.23), there is an unique annihilating polynomial \( P_G \), such that for all \( \psi \in \mathcal{D}(\mathbb{R}^+) \) and each \( t \geq 0 \)

\[
0 = P_G \left( \frac{d}{dt} \right) (f_{\text{stat}}(\psi))(t) = P_G \left( \frac{d}{dt} \right) (G \ast \psi)(t) = (P_G(D)(G \ast \psi))(t),
\]

where \( D \) is the distributional differentiation operator. In particular, we have that

\[
\left( P_G(D)G \right)(\psi) = 0,
\]

(5.26)

for each \( \psi \in \mathcal{D}(\mathbb{R}^+) \). Since \( G \in \mathcal{D}'(\mathbb{R}) \) with \( \text{supp}(G) \subseteq [0, \infty) \), this yields that \( \text{supp}(P_G(D)G) \subseteq \{0\} \). Distributions with point support are well characterized. In fact, there exists a polynomial \( Q \) such that (see [Schw2], p.100)

\[
P_G(D)G = Q(D)\delta_0,
\]

(5.27)

where the equation is in \( \mathcal{D}'(\mathbb{R}) \)-sense.

In the sequel, we deduce an explicit representation for the distribution \( G \). First, we need to introduce some technicalities.

It is readily checked that the operator \( (\frac{d}{dt} - \lambda) \) on \( \mathcal{D}_-(\mathbb{R}) \) is invertible. Indeed for each \( \lambda \in \mathbb{C} \), the inverse of \( (\frac{d}{dt} - \lambda) \) is defined by

\[
(\frac{d}{dt} - \lambda)^{-1}\psi(t) := - \int_t^\infty e^{\lambda(\tau - \cdot)} \cdot \psi(\tau) d\tau \quad (t \in \mathbb{R}, \psi \in \mathcal{D}_-(\mathbb{R})).
\]

(5.28)

Each \( (\frac{d}{dt} - \lambda)^{-1} \) is a continuous linear operator on \( \mathcal{D}_-(\mathbb{R}) \). Since the adjoint operator \( (\frac{d}{dt})^* \) equals \( -D \), the adjoint of \( -\lambda \), equals \( D + \lambda I \) is adjoint. Using an induction argument, we can obtain the following representations for the operators \( (\frac{d}{dt} - \lambda)^{-1} : = (\frac{d}{dt} - \lambda)^{-k} \) and \( (D + \lambda I)^{-1} = (D + \lambda I)^{-k} \):

Lemma 5.44 Let \( \lambda \in \mathbb{C}, k \in \mathbb{N} \). Then for each \( \psi \in \mathcal{D}_-(\mathbb{R}) \)

\[
(\frac{d}{dt} - \lambda)^{-k}\psi(t) = (-1)^k \cdot \int_t^\infty \frac{(\tau - t)^{k-1}}{(k-1)!} \cdot e^{\lambda(\tau - \cdot)} \cdot \psi(\tau) d\tau \quad (t \in \mathbb{R}),
\]

and

\[
(D + \lambda I)^{-k}\delta_0(\psi) = \int_0^\infty \frac{\tau^{k-1}}{(k-1)!} \cdot e^{-\lambda \tau} \cdot \psi(\tau) d\tau.
\]
Returning to equation (5.27), applying Lemma 5.44 yields that the degree of $P_\omega$ is strictly greater than the degree of $Q$.

**Lemma 5.45** Let $F \in \mathcal{D}_\omega(\mathbb{R})$ be such that $\text{supp}(F) \subseteq [0, \infty)$ and suppose that $F$ is regular in a neighborhood of $0$. If there are polynomials $P$ and $Q$ such that $P(D)F = Q(D)\delta_0$, then the degree of $P$ is strictly greater than the degree of $Q$.

**Proof.**
Without loss of generality, we may assume that $P(D) = \sum_{n=0}^{N} a_n D^n$ and $Q(D) = \sum_{l=0}^{L} b_l D^l$, where $a_N = 1$ and $L \geq N$. We will show that $b_l = 0$ for each $l = N, \ldots, L$.

Notice that

\[ \sum_{n=0}^{N} a_n D^{n-N}F = D^{-N}p(D)F = D^{-N}Q(D)\delta_0 \]
\[ = \sum_{l=0}^{N-1} b_l D^{l-N}\delta_0 + \sum_{l=N}^{L} b_l D^{l-N}\delta_0. \]  \hspace{1cm} (5.29)

Since $F$ is regular in a neighborhood of $0$ and $\text{supp}(F) \subseteq [0, \infty)$, there exists $\varepsilon > 0$ and $g \in L^1_{\text{loc}}(\mathbb{R})$, such that for each $\psi \in \mathcal{D}_\omega(\mathbb{R})$ with $\text{supp}(\psi) \subseteq (-\infty, \varepsilon)^c$

\[ F(\psi) = \int_{\mathbb{R}} \psi(\tau) \cdot g(\tau) \, d\tau. \]  \hspace{1cm} (5.30)

Applying Lemma 5.44 to equality (5.29), we obtain for each $\psi \in \mathcal{D}_\omega(\mathbb{R})$ with $\text{supp}(\psi) \subseteq (-\infty, \varepsilon)$

\[ D^{-N}p(D)F(\psi) \overset{[5.29]}{=} \int_{\mathbb{R}} \left( \sum_{n=0}^{N} \frac{(-1)^{N-n} a_n}{n!} D^{n-N} \psi \right)(\tau) g(\tau) \, d\tau \]
\[ = \int_{\mathbb{R}} \psi(\tau) g(\tau) \, d\tau + \int_{\mathbb{R}} g(\tau) \sum_{n=0}^{N} \frac{(\tau - \tau)^{N-n-1}}{(N-n-1)!} \cdot \psi(s) \, ds \, d\tau \]
\[ + \int_{\mathbb{R}} \psi(\tau) g(\tau) \, d\tau + \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \sum_{n=0}^{N} \frac{(\tau - \tau)^{N-n-1}}{(N-n-1)!} \cdot g(\tau) \, d\tau \right) \psi(s) \, ds. \]

We conclude that the distribution $D^{-N}p(D)F$ is regular on the interval $(-\infty, \varepsilon)$. By Lemma 5.44, we obtain for the right-hand side of (5.29)

\[ \left( \sum_{l=0}^{N-1} b_l D^{l-N}\delta_0 \right)(\psi) + \left( \sum_{l=N}^{L} b_l D^{l-N}\delta_0 \right)(\psi) = \]
\[ = \int_{0}^{\varepsilon} \sum_{l=0}^{N-1} b_l \frac{(\tau)^{l-1}}{(l-1)!} \psi(\tau) \, d\tau + \sum_{l=N}^{L} (-1)^{l-N} b_l \psi^{(l-N)}(0). \]

We see that $b_l = 0$ for each $l = N, \ldots, L$, which proves the assertion.

Applying Lemma 5.45 to the equality (5.27), $P_\omega(D)G = Q(D)\delta_0$, we have that

\[\int_{\mathbb{R}} \psi(\tau) g(\tau) \, d\tau + \int_{\mathbb{R}} g(\tau) \psi(\tau) \, d\tau = \int_{\mathbb{R}} \psi(\tau) \cdot g(\tau) \, d\tau.\]
5.2. FACTORIZATIONS: GENERAL THEORY

\text{degree}(Q) < \text{degree}(P_{k}) = \dim(X).

Suppose \( P_{k}(\lambda) = \prod_{i=1}^{b} (\lambda - \lambda_{i}^{m_{i}}) \), i.e. \( \lambda_{1}, \ldots, \lambda_{b} \) are the zeros of \( P_{k} \) with multiplicities \( m_{1}, \ldots, m_{b} \). Then there are constants \( a_{i,j} \), not necessarily non-zero, such that

\[
\frac{Q(\lambda)}{P_{k}(\lambda)} = \sum_{j=1}^{b} \sum_{i=1}^{m_{j}} a_{i,j} \cdot \frac{1}{(\lambda - \lambda_{j})^{i}}
\]  

(5.31)

Since \( P_{k}(D)^{-1} = \prod_{b=1}^{b} (D - \lambda_{i} I)^{-m_{b}} \), we have

\[
G = \prod_{b=1}^{b} (D - \lambda_{i} I)^{m_{b}} Q(D) b_{0} = \sum_{j=1}^{b} \sum_{i=1}^{m_{j}} a_{i,j} \cdot (D - \lambda_{i} I)^{j} b_{0}.
\]

So by Lemma 5.44, we have for all \( \psi \in \mathcal{D}(-\mathbb{R}) \)

\[
G(\psi) = \int_{0}^{\infty} \left( \sum_{j=1}^{b} \sum_{i=1}^{m_{j}} a_{i,j} \frac{e^{-(t-1)} \lambda^{j} \tau}{(t-1)!} \right) \psi(\tau) \, d\tau.
\]

We observe that \( G \) is a regular distribution.

**Lemma 5.46** Let \( \Sigma_{\text{ext}} = (V, \Omega, \Gamma, f_{\text{ext}}) \) be a strictly causal static SISO system with a finite-dimensional canonical factorization. Then the impulse response \( G \) is a regular distribution. In particular, \( G \) is of the form

\[
G(\psi) = \int_{\mathbb{R}} g(\tau) \psi(\tau) \, d\tau,
\]

where \( g \in L^{1}_{\text{loc}}(\mathbb{R}) \) and

\[
g(t) := \begin{cases} \sum_{j=1}^{b} \sum_{i=1}^{m_{j}} a_{i,j} t^{j-1} e^{-\lambda_{i} t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}
\]

(5.32)

for some \( \lambda_{1}, \ldots, \lambda_{b} \in \mathbb{C} \) and \( a_{i,j} \in \mathbb{C} \).

Conversely, if the impulse response of a strictly causal static SISO-system \( \Sigma_{\text{ext}} \) is a regular distribution of the form (5.32), then \( \Sigma_{\text{ext}} \) has a finite-dimensional canonical factorization.

**Proof.**

The first part of the Lemma follows from the above.

Define \( u_{i,k} \in L^{1}_{\text{loc}}(\mathbb{R}) \) for each \( \lambda \in \mathbb{C} \) each \( k \in \mathbb{N}_{0} \) by

\[
u_{i,k}(t) := \begin{cases} \sum_{j=1}^{b} \sum_{i=1}^{m_{j}} a_{i,j} t^{j-1} e^{-\lambda_{i} t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}
\]

Then for each \( \psi \in \mathcal{D}(\mathbb{R}^{-}) \) and each \( t \geq 0 \), we have

\[
(u_{i,k} * \psi)(t) = \int_{-\infty}^{0} (t - \tau)^{k} e^{\lambda_{i} (t-\tau)} \psi(\tau) \, d\tau
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \int_{-\infty}^{0} t^{k-i} e^{-\lambda_{i} \tau} \psi(\tau) \, d\tau.
\]
So, $v_{k,\psi} \in \text{span}\{u_{k,0}, \ldots, u_{k,k}\}$. So, if the impulse response $G$ of $\Sigma_{\text{stat}}$ is regular and of the form (5.32), then

$$f_{\text{stat}}(D(R^n)) \subseteq \text{span}\{u_{k,j} | i = 1, \ldots, k, j = 0, \ldots, m_j\}.$$  

We conclude that $f_{\text{stat}}(D(R^n))$ is finite-dimensional, so $\Sigma_{\text{stat}}$ has a finite-dimensional canonical factorization.

From the proof of Lemma 5.46, we obtain the following result for the canonical state space of a strictly causal static SISO-system with impulse response as in (5.32).

**Lemma 5.47** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system with impulse response $G$ as in Lemma 5.46. Then

$$f_{\text{stat}}(D(R^n)) \subseteq \ker \left( \prod_{i=1}^{k} \left( \frac{d}{dt} - \lambda_i \right)^{m_i} \right).$$

By a simple induction argument, it can be shown that, assuming that $a_{l,m_l} \neq 0$ for each $l = 1, \ldots, k$, the inclusion of Lemma 5.47 is an equality.

**Theorem 5.48** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system. Then $\Sigma_{\text{stat}}$ has a finite-dimensional canonical factorization $(X, (\Phi_1)_{l \geq 0}, g, h)$ if and only if the impulse response $G$ of $\Sigma_{\text{stat}}$ is a regular distribution. In particular, $G$ is of the form

$$G(\phi) = \int_{\mathbb{R}} g(t) \psi(t) \, dt,$$

where $g \in L^1_{\text{loc}}(\mathbb{R})$ and

$$g(t) = \begin{cases} \sum_{j=1}^{k} \sum_{l=0}^{m_l} a_{l,j} \left( \frac{d}{dt} - \lambda_j \right)^l e^{\lambda_j t}, & \text{for } t \geq 0 \\
0, & \text{for } t < 0, \end{cases}$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $a_{l,j} \in \mathbb{C}$. If $a_{l,m_l} \neq 0$ for each $j = 1, \ldots, k$, then the canonical state space $X$ of $\Sigma_{\text{stat}}$ satisfies

$$h(X) = \text{span}\{u_{k,j} | j = 1, \ldots, k, i = 0,1,\ldots,m_j-1\} = \ker \left( \prod_{j=1}^{k} \left( \frac{d}{dt} - \lambda_j \right)^{m_j} \right).$$

Systems having a finite-dimensional canonical factorizations, have also a finite dimensional realization (as is well known).

**Theorem 5.49** Let $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ be a strictly causal static SISO-system with impulse response $G$ which is a regular distribution of the form

$$G(\psi) = \int_{\mathbb{R}} g(t) \psi(t) \, dt,$$
where $g \in L^1_{\text{loc}}(\mathbb{R})$ and

$$g(t) := \begin{cases} \sum_{i=1}^{-k} \sum_{j=1}^{m_j} a_{ij} e^{\lambda_i t} e^{K_j t} & \text{for } t \geq 0 \\ \sum_{i=1}^{-k} \sum_{j=1}^{m_j} a_{ij} e^{\lambda_i t} e^{K_j t} & \text{for } t < 0, \end{cases}$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $a_{ij} \in \mathbb{C}$. If $a_{j,m_j} \neq 0$ for each $j = 1, \ldots, k$. Let $X = \mathbb{R}^n$. Then, the system $\Sigma_{\text{sys}}$ is described by the equations

$$\begin{align*}
\dot{x}(t) &= A(x(t)) + B(u(t)) & t > 0, \quad x(0) = x_0, \\
g(t) &= C(x(t)),
\end{align*}$$

which are to be read in $V$-sense, where

$$A := \begin{pmatrix}
\lambda_1 & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\lambda_k & 1 & \ddots & \ddots & 1 \\
\end{pmatrix}_{m_k \times m_k},$$

$$B = \begin{pmatrix}
0_{m_1} & \cdots & 0_{m_k}
\end{pmatrix}^T$$

and $C = (a_{1,m_1}, \ldots, a_{1,m_1}, \ldots, a_{k,m_k}, \ldots, a_{k,m_k})$. 

Proof. 
The proof follows by straightforward calculation, hence is omitted.

5.3 FACTORIZATIONS OF PSEUDO-RATIONAL INPUT/OUTPUT MAPPINGS

In the previous section, strictly causal static SISO-systems with a finite-dimensional canonical state space have been studied. We showed that this canonical state space is spanned by exponential-polynomials, and we characterized the impulse responses corresponding to such a system. In this section, we study systems with pseudo-rational impulse responses (cf. Definition 5.51). Sufficient conditions on pseudo-rational impulse responses are given such that the canonical state space for the corresponding SISO-system is spanned by exponential-polynomials.

The following definition is due to Yamamoto [Y4], Definition 2.5.
Definition 5.50 A distribution $G \in \mathcal{D}_0'(\mathbb{R})$ is called pseudo-rational if
$$G = Q^{-1} * P,$$
where $Q, P \in \mathcal{E}'(\mathbb{R}^+)$, i.e. the collection of all compactly supported distributions with support in $(-\infty, 0]$, and where $Q$ is invertible in $\mathcal{D}'_0(\mathbb{R})$ with respect to convolution.

Definition 5.51 A strictly causal static SISO-system $\Sigma_{\text{stat}}$ is called pseudo-rational if its impulse response $G \in \mathcal{D}'_0(\mathbb{R})$ is pseudo-rational.

Remark 5.51.1 In the behavioural approach, a smooth pseudo-rational SISO-system $\Sigma$ with impulse response $Q^{-1} * P$ can be expressed by the AR-like relation
$$(u, y) \in \Sigma \Leftrightarrow Q * y = P * u,$$
See, for example, Willems [Will] and Soethoudt, [Soe].

One of the advantages of pseudo-rational SISO-systems is that Fourier theory is available for the distributions $Q$ and $P$. We may speak of the transfer function of a pseudo-rational SISO-system $\Sigma$, namely if the impulse response of $\Sigma$ equals $G = Q^{-1} * P$, then $\mathcal{F}(Q^{-1})\mathcal{F}(P)$ is transfer function of $\Sigma$. In other words, pseudo-rational distributions are elements of $\mathcal{D}'_0(\mathbb{R})$ with a Fourier-Laplace transform.

Although not all static SISO-systems are pseudo-rational, delay systems are known to be pseudo-rational.

Example 5.52 Let $\Sigma_{\text{stat}} = (\mathcal{D}_0(\mathbb{R}), \mathcal{D}(\mathbb{R}^{-}), \mathcal{E}'(\mathbb{R}^+), f_{\text{stat}})$ be a strictly causal static SISO-system with impulse response $G \in \mathcal{D}_0'(\mathbb{R})$ defined by
$$G = \sum_{i=1}^{\infty} \delta_i.$$
The static input/output mapping $f_{\text{stat}}$ of $\Sigma_{\text{stat}}$ satisfies
$$f_{\text{stat}}(\psi)(t) = \sum_{i=1}^{\infty} \psi(t - i) \quad (\psi \in \mathcal{D}(\mathbb{R}^{-})).$$
$\Sigma_{\text{stat}}$ is a distributed time-delay system. Moreover, since $G$ decomposes in $Q^{-1} * P$, where $P = \delta_0 \in \mathcal{E}'(\mathbb{R}^{-})$ and $Q = \delta_{-1} - \delta_0 \in \mathcal{E}'(\mathbb{R}^{-})$, $\Sigma_{\text{stat}}$ is pseudo-rational.

Notice that, since $G * x \subseteq x$ for each $x \in V$, where $V$ any translatable strict LF-space of $\mathcal{D}_0(\mathbb{R})$-type, we could have replaced $\mathcal{D}_0(\mathbb{R})$, and correspondingly $\mathcal{D}(\mathbb{R}^{-})$ and $\mathcal{E}'(\mathbb{R}^+)$, by an arbitrary translatable strict LF-space $V$ of $\mathcal{D}_0(\mathbb{R})$-type, and corresponding $\Omega$ and $\Gamma$.

Next, we investigate the canonical state space of a pseudo-rational SISO-system. Since our aim is to classify those pseudo-rational systems, with canonical state space spanned by exponential-polynomials, we need to consider only smooth systems by Proposition 5.39.
5.3. Factorizations of pseudo-rational input/output mappings

First, we need some terminology. Let $\phi$ denote the quotient mapping from $D_+(\mathbb{R})$ into $D_+(\mathbb{R})/D(\mathbb{R}^-)$ and let the mapping $\pi : D_+(\mathbb{R})/D(\mathbb{R}^-) \to \mathcal{E}(\mathbb{R}^+)$ be the isomorphism defined by $\pi(x + D(\mathbb{R}^-)) := x|_{[0,\infty)}$ (see (3.27)). Recall from §3.3 that the operator $\sigma^+[(\sigma^*)^{-1}(F) \circ \pi]$ defines for each $F \in \mathcal{E}(\mathbb{R}^+)$ a continuous linear $(\sigma^+)^2$-invariant operator on $\mathcal{E}(\mathbb{R}^+)$. Now, let $F \in \mathcal{E}(\mathbb{R}^+)$. Then $\tilde{F} \in \mathcal{E}(\mathbb{R}^+)$, so, for each $x \in D_+(\mathbb{R})$ and $t \geq 0$, we have

$$\sigma^+[(\sigma^*)^{-1}(\tilde{F}) \circ \pi^-](x|_{[0,\infty)})(t) = \sigma^+[(\sigma^*)^{-1}(\tilde{F}) \circ \pi^-](\pi(\phi(x)))(t)$$

$$= \left((\sigma^+)^2(\tilde{F}) \circ \pi^-(\sigma^+)^2(\phi(x))\right)(t)$$

$$\equiv (\sigma^+)^2(\tilde{F})(\phi(\sigma(x))) = \tilde{F}(\sigma(x)) = (F \star x)(t).$$

**Definition 5.53** For each $F \in \mathcal{E}(\mathbb{R}^+)$, the convolution operator $C^+[F]$ on $\mathcal{E}(\mathbb{R}^+)$ is defined by $C^+[F] := \sigma^+[(\sigma^*)^{-1}(\tilde{F}) \circ \pi^-]$.

We mention some straightforward properties of the operators $C^+[F]$ on $\mathcal{E}(\mathbb{R}^+)$. 

**Lemma 5.54** Let $F_1, F_2 \in \mathcal{E}(\mathbb{R}^+)$. Then the following assertions hold true.

1. $C^+[F_1]$ is a continuous linear $(\sigma^+)^2$-invariant mapping on $\mathcal{E}(\mathbb{R}^+)$,

2. $C^+[F_1](x|_{[0,\infty)}) = (F_1 \star x)|_{[0,\infty)}$ for each $x \in D_+(\mathbb{R})$,

3. $C^+[F_1] \circ C^+[F_2] = C^+[F_1 \star F_2] = C^+[F_2] \circ C^+[F_1]$, and

4. $C^+[\delta_{x,t}] = \sigma^+ t$ for each $t \geq 0$. 

For the canonical state space of a pseudo-rational SISO-systems, we have the following result.

**Lemma 5.55** Let $(D_+(\mathbb{R}), D(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_{\text{max}})$ be a pseudo-rational SISO-system with response $Q^{-1} \star P$. Then

$$f_{\text{max}}(D(\mathbb{R}^-)) \subseteq \ker(C^+[Q]).$$

**Proof.**

For each $\psi \in D(\mathbb{R}^-)$, we have $Q \star ((Q^{-1} \star P) \star \psi) \in D(\mathbb{R}^-)$. Next, apply Lemma 5.54 ii.

**Definition 5.56** The distributions $Q, P \in \mathcal{E}(\mathbb{R}^-)$ are called coprime over the ring $(\mathcal{E}(\mathbb{R}^-), +, \star)$ if

$$Q \star R + P \star S = \delta_0,$$

for some $R, S \in \mathcal{E}(\mathbb{R}^-)$.
Definition 5.57 The distributions \( Q, P \in \mathcal{E}'(\mathbb{R}^-) \) are called approximately coprime over the ring \( (\mathcal{E}'(\mathbb{R}^-), +, *) \) if
\[
Q \ast R_n + P \ast S_n \rightarrow \delta_0,
\]
in weak-\(\sigma(\mathcal{E}(\mathbb{R}), \mathcal{E}(\mathbb{R}))\)-sense, for some sequences \((R_n)_{n \in \mathbb{N}}\) and \((S_n)_{n \in \mathbb{N}}\) in \( \mathcal{E}'(\mathbb{R}^-) \).

Remark 5.57.1 Let \( F_n = Q \ast R_n + P \ast S_n \) for each \( n \in \mathbb{N} \). Recall from Lemma 1.12, that the convergence of the sequence \((F_n)_{n \in \mathbb{N}}\) to \( \delta_0 \) is also in strong-sense. Since for each \( y \in \mathcal{E}(\mathbb{R}) \) and for each compact \( K \subseteq \mathbb{R} \), the collection \( \{ \sigma y \mid t \in K \} \) is bounded in \( \mathcal{E}(\mathbb{R}) \) by the local equicontinuity of the translation group \((\sigma_t)_{t \in \mathbb{R}}\) on \( \mathcal{E}(\mathbb{R}) \), we have for each \( y \in \mathcal{E}(\mathbb{R}) \) and each compact \( K \subseteq \mathbb{R} \)
\[
\max_{t \in K} |(F_n \ast y)(t) - y(t)| = \max_{t \in K} |(F_n - \delta_0)(\sigma_t y)|.
\]
(5.34)
Since the right-hand side of (5.34) converges to zero by the strong convergence of the sequence \((F_n)_{n \in \mathbb{N}}\), we obtain that for each \( y \in \mathcal{E}(\mathbb{R}) \)
\[
\lim_{n \to \infty} F_n \ast y = \lim_{n \to \infty} (Q \ast R_n + P \ast S_n) \ast y = y,
\]
in \( \mathcal{E}(\mathbb{R}) \)-sense.

The following result is suggested by Yamamoto, [Y5], Theorem 2.15.

Theorem 5.58 Let \((\mathcal{D}_+(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_{\text{sta}})\) be a pseudo-rational SISO-system with impulse response \( Q^{-1} \ast P \). Then \( f_{\text{sta}}(\mathcal{D}(\mathbb{R}^-)) = \ker(C^+[Q]) \) if and only if \( Q \) and \( P \) are approximately coprime.

Proof. Assume that \( f_{\text{sta}}(\mathcal{D}(\mathbb{R}^-)) = \ker(C^+[Q]) \). We need to construct sequences \((R_n)_{n \in \mathbb{N}}\) and \((S_n)_{n \in \mathbb{N}}\) in \( \mathcal{E}'(\mathbb{R}^-) \) such that
\[
Q \ast R_n + P \ast S_n \rightarrow \delta_0,
\]
in weak-\(\sigma(\mathcal{E}(\mathbb{R}), \mathcal{E}(\mathbb{R}))\)-sense. Let \((\psi_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{D}(\mathbb{R}) \) with \( \text{supp}(\psi_n) \subseteq [-1, 0] \), such that \( \psi_n \to \delta_0 \) in \( \mathcal{E}(\mathcal{E}(\mathbb{R}), \mathcal{E}(\mathbb{R})) \)-sense. Since, \( Q \ast Q^{-1} \ast \psi_n = \psi_n \in \mathcal{D}(\mathbb{R}^-) \), we have \( \langle Q^{-1} \ast \psi_n \rangle \rangle_{n \in \mathbb{N}} \in \ker(C^+[Q]) \). So, by the assumption, there is a sequence \((\omega_{n,m})_{m \in \mathbb{N}}\) in \( \mathcal{D}(\mathbb{R}^-) \) such that
\[
\langle Q^{-1} \ast \psi_n \rangle \rangle_{n \in \mathbb{N}} = \lim_{m \to \infty} f_{\text{sta}}(\omega_{n,m}) = \lim_{m \to \infty} (Q^{-1} \ast P \ast \omega_{n,m}) \rangle_{n \in \mathbb{N}}.
\]
Take \( \bar{r}_{n,m} \) in \( \mathcal{D}(\mathbb{R}^-) \) such that
\[
\bar{r}_{n,m}(t) := (Q^{-1} \ast P \ast \omega_{n,m})(t) - (Q^{-1} \ast \psi_n)(t) \quad (t \leq -1).
\]
Then, \( \text{supp}(Q^{-1} \ast P \ast \omega_{n,m} - \bar{r}_{n,m}) \subseteq [-1, 0] \subseteq [-N, \infty) \) for some \( N \in \mathbb{N} \) not depending on \( n \) and \( m \). Hence, there is a sequence \((\tilde{r}_{n,m})_{m \in \mathbb{N}}\) in \( \mathcal{D}(\mathbb{R}^-) \) with support in \([-1, \infty) \) such that
\[
Q^{-1} \ast \psi_n = \lim_{m \to \infty} \left( Q^{-1} \ast P \ast \omega_{n,m} - \bar{r}_{n,m} - \tilde{r}_{n,m} \right).
\]
in $D_s({\mathcal{R}})$-sense. Define $\tau_{n,m} := -(\tau_{n,m}^+ + \tau_{n,m}^-)$. Then
\[
\psi_n = Q \ast Q^{-1} \ast \psi_n = \lim_{n \to \infty} \left( P \ast \omega_{n,m} + Q \ast \tau_{n,m} \right).
\]
in $D_s({\mathcal{R}})$-sense. We observe that $\supp(P \ast \omega_{n,m} + Q \ast \tau_{n,m}) \subseteq (-\infty, 0]$ and $\supp(P \ast \omega_{n,m} + Q \ast \tau_{n,m}) \subseteq \supp(Q) = [-N, \infty]$ for each $n, m \in \mathbb{N}$. Hence, there is $M \in \mathbb{N}$ such that $\supp(P \ast \omega_{n,m} + Q \ast \tau_{n,m}) \subseteq [-M, 0]$ for each $n, m \in \mathbb{N}$. Take for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$ such that
\[
\max_{t \in [-M, 0]} \left| (P \ast \omega_{n,m(n)} + Q \ast \tau_{n,m(n)})(t) - \psi_n(t) \right| \leq \frac{1}{n},
\]
Define $R_n := \tau_{n,m(n)}$ and $S_n := \omega_{n,m(n)}$ and let $F_n := Q \ast R_n + P \ast S_n$ for each $n \in \mathbb{N}$. Then for each $x \in \mathcal{E}(\mathcal{R})$
\[
\left| (\psi_n - F_n) \ast x \right| \leq \frac{1}{n} \int_{-M}^0 |z(t)| \, dt
\]
where $(.,.)$ denotes the canonical duality between $D(\mathcal{R})$ and $\mathcal{E}(\mathcal{R})$. Since $\psi_n \to \delta_0$ in $\sigma(\mathcal{E}(\mathcal{R}), \mathcal{E}(\mathcal{R}))$-sense, $Q \ast R_n + P \ast S_n \to \delta_0$ in $\sigma(\mathcal{E}(\mathcal{R}), \mathcal{E}(\mathcal{R}))$-sense, proving the assertion.

Next, assume that $Q$ and $P$ are approximately coprime. Let $x \in D_s(\mathcal{R})$ be such that $x|_{[0,\infty)} \in \ker(C^+Q)$. Then $Q \ast x \in D(\mathcal{R}^+)$ by Lemma 5.54.11. Define $\omega_n \in D(\mathcal{R}^+)$ by $\omega_n := S_n + Q \ast x$. Then
\[
f_{\text{stat}}(\omega_n) = (Q^{-1} \ast P \ast S_n \ast Q \ast x)|_{[0,\infty)} = (P \ast S_n \ast x + Q \ast R_n \ast x)|_{[0,\infty)}.
\]
(5.35)

Since $P \ast S_n \ast x + Q \ast R_n \ast x \rightarrow x$ in $\mathcal{E}(\mathcal{R})$-sense (see Remark 5.57.1), the right-hand side of (5.35) converges to $x|_{[0,\infty)}$ in $\mathcal{E}(\mathcal{R}^+)$-sense, which proves the assertion.

Corollary 5.59 Let $\Sigma_{\text{stat}}$ be a smooth pseudo-rational SISO-system with impulse response $Q^{-1} \ast P$. If $Q$ and $P$ are approximately coprime, then any canonical state space of $\Sigma_{\text{stat}}$ is isomorphic to $\ker(C^+Q)$.

Next, we consider the main problem of this section: “Which pseudo-rational SISO-systems have a canonical state space spanned by exponential-polynomials?” We employ the following definition from Yamamoto [Y5], Definition 3.2.

Definition 5.60 A strictly causal static system $\Sigma_{\text{stat}} = (V, \Omega, \Gamma, f_{\text{stat}})$ is called spectrally complete if the canonical state space $f_{\text{stat}}(V)$ of $\Sigma_{\text{stat}}$ is the closed linear span of exponential-polynomials, i.e., functions of the form $\omega(\cdot) = e^{t} \cdot \eta$, $t \geq 0$.

Remark 5.60.1 The terminology spectrally complete will be explained in the latter part of this section. In fact, it turns out that the exponential-polynomials $\omega_{\lambda,\alpha}$ spanning the canonical state space $X$ of a spectrally complete static SISO-system correspond to the point spectrum of the infinitesimal generator of the translation semigroup $(\sigma_{\alpha})_{\alpha \in 0}$ restricted to $X$. 
In the search for spectrally complete static SISO-systems, we restrict ourselves to pseudo-rational SISO-systems with impulse response \( Q^{-1} * P \) such that \( Q \) and \( P \) are approximately coprime. Since the canonical state space of a static SISO-system \( \Sigma_{\text{stat}} \) is the completion of a canonical state space of the smooth static subsystem \( \Sigma^S_{\text{stat}} \) by Proposition 5.39, we may restrict ourselves to smooth systems.

Since \( E'(R^+) \) represents the dual of \( E(R^+) \) (see §3.3), \( E'(R^-) \) can be regarded as the dual of \( E(R^-) \).

**Lemma 5.61** The mapping \( x \in E(R^+) \mapsto \Phi(x_{\text{ext}}) \), where \( x_{\text{ext}} \in D_s(R) \) is such that \( x \cdot [g,0,0] = x \), defines for each \( \Phi \in E'(R^-) \) a continuous linear functional on \( E(R^+) \). Conversely, every continuous linear functional on \( E(R^+) \) can be written in this manner.

We denote the duality between \( E'(R^-) \) and \( E(R^+) \) introduced by Lemma 5.61 by \( \langle \cdot, \cdot \rangle \).

Having introduced a duality between \( E(R^+) \) and \( E'(R^-) \), the polar \( M^* \) of a subspace \( M \) of \( E(R^-) \) can be regarded as a subspace of \( E'(R^+) \). Being interested in subspaces of \( E(R^+) \) spanned by exponential-polynomials, we have the following result on the polar of such spaces.

**Lemma 5.62** Let \( M \) be a closed subspace of \( E(R^+) \). Let \( W \subset \mathbb{C} \times \mathbb{N} \). Then the following assertions are equivalent.

i. \( M = \text{span}\{u_{\lambda,j} \mid (\lambda,j) \in W\} \).

ii. \( M^* = \{ \Phi \in E'(R^-) \mid F(\Phi)^{ij}(-i\lambda) = 0 \text{ for all } (\lambda,j) \in W \} \).

where \( F \) denotes the Fourier transform on \( E(R) \) (cf. Definition 3.87).

**Proof.**

(i) \( \Rightarrow \) (ii). Let \( \Phi \in M^* \). Recall from Theorem 3.33 that \( e_{\lambda,j} \in E(R) \) is defined by \( e_{\lambda,j}(t) := e^{tj} \cdot e^{-i\lambda t} \) for each \( t \in R \). Hence, \( u_{\lambda,j} = e_{\lambda,j} \cdot [\mathbb{C},0,0] \). So, for each \( (\lambda,j) \in W \), we have

\[
0 = \langle \Phi, u_{\lambda,j} \rangle = (-1)^j \cdot \Phi(e_{-\lambda,j}) = (i)^j \cdot \left( \frac{d}{dt} \Phi(e_{\lambda,j}) \right)|_{t=-\lambda} = (i)^j \cdot F(\Phi)^{ij}(-i\lambda).
\]

So, \( M^* \subseteq \{ \Phi \in E'(R^-) \mid F(\Phi)^{ij}(-i\lambda) = 0 \text{ for all } (\lambda,j) \in W \} \).

Conversely, if \( \Phi \in E'(R^-) \) satisfies \( F(\Phi)^{ij}(-i\lambda) = 0 \) for all \( (\lambda,j) \in W \), then we have \( \langle \Phi, u_{\lambda,j} \rangle = 0 \) for each \( (\lambda,j) \in W \), so \( \Phi \in M^* \).

(ii) \( \Rightarrow \) (i). Since \( \langle \Phi, u_{\lambda,j} \rangle = 0 \) for each \( (\lambda,j) \in W \) and each \( \Phi \in M^* \), the closedness of \( M \) yields that each \( u_{\lambda,j} \in M \). So, \( M \supseteq \text{span}\{u_{\lambda,j} \mid (\lambda,j) \in W\} \). Conversely, if \( x \in M \) is such that \( x \notin \text{span}\{u_{\lambda,j} \mid (\lambda,j) \in W\} \), then the Hahn-Banach Theorem ensures the existence of \( \Phi \in E'(R^-) \) such that \( \langle \Phi, x \rangle = 1 \) and \( \langle \Phi, u_{\lambda,j} \rangle = (i)^j \cdot F(\Phi)^{ij}(-i\lambda) = 0 \) for each \( (\lambda,j) \in W \). Contradiction.

**Remark 5.62.1** Since a nonzero entire function has a countable number of zeros only (including multiplicities), an immediate observation of Lemma 5.62 is that a non-trivial closed subspace of \( E(R^+) \) contains only a countable number of exponential-polynomials.
5.3. Factorizations of pseudo-rational input/output mappings

Lemma 5.62 yields the following characterization of spectral completeness.

**Theorem 5.63** Let \( (\mathcal{D}_+, \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_{\text{stat}}) \) be a pseudo-rational SISO-system with impulse response \( Q^{-1} * P \). Suppose \( Q \) and \( P \) are approximately coprime. Let \( W_Q \subseteq \mathbb{C} \times \mathbb{N}_0 \) be defined by

\[
(\lambda, n) \in W_Q \iff \mathcal{F}(Q)^{ij}(-i\lambda) = 0 \quad \text{for all } i \leq n.
\]

Then the following assertions are equivalent.

i. \( \Sigma_{\text{stat}} \) is spectrally complete.

ii. \( f_{\text{stat}}(\mathcal{D}(\mathbb{R}^-)) = \ker(C^+[Q]) = \operatorname{span}\{u_{\lambda,j} \mid (\lambda, j) \in W_Q\} \)

iii. \( \ker(C^+[Q])^* = \{\Phi \in \mathcal{E}'(\mathbb{R}^-) \mid \mathcal{F}(\Phi)^{ij}(-i\lambda) = 0 \quad \text{for all } (\lambda, j) \in W_Q\} \).

**Proof.**

(iii) \(\Rightarrow\) (i). Obvious.

(i) \(\Rightarrow\) (ii). By assumption, we have

\[
\ker(C^+[Q]) = \operatorname{span}\{u_{\lambda,j} \mid (\lambda, j) \in W\}.
\]

Now, let \( (\lambda, j) \in W \). Then \( u_{\lambda,j} \in \ker(C^+[Q]) \). Since \( \ker(C^+[Q]) \) is \( (\sigma_i^+)_{Q} \)-invariant, we have \( u_{\lambda,l} \in \ker(C^+[Q]) \) for each \( l = 0, 1, \ldots, j \). Moreover, for each \( l = 0, 1, \ldots, j \), we have

\[
(i)^l \cdot \mathcal{F}(Q)^{ij}(-i\lambda) = (-1)^l \cdot Q(e_{\lambda,l}) = (Q * e_{\lambda,l})(0) = (C^+[Q])(u_{\lambda,j})(0) = 0.
\]

So, \( W \subseteq W_Q \) and \( \ker(C^+[Q]) \subseteq \operatorname{span}\{u_{\lambda,j} \mid (\lambda, j) \in W_Q\} \).

Conversely, let \( (\lambda, j) \in W_Q \). Since \( \sigma_{-}(e_{\lambda,j}) \in \operatorname{span}\{e_{\lambda,l} \mid l = 0, 1, \ldots, j\} \) for each \( t \geq 0 \), we have for each \( t \geq 0 \)

\[
C^+[Q](u_{\lambda,j})(t) = (Q * e_{\lambda,j})(t) = (-1)^j \cdot Q(\sigma_{-}(e_{\lambda,j}))(t) = 0,
\]

i.e. \( u_{\lambda,j} \in \ker(C^+[Q]) \), which proves the assertion.

The equivalence of (ii) and (iii) is an immediate consequence of Lemma 5.62.

By Theorem 5.63, we focus, in the search for spectrally complete pseudo-rational smooth SISO-systems, on the poles of their canonical state spaces.

**Theorem 5.64** Let \( Q \in \mathcal{E}'(\mathbb{R}^-) \) be invertible in \( \mathcal{D}_+(\mathbb{R}) \). Then

\[
\ker(C^+[Q])^* = \{Q \circ \Phi \mid \Phi \in \mathcal{E}'(\mathbb{R}^-)\} = \operatorname{ran}(C^+[Q]^*) \].
Proof. First, we prove the equality \( \ker(C^+[Q]^*) = \{ Q \ast \Phi \mid \Phi \in \mathcal{E}'(\mathbb{R}^-) \} \).
Let \( x \in \mathcal{D}_+(\mathbb{R}) \) and let \( \Psi \in \mathcal{E}'(\mathbb{R}^-) \). Then

\[
(\Psi, x|_{[0,\infty)}) := \Psi(\hat{x}) = (\Psi \ast x)(0) = (C^+[\Psi](x|_{[0,\infty]}))(0).
\] (5.36)

Now, let \( \Phi \in \mathcal{E}'(\mathbb{R}^-) \). Since the operators \( C^+[Q] \) and \( C^+[\Phi] \) commute by Lemma 5.54, we have

\[
y \in \ker(C^+[Q]) \quad \Rightarrow \quad C^+[\Phi]y \in \ker(C^+[Q]).
\]

Therefore, we have for each \( y \in \ker(C^+[Q]) \)

\[
(\Psi, y) \overset{(5.36)}{=} \left( C^+[\Phi \ast Q]y \right)(0) = \left( C^+[Q] \left( C^+[\Phi]y \right) \right)(0) = 0.
\]

We conclude that \( \{ Q \ast \Phi \mid \Phi \in \mathcal{E}'(\mathbb{R}^-) \} \subseteq \ker(C^+[Q])^* \).

Now, let \( \Psi \in \ker(C^+[Q]^*) \). Since \( (Q^{-1} \ast \Phi)|_{[0,\infty]} \in \ker(C^+[Q]) \) for each \( \phi \in \mathcal{D}(\mathbb{R}^-) \), we have

\[
0 = C^+[\Phi]|_{[0,\infty]} = (\Psi \ast Q^{-1} \ast \Phi)|_{[0,\infty]}.
\]

So, \( \Psi \ast Q^{-1} \ast \Phi \in \mathcal{D}(\mathbb{R}^-) \). Since \( \Phi \in \mathcal{E'(R^-)} \), we conclude that \( \Psi \ast Q^{-1} \in \mathcal{E'(R^-)} \), proving that \( \ker(C^+[Q]^*) = \{ Q \ast \Phi \mid \Phi \in \mathcal{E}'(\mathbb{R}^-) \} \).

To prove the latter equality, let \( \Phi \in \mathcal{E}'(\mathbb{R}^-) \). Then for each \( y \in \mathcal{E'(R^+)} \),

\[
(C^+[Q]^* \Phi, y) = (\Phi, C^+[Q]y) \overset{(5.36)}{=} \left( C^+[\Phi](C^+[Q]y)(0) = (C^+[\Phi \ast Q]y)(0)
\]

\[
\overset{(5.36)}{=} (\Phi, Q \ast y),
\]

i.e. \( C^+[Q]^* \Phi = Q \ast \Phi \).

\[\blacksquare\]

Corollary 5.65 Let \( (\mathcal{D}_+(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), \mathcal{F}_{\text{stat}}) \) be a pseudo-rational SISO-system with impulse response \( Q^{-1} \ast P \). Suppose \( Q \) and \( P \) are approximately coprime. Then

\[
\mathcal{F}_{\text{stat}}(\mathcal{D}(\mathbb{R}^-))^* = \{ Q \ast \Phi \mid \Phi \in \mathcal{E}'(\mathbb{R}^-) \}.
\]

In other words, \( \mathcal{F}_{\text{stat}}(\mathcal{D}(\mathbb{R}^-))^* \) is the ideal in \( \mathcal{E}'(\mathbb{R}^-) \) generated by \( Q \).

Proposition 5.66 Let \( \Sigma_{\text{stat}} \) be a pseudo-rational SISO-system with impulse response \( Q^{-1} \ast P \). Suppose \( Q \) and \( P \) are approximately coprime. Let \( W_Q \in \mathcal{C} \times \mathcal{N}_0 \) be defined by

\[
(\lambda, n) \in W_Q \iff \mathcal{F}(Q)^{(i)}(-i\lambda) = 0 \quad \text{for all} \quad i \leq n.
\]

Then

\[
\ker(C^+[Q]^*) \subseteq \{ \Phi \in \mathcal{E}'(\mathbb{R}^-) \mid \mathcal{F}(\Phi)^{(j)}(-i\lambda) = 0 \quad \text{for all} \quad (\lambda, j) \in W_Q \}.
\]
5.3. FACTORIZATION OF PSEUDO-RATIONAL INPUT/OUTPUT MAPPINGS

Proof.
Let \(-i\lambda\) be a zero of \(\mathcal{F}(\Phi)\) with multiplicity greater or equal to \(j\). Then, for each \(\Phi \in \mathcal{E}'(\mathbb{R}^-)\)
\[
\mathcal{F}(Q * \Phi)^{(j)}(-i\lambda) = \sum_{l=0}^{j} \binom{j}{l} \mathcal{F}(Q)^{(l)}(-i\lambda) \cdot \mathcal{F}(\Phi)^{(j-l)}(-i\lambda) = 0.
\]

So, \(Q * \Phi \in \{ \Phi \in \mathcal{E}'(\mathbb{R}^-) \mid \mathcal{F}(\Phi)^{(j)}(-i\lambda) = 0 \text{ for all } \lambda, j \in \mathbb{W}_Q \} \).

The following result is due to Malgrange [Mal]. A proof can also be found in Kahan [Kah2], p.133, or in the context of Laplace transformations in Yamamoto [Y3], Appendix B.

**Proposition 5.67** Let \(\Phi_1, \Phi_2 \in \mathcal{E}'(\mathbb{R})\). Then, the following two assertions are equivalent.

i. \(\mathcal{F}(\Phi_1)/\mathcal{F}(\Phi_2)\) is an entire function.

ii. There exists \(\Psi \in \mathcal{E}'(\mathbb{R})\), such that \(\Phi_1 = \Phi_2 * \Psi\).

Now, let \(\Phi \in \mathcal{E}'(\mathbb{R}^-)\) be such that \(\mathcal{F}(\Phi)^{(j)}(\lambda) = 0\) for all zeros \(\lambda\) of \(\mathcal{F}(\Phi)\) with multiplicity greater or equal to \(j\). Then by Proposition 5.67, there is \(\Psi \in \mathcal{E}'(\mathbb{R})\) such that
\[
\Phi = Q * \Psi. \tag{5.37}
\]

So, spectral completeness of a static SISO-system with pseudo-rational impulse response \(Q^{-1} * P\) is equivalent to the assertion that the distribution \(\Psi\) in [5.37] can always be chosen with support in \((-\infty, 0]\). It turns out that the latter assertion is equivalent to a condition on the support of \(Q\). To this extent, define \(r(\Phi)\) to be the supremum of the \(\mathcal{E}'(\mathbb{R})\)-distribution \(\Phi\) by
\[
r(\Phi) := \sup \{ t \mid t \in \text{supp}(\Phi) \}. \tag{5.38}
\]

Then for each \(\Phi_1, \Phi_2 \in \mathcal{E}'(\mathbb{R})\), we have
\[
r(\Phi_1 * \Phi_2) = r(\Phi_1) + r(\Phi_2).
\]

**Theorem 5.68** Let \((\mathcal{D}_s(\mathbb{R}), \mathcal{D}(\mathbb{R}^+), \mathcal{E}(\mathbb{R}^+), \mathcal{I}_{\text{stat}})\) be a pseudo-rational SISO-system with impulse response \(Q^{-1} * P\). Suppose \(Q\) and \(P\) are approximately coprime. Then \(\Sigma_{\text{stat}}\) is spectrally complete if and only if \(r(Q) = 0\).

**Proof.**
First, suppose \(r(Q) = 0\). Let \(W_Q \subseteq \mathbb{C} \times \mathbb{N}_0\) be defined by
\[
(\lambda, n) \in W_Q \iff \mathcal{F}(Q)^{(n)}(-i\lambda) = 0 \text{ for all } l \leq n.
\]
Let $\Phi \in \mathcal{E}'(\mathbb{R}^-)$ be such that $\mathcal{F}(\Phi)^{(Q)}(\lambda) = 0$ for all zeros $(\lambda, j) \in W_Q$. Then by Proposition 5.67, there is $\Psi \in \mathcal{E}'(\mathbb{R}^-)$ such that

$$\Phi = Q * \Psi.$$  
(5.39)

Since $r(\Phi) = r(Q) + r(\Psi) \leq 0$, we have that $\Psi \in \mathcal{E}'(\mathbb{R}^-)$. Applying Theorem 5.64, we conclude that $\Phi \in \ker(C^+|Q|)^*$. Hence, by Theorem 5.63 and Proposition 5.66, $\Sigma_{\text{stat}}$ is spectrally complete.

Conversely, suppose $r(Q) < 0$. Then $t > 0$ exists such that $Q * \delta_t \in \mathcal{E}'(\mathbb{R}^-)$ and $Q * \delta_t \not\in \ker(C^+|Q|)^*$ by Theorem 5.64. Since $\mathcal{F}(Q * \delta_t)^{(Q)}(\lambda) = 0$ for all $(\lambda, j) \in W_Q$, $\Sigma_{\text{stat}}$ is not spectrally complete by Theorem 5.63.

Since a static SISO-system $\Sigma_{\text{stat}}$ is spectrally complete if and only its smooth subsystem $\Sigma_{\text{stat}}^w$ is spectrally complete, we arrive at the following result.

**Theorem 5.69** Let $\Sigma_{\text{stat}}$ be a static SISO-system with pseudo-rational impulse response $Q^{-1} * P$. Suppose $Q$ and $P$ are approximately coprime. Then $\Sigma_{\text{stat}}$ is spectrally complete if and only if $r(Q) = 0$.

The pseudo-rational SISO-system from example 5.52 is spectrally complete.

**Example 5.70** Let $\Sigma_{\text{stat}}$ be the smooth pseudo-rational SISO-system from example 5.52. Recall that $\Sigma_{\text{stat}}$ has impulse response $G \in \mathcal{D}_c(\mathbb{R})$ defined by

$$G = \sum_{i=1}^{\infty} \delta_i = (\delta_{-1} - \delta_0)^{-1} * \delta_0,$$

and the static input/output mapping $f_{\text{stat}}$ of $\Sigma_{\text{stat}}$ satisfies

$$f_{\text{stat}}(\psi)(t) = \sum_{i=1}^{\infty} \psi(t - i) \quad (\psi \in \mathcal{D}(\mathbb{R}^-)).$$

Since $0 * (\delta_{-1} - \delta_0) + \delta_0 * \delta_0 = \delta_0$, the pair $(P, Q) = (\delta_0, \delta_{-1} - \delta_0)$ is (approximately) coprime over $\mathcal{E}'(\mathbb{R}^-)$. Hence, since $r(\delta_{-1} - \delta_0) = 0$, the system $\Sigma_{\text{stat}}$ is spectrally complete.

SISO-systems with a single time-delay are not spectrally complete.

**Example 5.71** Let $\Sigma_{\text{stat}} = (\mathcal{D}_c(\mathbb{R}), \mathcal{D}(\mathbb{R}^-), \mathcal{E}(\mathbb{R}^+), f_{\text{stat}})$ be the strictly causal static SISO-system with impulse response $\delta_1$. So, the static input/output mapping $f_{\text{stat}}$ of $\Sigma_{\text{stat}}$ satisfies

$$f_{\text{stat}}(\psi)(t) = \psi(t - 1) \quad (\psi \in \mathcal{D}(\mathbb{R}^-)).$$

Since $\delta_1 = (\delta_{-1})^{-1} * \delta_0$, $\Sigma_{\text{stat}}$ is pseudo-rational. Moreover, since $0 * \delta_{-1} + \delta_0 * \delta_0 = \delta_0$, the pair $(P, Q) = (\delta_0, \delta_{-1})$ is (approximately) coprime over $\mathcal{E}'(\mathbb{R}^-)$. Hence, since $r(\delta_{-1}) = -1$, the system $\Sigma_{\text{stat}}$ is not spectrally complete.
GLOSSARY OF VECTOR SPACES OF FUNCTIONS, MEASURES, AND DISTRIBUTIONS

In this appendix, we discuss briefly the function spaces, measure spaces, and distribution spaces appearing in this thesis.

Spaces of continuous functions

\( C(\mathbb{R}) \): The vector space consisting of all continuous functions on \( \mathbb{R} \). Equipped with compact-open topology, \( C(\mathbb{R}) \) is an F-space.

\( C_{\mathbf{+},n}(\mathbb{R}) \): The subspace of \( C(\mathbb{R}) \) consisting of all \( f \in C(\mathbb{R}) \) with support in \([-n, \infty)\). Equipped with relative \( C(\mathbb{R}) \)-topology, \( C_{\mathbf{+},n}(\mathbb{R}) \) is an F-space.

\( C_{\mathbf{+}}(\mathbb{R}) \): The subspace of \( C(\mathbb{R}) \) consisting of all \( f \in C(\mathbb{R}) \) with support bounded on the left, i.e. \( \text{supp}(f) \subseteq [-n, \infty) \) for some \( n \in \mathbb{N} \). Equipped with the strict inductive limit topology of the sequence \( (C_{\mathbf{+},n}(\mathbb{R}))_{n \in \mathbb{N}} \), \( C_{\mathbf{+}}(\mathbb{R}) \) is a strict LF-space.

\( C_{\mathbf{+}}(\mathbb{R}) \): The subspace of \( C(\mathbb{R}) \) consisting of all \( f \in C(\mathbb{R}) \) with support bounded on the right, i.e. \( \text{supp}(f) \subseteq (-\infty, n] \) for some \( n \in \mathbb{N} \). \( C_{\mathbf{+}}(\mathbb{R}) \) is a strict LF-space.

\( C_{\mathbf{\circ}}(\mathbb{R}) \): The subspace of \( C(\mathbb{R}) \) consisting of all \( f \in C(\mathbb{R}) \) with support in \([-\infty, \infty] \). Equipped with relative \( C(\mathbb{R}) \)-topology, \( C_{\mathbf{\circ}}(\mathbb{R}) \) is a Banach space.

\( C_{\mathbf{\circ}}(\mathbb{R}) \): The subspace of \( C(\mathbb{R}) \) consisting of all \( f \in C(\mathbb{R}) \) with bounded support, i.e. \( \text{supp}(f) \subseteq [-n, n] \) for some \( n \in \mathbb{N} \). Equipped with the strict inductive limit topology of the sequence \( (C_{\mathbf{\circ},n}(\mathbb{R}))_{n \in \mathbb{N}} \), \( C_{\mathbf{\circ}}(\mathbb{R}) \) is a strict LB-space.

\( C^{+}(\mathbb{R}) \): The vector space consisting of all continuous functions \( f \) on \( \mathbb{R}^+ \), such that \( \lim_{t \to 0^+} f(t) \) exists. Equipped with compact-open topology, \( C^{+}(\mathbb{R}) \) is an F-space.

Spaces of differentiable functions

Let \( k \in \mathbb{N} \).
\(C^k(\mathbb{R})\) : The vector space consisting of all \(k\)-times continuously differentiable functions on \(\mathbb{R}\). \(C^k(\mathbb{R})\) is an \(F\)-space.

\(C^k_{+\text{,n}}(\mathbb{R})\) : The subspace of \(C^k(\mathbb{R})\) consisting of all \(f \in C^k(\mathbb{R})\) with support in \([-n, \infty)\). Equipped with relative \(C^k(\mathbb{R})\)-topology. \(C^k_{+\text{,n}}(\mathbb{R})\) is an \(F\)-space.

\(C^k_{-\text{,n}}(\mathbb{R})\) : The subspace of \(C^k(\mathbb{R})\) consisting of all \(f \in C^k(\mathbb{R})\) with support bounded on the left. Equipped with the strict inductive limit topology of the sequence \((C^k_{-\text{,n}}(\mathbb{R}))_{n \in \mathbb{N}}\), \(C^k_{-\text{,n}}(\mathbb{R})\) is a strict LF-space.

\(C^k_{\text{+}(\mathbb{R})}\) : The subspace of \(C^k(\mathbb{R})\) consisting of all \(f \in C^k(\mathbb{R})\) with support bounded on the right. \(C^k_{\text{+}(\mathbb{R})}\) is a strict LF-space.

\(C^k_{\text{-}(\mathbb{R})}\) : The subspace of \(C^k(\mathbb{R})\) consisting of all \(f \in C^k(\mathbb{R})\) with bounded support. \(C^k_{\text{-}(\mathbb{R})}\) is a strict LF-space.

\(C^k(\mathbb{R}^+)\) : The vector space consisting of all \(k\)-times continuously differentiable functions \(f\) on \(\mathbb{R}^+\), such that \(\lim_{t \to 0^+} f^{(l)}(t)\) exists for \(l = 1, \ldots, k\). \(C^k(\mathbb{R}^+)\) is an \(F\)-space.

Spaces of smooth functions

\(\mathcal{E}(\mathbb{R})\) : The vector space consisting of all arbitrarily many times differentiable functions on \(\mathbb{R}\). Equipped with compact-open topology. \(\mathcal{E}(\mathbb{R})\) is an \(F\)-space.

\(\mathcal{D}_{+\text{,n}}(\mathbb{R})\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with support in \([-n, \infty)\). Equipped with relative \(\mathcal{E}(\mathbb{R})\)-topology. \(\mathcal{D}_{+\text{,n}}(\mathbb{R})\) is an \(F\)-space.

\(\mathcal{D}_{-\text{,n}}(\mathbb{R})\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with support bounded on the left. Equipped with the strict inductive limit topology of the sequence \((\mathcal{D}_{-\text{,n}}(\mathbb{R}))_{n \in \mathbb{N}}\), \(\mathcal{D}_{-\text{,n}}(\mathbb{R})\) is a strict LF-space.

\(\mathcal{D}_{\text{+}}(\mathbb{R})\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with support bounded on the right. \(\mathcal{D}_{\text{+}}(\mathbb{R})\) is a strict LF-space.

\(\mathcal{D}_{\text{-}}(\mathbb{R})\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with support bounded by \([-n, n]\). Equipped with relative \(\mathcal{E}(\mathbb{R})\)-topology. \(\mathcal{D}_{\text{-}}(\mathbb{R})\) is an \(F\)-space.

\(\mathcal{D}(\mathbb{R})\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with bounded support. Equipped with the strict inductive limit topology of the sequence \((\mathcal{D}_{\text{-}}(\mathbb{R}))_{n \in \mathbb{N}}\), \(\mathcal{D}(\mathbb{R})\) is a strict LF-space.

\(\mathcal{E}(\mathbb{R}^+)\) : The vector space consisting of all functions \(f\) on \(\mathbb{R}^+\) which are arbitrary many times differentiable on \((0, \infty)\), such that \(\lim_{t \to 0^+} f^{(k)}(t)\) exists for each \(k \in \mathbb{N}_0\). Equipped with compact-open topology. \(\mathcal{E}(\mathbb{R}^+)\) is an \(F\)-space.

\(\mathcal{D}(\mathbb{R}^-)\) : The subspace of \(\mathcal{E}(\mathbb{R})\) consisting of all \(f \in \mathcal{E}(\mathbb{R})\) with bounded support in \((-\infty, 0]\).
Spaces of Lebesgue measurable functions

Let $p \geq 1$.

$L^p_{loc}(\mathbb{R})$ : The vector space consisting of all locally $p$-Lebesgue integrable functions. $L^p_{loc}(\mathbb{R})$ is an F-space.

$L^p_{loc,+}(\mathbb{R})$ : The subspace of $L^p_{loc}(\mathbb{R})$ consisting of all locally $p$-Lebesgue integrable functions with support in $[-n, \infty)$. Equipped with relative $L^p_{loc}(\mathbb{R})$-topology, $L^p_{loc,+}(\mathbb{R})$ is an F-space.

$L^p_{loc,-}(\mathbb{R})$ : The subspace of $L^p_{loc}(\mathbb{R})$ consisting of all locally $p$-Lebesgue integrable functions with support bounded on the left. Equipped with the strict inductive limit topology of the sequence $(L^p_{loc,+n}(\mathbb{R}))_{n \in \mathbb{N}}$, $L^p_{loc,-}(\mathbb{R})$ is a strict LF-space.

$L^p_{loc,-}(\mathbb{R})$ : The subspace of $L^p_{loc}(\mathbb{R})$ consisting of all locally $p$-Lebesgue integrable functions with support bounded on the right. $L^p_{loc,-}(\mathbb{R})$ is a strict LF-space.

$L^p_{comp}(\mathbb{R})$ : The subspace of $L^p_{loc}(\mathbb{R})$ consisting of all locally $p$-Lebesgue integrable functions with bounded support. $L^p_{comp}(\mathbb{R})$ is a strict LB-space.

$L^p_{loc}(\mathbb{R}^+)$ : The vector space consisting of all functions $f$ on $\mathbb{R}^+$ which are locally $p$-Lebesgue integrable. $L^p_{loc}(\mathbb{R})$ is an F-space.

Measure spaces

$\mathcal{M}(\mathbb{R})$ : The vector space consisting of all Radon measures. $\mathcal{M}(\mathbb{R})$ is the dual of $C_c(\mathbb{R})$.

$\mathcal{M}_+(\mathbb{R})$ : The subspace of $\mathcal{M}(\mathbb{R})$ consisting of all $\mu \in \mathcal{M}(\mathbb{R})$ with support bounded on the left. $\mathcal{M}_+(\mathbb{R})$ is the dual of $C_c(\mathbb{R})$.

$\mathcal{M}_-(\mathbb{R})$ : The subspace of $\mathcal{M}(\mathbb{R})$ consisting of all $\mu \in \mathcal{M}(\mathbb{R})$ with support bounded on the right. $\mathcal{M}_-(\mathbb{R})$ is the dual of $C_c(\mathbb{R})$.

$\mathcal{M}_x(\mathbb{R})$ : The subspace of $\mathcal{M}(\mathbb{R})$ consisting of all $\mu \in \mathcal{M}(\mathbb{R})$ with compact support. $\mathcal{M}_x(\mathbb{R})$ is the dual of $C(\mathbb{R})$.

Distribution spaces

$\mathcal{D}'(\mathbb{R})$ : The space of all Schwartz-distributions on $\mathbb{R}$. $\mathcal{D}'(\mathbb{R})$ is the dual of $\mathcal{D}(\mathbb{R})$.

$\mathcal{D}'_{+,n}(\mathbb{R})$ : The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with support in $[-n, \infty)$.

$\mathcal{D}'_+(\mathbb{R})$ : The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with support bounded on the left. $\mathcal{D}'_+(\mathbb{R})$ is the dual of $\mathcal{D}_-(\mathbb{R})$. 
$\mathcal{D}_c(\mathbb{R})$: The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with support bounded on the right. $\mathcal{D}_c(\mathbb{R})$ is the dual of $\mathcal{D}_+((\mathbb{R})$.

$\mathcal{E}'(\mathbb{R})$: The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with bounded support. $\mathcal{E}'(\mathbb{R})$ is the dual of $\mathcal{E}(\mathbb{R})$.

$\mathcal{E}'(\mathbb{R}^+)$: The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with bounded support in $[0, \infty)$. $\mathcal{E}'(\mathbb{R}^+)$ is the dual of $\mathcal{E}(\mathbb{R}^+)$.

$\mathcal{E}'(\mathbb{R}^-)$: The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all $F \in \mathcal{D}'(\mathbb{R})$ with bounded support in $(-\infty, 0]$.
REFERENCES


INDEX

α-group, 20
(αt)t∈ℝ-invariant operator, 24
(αt)t∈ℝ-invariant subspace, 23
α-semigroup, 25
C∞-domain, 21
approximately coprime, 146
barrelled space, 20
behaviour, ii
behavioural approach, ii
Borel Theorem, 77
bounded subset, 5
canonical factorization, 136
causal IO-system, 121
Closed Graph Theorem, 9
closed operator, 6
compact support, 60
completion, 7
convolution algebra, 59
convolution operator on Dc(ℝ), 73
convolution operator on D∞(ℝ), 57
convolution operator on L(ℝ+), 80
convolution operator on L(ℝ), 63
convolution product, 61
convolution product on Dc(ℝ), 73
convolution product on D∞(ℝ), 57
coprime, 145
distribution, 60
dual system, 4
exponential-polynomial, 69
factorization, 135
factorization problem, 135
finite-dimensional factorization, 138
Fourier transform on L(ℝ), 65
Fourier transform on D(ℝ), 62
Fréchet space, 8
generating family of seminorms, 2
graph topology, 6
impulse response, 120
inductive limit topology, 16
inductive system, 16
infinitesimal generator, 21
locally convex topological vector space, 2
locally convex topology, 2
locally equicontinuous, 20
mean periodic functions, 68
MIMO-system, 118
observability mapping, 135
observable, 135
Open Mapping Theorem, 9
Paley-Wiener Theorem for L(ℝ), 66
Paley-Wiener Theorem for D(ℝ), 62
polar set, 4
pre-Fréchet space, 8
projective limit topology, 10
pseudo-rational distribution, 144
pseudo-rational SISO-system, 144
quasi-reachable, 135
quotient topology, 7
Radon measure, 22
reachability mapping, 135
reachable, 135
realization, 118
realization problem, 118
reflection operator, 57
regular distribution, 122
relative topology, 7
semigroup associated to a factorization, 135
separating set of seminorms, 2
signal alphabet, ii
SISO-system, 118
smooth static input/output mapping, 129
spectrally complete, 147
state space, 135
state vector, 135
static impulse response, 131
static input space, 123
static input/output mapping, 123
static mode, 132
static output space, 123
static SISO-system, 123
strict LF-space, 11
strict inductive limit, 11
strict inductive limit topology, 11
strict inductive system, 11
strict pre-LF-space, 11
strictly causal SISO-system, 122
strong convergence, 5
subsystem, 121
support of a distribution, 61
support of a function, 60
supremum of a support, 151
time-axis, ii
topological dual, 3
topologically observable, 135
translatably F-space of \(\mathcal{E}(\mathbb{R})\)-type, 100
translatably strict LF-space of \(\mathcal{D}(\mathbb{R})\)-type, 112
translatably strict LF-space of \(\mathcal{D}_s(\mathbb{R})\)-type, 112
translation group, 57
translation group on \(\mathcal{E}(\mathbb{R})\), 111
translation group on \(\mathcal{D}(\mathbb{R})\), 99
translation group on \(\mathcal{D}_s(\mathbb{R})\), 85
translation operator, 57, 61
translation-invariant, 55
weak topology, 4, 5
working mode, 120
SAMENVATTING

In dit proefschrift staat de vraag centraal of de keuze van de signaalruimte invloed heeft op de externe beschrijving van een tijdsvariant lineair continue-tijd-systeem. Het blijkt dat voor een grote klasse van signaalruimten deze vraag ontkennen kan worden beantwoord.

In de "behavioural approach" worden dergelijke continue-tijd systemen gezien als translatie-invariante deelruimten van een functieruimte of een distributieruimte. Voor een ingang/uitgangs-systeem is er sprake van een lineaire, translatie-invariante afbeelding van de ingangsruiente naar de uitgangsruiete. Onderwijselijk zal de keuze van de signaalruimte en haar topologie een rol spelen.

In dit proefschrift worden functieruimten beschouwd die kunnen dienen als signaalruimten in de "behaviour" beschrijving of in de "ingangs/uitgangs" beschrijving. De translatiegroep op deze functieruimten is een $c_0$-groep. Gekozen is voor functieruimten met een Fréchettopologie dan wel een stricte inductieruimte dan wel een stricte inductieruimte. Zo wordt de klasse van transleerbare Fréchetruimten van het $\mathcal{E}(\mathbb{R})$-type ingevoerd en de klasse van transleerbare stricte LF-ruimten van het $\mathcal{D}_+(\mathbb{R})$-type. Deze klassen zijn zo gecreëerd, dat het $c_0$-domijn van de translatiegroep respectievelijk $\mathcal{E}(\mathbb{R})$ en $\mathcal{D}_+(\mathbb{R})$ is. Eerstgenoemde klasse kan gebruikt worden in de "behaviour" beschrijving, laatstgenoemde klasse in de "ingangs/uitgangs" beschrijving.

Voor de elementen uit elk dezer klassen worden de gesloten translatie-invariante deelruimten en gesloten translatie-invariante operatoren gekarakteriseerd. De verwante functionaal analytische concepten, als $c_0$-groep en $c_0$-semigroep, gesloten invariante deelruimten en gesloten invariante operatoren, worden in de wijder context van rijvolledige lokaal convex topologische vectorruimten beschouwd. Voor een subklasse van transleerbare stricte LF-ruimten van $\mathcal{D}_+(\mathbb{R})$-type wordt een factorisatie-theorie ontwikkeld die een uitbreiding is van een bestaande factorisatie-theorie met $L^p_{\text{loc}}(\mathbb{R})$ als signaalruimte. Het blijkt dat onder zwakke voorwaarden de toestandsruiete in de factorisatie wordt opgespannen door exponentiele polynomen.
CURRICULUM VITAE


Stellingen

behorende bij het proefschrift

SEMIGROUPS, INVARIANCE AND TIME-INVARIANT LINEAR SYSTEMS

door

M.M.A. de Rijke
1. Laat $L$ een continue translatie-invariante operator zijn op de stricte LB-ruimte $L^1_+([R])$, bestaande uit alle Lebesgue-integreerbare functies op $[R]$ met links-begrensd de drager. Dan is er een unieke Radonmaat $\mu$ met links-begrensd de drager, zo dat:

$$ L(g) = \mu \ast g \quad (g \in L^1_+([R])). $$

Zij $F \in D'_c([R])$. Veronderstel dat $F \ast g \in L^\infty_+([R])$ voor alle $g \in L^\infty_+([R])$. Dan is $F$ een Radonmaat met links-begrensd de drager.

Literatuur: S.J.L. van Eijndhoven, M.M.A. de Rijke, [vEdR].

2. Zij $\{u_n\}$ een frame in een Hilbertruimte $\mathcal{H}$. Dan is er een Hilbertruimte $\mathcal{K}$ en een frame $\{v_n\}$ in $\mathcal{K}$, zo dat $\{u_n \oplus v_n\}$ een Rieszbasis is in de Hilbertruimte $\mathcal{H} \oplus \mathcal{K}$.

Deze stelling is een oneindig dimensionale uitbreiding van de volgende bewering:

Zij $U$ een $m \times n$ matrix, $m < n$, van volle rang, dan is er een $(n - m) \times n$ matrix $V$ zo dat de matrix

$$
\begin{pmatrix}
U \\
V
\end{pmatrix},
$$

inverteerbaar is.

Literatuur: Young [You].

3. Zij $M$ een gesloten translatie-invariante deelruimte van $C^\infty([R])$ en zij $p$ een polynoom. Veronderstel dat de operator $p(\frac{d}{dx})$ injectief is op $M$. Dan is voor iedere $f \in M$ de differentiaalvergelijking

$$ p(\frac{d}{dx})x = f $$

oplosbaar in $M$.

4. Gegeven een lokaal compacte topologische ruimte $(X, \mathcal{T})$, een lokaal convex vectorruimte $V$ en een onafhankelijk stelsel continue functies $\{f_1, \ldots, f_n\}$ van $X$ in $[R]$. Dan is de collectie

$$ \{ f_k \otimes y \mid y \in V, k = 1, \ldots, n \} $$

gesloten in $C(X, V)$. Hierbij zijn de functies $f_k \otimes y$ gedefinieerd door

$$ (f_k \otimes y)(x) := f_k(x)y \quad (x \in X, y \in V) $$

een is $C(X, V)$ de lokaal convex vectorruimte van alle continue functies van $X$ in $V$ met de compact-open topologie.
5. Zij $F$ een transleerbare Fréchetruimte van $\mathcal{E}(\mathbb{R})$-type en zij $(p_n)_{n \in \mathbb{N}}$ een rij seminormen op $F$ welke de topologie van $F$ genereert. Zij $I$ de functie op $\mathbb{R}$ gedefinieerd door $I(t) := 1$ en zij $I : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ de Volterra operator gedefinieerd door

$$(I x)(t) := \int_0^t x(\tau) \, d\tau \quad (x \in C^\infty(\mathbb{R})).$$

Veronderstel dat $I$ continu is ten opzichte van de geïnduceerde $F$-topologie op $C^\infty(\mathbb{R})$. Dan brengen de seminormen $(q_n)_{n \in \mathbb{N}}$, gedefinieerd door

$$q_n(x) := \inf_{\alpha \in \mathbb{C}} p_n(I x + \alpha 1) \quad (x \in C^\infty(\mathbb{R})), $$

een pre-Fréchettopologie $T$ op $C^\infty(\mathbb{R})$ voort. Zij $F^{(-1)}$ de completering van $(C^\infty(\mathbb{R}), T)$ in $\mathcal{D}'(\mathbb{R})$. Dan gelden de volgende beweringen:

- $F^{(-1)}$ is een transleerbare Fréchetruimte van $\mathcal{E}(\mathbb{R})$-type.
- $F^{(-1)} = \{Dx \mid x \in F\}$, waarbij $D$ de distributionele differentiatie is.
- $F \hookrightarrow F^{(-1)}$.

Literatuur: Hoofdstuk 4 van dit proefschrift.

6. De suggestie van Soethoudt in zijn proefschrift, hoofdstuk 6, dat zijn condities C1, C2, C3 en C4 nodig en voldoende zijn opdat een "behaviour" in $L^1_{w^\infty}(\mathbb{R})$ afkomstig is van een AR-systeem is een ware bewering. Ze geldt namelijk voor iedere transleerbare Fréchetruimte van $\mathcal{E}(\mathbb{R})$-type.


7. "Tractationem calculi differentialis integralium definitorum theoria antevia debet."

Vrij vertaald: Aan iedere invloed van een differentiaalcalculus dient een invoering van een integraalbegrip vooraf te gaan.

Literatuur: F.G. Frobenius Berolini [Pro].

8. "De vernietiging van het verleden, of liever gezegd, de sociale mechanismen die de eigen levensverwarring in verband brengen met die van vorige generaties, is een van de meest kenmerkende en angstenaanjagende verschijnselen van de 20e eeuw. De meeste jongeren aan het einde van deze eeuw groeien op in een soort onveranderlijk heden, dat geen enkele organische relatie vertoont met de voorafgaande periode."

Dit citaat van Eric Hobsbawm [Hob], p. 15, zal begin vorige eeuw - wat Nederland betreft - mede als gevolg van de invoering van "de tweede-fase" voortgezet onderwijs in verstevende mate geldig zijn.

Literatuur: E. Hobsbawm [Hob].
9. In de wiskunde bestaat de trend om wiskundige technieken te verwarren met wiskundige concepten. Het belang dat daardoor aan deze technieken gehecht wordt kan verergerd werken.

10. Een wielrenner kan beter dikke benen hebben dan een dikke buik.

Referenties


