EUCLIDEAN REPRESENTATIONS
AND SUBSTRUCTURES
OF DISTANCE-REGULAR GRAPHS

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OF DISTANCE-REGULAR GRAPHS
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Preface

In the late sixties, distance-regular graphs were introduced by Biggs as a combinatorial generalisation of distance-transitive graphs. On the other hand, motivated by problems in coding theory, Delsarte studied in his thesis [35] P-polynomial association schemes (or metrically regular graphs). But the P-polynomial schemes are exactly the distance-regular graphs.

One of the reasons to study distance-regular graphs is that most classical schemes are distance-regular, for example the Hamming schemes, the Johnson schemes and the dual polar schemes. Using the theory of association schemes, Delsarte was able to prove his linear programming bound, which is still one of the best bounds known in coding theory.

In this thesis we try to obtain information by looking at substructures and Euclidean representations of distance-regular graphs. It is known that the classical distance-regular graphs contain many nice substructures, like designs in the Johnson schemes and the binary Golay code in the 23-cube. By looking at the eigenvalues of distance-regular graphs it is possible to get Euclidean representations for them. In this way, Terwilliger and Neumaier were able to show that the Johnson schemes are uniquely determined by their intersection arrays. For more information about distance-regular graphs and association schemes, there are the following four books: Biggs [11], Bannai and Ito [8], Brouwer, Cohen and Neumaier [19] and Godsil [47].

The following preprints and papers are included in this thesis (in order of their appearance as preprint):


(F) J.H. Koolen, On a conjecture of Martin on the parameters of completely regular codes and the classification of the completely regular codes in the Biggs-Smith graph, preprint (1993), to appear in IAMA.

(H) J.H. Koolen and S.V. Shpectorov, *Distance-regular graphs the distance matrix of which has only one positive eigenvalue*, European Journal of Combinatorics 15 (1994) 269-275.


This thesis consists of two parts. In the first part we study substructures in distance-regular graphs and this part contains Chapters 2, 3, 4 and 5. In the second part we study Euclidean representations of distance-regular graphs and this part contains Chapters 6 and 7.

In Chapter 1 we will introduce the basic definitions and concepts. The other six chapters can be read independently.

In Chapter 2 we study distance-regular subgraphs of distance-regular graphs. It contains the papers [D] and [E].

In Chapter 3 we study completely regular codes. These codes are an algebraic generalisation of perfect codes. They were invented by Delsarte, and Neumaier has given a definition in terms of partitions. In the first three sections we study a conjecture that W.J. Martin made in his thesis [74]. We show that this conjecture is true for most classical schemes, but also that it is not true in general. Furthermore all the completely regular codes in the Higgs-Smith graph are determined. In the fourth section we study perfect codes with two distinct radii in the binary Hamming scheme. As was remarked by Brouwer [18], the class of codes we study is a subclass of the class of completely regular codes. The first three sections are based on [F], the fourth section on [A].

In Chapter 4 we give a characterisation of the Doob graphs, using completely regular codes. This chapter is based on [G].

In Chapter 5 we construct some new graphs. In the first section we give two constructions for uniformly geodetic graphs. In the second section we construct a new distance-regular graph. This new graph is a bipartite double of the complement of the Berlekamp-van Lint-Seidel graph and its existence solves a problem in the book of Brouwer, Cohen and Neumaier, [19, p360]. The first section is based on [C] and partially on [B] and the second section is based on [I].

In Chapter 6 we consider the metric hierarchy for graphs. In the first two sections an introduction and the basic definitions are given. In the third section we will define root graphs and classify the amply regular root graphs. (We closely follow [19, §§3.14-3.15]; however, since there are several mistakes in those sections we thought it appropriate to completely give this piece of theory.) This culminates in Theorems 6.30, 6.40 and 6.42. (The proof of the \( \mu = 4 \) case in Theorem 6.30 is given in [B].) In the last section we will classify the distance-regular graphs whose distance matrix has exactly one positive eigenvalue. This last section is based on [H].

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In Chapter 7 we will look at the standard representations of distance-regular graphs. In the first section we will develop some theory using ideas of Terwilliger and Godsil. In the second section we will apply this theory to distance-regular graphs with \( a_d = 0 \). This section is based on [J]. In the last section we will give the classification of distance-regular graphs with an eigenvalue of multiplicity 8. This section is joint work with W.J. Martin.

At the first place I would like to thank my supervisor Andries Brouwer for his continuous support and patience during the period of my research. Furthermore I would like to thank the Discrete Mathematics group for the stimulating working environment, especially I am grateful to Aart Blokhuis, Hans Cuypers and Henny Wilbrink. Also my thanks go to prof.dr. J.J. Seidel for numerous discussions and for the careful reading of (parts of) a first draft of my thesis.

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Finally I would like to thank my family and friends.

Eindhoven, July 1994, \hspace{5cm} \text{Jack Koolen.}
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Chapter 1

Introduction

In this chapter we will introduce distance-regular graphs. Also we will explain the subjects of this thesis. There is only one exception, the metric theory of graphs will be introduced in Chapter 6. Only Section 3 is new in this chapter. For the notations we follow [19].

1.1 Distance-regular graphs

First we introduce distance-regular graphs. For more information about distance-regular graphs I would like to refer to the following four books: Biggs [11], Bannai & Ito [8], Brouwer, Cohen & Neumaier [19] and Godsil [46]. For a graph $\Gamma$ and a vertex $x$ of $\Gamma$ define $\Gamma_i(x)$ as the set of vertices at distance $i$ from $x$. We usually write $\Gamma(x)$ instead of $\Gamma_0(x)$.

A connected graph $\Gamma$ is called distance-regular if it is regular of valency $k$ and if there are integers $b_i, c_i$ ($i \geq 0$) such that for any two vertices $x, y \in V\Gamma$ at distance $i = d(x, y)$ there are precisely $c_i$ neighbours in $\Gamma_{i-1}(x)$ and $b_i$ neighbours in $\Gamma_{i+1}(x)$. The sequence

$$i(\Gamma) := \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\},$$

where $d$ is the diameter of $\Gamma$, is called the intersection array of $\Gamma$; the numbers $c_i, b_i$ and $a_i$, where

$$a_i = k - b_i - c_i \quad (i = 0, 1, \ldots, d)$$

is the number of neighbours of $y$ in $\Gamma_i(x)$ for $d(x, y) = i$, are called the intersection numbers of $\Gamma$. Clearly

$$b_0 = k, \quad b_d = c_0 = 0, \quad c_1 = 1.$$ 

By counting edges $yz$ with $d(x, y) = i$ and $d(x, z) = i + 1$, we see that $\Gamma_i(x)$ contains $k_i$ vertices, where

$$k_0 = 1, \quad k_1 = k, \quad k_{i+1} = \frac{k_i b_i}{c_{i+1}} \quad (i = 0, 1, \ldots, d - 1).$$ 

A distance-regular graph with intersection array $\{b_0, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ has intersection diagram:

![Intersection Diagram](image-url)

Examples.
(1) The polygons. They have intersection array \(\{2,1,1,\ldots,1;1,1,\ldots,1,c_d\}\) where \(c_d = 1\) for an \((2d+1)\)-gon and 2 otherwise.

(2) The Platonic solids. Their vertices and edges form distance-regular graphs with intersection arrays \(\{3,1\}\) (tetrahedron), \(\{4,1;1,4\}\) (octahedron), \(\{3,2,1,1,2,3\}\) (cube), \(\{5,2,1,2,5\}\) (icosahedron) and \(\{3,2,1,1,1,1,1,2,3\}\) (dodecahedron).

(3) The Petersen graph and its line graph. Identification of the antipodal points in the dodecahedron leads to the Petersen graph, which is distance-regular with intersection array \(\{3,2,1,1,1\}\). The line graph of the Petersen graph is also distance-regular: its intersection array is \(\{4,2,1,1,1,1,1\}\).

(4) The Johnson graphs. The Johnson graph \(J(n,t)\) has as vertices the \(t\)-subsets of an \(n\)-set and two subsets are adjacent if and only if their symmetric difference has cardinality 2. Without loss of generality we may assume that \(n \geq 2t\). Then it has intersection array \(\{(n-t)t,(n-t-1)(t-1),\ldots,n-2t+1;1,1,1,\ldots,t^2\}\).

It is not difficult to verify by drawing that in the first three cases the intersection array determines the graph uniquely (up to isomorphism, of course). In the fourth case it is not easy to see it. The Johnson graphs are the graphs uniquely determined by their intersection array, except for the array \(\{12,5;1,4\}\) as was shown by Neumaier and Terwilliger, independently. In Chapter 8 we will give a proof for this fact.

For a graph \(\Gamma\) we can define the \(i\)th distance matrix \(A_i(\Gamma)\) of \(\Gamma\) as the \((v \times v)\) 01-matrix with an \(xy\)-entry an 1 if and only if \(d(x,y) = i\), where \(v\) is the number of vertices of \(\Gamma\).

The adjacency matrix \(A(\Gamma)\) is the matrix \(A_1(\Gamma)\). If it is clear which graph we mean, then we write \(A\) instead of \(A_1(\Gamma)\).

The eigenvalues of a graph are the eigenvalues of its adjacency matrix \(A\).

Let \(\Gamma\) be a distance-regular graph with intersection array \(\{b_0, b_1, \ldots, b_{d-1}; 1, c_2, c_3, \ldots, c_d\}\). Now one easily checks that the matrices \(A_i\) satisfy the relations

\[
\begin{align*}
A_0 &= I, \quad A_1 = A, \\
A_i &= c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1} \quad (i = 1, 2, \ldots, d-1), \\
A_0 + A_1 + \cdots + A_d &= J,
\end{align*}
\]

where \(A\) is the adjacency matrix of \(\Gamma\). It follows that the matrices \(A_i\) can be written as polynomials in \(A\) of degree \(i\),

\[
A_i = v_i(A) \quad (i = 1, 2, \ldots, d),
\]

where the \(v_i\) are polynomials of degree \(i\) defined recursively by

\[
\begin{align*}
v_0(x) &= 1, \quad v_1(x) = x, \\
c_{i+1}v_{i+1}(x) &= (x-a_i)v_i(x) - b_{i-1}v_{i-1}(x) \quad (i = 1, 2, \ldots, d-1)
\end{align*}
\]

By (1.1) and (1.2) we find that

\[
A_iA_j = \sum_{\ell=0}^{d} p_{\ell} A_{\ell}
\]

for certain numbers \(p_{\ell}\). By comparison of the \(xy\)-entries we see that for vertices \(x, y\) at distance \(t\), the number \(p_t\) equals the number of vertices \(z\) with \(d(x,z) = i\) and \(d(y,z) = j\).
for \( i, j, t = 0, 1, \ldots, d \). In particular the \( p_{ij}^t \) are non-negative integers.

Now we look at the eigenvalues of \( \Gamma \). The adjacency matrix \( A \) has at least \( d+1 \) different eigenvalues, since the matrices \( A_i = u_i(A), (i = 0, 1, \ldots, d) \) are linearly independent. But also \( A A_d = b_{d-1} A_{d-1} + a_d A_d \), so there is a polynomial \( \omega \) of degree \( d+1 \) such that \( \omega(A) = 0 \); therefore \( A \) has exactly \( d+1 \) distinct eigenvalues.

Define the tridiagonal \((d+1) \times (d+1)\) matrix

\[
L_1 = \begin{pmatrix}
0 & k & & & \\
1 & a_1 & b_1 & & \\
& a_2 & b_2 & & \\
& & \ddots & & \\
& & & a_{d-1} & b_{d-1} \\
& & & c_d & a_d
\end{pmatrix}
\]

The algebraic multiplicity is equal to the numerical multiplicity for all the eigenvalues of \( L_1 \), because tridiagonal matrices are similar to tridiagonal symmetric matrices. For all numbers \( \tau \in \mathbb{R} \), the matrix \( L_1 - \tau I \) has rank at least \( d \), because \( c_i \neq 0 \), \( i = 1, 2, \ldots, d \). Hence \( L_1 \) has \( d+1 \) distinct eigenvalues. Let \( \theta \) be an eigenvalue of \( L_1 \). Let \( (1 = u_0(\theta), u_1(\theta), \ldots, u_d(\theta))^T \) be a right eigenvector of \( L_1 \) corresponding to \( \theta \), i.e.

\[
u_0(\theta) = 1, \quad u_1(\theta) = \theta/k, \quad c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (i = 1, 2, \ldots, d-1).
\]

Let \( x \in V \Gamma \). Define the vector \( w \) by \( w_y = u_i(\theta) \) if \( d(x, y) = i \). It is easy to see that the vector \( w \) is an eigenvector of \( A \) corresponding to \( \theta \). The conclusion is that the eigenvalues of \( \Gamma \) can be calculated from the intersection array only. Also the multiplicities of these eigenvalues can be calculated from the intersection array only, cf. [19, Theorem 4.1.4]. The array \( u_i(\theta) \) is called the standard sequence corresponding to \( \theta \).

Now we give some easy restrictions on the intersection arrays for distance-regular graphs.

**Proposition 1.1.** For a distance-regular graph, the following restrictions on the intersection array hold.

(i) \( k = b_0 \geq b_1 \geq \ldots \geq b_{d-1} \).

(ii) \( 1 = c_1 \leq c_2 \leq \ldots \leq c_d \).

(iii) If \( i + j \leq d \), then \( c_i \leq b_j \).

(iv) All parameters \( p_{ij}^k \) (including the \( k_j \)) are nonnegative integers.

(v) All the multiplicities of the eigenvalues are integral. \( \Box \)

Indeed, \( p_{ij}^k \) and multiplicities can be calculated from the intersection array. The significance of this proposition is that it is a sieve for potential intersection arrays. An intersection array is sometimes called 'feasible' if it passes these five tests. Thus, the intersection array of a distance-regular graph is feasible, but a feasible array need not correspond to a distance-regular graph.

If there are induced quadrangles in the distance-regular graph, then Terwilliger had shown the following theorem. \((A strongly regular graph is a distance-regular graph of diameter 2. In the next section we see some properties of strongly regular graphs.)\)
Theorem 1.2 (Terwilliger [106, 104], cf. [19, Theorem 5.2.1, Corollary 5.2.4]) Let Γ be a distance-regular graph with a quadrangle as induced subgraph. Then the following properties hold.
(i) $c_i - b_i \geq c_{i-1} - b_{i-1} + a_i + 2$ for $i = 1, 2, \ldots, d$.
(ii) For the diameter $d$ we have
\[ d \leq \frac{k + c_d}{a_d + 2}, \]
and if equality holds, then Γ is a strongly regular graph with smallest eigenvalue $-2$, a Hamming graph, a Dode graph (i.e. a cartesian product of $4$-cliques and Shrikhande graphs), a locally Petersen graph, a Johnson graph, a halved cube or the Gossel graph. □

This theorem was a main tool of Terwilliger in his proof of the fact that the Johnson graphs are uniquely determined by their intersection array. The locally Petersen graphs have been classified by J. Hall.

Theorem 1.3 (Hall [57]) Up to isomorphism, there are precisely three connected locally Petersen graphs, namely:

(i) The complement of the triangular graph $T(7) = J(7, 2)$. It has 21 vertices and intersection array \{10, 6; 1, 6\}.
(ii) The Conway-Smith graph. It has 63 vertices and intersection array \{10, 6, 4, 1; 1, 2, 6, 10\}.
(iii) The Doro graph. It has 65 vertices and intersection array \{10, 6, 4, 1; 2, 5\}.

1.2 Strongly regular graphs and related graphs

A graph of diameter 1 is a clique and hence distance-regular with intersection array \{k; 1\}. A connected strongly regular graph is a distance-regular graph of diameter 2. If it has intersection array \{k, b_1; 1, c_2\} then we say that it is a strongly regular graph with parameters \{(v, k, \lambda, \mu)\}, where $v$ is the number of vertices, $\lambda = a_1$ and $\mu = c_2$. The following theorem gives some properties of strongly regular graphs.

Theorem 1.4 (cf. [19, Theorem 1.3.1]) Let $\Gamma$ be a connected strongly regular graph with parameters \{(v, k, \lambda, \mu)\}. Then the following holds.
(i) \(v = 1 + k + k(k - \lambda - 1)/\mu\).
(ii) The eigenvalues of $\Gamma$ are $k, r, s$ where $r \geq 0$, $s < -1$, where $r, s$ are the roots of the quadratic equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$.
(iii) The eigenvalues $r$ and $s$ are integers, unless $r$ and $s$ have the same multiplicity. Then $r, s = (\pm 1 \pm \sqrt{5})/2$ and $\Gamma$ is called a conference graph. Conference graphs have parameters \{(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)\} for some $t \geq 1$.

If $\mu < k$ then the complement of $\Gamma$ is again a connected strongly regular graph with parameters \{(v, \bar{k}, \bar{\lambda}, \bar{\mu})\}, where
\[
\bar{k} = v - k - 1,
\bar{\lambda} = v - 2k + \mu - 2,
\bar{\mu} = v - 2k - \lambda,
\]
and has eigenvalues $\bar{k}, -s - 1, -r - 1$. □
Examples of conference graphs are the *Pailey graphs* $QR(q)$, defined for prime powers $q \equiv 1 \pmod{4}$, which have $F_q$ as vertex set and where two vertices are adjacent if and only if their difference is a nonzero square, cf. Pailey [87]. ($QR$ is mnemonic for 'quadratic residue'.)

A generalisation of strongly regular graphs are *amply regular* graphs. A connected graph is amply regular with parameters $(v, k, \lambda, \mu)$ if it is regular with valency $k$, each edge lies in $\lambda$ triangles and each pair of vertices at distance 2 have $\mu$ common neighbours. We will see that if $\mu \geq 2$ then the regularity condition is superfluous.

A generalisation of the amply regular graphs are the $(s, c, a, k)$-graphs. For any integers $s, c, a, k$ with $a + 2, s, c, k \geq 2$, a graph $\Gamma$ is called an $(s, c, a, k)$-graph if it satisfies the following properties:

(i) the girth of $\Gamma$ is $2s$ if $a = 0$, and $2s - 1$ otherwise,

(ii) for two vertices at distance $s - 1$ there are $a$ paths of length $s$ connecting them,

(iii) for two vertices at distance $s$, there are $c$ paths of length $s$ connecting them,

(iv) the maximal valency in $\Gamma$ is $k$.

They were introduced by Terwilliger [103]. He showed the following proposition.

**Proposition 1.5** An $(s, c, a, k)$-graph $\Gamma$ is regular or bipartite and semiregular. If $s = 2$, then $\Gamma$ is regular. □

### 1.3 Uniformly geodetic graphs

A *uniformly geodetic graph* is a connected graph such that the number of geodesics (i.e. shortest paths) between any two vertices $x$ and $y$ only depends on the distance $d(x, y)$.

These graphs were introduced by Cook and Pryce [32]. They are also called $F$-geodetic graphs, see Ceccherini and Sappa [27] and Scapellato [92], where $F(j)$ is the number of geodesics between two vertices at distance $j$.

For $x, y$ vertices at distance $i$ in a graph $\Gamma$ we define $c_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma(y), a_i(x, y) = \Gamma_i(x) \cap \Gamma(y), b_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma(y)$. We say that the number $c_i$ (resp. $a_i, b_i$) exists if $c_i(x, y)$ (resp. $a_i(x, y), b_i(x, y)$) does not depend on the vertices $x, y$.

The following lemma gives a trivial consequence of this definition.

**Lemma 1.6** A connected graph is uniformly geodetic if and only if the numbers $c_i$ exist. □

Thus uniformly geodetic graphs generalise distance-regular graphs and trees. An easy property of uniformly geodetic graphs is the following.

**Lemma 1.7** For a uniformly geodetic graph with numbers $c_i, i = 1, 2, \ldots, d$ we have

\[ 1 = c_1 \leq c_2 \leq \ldots \leq c_d. \]

□

In the next subsection we will characterise bipartite uniformly geodetic graphs and give some examples of them.
1.3.1 Bipartite uniformly geodetic graphs

We consider the partition into the two classes of the vertex set of a bipartite graph as a colouring with the two colours red and green. So we can speak of the colour of a vertex to indicate its class. If $\Gamma$ is semiregular then the valency of a red vertex is denoted by $k_r$ and that of a green vertex is denoted by $k_g$. We define $d_x = \max\{d(x, y) \mid x, y \text{ vertices and } x \text{ red }\}$ and $d_z$ analogously. Without loss of generality we assume that $d_x \leq d_z$.

A connected bipartite graph $\Gamma$ is distance-biregular if $\Gamma$ is semiregular and for any two vertices $x$ and $y$ of $\Gamma$ at distance $i$ the number $c_i(x, y)$ only depends on the colour of $x$. We write $c^x_i$ for $c_i(x, y)$ if $x$ is red, and similarly for $c^y_i$, $d^x_i$ and $d^y_i$.

Proposition 1.8 A uniformly geodetic graph $\Gamma$ with (finite) even girth is a regular graph or a distance-biregular graph.

Proof. By Lemma 1.6 the numbers $c_i$ exist. Let girth($\Gamma$) = $2m$. So $c_m > 1$. Hence we obtain that $\Gamma$ is an $(s, c, a, k)$-graph with $s = m$, $e = c_m$, $a = 0$. By Proposition 1.5, an $(s, c, a, k)$ graph is regular, or bipartite and semiregular. But now it is obvious that $\Gamma$ is either a regular graph, or a distance-biregular graph. □

We shall now give a characterisation of bipartite uniformly geodetic graphs.

Theorem 1.9 A bipartite uniformly geodetic graph $\Gamma$ is a tree, a distance-regular graph or a distance-biregular graph.

By Lemma 1.6 the numbers $c_i$ exist. If they are all 1, then there cannot be a cycle in $\Gamma$. Otherwise, the girth of $\Gamma$ is even and by the previous proposition we are done. □

Remark. Recently Scapinato [93] has shown a special case of the above theorem.

In the rest of this subsection we consider uniformly geodetic distance-biregular graphs.

Lemma 1.10 (Delorme [35]) For a distance-biregular graph we have

(i) $c^x_{i}c^y_{i+1} = c^x_{i}c^y_{i+1}$ for $i = 1, 2, \ldots, \lfloor \frac{d_x-1}{2} \rfloor$.

(ii) $b^x_{i}b^y_{i-1} = b^x_{i}b^y_{i-1}$ for $i = 1, 2, \ldots, \lfloor \frac{d_x-1}{2} \rfloor$. □

Proposition 1.11 (Delorme [35]) If for a distance-biregular graph we have $d_x < d_y$, then $d_x + 1 = d_z$ and $d_y$ even. □

Lemma 1.12 (Van den Akker [1]) For a distance-biregular graph we have

(i) $c^x_{i} \leq b^x_{i}$ if $i + j$ even and $i + j \leq d_x$.

(ii) $c^x_{i} \leq b^x_{i}$ if $i + j$ even and $i + j \leq d_x$. □

Proposition 1.13 Let $\Gamma$ be a uniformly geodetic distance-biregular graph. Then:

(i) $d_x = d_x + 1$ is even.

(ii) $c_{2i} = c_{2i-1}$ for $i = 1, 2, \ldots, \lfloor \frac{d_y}{2} \rfloor$.

Proof. (i) By $k_c \neq k_r$ we have $d_r \neq d_g$. So $d_g = d_x + 1$. But now it is obvious that $d_y$ is even.

(ii) By Lemma 1.10 we have

$b^x_{i}b^y_{i-1} = b^y_{i}b^x_{i-1}$ for all $i = 1, 2, \ldots, \lfloor \frac{d_y-1}{2} \rfloor$. 6
So we get \((k_n - c_2)(k_0 - c_{2i-1}) = (k_n - c_2)(k_n - c_{2i-1})\) and hence \(c_{2i} = c_{2i-1}\) for \(i \leq (d_n - 1)/2\). Also \(c_{d_n} = k_0 = c_{d_n} = c_{d_n+1}\). □

We now give some examples of uniformly geometric distance-biregular graphs.

(i) Let \(X = \{1, 2, \ldots, n\}\) and \(t < n\). We define the graph \(J(n,t,t+1)\) as the graph with vertices all the \(t\)-subsets and the \((t+1)\)-subsets of \(X\) and \(U \sim W\) if \(|U \cap W| = 1\).

(ii) The point-block incidence graph of a 2-design with \(\lambda = 1\).

(iii) Let \(q\) be a prime power. Let \(V\) be an \(n\)-dimensional vector space over the field \(F_q\). Let \(G_q(n,t,t+1), n > t \geq 0\) be the graph with vertices the \(t\)-dimensional and the \((t+1)\)-dimensional subspaces of \(V\), and a \(t\)-dimensional subspace \(U\) is adjacent to a \((t+1)\)-dimensional subspace \(W\) if \(U \subseteq W\) and those are all the adjacencies. Note that \(c_{d}(G_q(n,t,t+1)) = q + 1\).

(iv) Let \(\Delta\) be a strongly regular graph with \(\mu = 1\). Then define the graph \(\Gamma\) with vertices the vertices of \(\Delta\) and the \((\lambda + 2)\)-cliques, and \(x \sim y\) if and only if \(x \in\) clique for a vertex \(x\) of \(\Delta\). Then \(\Gamma\) is a uniformly geometric graph of diameter 6, and \(c_1 = c_2 = c_3 = c_4 = 1, c_5 = c_6 = \lambda(\Gamma) + 2\).

Remarks. (i) For a proof that the graphs of example (iii) are uniformly geometric, see DELORME [35].
(ii) Example (iv) was first described by DELORME [35]. The only known strongly regular graphs with \(\mu = 1\) are the pentagon, the Petersen graph and the Hoffman-Singleton graph. They all have \(\lambda = 0\).

CUYPERS [33] has shown that the examples (i) and (iii) are special in some way.

Theorem 1.14 ([33, CUYPERS, Theorem 4.7]) Let \(\Gamma\) be a uniformly geometric distance-biregular graph with diameter \(d \geq 5\) and \(c_3 = q + 1\) for some integer \(q \geq 1\). The \(q\) is a prime power and one of the following holds.

(i) If \(q = 1\), then \(\Gamma\) is a \(J(n,t,t+1)\).
(ii) If \(q > 1\), then \(\Gamma\) is a \(G_q(n,t,t+1)\). □

This theorem generalises a result of RAY-CHAUDHURI AND SPRAGUE [88]. Now we look at \(d = 4\).

Proposition 1.15 Let \(\Gamma\) be a distance-biregular graph with \(d = 4\). Then \(\Gamma\) is uniformly geometric if and only if \(\Gamma\) is the block-point incidence graph of a non-symmetric \(2-(v,k,1)\)-design.

Proof. Straightforward. □

1.4 Partitions and completely regular codes

1.4.1 Partitions

Let \(\Gamma\) be a graph. A partition \(P\) of the vertex set \(V\) is called regular when for \(A, B \in P\) and \(a \in A\), the number \(e_{AB} = |\Gamma(a) \cap B|\) does not depend on the choice of \(a \in A\).

The partition into singletons is always regular; the partition \(\{V\}\) is regular precisely when \(\Gamma\) is regular. For any group of automorphisms of \(\Gamma\), the partition of \(V\) into \(\Gamma\)-orbits is regular. We already have seen that the partition \(\{\{z\}, \Gamma_1(z), \Gamma_2(z), \ldots, \Gamma_d(z)\}\)
of the vertex set of a distance regular graph \( \Gamma \), where \( z \) is a vertex of \( \Gamma \), is regular.

Regular partitions give information about the spectrum: the eigenvalues of the matrix \( (e_{AB})_{A,B \in \mathcal{P}} \) are eigenvalues of \( \Gamma \) (with at least the same multiplicity). Much more information can be found in Godsil & McKay [50] (who, following Schwenk [96], call such partitions equitable).

The distribution diagram of \( \Gamma \) with respect to a regular partition \( \mathcal{P} \) consists of a number of balloons \( b_A \), one for each element \( A \in \mathcal{P} \), and a number of lines \( l_{AB} \) joining two balloons \( b_A \) and \( b_B \), one for each pair \( \{A, B\} \) for which \( e_{AB} \neq 0 \) (then also \( e_{BA} \neq 0 \)). The lines \( L_{AA} \) are usually not drawn. This diagram is provided with numbers as follows: in the balloon \( b_A \) we write \( |A| \), and at the \( A \)-end of \( l_{AB} \) we write \( e_{AB} \). The number \( e_{AA} \) is just written to \( b_A \); sometimes, when \( e_{AA} = 0 \), we write \( -\).

**Examples.** The structure of the dodecahedron around a face is shown by

```
\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \node (4) at (3,0) {4};
  \node (5) at (4,0) {5};
  \node (6) at (0,-1) {6};
  \node (7) at (1,-1) {7};
  \node (8) at (2,-1) {8};
  \node (9) at (3,-1) {9};
  \node (10) at (4,-1) {10};

  \draw (1) -- (2) -- (3) -- (4) -- (5) -- (1);
  \draw (6) -- (7) -- (8) -- (9) -- (10) -- (6);

  \draw (1) -- (6);
  \draw (2) -- (7);
  \draw (3) -- (8);
  \draw (4) -- (9);
  \draw (5) -- (10);
\end{tikzpicture}
\end{center}
```

We obtain a more complicated picture by looking at the structure around the two vertices \( x, y \) of an edge in the Petersen graph

```
\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (-1,0) {3};
  \node (4) at (0,1) {4};
  \node (5) at (0,-1) {5};

  \draw (1) -- (2) -- (3) -- (4) -- (5) -- (1);
  \draw (1) -- (4);
  \draw (2) -- (5);
\end{tikzpicture}
\end{center}
```

As suggested by the two examples, we want to determine \( \mathcal{P} \) by giving a few of its elements. And indeed this can be done in a canonical way:

**Proposition 1.16** Let \( S \) be an arbitrary partition of the vertex set \( VT \) of a graph \( \Gamma \). Then there is a unique coarsest partition \( \mathcal{P} \) of \( VT \) finer than \( S \), that is regular in the above sense. \( \square \)

Thus, we have associated a distribution diagram with an arbitrary partition of \( VT \); the 'distribution diagram around \( C \)' for \( C \) a subset of \( VT \) is that associated with \( \{C, VT/C\} \).

### 1.4.2 Codes and completely regular codes

For us a code in a graph \( \Gamma \) is just a non-empty subset of the vertex set of \( \Gamma \). Let \( C \) be a code in a graph \( \Gamma \). For a vertex \( x \), the distance \( d(x, C) \) is defined by \( d(x, C) = \min \{d(x, c) | c \in C\} \). Define \( C_x = \{x | d(x, C) = 1\} \). We denote the cardinality of \( C_x \) by \( k_x \). The covering radius, \( \rho \), of a code is the maximal \( m \) such that \( C_m \neq \emptyset \).

A code in a connected graph is called completely regular if the partition

\[
\{C = C_0, C_1, C_2, \ldots, C_\rho\}
\]
is regular. We denote by $\gamma_i$ (resp. $\alpha_i, \beta_i$) the number $e_{c_i,c_{i-1}}$ (resp. $e_{c_i,c_i}, e_{c_i,c_{i+1}}$). A completely regular code has intersection diagram:

\[
\begin{array}{ccccccc}
\kappa_0 & \beta_0 & \gamma_1 & \kappa_1 & \beta_1 & \gamma_2 & \kappa_2 \\
\alpha_0 & \beta_1 & \gamma_2 & \kappa_2 & \beta_2 & \gamma_3 & \kappa_3 \\
\end{array}
\]

A completely regular code in a regular graph has intersection array

\[\{\beta_0, \beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_3\}\].

Remark that then $\alpha_i = k - \beta_i - \gamma_i$, if $k$ is the valency of $\Gamma$. This definition of completely regular codes is due to Neumaier and he has shown that this definition is equivalent with the original definition of completely regular codes in distance-regular graphs by Delsarte. Note that for a distance-regular graph and a distance-biregular graph each vertex is a completely regular code. It was shown by GODBIL AND SAWALHE-TAYLOR [51] that all connected graphs such that every vertex is a completely regular code are distance-regular or distance-biregular. Remark that completely regular codes come in pairs. A code $C$ is completely regular if and only if $C^\perp$ is completely regular. It is not true that for a completely regular code in a distance-regular graph, the array $\gamma_1, \gamma_2, \ldots, \gamma_3$ is an increasing array. We saw already that the pentagon in the dodecahedron is an example for this. In Chapter 3 we will show that this array is an increasing array for most classical graphs. Also in that chapter we will show that the array $\kappa_0, \kappa_1, \ldots, \kappa_3$ is not always unimodular for a completely regular code in a distance-regular graph.

A code $C$ is a perfect (e-error-correcting) code if for each $x \in VT$ there is a unique $c \in C$ with $d(x, c) \leq e$. Note that a perfect $e$-error-correcting code is completely regular in a distance-regular graph and has intersection array $\{k, b_1, \ldots, b_{e+1}; c_1, c_2, \ldots, c_4\}$.

If $C$ and $D$ are two codes then $d(C, D) := \min \{d(c, d) \mid c \in C, d \in D\}$. A regular partition is called uniformly regular if there are constants $e_0$ and $e_1$ such that $e_{CD} = e_0$ if $C = D$ and $e_{CD} = e_1$ if $d(C, D) = 1$ for all $C, D \in P$ (of course $e_{CD} = 0$ if $d(C, D) \geq 2$). A partition $P$ is called completely regular if it is regular, and all $C \in P$ are completely regular with the same intersection diagram. Note that a completely regular partition is uniformly regular. Let $P$ be a partition of the vertex set of $\Gamma$. The quotient graph $\Gamma/P$ is the graph with vertices the clauses of $P$ and two clauses $C, D$ are adjacent if $d_P(C, D) = 1$. The following theorem shows that if you have a completely regular partition $P$ in a distance-regular graph, then the quotient graph is also distance-regular.

**Theorem 1.17** ([19], Theorem 11.1.6.) Let $P$ be a uniformly regular partition of a distance-regular graph $\Gamma$. Then $\Gamma/P$ is distance-regular if and only if $P$ is completely regular.

Let $C$ be a linear code over $F_q$ (i.e. a vector space over $F_q$). We consider $C$ as a subset of the vertex set of the Hamming graph $H(n, q)$. If $C$ is completely regular in $H(n, q)$, then also all its translates $C + c, c \in F_q^n$ are completely regular with the same parameters as $C$. Hence the quotient graph $H(n, q)/P$ is distance-regular, where $P$ is the partition of $F_q^n$ into the cosets of $C$. In this case the quotient graph is called the coset graph belonging to $C$.

**Example.** The binary Golay code is perfect and linear, hence the coset graph $\Gamma$ belonging to this code is distance-regular and has intersection array $\{23, 22, 21; 1, 2, 3\}$. 

9
Chapter 2

On subgraphs in distance-regular graphs

2.1 Pappus subgraphs

In this section we give a sufficient condition to assure that there are Pappus subgraphs in a distance-regular graph. As an application of this we are able to rule an array as the intersection array of a distance-regular graph.

**Theorem 2.1** Let be a graph with \( a_1 = a_2 = a_3 = 0, \) \( c_i = i - 1 \) for \( i = 2, 3, 4 \). Then any pair of vertices at mutual distance 4 determines a unique Pappus graph: a geometrically closed subgraph on 18 vertices with intersection array \( \{3, 2, 2, 1; 1, 1, 2, 3\} \).

**Proof.** Let \( z_1, z_2 \) be two vertices at distance 4. Let \( \Gamma(z_1) \cap \Gamma(z_2) = \{y_1, y_2, y_3\} \), \( \Gamma_2(z_1) \cap \Gamma_2(z_2) = \{z_1, z_2, z_3, z_4, z_5, z_6\} \), \( \Gamma_3(z_1) \cap \Gamma(z_2) = \{y_4, y_5, y_6\} \). Without loss of generality we have \( z_1 \sim y_1 \sim z_2, z_3 \sim y_3 \sim z_4, z_5 \sim y_2 \sim z_6, z_1 \sim y_4 \sim z_3, z_2 \sim y_5 \sim z_4, z_6 \sim y_6 \sim z_5 \). Also we have \( d(y_1, y_6) = 4 \) (not 3, otherwise \( a_3 \neq 0 \)) and \( \Gamma(y_1) \cap \Gamma(y_6) = \{z_1, z_2, z_1\} \), \( \Gamma(z_1) \cap \Gamma(z_6) = \{z_4, z_5, z_2\} \), \( \{y_4, y_5, y_2, y_6\} \subseteq \Gamma_2(y_1) \cap \Gamma_2(y_6) \).

By \( c_3 = 2 \) we have \( d(z_1, z_4) = 4 \) (the distance is 2 or 4, but not 2, since in that case there would be more than two shortest paths joining \( z_4 \) and \( y_4 \) which are at distance 3 from each other). By looking at the geodesics between \( y_1 \) and \( y_6 \), there is a vertex \( v_3 \) such that \( z_1 \sim v_1 \sim z_3 \) and a vertex \( u_3 \) such that \( z_2 \sim u_3 \sim z_4 \). In the same way, we have a vertex \( u_3 \) with \( z_2 \sim u_3 \sim z_2 \). All these \( u_i \) are different. By looking at the pair \( (z_1, z_4) \) one can see that there is a vertex \( u_3 \) with \( u_1 \sim v_1 \sim u_2 \). Similarly, there is a vertex \( u_3 \) with \( u_1 \sim v_2 \sim u_4 \). We have \( d(y_2, u_3) = 4 \) and \( \{z_6, z_1, v_1, v_2\} \subseteq \Gamma(y_3) \cap \Gamma(v_1) \); hence \( u_1 = v_2 \).

Now the subgraph induced by \( \{z_1, z_2, z_1, z_2, \ldots, z_6, y_1, y_2, \ldots, y_6, u_1, u_2, v_3, v_4\} \) is the Pappus graph, the distance-regular graph with intersection array \( \{3, 2, 2, 1; 1, 1, 2, 3\} \). □

**Corollary 2.2** The numbers \( \frac{b_4}{4} \) and \( \frac{b_6}{36} \) are integral.

**Proof.** These numbers are the number of Pappus graphs on an edge and the total of Pappus graphs. □

The latter condition rules out the existence of distance-regular graphs with intersection array \( \{7, 6, 5, 4, 3; 1, 1, 2, 3, 4\} \), \( v = 686, \Delta = 210 \).
2.2 Distance-regular subgraphs of distance-regular graphs

2.2.1 Introduction

In this section we study distance-regular subgraphs of distance-regular graphs. In the second subsection we give some sufficient conditions to assure that the graph induced by the geodesics between two vertices is distance-regular. Terwilliger [103] has given the diameter bound \( d \leq (s - 1)(k - 1) + 1 \) for distance-regular graphs with girth \( 2s \) and valency \( k \geq 3 \). In the third subsection we show that the only distance-regular graphs with even girth which reach this bound are the hypercubes and the doubled Odd graphs (Theorem 2.7) and give a somewhat improved diameter bound for bipartite distance-regular graphs.

In the fourth subsection we study distance-regular subgraphs in a hypercube. In this subsection the subgraphs are not necessarily subgraphs. Woeginger [109] has studied them and conjectured that the only distance-regular subgraphs of a hypercube are the even polygons, the hypercubes and the doubled Odd graphs and proved this in the case of girth 4. We show that if the girth is 6, then it must be a doubled Odd graph (Theorem 2.14). If the girth is equal to 8 then the valency is at most 12 (Theorem 2.17).

2.2.2 Substructures

Let \( \Gamma \) be a graph. For two vertices \( x, y \) of \( \Gamma \), put \( C(x, y) := \{ z \mid d(z, x) + d(z, y) = d(x, y) \} \).

Let \( \Delta(x, y) \) denote the graph with vertex set \( C(x, y) \) and two vertices \( u, v \in C(x, y) \) are adjacent if and only if \( u \) is an edge in \( \Gamma \) and \( d(u, x) \neq d(v, x) \). In this subsection we investigate when \( \Delta(x, y) \) is distance-regular.

Proposition 2.3 (I) Let \( e \) be an integer, \( e \geq 2 \). Let \( \Gamma \) be a uniformly geoedetic graph (or, more generally, a graph such that \( c_i \) exists for \( 2 \leq i \leq e \)) such that \( c_{e-1} < c_e \).

Then we have the following:

(Ia) For all vertices \( u, u' \) at distance \( e \) there exists a bijective map \( \phi: \Gamma(u) \cap \Gamma_{e-1}(u') \to \Gamma_{e-1}(u) \cap \Gamma(u') \) such that \( d(v, \phi(v)) > e - 2 \).

(Ib) \( c_i + c_{e-1} \leq c_e \) for all \( i, 1 \leq i \leq e - 1 \).

(II) Let \( e \) be an integer, \( e \geq 1 \). Let \( \Gamma \) be a distance-regular graph such that \( b_{e+1} < b_e \).

Then we have the following:

(IIa) For all vertices \( u, u' \) at distance \( e \) there exists a bijective map \( \phi: \Gamma(u) \cap \Gamma_{e+1}(u') \to \Gamma_{e+1}(u) \cap \Gamma(u') \) such that \( d(v, \phi(v)) < e + 2 \).

(IIb) \( c_i + b_{e+1} \leq b_e \) for all \( i, 1 \leq i \leq d - e \).

Proof. (I) Let \( u, u' \) be two vertices at distance \( e \). Put \( S := \Gamma(u) \cap \Gamma_{e-1}(u') \) and \( S' := \Gamma_{e-1}(u) \cap \Gamma(u') \).

(i) Define the set \( P_s \) by \( P_s := \{ s' \in S' \mid d(s, s') \geq e - 1 \} \) for \( s \in S \). Analogously we define \( P_{s'} \) for \( s' \in S' \). Note that \( |P_s| = |P_{s'}| = c_{e-1} \). Let \( \Delta \) be a graph with vertex set \( S \cup S' \) such that \( \Delta(s) = P_s \) and \( \Delta(s') = P_{s'} \) for \( s \in S, s' \in S' \). Then \( \Delta \) is a regular bipartite graph and thus has a complete matching (cf. [86], Theorem 7.5.2). So we have (Ia).

(ii) Let \( v \) be a vertex such that \( d(u, v) = e \) and \( d(u', v) = e - i \) for some integer \( 1 \leq i \leq e - 1 \). Denote \( A = \{ s \in S \mid d(s, v) = e - 1 \} \) and \( B = \{ s' \in S' \mid d(s', v) = e - i - 1 \} \). It follows that \( \phi(A) \cap B = \emptyset \). Now we get \( c_{e-1} - c_e = |A| + |B| = |\phi(A) \cup B| \leq c_{e-1} \).

The proof for (II) is analogous to (I). So we are done. \( \Box \)

Theorem 2.4 Let \( e \) be an integer, \( e \geq 2 \). Let \( \Gamma \) be a uniformly geoedetic graph such that
(i) $c_i + c_{e-i} = c_e$ for all $i$, $0 < i < e$,
and
(ii) $\Gamma$ does not contain two edges $xy$ and $zw$ such that $d(x, z) = e$ and $d(x, w) = d(y, z) = d(y, w) = e - 1$.

Then for any two vertices $u, u'$ of $\Gamma$ at distance $e$, the subgraph $\Delta(u, u')$ is a bipartite distance-regular graph with intersection array $\{c_i, c_{e-1}, \ldots, 1, 1, c_2, \ldots, c_e\}$.

**Proof.** Let $u, u'$ be two vertices of $\Gamma$ at distance $e$. Let $S, S'$ be defined as in the proof of the previous lemma. Let $s \in S$, then $d(s', s) = e - 1$ and so there is a unique vertex in $S'$, say $s'$, such that $d(s, s') \geq e - 1$, because $c_{e-1} = c_e - 1$. We have $d(u, u') = e$, $d(u, s') = d(u', s) = e - 1$ and $d(u, s) = d(u', s') = 1$ and hence, by (ii), we get $d(s, s') \neq e - 1$. So $d(s, s') = e$.

Now we will show: $C(u, u') = C(s, s')$. Let $v \in C(u, u') \setminus \{u, u'\}$. Then $d(v, u) = i$ and $d(v, u') = e - i$ for an integer $i$, $0 < i < e$. Let $A := \{t \in S \mid d(v, t) = i - 1\}$, $B := \{t' \in S' \mid d(v, t') = e - i - 1\}$ and $A' := \{a \in S \mid d(a, t') = e \text{ for an } a \in A\}$. Now we get $A' \cap B = \emptyset$ and thus $c_e = c_i + c_{e-1} = |A| + |B| = |A'| + |B'| = |A' \cup B| \leq c_e$.

So we have shown that $s \notin A$ implies $s' \in B$ and therefore $v \in C(s, s')$ and thus we get $d(v, u) = i - 1$ or $d(v, u') = i + 1$. We conclude that $C(u, u') \subseteq C(s, s')$, but these two sets have the same cardinality and thus they are equal.

Let $vw$ be an edge in $\Delta(s, s')$. Let $d(v, u) = i$. Then $d(v, s) = i + 1$ or $d(v, s) = i - 1$. Thus if $d(v, u) = i$, then $d(w, s) = d(v, s)$, but this is impossible because $vw$ is an edge in $\Delta(s, s')$. So $vw$ is an edge in $\Delta(u, u')$. With induction on $\min\{d(w, u), d(w, u')\}$ it is easy to prove that for all vertices $w \in C(u, u')$ there is a unique $w' \in C(u, u')$ such that $d(w, u') = e$. Furthermore, for such a pair we have $\Delta(u, u') = \Delta(w, w')$.

So we have shown that the subgraph $\Delta(u, u')$ is a bipartite distance-regular graph with intersection array $\{c_i, c_{e-1}, \ldots, 1, 1, c_2, \ldots, c_e\}$. $\Box$

**Remark.** For $c_i = i$, Mulder [80, 81] has shown the previous theorem without assumption (ii). More examples are given below.

**Proposition 2.5** If $\Gamma$ is the collinearity graph of a near polygon then (ii) holds.

**Proof.** Let $e \geq 3$. Suppose there are two edges $xy$ and $zw$ in $\Gamma$ such that $d(x, z) = e$ and $d(x, w) = d(y, z) = e - 1$. Let $x$ and $y$ lie on line $l$. There is a vertex $u$ on $l$ such that $d(u, w) = e - 2$, but then $d(u, z) \leq d(u, w) + d(w, z) = e - 1$. So $d(x, z) = e$, $d(x, y) = e - 1$ and $d(z, w) \leq e - 1$ and thus $u = y$, contradiction. $\Box$

**Examples.** (i) For $e = 2$ we find the not very surprising statement that in graphs with $\mu = 2$ and without induced $K_{2, \lambda+1}$ any two vertices at distance 2 determine a quadrangle. In particular this holds for grids $m \times n$, so that $\lambda$ need not be small.

(ii) Graphs with $(c_i)c_{e-1} = (1, 1, 2, 2, 3, \ldots, e \text{ odd})$ contain doubled Odd graphs. For example, this holds for Odd graphs and doubled Odd graphs. Thus, apart from the obvious inclusions $O_m \subseteq O_{m+1}$ and $2O_m \subseteq 2O_{m+1}$ we have $2O_m \not\subseteq O_{2m} (e = 2m - 1, m \geq 1)$.

**Corollary 2.6** Let $\Gamma$ be a distance-regular graph with $c_i = 1$, $c_{i+1} = \ldots = c_{2i} = 2$, $c_{2i+1} = 3$ and $\alpha_1 = \ldots = \alpha_{2i-1} = 0$, $\alpha_{2i} \leq 2$. Then $i \leq 2$. Furthermore one of the following holds

(i) $i = 1$ and any two vertices at distance 3 determines a unique $3$-cube,
(ii) $i = 2$ and $\Gamma$ is a Odd graph or a doubled Odd graph.

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Proof. By the previous theorem and Proposition 3, for any pair of vertices \( x, y \) at distance \( 2i + 1 \) the subgraph induced by \( C(x, y) \) is a bipartite distance-regular graph with \( k = 3, c_1 = 1, c_{i+1} = \ldots = c_{2i} = 2 \) and \( c_{2i+1} = 3 \). By DAMERELL [34] and also by BANNAI & ITO [11] there are no Moore graphs with diameter at least 3 and valency at least 3. Hence we get \( i \leq 2 \).

By RAY CHAUDHURI & SPRAGUE [88] and KOOLEN [67] a distance-regular graph with parameters \( d \geq 5, c_2 = 1, c_3 = c_4 = 2 \) and \( a_1 = a_2 = a_3 = 0 \) is an Odd graph or a doubled Odd graph. □

Remark 2. The case \( i = 1 \) of the previous corollary is contained in Brouwer [17].

Remark 3. Related work is done by Brouwer & Wilbrink [25], CHIMA [28] (cf. [19], Proposition 4.3.14), IVANOV [62] and Brouwer [16] (cf. [19], Proposition 4.3.11) and Koolen [65]. Brouwer and Wilbrink have investigated when there are geodetically closed substructures in near polygons. Chima has shown that in a distance-regular graph with parameters \( c_2 = 2, c_1 = 4, a_1 = 0 \) and \( a_2 \leq 3 \) any two vertices at distance 2 determine a unique distance-regular graph, which is isomorphic to the incidence graph of the biplane \( (7, 4, 2) \). Ivanov and Brouwer have given conditions to assure that graphs contain geodetically closed Moore geometries. Koolen has shown that in a distance-regular graph with parameters \( c_2 = 1, c_3 = 2, c_4 = 3 \) and \( a_1 = a_2 = a_3 = 0 \) any two vertices at distance 4 determine a unique Pappus graph.

### 2.3 On the Terwilliger bound

In this subsection we give the proofs of the results on the Terwilliger bound we have mentioned in Subsection 2.2.1.

**Theorem 2.7** The only distance-regular graphs with girth \( 2s - 1 \) or \( 2s \), valency \( k \geq 3 \), \( c_i \geq 2, c_s \geq a_{s-1} \) and diameter \( d \geq (s - 1)(k - 1) + 1 \) are the hypercubes and the doubled Odd graphs.

**Proof.** First let \( s = 2 \). Then \( d \geq k \) and by TERWILLIGER [106] (cf. [19], Corollary 5.2.4) we must have a hypercube.

Now let \( s \geq 3 \). TERWILLIGER [103] has proved that for a distance-regular graph with girth \( 2s - 1 \) or \( 2s \), valency \( k \geq 3 \), \( c_s \geq 2 \) and \( c_i \geq a_{i-1} \) we have \( c_i \geq c_{i-1} + 1 \) for \( s \leq i \leq d \), and \( b_i \leq b_{i-1} - 1 \) for \( s - 1 \leq i \leq d \). If the graph is non-bipartite, then it easy follows that \( d \leq (s - 1)(k - 1) \). So the graphs must be bipartite. Now the only way to reach \( d = (s - 1)(k - 1) + 1 \) is that \( c_{2s-2} = 2 \) and \( c_{2s-1} = 3 \). By Corollary 2.6 the only possible graphs are the doubled Odd graphs. □

**Theorem 2.8** For a bipartite distance-regular graph with valency \( k \geq 3 \), and girth \( 2s \geq 6 \), not a doubled Odd graph, the diameter \( d \) is bounded by

\[
d \leq (s - 1)(k - 1) - \left\lfloor \frac{k - 3}{2} \right\rfloor.
\]

**Proof.** Recall that \( c_{i-1} + 1 \leq c_i \) for \( i = s, \ldots, d \).

Suppose that \( c_{2s-1} = 2 \), this means that \( d \geq 2(s - 1) \) and \( c_{2s-1} \geq c_s + 1 \geq 3 \). But if \( c_{2s-1} = 3 \) then by Corollary 2.6 we have a doubled Odd graph. Now there are two cases:

**Case 1:** \( c_{2s-1} \geq 3 \).
We shall show with induction that
\[ c_{d(s-1)-i+1} \geq 2 + 2i \text{ or } c_{d(s-1)-i+2} \geq 2 + 2i \text{ if } d > 2i(s-1) - i + 1. \]

For \( i = 1 \) it is true.

Let now \( d > 2i(s-1) - i + 1, c_{d(s-1)-i+1} \leq 2i \) and \( c_{d(s-1)-i+2} \leq 2i + 1 \) for some \( i \geq 2 \). If \( c_{d(s-1)-i+1} \geq 2i \), then \( c_{d(s-1)-i+1} \geq c_{d(s-1)-i+2} \geq 2i + 1 \), contradiction. So \( c_{d(s-1)-i+1} \geq 2i - 1 \) and thus \( c_{d(s-1)-i+2} \geq 2s - 2 \) and \( c_{d(s-1)-i+2} \geq 2i+1 \). We have shown that \( c_{d(s-1)-i+1} = 2i \) and \( c_{d(s-1)-i+2} = 2i + 1 \). Now \( c_{d(s-1)} + c_{d(s-1)-i+1} \geq c_{d(s-1)-i+2} \geq 3 + 2i - 1 = 2i + 2 \), but this is impossible by Proposition 2.3.

First let \( k = 2l \). If \( d > 2(l-1)(s-1) - l + 2 \), then \( c_{d(s-1)-i+1} \geq 2l - 1 \) or \( c_{d(s-1)-i+2} \geq 2l \). We get \( d \leq 2(l-1)(s-1) - l + 3 \), hence \( d \leq 2l - 1 \).

Now let \( k = 2l + 1 \). If \( d \geq 2l(s-1) - l + 2 \), then \( c_{d(s-1)-i+1} \geq 2l + 1 \) or \( c_{d(s-1)-i+2} \geq 2l + 2 \), but both are impossible. So \( d \leq 2l(s-1) - l + 1 \).

**Case 2:** \( c_{d(s-1)} \geq 4 \).

In the same way as in Case 1 we can show that
\[ c_{d(s-1)-i+1} \geq 2i + 2 \text{ or } c_{d(s-1)-i+3} \geq 2i + 3 \text{ if } d > 2i(s-1) - i + 2. \]

First let \( k = 2l \). If \( d \geq 2l(s-1) - l + 4 \), then \( c_{d(s-1)-i+1} \geq 2l \), or \( c_{d(s-1)-i+3} \geq 2l + 1 \), but both are impossible. Thus \( d \leq 2l(s-1) - l + 3 \leq 2l(s-1) - l + 2 \).

Now let \( k = 2l + 1 \). If \( d \geq 2l(s-1) - l + 4 \), then \( c_{d(s-1)-i+3} \geq 2l \), or \( c_{d(s-1)-i+4} \geq 2l + 1 \). So \( d \leq 2l(s-1) - l + 3 \leq 2l(s-1) - l + 1 \).

The conclusion is that
\[ d \leq (k-1)(s-1) - \left\lfloor \frac{k-3}{2} \right\rfloor. \]

\[ \square \]

**Remark 4.** The above bound is tight. The Foster graph with intersection array \[\{3,2,2,2,1,1,1,1,1,1,2,2,2,3\}\], has diameter 8, girth 10 and valency 3.

### 2.3.1 On distance-regular subgraphs of a cube

In this subsection subgraphs are not necessarily induced subgraphs. We show that the only distance-regular subgraphs of a hypercube with girth 6 are the doubled Odd graphs. Also we show that distance-regular subgraphs of a cube with girth 8 have valency at most 12. First we give some notation. From now on we say cube instead of hypercube. Let \( \Gamma \) be a subgraph of a cube. Let \( X \) be a vertex of \( \Gamma \). Without loss of generality we represent \( x \) with the vector \((000\ldots0)\). A vertex \( y \) lies on level \( r \) with weight \( s \) if \( d(y,x) = r \) and \( y \) is represented by a vector of weight \( s \) in the cube.

First we give two elementary lemmas.

**Lemma 2.9** Let \( \Gamma \) be a uniformly geodesic subgraph of a cube with parameters \( (c_i) \), and let \( y \) be a vertex on level \( i \) with weight \( i \). Then for each \( j \), \( 0 \leq j \leq i \), we have
\[
\binom{i}{j} \geq \frac{c_1 c_2 \ldots c_{j-1}}{c_1 c_2 \ldots c_j}.
\]
Proof. We calculate the number of vertices $z$ on level $j$ with $d(z, y) = i - j$. On the one hand we have that this number is

$$
\frac{c_i c_{i-1} \cdots c_{i-j+1}}{c_i c_j \cdots c_j},
$$

On the other hand we can consider such vertices as $j$-subsets of a $i$-set. So we are done.

\[\square\]

Lemma 2.10 Let $\Gamma$ be a uniformly geodetic subgraph of a cube with parameters $(c_i)$. Let $y$ be a vertex on level $i$ with weight $i$. If $c_{i-j} = c_i - e$, then

$$
(2.1) \quad \sum_{s=0}^{i} \binom{i-c_i}{j-s} c_s \geq \frac{c_i c_{i-1} \cdots c_{i-j+1}}{c_i c_j \cdots c_j}.
$$

Proof. We calculate the number of vertices $z$ on level $j$ with $d(z, y) = i - j$. This number equals the right side of (2.1).

Suppose that precisely $c_i - s$ neighbours of $y$ on level $i-1$ (considered as $(i-1)$-sets) contains the vertex $z$ (considered as a $j$-set). Then $0 \leq s \leq e$ and we find that the number of vertices $z$ is at most the left hand side of (2.1). \[\square\]

The proof of Weichsel [109], Theorem 5 shows

Lemma 2.11 Let $\Gamma$ be a distance-regular subgraph of a cube with valency $k$ and girth $2t \geq 6$.

(i) If $v$ is a vertex of $\Gamma$ on level $r$ and of weight $r$, then $2r - 1 < 2r - 1$.

(ii) If $k \geq r$, then $2r - 1 < 2r - 1$. \[\square\]

The next lemma is a modification of Theorem 5 of Weichsel [109].

Lemma 2.12 Let $\Gamma$ be a distance-regular subgraph of a cube with valency $k$ and girth $2t$.

(i) If $t \geq 3$ and $v$ is a vertex on level $r$ and of weight $r$ for an $r \geq 3$, then $c_{r-1} \leq \frac{2r-1}{3}$.

(ii) If $t \geq 3$ and $k \geq r \geq 3$, then $c_{r-1} \leq \frac{2r-1}{3}$.

Proof. (i) & (ii). Suppose $c_{r-1} = \frac{2r-1}{3}$. Then by Lemma 2.11, we have $c_r = c_{r-1}$. By Lemma 2.10 we get $c_r \leq \frac{7}{5}$ and thus $\frac{2r-1}{3} \leq \frac{7}{5}$. This implies $r \leq 2$. \[\square\]

Proposition 2.13 The Pappus graph is not a subgraph of a cube.

Proof. The Pappus graph has intersection array $\{3, 2, 2, 1; 1, 1, 2, 3\}$ and is the unique graph with this intersection array. Let $\Gamma$ be the Pappus graph. Let $w_i, i = 1, 2, 3$, be three vertices such that $d(w_i, v_j) = 4, i \neq j$. We have

$$
\Gamma_4(w_1) \cap \Gamma_3(v_2) = \Gamma_4(w_1) \cap \Gamma_3(v_3) = \{w_1, w_2, w_3, z_1, z_2, z_3\},
$$

such that $d(w_i, w_j) = d(z_i, z_j) = 4$ for $i \neq j$ and $d(w_i, v_j) = 2$. Let $w_1$ have weight $0$. If $v_2$ has weight $4$ then $\{w_1, w_2, w_3, z_1, z_2, z_3\}$ are two $3$-subsets of a $4$-set, and so there must be an $i$ and $j$ such that $d(w_i, v_j) = 3$, contradiction. Let now $v_2$ and $v_3$ be represented by $(100 \ldots 0)$ and $(1010 \ldots 0)$. But then at least five of $\{w_1, w_2, w_3, z_1, z_2, z_3\}$ are represented by a word of weight $2$ with an $1$ on the first position and this is impossible. \[\square\]

Theorem 2.14 Let $\Gamma$ be a distance-regular graph with valency $k$ and girth $6$. If $\Gamma$ is a subgraph of a cube then $\Gamma$ is the doubled odd graph with valency $k$. 

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Proof. Ray-Chaudhuri & Sprague [88] have shown that a bipartite distance-regular graph with $c_2 = 1$, $c_3 = c_4 = 2$ is a doubled Odd graph. If $k \geq 5$, then by Lemma 2.12 we get $c_4 \leq 2$ and hence $c_5 = c_6 = 2$, and so $\Gamma$ is a doubled Odd graph.

If $k = 4$, then by Lemma 2.11 we have $c_5 = 2$. Then we have one of the following possibilities.

$(i) \quad c_4 = 2,$
$(ii) \quad c_4 = 3, c_5 = 3, c_6 = 4,$
$(iii) \quad c_4 = 3, c_5 = 4,$
$(iv) \quad c_4 = 4.$

There are no bipartite distance-regular graphs with the parameters of cases $(ii)$, $(iii)$ and $(iv)$.

If $k = 3$, then we have the following possibilities.

$(i) \quad c_3 = c_4 = 2,$
$(ii) \quad c_3 = 2, c_4 = 3,$
$(iii) \quad c_3 = 3.$

The only possible graph in case $(ii)$ is the Pappus graph and this graph is ruled out by Proposition 2.13. Case $(iii)$ is not possible by [109], Theorem 7. □

Lemma 2.15 Let $\Gamma$ be a distance-regular subgraph of a cube with girth 8. If $c_{11} \leq 5$, then there is no vertex on level 11 with weight 11.

Proof. Suppose $c_{11} \leq 5$ and there is a vertex $y$ on level 11 with weight 11. By Lemma 2.10 we have $c_9 \leq 4$. Also by this lemma we have $c_{10} \leq 4$ or $c_8 \leq 3$ and thus $c_9 \leq 3$. Then we have $c_8 = 2$ or $c_9 = 3$. If $c_8 = 2$, then $c_9 = 1, c_4 = c_6 = 2$ and $c_7 = 3$. If $c_9 = 3$, then $c_8 = 1, c_4 = c_5 = 2, c_6 = c_7 = 3$ and $c_9 = 4$. There are no bipartite distance-regular graphs with $k = 3, c_3 = 1, c_4 = c_6 = 2$ and $c_7 = 3$, or with $k = 4, c_3 = 1, c_4 = c_5 = 2, c_6 = c_7 = 3$ and $c_8 = 4$. So, by Theorem 2, there are in both cases there are no bipartite distance-regular graph with these $c_i$'s. □

Lemma 2.16 Let $\Gamma$ be a distance-regular subgraph of a cube with girth 8. Then there is no vertex on level 14 with weight 14.

Proof. Suppose there is a vertex on level 14 with weight 14. If $c_{10} \geq 6$ then by Lemma 2.10 we have $c_{12} \geq 7$, and thus $c_{10}c_{11}c_{12} > \binom{12}{5}$. This is a contradiction with Lemma 2.9. So $c_{10} \leq 5$. By Lemma 2.10 we get $c_9 \leq 4$, $c_8 \leq 3$ and $c_4 \leq 2$.

Suppose $c_{11} \geq 6$. Then $c_{11} \geq 7$ by Lemma 2.10. If $c_{14} = 7$, then by Lemma 2.10 we get a contradiction. If $c_{14} \geq 8$, then by Lemma 2.9 we also get a contradiction. So $c_{11} \leq 5$ and by previous lemma we are done. □

Theorem 2.17 Let $\Gamma$ be a distance-regular subgraph of a cube with girth 8. Then its valency $k$ is at most 12.

Proof. If $k \geq 14$, then by [109], Lemma E, we have a vertex on level 14 with weight 14 and so by Lemma 2.16 we are done.

If $k = 13$, then we have a vertex, say $y$, on level 13 with weight 13. Suppose there is no vertex $x$ on level 14 with weight 14. We calculate now the number of vertices $u$ on level 10 with $d(u, y) = 3$. On one hand there must be a vertex $v$ on level 12 with $d(v, y) = 1$ and $d(v, u) = 2$. So the number is at most $\binom{13}{5} - \binom{13-c_{11}}{3}$. On the other hand we have that this number is equal to $c_{12}c_{11}c_{11}$. Suppose that $c_{11} \geq 6$. Then $c_{12} \geq 6$ and $c_{13} \geq 7$. 17
We have $\binom{10}{5} < 8.6.6$ and thus $e_{13} = 7$. This is also impossible. So $e_{11} \leq 5$ and we are done by Lemma 2.15. ∎

Remark 5. There are more examples of distance-regular subgraphs in distance-regular graphs. Some interesting examples are the Petersen graph in $J(6,3)$, the Shrikhande graph and the $4 \times 4$-grid in the halved 6-cube and the point-block incidence graph of the Fano plane in $J(7,3)$. The last example is not an isometric subgraph of $J(7,3)$.
Chapter 3

Completely regular codes

In the first three sections of this chapter we study a conjecture of Martin [74] on the parameters of completely regular codes in distance-regular graphs. In Section 1 we give some simple properties of completely regular codes. In Section 2 we show that this conjecture is not true in general, but for most classical graphs it is true. In Section 3 we show that there is a counterexample in the Biggs-Smith graph for a weakened version of Martin's conjecture. Furthermore in this section we classify the completely regular codes in the Biggs-Smith graph.

In Section 4 we study perfect codes in the Hamming scheme with two protective radii. We especially are interested in the subclass of bipartite perfect codes. This subclass is also a class of completely regular codes in the Hamming scheme. The main goal of this section is to give a non-existence result on this subclass of perfect codes.

The first three sections are based on the preprint of the author "On a conjecture of Martin on the parameters of completely codes and the classification of the completely regular codes in the Biggs-Smith graph", [68]. The fourth section is based on the paper with J.M. van den Akker and R.J.M. Vaessens, "Perfect codes with distinct protective radii", [2].

First we need some definitions and notation. Let $C$ be a subset of the vertex set of $\Gamma$. Then $d(x, C) = \min\{d(x, z) | z \in C\}$. We denote by $C_i$ the set $\{y | d(y, C) = i\}$. Write $\kappa_i$ for $|C_i|$. The covering radius $\rho$ of $C$ is the maximum $i$ with $C_i \neq \emptyset$. We say that $C$ is a completely regular code if there are constants $\alpha_i, \beta_i, \gamma_i$, $i = 0, \ldots, \rho$, such that each vertex in $C_i$ has $\alpha_i$ neighbours in $C_i$, $\beta_i$ neighbours in $C_{i+1}$ and $\gamma_i$ neighbours in $C_{i-1}$. The intersection matrix $A$ is the tridiagonal $(\rho + 1) \times (\rho + 1)$-matrix with $A_{i,i} = \beta_i, A_{i,i+1} = \alpha_i$, and $A_{i,i-1} = \gamma_i$ for $i = 0, \ldots, \rho$. Let $C$ be a completely regular code in a regular graph $\Gamma$ with valency $k$. Then the intersection array of $C$ is the array $(\beta_0, \beta_1, \ldots, \beta_{\rho-1}; \gamma_1, \gamma_2, \ldots, \gamma_\rho)$. A connected regular graph $\Gamma$ with valency $k$ is distance-regular if all codes $(x)$, for vertices $x$ of $\Gamma$, are completely regular codes with the same intersection array $(b_0, b_1, \ldots, b_{\rho-1}; c_1, c_2, \ldots, c_\rho)$. This array is called the intersection array of $\Gamma$. A graph $\Gamma$ is flag-transitive if the group of automorphisms is transitive on the vertices and the stabiliser of a vertex $x$ is transitive on $\Gamma_1(x)$. A connected graph is distance-transitive if for any two pairs of vertices $(x_1, y_1)$ with $d(x_1, y_1) = d(x_2, y_2)$ there is an automorphism $\pi$ such that $\pi(x_1) = x_2$ and $\pi(y_1) = y_2$. 19
3.1 Preliminaries and simple properties of completely regular codes

In this section we give some simple properties of completely regular codes. The straightforward proofs of most of the propositions, lemma's and theorems are omitted.

Proposition 3.1. For a completely regular code with covering radius $p$, in a regular graph $\Gamma$ of valency $k$ the following must hold.

(i) $\alpha_1 + \beta_1 + \gamma_1 = k$,
(ii) $\sum_{i=0}^{p} \alpha_i \leq (p+1)(k-2) + 2$, and
(iii) if $\alpha_i = 0$ for all $i$, then $\Gamma$ is bipartite.

Proposition 3.2. For a completely regular code $C$ in a connected regular graph with intersection array $\{\alpha_0, \alpha_1, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_p\}$ the subset $C_2$ is a completely regular code with intersection array $\{\gamma_0, \gamma_1, \ldots, \gamma_p; \beta_1, \beta_2, \ldots, \beta_p\}$.

Lemma 3.3. Let $C'$ be a completely regular code in a connected regular graph $\Gamma$, with covering radius $p$. Then $C' \cup C_2$ is a completely regular code if and only if $\alpha_i = \beta_2 -1$ and $\gamma_{p+1} = \beta$, for all $i$, $i < \frac{p+1}{2}$, and if $p$ odd, $\gamma_{\frac{p+1}{2}} = \beta_{\frac{p+1}{2}}$.

Theorem 3.4. (cf. [50], Corollary 2.3) Let $C$ be a completely regular code in a connected graph $\Gamma$ with covering radius $p$. Then the intersection matrix of $C$ has $p+1$ distinct eigenvalues. These eigenvalues are eigenvalues of $\Gamma$.

Theorem 3.5. (Neumaier [83], Theorem 3.2) Let $C$ be a completely regular code in a distance-regular graph $\Gamma$. Then there are constants $\eta_{ij}$ such that

$|V_i(x) \cap C_j| = \eta_{ij}$ for all $x \in C_i$.

Proposition 3.6. In a regular connected graph $\Gamma$, there is a 1-1 correspondence between the completely regular codes in $\Gamma$ with covering radius $1$ and the eigenvalues $\lambda > \rho < 1$ of $\Gamma$ such that there are integers $a, b$ with $v_i = a$ or $v_i = b$ for all vertices $i$ of $\Gamma$.

Proposition 3.7. (cf. Martin [74], Theorem 2.4.6). Let $\Gamma$ be a distance-regular antipodal code of a distance-regular graph $\Delta$ and let $\pi : \Gamma \to \Delta$ be the projection. Then for each completely regular code in $\Delta$ the code $D = \pi^{-1}(C)$ is a completely regular code in $\Gamma$ with the same intersection array as $C$.

This proposition has a partial converse.

Theorem 3.8. Let $\Gamma$ be a distance-regular antipodal 2-cover of a distance-regular graph $\Delta$. Let $x'$ be the antipodal of $x$ in $\Gamma$. Let $C$ be a completely regular of $\Gamma$ with covering radius $p$. Then either $x' \in C$ for all $x \in C$ or $C_2 = \{x' \mid x \in C\}$.

Proof. Let $x \in C$ with $x' \notin C$. There is an $i$ such that $x' \notin C_i$. Now by Theorem 3.5 for all $y \in C$ we have $y \notin C_i$. Let $I = \{i \mid \exists y \in \Gamma(y \in C_i)\}$. Again by Theorem 3.5 we have $\cup_{i \in I} C_i = \{y \mid y \in C\}$. Now since $\{y \mid y \in C\}$ is a completely regular code with the same parameters as $C$, it follows that its covering radius is $p$. Consequently $C_2 = \{y \mid y \in C\}$.

Remark 1. The above theorem is a generalisation of the fact that in a distance-regular 2-cover for a perfect code $C$ one has $x \in C$ if and only if $x' \in C$, cf. [19], Remark on the bottom of p. 439.
3.2 On a conjecture of Martin

In [74], Martin conjectured that \( \gamma_i \leq \gamma_{i+1} \) and \( \beta_i \geq \beta_{i+1} \) for a completely regular code in a distance-regular graph. In this section we prove this conjecture for several families of distance-regular graphs and give a family of counterexamples in the doubled Odd and Odd graphs showing that it does not hold in general. Even a weakened version of this conjecture is not true. It is not true that the numbers \( \kappa_i \) of a completely regular code in a distance-regular graph form a unimodal array. We shall give an example in Section 3.3.1. A very simple counterexample to Martin's conjecture is the following one. Take five vertices in the dodecahedron \( \Delta \) with induced subgraph a pentagon. This is a completely regular code in \( \Delta \) with intersection diagram

\[
\begin{array}{cccccc}
5 & 1 & 5 & 1 & 1 & 5 \\
2 & 1 & 2 & 5 & 1 & 1 \\
\end{array}
\]

and we see that \( \gamma_5 \leq \gamma_2, \beta_0 \leq \beta_1 \). Below we give an example where the array \( \{\gamma_i\} \) is not unimodal.

Lemma 3.9 Let \( \Gamma \) be a connected graph. Let \( C \) and \( D \) be two completely regular codes in \( \Gamma \) with the same parameters. Let \( s := \max\{1 : C \cap D \neq \emptyset\} \) Then \( c_i(C) \leq c_{i+s}(C) \) and \( b_i(C) \geq b_{i+s}(C) \).

Proof. Let \( x \in C_{i+s} \) such that \( d(x, D) = i \). Let \( y \in \Gamma(x) \) with \( d(y, D) = i - 1 \). Then \( y \in C_{i+s-1} \) by the triangle inequality. Hence \( c_{i+s}(C) \geq c_i(D) \) and thus \( c_{i+s}(C) \geq c_i(C) \). The proof for the second inequality is analogous. \( \square \)

The above example also shows that this lemma becomes false if we replace \( s \) by \( s - 1 \) in the inequalities.

Lemma 3.10 Let \( \Gamma \) be a connected graph and \( C \) be a completely regular code in \( \Gamma \). Suppose that \( \Gamma \) has an automorphism \( \phi \) such that \( d(\phi(c), C) \leq s \) for each \( c \in C \) and \( \phi(C) \neq C \). Then \( \gamma_i(C) \leq \gamma_{i+s}(C) \) and \( \beta_i(C) \geq \beta_{i+s}(C) \).

Proof. By the fact that \( \phi \) is an automorphism, it follows that \( \phi(C) \) is a completely regular code in \( \Gamma \) with the same parameters as \( C \). By the previous lemma we are done. \( \square \)

For a graph \( \Gamma \), denote by \( \Gamma^e \) the graph whose vertices are the symbols \( x^+, x^-, x \) a vertex of \( \Gamma \), and whose edges are the 2-sets \( \{x^+, y^+\} \), with \( x = y \) or \( x \sim y \), and \( x \neq y \).

The graph \( \Gamma^e \) is called the extended bipartite double of \( \Gamma \).

Lemma 3.11 Let \( \Gamma \) be a direct product of \( \Gamma_i \) (\( i = 1, \ldots, n \)), where \( \Gamma_i \) is a flag-transitive graph and has a non-identity automorphism \( \phi_i \) such that \( d(\phi_i(x), x) \leq 1 \) for all vertices \( x \) of \( \Gamma_i \), for all \( i \). Then \( \gamma_i(C) \leq \gamma_{i+1}(C) \) and \( \beta_i(C) \geq \beta_{i+1}(C) \) for all completely regular codes \( C \) in \( \Gamma \). In particular, this holds when \( n = 1 \) and \( \Gamma \) is a distance-transitive extended bipartite double of a graph \( \Delta \).

Proof. For a vertex \( x \) of \( \Gamma \) we have \( x = (x_1, x_2, \ldots, x_n) \), where \( x_i \) is a vertex of \( \Gamma_i \). An automorphism \( \pi \) of \( \Gamma \) can be extended to an automorphism \( \Pi \) of \( \Gamma \) by

\[
\Pi(x) = (x_1, \ldots, x_{i-1}, \pi(x_i), x_{i+1}, \ldots, x_n)
\]
for all vertices $x$ of $\Gamma$. We write $\Phi$ for the extended automorphism of $\phi$. Suppose that there is a vertex $x$ of $\Gamma$ such that $x \notin C$. Then there are vertices $y, z$ such that $d(y, z) = 1$ and $y \in C$. There is exactly one $y_0$ such that $c_0 \neq y_0$. There is a vertex $x$ in $\Gamma_{y_0}$ such that $\phi y_0(z) \sim z$ in $\Gamma_{y_0}$. By the flag-transitivity of $\Gamma_{y_0}$, there is an automorphism $\psi$ of $\Gamma_{y_0}$ such that $\psi(c_0) = z$ and $\psi(y_0) = \phi y_0(x)$. Let $x = \psi^{-1} \phi y_0$. Then $\psi(c_0) = y_0$ and for all vertices $y$ of $\Gamma_{y_0}$ we have $d(\psi(y), y) \leq 1$. Let $\Pi$ be the extended automorphism of $\pi$. The fact that $\Pi$ is an automorphism of $\Gamma$ implies that $\Pi(C)$ is a completely regular code of $\Gamma$ with the same parameters as $C$. By the previous lemma we are done. \qed

Lemma 3.12 Let $\Gamma$ be a distance-transitive extended bipartite double of a graph $\Delta$. Then $\gamma_1(C) \leq \gamma_{i+1}(C)$ and $\beta_1(C) \geq \beta_{i+1}(C)$ for all completely regular codes $C$ in $\Gamma$.

Proof. Directly from the previous lemma. \qedsymbol

Lemma 3.13 Let $\Gamma$ be an extended bipartite double of a graph $\Delta$. If for all completely regular codes $C$ in $\Delta$ we have $\gamma_1(C) \leq \gamma_{i+1}(C)$ and $\beta_1(C) \geq \beta_{i+1}(C)$, then $\gamma_i(C) \leq \gamma_{i+1}(C)$ and $\beta_i(C) \geq \beta_{i+1}(C)$ for all completely regular codes $C$ in $\Gamma$.

Proof. The fact that $\Gamma$ is an extended bipartite double of $\Delta$ is equivalent with that $\Gamma$ is bipartite has an automorphism $\phi$ of order 2 such that $\phi(x) \sim x$ for all vertices $x$ of $\Gamma$ and $\Gamma / \phi \cong \Delta$. Let $C_1$ be a completely regular code in $\Gamma$. Suppose that $\phi(C_1) = C_1$. Then $C = C_1 / \phi$ is a completely regular code in $\Delta$ with $\gamma_i(C_1) = \gamma_i(C)$ and $\beta_i(C) = \beta(C)$.

Theorem 3.14 Let $\Gamma$ be a connected flag-transitive graph. Let $\Gamma$ have a non-identity automorphism $\phi$, such that $d(\phi(x), x) \leq 1$ for all vertices $x$ of $\Gamma$. Let $\Delta$ be the distance $1$-or-2 graph of $\Gamma$. Then for each completely regular code $C$ in $\Delta$ we have $\gamma_i \leq \gamma_{i+1}$ and $\beta_i \geq \beta_{i+1}$.

Proof. Let $w$ be a vertex such that $\phi(w) \neq w$. Let $\{y, z\}$ be an edge of $\Gamma$. By the flag-transitivity, there is an automorphism $\pi$ with $\pi(w) = y$ and $\phi(\pi(w)) = z$. Then $\pi^{-1} \phi \pi$ is an automorphism of $\Gamma$ such that $\psi(y) = z$ and $d(\psi(x), z) \leq 1$ for all vertices $x$. So we showed that for each edge $\{y, z\}$ there is an automorphism $\psi$ such that $\psi(y) = z$ and $d(\psi(x), z) \leq 1$ for all vertices $x$ of $\Gamma$. Let $v_1, v_2$ be two vertices at distance 2. Let $v_1$ be a common neighbour of $v_1$ and $v_2$. Then there are automorphisms $\pi_1$ and $\pi_2$ such that $\pi_1(v_1) = w$ and $\pi_2(v_1) = v_2$, and $d(\pi_i(x), z) \leq 1$ for all vertices $x$ and $i = 1, 2$.

Theorem 3.15 Each completely regular code $C$ in $\Gamma$ has $c_i(C) \leq c_{i+1}(C)$ and $b_i(C) \geq b_{i+1}(C)$, where $\Gamma$ is a member of one of the following families.
(i) The Hamming graphs, $H(n,d)$,
(ii) the Johnson graphs, $J(n,d)$,
(iii) the Grassmann graphs, $G_q(n,d)$,
(iv) the symplectic dual polar graphs on $[C_d(q)]$,
(v) the orthogonal dual polar graphs on $[B_d(q)]$,
(vi) the orthogonal dual polar graphs on $[D_d(q)]$,
(vii) the orthogonal dual polar graphs on $[I^2D_{d+1}(q)]$,
(viii) the unitary dual polar graphs on $[A_{2d}(r)]$,
(ix) the unitary dual polar graphs on $[A_{2d-1}(r)]$,
(x) the bilinear forms graphs, $H_q(n,d)$,
(xi) the alternating forms graphs,
(xii) the Hermitian forms graphs,
(xiii) the symmetric bilinear forms graphs,
(xiv) the quadratic forms graphs,
(xv) the folded Johnson graphs, $J(2m,m)$,
(xvi) the folded cubes,
(xvii) the halved cubes,
(xviii) the Doob graphs, the direct products of Shrikhande graphs and 4-cliques,
(xix) the half dual polar graphs, $D_{m,m}(q)$,
(xx) the Ustimenko graphs, which are the distance 1-or-2 graphs of dual polar graphs on $[C_d(q)]$, and
(xxi) the Hammetter graphs, the extended bipartite doubles of the dual polar graphs on $[C_d(q)]$.

**Proof.** Only symmetric bilinear forms graphs with diameter at least 3 and characteristic odd, or with diameter at least 4 and characteristic even are not distance-regular, all the other graphs are distance-regular, cf. [19], p. 286 and the errata of that book. The graphs of families (xiv), (xviii), (xix) and (xxi) are not flag-transitive. The other graphs are flag-transitive and by Lemma 3.11 for any graph not a member of family (xiv), (xviii), (xix) or (xxi), we only have to find an automorphism φ, not the identity, such that $d(\phi(x),x) \leq 1$ for all vertices $x$. Let $\Gamma$ be a graph of one of the above families. If $\Gamma$ is a Hamming graph then you may take for $\phi$ a translation over a one-weight vector. If $\Gamma$ is a Johnson graph then $\phi$ is a transposition. If $\Gamma$ is a Grassmann graph then $\phi$ is a transvection. If $\Gamma$ is a symplectic dual polar graph then $\phi$ is a symplectic transvection. If $\Gamma$ is an orthogonal dual polar graph and the characteristic is not two then $\phi$ is a reflection. An orthogonal dual polar graph on $[A_d(q)]$ with $q$ even is isomorphic to a symplectic dual polar graph. If $\Gamma$ is an orthogonal dual polar graph on $[D_d(q)]$ or $[I^2D_{d+1}(q)]$ with $q$ even then $\phi$ is a transvection. If $\Gamma$ is a unitary dual polar graph then $\phi$ is a unitary transvection. The bilinear forms graph $H_q(n,d)$ is isomorphic to the induced subgraph of $G_q(n+d,d)$ on $[V \setminus W \cap W = 0]$ where $W$ is a $n$-subspace. So we can take for $\phi$ a transvection of $G_q(n+d,d)$ which is the identity on $W$. Let $\Gamma$ be the alternating forms graph on $G(q)^d$. Let $\Delta$ be an orthogonal dual polar graph on $[D_d(q)]$ and $x$ a vertex of $\Delta$. Then the induced subgraph of the distance 2 graph of $\Delta$ on $\Delta_d(x)$ is isomorphic to $\Gamma$. So take for $\phi$ a Siegel transformation which fixes $x$. Let $\Gamma$ be the Hermitian forms graph on $G(q)^d$. Let $\Delta$ be the unitary dual polar space on $[A_{2d-1}(r)]$ and $x$ a vertex of $\Delta$. Then $\Gamma$ is isomorphic to the induced subgraph of $\Delta$ on $\Delta_d(x)$. Now take for $\phi$ a unitary transformation which fixes $x$. Let $\Gamma$ be the symmetric bilinear forms graph on $G(q)^d$. Let $\Delta$ be the symplectic dual polar space on $[C_d(q)]$ and $x$ a vertex of $\Delta$. Then $\Gamma$ is isomorphic to the induced subgraph of $\Delta$ on $\Delta_d$. Now take for $\phi$ a symplectic transvection which fixes $x$. If $\Gamma$ is a folded Johnson graph or a folded cube, we are done.
by Proposition 3.7. If $\Gamma$ is a halved cube, then take for $\psi$ a transposition. If $\Gamma$ is a half dual polar graph, then take for $\psi$ a Siegel transformation.

Let $\Gamma$ be a quadratic forms graph of odd characteristic. Then $\Gamma$ is the distance 1-or-2 graph of a symmetric bilinear forms graph $\Delta$. By Lemma 3.14 we are done. If $\Gamma$ is a quadratic forms graph of even characteristic, then $\Gamma$ has a partition in cliques of order $q^2$, such that each clique belongs to the same alternating form. If $C$ is not the union of such cliques, then there is an automorphism $\phi$, which sends a quadratic form to another but leaves the bilinear form the same and $\phi(C) \neq C$. Otherwise $C$ covers a completely regular code in an alternating forms graph. So again we are done.

The Shr"{a}ehde graph $H$ is flag-transitive and has a non-identity automorphism $\phi$ such that $d(\phi(x), x) \leq 1$ for all vertices of $H$. So by Lemma 3.11, Martin's conjecture holds for the Dnsh graphs.

Let $\Delta$ be a dual polar graph on $[C_2(q)]$. The group $Sp(2d, q)$ is generated by symplectic transvections, and therefore has an element $\phi$ such that $d(\phi(x), x) \leq 2$ for all vertices $x$ of $\Delta$ and $(x, \phi(x)) = 2$ for a vertex $x$. By Lemma 3.14, the conjecture of Martin holds for the Unit-mer graphs. By Lemma 3.13, Martin's conjecture holds for the Hammet graphs. $\square$

**Remark 2.** Neumaier [33] asked for feasibility conditions for the parameters $\mu_1 = \mu_2 = \gamma_1$ of a completely regular code in a Hamming graph and the above theorem gives such a condition. Is it true that for a completely regular code in a Hamming graph the array $\{\gamma_1\}$ is a strictly increasing array?

**Remark 3.** More information on the families of classical graphs can be found in [19], Chapter 9. The results on the classical groups we use can be found in [101].

As we shall see later, for the Odd and doubled Odd graphs with valency at least 4, Martin's conjecture is not true. But for those graphs we can show a weaker version of this conjecture.

**Proposition 3.16** Let $\Gamma$ be a doubled Odd, Odd or a doubled Grassmann graph. Then for any completely regular code in $\Gamma$ we have $\gamma_1 \leq \gamma_{t+2}$ and $\beta_i \geq \beta_{t+2}$.

**Proof.** Let $\Gamma$ be a completely regular code in $\Gamma$. If $\Gamma$ is a doubled Odd or Odd graph and suppose that there is a vertex $x \notin C$. Then there is a transposition $\psi$ such that $\psi(C) \neq C$ and byLemma 3.10 we are done.

If $\Gamma$ is a doubled Grassmann graph and suppose that there is a vertex $x \notin C$. Then there is a transposition $\psi$ such that $\psi(C) \neq C$ and again by Lemma 3.10 we are done. $\square$

Let $N = \{1, 2, \ldots, 2n + 1\}$. Let $C'$ be the set of $n$-subsets and $(n + 1)$-subsets containing exactly $i$ elements of $\{1, 2, \ldots, 2n\}$.

**Lemma 3.17** The set $C'$ is a completely regular code in $DO_{n+1}$ with covering radius $\min\{2i, 2(n-i)+1\}$. For the parameters we have $\kappa_{2i-1} = 2i^{(2i)(\frac{(n-i)+1)}{2}}, \kappa_{2i} = 2i^{(2i)(\frac{(n-i)+1)}{2}}$. $\gamma_{2i} = n + i + j + 1, \gamma_{2i-1} = i + j, \beta_{2i} = i - j, \beta_{2i-1} = n - i - j + 1, \alpha_{i} = 0$ and $\alpha_{0} = n + 1 - i$.

**Proof.** For the $C_i$ we have

$$C_{2i-1} = \{V \mid V \text{ a } (n + 1) \text{-subset of } N \text{ and } |V \cap i| = i + j\} \cup \{W \mid W \text{ a } n \text{-subset of } N \text{ and } |W \cap i| = i - j\}$$

and

$$C_{2i} = \{V \mid V \text{ a } (n + 1) \text{-subset of } N \text{ and } |V \cap i| = i - j\} \cup \{W \mid W \text{ a } n \text{-subset of } N \text{ and } |W \cap i| = i + j\}.$$
Now the parameters follow easily. □

For $n \geq 3$ and $i = 1$ you get $\beta_2 > \beta_1$. The array $\kappa_i$ is decreasing for $C$. It is an open question whether the array $\kappa_i$ of a completely regular code in a doubled Odd graph has to be unimodal. For all $c \in C$ we have $N \setminus c \in C$ and this implies that $C$ is a completely regular code in $O_{n+1}$ with the same intersection array.

For $n = 7$ and $i = 3$ you get $\gamma_1 = 4, \gamma_2 = 6, \gamma_3 = 5, \gamma_4 = 7$ and hence the array $\{\gamma_i\}$ is not unimodal.

There is still no example known of a completely regular code in a doubled Grassmann graph with $\gamma_i > \gamma_{i+1}$, but I conjecture that there must be such an example.

### 3.3 Completely regular codes in the Biggs-Smith graph

#### 3.3.1 Examples

In this subsection we give the original construction of the Biggs-Smith graph $\Gamma$, due to Biggs & Smith [13], and construct some completely regular codes in $\Gamma$. In the next section we show that there are no other completely regular codes in $\Gamma$.

Let the vertex set of $\Gamma$ be the set $\{x_i, y_i, z_i^j \mid i \in \mathbb{Z}_{17}, j = 1, 2, 4, 8\}$. In $\Gamma$ we have the following edges, $\{x_i, y_i\}, \{x_i, z_i^j\}, \{x_i, z_i^j\}, \{y_i, z_i^j\}$ and $\{y_i, z_i^j\}$ for $i \in \mathbb{Z}_{17}, \{z_i^j, z_i^j\}$ if $|i - j| = j$. It is easy to see that the induced graph on $\{x_i, y_i, z_i^j \mid j = 1, 2, 4, 8\}$ has the form of an $H$.

The graph $\Gamma$ is a distance-regular graph with intersection array $\{3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 3\}$.

The following theorem contains some known facts about the Biggs-Smith graph. For more information about the Biggs-Smith graph and other constructions, see [19], Section 13.4 and [17].

**Theorem 3.18** (cf. [19], Theorem 13.4.1) There is a unique distance-regular graph with intersection array $\{3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 3\}$. It has 102 vertices and full automorphism group $PSL(2,17)$, acting distance transitively with point stabilizer $Sym(4)$. It has diameter 7, girth 9 and does not contain 10-cycles. Its spectrum is $3^1, 2^{18}, 0^{17}, (1 \pm \sqrt{17})^9, 0^1, \varepsilon (j = 1, 2, 3)$ where the $\varepsilon$ are the three roots of $\theta^2 + 3\theta - 3 = 0$. □

Let $\pi : \Gamma \to \Gamma$ be the map such that $\pi(x_i) = x_{i+1}$, $\pi(y_i) = y_{i+1}$ and $\pi(z_i^j) = z_{i+1}^j$. If $i = 1, 2, 4, 8$ and $i \in \mathbb{Z}_{17}$. It is easy to see that $\pi$ is an automorphism of $\Gamma$. Let $H = \langle \pi \rangle$ be the group generated by $\pi$. Let $X = \{x_i \mid i \in \mathbb{Z}_{17}\}$, $Y = \{y_i \mid i \in \mathbb{Z}_{17}\}$ and $Z_j = \{z_i^j \mid i \in \mathbb{Z}_{17}\}$, for $j = 1, 2, 4, 8$. The orbits under $H$ are $X, Y$ and $Z_j$ ($j = 1, 2, 4, 8$) and therefore they form an equitable partition with the following intersection diagram.

![Intersection Diagram](image_url)
Now it is easy to see that $Z_4 \cup Z_4$ is a completely regular code with intersection diagram

$$
\begin{array}{c}
34 & 2 & 17 & 2 & 17 & 2 & 34 \\
\end{array}
$$

$Z_1 \cup Z_2$ and $Z_4 \cup Z_4$ completely regular codes with intersection diagram

$$
\begin{array}{c}
34 & 2 & 34 & 1 & 34 & 2 & 68 \\
\end{array}
$$

and $X \cup Y$ a completely regular code with intersection diagram

$$
\begin{array}{c}
34 & 2 & 68 & 1 & 2 \\
\end{array}
$$

Note that $\kappa_0 = \kappa_1 = 34$ and $\kappa_2 = 17$ for the completely regular code $Z_1 \cup Z_1$ and thus the array $\{\kappa_i\}, i = 0, \ldots, 6$, is not a unimodal array. Let $\psi : \Gamma \rightarrow \Gamma$ be a map, such that $\psi(x_i) = y_{2i}$, $\psi(y_i) = x_{2i}$, $\psi(z_i^j) = z_{2i}^j$, if $j = 1, 2, 4$, and $\psi(z_i^3) = z_{2i}^3$ for $i \in Z_{17}$. It is easy to see that $\psi$ is an automorphism of $\Gamma$ of order $8$. The group $K = \langle \psi \rangle$ is isomorphic to $17 : 8$ (this is the semi-product of a $C_{17}$ and a $C_8$, using the notation of [39]). Now $K$ has two orbits on $\Gamma$ and one of them is $X \cup Y$. By the fact that $K$ is a maximal subgroup of $PSL(2, 17)$ it follows that $H$ is the stabilizer of $X \cup Y$ and of $Z_1 \cup Z_2 \cup Z_4 \cup Z_8$. It is obvious that $\psi$ is the only non-identity element of $\langle \psi \rangle$ which preserves the partition $\Pi = \{X, Y, Z_1, Z_2, Z_4, Z_8\}$. Also obvious is the fact that $\langle \psi^4 \rangle$ is the maximal subgroup of $\langle \psi \rangle$ which preserves the partition $\Pi_1 = \{X, Y, Z_1 \cup Z_4, Z_2 \cup Z_8\}$. In the same way one can see that only the identity of $\langle \psi \rangle$ stabilizes $Z_1 \cup Z_2$. This is also the only element of $\langle \psi \rangle$ which stabilizes $Z_1 \cup Z_8$. So we have shown the following proposition.

**Proposition 3.19** (i) The stabiliser of the partition $\{X \cup Y, Z_1 \cup Z_2 \cup Z_4 \cup Z_8\}$ is isomorphic to $17 : 8$ and has two orbits on $\Gamma$.

(ii) The stabiliser of the partition $\{X, Y, Z_1 \cup Z_4, Z_2 \cup Z_8\}$ is isomorphic to $17 : 4$ and has four orbits on $\Gamma$.

(iii) The stabiliser of the partition $\{X \cup Y, Z_1 \cup Z_2, Z_4 \cup Z_8\}$ is isomorphic to $D_{34}$ and has six orbits on $\Gamma$.

(iv) The stabiliser of the partition $\{X \cup Y, Z_1 \cup Z_8, Z_4 \cup Z_2\}$ is isomorphic to $D_{34}$ and has six orbits on $\Gamma$.

3.3.2 Classification of the completely regular codes in the Biggs-Smith graph

In this subsection we classify the completely regular codes in the Biggs-Smith graph. We say that a completely regular $C$ with covering radius $\rho$ in a distance-regular graph $\Delta$ is trivial if $\rho \leq 1$, $|C| \leq 1$ or all the vertices of $\Delta$ are in $C$. Let $\Gamma$ be the Biggs-Smith graph and $C$ a non-trivial completely regular code in $\Gamma$. Define $w_i = \eta_{0i}$ where $\eta_{ij}$ are the numbers of Theorem 3.5.

**Theorem 3.20** A non-trivial completely regular code with covering radius at most 3 in the Biggs-Smith graph has, up to reversal, one of the following intersection arrays.

(i) $\{2; 1\}$

(ii) $\{1, 1, 1, 1\}$

(iii) $\{1, 1, 2, 2, 1, 1\}$
Proof. Let $\rho$ be the covering radius of $C$ and let $A$ be the intersection matrix of $C$. We have to handle three cases.

**Case 1:** $\rho = 1$. By Theorems 3.4, 3.18 and Proposition 3.1(ii), the only two eigenvalues for the intersection array are 3 and 0. The only possibility is intersection array (i).

**Case 2:** $\rho = 2$. By Theorems 3.4 and 3.18 there are two possibilities for the eigenvalues for $A$. The first possibility is that the eigenvalues of $A$ are 3, 2, 0. Then $\text{tr}(A)$ equals 4. Then $\rho = 2$. Then $\text{tr}(A)$ equals 4. Without loss of generality $a_0 = 2$ and $\beta_0 = 1$. If $a_1 = 0$, then by Lemma 3.3 we have a completely regular code with $a_0 = 0$ and covering radius 1. This is impossible. So $a_1 = 1 = a_2 = \beta_1 = \gamma_1$ and $\gamma_2 = 2$. But then $\kappa_0 + \kappa_1 + \kappa_2 = 5\xi_2$ and 5 does not divide 102, contradiction.

**Case 3:** $\rho = 3$. By Theorems 3.4 and 3.18 and Proposition 3.1(iii), there are two possibilities for the eigenvalues of $A$. The first possibility is that the eigenvalues of $A$ are 3, 2, 0 and 2. Then $\text{tr}(A) = 6$ and therefore the intersection array has to be $[1, 1, 1; 1, 1, 1]$. But then $\sum_{i=0}^{3} \kappa_i = 4\xi_0$ and 4 does not divide 102, contradiction. The second possibility is that the eigenvalues are 3, 2, 0. Then $\text{tr}(A) = 4$ and $\det(A) = 0$. So $\kappa_0 = 0$ and $\kappa_3 > 0$. There are without loss of generality four possibilities for the vector $a = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and these are

(i) $a = (1, 1, 1, 1)$,  
(ii) $a = (1, 1, 0, 2)$,  
(iii) $a = (1, 0, 1, 2)$ and  
(iv) $a = (2, 0, 0, 2)$.

The only possible intersection array for (i) is $[1, 1, 2; 1, 1, 1]$ and then $\sum_{i=0}^{3} \kappa_i = 6\xi_0$ and so $\kappa_0 = 17$. But now $\kappa_0$ is odd, which is a contradiction with [74], Proposition 2.24(ii).

By $\det(A) = 0$, the only possible intersection array for (ii) is $[1, 1, 1; 1, 2, 2]$ and $\sum_{i=0}^{3} \kappa_i = 3\xi_0$ and hence such a code can not exist.

By $\det(A) = 0$, there are no possible intersection arrays for (iii).

By $\det(A) = 0$, the only possible intersection array for (iv) is (iii). ⊖

Lemma 3.21. The minimum distance of $C$ is less than 6.

Proof. Suppose that the minimum distance is 7. By [74], Theorem 2.3.3 it follows that $|C| > 2$. Because of $p_{2, 2} = 1$ we get $|C| = 3$. But it is easy to see that such a code is not completely regular. Suppose now that the minimum distance is 6. Then we have $\beta_0 = 3$, $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$, and $\alpha_0 = \alpha_1 = \alpha_2 = 0$. If $\alpha_3 \neq 0$ then $\alpha_3 = 1$ and $\sum_{i=0}^{3} \kappa_i = 3\xi_0$, but 16 does not divide 102. So $\alpha_3 = 0$. By calculating the number of 9-cycles in through one vertex we get that this number is at most $\beta_0 \beta_1 \beta_2 \beta_3 < 24$, because there are no 6-, 8- and 10-cycles in $\Gamma$. But the number of 9-cycles through one vertex in $\Gamma$ equals 24. ⊖

Lemma 3.22. The minimum distance of $C$ is not equal to 5.

Proof. Let the minimum distance be 5. By Theorem 3.20, we may assume that the covering radius be at least 4. Then we have $\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = \beta_1 = 1$, $\beta_0 = 3$ and $\beta_1 = 2$. In $\Gamma$ each pair of vertices at distance 5 are connected by exactly one path of length 5. By counting paths of length 5 with endpoints in $C$ it follows $w_5 = 6$ and hence $|C| \geq 7$.

If $\gamma_1 = 1$, then $\sum_{i=0}^{3} \kappa_i = 16\xi_0$, and hence $\kappa_0 \leq \frac{16}{102}$. So $|C| = \kappa_0 \leq 6$, contradiction.

So $\gamma_1 = 2$ and hence $\alpha_3 = 0$ and $\beta_3 = 1$, by the assumptions. If $\gamma_1 = 3$, then there exists also a completely regular code with minimum distance 5 and covering radius 2, by
folding. By Theorem 3.20 such a code does not exist.
If γ4 = 2, then \( \sum_{i=0}^{4} \kappa_i = 14 \frac{1}{2} \kappa_0 \) and \( 2 \gamma_4 = 3 \kappa_0 \). Hence \( \kappa_0 \) is even and thus \( |C| = \kappa_0 \leq 6 \), contradiction.
If γ4 = 1, then \( \sum_{i=0}^{4} \kappa_i = 16 \kappa_0 \). Hence \( |C| = \kappa_0 \leq 6 \), contradiction. 11

Lemma 3.23 The minimum distance of \( \Gamma \) is not 4.

Proof. Suppose that the minimum distance is 4. Then \( \alpha_2 = \alpha_0 = 0, \beta_0 = 3, \beta_1 = 2 = \gamma_2 \) and \( \gamma_1 = 1 \). In \( \Gamma \) every two vertices at distance 4 there is a path of length 5 in \( \Gamma \) such that the two vertices are endpoints in this path, by \( \alpha_2 = 1 \). Hence we get \( \alpha_2 \geq 1 \). But then the covering radius of \( C \) equals 2. By Theorem 3.20 this is impossible. 13

Lemma 3.24 The minimum distance of \( \Gamma \) is not 2.

Proof. Suppose that the minimum distance of \( C \) is 2. Then \( \alpha_0 = 0, \beta_0 = 3 \) and \( \gamma_1 = 2 \).
This implies \( w_2 = 3 \). There three vertices do not lie at mutual distance 2 and so by \( p_{1,2}^{2} = p_{2,1}^{2} = 0 \) we get \( w_4 > 0 \).
If \( \alpha_1 \neq 0 \), then \( \alpha_1 = 1 \) and because \( -2 \) is not an eigenvalue of \( \Gamma \) such a completely regular code does not exist.
So \( \alpha_1 = 0 \) and by \( w_4 > 0 \) and girth 9 we get that \( \alpha_2 \neq 0 \). But then \( p_{1,2}^2 \neq 0 \), but in \( \Gamma \) this number equals 0, contradiction. 1

Lemma 3.25 There are no non-trivial completely regular codes in \( \Gamma \) with cardinality at most 3.

Proof. By [74], Theorem 2.3.3 there are no completely regular codes of cardinality 2 in \( \Gamma \).
In the proof of [74], Proposition 2.3.1, Martin shows that a completely regular code with minimum distance \( t \) and cardinality 3 in a distance-regular graph with odd valency only exists if \( t = d \) or \( t \leq 2 \), where \( d \) is the diameter of the graph. By this and because the girth of \( \Gamma \) equals 9 the minimum distance of \( C \) must be 7.
But this is impossible by Lemma 3.21. 1

Lemma 3.26 It is impossible that the minimum distance of \( |C| \) equals 3, the covering radius \( \rho \) of \( C \) is at least 4 and \( \alpha_2 \geq 1 \).

Proof. Suppose there exists a completely regular code with these properties. For the parameters we have \( \alpha_0 = 0, \beta_0 = 3, \alpha_1 = \beta_1 = \gamma_2 = 1 \) and \( \gamma_1 \leq 2 \).
If \( \gamma_2 > 1 \), then by the fact that the girth equals 9, it follows that \( \alpha_2 > 6 \), contradiction. It follows that \( \sum_{i=0}^{3} \kappa_i \geq \kappa_0 \).
By \( \alpha_0 = 0 \) and \( \alpha_2 \neq 0 \) we get \( 3 \mid \kappa_i \) for \( i \neq 0 \). By \( e = 102 \), we have \( 3 \mid \kappa_2 \), and hence \( 9 \mid \kappa_i \) for \( i \neq 0 \). Thus \( \kappa_0 = 3(\text{mod } 9) \).
But now we have
\[
\sum_{i=0}^{3} \kappa_i \geq \sum_{i=0}^{3} \kappa_i \geq 8 1 \kappa_0
\]
and therefore \( \kappa_0 < 11 \). So \( \kappa_0 = 3 \), but there are no codes with \( |C| = 3 \), by Lemma 3.25.


Proposition 3.27 Let \( \rho \) be the covering radius of \( C \). Let \( \rho \geq 4 \). Then we have
(i) \( 2 \leq \alpha_0 + \alpha_1 + \alpha_{\rho-1} + \alpha_{\rho} \leq \text{tr}(A) \) and
(ii) \( \alpha_0 = 0 \) if and only if \( \alpha_2 = 0 \).
\[ \square \]
Lemma 3.28 There are no completely regular codes in the Biggs-Smith graph with covering radius 4.

Proof. Let the covering radius, ρ, be 4. By Theorems 3.4, 3.18 and Proposition 3.27 (i) there are two possibilities for the eigenvalues of A. The first case is when \(3, \theta_i, i = 1, 2, 3\) and 2 are the eigenvalues of A. Then \(\text{tr}(A) = 2\) and \(\text{det}(A) = 18\). Therefore, by Proposition 3.27 (i), we get \(\alpha_2 = 0\). If \(\alpha_0 = 0\), then Proposition 3.27 (ii) implies \(\text{det}(A) = 0\), contradiction. So \(\alpha_0 = \alpha_4 = 1\) and \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). Calculating the determinant of A we get \(18 = \text{det}(A) = 2(\beta_1 \gamma_2 \gamma_3 + \gamma_1 \beta_2 \gamma_3)\). Without loss of generality we get the intersection array \(\{2, 2, 1, 2; 1, 2, 1, 2\}\) and then \(\sum_{i=0}^{5} \kappa_i = 9\alpha_0\). But 9 is not a divisor of 102, contradiction.

The second possibility for the eigenvalues is that these are \(3, 2, 0, 1\sqrt{17}\). Then \(\text{tr}(A) = 6\), and hence without loss of generality \(\alpha_0 = 2\) and \(\alpha_1 = 1\). By \(\text{det}(A) = 0\) there are only two possible intersection arrays for C. The first one is \(\{1, 1, 1, 1, 1, 1, 1\}\), but then \(\sum_{i=0}^{5} \kappa_i = 9\alpha_4\) and 9 does not divide 102, contradiction. The second is \(\{1, 1, 1, 2, 1, 1, 1\}\) and then \(\sum_{i=0}^{5} \kappa_i = 6\alpha_3\) and thus \(\kappa_3 = 17\). But \(\alpha_1 \kappa_1\) is odd, contradicting [74, 2.2.4 (ii)].

Lemma 3.29 There are no completely regular codes in the Biggs-Smith graph with covering radius 5.

Proof. Let the covering radius, ρ, be 5. By Theorems 3.4, 3.18 and Proposition 3.27 (i) the eigenvalues of A are \(3, \theta_i, i = 1, 2, 3, 0\) and 2. Then \(\text{tr}(A) = 2\) and \(\text{det}(A) = 0\). Therefore, by Proposition 3.27 (i), we get \(\alpha_0 = \alpha_2 = 0\). If \(\alpha_0 = 0\), then Proposition 3.27 (ii) implies \(\text{det}(A) \neq 0\), contradiction. So \(\alpha_0 = \alpha_4 = 1\) and \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). Calculating the determinant of A we get \(0 = \text{det}(A) = \delta_1 \gamma_2 \gamma_3 + \gamma_1 \beta_2 \gamma_3\). Without loss of generality we get \(\gamma_4 = \beta_2 = 2\). There are now two possible intersection arrays. The first one is \(\{2, 2, 2, 1, 1, 1, 1, 2, 2\}\) and then \(\sum_{i=0}^{6} \kappa_i = 2\alpha_0\). But 27 does not divide 102, contradiction. The second possibility is \(\{2, 2, 1, 1, 2, 2, 2, 1\}\) and then \(\sum_{i=0}^{6} \kappa_i = 15\alpha_5\). But 15 does not divide 102, contradiction.

Lemma 3.30 There are no completely regular codes in the Biggs-Smith graph with covering radius 6.

Proof. By Theorems 3.4, 3.18 and Proposition 3.27 (i) the eigenvalues of A are \(3, \theta_i, i = 1, 2, 3, 0\) and \(1\sqrt{17}\), and hence \(\text{tr}(A) = 3\) and \(\text{det}(A) = -18\). Suppose that \(\alpha_0 \geq 1\). Then \(\alpha_0 \neq 0\), by Proposition 3.27. Without loss of generality we may assume that \(\alpha_0 = 1, \alpha_1 = \alpha_2 = 0\).

Let \( E \) be an edge in \( G \), say \( E = \{z, y\} \). By \( p_{z,y}^2 = 0 \), it follows that the covering radius of \( E \) equals 6. Moreover \( \Gamma_6(E) = \{p_{z,y}^2 + p_{z,y}^6 + p_{z,y}^6 \} \). By \( \kappa_{2,2} = 1 \) the vertices in \( \left( \Gamma_6(x) \cap \Gamma_6(y) \right) \cup \left( \Gamma_6(y) \cap \Gamma_6(x) \right) \) have valency two in the subgraph induced by \( \Gamma_6(E) \).

Suppose that \( \gamma_1 = 2 \). Then by the above reasoning it follows that \( |C| \leq 16 \). If \( \gamma_2 = 2 \), then \( |C| \geq w_0 + w_1 + \ldots + w_3 = 1 + 1 + 2 + 4 + 8 + 8 > 16 \), contradiction. So \( \gamma_2 = 1 \) and \( |C| \geq w_0 + w_1 + \ldots + w_3 = 1 + 1 + 2 + 4 + 4 + 4 = 16 \) and thus \( C \subseteq \Gamma_6 \). If \( \gamma_3 = 1 \), then \( \sum_{i=1}^{6} \kappa_i \geq 16 + 16 + 16 + 4 + 8 + 8 + 8 = 106 \), contradiction. So \( \gamma_3 = 2 \) and thus \( w_0 \geq 2 \), from which follows that \( |C| \geq 18 \), contradiction.

Hence \( \gamma_1 = 1 \). If \( \gamma_2 = 2 \), then \( |C|/2 \leq 24/5 \) and thus by \( |C| \), it follows \( |C| \leq 8 \). But \( |C| \geq w_0 + w_1 + w_3 = 1 + 4 + 4 \), contradiction. So \( \gamma_2 = 1 \). Now we have \( |C|/2 \leq 24/7 \), and hence \( |C| \leq 6 \). But then \( \gamma_3 = 2, \alpha_2 = 1, \) or \( \gamma_3 = 1, \alpha_3 = 0 \) and \( \gamma_4 = 2 \). In each of these cases you easily see that \( |C| \geq 8 \), contradiction. Hence \( \alpha_0 = 0 \) and by Proposition 3.27
(iii) we have \( \alpha_0 = \alpha_0 = 0 \) and \( \alpha_1 = \alpha_5 = 1 \). Without loss of generality \( \alpha_2 = 0 \) and by \( \det(A) \neq 0 \) we get \( \alpha_3 = 0 \) and \( \alpha_4 = 1 \). If \( \gamma_2 = 2 \) then \( v_4 > 0 \) and thus \( \alpha_2 \neq 0 \), because there must be a path of length 5 between two members of \( C \) at distance 4. So, by \( \det(A) = -18 \), the only possible intersection array is \( \{3,1,2,2,1.1;1,1,1,1,1,1,3\} \) and then \( \sum_{i=0}^{4} \alpha_i = 41 \). But 41 does not divide 102, contradiction.

Lemma 3.31 There are no non-trivial completely regular codes in the Biggs-Smith graph with covering radius 7.

**Proof.** The eigenvalues of the Biggs-Smith graph are also eigenvalues of \( A \), and hence \( \det(A) = 0 \) and \( \text{tr}(A) = 3 \). Because of \( \alpha_2 = 0 \) it follows that \( \alpha_7 = \alpha_0 = 0 \) and \( \alpha_1 = \alpha_0 = 1 \). Now without loss of generality \( \alpha_2 = \alpha_4 = 0 \), by \( \sum_{i=2}^{5} \alpha_i = 1 \). But then \( \det(A) \neq 0 \), contradiction.

Lemma 3.32 The only completely regular codes in the Biggs-Smith graph with intersection array \( \{2;1\} \) are the images of \( X \cup Y \) under \( PSL(2,17) \).

**Proof.** By Proposition 3.19, the stabiliser of \( X \cup Y \) is \( 17:8 \) and has two orbits on \( X \) and one of them is \( X \cup Y \). The vector \( v \) defined by

\[
    v = \begin{cases} 
        2 & \text{for } x \in X \cup Y, \\
        -1 & \text{otherwise}
    \end{cases}
\]

is an eigenvector of \( \Gamma \) for the value 0. Let \( O := \varphi_{PSL(2,17)} \). The cardinality of \( O \) equals 18. The space \( V \) spanned by \( O \) is a subspace of the 0-eigenspace of \( \Gamma \). Using the fact that the permutation representation of \( PSL(2,17) \) with point stabiliser \( 17 : 8 \) is the direct sum of two irreducible representations, one of dimension 1 and one of dimension 17, see [30], p.9, \( V \) is a 1-dimensional space and the 0-eigenspace is 17 dimensional. We get that \( V \) is the 0 eigenspace. There is a 1:1 relation between completely regular codes with intersection array \( \{2;1\} \) and 0-eigenvectors with on 34 positions a 2 and on 68 positions a -1. Using GRAPE [90], GAP [99] and Magma [76] we found that there are only 18 0-eigenvectors with on 34 positions a 2 and on 68 positions a -1. So we are done.

Lemma 3.33 The only completely regular codes in the Biggs-Smith graph with intersection array \( \{1,1;1,1\} \) are the images of \( Z_1 \cup Z_4 \) under \( PSL(2,17) \).

**Proof.** Let \( C \) be a completely regular code with intersection array \( \{1,1;1,1\} \). Then \( C \cup C_2 \) is a completely regular code with intersection array \( \{1;2\} \). By the previous lemma we have that \( C_1 \) is an image of \( X \cup Y \) and therefore \( C \) must be an image of \( Z_1 \cup Z_4 \), \( i < j \). The only possibilities for \( (i,j) \) are \( (1,4) \) and \( (2,8) \). But \( Z_2 \cup Z_6 \) is an image of \( Z_1 \cup Z_4 \) under \( PSL(2,17) \).

Lemma 3.34 The only completely regular codes in the Biggs-Smith graph with intersection array \( \{1,1,2;2,1,1\} \) are the images of \( Z_1 \cup Z_5 \) under \( PSL(2,17) \).

**Proof.** Let \( C \) be a completely regular code with intersection array \( \{1,1,2;2,1,1\} \). Then \( C \cup C_2 \) is a completely regular code with intersection array \( \{1;2\} \). By the previous lemma we have that \( C_1 \cup C_2 \) is an image of \( X \cup Y \) and therefore \( C \) must be an image of \( Z_1 \cup Z_4 \), \( i < j \). The only possibilities for \( (i,j) \) are \( (1,2) \), \( (1,8) \), \( (2,4) \) and \( (4,8) \). But \( Z_4 \cup Z_6 \) are images of \( Z_1 \cup Z_4 \) under \( PSL(2,17) \).
Theorem 3.35. A non-trivial completely regular code in the Biggs-Smith graph has one of the following intersection arrays, up to reversal.

(i) \{2; 1\}
(ii) \{1, 1; 1, 1\}
(iii) \{1, 1, 2; 2, 1, 1\}

And for each intersection array there is a completely regular code with this intersection array. Furthermore the only examples are the examples we construct in Section 3.3.1. The eigenvalues of the intersection matrix are in case (i), 3 and 0, in case (ii), 3, 2 and 0, and in case (iii), 3, $\frac{9\pm\sqrt{17}}{2}$ and 0. □

3.4 Perfect codes with distinct protective radii

3.4.1 Introduction

We shall use standard terminology. A code $C$ of length $n$ over an alphabet $Q$ with $q$ symbols is a subset of $Q^n$, i.e., we consider codes in a Hamming scheme. We denote the cardinality $|C|$ of the code by $M$; $d$ is the minimum (Hamming-)distance of the code. A ball $B_r(c)$ with center $c$ and radius $r$ is defined by

$$B_r(c) := \{x \in Q^n | d(x, c) \leq r\}.$$  

Suppose that $C$ is the union of two disjoint subcodes $C_1$ and $C_2$ such that the following holds. There are integers $r_1$ and $r_2$ such that the balls $B_r(c)$ where $r = r_i$ if $c \in C_i$ ($i = 1, 2$) are disjoint. We then call $C$, with the specified subcodes $C_1$ and $C_2$, a $(r_1, r_2)$-error-correcting code. Let $M_i := |C_i|$ and let $d_i$ be the minimum distance of $C_i$. Then $d_i \geq 2r_i + 1$ ($i = 1, 2$). If we also define

$$d_{1,2} := \min\{d(c_1, c_2) | c_1 \in C_1, c_2 \in C_2\},$$

then it is also clear that

$$d_{1,2} = r_1 + r_2 + 1.$$  

Define $r(c) = r_i$ if $c \in C_i$. If $c$ and $c'$ are codewords with

$$d(c, c') = r(c) + r(c') + 1,$$

then $c$ and $c'$ will be called adjacent. In this way a graph is defined on the vertex set consisting of codewords of $C$. The code $C$ is called bipartite if the two sets $C_1$ and $C_2$ are cliques (independent sets) in this graph.

If

$$Q^n = \bigcup_{c \in C_1} B_{r_1}(c) \cup \bigcup_{c \in C_2} B_{r_2}(c),$$

then $C$ is called a perfect $(r_1, r_2)$-error-correcting code.

Let $V(n, r)$ denote the volume of a ball of radius $r$ in $Q^n$, i.e.,

$$V(n, r) = \sum_{i=0}^{r} \binom{n}{i} (q-1)^i.$$
Then obviously

\begin{equation}
V(n, r) = V(n, r - 1) + \left( \begin{array}{c} n \\ r \end{array} \right)(q - 1)^r.
\end{equation}

and from well-known relations for binomial coefficients we find

\begin{equation}
qV(n, r) = V(n, r + 1) + \left( \begin{array}{c} n \\ r \end{array} \right)(q - 1)^{r+1}.
\end{equation}

If \( C \) is a perfect \( (r_1, r_2) \)-error-correcting code in \( Q^n \), then by (3.3) we have

\begin{equation}
M_1V(n, r_1) + M_2V(n, r_2) = q^n.
\end{equation}

Perfect codes with distinctive radii were introduced by Cohen & Montaron [29, 78]. They were also studied by Gundlach [56] (see also [54, 55]). The examples we mention below were already found by Cohen & Montaron (all of which were also found by the present authors, before Prof. J.H. van Lint called Gundlach's work to their attention; no new examples were found). The aim of this section is to prove a non-existence result that does not occur in [56]. This theorem states that Example 4 (below) is unique (in a certain sense).

**Example 1.** Let \( C \) be any \( r \)-error-correcting code and let \( C \) consists of the words of \( C \) and all words in \( Q^n \) with distance \( r \) to \( C \). Then \( C \) is a trivial example of a perfect code with distinct protective radii, namely an \( (r, 0) \)-error-correcting code.

**Example 2.** Let \( C \) be the binary repetition code of length \( n \), and \( C_1 = \{0\} \). If \( r_1 + r_2 + 1 = n \) then this is a perfect \( (r_1, r_2) \)-error-correcting code.

**Example 3.** Let \( C \) be a perfect \( r \)-error-correcting code of length \( n \). Let \( C' \) be the code of length \( n - 1 \) obtained from \( C \) by puncturing (on the last position). We define \( C_1 \), respectively \( C'_1 \), to be the subcodes of \( C \) obtained from the words of \( C \) ending in \( 0 \), respectively not ending in \( 0 \). Then \( C' \) is a perfect \( (r, r - 1) \)-error-correcting code (since we have \( |C'| = q^{-1}|C'| \)). Examples can be made using the known perfect codes.

**Example 4.** Let \( P \) be a code with the parameters of a Preparata code of length \( n = 2^k - 1 \). Let \( H \) be the union of \( P \) and all words in \( Z_2^n \) that have distance \( \delta \) to \( P \). In [112] it is shown that \( H \) is a perfect \( 1 \)-error-correcting code. Now take the extended codes \( C := H \) and \( C_1 := P \). Then \( C \) is a perfect \( (2, 1) \) error-correcting code. Note that if \( P = Q \) is the Preparata code, then \( H \) is the Hamming code (see [6]).

We can now state the main theorem of this section.

**Theorem 3.36.** If \( C \) is a bipartite perfect \( (r, 1) \)-error-correcting code with \( r \geq 2 \), then \( r = 2 \) and \( C \) belongs to the family of codes mentioned in Example 4.

**3.4.2 Perfect codes**

We shall prove several elementary results on (bipartite) perfect codes. In many cases the roles of \( C_1 \) and \( C_2 \) can be interchanged. From now on we usually assume that \( r_1 \geq r_2 \).

**Lemma 3.37.** Let \( C \) be a perfect \( (r_1, r_2) \)-error correcting code. If \( d_1 \geq 2r_1 + 1 \), then each word in \( C_1 \) is adjacent to some word in \( C_2 \) and hence

\begin{equation}
d_{1,2} = r_1 + r_2 + 1.
\end{equation}
Proof. Let $c \in C_1$. Let $x \in Q^n$ be a word with $d(c, x) = r_1 + 1$. There is a codeword $c'$ such that $d(x, c') \leq d(c')$. This implies that $c$ and $c'$ are adjacent, and hence $c' \in C_2$. By (3.8) we are done. □

The next lemma shows that nontrivial bipartite perfect codes are binary.

**Lemma 3.38** Let $q \geq 3$ and let $C$ be a perfect $q$-ary $(r_1, r_2)$-error-correcting code with $r_1 > r_2 > 0$. Then $d_1 = 2r_1 + 1$ or $d_2 = 2r_2 + 1$.

**Proof.** Assume that $d_1 > 2r_1 + 1$. Consider a word $c \in C_1$: without loss of generality we assume that this word is $0$. Let $x = (1 \cdot \cdot \cdot 10 \cdot \cdot \cdot 0)$, a word of weight $r_1 + 1$. By Lemma 3.37 there is a codeword $c' \in C_2$ such that $d(c, c') = r_2$, and we may assume without loss of generality that $c' = (11 \cdot \cdot \cdot 1 * \cdot \cdot \cdot 00 \cdot \cdot \cdot 0)$, where the first $r_1 + 1$ coordinates are $1$, the next $r_2$ are nonzero, and the remaining coordinates are $0$. Now define $y := (a11 \cdot \cdot \cdot 100 \cdot \cdot \cdot 0)$, a word of weight $r_1 + 1$ with $a \notin \{0, 1\}$. By the same argument as above there is a codeword $c''$ of weight $r_1 + r_2 + 1$ with $d(y, c'') = r_2$. The words $c'$ and $c''$ are both in $C_2$, distinct, and there distance is clearly at most $2r_2 + 1$ and so we are done. □

**Lemma 3.39** Let $C$ be a perfect $(r_1, r_2)$-error-correcting code with $d_1 > 2r_1 + 1$. Then the covering radius $\rho(C_2)$ of $C_2$ satisfies

\[(3.9) \quad \rho(C_2) = d_1, 2 = r_1 + r_2 + 1.\]

**Proof.** To prove the lemma, we consider a perfect $(r_1, r_2)$-error-correcting code $C$ with $\rho(C_2) = \rho > r_1 + r_2 + 1$. We consider words $x, c$ with $d(x, c) = d(x, C_2) = \rho$. Without loss of generality $c = 0$ and $x = (z_1 \ldots z_{r_1}00 \ldots 0)$. We define

\[y_1 := (z_1 \ldots z_{r_2+1}00 \ldots 0), \quad y_2 := (z_1 \ldots z_{r_2+2}00 \ldots 0).\]

Clearly $d(y_1, C_2) = r_2 + i (i = 1, 2)$. There must be a codeword $u \in C_1$ such that $d(y_1, u) \leq r_1$, hence $u = (z_1 \ldots z_{r_2+1} * \ldots * )$, a word of weight $r_1 + r_2 + 1$. This implies that there is a $j$ with $r_3 < j \leq \rho$ such that $z_j \neq u_j$, and we assume that the numbering of coordinates is such that $j = 2r_2 + 2$. Now we have $d(y_2, u) = r_1 + 1$. So, there is a codeword $u' \neq u$ in $C_1$ such that $d(y_2, u') = r_1$. Hence $d(u, u') = 2r_1 + 1$. □

**Lemma 3.40** Let $C$ be a perfect $(r_1, r_2)$-error-correcting code with $r_1 > r_2 > 0$. Then we have:

(i) If $d_1 \geq 2r_1 + 2$ then

\[(3.10) \quad (q - 1)^{r_1 + 1} \binom{n - 1}{r_1} M_1 \leq (q - 1)^{r_2} \binom{n - 1}{r_2} M_2;\]

(ii) If $d_2 \geq 2r_2 + 2$ then

\[(3.11) \quad (q - 1)^{r_2 + 1} \binom{n - 1}{r_2} M_2 \leq (q - 1)^{r_1} \binom{n - 1}{r_1} M_1.\]

**Proof.** It is sufficient to prove the first assertion. So, assume that $d_1 \geq 2r_1 + 2$. From Lemma 3.37 it follows that the punctured code $C' \ (\text{delete the last symbol})$ is an $(r_1, r_2 - 1)$-error-correcting code. So not only does (3.14) hold but we also have

\[(3.12) \quad M_1, V(u - 1, r_1) + M_2, V(n - 1, r_2 - 1) \leq q^{n + 1}.\]
If we multiply both sides of (3.12) by \( q \), subtract from (3.7), and apply (3.5) and (3.6), the result follows. \( \square \)

**Remark.** Since (3.10) and (3.11) cannot hold simultaneously unless \( q = 2 \), we have a second proof of Lemma 3.38.

**Lemma 3.41** If \( C \) is a (binary) bipartite perfect \((r_1, r_2)\)-error-correcting code with \( r_1 > r_2 > 0 \), then

\[
\binom{n-1}{r_1} M_1 \leq \binom{n-1}{r_2} M_2,
\]

\[
d_i = 2r_i + 2 \quad (i = 1, 2) \text{ if } d_{1,2} < n. \tag{3.14}
\]

**Proof.** The first assertion is a direct consequence of Lemma 3.40. To prove the second assertion, assume \( d_1 \geq 2r_1 + 3, d_2 \geq 2r_2 + 2 \). Now \( C \) is also a \((r_1 + 1, r_2 - 1)\)-error-correcting code. So we have (as in [forin?])

\[
M_1 \cdot V(n, r_1 + 1) + M_2 \cdot V(n, r_2 - 1) \leq 2^n. \tag{3.15}
\]

From (3.15), (3.7) and (1.5) we find

\[
M_1 \binom{n}{r_1 + 1} \leq M_2 \binom{n}{r_2}. \tag{3.16}
\]

Combining (3.13) and (3.16) we find \( n \leq r_1 + r_2 + 1 = d_{1,2}. \) \( \square \)

**Lemma 3.42** If \( C \) is a (binary) bipartite perfect \((r_1, r_2)\)-error-correcting code, then the punctured code \( C' = C'_1 \cup C'_{2} \) is a perfect \((r_1, r_2 - 1)\)-error-correcting code and also a perfect \((r_1 - 1, r_2)\)-error-correcting code.

**Proof.** If (3.10) and (3.11) both hold, i.e. (3.13) holds, then we must have equality in (3.12), showing that \( C' \) is perfect for \((r_1, r_2 - 1)\). The second assertion follows in the same way. \( \square \)

There is a partial converse to Lemma 3.42. We state it, although we shall not need it later.

**Lemma 3.43** If \( C = C_1 \cup C_2 \) is a binary perfect \((r_1, r_2 - 1)\)-error-correcting code and also a perfect \((r_1 - 1, r_2)\)-error-correcting code, with \( d_{1,2} < n \), then if \( d_{1,2} \) is odd the punctured code \( C' = C'_{1} \cup C_{2} \) is a perfect bipartite \((r_1, r_2)\)-error-correcting code.

**Proof.** For the distances \( d_{1,1}, \overline{d}_1, \overline{d}_2, \overline{d}_{1,2} \) of \( C \) we find \( d_{1,1} \leq 2r_1 + 2, \overline{d}_1 \leq 2r_2 + 2, \overline{d}_{1,2} \geq d_{1,2} + 1 > r_1 + r_2 + 1 \) (by Lemma 3.37). So, \( C' \) is certainly an \((r_1, r_2)\)-error-correcting code. That \( C' \) is perfect now again easily follows by using (3.7) three times. \( \square \)

3.4.3 **Proof of Theorem 3.36**

We remind the reader of the Johnson bound for binary codes (cf. [73]). If \( C \) is a binary \( e \)-error-correcting code of length \( n \), then

\[
|C| \left( V(n, e) + \frac{n^n - |C|}{n^n} \right) \leq 2^n. \tag{3.17}
\]

34
If equality holds in (3.17), then either $e + 1$ divides $n + 1$ in which case $C$ is a perfect code, or otherwise $C$ is called nearly perfect. Note that if the fraction on the left-hand side of (3.17) is not 0, then it is at least $\frac{1}{2} \binom{n}{e}$. Therefore

$$|C| \left\{ V(n, e) + \frac{1}{n} \binom{n}{e} \right\} = 2^n$$

implies that $e + 1$ divides $n$ and that $C$ is a nearly perfect code.

Now assume that $C$ is a bipartite perfect $(r, 1)$-error-correcting code with $r \geq 2$. Apply Lemmas 3.41 and 3.42. We find that $C'$ is a perfect $(r, 0)$-error-correcting and that

$$M_2 = \frac{1}{n - 1} \binom{n - 1}{r} M_1.$$ 

It follows that $C'_1$ is an $r$-error-correcting code with equality in (3.18). Therefore $C'_1$ is nearly perfect. It is well known (cf. [72]) that $(r \geq 2)$ this implies that $C'_1$ has the parameters of a Preparata code. This completes the proof. □

**Remark.** The Preparata codes are not unique for length $> 16$, but for length 16 it is known as the Nordstrom-Robinson code.
Chapter 4

A characterisation of the Doob graphs

For a graph $\Gamma$ we say that $\lambda$ exists if every edge lies in exactly $\lambda$ triangles. EGAWA [43] has shown that the Hamming graphs are determined by their parameters, unless $\lambda$ equals 2. In that case the only possible graphs are the Doob graphs. RIPÀ & HUGUET [90] and NOMURA [85] generalised this result by Egawa for $\lambda \neq 2$. We generalise Egawa’s result for $\lambda = 2$.

The main part of the proof is a characterisation of cartesian products of graphs. This chapter is based on the preprint “A characterisation of the Doob graphs” [89].

4.1 Introduction

Graphs with valency $k$, $c_1 = 2$, $c_2 = 3$ and $a_2 = 2\lambda$ are studied by BROUWER [14], LAMBECK [71], NOMURA [85] and RIPÀ & HUGUET [90]. They found that if $\lambda \neq 2$, then such a graph is covered by the Hamming graph $H(n, \lambda + 2)$, where $n = k/(\lambda + 1)$.

In this paper we look at the case $\lambda = 2$. We shall prove the following result.

Theorem 4.23 Let $\Gamma$ be a connected graph with $\lambda = 2$, $a_2 = 4$, $c_2 = 2$, $c_3 = 3$ and with valency $k$. Then there is a Doob graph $\Delta$ with the same valency as $\Gamma$ and a local isomorphism $\pi : \Delta \rightarrow \Gamma$. □

Also we give the following characterisation of cartesian products of cliques.

Theorem 4.21 Let $\Gamma$ be a connected regular graph without induced $K_{2,1,1}$'s or induced pentagons. Let $c_3 = 3$ and $c_2 = 2$. Then there is a cartesian product of cliques $\Delta$ and a local isomorphism $\pi : \Delta \rightarrow \Gamma$. □

A consequence of this theorem is the following theorem of NOMURA [85] and RIPÀ AND HUGUET [90].

Theorem 4.22 Let $\Gamma$ be a connected graph with $a_1 = \lambda \neq 2$, $a_2 = 2\lambda$, $c_2 = 2$ and $c_3 = 3$. Then there is a Hamming graph $H(n, d)$ and a local isomorphism $\pi : H(n, d) \rightarrow \Gamma$. □

Using a result of HONG [61] we get the following result as a corollary of the Theorems 4.23 and 4.22.
Corollary 4.29 Let $\Gamma$ be a distance-regular graph with diameter $d, d \geq 3$, $c_i = i$ ($1 \leq i \leq d$) and $a_i = i\lambda$ ($1 \leq i \leq d - 1$). Then $\Gamma$ is a Doob graph, a Hamming graph, a folded $(2t + 1)$-cube or the root graph of the binary Golay code. \hfill \Box

This corollary was already shown by Nomura [88] and Rifà and Huguet [99] when $\lambda \neq 2$. The paper is organised as follows: In Section 2 we give the definitions and notations. In Section 3 we discuss when a graph is covered by a cartesian product of graphs. In Section 4 we look at the local structure of a graph with $c_2 = 2, c_3 = 3, \lambda = 2$ and $a_2 = 4$. In Section 5 we give the proofs of the above results.

4.2 Definitions and notation

For two graphs $\Gamma, \Delta$, a graph homomorphism $\psi$ is a map $\psi : V \Gamma \to V \Delta$ such that if $xy$ is an edge in $\Gamma$, then $\psi(x)\psi(y)$ is an edge in $\Delta$. A graph homomorphism $\psi$ is a graph isomorphism if $\psi$ is bijective and $\psi^{-1}$ is also a graph homomorphism. For $\psi$ a graph homomorphism from $\Gamma$ to $\Delta$ we also write $\psi : \Gamma \to \Delta$. A graph homomorphism $\psi : \Gamma \to \Delta$ is called a local isomorphism if $\psi(\Gamma(x)) = \Delta(\psi(x))$ for all $x \in V \Gamma$. Note that $\psi(V\Delta) = V\Gamma$, if $\Gamma$ is connected.

A partition $P$ of the vertex set of a graph $\Gamma$ into (non-empty sets) is called regular, or equitable if, for any two $C, C' \in P$, the number of vertices $y \in C$ adjacent to $x \in C'$ is a constant $c(C, C')$ independent of $x \in C'$. A regular partition $P$ is uniformly regular if there are constants $c_0$ and $c_1$ such that

$$c(C, C') = \begin{cases} c_0 & \text{if } C = C' \\ c_1 & \text{if } C \neq C' \text{ and there is a } x \in C \text{ such that } d(x, C') = 1. \end{cases}$$

A regular partition $P$ is completely regular if all $C \in P$ are completely regular with the same parameters. Note that a completely regular partition is uniformly regular.

A subgraph $\Delta$ of a graph $\Gamma$ is $i$-convex if for all $x, y$ at distance $j \leq i$, the paths between $x$ and $y$ of length $j$ in $\Gamma$ are also in $\Delta$. It is $i$-superconvex if, for all $x, y$ at distance $j \leq i$, all paths between $x$ and $y$ of length at most $j + 1$ in $\Gamma$ are in $\Delta$. An $i$-convex subgraph is $i$-closed if it is also $(i + 1)$-superconvex.

The weight $w(u)$ of a vector $u$ is the number of $i$ such that $u_i \neq 0$. If $u$ and $v$ are two vectors of the same length and $A$ is a set of coordinates, then $u|_A, v$ if and only if $u_i = v_i$ for $i \notin A$. For $A = \{i\}$ we write $u, v$ instead of $u|_i, v$.

The cartesian product of graphs $F^i, i = 1, 2, \ldots, n$, has as vertices vectors $u$ of length $n$ with $u_i, a$ vertex of $F^i$. The vertices $u$ and $v$ are adjacent if and only if $u|_i, v$ and $u_i = v_i$ for some $i$. This graph is denoted by $F^n$. If $\Delta_1, \Delta_2$ are subgraphs of a graph $\Gamma$, then we say that a subgraph $\Delta$ of $\Gamma$ splits into the cartesian product $\Delta_1 \times \Delta_2$ if $\Gamma$ contains $\Delta_1$ and $\Delta_2$ as subgraphs and there exists an isomorphism $\phi : \Delta_1 \to \Delta_1 \times \Delta_2$ such that $\phi$ is the identity on $V \Delta_1$ and $V \Delta_2$. A $q$-clique is a graph on $q$ vertices such that any two vertices are adjacent. The Hamming graph $H(n, q)$ is the cartesian product of $n$ $q$-cliques.

Now we define the Shrikhande graph (cf. [98]).

For $i, j$ the vertex set is the set $Z_4 \times Z_4$ and $(i, j) \sim (k, l)$ if and only if $(i - k, j - l) \in \{(0, 0), (0, 1), (1, 1)\}$. The graph $D(c, s)$ is the cartesian product of $c$ 4-cliques and $s$ Shrikhande graphs and is distance regular with the same parameters as the graph $H(c + 2s, 4)$, cf. Doob [41].

A Doob graph is a graph isomorphic to $D(c, s)$ for some integers $c$ and $s$. We consider
the Hamming graph $H(n, 4)$ also as Doob graphs.

We now give some examples of some of the above definitions.
(i) Each maximal clique in a cartesian product of cliques is 2-closed.
(ii) The Shrikhande graph has no proper 2-closed subgraphs containing an edge.
(iii) $i$-superconvex $\Rightarrow$ i-convex $\Rightarrow$ i-closed $\Rightarrow$ $(i-1)$-superconvex.
(iv) The i-cube is an i-convex subgraph of $H(n, 4)$, $n \geq 1$, but is neither 1-superconvex nor 2-closed.

4.3 On products of graphs

In this section we look at when a graph is covered by a cartesian product of graphs.

**Theorem 4.1** Let $\mathcal{F}$ be a family of connected graphs without proper 2-closed subgraphs containing an edge. Let $\Gamma$ be a connected graph such that:
(i) For each edge $e$ the minimal 2-closed subgraph $C(e)$ in $\Gamma$ containing $e$ is isomorphic to a member of $\mathcal{F}$.
(ii) For each pair of intersecting edges $e$ and $f$ with $C(e) \neq C(f)$, the minimal 2-closed subgraph of $\Gamma$, which contains $C(e)$ and $C(f)$ as subgraphs, splits into the cartesian product $C(e) \times C(f)$.
(iii) If $x$ is a vertex and $e, f, g$ are three edges on $x$, such that $C(e), C(f), C(g)$ are pairwise distinct, then $e, f, g$ lie in a unique induced 3-cube of $\Gamma$.

Then there are members $F_1, \ldots, F_s$ of $\mathcal{F}$ such that there exists a local isomorphism $\pi : \Pi F^i \rightarrow \Gamma$.

**Proof.** Let $x \in V \Gamma$. Let $\Gamma(x)$ be partitioned in parts $\Psi_i$, for $i = 1, 2, \ldots, s$, for a suitable integer $s$, such that the elements of $\Psi_i$ are the neighbours of $x$ in a subgraph $\Delta_i$, which is isomorphic to $F^i \in \mathcal{F}$. We first prove the following lemma.

**Lemma 4.2** Each vertex $y$ of $\Gamma$ lies in exactly $s$ different subgraphs (with at least two vertices) isomorphic to a member of $\mathcal{F}$. Let these subgraphs be $\Sigma_i$, $i = 1, \ldots, s$. Then there is a permutation $\gamma$ such that $F^i$ is isomorphic to $\Sigma_{\gamma(i)}$.

**Proof of Lemma 4.2.** Let $z \in \Gamma(x)$. Then the edge $\{x, z\}$ is in $\Delta_j$ for some $j$. Then $\Delta_i$ and $\Delta_i$ with $i \neq j$ are in a 2-closed subgraph which splits into the cartesian product of $\Delta_i$ and $\Delta_i$. So $z$ lies in subgraphs $\Sigma_i$, $i = 1, \ldots, s$ such that $\Sigma_i$ is isomorphic to $\Delta_i$ for $i = 1, \ldots, s$. It is easy to see that all $\Sigma_i$ are different subgraphs of $\Gamma$, by the fact that all $\Delta_i$ are different. By (ii), $z$ does not lie in more subgraphs isomorphic to a member of $\mathcal{F}$. Now we are done by the fact that $\Gamma$ is connected. $\square$

In the rest of the proof we write $\Pi$ for $\Pi F^i$. In the following we will define the graph homomorphism $\pi : \Pi \rightarrow \Gamma$. We will show that $\pi$ will satisfy the following two conditions.
(I) $\pi$ preserves the distances $\leq 2$.
(II) $k(u) = k(\pi(u))$.

Conditions (I) and (II) imply that $\pi$ is a local isomorphism. Let $\psi_i : \Delta_i \rightarrow F^i$ be an isomorphism between $\Delta_i$ and $F^i$. Give the vertex $\psi_i(x)$ the label 0. Put $\pi(0) = x$. Map the vectors $u$ of weight 1, with $u_0 \neq 0$ to the vertices of $\Delta_0$ in such a way that the map $\pi$ restricted to $\{u \in \Pi \mid u_0 = 0\}$ is an isomorphism. By assumption (ii) it follows that $\pi$ restricted to the vectors of weight $\leq 1$ preserves distances $\leq 2$. For the definition of $\pi(u)$ on the set of vectors of weight at least 2 we proceed by induction on $d(0, u)$. Let
\[ u_i \neq 0 \neq u_j \text{ for } i \neq j. \]
Let \( a_i \in (F^2)_{d(u_i,0)-1}(0) \cap F^2(u_i) \) and \( a_j \in (F^2)_{d(u_j,0)-1}(0) \cap F^2(u_j). \)
Let \( \Pi \), \( u \), \( u_i \), \( u_j \), \( u \) and \( x_i \) \( x_j \) \( x \) with \( u_i - x_i = a_i \) and \( u_j - x_j = a_j. \)

By (ii) the vertices \( \pi(u) \) and \( \pi(x) \) have exactly common neighbours. Define \( \pi(x) \) as the common neighbour of \( \pi(u) \) and \( \pi(u) \) different from \( \pi(x). \)

We first show that \( \pi \) is well-defined. By (ii) we know that \( \phi: (u) \rightarrow \phi(u) \) is an isomorphism of graphs such that the whole set \( \{u \in (F^2) | u \neq 0 \} \)

Now we get \( \pi \) is well-defined for all vectors of weight at most two. We have shown that \((II)\) is satisfied.

Now we have to show that \( \pi \) is well-defined for all vectors of weight at most two. We have shown that \((III)\) is satisfied.

Let \( u \) be a vector with \( d(u,0) = t+1 \). We may assume that the whole set \( \{u \in (F^2) | u \neq 0 \} \)

Next we get \( \pi \) is well-defined for all vectors of weight at most two. We have shown that \((IV)\) is satisfied.

The above and \((V)\) imply \((II)\) is satisfied. So we are done. \( \square \)

Lemma 3.3 Let \( \mathcal{F} \) be a family of connected graphs without proper 2-closed subgraph containing an edge. Let \( \Gamma \) be a connected graph such that:

(i) Each edge \( e \) lies in a unique 2-closed subgraph \( C(e) \) isomorphic to a member of \( \mathcal{F} \) and \( C(e) \) is 2-supervertex.

(ii) For each pair of intersecting edges \( e \) and \( f \) with \( C(e) \neq C(f) \), the minimum 2-supervertex subgraph of \( \Gamma \) which contains \( C(e) \) and \( C(f) \) as subgraphs, splits into the cartesian product \( C(e) \times C(f) \).

(iii) \( r_3 = 3 \).

Then for a vertex and \( r_1, r_2, r_3 \) are three edges on \( x \), such that \( C(r_1), C(r_2), C(r_3) \) are pairwise distinct, the three edges \( r_1, r_2, r_3 \) lie in a unique induced 3-cube of \( \Gamma \).

Proof. Let the edge \( c_i \) contain the vertices \( x_i, y_i \) for \( i = 1,2,3 \). The edges \( c_i \) and \( c_j \), \( i \neq j \), lie in a subgraph \( \Phi_{ij} \) of \( \Gamma \), which splits into the cartesian product \( C(c_i) \times C(c_j) \), therefore \( \rho(\Phi_{ij}) = 2 \). Let \( z_k \) be the common neighbour of \( y_i \) and \( y_j \), different from \( x_i, i \neq j \neq k \neq i. \) Since \( C(c_i) \) and \( C(c_j) \) are the only proper connected 2-closed of \( \Phi_{ij} \) containing \( x \), the vertex \( y \) is not a vertex of \( \Phi_{ij}. \) It follows that \( d(y_k, z_k) = 3 \), by the
2-superconvexity of $\Phi_{ij}$. Let $f_a$ be the edge $\{y_i, z_k\}$, where $\{a, b\} = \{j, k\}$. For the edge $f_j$ the subgraph $C(f_j)$ is isomorphic to the subgraph $C(e_j)$. Since $y_k$ does not belong to $\Phi_{ij}$, it follows from the 2-convexity of $\Phi_{ij}$ that $z_j$ does not belong to $\Phi_{ij}$, and hence $C(f_j)$ and $C(f_a)$ are distinct. Let $t$ be the common neighbour of $z_i$ and $z_j$. It follows from $d(z_2, y_1) = 3$, that $t \neq y_2$. We have $d(z_1, y_1) = 3$, $\Gamma(z_1) \cap \Gamma(z_2) = \{t, y_3, y_4\}$, $\Gamma_2(z_1) \cap \Gamma_2(z_2) = \{z_1, z_2, z_3\}$ and $\mu(z_1, z_2, z_3) = 2$. Since $\Gamma_2(z_1) \cap \Gamma(z_2) \subseteq \Gamma_2(z_1) \cap \Gamma(z_2)$, it follows that $z_2 \sim t$, and the subgraph induced by $\{x, y_1, y_2, y_3, z_1, z_2, z_3, t\}$ is a 3-cube. □

**Theorem 4.4** Let $F$ be a family of connected graphs without proper 2-closed subgraphs containing an edge. Let $\Gamma$ be a connected graph such that:

(i) For each edge $e$ the minimal 2-closed subgraph $C(e)$ in $\Gamma$ is isomorphic to a member of $F$ and this subgraph is 2-superconvex in $\Gamma$.

(ii) For each pair of intersecting edges $e$ and $f$ with $C(e) \neq C(f)$, the minimal 2-closed subgraph of $\Gamma$ which contains $C(e)$ and $C(f)$ as subgraphs is isomorphic to the cartesian product $C(e) \times C(f)$, and this subgraph is 2-superconvex in $\Gamma$.

(iii) $c_3 = 2$.

Then there are members $F_1, \ldots, F_n$ of $F$ and a local isomorphism $\pi : \Pi F_i \to \Gamma$.

**Proof.** This follows directly from Lemma 4.3 and Theorem 4.1. □

### 4.4 On the local structure

Let $\Gamma$ be a connected graph with $\mu = 2$, $c_3 = 3$, $\lambda = 2$ and $a_2 = 4$. In this section we look at the local structure of $\Gamma$. First we give two lemmas.

**Lemma 4.5** Let $x, y$ be two vertices of $\Gamma$ at distance 3. For $x_1, x_2$ two distinct vertices in $\Gamma(x) \cap \Gamma(y)$ their common neighbour distinct from $x$ lies in $\Gamma_2(x) \cap \Gamma_2(y)$.

**Proof.** The vertices $x_1$ and $y_1$ have two common neighbours in $\Gamma_2(y) \cap \Gamma_2(x)$, for $i = 1, 2$. But $|\Gamma_2(y) \cap \Gamma_2(x)| = 3$, and hence $x_1$ and $x_2$ have a common neighbour in $\Gamma_2(x) \cap \Gamma_2(y)$. Also $x_1$ and $x_2$ have exactly two common neighbours. □

**Lemma 4.6** (i) Every Shrikhande subgraph of $\Gamma$ is 2-superconvex.

(ii) Any two Shrikhande subgraphs of $\Gamma$ have at most one vertex in common.

**Proof.** (i) Directly from $\lambda = 2 = c_3$ and (ii) follows from (i). □

Let $z \in VT$. Since $\lambda = 2$ the subgraph induced by $\Gamma(z)$ is the disjoint union of cycles. Let $y_1, \ldots, y_n$ be a cycle in $\Gamma(z)$, such that $y_1 \sim y_i$ if and only if $|i - j| = 1$ or $|i - j| = n - 1$.

**Lemma 4.7** Let $n \geq 4$. Let $z$ be the common neighbour of $y_1$ and $y_2$, different from $z$. If there is an $i$ such that $d(y_i, z) = 3$, then

$$\{|j \mid d(z, y_i) = 1\} \cup \{|j \mid d(z, y_i) = 2\} = 2.$$ 

**Proof.** $n$ cannot be 4, by $\mu = 2$. Let $d(y_i, z) = 3$. If $y_1 \sim y_i$ and $j \notin \{1, 2, 3, n\}$, then $d(y_j, z) = 3$ (We have $d(y_i, y_1) = d(y_i, y_2) = 2$. If $d(y_j, z) = 2$, then $y_j \sim y_1$ or $y_j \sim y_2$, since Lemma 4.5. So $j \notin \{3, n\}$). □

**Lemma 4.8** If $n \geq 4$, then $\{|j \mid d(y_j, z) = 2\} = 4$.

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Proof. Suppose that \( |z| = \text{deg}(y, z) = 2| \leq 3 \). Then there is a vertex \( v_1 \sim z \) such that \( d(v_1, z) = 2 \), and \( u_1 \not\in \{y, z\} | \sim 1, \ldots, n \). Let \( u_2, u_3 \) be the two common neighbours of \( x \) and \( u_1 \). Then \( d(y_1, z) = d(y_2, z) = 2 \) and \( y_1 \rightarrow z \sim y_2 \), so without loss of generality \( d(u_2, z) = 3 \), by \( \mu = 2 \) and \( \alpha_2 = 4 \). But \( \{y_1, y_2\} \subseteq \Gamma(z) \cap \Gamma(z) \) and \( d(u_1, z) = 2 \) and therefore \( \lambda_2(u_2, z) \geq 4 \). Contradiction. By \( \alpha_2 = 4 \) we are done. \( \square \)

Lemma 4.9 We have \( n = 6 \) or \( n = 3 \).

Proof. Suppose \( n \geq 4 \). By previous lemma we get \( n \geq 6 \). If \( n \geq 7 \), then there is a \( j \) such that \( d(y, z) = 4 \), by \( d(x, z) = 2 \), \( \mu = 2 \) and \( \alpha_2 = 4 \). By Lemmas 4.7 and 4.8 we get a contradiction. \( \square \)

Proposition 4.10 If \( n = 6 \), then \( d(z, y) \leq 2 \). \( \square \)

From this proposition it follows that the vertices \( x, y_1, \ldots, y_6 \) lie in a unique Shrikhande subgraph of \( \Gamma \).

In fact we have shown the following theorem.

**Theorem 4.11** Each induced \( K_{2,1} \) lies in a unique Shrikhande subgraph. \( \square \)

Let \( \Delta \) be a Shrikhande subgraph in \( \Gamma \). Let \( z_1 \in \Delta \) and let \( y_1 \in \Gamma(z_1) \setminus \Delta \). Then for all \( z_1 \in \Delta(z_1) \) and for all \( z_2 \in \Delta(z_2(z_1)) \) we have \( d(z_1, y_1) = 2 \) and \( d(z_1, z_2) \geq 3 \) by respectively \( \lambda = 2 \) and \( \alpha_2 = 4 \).

Let \( x_1 \in \Delta(z_1) \). Let \( y_1 \) be the common neighbour of \( x_2 \) and \( y_1 \), different from \( x_1 \). It is obvious that \( y_1 \) is not in \( \Delta \). Let \( x_3, x_4 \) be the common neighbours of \( x_1 \) and \( x_2 \) and let \( y_2 \) be a common neighbour of \( y_1 \) and \( y_2 \).

If \( d(y_2, y_3) = 3 \), then it follows that \( d(y_2, x_2) = d(y_2, x_1) = 2 \) and the other common neighbour of \( y_2 \) and \( y_3 \) must be a neighbour of \( y_2 \), by \( \lambda_2 = 3 \). Hence \( y_2 \sim x_4 \).

Suppose that \( d(y_2, x_1) = 2 \). Let \( x_2 \) be the common neighbour of \( x_1 \) and \( x_3 \), distinct from \( x_2 \). Clearly \( d(x_2, y_3) \leq 3 \). By \( d(y_2, x_3) = 3 \), we get \( d(x_1, y_3) \geq 2 \). Then \( d(y_2, x_3) = 3 \), by \( d(y_2, x_3) = d(y_2, x_2) = 2 \) and \( d(x_1, y_3) = 3 \). By \( \lambda_2 = 3 \) it follows that \( y_3 \sim x_1 \) or \( y_3 \sim x_3 \), contradiction.

So we have shown that \( y_2 \sim x_2 \) or \( y_2 \sim x_3 \). Without loss of generality we assume that \( y_2 \sim x_2 \). Let \( y_3 \) be the other common neighbour of \( y_1 \) and \( y_2 \). Then \( y_3 \sim x_3 \). If you construct the unique Shrikhande subgraph \( \Delta_1 \) with \( y_1, y_2, y_3 \) and \( y_4 \) as vertices you easily see that for all vertices \( z \in \Delta \) there is a unique vertex \( z_1 \) of \( \Delta_1 \), such that \( z \sim z_1 \).

Let \( \Delta \) and \( \Delta_1 \) be two distinct Shrikhande subgraphs such that for each vertex \( x \) of \( \Delta \) there is a unique vertex \( x_1 \) of \( \Delta_1 \) such that \( x \sim x_1 \). It is obvious that no vertex of \( \Delta \) is a vertex of \( \Delta_1 \). Let \( x \in \Delta \) and \( y \in \Delta_1 \) such that \( x \sim y \). Let \( y \in \Delta_1 \) and \( y \in \Delta_1 \) such that \( y \sim y \). If \( d(x, y) = 2 \), then \( a_2(x, y) = 5 \), what is impossible. So \( d(x, y) = 1 \). So we have shown the following lemma.

**Lemma 4.12** Let \( \Delta \) be a Shrikhande subgraph of \( \Gamma \) and let \( x \) be a vertex, not in \( \Delta \), such that there is a vertex \( x_1 \) in \( \Delta_1 \) such that \( x \sim x_1 \). Then there is a unique Shrikhande subgraph \( \Delta_1 \) with \( x \) as a vertex and for each vertex \( z \) of \( \Delta \) there is a unique vertex \( z_1 \) of \( \Delta_1 \) such that \( z \sim z_1 \). Furthermore the map \( \phi : \Delta \rightarrow \Delta_1 \) with \( \phi(z) = z_1 \) is an isomorphism between \( \Delta \) and \( \Delta_1 \). \( \square \)

We have a similar result for \( 4 \)-cliques.

**Lemma 4.13** Let \( \Delta \) be a \( 4 \)-clique of \( \Gamma \) and let \( x \) be a vertex, not in \( \Delta \), such that there is a vertex \( x_1 \) in \( \Delta \) such that \( x \sim x_1 \). Then there is a unique \( 4 \)-clique \( \Delta_1 \) with \( x \) as a vertex and for each vertex \( z \) of \( \Delta \) there is a unique vertex \( z_1 \) of \( \Delta_1 \) such that \( z \sim z_1 \).
Proof. Let the subgraph induced by $x_1, i = 1, 2, 3, 4$ be a 4-clique. Let $x \sim x_1$. This implies $d(z, x_j) = 2$ for $j = 2, 3, 4$. Let $y$ be the common neighbour of $x_2$ and $z$ different from $x_1$, and let $z$ be a common neighbour of $x$ and $y$. We want to show that $z \sim x_3$ or $z \sim x_4$. If $d(z, x_3) = 3$, then by Lemma 4.5, $x_3$ is adjacent to $z$ (which, in fact, is a contradiction). Thus we may assume that $d(z, x_4) = d(z, x_3) = 2$. Let $\{u_1, y\} = \Gamma(z) \cap \Gamma(x_1)$ and $\{u_2, u_3\} = \Gamma(z) \cap \Gamma(x_2)$. It follows that $u_i \neq x_j$ for all $i, j$. Also the set $A := \{u_1, u_2, u_3, x, y\}$ has cardinality 5, by $\lambda = 2$. Then $d(a, x_4) = 2$ for all $a \in A$. But this implies $d_2(x_4, x) \geq 5$, contradiction.

So we have shown that $z \sim x_3$ or $z \sim x_4$. But now by $\mu = 2$ it easily follows that $x, y, z$ lie in a 4-clique $\Delta_1$ such that for each vertex $u$ of $\Delta_1$ there is exactly one $i$ such that $u \sim x_i$. □

As an immediate corollary from Lemma 4.12 and 4.13 we have the following lemma.

Lemma 4.14 The configuration

is not an induced subgraph of $\Gamma$. □

Let $x_1$ be a vertex of $\Gamma$ and suppose that $x_1$ is in a Shrikhande subgraph, say $\Delta$. Let $y_1 \in \Gamma(x_1)$, such that $y_1$ is not in $\Delta$. By Lemma 4.12, there is a Shrikhande subgraph $\Delta_1$ of $\Gamma$ such that for each vertex $z$ of $\Delta$ is a unique vertex $z_1$ of $\Delta_1$ such that $z \sim z_1$. Let $x_1, x_2, \ldots, x_4$ be the vertices of $\Delta$ and $y_1, y_2, \ldots, y_4$ the vertices of $\Delta_1$ such that $x_i \sim y_i$. Suppose that the edge $\{x_1, y_1\}$ lies in a Shrikhande subgraph. Then by Lemma 4.12, each edge $\{x, y\}$ lies in a unique Shrikhande subgraph, say $\Pi_i$ for $i = 1, 2, \ldots, 16$.

Lemma 4.15 $\Pi_i$ and $\Pi_j$ do not have a vertex in common, if $i \neq j$.

Proof. Suppose that $z$ is a vertex of $\Pi_i$ and $\Pi_j$, where $i \neq j$. Let $u$ be a common neighbour of $x_i$ and $x_j$. It is obvious that $u$ does not belong to $\Pi_i$ and $\Pi_j$. It follows that $d(z, x_i) = d(z, x_j) = d(u, z) - 1$. Suppose that $d(z, x_i) = d(z, x_j) = 1$. It follows that $z$ is in $\Delta$, because the common neighbours of $x_i$ and $x_j$ lie in $\Delta$. Contradiction. Suppose now that $d(z, x_i) = d(z, x_j) = 2$. But then $d(u, z) = 3$ and thus the other common neighbour of $x_i$ and $x_j$, say $u_1$, is a neighbour of $z$. But then $\{u_1, z\}$ is an edge of $\Pi_i$ and $\Pi_j$, so $\Pi_i = \Pi_j$, and thus $i = j$, contradiction. □

Lemma 4.16 Let $u_i, u_j, u_k, u_l \in V\Gamma$, such that $u_i \sim u_j \sim u_k \sim u_l$, $u_i, u_l \in V\Pi_i$, $u_j \in V\Pi_j$, and $u_k \in V\Pi_k$, and $i \neq j \neq k \neq l$. Then $u_i = u_l$.

Proof. If $u_i \sim u_l$, then $j = k$. If $d(u_i, u_l) = 2$, then $i = j$, by $\alpha_2 = 4$. □

Lemma 4.17 Let $u \in V\Pi_i$ and $u_j \in V\Pi_j$ such that $u \sim u_1$. Then $x_1 \sim x_j$ or $i = j$.

Proof. Suppose that $d(x_i, x_j) = 2$. Let $x_2$ be a common neighbour of $x_1$ and $x_j$. There is a vertex $u_2$ of $\Pi_i$, which is a neighbour of $u_1$ and also a vertex $u_3$ of $\Pi_i$, such that $u_2 \sim u_3$. By Lemma 4.16, $u = u_3$. Now by Lemma 4.14 we get a contradiction. □

Lemma 4.18 The induced subgraph, say $\Gamma_1$, on the vertices of $\Pi_1, \ldots, \Pi_{16}$ is isomorphic to a cartesian product of two Shrikhande subgraphs.
Proof. The number of vertices is $16^2 = 256$. The valency of $\Gamma_1$ equals to 12 by Lemma 4.17. Let $x, y, z, u$ be a triangle in $\Delta$. Let $u_1$ be vertex of $\Pi_1$. Then there are neighbours $u_1, u_2$ of $u$, such that $u_1$ is a vertex of $\Pi_1$ and $u_2$ is a vertex of $\Pi_2$. So $u_1 \sim u_2$, by Lemma 4.16. It follows that the Shrikhande subgraph with $u_1, u_2$ and $u_2$ is a subgraph of $\Pi_1$. □

Now we have shown the following lemma.

Lemma 4.20 Let $e, f$ be two intersecting edges such that the minimal 2-closed subgraphs $C(e)$ and $C(f)$, containing respectively $e$ and $f$, are isomorphic to the Shrikhande graph. Suppose that $C(e) \neq C(f)$. Then the minimal 2-closed subgraph $\Gamma_1$, containing $e$ and $f$, is isomorphic to the cartesian product $C(e) \times C(f)$. □

The following lemma shows a similar result if the edge $e$ lies in a 4-clique.

Lemma 4.21 Let $e, f$ be two intersecting edges of $\Gamma$. Let $C(e)$ be the minimal 2-closed subgraph of $\Gamma$ containing $e$ and let $C(f)$ be the minimal 2-closed subgraph containing $f$. Suppose that $C(e) \neq C(f)$. If $C(e)$ is a 4-clique, then the minimal 2-closed subgraph containing the edges $e$ and $f$ is isomorphic to the cartesian product $C(e) \times C(f)$.

Proof. Suppose that $C(f)$ is a Shrikhande subgraph. Let $z \in e \setminus f$. Then $z$ lies in Shrikhande subgraph $\Delta$, such that for each vertex $u$ of $C(e)$ there is a unique $u_1$ of $\Delta$ such that $z \sim u_1$. By Lemma 4.13, the edge $\{u, u_1\}$ lies in a 4-clique for all vertices $u$ of $C(e)$. Let $V(C(f)) = \{x_i \mid i = 1, 2, \ldots, 16\}$ and $V(\Delta) = \{x_i \mid i = 1, 2, \ldots, 16\}$, such that $w_1 \sim z$. The edge $\{x_i, w_1\}$ lies in a 4-clique, say $C_1$. The subgraphs $C_1$ and $C_2$ do not have a vertex in common by $\lambda = 2$ and $\nu = 2$.

If $u$ is a vertex of $C_1$, $u_1$ a vertex of $C_2$ and $u \sim u_1$, then $u \sim u_i$ or $i = j$, by $a_2 = 4$ and the Shrikhande graph has $a_2 = 4$.

By Lemma 4.13, if $z_i \sim z_1$ then for each vertex $u$ of $C_1$ there is a unique vertex of $C_2$ which is a neighbour of $u$. With Lemma 4.14 it follows that the subgraph induced by the vertices of all the $C_i$ is a direct product of a 4-clique and a Shrikhande subgraph.

If $C(f)$ is a 4-clique, then follows with the same reasoning that $x, y, z$ lies in a direct product of two 4-cliques. □

4.5 The characterisation of Doob graphs

In this section we prove the results we mentioned in the introduction.

Theorem 4.22 Let $\Gamma$ be a connected graph with $c_2 = 2, c_3 = 3$ and without induced $K_{2, 2, 1, 1}$s and pentagons. Then there is a cartesian product of cliques $\Delta$ and a local isomorphism $\pi : \Delta \rightarrow 1$.

Proof. Each edge $e$ is in a unique minimal clique $C(e)$ by the fact that there are no induced $K_{2, 2, 1, 1}$. Let $e$ and $f$ be two intersecting edges such that $C(e) \neq C(f)$, and let $\Phi$ be the minimal 2-closed subgraph containing $e$ and $f$. We show that $\Phi$ splits into the grids $C(e) \times C(f)$. Let $x$ be the common vertex of $e$ and $f$. Let $x_1, x_2$ be two distinct vertices of $C(e)$ and let $C(f)$ be a vertex of $C(f)$. For each $i = 1, 2$, let $y_i$ be the common neighbour of $x_i$ and $y$ different from $x$. It suffices to show that $y_1 \sim y_2$. By assumption, the pentagon $x_1 x_2 y_1 y_2$ has an additional edge $e_0$. From the assumption that $\Gamma$ has no induced $K_{2, 2, 1, 1}$, it follows that $e_0$ cannot be adjacent to $x_1$ or $x_2$, hence $e_0 = \{y_1, y_2\}$, as desired. Since there are no induced pentagons this grid is 2-superconvex. Now the theorem follows from Theorem 4.4. □
Remark. A related characterisation of the cartesian product of cliques is given by Mollard [87], Theorem 1.

Nomura [84] has shown that connected graphs with \( c_2 = 2, c_3 = 3, a_2 = 2\lambda \neq 4 \) do not have induced \( K_{2,1,1} \). The following theorem of Nomura [85] and Rifa & Huguet [90] is now a direct consequence of the previous theorem.

**Theorem 4.22** Let \( \Gamma \) be a connected graph with \( a_1 = \lambda \neq 2, a_2 = 2\lambda, c_2 = 2 \) and \( c_3 = 3 \). Then there are a Hamming graph \( H(n, d) \) and a local isomorphism \( \pi : H(n, d) \to \Gamma \).

**Proof.** We only have to show that \( \Gamma \) does not contain induced pentagons. Let \( x_i, i \in \mathbb{Z}_5 \) such that \( x_i \sim x_j \) if and only if \( i - j = \pm 1 \). Let \( y \) be the common neighbour of \( x_1 \) and \( x_3 \) distinct from \( x_2 \). By \( a_2 = 2\lambda \) we have \( x_4 \sim y \), since \( x_4 \neq x_2 \), and \( x_5 \sim y \), since \( x_5 \neq x_2 \). But then the graph induced by \( \{x_3, x_4, x_5, y\} \) is a \( K_{2,1,1} \), contradiction. \( \Box \)

**Theorem 4.23** Let \( \Gamma \) be a connected graph with \( a_1 = 2, a_2 = 4, c_2 = 2, c_3 = 3 \) and with valency \( k \). Then there is a Doob graph \( \Delta \) with the same valency as \( \Gamma \) and a local isomorphism \( \pi : \Delta \to \Gamma \).

**Proof.** Let \( \mathcal{F} \) be a family of graphs which only contains the Shrikhande graph and the 4-clique. By Theorem 4.11 we know that each edge of \( \Gamma \) lies in a Shrikhande subgraph or in a 4-clique. It is obvious that these graphs are 2-superconnected.

Let the edge \( \{x, y\} \) lie in a subgraph \( \Delta_1 \) isomorphic to a 4-clique or a Shrikhande graph, and the edge \( \{x, z\} \) lie in a subgraph \( \Delta_2 \) isomorphic to a 4-clique or a Shrikhande graph, such that \( \Delta_1 \neq \Delta_2 \). By Lemmas 4.19 and 4.20 we have that the edges \( \{x, y\} \) and \( \{x, z\} \) lie in a subgraph isomorphic to a direct product of \( \Delta_1 \) and \( \Delta_2 \). This subgraph is 2-superconnected and so is the minimal 2-superconnected subgraph of \( \Gamma \) with \( x, y, z \) as vertices.

By Theorem 4.4 we are done. \( \Box \)

Before we state the next theorem we first define quotient graphs. Let \( \Gamma \) be a graph and let \( \mathcal{P} \) be a partition of \( V \Gamma \). Then the quotient graph \( \Gamma / \mathcal{P} \) is the graph with vertices the classes of \( \mathcal{P} \) and \( CD \) is an edge if \( d_{\Gamma}(C, D) = 1 \).

**Corollary 4.24** The partition \( \mathcal{P} = \{C_n \mid x \text{ vertex of } \Gamma \} \) of the Doob graph \( \Delta \), where \( C_n = \pi^{-1}(\pi) \) is a uniformly regular partition of \( \Delta \) and \( \Gamma \) is isomorphic to \( \Delta / \mathcal{P} \). If \( \Gamma \) is distance-regular, then \( \mathcal{P} \) is a completely regular partition.

It is obvious that \( \Pi \) is a uniformly regular partition. The graph \( \Delta \) is a distance-regular graph and therefore by [19], Theorem 11.1.6 the graph \( \Gamma \) is distance-regular if and only if \( \Pi \) is a completely regular code. \( \Box \)

First we state Lloyd’s theorem.

**Theorem 4.25** (cf. [19], Corollary 2.5.4.) Let \( C \) be a perfect \( e \)-code in a distance-regular graph \( \Gamma \). Let this code have inner distribution \( \alpha \). Then \( \sum_{i=0}^e P_{ij} = 0 \) for all \( j \neq 0 \) such that \( n_{\alpha(j)} \neq 0 \) and there are exactly \( e \) of such \( j \)’s. \( \Box \)

The matrices \( P \) and \( Q \) depend only on the parameters of a distance-regular graph.

For Doob graphs we have that \( \sum_{i=0}^e P_{ij} = F_{\alpha}(j) \) where

\[
F_{\alpha}(x) = \sum_{i=0}^e (-q)^i(q-1)^{x-i} \binom{n-i-1}{x-i} \binom{z-1}{i}.
\]

Horton [61] has shown the following result.
**Theorem 4.26** The polynomial $F_{n,q}(x)$ has a non-integral zero if $e \geq 3$, $n \geq e + 1$ and $q \geq 3$. □

As a consequence of this theorem we have the following theorem.

**Theorem 4.27** There are no perfect $e$-codes with at least two vertices in the Doob graphs. □

The following theorem is the work of several people like Tietäväinen, van Lint, Best.

**Theorem 4.28** The only $e$-perfect codes in a Hamming graph with at least two members and $e \geq 3$ are the following ones.

(i) An antipodal class in a $(2t+1)$-cube, where $t \geq 3$.

(ii) The binary Golay code. □

By Corollary 4.24 and Theorems 4.27 and 4.28 we have the following corollary.

**Corollary 4.29** Let $G$ be a distance-regular graph, with diameter $d$, $d \geq 3$, $c_i = i, 1 \leq i \leq d$, and $a_i = i \lambda$, $1 \leq i \leq d - 1$. Then $G$ is a Doob graph, a Hamming graph, a folded $(2t+1)$-cube or the coset graph of the binary Golay code. □

This corollary was already shown by Nomura [85] and Rifà & Uguet [90], when $\lambda \neq 2$. 
Chapter 5

Some new graphs

In the first section we construct a bipartite distance-regular graph with intersection array \{45, 44, 36, 5; 1, 9, 40, 45\} and automorphism group \(3^5 : (2 \times M_{10})\) (acting edge-transitively) and discuss its relation to previously known combinatorial structures. This section is based on joint work with A.E. Brouwer and R.J. Riebeek, [24]. In the second section we give two constructions for uniformly geodetic graphs. This section is based on [67].

5.1 A new distance-regular graph associated to the Mathieu group \(M_{10}\)

5.1.1 Introduction

Let \(G\) be the perfect ternary Golay code generated by the rows of the circulant \((- + - + + - + + - - - - - - -)_{11}\). Then \(G\) is a ternary [11, 6, 5] code. Let \(\Gamma\) be the coset graph of \(G\), that is, the graph with as vertices the \(3^5\) cosets of \(G\) in \(F_3^{11}\), where two cosets are adjacent when their difference contains a vector of weight one. Then \(\Gamma\) is a strongly regular graph with parameters \((v, k, \lambda, \mu) = (243, 22, 1, 2)\), known as the Beslekamp-van Lint-Seidel graph. (See Beslekamp, van Lint & Seidel [10], and Brouwer, Cohen & Neumaier [19], Section 11.3B.)

In [19], p. 360, the question was raised whether the complementary graph of the graph \(\Gamma\) is the halved graph of a bipartite distance-regular graph \(\Delta\) of diameter 4. In this section this question is answered affirmatively: R.J. Riebeek and the author constructed such a graph \(\Delta\). (This also settles the last open case in Riebeek [89, Chapter 7]).

5.1.2 Construction

Put \(Q := \{1, 3, 4, 5, 9\}\), the set of (nonzero) squares mod 11, and \(N := \{2, 6, 7, 8, 10\}\), the nonsquares. Consider in the graph \(\Gamma\) the set \(D\) consisting of the following 45 cosets (we write \(u\) instead of \(u + G\)): \(-e_0 - e_j, e_j, -e_0 - e_j (j \in N)\)

\[c_0 - e_i, c_i + e_0, \pm(e_i - e_0), c_i - e_0, -e_i - e_0, -e_i - e_{10} (i \in Q)\].

Then \(D\) is a 45-coclique, and the point-coclique incidence graph \(\Delta\) on cosets and translates of \(D\) is distance-regular with intersection array \(\{45, 44, 36, 5; 1, 9, 40, 45\}\) and distance distribution diagram

\[
\begin{array}{cccccccccc}
1 & 45 & 1 & 44 & 9 & 226 & 36 & 40 & 198 & 5 & 45 & 23 & v = 486
\end{array}
\]
The automorphism group of $\Delta$ has shape $3^5 : (2 \times M_{19})$, and acts edge-transitively with point stabilizer isomorphic to $M_{19}$. The orbit diagram is

$$\vdots$$

5.1.3 Structure of the group; related graphs

In order to describe the group of automorphisms more precisely, we have to specify the representation of $2 \times M_{19}$ inside $GL(3,3)$. The direct factor 2 may be represented by $\pm I$, and then it remains to look at the group $H := 3^5 : M_{19}$, the stabilizer of the bipartition of $\Delta$. This group has a centre of order 3, acting fixed point freely on $\Delta$. The quotient graph is a bipartite graph $E$ of valency 45 on 162 vertices that can be found inside the McLaughlin graph $M$ as follows.

Let $x$, $y$ be two adjacent vertices of $M$. Let $X$ and $Y$ be the sets of vertices of $M$ adjacent to $x$ but not to $y$, and to $y$ but not to $x$, respectively. Then $|X| = |Y| = 81$ and $E$ is isomorphic to the graph with vertex set $X \cup Y$, where $X$ and $Y$ are cliques, and the edges between $X$ and $Y$ are precisely those present in $M$. (Thus, $E$ is not the graph induced by $X \cup Y$; in $M$ the sets $X$ and $Y$ induce subgraphs of valency 20. See also Brouwer & Haemers [21], Construction D.)

A larger graph. Let $Z$ be the set of 81 vertices in $M$ nonadjacent to both $x$ and $y$. The graph induced by $M$ on $X \cup Y \cup Z$, after switching with respect to $Z$, is isomorphic to the Delaunay graph, a strongly regular graph with parameters $(v,b,k,\lambda, \mu) = (243,110,37,60)$.

If we remove from this graph the edges inside $X$, $Y$ or $Z$, we obtain a tripartite graph $F$ of valency 90 on 283 vertices such that the subgraph induced on the union of any two of its parts is isomorphic to $E$. We have $\text{Aut}(F) \cong 3^5 : (2 \times M_{19})$.

This latter graph has a triple cover $\Sigma$, of course again tripartite, such that the subgraph induced on the union of any two of its parts is isomorphic to $\Delta$. We have $\text{Aut}(\Sigma) \cong 3^6 : (2 \times M_{19})$.

The graph $\Sigma$ can be constructed as follows:
Let \( A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \end{pmatrix} \) and \( B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \). Let \( M := \langle A, B \rangle \) be the group generated by \( A \) and \( B \). Then \( M \cong M_{10} \), and \( M \) has orbits of sizes 1, 1, 20, 20, 20, 72, 72, 90, 90, 90, 180 on \( F_9^d \). Let \( N := \langle A, B, -I \rangle \). Then \( N \cong 2 \times M_{10} \) and \( N \) has orbits of sizes 1, 2, 20, 40, 72, 144, 90, 180, 180. The vector \( (000001) \) is a representative of the \( N \)-orbit \( O \) of size 90. The graph \( \Sigma \) is the graph with vertex set \( F_9^d \), where two vertices are adjacent when their difference lies in \( O \). Now the graph \( \Delta \) is the subgraph of \( \Sigma \) induced on the set of vectors with nonzero last coordinate.

### 5.2 Uniformly geodetic graphs

In this we give two constructions for uniformly geodetic graphs. Also we give sufficient conditions to assure that the graphs constructed are uniformly geodetic.

#### 5.2.1 Construction 1

In this subsection we give a construction for uniformly geodetic graphs. Let distance 1-or-2 graph \( G_1 \cup G_2 \) of a graph \( G \) has \( \forall \gamma \) as vertex set and edges \( xy \) if \( d_{\Gamma}(x, y) \in \{1, 2\} \).

**Proposition 5.1** Let \( G \) be a connected graph such the numbers \( a_i \) and \( c_i \) exist for \( i = 1, 2, \ldots, d \), where \( d \) is the diameter of \( G \). Then \( G_1 \cup G_2 \) is uniformly geodetic if and only if

\[
\begin{align*}
  c_{2i} &= i(c_2 - a_1) + \sum_{j=1}^{2i-1} a_j, \\
  c_{2i+1} &= c_2 + \sum_{j=i}^{2i} a_{j+1}.
\end{align*}
\]

for \( i, 1 \leq i \leq d/2 \).

**Proof.** Denote: \( c'_i := c_i(G_1 \cup G_2) \). Then

\[
\begin{align*}
  c'_i &= \frac{c_{2i}c_{2i-1}}{c_2} = c_{2i-1} + \frac{c_{2i-1}c_{2i-2}}{c_2} + \frac{c_{2i-1}(a_{2i-1} + a_{2i-2} - a_1)}{c_2}.
\end{align*}
\]

This is equivalent with:

\[
\begin{align*}
  c_{2i} &= i(c_2 - a_1) + \sum_{j=1}^{2i-1} a_j.
\end{align*}
\]

\( \Box \)

Now we give some examples of graphs \( G \) such that the distance 1-or-2 graph is a uniformly geodetic graph.

(A) The distance 1-or-2 graph of a folded \((4m+2)\)-cube.

(B) The distance 1-or-2 graph of a \( J(n,1,1) \).

(C) The distance 1-or-2 graph of a Hadamard graphs.

(D) The distance 1-or-2 graph of the coset graph of a subcode of the doubly punctured binary Golay code with intersection array \( \{21,20,16,9,2,1;1,2,3,16,20,21\} \).
5.2.2 Construction 2

We give now a second construction, which was first described in Brouwer & Koolen [29]. Let $\Gamma$ be a graph. Construct a new graph $B(\Gamma)$ with vertices $(x,0)$ and $(x,1)$ where $x \in \Gamma$. Two vertices of $B(\Gamma)$, say $(x,l),(y,m)$ are adjacent if either $l = m$ and $1 \leq d(x,y) \leq 2$, or $l \neq m$ and $d(x,y) \leq 1$. Below in Theorem 5.7 we shall give necessary and sufficient conditions for $B(\Gamma)$ to be uniformly genetid. But first we need to give some results on distance-regular graphs.

**Lemma 5.2** Let $\Gamma$ be a distance-regular graph with $d \geq 3$, $c_1 = c_2 = 1$, $c_3 = 2$, $a_1 = a_2 = 0$ and valency $k$. Define $\Delta$ as the distance-2 graph of $\Gamma$. Then for each $y \in V(\Delta)$, the set $\Delta(y)$ is partitioned into a family of $k$ cliques of size $k-1$: say $\{M_i(y) | 1 \leq i \leq k\}$ such that each point in $M_i(y)$ is adjacent to a unique point of $M_j(y)$ for all $j \neq i$.

**Proof.** We use a bar to distinguish the parameters of $\Delta$ from those of $\Gamma$. Now we find: $\bar{k} = k(k-1)$. Let $u \in \Gamma(1)$, then $\Gamma(u) = \{u_i | i = 1, 2, \ldots, k\}$. Now for all $i = 1, 2, \ldots, k$, the set $\Gamma_i(u) \cap \Gamma(v_i)$ is a $(k-1)$-clique of $\Delta$ and $\Delta(u) = \Gamma_2(u) = \bigcup_{i=1}^{k} (\Gamma_i(u) \cap \Gamma(v_i))$. Let $x \in \Gamma_2(u) \cap \Gamma(v_i)$. Then there are unique $w_1, w_2 \in \Gamma_1(u)$ such that $d(w_1, x) = d(w_2, x) = d(w_3, y) = \bar{d}(w_3, y) = 1$. We have $w_3 \neq w_2$, since $c_2 = 2$. We obtain $\bar{d}(x, y) = 2$ and $\Gamma_2(u) \cap \Gamma(v_i) \supseteq \{v_1, w_1, w_2, v_2\}$, but this contradicts $c_3 = 2$. Thus there is a unique vertex $y \in \Gamma_2(u) \cap \Gamma(v_i)$, $j \neq i$ such that $d(x, y) = 2$. So we are done.

**Lemma 5.3** Let $\Gamma$ be a non-bipartite distance-regular graph with diameter $d \geq 4$, $c_1 = c_2 = 1$, $c_3 = 2$, $a_1 = a_2 = a_3 = 0$ and valency $k$. Define $\Delta$ as the distance-2 graph of $\Gamma$. Then $\Delta$ is the Johnson graph $J(2k-1, k-1)$.

**Proof.** We use a bar to distinguish the parameters of $\Delta$ from those of $\Gamma$. Note that $\Delta$ is connected. We have: $\bar{k} = k(k-1)$ and $\bar{c}_2 = c_2c_4 = 4$. By Lemma 5.2 we can partition the set $\Delta(x)$ into a family of $k$ cliques of size $k-1$, say $\{M_i(x) | 1 \leq i \leq k\}$, such that each $x \in M_i(x)$ is adjacent to a unique point of $M_j(x)$ for each $j \neq i$.

From [19, Theorem 9.1.3] we find that $\Delta$ is a Johnson graph or is doubly covered by a $J(2k-1, n)$. In this last case $\Delta$ has valency $n^2$, but its valency is also $k(k-1)$, contradiction. So $\Delta$ is a Johnson graph. By the partition of $\Delta(x)$ into $(k-1)$-cliques we obtain that $\Delta$ has maximal cliques of size $k$. Hence $\Delta$ is a $J(2k-1, k-1)$.

Let $G$ be a group of automorphisms of a graph $\Gamma$. The quotient graph $\Gamma/G$ has as vertices the $G$ orbits $Gx$ of the vertices of $\Gamma$ and edges $\{(A, B) | \text{whenever there are } a \in A, b \in B \text{ such that } a \sim b, \text{if } \rho \text{ is an automorphism of } \Gamma \text{ we also write } \Gamma/\rho \text{ instead of } \Gamma(\rho)\}$.

**Lemma 5.4** Let $\rho$ be an automorphism of the doubled Odd graph with valency $k$, say $\Gamma$, such that $\rho^2 = 1$ and $d(x, \rho(x)) \geq 2$ for all vertices $x \in \Gamma$. If $\Gamma/\rho$ is distance-regular, then $\rho$ is the map sending a vertex to its antipode and $\Gamma/\rho$ is the Odd graph with valency $k$.

**Proof.** Define $C_x = \{z \in \Gamma | x \in \Gamma(z)\}$ for vertex $x$. Then $\Pi = \{C_x \ | \ x \in \Gamma\}$ is a uniformly regular partition of $\Gamma$ with $\eta_0 = 0$, $\eta_1 = 1$. Because $\text{Aut}(\Gamma) \cong \text{Sym}(2k - 1) \times 2$, there is a vertex $y$ of $\Gamma$ such that $\rho(y)$ is the antipode of $y$. So $\{y, \rho(y)\}$ is a perfect $(k-1)$-error-correcting code of $\Gamma$. By [10, Theorem 11.1.6] we are done.

The following theorem is a generalisation of the fact that the Odd graphs are uniquely determined by their parameters, see Mool [79].
Theorem 5.8 Let $\Gamma$ be a non-bipartite distance-regular graph with diameter $d \geq 4$, valency $k$ and $c_1 = c_2 = 1$, $c_3 = c_4 = 2$, $a_1 = a_2 = a_3 = 0$. Then $\Gamma$ is the Odd graph with valency $k$.

Proof. Let $\Delta$ be the distance-2 graph of $\Gamma$. It follows from Lemma 5.3 that $\Delta \cong J(2k - 1, k - 1)$. If $x$ is a vertex of $\Delta$, then we denote by $\overline{x}$ the corresponding vertex of $\Delta$. We construct now the graph $\Pi$ from $\Delta$ as follows. The vertices of $\Pi$ are the vertices of $\Delta$ and the maximal $k$-cliques of $\Delta$ and edges of $\Pi$ have the form $(\overline{x}, C)$ where $C$ is a maximal $k$-clique of $\Delta$ and $\overline{x}$ is a vertex of $\Delta$ such that $\overline{x} \in C$. Note that $\Pi$ is the doubled Odd graph with valency $k$. If $\overline{x}, \overline{y}$ are two adjacent vertices of $\Delta$ then there is a unique maximal $k$-clique $C$ in $\Delta$ such that $\overline{x}, \overline{y} \in C$. Define now the function $\phi : V\Pi \to V\Gamma$ by

\begin{align}
\phi(\overline{x}) := x & \quad \text{if } \overline{x} \in \Delta, \\
\phi(C) := x & \quad \text{if } C \text{ is a maximal } k\text{-clique of } \Delta \text{ and } \phi(y) \sim x \text{ for all } y \in C.
\end{align}

The function $\phi$ is well-defined, because if $C$ is a maximal $k$-clique in $\Delta$ and $\overline{x}, \overline{y} \in C$, then there is a unique vertex $z$ such that $\phi(\overline{y}) \sim z \sim \phi(\overline{x})$ and in Lemma 5.2 we have shown that $\{\overline{x} \mid u \in \Gamma(\phi(\overline{y})) \cap \Gamma(z)\}$ is a maximal $(k - 1)$-clique in $\Delta(\overline{y})$. So $C = \{\overline{x} \mid u \in \Gamma(\phi(\overline{y})) \cap \Gamma(z)\} \cup \{\overline{y}\}$. We denote by $C_2$ the maximal $k$-clique $C$ such that $\phi(C) = x$. So $\phi^{-1}(x) = \{\overline{x}, C_2\}$. Now construct a new graph $\Lambda$ with vertices $\{\overline{x}, C_2\}$ and $\{\overline{y}, C_2\} \sim \{\overline{y}, C_2\}$ if $\overline{y} \in C_2$. First we have to show: if $\overline{z} \in C_2$ then $\overline{z} \in C_2$. If $\overline{z} \in C_2$ then $z \sim x$. Let $\overline{u} \in \Gamma(\overline{y}) \cap \Gamma_2(\overline{x})$. So $\overline{C}_2$ is the maximal $k$-clique $C$ in $\Delta$ such that $\overline{x}, \overline{y} \in C$.

Define the map $\psi : V\Gamma \to V\Lambda$ by

$$\psi(\overline{x}) := \{\overline{x}, C_2\}.$$

We show that $\psi$ is an isomorphism of graphs. The map is obviously a bijection. Let $x, y$ be two adjacent vertices in $\Gamma$. We already have seen $C_2$ is the maximal $k$-clique $C$ such that $\overline{y} \in C$ where $u \in \Gamma(x) \cap \Gamma_2(y)$. So $\overline{y} \sim C_2$ and thus $\{\overline{y}, C_2\} \sim \{\overline{y}, C_2\}$. We have proved that $\Gamma \cong \Delta$. Define the function $\pi : V\Pi \to V\Pi$ by

\begin{align}
\pi(\overline{x}) := C_2 & \quad \text{if } \overline{x} = x, \\
\pi(C) := \overline{x} & \quad \text{if } C \text{ is a maximal } k\text{-clique of } \Delta \text{ and } \phi(y) \sim x \text{ for all } y \in C.
\end{align}

This function is an automorphism of $\Pi$ of order 2 and $\Pi/\pi \cong \Gamma$. Now $\pi$ satisfies the conditions of Lemma 5.4 and so we are done. $\square$

A distance-regular graph $\Gamma$ with diameter $d$ is called a generalized Odd graph if $a_1(\Gamma) = a_2(\Gamma) = \ldots = a_{d - 1}(\Gamma) = 0$ and $a_d(\Gamma) \neq 0$.

Riček & Huguet [90] have shown the following theorem, but they forgot to mention the folded cubes.

Theorem 5.8 The only generalized Odd graphs with $c_i = i$ for $1 \leq i \leq d$ and $d \geq 3$ are the folded $(2m + 1)$-cubes, and the coset graph of the binary Golay code with intersection array $\{23, 22, 21; 1, 2, 3\}$. $\square$

Theorem 5.7 Let $\Gamma$ be a uniformly geodetic graph with diameter $d$ at least 4, and such that $\lambda$ exists. Let $\Delta = B(\Gamma)$ be the graph obtained from $\Gamma$ by the second construction. Then $\Delta$ is uniformly geodetic if and only if $\Gamma$ is one of the following graphs.
(i) A $J(n,t,1)$.  
(ii) An $m$-cube.  
(iii) An Odd graph $O_n$, with odd valency.  
(iv) A folded $(4m+1)$-cube.

**Proof.** We use a prime to distinguish the parameters of $\Delta$ from those of $\Gamma$. We shall first show that for all $j \leq d$ the numbers $q_j$ exist and that for all $i$, $2i+1 \leq d$ we have

\begin{align}
(5.5) \quad a_{2i} + a_{2i+1} &= \lambda \\
(5.6) \quad c_{2i} &= i\mu \\
(5.7) \quad c_{2i+1} &= i\mu + 1 + a_{2i}
\end{align}

and $c_d = k\mu$ if $d = 2t$.

For $i = 0$, the Equations (5.5), (5.6) and (5.7) hold.

Suppose that for all $j < k$ with $k \leq (d-1)/2$ the numbers $a_j$ and $a_{2j+1}$ exist and satisfy Equations (5.5), (5.6) and (5.7).

Then

\[ c'_{k} = \frac{c_k c_{2k-1}}{\mu}. \]

This is equivalent with $c_k = \mu + c_{2k-2} = k\mu$.

Let $x, y$ be two vertices of $\Gamma$ at distance $2k$. Then

\[ c'_{k+1} = \frac{c_k c_{2k-1}}{\mu} + c_{2k} + a_{2k}(x, y) + 1 + \frac{c_k (a_{2k}(x, y) + a_{2k-1} - \lambda)}{\mu} \]

\[ = (k+1)(k\mu + 1 + a_{2k}(x, y)). \]

On the other hand we have

\[ c'_{k+1} = \frac{c_k c_{2k+1}}{\mu} + c_{2k+1} = (k+1)c_{2k+1}. \]

It follows that $c_{2k+1} = 1 + k\mu + a_{2k}(x, y)$ and thus $a_{2k}$ exists and $c_{2k+1}$ satisfies Equation (5.7).

Let $u, v \in \Gamma$ at distance $2k + 1$ if $2k + 1 \leq d$. Then

\[ c'_{k+1} = c_{2k+1} + c_{2k+1}(a_{2k+1}(u, v) + a_{2k} - \lambda)/\mu + (c_{2k+1} c_{2k})/\mu. \]

We already have $c'_{k+1} = (k+1)c_{2k+1}$, hence $a_{2k+1}(u, v) + a_{2k} = \lambda$. Now we showed that $a_{2k+1}$ exists and $a_{2k+1} + a_{2k} = \lambda$.

Let $x$ and $y$ be two vertices at distance $3$ in $\Gamma$. Let $x_1, x_2$ be two vertices of $\Gamma$ such that $x \sim x_1 \sim x_2 \sim y$, then the set $\Gamma(x_2) \cap \Gamma(y)$ of size $\lambda$ is covered by the two sets $\Gamma(x_2) \cap \Gamma(x_1)$ and $\Gamma(x_1) \cap \Gamma(y)$ both of size $a_2$. Thus $2a_2 \geq \lambda$.

It follows from $1 + a_2 + \mu = c_2$ that $c_2 = 2\mu$ that $\mu \geq a_2 + 1$. Thus, if $\mu = 1$, then $a_2 = 0$, whence $\lambda = 0$ and $c_0 = 2$. So $\Gamma$ is a $(k, c, \ell, a, k)$-graph so that $\Gamma$ is either regular or semi-regular. It follows that $\Gamma$ is distance-regular or distance-biregular. If $\mu \geq 2$, then $\Gamma$ is regular and thus distance-regular.

If $\Gamma$ is distance-regular then by Proposition 5.5.1 of [19] we have $a_2 + a_3 = \lambda + 1$ if $\lambda = 0$. Since (5.7) we find $a_2 + a_3 = \lambda$, so $\lambda = 0$. It follows that $c_3 = \mu + 1$. By Theorem 5.4.1 of [19] we obtain $c_3 \geq \frac{2\mu}{\mu}$ if $\mu \geq 2$. Thus if $\mu \geq 2$, then $\mu + 1 \geq \frac{2\mu}{\mu}$, which is equivalent with $\mu \leq 2$ and hence $\mu = 2$.

There are now five cases for $\Gamma$:

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CASE 1. $\Gamma$ is distance-biregular. The girth of $\Gamma$ is not 4, so $\mu = 1$. We obtain $c_i = \{\frac{1}{2}\}$. The only distance-biregular graphs with these $c_i$ are the graphs $J(n, t, t + 1)$.

CASE 2. $\Gamma$ is a bipartite distance-regular graph with $\mu = 1$. We find that $c_i = \{\frac{1}{2}\}$. The only bipartite distance-regular graph with these $c_i$ are the doubled Odd graphs, which are the graphs $J(2t + 1, t, t + 1)$.

CASE 3. $\Gamma$ is a bipartite graph with $\mu = 2$. For the numbers $c_i$ we have $c_i = i$. The only bipartite distance-regular graph with these $c_i$ are the hypercubes.

CASE 4. $\Gamma$ is a non-bipartite graph with $\mu = 1$. By (5.5) we obtain that $d$ is even and $a_i = 0$ for $i < d$ and $a_d \neq 0$. The numbers $c_i = \{\frac{1}{2}\}$. By Theorem 5.5, the only distance-regular graphs with these parameters are the Odd graphs with odd valency.

CASE 5. $\Gamma$ is a non-bipartite graph with $\mu = 2$. Then $d$ is even and $a_i = 0$ for $i < d$, $a_d \neq 0$, and $c_j = j$ for all $j$. By Theorem 5.6, the only distance-regular graphs with these parameters are the folded $(4t + 1)$-cubes. $\square$

Remarks.

(i) For $\Gamma \cong J(2t - 1, t - 1, t)$ the graph $B(\Gamma)$ was constructed by Brouwer and Koolen [22]. The graph $(B(\Gamma))$ is regular with valency $t^2 + t + 1$. The full automorphism group of $B(\Gamma)$ is isomorphic to $D_8 \times \text{Sym}(2t - 1)$ acting transitively on the vertices, with vertex stabiliser $2 \times \text{Sym}(t - 1) \times \text{Sym}(t)$. The numbers $c_i(B(\Gamma)) = i^2$ and hence the Johnson graphs are not the only uniformly geodesic regular graphs with $c_i = i^2$. More information about these graphs is in [22].

(ii) For $\Gamma \cong m$-cube the graph $B(\Gamma)$ is the distance 1-or-2 graph of the $(m + 1)$-cube.

(iii) If $\Gamma \cong O_{2k+1}$, then $c_i(B(\Gamma)) = i^2$ for $i \leq k$, $c_{k+1} = 2(k + 1)^2$ and $d(B(\Gamma)) = k + 1$.

(iv) If $\Gamma$ is the folded $(4t + 1)$-cube, then $c_i(B(\Gamma)) = i(2i - 1)$ for $i \leq t$ and $c_{t+1} = 2(t + 1)(2t + 1)$ and $d(B(\Gamma)) = t + 1$.

(v) The construction also works on Taylor graphs.
Chapter 6

The Metric Hierarchy

In this chapter we will consider the metric hierarchy for graphs, recalling work by J.B. Kelly, Deza, Assouad, Terwilliger and others. In the second section we will define lattices and Euclidean representations of graphs. Also in this section we will define root graphs. In the third section we will classify the amply regular root graphs. (We closely follow [18, §§6.14–6.15]; however, since there are several mistakes in those sections we thought it appropriate to completely redo this piece of theory.) This culminates in Theorems 6.30, 6.40 and 6.42, where the case $\mu \geq 2$ is completely settled. In the last section we will classify the distance-regular graphs whose distance matrix has exactly one positive eigenvalue. This section is joint work with S.V. Shpectorov, [70].

6.1 Introduction

In this section we will give an introduction to the metric theory. First we will give some definitions.

Definition 1. Let $(X, d)$ be a metric space.

(i) $(X, d)$ is called hypermetric if all weight functions $w : X \to \mathbb{Z}$ with $\sum_{x \in X} w(x) = 1$ satisfy the following inequality

$$\sum_{x \in X} \sum_{y \in X} w(x)w(y)d(x, y) \leq 0$$

(ii) $(X, d)$ satisfies the $(2m + 1)$-gonal inequality if all weight functions $w : X \to \mathbb{Z}$ with $\sum_{x \in X} w(x) = 1$ and $\sum_{x \in X} |w(x)| \leq 2m + 1$ satisfy inequality (6.1). Clearly, the $(2m + 1)$-gonal inequality implies the $(2m - 1)$-gonal inequality. We say pentagonal inequality instead of $5$-gonal inequality.

(iii) $(X, d)$ is of negative type if all weight functions $w : X \to \mathbb{Z}$ with $\sum_{x \in X} w(x) = 0$ satisfy inequality (6.1).

(iv) $(X, d)$ satisfies the $(2m)$-gonal inequality if all weight functions $w : X \to \mathbb{Z}$ with $\sum_{x \in X} w(x) = 0$ and $\sum_{x \in X} |w(x)| \leq 2m$ satisfy inequality (6.1). Clearly, the $(2m + 2)$-gonal inequality implies the $2m$-gonal inequality. We say hexagonal inequality if $2m = 6$.

First we define the halved cubes ([19] calls them half-cubes). The vertices of the halved $n$-cube, also denoted by $\frac{1}{2}2^n$, are the $(2r)$-subsets of a $n$-set $N$ and $A \sim B$ if and only if $|A + B| = 2$.

Now we will give various facts about the connections between the different metric spaces.

Proposition 6.1 If $(X, d)$ is isometrically embeddable in a hypercube, then it is isometrically embeddable in a halved cube.
Proof. The hypercube $2^n$ is an isometric subgraph of the halved cube $A(2n)$. □

**Theorem 6.2** Let $(X, d)$ be a metric space. Then we have:

(i) (Deza [38]) If $(X, d)$ satisfies the $(2m - 1)$-gonal inequality, then $(X, d)$ satisfies the $2m$-gonal inequality.

(ii) (Kelly [63]) If $(X, d)$ is hypermetric, then $(X, d)$ is of negative type.

Proof. (i) Let $(X, d)$ satisfy the $(2m - 1)$-gonal inequality. We have to show that if $(x_1, x_2, \ldots, x_{2m})$ is a sequence of points (possibly with repetitions), then

$$\sum_{i=1}^{2m} \sum_{j=1}^{2m} \varepsilon_i \varepsilon_j d(x_i, x_j) \leq 0,$$

where $\varepsilon_i = 1$ for $i \leq m$ and $-1$ otherwise. Now

$$\sum_{i=1}^{2m} \sum_{j=1}^{2m} \varepsilon_i \varepsilon_j d(x_i, x_j) = \frac{1}{2m - 2} \sum_{h=1}^{2m} S_h,$$

where

$$S_h := \sum_{i=1}^{2m} \sum_{j=1}^{2m} \varepsilon_i \varepsilon_j^h d(x_i, x_j)$$

and

$$\varepsilon_i^h = \begin{cases} 
0 & \text{if } i = h \\
\varepsilon_i & \text{if } i \neq h \text{ and } i \leq m \\
-\varepsilon_i & \text{if } i \neq h \text{ and } i > m
\end{cases}$$

We find $\sum \varepsilon_i^h = 1$ and $\sum_i |\varepsilon_i^h| = 2m - 1$. For all $h$ we have $S_h \leq 0$, by the $(2m - 1)$-gonal inequality. So we are done.

(ii) Follows directly from (i). □

The distance-matrix $D = D(X)$ of a metric space $(X, d)$ is the matrix with on the $xy$-th entry the distance $d(x, y)$.

**Proposition 6.3** (cf. Graham and Winkler [53]) If a metric space $(X, d)$ is of negative type and $|X| > 1$, then the distance-matrix $D = (d(x, y))_{x, y \in X}$ has exactly one positive eigenvalue.

Proof. Let $w$ be a weight function with $\sum_{x \in X} w(x) = 0$. Define the column vector $w$ by $w_x = w(x)$. The fact that $(X, d)$ is of negative type becomes:

$$w^T D w \leq 0 \text{ for all } w \text{ with } \sum_p w_x = 0.$$

This means that $D$ is negative semidefinite on a hyperplane. From the Perron-Frobenius Theorem, cf. [19, Theorem 3.1.1] it follows that $D$ has at least one positive eigenvalue. So we are done. □

The conclusion of the above theorem and propositions is the so-called metric hierarchy for finite metric spaces $(X, d)$ with $|X| \geq 2$ (and in particular for finite connected graphs, with the shortest path metric, with at least two vertices):
\[ X \text{ is isometrically embeddable in a hypercube} \]
\[ X \text{ is isometrically embeddable in a halved cube} \]
\[ X \text{ is hypermetric} \Rightarrow X \text{ satisfies the pentagonal inequality} \]
\[ X \text{ is of negative type} \Rightarrow X \text{ satisfies the hexagonal inequality} \]
\[ D = D(X) \text{ has exactly one positive eigenvalue} \]

For connected bipartite graphs almost all of the above properties are equivalent.

**Theorem 6.4.** For a connected bipartite graph \( \Gamma \) on at least two vertices the following four properties are equivalent.

(i) \( \Gamma \) is an isometric subgraph of a hypercube.
(ii) The subset \( \{ z \mid d(x, z) < d(y, z) \} \) of \( V \Gamma \) is geodesically closed.
(iii) The distance-matrix \( D(\Gamma) \) has exactly one positive eigenvalue.
(iv) \( \Gamma \) satisfies the pentagonal inequality.

**Proof.** (i) ĐOKOVIĆ [40] showed that (i) is equivalent with (ii), ROHT & WINKLER [91] showed that (ii) is equivalent with (iii) and AVIS [5] showed that (ii) is equivalent with (iv). \( \Box \)

**Remark.** It is not known whether there exists a bipartite graph satisfying the hexagonal inequality, but not satisfying the pentagonal inequality.

SCHÖENBERG [95] gives the following characterisation of metric spaces of negative type.

**Theorem 6.5.** (SCHÖENBERG [95]) A metric space \((X, d)\) is of negative type if and only if \((X, \sqrt{d})\) is isometrically embeddable in a Euclidean space. \( \Box \)

Before we can formulate the following theorem we define a hole in a lattice. A hole in a lattice \( L \subseteq \mathbb{R}^n \) is a point of \( \mathbb{R}^n \) whose distance to the lattice is a local maximum. Let \( H(x) \) be the set of elements of \( L \) at minimal distance from \( x \). The following two theorems give two characterisations of hypermetric spaces:

**Theorem 6.6.** (ASSOUAD [3]) Let \((X, d)\) be a finite metric space. Then the space \((X, d)\) is hypermetric if and only if there is a lattice \( L \) with a hole \( x \) and a map \( \phi : X \rightarrow B(x) \), such that \( d(x, y) = \frac{1}{2} ||\phi(x) - \phi(y)||^2 \). \( \Box \)

and

**Theorem 6.7.** (TERWILLIGER & DEZA [108]) A connected metric space \((X, d)\) is hypermetric if and only if \((X, d)\) is isometrically embeddable in a cartesian product of Gosset graphs, halved cubes and cocktail party graphs. \( \Box \)

(Here a metric space \((X, d)\) is called connected when \( d \) is integer-valued, and the graph \( \Gamma \) on \( X \) defined by \( x \sim y \) when \( d(x, y) = 1 \) is connected. For a hypermetric representation of the Gosset graph see Section 6.3. The Schläfli graph and Johnson graphs that also occur in Theorem 2 of [108] can be embedded in the Gosset graph and the halved cubes,
respectively. We will see this in Section 6.3. The cocktail party graphs are the \( K_{n \times 2} \). As an application of the above theorem of Schoenberg we have the following partial collapse of the metric hierarchy:

**Theorem 6.8** Let \((X, d)\) be a finite metric space such that the distance matrix \(D\) has a constant row sum, say \(\alpha\). Then the space \((X, d)\) is of negative type if and only if \(D\) has only one positive eigenvalue.

**Proof.** Note that the positive eigenvalue is \(\alpha\) belonging to eigenvector \(1\). So \(\alpha I - D\) is a positive semidefinite matrix, and hence is a Gram matrix. Thus there exists a \((f \times v)\)-matrix \(W\) such that \(\alpha I - D = W^T W\) for a suitable integer \(f\). Let \(S\) be the set of column vectors of \(W\). For \(x, y \in S\), we have

\[
(x - y, x - y) = (x, x) + (y, y) - 2(x, y) = 2d(x, y).
\]

By the theorem of Schoenberg we are done. \(\Box\)

A further partial collapse of the metric hierarchy is given by

**Theorem 6.9** Let \((X, d)\) be a finite metric space with \(v\) elements such that the distance matrix \(D\) has a constant row sum, say \(\alpha\). Suppose that \(\alpha < 2v\) and that the matrix \(D\) has only integer elements. Then the space \((X, d)\) is hypermetric if and only if \(D\) has only one positive eigenvalue.

**Proof.** Assume that \(D\) has only one positive eigenvalue. Note that the positive eigenvalue is \(\alpha\) belonging to eigenvector \(1\). So \(2I - D\) is a positive semidefinite matrix, and hence is a Gram matrix. Thus there exists an \((f \times v)\)-matrix \(W\) such that \(2I - D = W^T W\) for a suitable integer \(f\). Let \(S\) be the set of column vectors of \(W\), and let \(L\) be the lattice spanned by \(S\). Let \(H := \{x \in \mathbb{R}^f \mid (x, x) < 2\}, \text{ then } S \subseteq \partial H\). For \(x, y \in S\), we have

\[
(x - y, x - y) = (x, x) + (y, y) - 2(x, y) = 2d(x, y).
\]

We have \(H \cap L = \{0\}\), since \(L\) is an even lattice. Let \(x \in S\) and let \(L_x\) be the lattice spanned by \(\{y - x \mid y \in S\}\). Suppose \(-x \in L_x\). Then

\[
-x = \sum_{y \in S, y \neq x} v_y(y - x).
\]

Define the vector \(w\) by

\[
w_x = \begin{cases} 
  v_y & \text{if } y \neq x \\
  1 - \sum_{y \in S, y \neq x} v_y & \text{if } y = x 
\end{cases}
\]

Then \(w\) is an eigenvector of \(W\) and thus of \(2I - D\) with eigenvalue \(0\). Also \(1\) is an eigenvector of \(2I - D\) with eigenvalue \(2v - \alpha \neq 0\). The vector \(w\) is not orthogonal to \(1\), contradiction. So \(-x \notin L_x\). We get \(S \subseteq (L_x + x) \subseteq L\) and \((L_x + x) \cap H = \emptyset\). So by Theorem 6.6 it follows that \(L\) is hypermetric. \(\Box\)

We can use the last theorem to show that the Gosset graph is hypermetric.

6.1.1 Local characterisations

In this subsection we give local information about graphs in the metric hierarchy.

The most obvious consequence of the pentagonal inequality is that the subgraphs
are forbidden (where the last graph is forbidden only when \(d(a,b) = 2\)) - indeed if we give the left vertices weight -1 and the right vertices weight +1 we see that the pentagonal inequality is violated. This gives a characterisation for graphs with diameter two satisfying the pentagonal inequality:

**Proposition 6.10** (Avis [3]) A connected graph with diameter 2 satisfies the pentagonal inequality if and only if it does not contain any one of the above four forbidden graphs.

**Proof.** For the pentagonal inequality we only have to check it for weight functions with weights \(+1, 0, -1\). The other weight functions are implied by the triangle inequality (regardless of the diameter assumption). Now it is obvious that the only forbidden subgraphs are those above. \(\square\)

**Proposition 6.11** (Assouad & Deza [4]) A connected graph, which satisfies the 10-gonal inequality, does not contain the first, third and fourth forbidden subgraph.

**Proof.** The weight functions \(-1, -2, 1, 1, 1\) and \(-2, -2, 1, 1, 2\) and \(-3, -2, 2, 2, 1\) (in order from top to bottom, first left then right) show that the hexagonal inequality forbids the first graph, the 8-gonal inequality also the third, and the 10-gonal inequality also the fourth. \(\square\)

**Proposition 6.12** Let \(\Gamma\) be a connected amply regular graph. If \(\Gamma\) has two non-adjacent vertices \(x\) and \(z\), lying in a induced quadrangle and does not contain the first, third and the fourth forbidden graph, then \(a_2(x,z) = 2(\lambda - \mu + 2)\).

**Proof.** Suppose \(Q = (x,y,z,w)\) is an induced quadrangle of \(\Gamma\) (i.e. \(x \sim y \sim z \sim w \sim x, z \not\sim y, y \not\sim w\)). No vertex is adjacent to precisely 3 vertices of \(Q\) (otherwise we would see the third forbidden graph), or to two opposite vertices of \(Q\) (otherwise we would see the first forbidden graph). Hence the neighbours of \(x\) are \(y\) and \(w\), the vertices adjacent to all 4 points of \(Q\), those adjacent only to \(x\) and \(w\), those adjacent only to \(x\) and \(y\), and those adjacent only to \(z\). The common neighbours of \(x\) and \(z\) are \(y, w\) and those adjacent to all points of \(Q\). The common neighbours of \(x\) and \(y\) are those adjacent to only \(x\) and \(y\), and those adjacent to all points of \(Q\). The vertices only adjacent to \(x\) have distance 3 to \(z\) (otherwise we see the fourth forbidden graph). So we find \(a_2(x,z) = 2(\lambda - \mu + 2)\). \(\square\)

An easy corollary of the above propositions is:

**Proposition 6.13** Let \(\Gamma\) be a connected graph satisfying the 10-gonal inequality or the pentagonal inequality. Then the following holds:

(i) If \(x \in V\Gamma\), then two nonadjacent neighbours of \(x\) have at most one common neighbour in \(\Gamma(x)\).

(ii) If \(x, y\) are two vertices at distance 2, then the graph induced by \(\Gamma(x) \cap \Gamma(y)\) is complete multipartite with classes of size 1 or 2.

(iii) If \(\mu(\Gamma) = \mu\) exists, then any two nonadjacent vertices \(y, z \in \Gamma(x)\), where \(x \in V\Gamma\),
have either \( \mu - 1 \) or \( \mu - 2 \) common neighbours in \( \Gamma(x) \).

(iv) If \( \Gamma \) is unicyclic graph with \( \mu(\Gamma) = \mu \) and \( \lambda(\Gamma) = \lambda \), then for any two nonadjacent vertices \( x, y \), lying in an induced quadrangle of \( \Gamma \), then we have \( a_2(x, y) = 2(\lambda - \mu + 2) \).

**Proof.**

(i) By the first and third forbidden subgraph this follows easily.

(ii) follows directly from (i) and the non-occurrence of the first forbidden graph.

(iii) Among the \( \mu \) common neighbours of \( y \) and \( z \), one is \( z \) and at most one is in \( \Gamma_2(x) \).

The remaining neighbours must be in \( \Gamma(x) \).

(iv) follows directly from Proposition 6.12. \( \Box \)

6.1.2 The metric hierarchy for strongly regular graphs

In this subsection we study the metric hierarchy for strongly regular graphs.

The suspension of a graph \( \Gamma \) is the complete union \( \Gamma \oplus K_1 \). If \( \Gamma \) is a graph, then define the truncated distance \( \delta' \) by \( \delta'(x, y) := \min\{d(x, y), 2\} \) for \( x, y \in V\Gamma \). The truncated distance matrix \( D' \) of a graph \( \Gamma \) is the distance matrix belonging to \( (V\Gamma, \delta') \). Note that \( D' = 2J - 2I - A \), where \( A \) is the adjacency matrix of \( \Gamma \).

Assouad & Duval [6] investigated when the suspension of a graph \( \Gamma \) is of negative type and found that this happens if and only if the adjacency matrix \( A \) of \( \Gamma \) has smallest eigenvalue \( \geq -2 \). The following proposition gives a similar result.

**Proposition 6.14** Let \( \Gamma \) be a regular graph with adjacency matrix \( A \). Then the following three properties are equivalent:

(i) \( \Gamma \) (that is, \( A \)) has smallest eigenvalue at least \(-2\),

(ii) \( \Gamma \) with truncated distance \( \delta' \) is hypermetric,

(iii) the truncated distance matrix \( D' \) has exactly one positive eigenvalue.

**Proof.** (i) \( \Rightarrow \) (iii): The largest eigenvalue of \( D' \) is that for the eigenvector 1, namely \( 2n - 2 - k \). The second largest eigenvalue of \( D' \) is \(-2 - \theta \), where \( \theta \) is the smallest eigenvalue of \( A \). Hence \( \theta \geq -2 \) if and only if \( D' \) has one positive eigenvalue.

(iii) \( \Leftrightarrow \) (ii): This follows directly from Theorem 6.9. \( \Box \)

Note that regularity is required in the above proposition: all graphs \( K_{1,m} \) are hypermetric (indeed, isometrically embeddable in a hypercube), but these have smallest eigenvalue \(-\sqrt{m}\).

As an immediate consequence of the above proposition we have for strongly regular graphs a partial collapse of the metric hierarchy:

**Theorem 6.15** Let \( \Gamma \) be a strongly regular graph. Equivalent are:

(i) \( \Gamma \) has smallest eigenvalue at least \(-2\),

(ii) \( \Gamma \) is hypermetric,

(iii) the distance matrix \( D(\Gamma) \) has exactly one positive eigenvalue.

\( \Box \)

A characterisation of graphs with smallest eigenvalue \( \geq -2 \) is given by Bussemaker & Neumaier [26]. For a computer-free proof of this theorem, see Brouwer, Cohen & Neumaier [19, Theorem 3.12.2].
Theorem 6.16 Let $\Gamma$ be a connected regular graph with $v$ vertices, valency $k$, and smallest eigenvalue $\geq -2$. Then one of the following holds:

(i) $\Gamma$ is the line graph of a regular or a bipartite semi-regular connected graph $\Delta$.
(ii) $v = 2(k + 2) \leq 28$ and $\Gamma$ is a subgraph of the Gosset graph, switching equivalent to the line graph of a graph $\Delta$ on 8 vertices, where the valencies of all vertices of $\Delta$ have the same parity.
(iii) $v = \frac{2}{3}(k + 2) \leq 27$ and $\Gamma$ is a subgraph of the Schlafli graph.
(iv) $v = \frac{5}{3}(k + 2) \leq 16$ and $\Gamma$ is a subgraph of the Clebsch graph.
(v) $v = k + 2$ and $\Gamma \cong K_{m\times 2}$ for some $m \geq 3$.

A corollary of this theorem is the following characterisation of strongly regular graphs with smallest eigenvalue $\geq -2$.

Theorem 6.17 (Seidel [97]) A strongly regular graph has smallest eigenvalue $\geq -2$ if and only if $\Gamma$ is a triangular graph, $T(n)$, a $(n \times n)$-grid, a Cocktail Party graph, $K_{n\times 2}$, the Petersen graph, the Shrikhande graph, the Clebsch graph (the halved 5-cube), the Schlafli graph or one of the three Chang graphs.

(The three Chang graphs $T^*(8)$, $T^*(8)$, and $T^*(8)$ with 28 vertices and valency 12, are obtained from the Triangular graph $T(8) = J(8, 2)$ with vertices $e_i + e_j$, ($1 \leq i < j \leq 8$) by switching with respect to the set $\{e_i + e_{i+1} \mid i = 1, 2, 3, 4\}$ for $T^*(8)$, the set $\{e_i + e_{i+1} \mid i \in Z_8\}$ for $T^*(8)$ and the set $\{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4 + e_5, e_5 + e_6, e_6 + e_7, e_7 + e_8, e_8 + e_1\}$ for $T^*(8)$.)

6.2 Lattices and graph representations

In the first we define lattices and give some properties we need later in this chapter. For more information on lattices, see for example Conway and Sloane [31] or Finkel [42].

6.2.1 Lattices

In this section we define the squared norm of a vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ to be $\langle x, x \rangle = \sum_{i=1}^{n} x_i^2$. A lattice in $\mathbb{R}^n$ is a discrete set of vectors in $\mathbb{R}^n$ which is closed under addition and subtraction (and hence under scalar multiplication by integers). A lattice $L$ is generated by a set $X$ of vectors if

$$L = \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{Z} \right\}.$$

We shall only look at lattices which are generated by a finite set. The dimension of a lattice $L$, denoted by $\text{dim} L$, defined by the dimension of the vector space spanned by the elements of $L$. An integral lattice is a lattice in which the inner product of any two vectors is integral. An integral lattice is called even if it contains only vectors of even squared norm. Note that an integral lattice is even when it is generated by a set $X$ of vectors of even squared norm. The direct sum of two lattices $L_1$ and $L_2$ is defined if $L_1$ and $L_2$ are orthogonal, i.e., if $\langle x_1, x_2 \rangle = 0$ for all $x_1 \in L_1, x_2 \in L_2$, as

$$L_1 \oplus L_2 = \{ x_1 + x_2 \mid x_1 \in L_1, x_2 \in L_2 \}.$$
A lattice \( L \) is called \emph{irreducible}, if \( L = L_1 \oplus L_2 \) implies \( L_1 = \{0\} \) or \( L_2 = \{0\} \). Otherwise the lattice is called \emph{reducible}. The vectors of of squared norm 2 in \( L \) are called the \emph{roots} of \( L \). A \emph{root lattice} is an integral lattice \( L \) in \( \mathbb{R}^n \) generated by a set of vectors of squared norm 2. There are connections between root systems and root lattices, for more information see for example [19, Chapter 3] and [42].


Witt [110] has determined all irreducible root lattices:

\textbf{Theorem 6.18} The only irreducible root lattices, up to isomorphism, are \( A_n \) (\( n \geq 1 \)), \( D_n \) (\( n \geq 4 \)) and \( E_6, E_7, E_8 \). \( \Box \)

\textbf{Remark.} If a root lattice \( L \) is the direct sum of the lattices \( L_1 \) and \( L_2 \), then \( L_1 \) and \( L_2 \) are also root lattices.

We now describe the irreducible root lattices which arise in the above theorem. We write \( e_i \) for the unit vector in \( \mathbb{R}^n \) having an one in the \( i \)-th coordinate and zeros elsewhere.

(i) For \( n \geq 1 \), define
\[
A_n := \left\{ x \in \mathbb{R}^n \mid x \in \mathbb{Z}, \sum_{i=1}^{n+1} x_i = 0 \right\}.
\]
The roots of \( A_n \) are the \( n(n+1) \) vectors \( e_i - e_j \) (\( i \neq j \)).

(ii) For \( n \geq 4 \), define
\[
D_n := \left\{ x \in \mathbb{R}^n \mid x \in \mathbb{Z}, \sum_{i=1}^n x_i \in 2\mathbb{Z} \right\}.
\]
The roots of \( D_n \) are the \( 2n(n-1) \) vectors \( \pm e_i \pm e_j \), \( i < j \). Note that our first example \( A_1 \) is represented on a hyperplane in \( \mathbb{R}^{n+1} \) rather than in \( \mathbb{R}^n \) in order to get a nicer representation, and in order to make the inclusion \( A_n \subset D_{n+1} \) obvious.

(iii) Define \( E_8 \) as the lattice spanned by \( D_8 \) and \( \frac{1}{2}(e_1 + \cdots + e_8) \). The roots of \( E_8 \) are the \( 240 = 112 + 128 \) vectors \( \pm e_i \pm e_j \) and \( \frac{1}{2}(\pm e_1 \pm e_2 \cdots \pm e_8) \), where in a root of the second type the numbers of minus signs is even.

(iv) Define
\[
E_7 = \{ x \in E_8 \mid x_1 + \cdots + x_8 = 0 \}.
\]
\( E_7 \) contains \( 126 = 56 + 70 \) roots.

(v) Define
\[
E_6 = \{ x \in E_8 \mid x_1 + \cdots + x_6 = x_7 + x_8 = 0 \}.
\]
\( E_6 \) contains \( 72 = 32 + 40 \) roots.

\textbf{6.2.2 Graph representations and lattices}

In this subsection we mean by 'squared distance' the squared Euclidean distance in \( \mathbb{R}^n \), i.e. the inner product of the difference with itself, and 'distance' means graph distance. A \emph{(Euclidean) representation of a graph} \( G \) is a map \( \rho : V^G \to \mathbb{R}^n \) such that the images
of adjacent \( x, y \in V \Gamma \) have constant squared distance. We can consider an undirected graph also as a directed graph in which both arcs exist between vertices \( x \) and \( y \) if \( xy \) is an edge. Remark that if a graph has a representation, then in fact we are labeling the arcs of \( \Gamma \) with constant norm vectors in such a way that the sum of the labels in a directed cycle (i.e. a connected graph in which every vertex has one incoming arc and one outgoing arc) add to \( 0 \), the all-zero vector.

We usually write \( \Xi \) for the image \( \rho(x) \) of a vertex \( x \) under \( \rho \), and

\[
[x, y] := (\Xi - \eta, \Xi - \nu)
\]

for the squared distance of the images of \( x, y \in V \Gamma \). The representation is faithful when \( \rho \) is injective.

**Lemma 6.19** (cf. [19, Lemma 3.5.1]). Let \( z \rightarrow \Xi (z \in V \Gamma) \) be a representation of a graph \( \Gamma \). Then, for \( z, z_1, z_2, \eta, \eta_1, \eta_2 \in V \Gamma \), we have

\[
\begin{align*}
(\Xi, \eta) &= \frac{1}{2}((\Xi, \Xi) + (\eta, \eta)) - [z, \eta], \\
(\Xi - \Xi_1 - \eta, \eta_1 - \eta_2) &= \frac{1}{2}((\Xi, \Xi_1) - (\Xi_1, \Xi_2) + [z_1, \eta] + [z_2, \eta]), \\
(\Xi - \Xi_1, \eta - \eta_2) &= \frac{1}{2}([z_1, \eta] + [z_2, \eta_1] - [z_1, \eta_1] - [z_2, \eta_2]).
\end{align*}
\]

**Proof.** Straightforward. □

We say that a representation \( z \rightarrow \Xi (z \in V \Gamma) \) of a graph \( \Gamma \) is integral when the squared distance \( [z, \eta] \) of the images of any two vertices \( z, \eta \in V \Gamma \) is an even integer. Associated with an integral representation is the lattice \( L(\Gamma) \) generated by all vectors \( \Xi - \eta, \eta \in V \Gamma \); equivalently,

\[
L(\Gamma) = \left\{ \sum_{z \in V \Gamma} \alpha_z \Xi \mid \alpha_z \in \mathbb{Z}, \sum_{z \in V \Gamma} \alpha_z = 0 \right\}.
\]

Of course, \( L(\Gamma) \) depends on the representation of \( \Gamma \), but it will be clear from the context which representation is meant. Another lattice which contains \( L(\Gamma) \) as a sublattice is:

\[
L^*(\Gamma) = \left\{ \sum_{z \in V \Gamma} \alpha_z \Xi \mid \alpha_z \in \mathbb{Z} \right\}.
\]

Let \( \Gamma \) be a graph with a representation. Let \( x \) be a vertex of \( \Gamma \). Without loss of generality we may assume that \( \Xi = 0 \). Let \( \Delta \) be the graph induced by \( \Gamma(x) \). Then \( L^*(\Delta) \subseteq L(\Gamma) \), where the representation of \( \Delta \) is the representation of \( \Gamma \) restricted to \( \Delta \).

**6.2.3 Root representations and root graphs**

A root representation of a graph is a representation such that squared distances \( [x, y] = (\Xi - \eta, \Xi - \nu) \) of the images of all \( x, y \in V \Gamma \) are even integers and satisfy

\[
[x, y] = 2 \text{ if } x \sim y \quad \text{and} \quad [x, y] = 4 \text{ if } d(x, y) = 2.
\]

A connected graph having a root representation is called a root graph. The name reflects the fact that the lattice \( L(\Gamma) \) of a root representation is a root lattice.
Proposition 6.20 Let $\Gamma$ be a root graph. Then $L(\Gamma)$ is a root lattice.

Proof. Since $\Gamma$ is connected, $L(\Gamma)$ is generated by the norm 2 vectors $\vec{r} - \vec{y}$ for $(x, y) \in V \Gamma, x \sim y$, and therefore is a root lattice. \qed

Recall from Chapter 4 that a subgraph $\Delta$ of a graph $\Gamma$ is 2-closed if for any two vertices in $\Delta$ their common neighbours in $\Gamma$ are also in $\Delta$.

Lemma 6.21 Let $\Gamma$ be a connected root graph with $L(\Gamma) \cong L_1 \oplus L_2$. Let $\Delta$ be the graph with vertex set $V \Gamma$ and edge $xy$ if $xy$ is an edge of $\Gamma$ and $\vec{r} - \vec{y} \in L_1$. Then each component of $\Delta$ is a 2-closed subgraph of $\Gamma$.

Proof. For every edge $xy$ of $\Gamma$, $\vec{r} - \vec{y}$ is a root of $L(\Gamma)$, hence contained in $L_1 \cup L_2$; we call a edge $xy$ of type $j$ if $\vec{r} - \vec{y} \in L_j (j = 1, 2)$. Since $(\vec{r} - \vec{y}) + (\vec{y} - \vec{z}) + (\vec{z} - \vec{r}) = 0$, all edges of a triangle have the same type. Now let $xy$ be an edge of type 1 and let $E$ be the connected component of $\Delta$ containing the edge $xy$. It is obvious that $E$ is an induced subgraph of $\Gamma$. Then $L(E) \subseteq L_1$. If $E$ has two vertices $u, v$ with a common neighbour $w$ in $\Gamma$, then $r = \vec{w} - \vec{u}$ and $s = \vec{w} - \vec{v}$ are roots with $r - s \in L_1$; since $r, s \in L_1 \cup L_2$, this forces $r, s \in L_1$, so that $w \in E$. It follows that $E$ is 2-closed. \qed

A direct consequence of the above lemma is:

Proposition 6.22 Let $\Gamma$ be a locally connected root graph. Then $L(\Gamma)$ is an irreducible root lattice. \qed

Root graphs generalise graphs of negative type:

Theorem 6.23 Any graph of negative type is a root graph. Any subset of the vertex set of a root graph with diameter at most 2 is with the shortest path distance a metric space of negative type.

Proof. Using the characterisation of Schoenberg, Theorem 6.5, the theorem follows easily. \qed

As we shall see later a root graph does not need to be of negative type.

Examples. Now we shall give some examples of root graphs. All these examples have a root representation such that $x \sim y$ if and only if $[x, y] = 1$.

(i) The Gosset graph can be viewed as the graph with the 56 labels $e_i + e_j$, $e_i - e_j$, ($1 \leq i < j \leq 8$), where $e_6 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_7 + e_8)$. The Gosset graph has valency 27.

(ii) The Schlaffi graph can be viewed as the graph with the 27 labels $e_i + e_7$, $e_i + e_8$ ($1 \leq i \leq 6$), $e_8 - e_i - e_7$, ($1 \leq i < 6$) and has valency 15. The Gosset graph is locally the Schlaffi graph.

The following examples are examples of code graphs $\Gamma(C)$, defined by the words of a binary code $C$ of length $n$, words are adjacent if they differ in exactly two positions (i.e. if their Hamming distance is 2); they have a natural root representation by (0, 1)-vectors in $\mathbb{R}^n$.

(iii) The halved cube $\Delta(n)$ is the code graph of the binary code consisting of all words
with even weight (i.e. the number of positions with a one is even). It has $2^{n-1}$ vertices, valency $\binom{n}{2}$ and $\mu = 6$. $\Lambda_5$ is known as the Chubes graph.

(iii) The Johnson graph $J(n,t)$ is the code graph of the binary code consisting of all words of length $n$ and weight $t$. $J(n,t)$ has $\binom{n}{t}$ vertices, valency $t(n-t)$ and $\mu = 4$. In particular ($t = 1$), cliques of size $n$ are code graphs.

(iv) The graphs $B(n,t)$. Say that a binary vector $x = (x_1, x_2, \ldots, x_{n+2})$ has weight $(w_1, w_2)$ where $w_1 = x_1 + x_2$ and $w_2 = x_3 + x_4 + \ldots + x_{n+2}$, where we add in $\mathbb{Z}$. Now $B(n,t)$ is the code graph of the binary code of all words of length $n+2$ with weights $(2, t)$, $(1, t-1)$, and $(0, t)$ and has $\mu = 4$. An $B(n,t)$ is regular if and only if $n = 2t - 1$. The graphs $B(2t - 1, t)$ are first constructed by Brouwer and Koolen [22]. In that paper you can find more information about these graphs. In Chapter 5 we construct the graphs $B(n,t)$ with a more general construction.

(vi) The icosahedron with 12 vertices, valency 5 and $\mu = 2$ can be viewed as the code graph of the binary code consisting of the words $000000$, $110000$, $001111$, $111111$ and those obtained by a cyclic permutation of the first five entries.

(vii) The dodecahedron with 20 vertices, valency 3 and $\mu = 1$ can be viewed as the code graph consisting of the words $1100000000$, $1100011000$ and those obtained by permutations $[(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)]$, $(i = 1, 2, 3, 4, 5)$ of the entries and by complementation.

(viii) The Shrikhande graph with 16 vertices, valency 6 and $\mu = 2$ can be viewed as the code graph of the binary code consisting of the words $000000$, $110000$, $010111$, $011011$ and those obtained by a cyclic permutation of the six entries.

(ix) The Petersen graph with 10 vertices, valency 3 and $\mu = 1$ can be viewed as the code graph of the binary code consisting of the words $000000$, $100100$, $001111$ and those obtained by a cyclic permutation of the six entries.

(x) A polygon: a $g$-gon can be viewed as the code graph of the binary code consisting of the word of length $g$ with on the first $\lfloor \frac{g}{2} \rfloor$ entries ones and zeroes on the other entries, and those obtained by a cyclic permutation of the $g$ entries.

(xi) If $\Gamma$ and $\Delta$ are root graphs, then the cartesian product $\Pi = \Gamma \times \Delta$ is again a root graph: simply represent $(x, y)$ by the sum of the images of $x$ and $y$. In particular, Hamming graphs (cartesian product of cliques) and Doob graphs (cartesian product of $4$-cliques and Shrikhande graphs) are root graphs.

We first prove some easy results on the local structure of root graphs.

**Lemma 6.24** Let $\Gamma$ be a root graph and $x, y, z \in V\Gamma$, where $d(x, y) \leq 2$. For the associated root representation we have

\[
(x - \tilde{x}, y - \tilde{y}) = \begin{cases} 
2 - d(x, y) \geq 0 & \text{if } x, y \in \Gamma(z), \\
3 - d(x, y) \geq 0 & \text{if } x \in \Gamma(z), y \in \Gamma_2(z), \\
4 - d(x, y) \geq 0 & \text{if } x, y \in \Gamma_2(z). 
\end{cases}
\]

In particular, if $x = 0$, then the vertices in $\Gamma(z)$ are represented by norm 2 vectors, and vertices of $\Gamma_2(z)$ are represented by norm 4 vectors.
Proof. It is straightforward that \( (x - x, y - z) = \frac{1}{2}(|x, y| + |z, z| - |x, y| - |z, z|) = d(x, z) + d(y, z) - d(x, y) \). This implies the first statement. The second statement follows by putting \( x = y \). □

A direct consequence of Theorem 6.23 and Proposition 6.13 (note that graphs of negative type satisfy the 10-gonal inequality) is the following proposition.

Proposition 6.25 Let \( \Gamma \) be a root graph. Then the following holds:

(i) If \( x \in V\Gamma \), then two nonadjacent neighbours \( y, z \) of \( x \) have at most one common neighbour in \( \Gamma(x) \) and such a neighbour has label \( y + z - x \).

(ii) If \( x, y \) are two vertices at distance 2, then the graph induced by \( \Gamma(x) \cup \Gamma(y) \) is complete multipartite with classes of size 1 or 2.

(iii) If \( \mu(\Gamma) = \mu \) exists, then any two nonadjacent vertices \( y, z \in \Gamma(x) \), where \( x \in V\Gamma \), have either \( \mu - 1 \) or \( \mu - 2 \) common neighbours in \( \Gamma(x) \).

(iv) If \( \Gamma \) is an amply regular graph with \( \mu(\Gamma) = \mu \) and \( \lambda(\Gamma) = \lambda \), then for any two nonadjacent vertices \( x, y \), lying in an induced quadrangle of \( \Gamma \), we have \( a_2(x, y) = 2(\lambda - \mu + 2) \).

Proof. (i) Let \( u \) be a common neighbour of \( y \) and \( z \) at distance 2 from \( x \). Then \( (x - y, x - u) = \frac{1}{2}(|x, u| + |y, z| - |x, y| + |y, z|) = 2 \), and thus \( x - y = x - u \). So we are done. □

6.3 Classification of root graphs

Using the classification of root lattices we shall give a complete characterisation of distance-regular graphs with \( \mu \geq 2 \) later in this section. In addition, in Theorem 6.40, we give a complete characterisation of amply regular root graphs with \( \mu \geq 2 \), by use of computer results.

Remark. Related questions are considered by TERWILLIGER [107], who considers structure theorems for (not necessarily regular) graphs having a spherical representation, i.e. \( (x, y) \) does not depend on \( y \), such that \( (x, y) \) is an even integer for all \( x, y \in V\Gamma \) and \( |x, y| \leq 2 \) if and only if \( x \sim y \).

A graph \( \Gamma \) has an integral root representation if it has a root representation such that \( L(\Gamma) \subseteq \mathbb{Z}^n \) for some \( n \), i.e. the label are elements of \( \mathbb{Z}^n \). First we will consider root graphs with an integral root representation.

6.3.1 Integral root representations

In this subsection we classify the amply regular root graphs with an integral root representation.

Lemma 6.26 Let \( \Gamma \) be a root graph with \( L(\Gamma) \subseteq \mathbb{Z}^n \) and \( x, y \) vertices at distance 2 with \( y - x = 2e_i \), where \( e_i = (1, 0, \ldots, 0) \). Then no vertex of \( \Gamma(x) \cap \Gamma(y) \) is adjacent to a vertex of \( \Gamma(x) \setminus \Gamma(y) \).

Proof. Without loss of generality we may assume that \( x = 0 \) and \( y = 2e_i \). Lemma 6.24 shows that common neighbours of \( x \) and \( y \) are represented by suitable vectors \( e_i \pm e_i \), and, by permuting the \( e_i \) and changing their signs if necessary, we may assume that \( M := \Gamma(x) \cap \Gamma(y) \) is represented by the \( n = s + t \) vectors

\[
e_i \pm e_i (l = 2, \ldots, s + 1), e_i + e_i (l = s + 2, \ldots, s + t + 1).
\]

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Let $u \in M$ so that (for some $i$) $\overline{u} = e_i \pm e_i$, and suppose that $x$ has a neighbour $v \notin M$ adjacent to $u$. Then $d(y, v) = 2$, but either $\overline{v} = e_i \pm e_j$, $(\overline{y}, \overline{v}) = 2$, or $\overline{v} = 2e_i \pm e_i \pm e_j$, $(\overline{y}, \overline{v}) = 0$, both contradicting Lemma 6.24. Hence no vertex of $\Gamma(x) \setminus \Gamma(y)$ is adjacent to a vertex of $M$. □

**Proposition 6.27** Let $\Gamma$ be a root graph with $L(\Gamma) \subseteq \mathbb{Z}^n$ for some $n$, and $(\ast)$ for no $x, y \in V\Gamma$ at mutual distance two do we have $x - y = \pm 2e_i$ for some $i$, $1 \leq i \leq n$. Then $\Gamma$ has a root representation in $(0, 1)^n$ sending an arbitrary $x \in V\Gamma$ to 0.

**Proof.** Given a root representation $\rho : x \in \mathbb{Z}^n \to \mathbb{Z}^n$ satisfying $(\ast)$, and a vector $u \in \mathbb{Z}^n$, we find new root representations $\rho - u : x \in \mathbb{Z}^n \to \mathbb{Z}^n$ and $\rho + u : x \in \mathbb{Z}^n \to \mathbb{Z}^n$ both satisfying $(\ast)$. (Indeed, if $x_i, y_i \in \{-1, 0, 1\}$ then $(x_i - y_i)^2 = (x_i - y_i)^2$; in particular this holds for $d(x, y) \leq 2$.) Let $\rho$ be a root representation satisfying $(\ast)$ in $(0, \ldots, m)^n$, where $m$ is chosen minimal. Then $m \leq 1$, otherwise $|\rho - 1|$ would be a representation in $(0, 1, \ldots, m - 1)^n$. Now $|\rho - 1|$ is the required representation. □

**Lemma 6.28** Let $\Gamma$ be a root graph with a root representation in $(0, 1)^n$ and $\mu(u, v) \geq 2$ for any two vertices $u, v \in \Gamma$ at distance two. Suppose $d(x, y) \leq 4$.

(i) If $x = y$, then $x = y$.

(ii) If $d(x, y) = 2$ then $x \sim y$.

**Proof.** If $d(x, y) \leq 2$, this is clear. We first prove part (ii). Assume $d(x, y) = 2$. If $d(x, y) = 3$, then choose $v \in \Gamma(x) \cap \Gamma(y)$ and shift the representation such that $\overline{v} = 0$. We may assume that $x = e_i + e_j$ and $y = e_i + e_j$ with distinct $i, j, k, l$. The common neighbours of $y$ and $v$ are represented by certain $e_a + e_b$ with $\{a, b\} \subseteq \{i, j, k, l\}$, and since there are at least two of those, it follows that $x$ has distance at most one to (at least one) of them, so that $d(x, y) \leq 2$, contradiction. If $d(x, y) = 4$, then choose $v \in \Gamma_2(x) \cap \Gamma_2(y)$ and shift the representation such that $\overline{v} = 0$. We may assume that $x = e_i + e_j + e_k + e_l$ and $y = e_i + e_j + e_k + e_l$. By the foregoing, no common neighbour of $x$ and $v$ is represented by $e_a + e_b$ with $\{a, b\} \subseteq \{i, j, k, l\}$, so the (at least two) vertices in $\Gamma(x) \cap \Gamma(y)$ are represented by $e_a + e_b$ for certain $a \in \{i, j, k\}$, and similarly those in $\Gamma(y) \cap \Gamma(v)$ by $e_a + e_b$ for certain $a \in \{i, j, k\}$. But then some vertex in $\Gamma(x) \cap \Gamma(y)$ is adjacent to some vertex in $\Gamma(y) \cap \Gamma(v)$ so that $d(x, y) \leq 3$, contradiction. This proves part (ii). If $x = y$ and $j = d(x, y) \in \{3, 4\}$, then choose $v \in \Gamma(x) \cap \Gamma_{j+1}(y)$. Now $d(x, y) = 2$ and hence by the above $x \sim v$, contradiction. This proves part (i). □

**Proposition 6.29** Let $\Gamma$ be a root graph with a root representation in $(0, 1)^n$ and such that all $\mu$-graphs are non-complete. Then for any vertex $x \in V\Gamma$ the root representation of $\Gamma$ is uniquely determined by its restriction to $\{x\} \cup \Gamma(x)$. If moreover $\mu(\Gamma) = \mu$ is constant, then $\Gamma$ itself is uniquely determined by the representation of $\{x\} \cup \Gamma(x)$.

**Proof.** The first claim follows immediately from Proposition 6.25 (ii). If $\mu$ is constant, then the representation of $\{x\} \cup \Gamma(x)$ determines the structure and representation of $\Gamma(x)$: for any two nonadjacent vertices $u, v \in \Gamma(x)$ that have only $\mu$-2 common neighbours in $\Gamma(x)$, we find a vertex $y$ represented by $\overline{u} + \overline{v} - \overline{x}$. But the restriction of the representation to $\Gamma_2(x)$ is injective, and determines the structure of this subgraph, by Lemma 6.28 above. Since root graphs are connected, we are done. □

Before we can state the next theorem we first have to define the graphs $L(s, t)$. An $L(s, t)$ is the graph with vertex set $V = S \cup T$, $|S| = 3s + t$, $|T| = t$. The subgraph induced by $S$ is a clique. A vertex $t \in T$ is adjacent to exactly 3 vertices in $S$ and the distance between any two $t_1, t_2 \in T$ is three.

Note that $\mu(L_{s,t}) = 3$. The graph $L_{s,t}$ is a root graph with labels $e_i, 1 \leq i \leq 3s + t$ and $e_j + e_{j+1} + e_{2j+1}, 1 \leq j \leq s$. 67
Theorem 6.30 Let $\Gamma$ be a non-complete root graph with $\mu(\Gamma) = \mu \geq 3$ and with an integral root representation. Then $\Gamma$ is locally connected and one of the following holds:

(i) $\Gamma$ is a complete multipartite graph with classes of size 1 or 2.
(ii) $\mu = 6$ and $\Gamma$ is a halved cube.
(iii) $\mu = 5$ and $\Gamma$ is the suspension of a triangular graph.
(iv) $\mu = 4$ and either $\Gamma$ is a Johnson graph or $\Gamma \cong B(n, t)$ for some $n > t \geq 1$.
(v) $\mu = 3$ and either $\Gamma$ is the suspension of the $(p \times q)$-grid, $\Gamma$ is an $L(s, t)$, or $\Gamma \cong K_{3,1,1}$.

Furthermore, all the above graphs have an integral root representations.

Proof. By Proposition 6.25 (iii), $\Gamma$ is locally connected. Suppose that $\Gamma$ contains vertices $\pi, \eta$ at distance 2 such that the norm 4 vector $\overline{ar} - \overline{c} - \overline{e}$ has the form $2e_j$ for some $j$. Since $\Gamma$ is locally connected, it follows from Lemma 6.26 that $\Gamma(y) \cap \Gamma(z) = \Gamma(x)$. First assume that $\Gamma_2(x) \neq \{y\}$ and let $z \in \Gamma_2(x)$, distinct from $y$. Then $\Gamma(z) = \Gamma(y) \cap \Gamma(x)$, since $\Gamma(x)$ has cardinality $\mu$. So we may assume without loss of generality that $\overline{a} = 0$ and $\overline{c} = e_1 + e_s + e_t$, for some $s, t$. The vertices of $\Gamma(z)$ are represented by $e_1 + e_i, \overline{c}, e_j$ with $i \in \{r, s, t\}$. So $\mu = 3$ and $\Gamma$ is $K_{3,1,1}$.

If $\Gamma_2(x)$ consists only of $y$, then it is easy to see that we are in case (i).

We may now assume that $\Gamma$ does not contain vertices $\pi, \eta$ at distance 2 such that $\overline{a} = \overline{c} = \overline{e} = \overline{f} = 0$.

Fix a vertex $\pi$ of $\Gamma$. By Proposition 6.27 we may assume that $\overline{y} = \sum y_i e_i$ with $y_i \in \{0, 1\}$ for all $i$ and $y \in V(\Gamma)$, and that $\overline{f} = 0$.

Now let $\Delta$ be the graph whose vertices are the $n$ indices $1, \ldots, n$, and whose edges are the pairs $ij$ such that some point of $\Gamma(x)$ is represented by $e_i + e_j$. Since distinct vertices of $\Gamma(x)$ are represented by distinct vectors (their difference must have norm 2 or 4) we may identify the point represented by $e_i + e_j$ with the edge $ij$ of $\Delta$.

Let us denote by $S(y)$ the set of indices $i$ with $y_i = 1$ for $y \in V(\Gamma)$. By Lemma 6.24, the vertices $y \in \Gamma_2(x)$ are represented by norm 4 vectors $\overline{y} = \overline{a} - \overline{c}$, and these must have the form $e_1 + e_s + e_t$, i.e., $S(y)$ is a 4-set. The $\mu$ common neighbours of $x$ and $y$ are certain edges of the subgraph of $\Delta$ induced on $S(y)$. By Lemma 6.28, the intersection $\Gamma(x) \cap \Gamma(y)$ coincides with the set of edges of $\Delta$ contained in $S(y)$, and $y \in \Gamma_2(x)$ is uniquely determined by $S(y)$. To simplify the notation we shall use the abbreviation $\alpha \beta \gamma \delta$ for a 4-set $\{\alpha, \beta, \gamma, \delta\}$.

Let us call a set $S \subseteq \{1, 2, \ldots, n\}$ special if there is a vertex $y \in \Gamma$ with $S(y) = S, d(x, y) = \frac{1}{2}|S|$.

Clearly, the empty set is special. Application of Lemma 6.24 to pairs of nonadjacent points in $\Gamma_2(x)$ yields the following facts (here distinct Greek letters denote distinct indices taken from $\{1, \ldots, n\}$).

S1. If $y \in \Gamma_2(x)$, then $S(y)$ is special.

S2. If $\alpha \beta \gamma \delta$ is special then precisely $\mu$ of the sets $\alpha \beta, \alpha \gamma, \alpha \delta, \beta \gamma, \beta \delta, \gamma \delta$ are special.

S3. If $\alpha \beta$ and $\gamma \delta$ are special then precisely $\mu$ of the sets $\emptyset, \alpha \gamma, \alpha \delta, \beta \gamma, \beta \delta, \gamma \delta \emptyset$ are special.

S4. If $\alpha \beta \gamma \delta$ and $\emptyset$ are special then none or $\mu$ of the sets $\alpha \beta, \beta \gamma, \gamma \delta, \alpha \beta \emptyset, \alpha \gamma \emptyset, \beta \gamma \emptyset$ are special.
(Indeed, if \( a \beta \gamma \delta \) and \( \epsilon \zeta \) represent vertices at distance two, then they have \( \mu \) common neighbours, and these must be among the six sets listed. On the other hand, if one of these six sets is special, we must show that it represents a common neighbour of \( a \beta \gamma \delta \) and \( \epsilon \zeta \). But this follows immediately from Lemma 6.28.)

By definition, the edges of \( \Delta \) are just the special 2-sets. Since \( \mu \geq 3 \), S3 implies that \( \Delta \) has no subgraph consisting of two disjoint edges. Therefore \( \Delta \) consists of a connected graph \( \Delta_0 \) and a (possibly empty) set of isolated vertices. S3 implies

S3. Any two disjoint edges of \( \Delta \) have \( \mu = 1 \) or \( \mu = 2 \) transversals (i.e. edges meeting both given edges).

We now consider each value of \( \mu \) separately. Clearly, S3 implies that \( \mu \leq 6 \).

**CASE 1.** \( \mu = 6 \).

By S1 and S2, the induced subgraph of \( \Delta_0 \) on the label of a vertex of \( \Gamma_2(x) \) is a \( K_4 \).

Let \( K \) be a complete subgraph \( K_m \) of \( \Delta_0 \) with maximal \( m \geq 4 \). If \( \Delta_0 \not= K \) then there is a vertex \( \beta \in V \Delta_0 \) at distance 1 from \( K \), and a vertex \( \alpha \in K \) adjacent with \( \beta \). By maximality of \( K \), there is a vertex \( \gamma \in K \) with \( \gamma \not= \beta \). Let \( \delta \in K \setminus \{ \alpha, \gamma \} \). Then the edges \( \alpha \beta, \gamma \delta \) have at most 3 transversals, contradicting S5. Hence \( \Delta_0 = K \) is complete and \( \Gamma(x) \cong T(m) \). By Proposition 6.29 this determines the representation uniquely, and we find that \( \Gamma \) is the halved \( m \)-cube.

**CASE 2.** \( \mu = 5 \).

We may assume that \( \Gamma_2(x) \not= \emptyset \). By S1 and S2, a vertex of \( \Gamma_2(x) \) is a subgraph \( K_{m,1,1} \) of \( \Delta_0 \). Let \( K \) be a subgraph of \( \Delta_0 \) isomorphic to \( K_{m,1,1} \) with a maximal number \( m + s \) of vertices, \( m \geq 2 \), \( s \geq 2 \). Using S5 we find as before that \( \Delta_0 = K_{m,1,1} \). Let \( L \) be the \( m \)-coclique of \( \Delta_0 \). From S3 we find that, for \( \alpha, \gamma \in K \setminus L \) and \( \beta, \delta \in L \), the set \( \alpha \beta \gamma \delta \) is special. If \( \epsilon \in K \setminus \{ \alpha, \gamma \} \), then S4 yields that \( \alpha \beta \epsilon \gamma \) is special, contradicting \( \mu < 6 \). Hence \( \Delta_0 = K_{m,1,1} \), and we find by Proposition 6.29 that \( \Gamma \) is the suspension of the triangular graph \( T(m + 2) \).

**CASE 3.** \( \mu = 4 \).

By S5, each pair of disjoint edges has 2 or 3 transversals. By S1 and S2, a vertex of \( \Gamma_2(x) \) is a special 4-set on which \( \Delta_0 \) induces

- or -

and by S3, all subgraphs of \( \Delta_0 \) of these shapes are special sets.

Let \( E = A \cup B \) be a largest subset of \( V \Delta_0 \) such that \( \alpha \sim \beta \) for all \( \alpha \in A \) and \( \beta \in B \), where \( A \) and \( B \) are nonempty.

(i) \( E \) contains all vertices of \( \Delta_0 \). Indeed, if not then let \( \gamma \in V \Delta_0 \), \( \gamma \not\in E \). We may assume \( d(\gamma, E) = 1 \), say \( \gamma \sim \alpha \in A \). By maximality of \( E \) there is a \( \beta \in B \) such that \( \beta \not\sim \gamma \). Put \( A_0 := A \cap \Delta(\gamma) \) and \( A_1 := A \setminus \Delta(\gamma) \), then for any \( \alpha \in A_0 \) and \( \delta \in A_1 \), we have \( \alpha \sim \delta \) since \( \alpha \gamma \) and \( \beta \delta \) have at least two transversals. But then \( E \cup \{ \gamma \} = A_0 \cup (A_1 \cup B \cup \{ \gamma \}) \)
is a larger subset of $V\Delta$ with the required property, contradiction.

(ii) $\Delta_0$ does not contain a complete graph $K_4$ or a wheel $K_{1,2,2}$. Indeed, two disjoint edges in a $K_4$ have 4 transversals, which is impossible. If $\Delta_0$ contains an induced subgraph $K_{2,2,1}$, say $\alpha \sim \beta \sim \gamma \sim \delta \sim \alpha$, $\varepsilon \sim \alpha, \beta, \gamma, \delta$, then $\alpha \beta \gamma \delta$ is special, and by S4 we find a contradiction.

(iii) $\Delta_0$ is either complete bipartite or complete bipartite with one additional edge. Indeed, if $A$ contains two disjoint edges, then (since there must have at least two transversals) $A$ contains a triangle or a quadrangle, and since $H \neq \emptyset$ this is forbidden by (ii). Thus, all edges in $A$ pass through one vertex $\delta$. If both $A$ and $B$ contain an edge then $\Delta_0$ contains a $K_4$. So we may assume that $B$ is a coclique. If the cardinality of $B$ is at least 2 and $A$ contains two intersecting edges, then $\Delta_0$ contains a $K_{1,2,2}$, contradiction. So we may assume that $A$ has at least two edges $\gamma \delta$ and $\varepsilon \delta$ and an isolated point $\alpha$. Now S4 yields a contradiction.

(iv) If $\Delta_0$ is complete bipartite, say $K_{s,t}$, then $\Gamma \cong J(s+t,t)$. Indeed, this follows from Proposition 6.29.

Now assume that $\Delta_0$ is a complete bipartite graph $K_{s,t}$ with one additional edge in the $t$-coclique, so that $\Gamma(x)$ is a grid $s \times t$ with one additional vertex adjacent to a $s \times 2$ subgrid. Let us temporarily call this latter graph $s \times t$ in the figure below the graph $4 \times 3$ is drawn; a label $ij$ denotes the vector $a = ri + tj$.

(v) If $\Gamma(x) \cong s \times t$, then $\Gamma \cong H(s+1, 2,s)$. Indeed, this follows from Proposition 6.29.

Case 4. $\mu = 3$. By S3, each pair of disjoint edges has 1 or 2 transversals. By S1 and S2, a vertex of $\Gamma(x)$ is a special 4-set on which $\Delta_0$ induces either

or

or

Assume first that $\Delta_0$ contains an induced path $P$ of length 3 with vertices $\alpha \sim \beta \sim \gamma \sim \delta$. By S3, $\alpha \beta \gamma \delta$ is special. Let $e$ be at distance 1 from $P$. If $e \sim \delta$, then S4 implies that precisely two of the sets $\alpha \beta \delta e$, $\alpha \gamma \delta e$ and $\beta \gamma \delta e$ are special. However, this is impossible.
If \( \varepsilon \sim \alpha \), then \( \varepsilon \sim \beta \) implies that \( \alpha \delta \varepsilon \) and \( \beta \gamma \varepsilon \) are not special. So \( \varepsilon \not\sim \beta \), and by symmetry \( \varepsilon \not\sim \gamma \). But then \( \alpha \delta \varepsilon \), \( \alpha \gamma \varepsilon \) and \( \beta \delta \varepsilon \) are all special, contradiction. And if \( \varepsilon \not\sim \alpha \) then \( \varepsilon \sim \beta \) since \( \alpha \delta \) and \( \delta \varepsilon \) must have a transversal, and \( \varepsilon \not\sim \gamma \) since \( \beta \gamma \) and \( \delta \varepsilon \) cannot have 3 transversals. But then \( \alpha \gamma \varepsilon \) and \( \beta \delta \varepsilon \) are not special, contradiction. Hence \( \varepsilon \not\sim \delta \), and by symmetry, \( \varepsilon \not\sim \alpha \).

If \( \varepsilon \sim \beta \) and \( \varepsilon \sim \gamma \), then \( S\delta \) with \((\beta, \delta)\) interchanged gives a contradiction. Therefore \( \varepsilon \) is adjacent to just one of \( \beta \) and \( \gamma \) (and if \( \varepsilon \sim \beta \) then \( \varepsilon \beta \delta \gamma \) is special). Hence the neighbours distinct from \( \gamma \) of \( \beta \) form a coclique \( A \), and the neighbours distinct from \( \beta \) of \( \gamma \) form a coclique \( D \). Now it is easy to see that \( \Delta_0 = A \cup \{\beta, \gamma\} \cup D \) (the coclique extension of \( P \) where \( \alpha \) and \( \delta \) are blown up to \( A \) and \( D \), respectively). The set \( \alpha \beta \delta \gamma \) is special for all \( \alpha \in A \) and \( \delta \in D \). Let \( \alpha \in A \), and \( \beta \delta \in D \). Then by \( S\delta \) with \((\gamma, \delta)\) interchanged we find that \( \gamma \varepsilon \delta \gamma \) is not special for all \( \eta \in D \). It follows that there are no more special 4-sets. With the same argument and Proposition 6.39 we see that there are no special 2-sets or non-special 4-sets at distance 3. We obtain that \( \Gamma \) is the suspension of the \(((|A| + 1) \times (|B| + 1))\)-grid.

So we may assume that \( \Delta_0 \) does not contain induced paths of length 3. So there are no induced quadrangles in \( \Gamma \). Let \( C \) be a clique in \( \Gamma \) of maximal size \( \varepsilon \) and without loss of generality \( x \in C \) and there is a \( \delta \) such that \( \delta \in S(x) \) for all \( y \in C \setminus \{x\} \). Let \( \alpha \beta \in \Gamma(x) \setminus C \). Then \( \delta \notin \{\alpha, \beta\} \), otherwise \( C \) was not maximal. We may assume that \( \alpha \delta \in C \). By the fact that \( \Delta_0 \) does not contain induced 3-paths we obtain \( \delta \beta \notin C \) by \( S\delta \). By \( S\delta \) we get \( \Gamma(x) \setminus \{x\} = C \cup \{\alpha \delta\} \).

Let \( x \) be a vertex at distance 2 from \( x \). Then \( S(x) = \{x, \delta \} \) such that the induced subgraph by \( S(x) \) is a 3-claw. It follows \( d(y, z) = 3 \) and \( \Gamma \) is an \( I(3, 1) \).

This completes the proof. \( \square \)

Now we look at the case \( \mu = 2 \). First we derive some properties.

**Lemma 6.31.** Let \( \Gamma \) be a root graph with \( \mu(\Gamma) = 2 \). Let \( \Delta_1, \Delta_2 \) be two \( 2 \)-closed connected subgraphs of \( \Gamma \) such that the root representation of \( \Gamma \) is injective restricted to \( V \Delta_1 \). If there is a vertex \( x \) with \( x \in V \Delta_1 \cap V \Delta_2 \) and \( \Delta_1(x) \cap \Delta_2(x) = \emptyset \), then the cartesian product \( \Delta_1 \times \Delta_2 \) is a 2-closed subgraph of \( \Gamma \).

**Proof.** Shift the representation such that \( \mathfrak{F} = 0 \). Let \( y \) be a vertex of \( \Delta_2 \). For each vertex \( z \) of \( \Delta_1 \) there is a unique vertex \( \phi(z) \in \Gamma(z) \) with \( \phi(z) = \mathfrak{F} + \mathfrak{Z} \). This follows by induction on \( d \Delta_1(x, z) \). Now define the map \( \phi : V \Delta_1 \rightarrow V \Gamma \), by \( z \rightarrow \phi(z) \). Thus by the \( 2 \)-closedness of \( \Delta_1 \) it follows that the induced subgraph \( I_1 \) on \( \phi(V \Delta_1) \) is \( 2 \)-closed, looking at the inverse of \( \phi \). The subgraph induced by \( \phi(V \Delta_1) \cup V \Delta_1 \) is isomorphic to the cartesian product \( \Delta_1 \times \Delta_2 \). It follows that \( \mathfrak{Z} \) is orthogonal to \( L(\Delta_1) \). By the \( \Delta_2 \)-closedness of \( \Delta_2 \) and \( \Delta_1(x) \cap \Delta_2(x) = \emptyset \) it follows that \( \Pi(y) \cap \Delta_2(y) = \emptyset \). By connectedness of \( \Delta_2 \) we get \( L(\Delta_2) \) is orthogonal to \( L(\Delta_2) \). It is easy to see that the cartesian product \( \Delta_1 \times \Delta_2 \) is a subgraph such that \( \mathfrak{V} \) with \( \mathfrak{V} \notin V \Delta_1 \times V \Delta_2 \) has label \( \mathfrak{V} + \mathfrak{V} \). Since the labelling of \( \Delta_1 \) is injective and \( L(\mathfrak{V}) \) is orthogonal to \( L(\mathfrak{V}) \) for \( x \in V \Delta_1 \). Now it is obvious that this subgraph is a connected 2-closed subgraph of \( \Gamma \). \( \square \)

**Proposition 6.32.** Let \( \Gamma \) be a root graph with \( \mu(\Gamma) = 2 \). Let \( C \) be a clique in \( \Gamma \). If \( C \) is a \( 2 \)-closed subgraph of \( \Gamma \) then \( \Gamma \) is the cartesian product \( \Delta \times \Delta, \) where \( \Delta \) is a \( 2 \)-closed subgraph of \( \Gamma \).

**Proof.** Let \( x \in C \) and shift the representation such that \( \mathfrak{F} = 0 \). Let \( y \in \Gamma(x) \setminus C \). Then, by the \( 2 \)-closedness of \( C \), \( \mathfrak{Z} \) is perpendicular to \( \mathfrak{V} \) for all \( e \in C \). There is a set \( T \)
of neighbours of \( y \) such that for each \( e \in C \setminus \{ x \} \), there is a \( f \in T \) with \( f \equiv y + e \). The subgraph induced by \( T \cup \{ y \} \) is a 2-closed clique, otherwise with the inverse procedure we get that \( C \) is not 2-closed. Therefore \( f - y \) is perpendicular to \( v - y \) for all \( v \in \Gamma(y) \setminus T \) and all \( f \in T \).

Let \( H \) be the graph with vertex set \( V_T \) and \( uv \) is an edge in \( H \) if \( uv \) is an edge in \( \Gamma' \) and \( u - v \) is orthogonal to \( \overline{y} \) for all \( e \in C \). Let \( \Delta \) be the component which contains \( x \). With induction to the distance of two vertices in \( \Delta \) it is easy to see that \( \Delta \) is a 2-closed subgraph. By the previous lemma \( \Gamma' \) has an subgraph isomorphic to the cartesian product \( C \times \Delta \). But this must be \( \Gamma' \) itself. \( \square \)

Before we can state the following proposition we first need to define the graph \( L \).

The vertices of \( L \) are the 192 subsets \( A \) of the vertex set of a Petersen graph \( \Pi \) such that the induced subgraph on \( A \) in \( \Pi \) has one of the following types, and \( A \sim B \) if and only if \( |A \oplus B| = 2 \):

![Graph Diagram](image)

The graph \( L \) is locally the line graph of Petersen and with respect to the above partition of the vertex set of \( L \) we have the following intersection diagram for \( L \):

![Intersection Diagram](image)

**Remark.** The graph \( L \) is an antipodal 2-cover of a graph \( \Delta \). This graph \( \Delta \) is a distance-regular graph with intersection array \( \{15, 10, 1; 1, 2, 15 \} \) and is locally the line graph of Petersen. See [23], for more information on \( L \) and related structures.
Proposition 6.33 Let \( \Gamma \) be an amply regular root graph with \( \mu(\Gamma) = 2 \). If \( L(\Gamma) \subset \mathbb{Z}^n \), then one of the following holds.

(i) \( \Gamma \) is a Hamming graph,

(ii) \( \lambda = 2 \) and \( \Gamma \) is a direct product of 4-cliques, icosahedra and Shrikhande graphs, and

(iii) \( \lambda = 4 \) and \( \Gamma \) is the direct product of 6-cliques and some copies of \( L \).

Proof. If \( \lambda \leq 1 \), then every edge lies in a \((\lambda + 2)\)-clique. Hence, by Lemma 6.32, we are in case (i). So we may assume that \( \lambda \geq 2 \).

The graph \( \Gamma \) does not contain vertices \( x, y \) at distance 2 such that the norm 4 vector \( \bar{x} - \bar{y} \) has shape \((\pm 1)^n\). Indeed, then Lemma 6.26 would show that \( \Gamma(x) \) has a component of size at most 2 and this contradicts \( \lambda \geq 2 \). As a consequence, for any two vertices \( x, y \) at distance two, the vector \( \bar{x} - \bar{y} \) must have shape \((\pm 1)^n\). We can proceed as the proof of Theorem 6.30. By Proposition 6.27, we fix a vertex \( x \) and shift the representation such that \( \bar{x} = 0 \) and all vertices of \( \Gamma \) are represented by \( \sum \alpha_i \epsilon_i \), where \( \alpha_i \in \{0, 1\} \). Let \( \rho \) be the map which sends \( x \) to \( \bar{x} \).

We define the graph \( \Delta \) and the special sets as before and find that \( S_1 \) and \( S_2 \) remain valid; in particular, special 4-sets contain precisely two edges, and the union of two disjoint edges without a transversal is special. Let \( xy \) be an edge of \( \Gamma \). We will show that the minimal 2-closed subgraph, say \( \Pi \), of \( \Gamma \) containing the edge \( xy \) is isomorphic to either a \((\lambda + 2)\)-clique, the Shrikhande graph, the icosahedron or the graph \( L \).

Let \( \Delta_0 \) be the component of \( \Delta \) containing \( x \), where \( \bar{y} = c_0 + c_\beta \). Since \( \Gamma(x) \) is regular of valency \( \lambda \), every edge of \( \Delta_0 \) intersects precisely \( \lambda \) other edges of \( \Delta_0 \). If \( \Delta_0 \) contains a triangle, then \( S_3 \) implies that \( \Delta_0 \) is a triangle, and thus \( \lambda = 2 \). This means that \( xy \) lies in a \( 4\)-clique.

By Proposition 6.32 we may assume that \( \Delta_0 \) does not contain a triangle. Also \( \Delta_0 \) does not contain a quadrangle, by \( S_5 \).

We need some further facts. Applying \( S_4 \) to a path of length 4 in \( \Delta_0 \) we find:

\( S_6 \). If \( \gamma \sim \delta \sim \epsilon \sim \beta \sim \alpha \) is an induced path in \( \Delta_0 \), then \( \alpha \gamma \delta \epsilon \) is special.

\( S_7 \). \( \Delta_0 \) contains no induced subgraph of the form \( D_0 \).

Indeed, \( S_6 \) shows that 1345 and 1346 are special, so that \( S_4 \) applied to 1345 and 46 gives a contradiction. We now distinguish three cases.

**Case 1.** \( \Delta_0 \) is a tree. Since \( L(\Delta_0) \) must be regular of valency \( \lambda \), we find \( \Delta_0 \cong K_{n,1} \). But then the edge \( xy \) lies in a \((\lambda + 2)\)-clique.

**Case 2.** \( \Delta_0 \) has girth \( g \) at least 6. Then \( S_8 \) implies that no vertex of a \( g \)-gon has a further neighbour. Since \( \Delta_0 \) is connected, it is a \( g \)-gon with vertices labelled in \( Z_g \) and edges \( \{i, i+1\} \) (\( i \in Z_g \)), say. In particular, \( \lambda = 2 \). By \( S_3 \) and \( S_7 \), the sets \( \{i, i+1, j, j+1\} \) (\( |i-j| \neq 0, 1, 2 \) and \( i-1, i, i+1, i \pm 3 \) are special. Now the vertices represented by the \( 0, 022, 2846 \) are neighbours of the vertices labelled with \( 34 \) and \( 23 \). By \( \lambda = 2 \) this
forces $g = 6$. The 16 special sets found so far represent the Shrikhande graph, a graph $E$ with $\lambda(E) = 2 = \lambda$, $\mu(E) = 2$ and of diameter 2, this means that $\Pi$ is the Shrikhande graph.

Case 3. $\Delta_0$ has girth 5. Then $\Delta_0$ is not bipartite, and since $\lambda(\Delta_0)$ must be regular of valency $\lambda$, $\Delta_0$ is regular of valency $\frac{1}{2} \lambda + 1$. If $\lambda = 2$, then $\Delta_0$ is the pentagon. Now it is clear that the edge $xy$ lies in an induced subgraph isomorphic to the icosahedron. It is easy to see that this icosahedron is an isometric subgraph of $\Gamma$ and is a 2-closed subgraph. It follows that $\Pi$ is the icosahedron.

If $\lambda > 2$, then for every vertex $i$ of a pentagon $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 1$ of $\Delta_0$, the set $\Delta_0(i)$ of neighbours of $i$ is a $(\frac{1}{2} \lambda - 1)$-clique, and by $S7$, the induced subgraph on $\Delta_0(i) \cup \Delta_0(i + 2)$ ($i \in \mathbb{Z}_5$) is complete bipartite. Since there are no quadrangles, this forces $\lambda = 4$, and $\Delta_0$ is the Petersen graph. Now it follows that the component in $\Gamma(x)$ containing $y$ is the line graph of the Petersen graph. Now every $K_1 + K_{1,2}$ of $\Delta_0$ is in some path of length 4, and $S6$ shows that every 4-set of $\Delta_0$ such that the induced subgraph on this 4-set is a $K_1 + K_{1,2}$ is special. The component of $\Gamma(x)$ containing $y$ is labelled with the special 0-2- and 4-sets already found. By induction on the distance, it is easy to see that for all vertices $x \in \mathbb{V}$ there is a component in $\Pi(x)$ whose labels are subsets of $\mathbb{V} \Delta_0$. It follows that $\Pi$ must be locally the line graph of the Petersen graph. Also by induction on the distance you easily see that the labelling of $\Pi$ is unique. We already saw that the graph $L$ is a locally $L($Petersen$)$ root graph and therefore $\Pi$ must be $L$.

This completes the proof of the proposition.\[\square\]

6.3.2 Amply regular Terwilliger root graphs

In this subsection we classify the amply regular root graph without an induced quadrangle.

A Terwilliger graph is a non-complete graph $\Gamma$ such that, for any two vertices $\gamma, \delta$ at distance two, $\Gamma(\gamma) \cap \Gamma(\delta)$ is a clique of size $\mu$ (for some fixed $\mu \geq 0$). In other words, a Terwilliger graph is a non-complete graph $\Gamma$ without induced quadrangles such that any two nonadjacent vertices have $\mu$ or $\mu$ common neighbours. They were studied by Terwilliger [104].

For a Terwilliger graph you can define an equivalence relation $\equiv$ on the set $\Gamma$ as follows: $x \equiv y$ if $\{x\} \cup \Gamma(x) = \{y\} \cup \Gamma(y)$. We shall write $\overline{\Gamma}$ for the quotient $\Gamma/\equiv$. For an amply regular Terwilliger root graph, the reduced graph $\overline{\Gamma(x)}$ of $\Gamma(x)$ is a strongly regular root graph without induced quadrangles, cf. [19, Theorem 1.16.3]. The only strongly regular root graphs without induced quadrangles are the pentagon and the Petersen graph. Because of the fact that they have lines of size 2, it follows that if they occur as reduced graph they will occur as neighbourhood graph. The only connected graph with is locally a pentagon is the icosahedron, which is a root graph. There are 3 connected graphs which are locally Petersen and only two of them are Terwilliger. Now we look at the locally Petersen case.

**Lemma 6.34** Let $\Gamma$ be the suspension of the Petersen graph. Then $L(\Gamma) = E_6$.

**Proof.** It is trivial that $\Gamma$ has no integral root representation, because the Petersen graph contains a 3-claw and you can label the vertices with roots in $E_6$.\[\square\]

**Lemma 6.35** The icosahedron has exactly four non-isomorphic root representations.
Proof. Let \( x \) be a vertex of the icosahedron. Without loss of generality we may assume that the elements of \( \{x\} \cup \Gamma(x) \) are represented by \( 0 \) and \( v_i + e_{i+1}, (i \in \mathbb{Z}_3) \). Now it is straightforward that you can extend this representation in exactly four ways. \( \Box \)

Lemma 6.36 The locally Petersen distance-regular graph \( \Gamma \) with intersection diagram

```
1 10 1 10 3 2 30 4 6 20 1 10 2
v=63
```

is not a root graph.

Proof. \( \Gamma \) is locally connected and so \( L(\Gamma) \) is irreducible. By the previous lemma we have \( L(\Gamma) = E_8 \), for some \( i \in \{6, 7, 8\} \).

Let \( x, y \) be two vertices at distance 3. We may assume that \( x = 0 \). Let \( \Delta \) be the graph induced by \( C(x, y) \), the set of vertices lying on a geodesic from \( x \) to \( y \). The graph induced by \( \Delta(x) \) is hexagon. Let \( \Delta(x) = \{x_1, x_2, \ldots, x_6\} \) be such that \( x_i \sim x_{i+1}, i \in \mathbb{Z}_6 \). Let \( \Delta(y) = \{y_1, y_2, \ldots, y_6\} \) be such that \( y_i \sim y_{i+1}, i \in \mathbb{Z}_6 \). Without loss of generality we may assume that \( x_i = e_i + e_{i+1}, i \in \mathbb{Z}_6 \), where \( e_j \) is a binary vector of length 8 where only the \( j \)-th entry is a one. The vertices \( x_i \) and \( x_{i+3} \) have a common neighbour \( x \) in \( \Gamma(x) \). Without loss of generality we may assume that \( x_i = e_i + e_{i+3} \). Also without loss of generality we may assume that \( y_i, y_{i+1} \in \Delta(x) \) for \( i \in \mathbb{Z}_6 \). The label of \( y \) must be integral by the labels of \( x_i \) and \( x_{i+1} \). If \( y \) has an integral label then \( y = e_i + e_{i+1} + e_{i+2} + e \) where \( e \in \{\pm e_1, \pm e_3\} \). But then follows a contradiction by looking at \( y_i \) and \( y_{i+1} \). \( \Box \)

Lemma 6.37 Let \( \Gamma \) be the distance-regular graph with intersection diagram

```
1 10 1 10 3 2 30 4 6 20 1 10 2
v=65
```

Then \( \Gamma \) is not a root graph.

Proof. First we give some properties of \( \Gamma \). The graph is locally Petersen, \( Aut(\Gamma) \cong PSL(2, 25) \) acts distance-transitive with point stabiliser \( Sym(5) \times 2 \). Already \( PSL(2, 25) \) acts distance-transitive, with point stabiliser \( T \cong Sym(5) \).

For \( x, y \) two vertices at distance 3, the subgraph induced by \( C(x, y) \) is the icosahedron. From this it is easy to see that the subgraph \( \Delta \) induced on \( \Gamma(x) \) is the disjoint union of two icosahedra. Let \( G \) be the group fixing both icosahedra in \( \Gamma(x) \). Then \( G \cong Alt(5) \times 2 \). It is obvious that \( T \times G \cong Alt(5) \) and therefore there is an automorphism \( \pi \) of \( Aut(\Gamma) \) fixing the set \( \{x\} \cup \Gamma(x) \) and sends each \( u \in \Gamma(x) \) to its antipodal in the component of \( \Delta \) where \( u \) is a vertex of.

Suppose that \( \Gamma \) has a root representation. The lattice \( L(\Gamma) \) contains \( E_8 \) as sublattice, because this is the lattice belonging to the suspension of the Petersen graph. Also \( L(\Gamma) \) is irreducible by the locally connectedness, so \( L(\Gamma) \leq E_8 \).

Then there are vertices \( x, y \) at distance 3 with \( (x - y, y - y) = 4 \), because the distance matrix \( D \) has more then one positive eigenvalue and the subgraph induced by \( C(x, y) \) is an icosahedron. Shift the representation such that \( x = 0 \), the five vertices of \( \Gamma(x) \cap \Gamma_2(y) \) are represented by \( (1000000) \) and those vectors obtained by a cyclic permutation of the first five entries and \( y' = \frac{1}{3}(1111111) \). Let \( z = \pi(y) \). By the fact that \( \pi \) fixes \( \Gamma(x) \) pointwise we obtain \( \Gamma(x) \cap \Gamma(y) = \Gamma(x) \cap \Gamma_2(z) \). From the fact that at least three elements of \( \Gamma(y) \cap \Gamma_2(z) \) must have an integral label where \( v \in \{y, \pi(y)\} \), there is a vertex \( u \in \Gamma(y) \cap \Gamma_2(z) \) such that both \( u \) and \( \pi(u) \) have an integral label. But this means that the sixth entry of \( \pi(u) \) must be \(-1 \). Therefore \( (y - \pi(u), y - \pi(u)) > 0 \), what is
impossible, because they lay in an icosahedron. □

The conclusion is:

**Proposition 6.38** The only amply regular root graph without induced quadrangles is the icosahedron. □

6.3.3 Classification of root graphs

In this subsection we classify first the distance-regular root graphs. We also classify the amply regular root graphs, using some computer results.

Distance-regular graphs

**Theorem 6.39** Let \( \Gamma \) be a distance-regular root graph of diameter \( d \). If \( L(\Gamma) \cong E_6, E_7, E_8 \) and \( \mu \geq 2 \), then \( \Gamma \) is the icosahedron, one of the Chang graphs, the Schl"afli graph or the Gosset graph.

**Proof.** Fix a vertex of \( \Gamma \), and shift the representation such that \( x = 0 \). Then \( L^2(\Gamma(x)) \leq L(\Gamma) \leq E_6 \), and the induced subgraph induced by \( \Gamma(x) \) has smallest eigenvalue \(-2\), since Proposition 6.14. Looking in the list of the strongly regular root graphs we find that if \( d = 2 \) then \( \Gamma \) must be one of the Chang graphs, or the Schl"afli graph. (Note that \( L(\Gamma) \) does not depend on the representation if the diameter is at most 2.)

From now on we assume that \( d \geq 3 \). If \( \Gamma \) does not contain induced quadrangles, then by Proposition 6.38 we find that \( \Gamma \) must be the icosahedron. So we may assume that \( \Gamma \) has an induced quadrangle and thus by Proposition 6.25 (iv), we have \( k = b_2 + 2\lambda + 4 - \mu \). Also we may assume that there are vertices \( y, z, u \) such that \( x \sim y \sim z \sim u \sim x \) and \( d(x, z) = d(y, u) = 2 \).

By Proposition 6.32 we see that \( \Gamma \) does not contain \((\lambda+2)\)-cliques. By Proposition 6.25 (iii), \( \Gamma \) is locally connected, except possibly when \( \mu = 2 \). But then \( \lambda \geq 2 \) and each component of \( \Gamma(x) \) is a regular graph containing an induced \( g \)-gon, with \( g \geq 5 \), because \( \Gamma(x) \) does not contain an induced quadrangle, otherwise \( \mu \geq 3 \). But then each component of \( \Gamma(x) \) contributes at least 5 to the dimension of \( L^2(\Gamma(x)) \) and thus there can be only one component in \( \Gamma(x) \). Therefore \( \Gamma(x) \) is connected regular graph with \( k \) vertices and valency, and \( k \geq \lambda + \mu + 1 \), by [19], Theorem 1.5.5. We now use Theorem 6.16 to determine the possibilities for \( \Gamma(x) \). Note that \( \lambda \geq 2 \) by locally connectedness.

Suppose first that \( \Gamma(x) \) is not a line graph. Then we have to consider 4 cases.

**Case 1.** \( k = 2(\lambda + 2) \leq 28 \). We proceed this case in 10 steps.

**Step 1.** \( \lambda \leq 3(\mu - 1) \).

Let \( v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \). Then \( \lambda = \lambda(x, v) \leq |\Gamma(v) \cap \Gamma(y)| - 1 + |\Gamma(v) \cap \Gamma(z)| - 1 = 2(\mu - 1) \), because \( b_2(n, x) = \mu \).

Let \( \Pi \) the graph induced by \( \Gamma(x) \cap \Gamma(y) \). Let \( t \) be the number of edges in \( \Pi \).

**Step 2.** \( \lambda(\lambda - 1) \geq (\lambda + 3)(\mu - 2) + 2t \).

We count the edges between \( \Gamma(x) \cap \Gamma(y) \) and \( \Gamma(x) \cap \Gamma(z) \). Note that any non-adjacent vertices in \( \Gamma(x) \) have at least \( \mu - 2 \) common neighbours in \( \Gamma(x) \).

**Step 3.** The graph \( \Pi \) has minimal valency at least \( \mu - 4 \).
Let \( w \) be a vertex of \( \Pi \). If there is no vertex at distance 2 in \( \Pi \), then by \( \lambda \geq \mu - 2 \), the
valency of \( w \) is at least \( \mu - 3 \). Otherwise let \( a \) be a vertex of \( \Pi \) at distance 2 from \( w \).
The vertices \( a, w, x \) lie in at most one induced quadrangle of \( \Gamma \). The same holds for the
vertices \( a, w, y \). Now this step follows easily.

**Step 4.** If \( \mu \geq 5 \), then \( t \geq \min \{ (\mu - 4)(\lambda - \mu + 3) + 2(\lambda - \mu + 1), \\
(\mu - 3)(\lambda - \mu + 2) + 2(\lambda - \mu - 2), \\
(\mu - 2)(\lambda - \mu + 1) \} \). Two non-adjacent vertices in \( \Pi \) have at least \( \mu - 4 \) common neighbours.
Let \( v \) be a vertex of \( \Pi \) with minimal valency, say \( l \). Then there are at least \( (\mu - 4)(\lambda - l - 1) \\
\) edges between \( \Pi(\nu) \) and \( \Pi(l)(v) \). Let \( w \in \Pi \) be such that \( |\Pi(w) \cap \Pi(\nu)(v)| \) is maximal. This
last number, call it \( m \), is at least \( (\mu - 4)(\lambda - l - 1) \). The number of edges in \( \Pi(w) \cap \Pi(\nu)(v) \) is
at least \( 2(m - 2) \), because in the Gossen graph each path of length 2 lies in exactly
one 3-claw and \( \Pi \) is an induced subgraph of the Gossen graph. If we consider \( l = \mu - 4, \\
l = \mu - 3 \) and \( l \geq \mu - 2 \) we obtain this step.

**Step 5.** For the pairs \( (\lambda, \mu) \) we have the following possibilities.
(i) \( \mu = 2 \) and \( \lambda = 3 \).
(ii) \( \mu = 3 \) and \( \lambda \in \{4, 6\} \).
(iii) \( \mu = 4 \) and \( 5 \leq \lambda \leq 9 \).
(iv) \( \mu = 5 \) and \( \lambda \in \{8, 12\} \).
(v) \( \mu = 6 \) and \( \lambda = 12 \).

It is straightforward to see that, by steps 1, 2 and 3, the following pairs \( (\mu, \lambda) \) are
still possible.

(1) \( (2, \lambda) \) with \( \lambda \in \{2, 3\} \).
(II) \( (3, \lambda) \) with \( 3 \leq \lambda \leq 6 \).
(III) \( (4, \lambda) \) with \( 5 \leq \lambda \leq 9 \).
(IV) \( (5, \lambda) \) with \( 7 \leq \lambda \leq 12 \).
(V) \( (6, \lambda) \) with \( 9 \leq \lambda \leq 12 \).
(VI) \( (7, \lambda) \) with \( \lambda \in \{11, 12\} \).

The number \( n = \frac{\lambda k - \lambda - 11}{\mu} \) must be an integer. This rules out the possibility of \( (3, 5), \\
(3, 9), (5, 10), (5, 11) \) and \( (6, 11) \).

For \( \mu = 2 \) and \( \lambda = 2 \), we get that \( \Gamma \) is locally the 8-gon, what is a line graph.
If \( \mu = 3 \) and \( \lambda = 3 \) then \( \Gamma(\lambda) \) has valency 3 and each two non-adjacent vertices in \( \Gamma(\lambda) \)
must have at least one common neighbour in \( \Gamma(\lambda) \). Therefore \( \Gamma \) must be locally the
Petersen graph, but none of the locally Petersen graphs have \( \mu = 3 \), see Theorem 1.3.
Steps 2 and 4 rule out the possibility of \( (7, 12), (7, 11), (6, 10), (6, 9) \) and \( (5, 7) \).

**Step 6.** If \( \Gamma \) has intersection array \( \{k, k/2 + 1, 1, \mu, k/2\} \), then the graph induced by
\( \Gamma(x) \), where \( x \) is a vertex, is a Taylor graph with intersection array \( \{k/2, \mu, 1, 1, \mu, k/2\} \).
This follows direct from \( p^3_{2,2} = 1, p^3_{2,3} = k/2 = p^3_{3,3} \).

**Step 7.** The diameter of \( \Gamma \) is at most 4 and \( c_3 \geq \lambda + 2 \). If the diameter is four,
then \( \Gamma \) has intersection array \( \{k, \lambda + 3, \mu, 1, 1, \mu, \lambda + 3, k\} \).
The graph \( \Gamma \) has an induced quadrangle and \( b_1 = \mu \). We have \( c_1 - b_1 \geq c_i - b_i = c_{i-1} - b_{i-1} + \lambda + 2 \),
by a result of Terwilliger[105], cf. [19], Theorem 5.2.1. Therefore the number \( c_3 \) is at
least \( \lambda + 2 \), and equality holds, then the diameter is 3. If \( c_3 \geq \lambda + 3 \) and \( b_3 \geq 1 \), then
since \( b_3 = \lambda + 3 \) and \( c_{i-1} \leq b_i \) we have \( c_3 = \lambda + 3 \). But then \( b_3 = 1 \) and \( c_4 = \lambda \).
STEP 8. \( c_3 \neq \lambda + 2 \).
Suppose that \( c_3 = \lambda + 2 \). Then by Step 6, there is also a root Taylor graph, say \( \Psi \), with interaction array \( (\lambda + 2, \mu, 1; 1, \mu, \lambda + 2) \). Then \( \Psi \) is locally a strongly regular root graph, say \( \Phi \), with \( k(\Phi) = 2\mu(\Phi) \), cf. [19], Theorem 1.5.3. Looking in the list of possible strongly regular graphs on at most 14 vertices, we find that \( \Phi \) is either the pentagon, or the \((3 \times 3)\)-grid. In the first case we get that \( \Gamma \) is locally the Petersen graph, which is not possible, and in the second case we find that \( \Gamma \) is \( J(9, 3) \). But this last graph is locally the \((6 \times 3)\)-grid, hence \( L(\Gamma) \neq E_i \), for \( i = 6, 7, 8 \).

STEP 9. \( \Gamma \) has not interaction array \((2\lambda + 4, \lambda + 3, \mu, 1; 1, \mu, \lambda + 3, 2\lambda + 4) \).
Then \( \Gamma \) is a 2-cover of a strongly regular graph with interaction array \( (2\lambda + 4, \lambda + 3, 1, 1, \mu, \lambda + 2, 1, 2\mu) \).
Looking at the ten possibilities of Step 5, the only possible pair \((\mu, \lambda)\) for which such a strongly regular graph exists is \((2, 3)\). But the intersection array \( \{10, 6, 2, 1, 1, 2, 6, 10\} \) is not possible as a distance-regular graph by the fact that the eigenvalue 5 should have multiplicity 52/7.

STEP 10. \( \Gamma \) has not interaction array \((2\lambda + 4, \lambda + 3, \mu, 1; 1, \mu, c_3) \) with \( c_3 \geq \lambda + 3 \).
The numbers \( k_3 = \frac{2(\lambda + 2)(\lambda + 3)}{\mu} \) and \( p_{\lambda, 3}^\mu \) must be integers.
If \( c_3 = 2(\lambda + 2) \), then \( p_{\lambda, 3}^\mu = \frac{\mu - 1}{2(\lambda + 2)} \) is an integer less than \( k_3 = \lambda + 3 \), it follows that \( \mu = 2 \). But this array has no integral eigenvalues, what an impossibility for distance-regular of diameter 3, not the heptagon.
If \( c_3 = \lambda + 3 \), then \( p_{\lambda, 3}^\mu = \frac{\mu - 1}{2(\lambda + 2)} \) is an integer. In this case we have the following five possibilities: \((\mu, \lambda) \subset \{(2, 3), (3, 6), (4, 5), (4, 9), (5, 12)\}\). The last four arrays do not have an integral eigenvalue. The first array has \( p_{\lambda, 3}^\mu = 4, p_{\lambda, 4}^\mu = 3 \) and \( p_{\lambda, 5}^\mu = 2 \). Therefore, if \( \Gamma \) has this array, then for each \( y \in \Gamma_3(x) \) there is a unique neighbour \( z \in \Gamma_3(x) \) of \( y \) with the same neighbours in \( \Gamma_3(x) \). But this means that \( c_3 \) must be odd, contradiction.
If \( \lambda + 3 < c_3 < 2\lambda + 4 \), then we have the following possibilities for \((\mu, \lambda, c_3)\): (i) \((\lambda, \mu, c_3) = (3, 6, 12)\), (ii) \((\lambda, \mu, c_3) = (4, 6, 12)\), (iii) \((\lambda, \mu, c_3) = (4, 7, 12)\), (iv) \((\lambda, \mu, c_3) = (4, 7, 15)\), (v) \((\lambda, \mu, c_3) = (5, 12, 20)\), (vi) \((\lambda, \mu, c_3) = (5, 12, 21)\), (vii) \((\lambda, \mu, c_3) = (6, 12, 20)\), (viii) \((\lambda, \mu, c_3) = (6, 12, 21)\). For possibilities (vi), (vii), (viii) and (viii) the number \( p_{\lambda, 3}^\mu \) is not integer. The interaction arrays of (i), (iii) and (v) have no integral eigenvalue, and the intersection array of (ii) has eigenvalue \(-4\) with multiplicity 52/3.

So we are done in this case.

CASE 2. \( k = \frac{1}{3}(\lambda + 2) \leq 27 \).
By \( k \geq \lambda + \mu + 1 \) we have \( \mu \leq \frac{1}{3}\lambda + 2 \). As a consequence of \( k = b_2 + 2\lambda + 4 - \mu \) and \( k = \frac{1}{3}(\lambda + 2) \) we have \( \mu = \frac{1}{3}\lambda + b_2 + 1 \). By \( d \geq 3 \) we have \( b_2 \geq 1 \). So \( b_2 = 1 \) and \( \Gamma \) is a Taylor graph, by Theorems 1.5.5 and 1.5.3 of [19], and hence \( \Gamma(x) \) is a strongly regular root graph \( \Delta \) with \( v(\Delta) = \frac{1}{2}(k(\Delta) + 2) \leq 27 \), \( 2\mu(\Delta) = k(\Delta) \) and \( \delta(\Delta) = \frac{1}{3}(k(\Delta) - v(\Delta) - 1) = \frac{1}{3}(k(\Delta) - 8) \). By the fact that \( \Delta \) is not a line graph the only possibility we get is \( k = 16 \) and \( \Gamma \) is the Gosset graph.

CASE 3. \( k = \frac{1}{3}(\lambda + 2) \leq 16 \).
By \( k \geq \lambda + \mu + 1 \) we have \( \mu \leq \frac{1}{3}(\lambda + 5) \). As a consequence of \( k = b_2 + 2\lambda + 4 - \mu \) and \( k = \frac{1}{3}(\lambda + 2) \) we have \( \mu = \frac{1}{3}(2\lambda + 4) + b_2 \). Hence by the fact that \( \lambda \geq 2 \), we get a contradiction.

CASE 4. \( k = \lambda + 2 \).
By \( k \geq \lambda + \mu + 1 \) we have \( \mu \leq 1 \).
From now on we suppose that $\Gamma(x)$ is a line-graph. Hence we may assume that $\Gamma(x)$ is the line graph of a graph $\Delta$ with $n$ vertices. By Theorem 6.16, $\Delta$ is regular, or bipartite and semiregular. Moreover, in the bipartite case $L^+(\Gamma(x)) \cong A_{n-1}$ and hence $n \leq 9$, and in the other case $L^+(\Gamma(x)) \cong D_n$ and hence $n \leq 8$. By Proposition 6.26 (iii), non-adjacent vertices of $\Gamma(x)$ have $\mu - 1$ or $\mu - 2$ common neighbours in $\Gamma(x)$, i.e., disjoint edges of $\Delta$ have $\mu - 1$ or $\mu - 2$ transversals (i.e., edges intersecting the given edges). In particular $\mu \leq 6$. Using the fact that $x$ lies in an induced quadrangle, it is a simple exercise (cf. Proposition 5 of Neumaier [82]) to show that we have one of the following cases.

(i) $\mu = 2$ and $\Delta$ is an $n$-gon, with $n \in \{6,7,8\}$
(ii) $\mu = 3$ and $\Delta$ is the complement of the hexagon,
(iii) $\mu = 4$ and $\Delta$ is either $K_{3,n}$ or $K_{n,n}$ with $n + 2 \leq 8$.
(iv) $\mu = 6$ and $\Delta = K_n$ with $n \leq 8$.

**Case 1.** $\mu = 2$. Then $\Gamma(x)$ is a $k$-gon, with $k \in \{6,7,8\}$. So $\lambda = 2$, and hence $\Gamma$ is locally a $k$-gon. If $k = 6$, then by $k = 2\lambda + 4 - \mu$ the diameter is 2. So we may assume that $k \in \{7,8\}$. Represent $\Gamma(x)$ by the vectors $c_i + c_{i+1}$, $i \in \mathbb{Z}_k$. Let $x$ be the neighbour of $x$ represented by $c_i + c_{i+1}$, and let $x_{ij} = x_{j}j$ be the common neighbour of $x_i$ and $x_{j}$ distinct from $x$, for $i - j \neq \pm 2$. Then $x_{ij}$ must be represented by $c_i + c_{i+1} + c_j + c_{j+1}$, for $i - j \neq \pm 1$. The common neighbours of $x_4$ and $x_{3,0}$ are $x_3$ and $x_{4,0}$, so that $x_{3,0} \neq x_{3,2}$. In the same way we see that $x_{3,4} \neq x_{3,2}$, therefore $x_{3,0}$ has valency at most one in $\Gamma(x_3)$, Contradiction.

**Case 2.** $\mu = 3$. Then $k = 2\lambda + 4 - \mu$, and so the diameter equals two.

**Case 3.** $\mu = 4$. If $\Delta = K_{3,n}$, then again the diameter is two. So $\Gamma(x)$ is the $(p \times q)$-grid, with $k = pq$ vertices and therefore $\Gamma$ is locally the $(p \times q)$-grid. Represent the vertices of $\Gamma(x)$ with the vectors $c_i + c_j$ ($i \in P, j \in Q$), where $P$ and $Q$ are the two colour classes of $\Delta$. We mean with vertex $ab$ the vertex in $\Gamma(x)$ represented with $a_0 + b_0$. Any two non-adjacent vertices $ab$ and $cd$ in $\Gamma(x)$ have 2 common neighbours in $\Gamma(x)$, hence they have a common neighbour $u$ in $\Gamma(x)$, represented with $a_0 + c_0 + e_0 + f_0$. Since we may choose $a, c \in P$ and $b, d \in Q$, arbitrary but distinct, this yields $\binom{2}{2}$ distinct norm 1 vectors, and this exactly the number $k_0$. So all vertices of $\Gamma(x)$ are represented by vectors of the shape $1^{2p+q-4}$. An easy induction argument shows that $\Gamma$ is represented by $(0,1)$-vectors of even weight, i.e., $L(\Gamma) \subseteq D_{n+4}$, Contradiction.

**Case 4.** $\mu = 6$. Then $\Gamma(x)$ is a triangular graph $T(n)$ with $k = \binom{n}{2}$ vertices represented by the vectors $c_i + c_j$, ($1 \leq i < j \leq n$). With the same argument as in Case 3 we find that $L(\Gamma) \subseteq D_n$, Contradiction.

This completes the proof. \( \Box \)

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**Theorem 6.40** Let $\Gamma$ be a distance-regular root graph with $\mu \geq 2$. Then one of the following holds.
(i) \( \mu = 10 \) and \( \Gamma \) is the Gosset graph.

(ii) \( \mu = 8 \) and \( \Gamma \) is the Schl"afli graph.

(iii) \( \mu = 6 \) and \( \Gamma \) is a halved cube.

(iv) \( \mu = 4 \) and \( \Gamma \) is either a Johnson graph, or one of the Chang graphs.

(v) \( \mu = 2 \) and either \( \Gamma \) is a Hamming graph, a Doob graph, i.e. a cartesian product of 4-cliques and Shrikhande graphs, or the icosahedron.

**Proof.** Suppose first that \( \mu \geq 3 \). Then \( \Gamma \) is locally connected and by Proposition 6.22 \( L(\Gamma) \) is irreducible. If \( L(\Gamma) \subset D_n \) for some \( n \) then \( \Gamma \) has an integral root representation and by Theorem 6.30 we have cases (iii) and (iv). If \( L(\Gamma) = E_1 \) then by Theorem 6.30 we are in the the cases (i), (ii) or (iv).

So we may suppose that \( \mu = 2 \). If \( \Gamma \) does not contain an induced quadrangle then \( \Gamma \) is an Terwilliger amply root graph and by Proposition 6.38 \( \Gamma \) must be the icosahedron.

Suppose now that \( \Gamma \) contains an induced quadrangle. Then \( \lambda = 2 \), by Proposition 6.25 (iv). Therefore the icosahedron is not a 2-closed subgraph of \( \Gamma \). Let \( L(\Gamma) = L_1 \oplus L_2 \), where \( L_1 \) is irreducible. Let \( \Delta_j \) be the graph with vertex set \( V\Gamma \) and edge \( xy \) if \( xy \) is an edge of \( \Gamma \) and \( x \neq y \) in \( L_j \), \( j = 1, 2 \). Let \( x \) be a vertex of \( \Gamma \). Let \( \Pi_j \) be the component of \( \Delta_j \) which contains the vertex \( x \). By Lemma 6.21 the subgraph \( \Pi_j \) is a 2-closed subgraph.

If \( L_1 \subset D_n \) for some \( n \), then \( \Pi_1 \) is a Shrikhande graph of a \( (\lambda + 2) \)-clique. By Lemma 6.31 and the regularity of \( \Gamma \), the graph \( \Gamma \) is the cartesian product \( \Pi_1 \times \Pi_2 \). Therefore \( \Gamma \) is a Hamming graph or a Doob graph. \( \square \)

**Ampl-regular graphs**

Brouwer, Cohen & Neumaier, [19, Proposition 3.15.2] classified the amply-regular root graphs with \( L(\Gamma) \cong E_\mu \), using a computer search by Busemacker and the tables of regular graphs with smallest eigenvalue \(-2\), but they forgot to mention the icosahedron.

**Proposition 6.41** Let \( \Gamma \) be an amply regular root graph of diameter \( d \) with parameters \((n, k, \lambda, \mu)\). If \( L(\Gamma) \cong E_\mu \), then either we have \( d > 2 \), \( k \leq 8 \) and \( \mu = 1 \), or \( \Gamma \) is isomorphic to the icosahedron, one of the Chang graphs, the Schl"afli graph or the Gosset graph. \( \square \)

Now we are ready for the classification of the amply regular root graphs with \( \mu = 2 \).

**Theorem 6.42** Let \( \Gamma \) be an amply regular root graph with \( \mu = 2 \). Then one of the following holds.

(i) \( \lambda = 2 \) and \( \Gamma \) is the cartesian product of 4-cliques, icosahedra and Shrikhande graphs.

(ii) \( \lambda = 4 \) and \( \Gamma \) is a cartesian product of 6-cliques and copies of graph \( L \).

(iii) \( \Gamma \) is a Hamming graph.

**Proof.** If \( \Gamma \) has an integral root representation, then, by Proposition 6.33, we are in one of the three cases. Furthermore in all three cases \( \Gamma \) has an integral root representation. We may suppose that \( L(\Gamma) \cong L_1 \oplus L_2 \) where \( L_i \cong E_\mu \) for some \( i \in \{6, 7, 8\} \). Let \( x \) be a vertex of \( \Gamma \). Let \( \Delta_j \) be the graph with vertex set \( V\Gamma \) and edge \( xy \) if \( xy \) is an edge of \( \Gamma \) and \( x \neq y \) in \( L_j \), \( j = 1, 2 \). Let \( \Pi_j \) be the component of \( \Delta_j \) which contains \( x \) as a vertex.

By Lemma 6.21 \( \Delta_j \) is a 2-closed subgraph and so is an amply regular root graph. If \( L(\Pi_j) \subset D_n \), then we may assume that the representation restricted to \( V\Pi_1 \) is integral.
But then the root representation restricted to $\Pi_i$ is injective and so by Lemma 6.31 it follows that the $\Gamma$ is the cartesian product $\Pi_1 \times \Pi_2$. This contradicts the assumption that $L_i = E_i$ for some $i$. So $L(\Delta_1) = E_i$ for some $i$ and so by Proposition 6.41 we obtain that $\Delta_1$ is the icosahedron. But then $\Gamma$ is the cartesian product $\Delta_1 \times \Delta_2$, where $\Delta_1$ is the icosahedron. So we can proceed by induction to the dimension of $L(\Gamma)$. □

6.4 Distance-regular graphs and the metric hierarchy

In this section we will show the following theorem.

Theorem 6.43 The distance matrix of a distance-regular graph $\Gamma$ has exactly one positive eigenvalue if and only if $\Gamma$ is one of the following graphs:

(1) a cocktail party graph,

(II) the Gosset graph,

(III) the Schl"afli graph,

(IV) a hypercube,

(V) a Johnson graph, $J(n, t)$,

(VI) one of the three Chang graphs,

(VII) a Hamming graph, $H(d, n)$,

(VIII) a Doob graph,

(IX) the icosahedron,

(X) a polygon,

(XI) a doubled Odd graph,

(XII) the Petersen graph, and

(XIII) the dodecahedron. □

By Theorem 6.8 the distance-regular graphs with a distance matrix having exactly one positive eigenvalue are of negative type. Hence in the rest of this section we may restrict ourselves to distance-regular graphs of negative type. From now on we always consider $\Gamma$ to be a distance-regular graph with the vertex set $V\Gamma$ and the path distance $d$, embedded (via an embedding $x \mapsto \overline{x}$) in a Euclidean space $E$, so that $\| \overline{x} - \overline{y} \|^2 = 2d(x, y)$. A graph of negative type is a root graph. The distance-regular root graphs with $\mu \geq 2$ are classified in Theorem 6.40 and we see that they are all of negative type.

From now on we assume that $\mu(\Gamma) = 1$.

6.4.1 Properties

In this subsection we will give some properties of distance-regular graphs of negative type.

Lemma 6.44 If $x, y, z, t \in X$ then $(x, y, z, t) = d(x, t) - d(x, z) - d(y, t) + d(y, z)$.

Proof. Straightforward computation, by use of $d(a, b) = \frac{1}{2} \| \overline{a} - \overline{b} \|^2 = \frac{1}{2} \| \overline{a} - \overline{b} \|^2$ for all $a, b \in X$. □

Corollary 6.45 Let $x, y \in X$ be at distance $s$ in $\Gamma$. Let $x = x_0 \sim x_1 \sim \ldots \sim x_s = y$ be a shortest path between $x$ and $y$. Then $(\overline{x}_i - \overline{x}_{i-1}, \overline{x}_j - \overline{x}_{j-1}) = 0$ for all $0 < i < j \leq s$. □

Recall that a root lattice is a direct sum of irreducible root lattices, and every irreducible root lattice is one of the lattices $A_n, D_n$, or $E_n$. 81
Lemma 6.46 Suppose that \( L(\Gamma) \) is a sublattice of \( D_n \). Suppose also that, whenever \( x_0 \sim x_1 \sim \ldots \sim x_s \) is a geodetic path and \( \bar{x}_i - \bar{x}_{i-1} = e_i - e_j \), we have that \( \bar{x}_i - \bar{x}_{i-1} \neq \pm (e_i + e_j) \). Then \( \Gamma \) is an isometric subgraph of a halved cube. In particular, if \( L \) is a sublattice of \( A_n \), then \( \Gamma \) is an isometric subgraph of a halved cube.

Proof. If \( x_0 \sim x_1 \sim \ldots \sim x_s \) is a geodetic path then by Lemma 6.44, for any \( 1 \leq i < j \leq s \), the roots \( \bar{x}_i - \bar{x}_{i+1} \) and \( \bar{x}_j - \bar{x}_{j+1} \) are perpendicular. By assumption, \( \bar{x}_i - \bar{x}_{i+1} = \pm e_i \pm e_k \) and \( \bar{x}_j - \bar{x}_{j+1} = \pm e_j \pm e_l \) for disjoint \( \{a, b\} \) and \( \{c, d\} \). It means that for all vertices \( x \) and \( y \) in \( \Gamma \) every coordinate of the vector \( \bar{x} - \bar{y} \) (in the base \( \{e_1, \ldots, e_n\} \)) is equal to zero or \( \pm 1 \). It implies that up to a shift the vectors \( x \in X \) belong to the natural cube, and this, clearly, provides an isometric embedding of \( \Gamma \) into the associated halved cube. \( \square \)

As we already observed, we may restrict ourselves to the case \( \mu = 1 \). This implies, in particular, that every edge of \( \Gamma \) is contained in exactly one maximal clique. Let \( g \) denote the geometric girth of \( \Gamma \), i.e., the minimal length of an induced cycle, other than a triangle.

6.4.2 The geometric girth is even

In this subsection we look at the case that the geometric girth is even.

Proposition 6.47 Suppose \( g = 2l \) is even. Suppose also that every geodetic path of length \( l \) in \( \Gamma \) is contained in at least one induced cycle of length \( g \). Then \( \Gamma \) is an isometric subgraph of a halved cube.

Proof. Put \( n = l - 2 \). Since \( n \geq 1 \), we have \( g \geq 6 \) and therefore \( s \geq 1 \). Let us define a graph \( \Sigma \) having as vertices all geodetic paths of length \( s \) in \( \Gamma \) (we consider them here as unoriented). Two paths \( x_0 \sim x_1 \sim \ldots \sim x_s \) and \( y_0 \sim y_1 \sim \ldots \sim y_s \) are adjacent whenever \( x_i = y_i+1 \) for \( i = 1, \ldots, s \), or symmetrically, \( y_i = x_{i-1} \) for \( i = 1, \ldots, s \). Clearly, \( \Sigma \) is connected.

For a path \( p \) define \( R(p) \) to be the set of all root vectors \( \bar{x} - \bar{y} \in L(\Gamma) \) corresponding to all edges \( \{x, y\} \) where at least one of \( x \) and \( y \) belongs to \( p \). We claim that \( R(p) \) does not depend on \( p \). By connectivity of \( \Sigma \) it suffices to check the claim for two adjacent paths.

Let \( x_0 \sim x_1 \sim \ldots \sim x_s \) and \( z_0 \sim z_1 \sim \ldots \sim z_{s+1} \) be such paths (call them \( p_1 \) and \( p_2 \)). If \( \{x, y\} \) is an edge such that \( x = x_0 \) and \( y \neq x_1 \), then the path \( y \sim x_0 \sim x_1 \sim \ldots \sim x_{s+1} \) is geodetic and by assumption there exists a cycle \( y \sim x_0 \sim \ldots \sim x_{s+1} \sim \ldots \sim x_0 = y \). Since this cycle is minimal, we know the distances between vertices on it. Applying Lemma 6.44, we establish that \( \bar{y} - \bar{y} = \bar{x}_1 - \bar{x}_{s+1} \). Hence \( R(p_1) \subseteq R(p_2) \). By symmetry, it implies the equality.

Now, for a particular path \( p \) of length \( s \), the subgraph induced by the set of vertices at distance \( 1 \) from \( p \) is a disjoint union of cliques. Lemma 6.44 easily implies that \( v, z \in L(\Gamma) \) are perpendicular if and only if the corresponding edges do not belong to the same clique. It follows that \( L \) is a sum of lattices \( A_n \)'s and hence \( \Gamma \) is a sublattice of \( A_n \) for a sufficiently large \( n \). By Lemma 6.46 we establish that \( \Gamma \) is an isometric subgraph of a halved cube. \( \square \)

Proposition 6.48 Let \( \Gamma \) be a distance-regular graph with even geometric girth \( g \geq 6 \). If \( \Gamma \) satisfies the pentagonal inequality, then \( \Gamma \) is a doubled odd graph or a polygon.

Proof. Let \( g = 2l \) and \( h = \lceil \frac{g}{2} \rceil \). We proceed in 4 steps.

Step 1. \( a_1 = 0 \) and \( e_1 = 2 \), and thus \( \lambda = 0 \).

Suppose that \( a_1 + e_1 \geq 3 \). Then we can choose vertices \( x_1, x_2, y_1, y_2, y_3 \) such that
\[ d(x_1, x_2) = d(y_1, y_2) = d(y_2, y_3) = t, \quad d(y_1, z_1) = d(y_2, z_1) = d(y_3, z_2) = h, \quad d(y_1, z_2) = d(y_2, z_3) = t, \quad d(y_1, y_2) = 2h \text{ and } d(x_2, y_2) \in \{h, h + 1\}. \] Now if we give \( x_1 \) and \( x_2 \) weight 1 and \( y_1, y_2 \) and \( y_3 \) weight \(-1\) then we see that \( \Gamma \) does not satisfy the pentagonal inequality.

**Step 2.** If \( t \geq 4 \), then \( \Gamma \) is a polygon.

By step 1, if the diameter of \( \Gamma \) is \( t \), then \( \Gamma \) is the polygon. So we may assume that the diameter is at least \( t + 1 \). But then \( c_{t+1} \geq 2 \) and we can choose vertices \( x_1, x_2, y_1, y_2, y_3 \) such that \( d(x_1, x_2) = d(y_1, y_2) = d(y_2, y_3) = t, \quad d(x_1, y_1) = d(x_1, y_2) = d(x_2, y_3) = h, \quad d(x_1, y_2) = d(x_2, y_2) = t - h, \quad d(y_1, y_2) = 2h \) and \( d(x_1, y_2) \leq t + 1 - h \). Now if we give \( x_1, x_2 \) weight 1 and \( y_1, y_2, y_3 \) weight \(-1\), then we see that \( \Gamma \) does not satisfy the pentagonal inequality.

**Step 3.** If \( g \) equals 6, then \( \Gamma \) is bipartite.

Suppose that \( a_r \neq 0 \) for minimal \( r \). We can choose vertices \( x_1, x_2 \) and \( y_1, y_2 \) such that \( d(x_1, x_2) = r, \quad d(x_1, y_1) = r - l \) \((i = 1, 2)\), \( d(y_1, y_2) = 3, \quad d(x_2, y_2) = 2 \) and \( y_1 \sim x_2 \). By \( c_3 = 2 \) and \( r \) minimal we can find a vertex \( z \) with \( d(z, x_1) = r, \quad z \sim y_2, \quad d(y_1, x_2) = 2 \) and \( d(z, x_2) = 3 \). If we give \( x_1, x_2, z \) weight \(-1\) and \( y_1, y_2 \) weight 1, then we see that \( \Gamma \) does not satisfy the pentagonal inequality.

**Step 4.** If \( g \) equals 6, then \( \Gamma \) is a doubled Odd graph.

The graph \( \Gamma \) is bipartite, by step 3, and hence isometrically embeddable in a hypercube. So the halved graph is an isometric distance-regular subgraph of the halved cube with \( \mu \geq 4 \). By Theorem 6.40 we get that the halved graph of \( \Gamma \) is a Johnson graph or a halved cube.

**Hemmeter [58, 59] found that a Johnson graph is only the halved graph of a doubled Odd graph and a halved cube is the halved graph of a distance-regular graph with \( \mu = 2 \).**

This completes the proof of the proposition. \( \Box \)

The conclusion of this subsection is that the distance-regular graphs with even geometric girth and of negative type are exactly the doubled Odd graphs and the even polygons.

### 6.4.3 The geometric girth is odd

In this subsection we look at the case that the geometric girth is odd.

**Lemma 6.49** Suppose \( g = 2s + 1 \) is odd. Suppose also that every geometric path of length \( s \) is contained in an induced cycle of length \( g \). Then \( \mathbf{L}(\Gamma) \) is irreducible.

**Proof.** It suffices to prove that for every two adjacent edges the corresponding roots belong to the same irreducible component. By assumption these two edges are contained in a cycle \( C \) of length \( g \). By Lemma 6.44 and by the minimality of \( C \), the roots corresponding to two edges of \( C \) at distance \( s + 1 \) (maximal) from each other are not perpendicular.

Clearly, it implies that all the roots corresponding to the edges of \( C \) belong to the same irreducible component. \( \Box \)

By this lemma, in case of odd geometric girth, \( \mathbf{L}(\Gamma) \) is one of the lattices \( A_n, D_n \) or \( E_n \).

The case of \( A_n \) is covered by Lemma 6.46.

**Lemma 6.50** Suppose \( g = 2s + 1 \) is odd. Suppose also that every geodetic path of length \( s \) is contained in an induced cycle of length \( g \). If \( \mathbf{L}(\Gamma) = D_n \) then \( \Gamma \) is an isometric subgraph of a halved cube.
Proof. By Lemma 6.46 we may assume without loss of generality that for some geodetic path \( x_0 \sim x_1 \sim \ldots \sim x_t \) the vector \( \overrightarrow{x_0 - x_t} = e_1 - e_2 \) and \( \overrightarrow{x_t - x_i} = e_1 + e_2 \). We may assume that \( t \) is taken minimal. If \( t \leq s \) then there is a cycle \( x_0 \sim \ldots \sim x_t \sim x_0 \) (call it \( C \)). By Lemma 6.44, \( \overrightarrow{x_t + x_1} \) is perpendicular to \( \overrightarrow{x_t} - \overrightarrow{x_1} = e_1 + e_2 \) and non-perpendicular and non-equidistant to \( \overrightarrow{x_t} - \overrightarrow{x_0} = e_1 - e_2 \). Such a root in \( D_n \) does not exist. Therefore, \( t > s \). Let now \( C \) be a cycle \( x_0 \sim \ldots \sim x_s \sim y_{s+1} \sim \ldots \sim y_s = x_0 \). Since \( \overrightarrow{y_{s+1} - x_1} \) is not perpendicular to \( e_1 - e_2 \), it is not perpendicular to \( e_1 + e_2 \), either. Since \( d(x_i, x_j) = d(x_{i-1}, x_j) + 1 \), Lemma 6.44 implies that \( d(x_{i-1}, y_{s+1}) = d(x_{i-1}, y_{s+1}) \), hence, both these distances are equal to \( t - s \). It follows that the edge \( \{y_{s+1}, x_1\} \) lies on a geodetic path connecting \( x_0 \) and \( x_t \). Hence \( d(y_{s+1}, x_t) = t - s + 1 \). Now \( \overrightarrow{y_{s+1} - x_1} \) is not perpendicular to \( e_1 - e_2 \), and, therefore, it is not perpendicular to \( e_1 + e_2 \). It follows from Lemma 6.44 that \( d(x_{i-1}, y_{s+1}) \neq t - s + 1 \) and hence this distance is equal to \( t - s \). Hence there is a geodetic path \( p \) between \( x_{i-1} \) and \( x_t \) which passes through \( x_{i-1} \) and \( y_{s+1} \). Since \( C \) has minimal length, \( p \) does not pass through \( x_i \). On the other hand, considering \( \overrightarrow{x_t - x_0} \), we establish that there is an edge on \( p \), other than \( \{x_{i-1}, x_i\} \), such that the corresponding root has form \( e_1 \pm e_2 \) for some \( t \). Since it must be perpendicular to \( e_1 + e_2 \), we obtain that \( t = 2 \). This gives a contradiction with the minimality of \( t \). \( \Box \)

We now can summarize.

Theorem 6.51 Every distance-regular graph \( \Gamma \) of negative type with odd geometric girth, such that \( L \neq E_1, E_2 \) and \( E_3 \), is an isometric subgraph of a halved cube.

Proof. Directly from Lemmas 6.46, 6.49, 6.50. \( \Box \)

The dimension of the remaining lattices does not exceed 8. From now on we consider a distance-regular graph \( \Gamma \) of negative type, such that \( \mu = 1 \) and \( \dim L(\Gamma) \leq 8 \). We estimate the valency and the number of vertices of \( \Gamma \) and then use the tables of intersection arrays from \([19]\) and various known characterizations, to finally check that all such \( \Gamma \)'s have already been listed in Theorem 6.43. Notice that in case when the intersection array of \( \Gamma \) is known, we can easily express the eigenvalues of the distance matrix of \( \Gamma \) via the eigenvalues of \( \Gamma \) itself.

Lemma 6.52 The valency \( k \) of \( \Gamma \) is not greater than \( \dim L \leq 8 \).

Proof. Since \( \mu = 1 \), the neighborhood of a vertex is a disjoint union of cliques. It is easy to check that the roots corresponding to the edges adjacent to any particular vertex of \( \Gamma \) are linearly independent. \( \Box \)

Lemma 6.53 (i) \( b_0 \leq \dim L(\Gamma) \).

(ii) If \( \lambda = 0 \) then \( k = r_i \leq \dim L(\Gamma) - i + 1 \) for all \( i \). Moreover, \( k = r_i \leq \dim L(\Gamma) - i \) if \( a_i \leq 3 \).

Proof. Let \( x_0 \sim x_1 \sim \ldots \sim x_t \) be a geodetic path of length \( t \). Consider the sets of roots \( \Sigma_1 = \{ x_j \sim x_j \} \) (j = 1, \ldots, t), \( \Sigma_2 = \{ \overrightarrow{x_j - x_{j-1}} \in \Gamma(x_j) \cap \Gamma(x_{j-1}) \} \) and \( \Sigma_3 = \{ \overrightarrow{x_j - x_i} \in \Gamma(x_j) \cap \Gamma(x_i) \cap \Gamma(x_0) \} \). The cardinalities of these sets are as follows: \( |\Sigma_1| = t, |\Sigma_2| = a_t \) and \( |\Sigma_3| = b_t \). Consider first the general case. As in the preceding lemma, \( \mu = 1 \), \( |\Sigma_2| = b_t \) and \( |\Sigma_3| \leq b_t \). Consider first the general case. As in the preceding lemma, \( \mu = 1 \), \( |\Sigma_2| = b_t \) and \( |\Sigma_3| \leq b_t \). Consider first the general case. As in the preceding lemma, \( \mu = 1 \), \( |\Sigma_2| = b_t \) and \( |\Sigma_3| \leq b_t \). Consider first the general case. As in the preceding lemma, \( \mu = 1 \), \( |\Sigma_2| = b_t \) and \( |\Sigma_3| \leq b_t \).

Lemma 6.44 imply that the roots from \( \Sigma_3 \) are linearly independent. Also they are perpendicular to the roots from \( \Sigma_1 \), which in their turn are pairwise perpendicular. So (i) follows. Now assume \( \lambda = 0 \). We put \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \) and claim that the dimension of the span of \( \Sigma \) is at least \( |\Sigma| - 1 \) (clearly, this would imply the statement of the lemma).

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Indeed, by Lemma 6.44, every root from \( \Sigma_3 \) is perpendicular to every other root from \( \Sigma \) (recall that \( \lambda = 0 \)). The roots from \( \Sigma_1 \) (resp. \( \Sigma_2 \)) are pairwise perpendicular by the same Lemma 6.44. Finally, if \( x \in \Pi(x_i) \cap \Pi'(x_j) \) then there exists an \( s \leq i \) such that \( d(x, x_i) = i - j + 1 \) if \( s \leq j \leq i \), and \( i - j \) if \( 0 \leq j < s \). By Lemma 6.44 we compute that the inner product of \( \mathbf{x} - \pi_1 \) and \( \pi_2 - \pi_{j+1} \) is equal to \(-1\) for \( j = s \), and to 0 for all other \( j \). It means that \( \Sigma_1 \cup \Sigma_2 \) falls apart into a collection of pairwise orthogonal subsets of the form \( \{ r \} \cup \{ r_1, \ldots, r_s \} \), where \( r \in \Sigma_1 \) and \( r_1, \ldots, r_s \in \Sigma_2 \). Checking the Gram matrix, we see that \( t \leq 4 \) and that every such subset spans a subspace of dimension \( t + 1 \) if \( t < 4 \), and of dimension 4 if \( t = 4 \). Since the valency of \( \Gamma \) is at most 8, there cannot be two subspaces with \( t = 4 \). It means that the total dimension of the span of \( \Sigma \) is at least \( |\Sigma| - 1 \). The second claim of (ii) follows as well. \( \square \)

**Theorem 6.54** If \( \Gamma \) is a distance-regular graph of negative type, such that \( \mu = 1 \) and \( \dim L \leq 8 \) then \( \Gamma \) belongs to the list from Theorem 6.43.

**Proof.** The distance-regular graphs of valency 3 were determined by Biggs, Bosnier & Shawe-Taylor [12]. Checking their list, it is easy to establish the claim in this case. So we may restrict ourselves to the case \( 4 \leq k \leq 8 \). If \( \Gamma \) has diameter 2 then it is a strongly regular graph. As we said, these graphs with the distance matrix having only one positive eigenvalue are all known after Snied [97], see Theorem 6.17. So we may assume without loss of generality that the diameter \( d \) of \( \Gamma \) is at least 3. By Proposition 6.47, we may also restrict ourselves to the case of odd geometric girth.

We have to handle three cases, namely \( \lambda = 0 \), \( \lambda = 1 \) and \( r \geq 3 \), and \( r = 2 \), where \( r = k/(\lambda + 1) \) is the number of maximal cliques on a vertex of \( \Gamma \).

**Case 1.** Assume \( \lambda = 0 \). Applying Lemmas 6.52 and 6.53, we easily establish that the number of vertices of \( \Gamma \) can never exceed 2090 (the estimate for \( k = 5 \)) if \( d \geq 4 \), and 302 (the estimate for \( k = \tau \)) if \( d = 3 \). It means that we can use the tables from [19, Chapter 14].

This gives us (after checking the conditions from Lemma 6.53 and the odd girth condition) the following list of intersection arrays for which new graphs may occur:

1. \( \{5, 4, 2, 1, 1, 4\} \)
2. \( \{5, 4, 2, 1, 1, 2\} \)
3. \( \{6, 5, 2, 1, 1, 3\} \)
4. \( \{8, 1, 1, 1, 1, 4\} \)
5. \( \{8, 5, 1, 1, 1, 4\} \)
6. \( \{6, 4, 2, 1, 1, 4, 0\} \)
7. \( \{5, 4, 3, 1, 1, 1, 1, 2\} \)

**Fon-der-Flaass** [44] [45] has shown that there are no distance-regular graphs with intersection arrays (ii) and (iv). For the remaining intersection arrays one can find the eigenvalues of the distance matrix, and see that in each case there exists a second positive eigenvalue.

**Case 2.** Suppose \( \lambda = 1 \) and \( r \geq 3 \). Then \( k = 6 \) or 8.

The distance-regular graphs with valency 6 and \( \lambda = 1 \) were classified by Hiraki, Nomura & Suzuki [60]. The only graph they found with odd geometric girth is the graph with intersection array \( \{6, 4, 2, 1, 1, 1, 4, 0\} \). It is easy to see that the distance matrix of this graph has more than one positive eigenvalue.

Let \( k = 8 \). Consider the roots corresponding to the edges incident to one of two adjacent vertices \( x \) and \( y \). If \( g \geq 6 \) all these roots are linearly independent, which contradicts the condition \( \dim L \leq 8 \). Hence, \( g = 5 \), i.e., \( \alpha_2 > 1 \). If \( \alpha_3 = 1 \), then [19, Proposition 4.3.11] implies that \( \Gamma \) contains a strongly regular subgraph \( \Delta \) with parameters \( \delta_\Delta = \delta_\Delta + 1, \lambda_\Delta = \),

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1 and \( \mu_\Delta = 1 \). Then \( a_2 \) must be a square (cf. [19, Theorem 1.3.1]), i.e., \( a_2 = 4 \). But \( \lambda_\Delta = \mu_\Delta = 1 \) implies that the valency of \( \Delta \) is even; a contradiction.

Thus, \( c_2 \geq 2 \). Applying Lemma 6.53 (i) and using \( a_2 > 1 \) and \( c_3 > 1 \), we see that the number of vertices of \( \Gamma \) does not exceed 3327. Checking the tables from [19, Chapter 14], we find that there are no distance-regular graphs under our assumptions \( (k = 8, a_1 = 1, a_2 > 1 \) and \( \mu = 1 \)).

**Case 3.** Suppose now \( r = 2 \). By [19, Proposition 4.3.4 and Theorem 4.2.16] we find that \( \Gamma \) must be the line graph of a Moore graph. Since \( k \leq 8 \), this Moore graph can only be the Petersen graph. Now we check the eigenvalues and establish a contradiction. \( \square \)

**Theorem 6.55 (Specterov [100])** If \( \Gamma \) is an isometric distance-regular subgraph of a halved cube with odd geometric girth \( g \) and with valency at least 3, then \( \Gamma \) is the dodecahedron or the Petersen graph. \( \square \)

**Remark.** The only graphs of Theorem 6.43 that are not isometric subgraphs of a halved cube are the Gosset graph, the Schl"{u}fli graph, the three Chang graphs and the Cocktail Party graphs with \( \mu \geq 7 \). But these graph are all hypermetric by Theorem 6.9. So all the graphs of Theorem 6.43 are hypermetric. Maybe it is possible to show this directly.
Chapter 7

Standard representations

In Section 7.1 we discuss the existence of the standard representations of distance-regular graphs and develop some theory using ideas of Terwilliger and Godsil. Main results of this section are an improvement of Godsil’s diameter bound in Theorem 7.17 and a characterisation of the cubes in Theorem 7.24.

In the second section we will look at distance-regular graphs with $a_d = 0$ and discuss a new feasibility condition for distance-regular graphs with $a_d = 0$. Using this condition we will show that as a consequence of Theorem 7.12 there do not exist distance-regular graphs for an infinite family of possible intersection arrays. This section is based on joint work with C.D. Godsil, [49].

In the final Section 7.3 we will classify the distance-regular graphs with an eigenvalue with multiplicity 8 (Theorem 7.33). This is joint work with W.J. Martin.

7.1 General theory

We start by giving the explanation for the existence of standard representations.

Proposition 7.1 (cf. [19, Proposition 4.4.1]) Let $\Gamma$ be a distance-regular graph with $v$ vertices, valency $k$, and intersection numbers $a_i, b_i, c_i$, and let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity $m$. Then $\Gamma$ has a spherical representation in $\mathbb{R}^m$ such that

$$(\overline{x}, \overline{y}) = u_i \quad \text{for all } x, y \in V \Gamma \text{ with } d(x, y) = i,$$

where $(u_0, u_1, \ldots, u_d)$ is the standard sequence corresponding to $\theta$, i.e.,

$$u_0 = 1, \quad u_i = \frac{\theta}{k}, \quad c_iu_{i-1} + a_iu_i + b_iu_{i+1} = \theta u_i \quad (i = 1, 2, \ldots, d - 1).$$

For an eigenvalue $\theta$, the representation we have from Proposition 7.1 is called the standard representation associated to $\theta$. The following proposition gives some simple properties of this standard representation.

Proposition 7.2 (cf. [19, Proposition 4.4.4]) Let $\Gamma$ be a distance-regular graph with eigenvalue $\theta$. Let $(u_0, u_1, \ldots, u_d)$ be the standard sequence associated to $\theta$. Then $|u_i| \leq 1$ for all $i$. Furthermore, if $u_i = \pm 1$ for some $i > 0$, then one of the following holds.

(i) $\theta = k, \ u_i = \frac{1}{k}$ for all $i$.

(ii) $\theta = -k, \ u_i = (-1)^i$ for all $i$, and $\Gamma$ is bipartite.

(iii) $i = d, \ u_i = u_{d-i}$ for all $i$ and $\Gamma$ is antipodal.

(iv) $i = d, \ u_i = -u_{d-i}$ for all $i$ and $\Gamma$ is an antipodal $\mathbb{Z}$-cover. □
In this chapter we will use \( \mathcal{F} \) for the label of \( x \) in the standard representation corresponding to a given eigenvalue. If \( A \) is set of vertices, we will use \( \overline{A} \) instead of \( \{a \mid a \in A\} \).

Now we will show a lower bound on the multiplicity of an eigenvalue in a distance-regular graph.

A chordal graph is a graph such that between any two vertices there is a unique path connecting them. In a chordal graph, \( \Gamma \), a vertex \( x \) is called an end vertex if the induced graph on \( \Gamma \setminus \{x\} \) is still connected. If the induced graph on \( \Gamma \setminus \{x\} \) is not connected, then \( x \) is called a cut vertex.

**Proposition 7.3** Let \( \Gamma \) be a distance-regular graph and let \( \Delta \) be an isometric non-complete chordal subgraph of \( \Gamma \). Define

\[
\begin{align*}
  w := |V\Delta|,
  c := \text{the number of end vertices in } \Delta.
\end{align*}
\]

For an \( s \in \mathbb{R} \cup \{\infty\} \) define

\[
\begin{align*}
  c_s := \text{the number of maximal cliques of size } s + 1 \text{ in } \Delta,
  u_s := \text{the number of cut vertices of } \Delta, \text{ lying in at most one maximal clique of size } s + 1.
\end{align*}
\]

Then for \( \theta \neq k \) an eigenvalue of \( \Gamma \) with multiplicity \( m \) and \( s := -\frac{\lambda_s}{\theta + 1} \) we have

\[
\dim(\overline{\mathcal{V}\Delta}) \geq w - u_s - c_s.
\]

Thus, if \( s \) is not a positive integer (and in particular if \( \theta = -1 \)), then \( m \geq c \).

**Proof.** We proceed by induction on \( w - c \), the number of cut vertices of \( \Delta \).

Let \( x \in V\Delta \) such that the graph \( \Pi \) induced by \( V\Delta \setminus \{x\} \) is disconnected and has at most one non-complete component, say \( \Pi_1 \). Let \( A = \cap \Pi_1 \cup \{x\} \) and \( B = V\Delta \setminus A \). Let \( W \) be the vector space spanned by vectors \( \{\alpha_b\}_{b \in B} \) such that

\[
\sum_{b \in B} \alpha_b \beta_b \in (\overline{\mathcal{A}}).
\]

It is easy to see that \( \dim(\overline{\mathcal{V}\Delta}) = \dim(\overline{\mathcal{A}}) + |B| - \dim(W) \). For \( b \in B \) we define \( s_b \) such that \( s_b + 1 \) is the size of the maximal clique in \( \Delta \) containing \( b \). Let \( w = \sum_{b \in B} \alpha_b \beta_b = \sum_{a \in A} \gamma_a \alpha_a \). The inner product \( (b, w) \) does not depend on \( b \in B \), by the fact that for all \( a \in A \), the distance \( d(a, b) \) does not depend on \( b \in B \). For \( b \in B \) we have

\[
(w, b) = \beta_b + u_1 \sum_{b \in B, b \neq b} \beta_b + u_2 \sum_{b \in B, b \neq b} \beta_b.
\]

Let \( b, f \in B, b \neq f \). Suppose first that \( b \sim f \). From (7.1) we obtain that

\[
\beta_b + u_1 \beta_f = \beta_f + u_2 \beta_b.
\]

because of \( (b, w) = (f, w) \). Since \( \theta \neq k \), we have \( u_1 \neq 1 \) and it follows that \( \beta_b = \beta_f \). Suppose now that \( b \neq f \). Again by (7.1) we obtain now

\[
(1 - u_2)\beta_b + (u_2 - 1)\beta_f + (u_1 - u_2) \sum_{b \in B, b \neq b} \beta_b + (u_2 - u_1) \sum_{b \in B, b \neq f} \beta_b = 0.
\]

We just saw that \( b \sim c \) implies \( \beta_b = \beta_c \). By substitution we obtain:

\[
\beta_b(1 + (a_b - 1)u_1 - a_f u_2) = \beta_f(1 + (a_f - 1)u_1 - a_f u_2).
\]

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It follows that one simple nonzero $\beta_k$ determines the others (so $\dim W \leq 1$), unless $1 + (s_k - 1)u_1 = s_k u_2 = 0$, i.e., $s_k = -b_1/(\theta + 1)$, (since $u_1 = \theta/k$ and $u_2 = (\theta^2 - (\theta \lambda - k)/(kb_1)$) and in this case $\dim W \leq t$, where $t$ is the number of maximal $(\lambda + 1)$-cliques in $\Delta$ containing only $x$ as cut vertex.

We have shown that if $\theta = \frac{b_1 k}{b_1 - 1}$, then $\dim(W) \leq \max\{1, t\}$, where $t$ is the number of maximal $(\lambda + 1)$-cliques in $\Delta$ containing only $x$ as cut vertex, and otherwise $\dim(W) \leq 1$.

So with the induction assumption we are done. $\Box$

A corollary of this proposition is the following.

**Proposition 7.4 (Terwilliger [102])** Let $\theta$ be an eigenvalue of a corconnected distance-regular graph $\Gamma$ and suppose $\theta \neq \pm k$. Then $\theta \neq -1 - b_1$ and hence the multiplicity of $\theta$ is at least the number of leaves of an isomorphic tree in $\Gamma$.

**Proof.** By Proposition 7.3 we may assume that $\theta = -(1 + b_1)$. But this means that $u_2 = 1$ and since $\theta \neq \pm k$ it follows from Proposition 7.2 that $\Gamma$ is complete multipartite. But that is excluded by the corconnectedness. $\Box$

Another consequence of Proposition 7.3 is:

**Proposition 7.5** Let $\Gamma$ be a distance-regular graph such that each edge lies in a $(\lambda + 2)$-clique. Let $g$ be the geometric girth of $\Gamma$ and let $\theta \neq k$ be an eigenvalue of $\Gamma$.

(i) If $g \geq 4r$ and $\theta = -\frac{k}{\lambda + 1}$, then

$$m \geq 1 + \frac{\lambda b_1 b_1 - 1}{(\lambda + 1)(b_1 - 1)}.$$ 

(ii) If $g \geq 4r$ and $\theta = -\frac{k}{\lambda + 1}$, then

$$m \geq \lambda + 1 + \frac{b_1(\lambda + 2)b_1 - 1}{(\lambda + 1)(b_1 - 1)}.$$ 

(iii) If $g \geq 4r$ and $\theta \neq -\frac{k}{\lambda + 1}$, then

$$m \geq k(b_1)^{b_1 - 1}.$$ 

(iv) If $g \geq 4r$ and $\theta \neq -\frac{k}{\lambda + 1}$, then

$$m \geq (\lambda + 2)(b_1)^r.$$ 

$\Box$

**Remark.** This proposition generalises a result of Zhu [114] who obtained $2(m - 1) \geq k$ for $r = 1$. It also generalises a result of Bannai & Ito [9]. They showed that if $g \geq 4r$, then $m \geq (k/2)^r$.

Now we will use the following fact. If $A$ is a basis of a vector space $V$, then every $v \in V$ can be written in a unique way as a linear combination of the elements of $A$. The following proposition is an easy application of this fact.

**Proposition 7.6** Let $\Gamma$ be a distance-regular graph with eigenvalue $\theta$ of multiplicity $m \geq 2$. Let $A$ be a subset of vertices such that $|A| = \dim(A)$. If $x$ is a vertex with $x \in (A)$, then $x$ is the unique vertex $y$ with $d(y, a) = d(x, a)$ for all $a \in A$, unless perhaps when $a_0 = 1$ and $d(x, a) \geq d/2$ for all $a \in A$.
Proof. Assume that there are two vertices \( x_1 \) and \( x_2 \) with \( d(x_1, a) = d(x_2, a) \) for all \( a \in A \). Then \((x_1 \sim x_2) = (x_1 \sim x) - (x_2 \sim x) = u_1 - u_2 = 0\) if \( d(x_1, a) = i \). By the fact that \( x_1 \) is a linear combination of the elements in \( A \), it follows that \( x_1 \sim x_2 \). So \( u_1 = 1 \), where \( i = d(x_1, x_2) \). By \( m \geq 2 \), the proposition follows. \( \square \)

A spherical \( t \)-distance set is a set \( S \) of vectors of length 1 in a Euclidean space \( V \) such that the inner product \((x, y) = (x, y) \in S, x \neq y\) takes at most \( t \) different values.

**Proposition 7.7.** Let \( V \) be an Euclidean space and let \( W \) be an affine subspace of dimension \( m \geq 1 \). Let \( S, S \subseteq W \) be a spherical \( 2 \)-set in \( V \). Then the size of \( |S| \leq m(m+3)/2 \) (If \( \dim(W) = 0 \), then \( |S| \leq 1 \)).

**Proof.** Let \( U \) be an \( m \)-dimensional subspace of \( V \) parallel to \( W \). Let \( w \in W \) be such that the inner product \((w, a)\) is less than 1 and does not depend on \( w \in S \). If we translate the set \( S \) over the vector \(-w\) we find a \( 2 \)-distance set in \( U \). The so-called absolute bound of Delsarte, Goethals and Seidel [37] states that the size of a \( 2 \)-distance set in a \( m \)-dimensional vector space is at most \( m(m+3)/2 \). \( \square \)

In [19, Proposition 4.4.8] there are some little errors. The following proposition gives a corrected and slightly extended version.

**Proposition 7.8 (cf. [40]).** Let \( \Gamma \) be a reconnected distance-regular graph of valency \( k \) and diameter \( d \). Let \( \theta \) be an eigenvalue of \( \Gamma \) with multiplicity \( m \geq 3 \). Then the following holds.

(i) If \( b_i > 1 \), then \( m > i + 1 \), and

\[
(m - i - 1)(m - i + 2) \geq 2b_i,
\]

\[
m \geq i + b_i \quad \text{if} \quad \lambda = 0,
\]

\[
m \geq i + 1 + \frac{1}{2}b_i \quad \text{if} \quad \lambda = 1 \quad \text{or} \quad \mu = 1.
\]

\[
m \geq i + 1 + \frac{\lambda}{\lambda + 1}b_i \quad \text{if} \quad \lambda > \mu = 1 \quad \text{and} \quad i \leq 1.
\]

(ii) \( 2b_i \leq (m - 2)(m + 1) \).

(iii) Assume that either \( \Gamma \) is not an antipodal cover, or \( \Gamma \) is an antipodal \( t \)-cover with \( t \geq 3 \) and \( \theta \) is not an eigenvalue of the folded graph. If \( c_{d-i} \geq 2 \), then \( m > i + 2 \), and

\[
(m - i - 2)(m + 1 - i) \geq 2c_{d-i},
\]

\[
m \geq i + 1 + c_{d-i} \quad \text{if} \quad \lambda = 0.
\]

\[
m \geq i + 2 + \frac{1}{2}c_{d-i} \quad \text{if} \quad \lambda = 1 \quad \text{or} \quad \mu = 1.
\]

**Proof.** (i) Let \( x_0 \sim x_1 \sim \ldots \sim x_i \) be a path of maximal length such that \( \{x_i \mid i = 0, 1, 2, \ldots, t\} \) is an independent set of vectors and \( d(x_i, x_j) = |i - j| \). By Proposition 7.6 we obtain \( b_i = 1 \). Let \( b_i \geq 2 \). If \( y_0 \sim y_1 \sim \ldots \sim y_i \) is a path with \( d(y_0, y_i) = s \), then \( \{y_0, y_1, \ldots, y_i\} \) is an independent set of vectors. Let \( V = \{x \in \mathbb{R}^m \mid \).
\((x,y) = u_{i+1}, \ldots , x_i \). It is clear that \(V\) is an \((m-s-1)\)-dimensional affine space. Let \(A = \Gamma_{\mu}(y) \cap \Gamma(y)\). Then \(A\) is a spherical 2-distance set in \(V\). So we can apply Proposition 7.7 and get \(2 \geq (m - s + 2)(m - s - 1)\). If \(\mu = 0\), then \(A\) is a simplex in an affine subspace of dimension \((m - s - 1)\). If \(\mu = 1\) or \(\mu = 2\), the result follows from Propositions 7.3 and 7.4. The last inequality follows also from these propositions.

(ii) If \(d(x,y) = 2\), then we see that the set \(\Gamma_{\mu}(x) \cap \Gamma_{\mu}(y)\) is a spherical 2-distance set in \(V\). Since \(m > 2\) the result follows.

(iii) We only have to show that if \(u_i \neq \pm 1\) and \(c_{\mu-s} \geq 2\), then the set \(\{x, y, y_1, y_2, \ldots , y_s\}\) is an independent set of vectors if \(d(y_i, y_j) = |i - j|\) and \(d(y_i, x) = d - i\). The rest follows in the same way as in (i).

By assumption \(u_i \neq \pm 1\) (cf. Proposition 7.2). Let \(x, y\) be two vertices at mutual distance \(d\). Since \(u_i \neq \pm 1\) the vectors \(x\) and \(y\) are independent. Let \(y = y_0 \sim y_1 \sim \cdots \sim y_s\), be a maximal path such that \(d(y_i, y_s) = s, d(y_i, x) = d - s\) and \(\{x, y_0, y_1, \ldots , y_s\}\) is an independent set of vectors. Now it follows easily from the Proposition 7.6 that \(c_{d-s} = 1\).

Using \(b_{m-1} = 1\), Godsil [46] showed that there are only finitely many coconnected distance-regular graphs with an eigenvalue with a given multiplicity.

**Theorem 7.9 (Godsil [46]).** There are only finitely many coconnected distance-regular graphs with an eigenvalue with a fixed multiplicity \(m > 2\). Any such graph has diameter \(d \leq 3m - 4\) and valency \(k \leq \frac{1}{2}(m - 1)(m + 2)\). If \(\lambda > 0\), then \(d \leq 2m - 2\). If \(\lambda = 0\), then \(k \leq m\).

Later, in Theorem 7.17, we will improve this diameter bound. First we give an improvement of a result of Terwilliger.

**Proposition 7.10.** Let \(\Gamma\) be a distance-regular graph of diameter \(d \geq 2\) with eigenvalues \(k = \theta_0 > \theta_1 > \cdots > \theta_d\). Let \(\Delta\) be an induced regular subgraph, say with valency \(l\), of \(\Gamma\), such that for all \(x, y \in V\Delta\) the distance \(d_{\Delta}(x, y)\) is at most 2. Let \(\theta\) be an eigenvalue of \(\Gamma\) distinct from \(k\) and \(\eta\) an eigenvalue of \(\Delta\), where \(\eta \neq l\) if \(\Delta\) is connected. Then we have the following inequality:

\[
(\theta + k - a_1) + (\theta + 1)\eta \geq 0.
\]

In particular

\[
-1 - \frac{b_1}{\theta_1 + 1} \geq \eta \geq -1 + \frac{b_1}{\theta_1 + 1},
\]

and equality in (7.2) can occur only for \(\theta = \theta_1\) or \(\theta = \theta_d\). Moreover, equality in (7.2) occurs precisely when there is a vector \(x \neq 0\) orthogonal to the all-one vector \(j\) with \(Gx = 0\), where \(G\) denotes the Gram matrix for \(V\Delta\).

**Proof.** Let \(\theta\) be an eigenvalue of \(\Gamma\) different from \(k\), and let \(u_0, u_1, u_2, \ldots , u_d\) be the standard sequence associated to \(\theta\). We denote the adjacency matrix of \(\Delta\) by \(B\). The Gram matrix \(G\) of \(V\Delta\) equals \(I + u_1 B + u_2 (J - I - B)\) and all its eigenvalues are non-negative. It follows:

\[
G = (1 - u_2)I + (u_1 - u_2)B + u_2 J.
\]

Let \(\eta\) be an eigenvalue of \(B\) with eigenvector \(x\) orthogonal to the all-one vector \(j\). Then

\[
Gx = ((1 - u_2) + (u_1 - u_2)\eta)x.
\]
whence follows that
\[(1 - u_2) + (u_1 - u_2)\eta \geq 0.\]
Using
\[
(7.4) \quad 1 - u_2 = \frac{(k - \theta)(k - a_1 + \theta)}{kb_1},
\]
\[
(7.5) \quad u_1 - u_2 = \frac{(k - \theta)(1 + \theta)}{kb_1}
\]
we deduce inequality (7.2). This holds for all \(\theta \neq k\) of \(\Gamma\) and all eigenvalues \(\eta\) of \(\Delta\) with an eigenvector orthogonal to the all-one vector \(j\). In particular, inequality (7.3) follows by use of \(\theta_d < -1\) and \(\theta_d \geq 0\) and equality in (7.2) can occur only for \(\theta = \theta_d\) and for \(\theta = \theta_1\). ☐

**Theorem 7.11** Let \(\Gamma\) be a distance-regular graph with eigenvalues \(k = \theta_0 > \theta_1 > \cdots > \theta_d\). Let \(\Delta\) be an induced regular subgraph of \(\Gamma\) with valency \(l > 0\) such that for all vertices \(x, y \in V\Delta\) the distance \(d_{\Gamma}(x, y)\) is at most 2. Let \(\theta\) be an eigenvalue of \(\Gamma\) distinct from \(k\) and let \(\bar{x}\) be the representation of \(x\) belonging to \(\theta\). Let \(w\) be the cardinality of \(V\Delta\).

Define
\[
t := (w - l - 1)\theta^2 + \theta(a_1(1 - w) + (k - 1)l) + k(b_1 - w + l + 1).
\]

Let \(\varepsilon = 1\) if \(t = 0\) and \(\varepsilon = 0\) otherwise.

\[
w > \varepsilon + \dim(V\Delta),
\]
then
(i) \(\theta \in \{\theta_1, \theta_d\}\),
(ii) \(\theta + 1\) is an integer dividing \(b_1\), or \(\theta_1 + 1\) and \(\theta_0 + 1\) are conjugate algebraic integers dividing \(b_1\),
(iii) \(-1 = \frac{b_1}{\theta + 1}\)

is an eigenvalue of \(\Delta\) with multiplicity \(w - \dim(V\Delta) - \varepsilon\), unless \(-1 - \frac{b_1}{\theta + 1} = l\) is the valency of \(\Delta\). In that case the number of components of \(\Delta\) equals \(w - \dim(V\Delta) - \varepsilon + 1\).

**Proof.** Let \(u_0, u_1, u_2, \ldots, u_\theta\) be the standard sequence associated to \(\theta\). We denote the adjacency matrix of \(\Delta\) by \(B\). By assumption \(B_{j}^{T} = j_{j}^{T}\). The Gram matrix \(G\) of \(V\Delta\) equals \(I + u_1 B + u_3 (J - I - B)\) and all its eigenvalues are non-negative. By evaluating, we obtain that \(G_{j}^{T} = 0\) is equivalent with \(t = 0\), where \(j\) is the all-one vector.

Let \(U\) denote the null space of \(G\). By Proposition 7.10, equality holds in (7.2) if and only if \(U\) contains a non-zero vector orthogonal to \(j\) and either \(\theta = \theta_1\) or \(\theta = \theta_d\). Let \(W\) be the subspace of \(U\) orthogonal to \(j\). As \(U\) is an eigenspace of \(G\) and \(j\) an eigenvector we see that \(W = U\) unless \(j \in W\). In this last case we have \(\dim(W) = \dim(U) - 1\). The condition \(j \in W\) is equivalent with \(t = 0\). So \(\dim(W) = w - \dim(V\Delta) - \varepsilon > 0\). We find that \(W\) is an eigenspace of \(B\) with eigenvalue \(\eta\) for which equality holds in (7.2). Let \(\eta'\) be an algebraic conjugate of \(\eta\). Then \(\eta'\) is an eigenvalue of \(B\) with the same multiplicity as \(\eta\).

Let \(\theta\) satisfy the equation \((1 + \theta)\eta' = -k + a_1 - \theta\). Then \(\theta'\) is an algebraic conjugate of \(\theta\) and so is an eigenvalue of \(\Gamma\). Hence if \(\theta\) is not an integer then \(\theta_1\) and \(\theta_d\) are the zeros of a quadratic equation over the integers. By the fact that \(\eta\) is an eigenvalue it is an algebraic integer and therefore it follows that \(\theta + 1\) divides \(b_1\). If \(\eta = l\), then also \(j\) is an eigenvector of \(\eta\).

So we are done. ☐
Example. The dodecahedron has eigenvalues 0 and −2 both with multiplicity 4, and eigenvalues ±\(\sqrt{5}\) with multiplicity 3. Looking at an induced pentagon of the dodecahedron, we see that the case with \(\varepsilon = 0\) occurs with the eigenvalues ±\(\sqrt{5}\), and indeed \(\theta_4 = \sqrt{5}, \theta_5 = -\sqrt{5}\). For the eigenvalues 0 and −2 we have \(\varepsilon = 1\) and \(\varepsilon = 1 + \dim(V\Delta)\).

Remark. Probably is it possible to generalise this theorem in the case that the subgraph has larger diameter.

**Theorem 7.12 (Terwilliger [105])** Let \(\Gamma\) be a distance-regular graph with diameter \(d \geq 3\) and with eigenvalues \(k = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_d\) with multiplicities \(m_0, m_1, \ldots, m_d\), respectively. If \(\dim(V(x)) < k\) for some \(\theta_i\) with \(i \geq 1\) and vertex \(z\) (in particular if \(m_i < k\)), then

(i) \(i \in \{1, d\}\),

(ii) \(\theta_i + 1\) is an integer dividing \(b_1\), or \(\theta_i + 1\) and \(\theta_d + 1\) are conjugate algebraic integers dividing \(b_1\),

(iii) 

\[
-1 - \frac{b_1}{\theta_i + 1}
\]

is an eigenvalue of the local graph \(\Gamma(z)\) with multiplicity at least \(k - m_i\). If

\[
-1 - \frac{b_1}{\theta_i + 1} = \lambda
\]

then the number of components of \(\Gamma(z)\) is at least \(k - m_i + 1\).

**Proof.** Assume \(k > \dim(V(x))\) for some eigenvalue \(\theta\) and vertex \(x\), and apply now Theorem 7.11 to \(\Delta = \Gamma(x)\). The fact that \(\varepsilon = 1\) is equivalent with \(\theta = 0\). But if \(\theta = 0\), then \(u_1 = 0\) and so \(\dim(V(x)) \leq m - 1\), where \(m\) is the multiplicity of \(\theta\). ∎

In the next section we will see a generalisation of the above theorem.

Now we will give an improvement of the diameter bound. First we need some lemmas and propositions. Using the theory developed by A.A. Ivanov and A.V. Ivanov, cf. [19, Section 5.9], we find the following proposition.

**Proposition 7.13** Let \(\Gamma\) be a distance-regular graph. Let \(t\) be the maximal number with \(c_1 = 1\) and \(a_t = a_1\). If \(c_j = 1\), then \(j \leq (k - 1 - a_1)j + 1\).

**Proof.** Repeatedly apply [19, Theorem 5.9.9 (i)]. ∎

**Lemma 7.14** Let \(\Gamma\) be a distance-regular graph with \(\lambda = 0\) and \(a_2 \neq 0\). Let \(\theta\) be an eigenvalue of multiplicity \(m \geq 3\) of \(\Gamma\) and let \(\Pi\) be an induced pentagon in \(\Gamma\). If \(\Gamma\) is not the dodecahedron, then \(\dim(V\Pi) \geq 4\).

**Proof.** If \(\dim(V\Pi) \leq 3\), then by Theorem 7.11 we obtain that

\[
\frac{1}{2}(-1 \pm \sqrt{5}) = -1 - \frac{b_1}{\theta + 1}.
\]

This implies that

\[
\theta = \frac{b_1(1 \pm \sqrt{5})}{2} - 1.
\]
The two numbers on the right side of the last equation are algebraic conjugates and this implies that we may assume that

\[ \theta = \frac{b_1(1 + \sqrt{3})}{2} - 1. \]

Since \( b_1 = k - 1 \) and \( \theta < k \) we find that \( k \leq 4 \). By Proposition 7.13 we find that \( c_3 \geq 2 \). Hence \( a_3 = 0 \) or \( b_3 = 1 \), so we have \( d \leq 5 + 2 = 7 \) (by [19, Proposition 5.5.7]) or \( d < 5 + 5 = 10 \). So the number of vertices is bounded by 9.24 = 216. By looking in the tables of [19, Chapter 14] we find that \( \Gamma \) must be the dodecahedron. 

**Lemma 7.15** Let \( \Gamma' \) be a distance-regular graph with \( a_3 = a_2 = 0, a_3 \geq 1 \) and \( \mu = 1 \). Let \( \theta \) be an eigenvalue of multiplicity \( m \geq 3 \) of \( \Gamma' \) and let \( \Pi \) be an induced heptagon in \( \Gamma' \). Then \( \dim(\Pi') \geq 5 \).

**Proof.** We will show that \( \dim(\Pi') \) is either 6 or odd. With the fact that \( \dim(\Pi') \geq 4 \) \((b_2 \geq 2)\) the proposition follows. Let \( H_i \) be the \( i \)-adjacency matrix of \( \Pi \), i.e. \((B_i)_{xy} = 1 \)

if \( d(x, y) = i \) and 0 otherwise. Then \( B_1 = B_1^2 = 2I \) and \( B_2 = J - B_1^2 - B_1 + I \). So the Gram matrix \( M \) of \( \Pi \) corresponding to the standard representation satisfies:

\[ M = u_3 J + (u_2 - u_3)B_1^2 + (u_1 - u_3)B_1 + (1 - 2u_2 + u_3)I. \]

The eigenvalues of \( B_1 \) are the three zeroes of \( x^3 + x^2 - 2x - 1 \), say \( \eta_1, \eta_2, \eta_3 \), all with multiplicity 2, and 2 with multiplicity 1. Thus all eigenvalues of \( M \) have even multiplicity, except for the one with eigenvalue \( \eta_0 \).

Suppose \( M_{\eta_0} = 0 \). This means \( 2u_3 + 2u_2 + 2u_1 + 1 = 0 \). By evaluating \( k(k - 1)^2(2u_3 + 2u_2 + 2u_1 + 1) = 0 \), we find that

\[ 2\theta^3 - 2(k - 1)\theta^2 + (2k^2 - 8k + 4)\theta + k(k - 1)(k - 3) = 0. \]

For \( k \geq 4 \) this expression has only one real zero, hence \( \theta \) must be integral in this case. For \( k = 3 \) the zeroes are \( 0, 1 \pm \sqrt{2} \). It follows that \( \theta \in \mathbb{Q}(\sqrt{2}) \). If \( \dim(\Pi') \leq 6 \), then it follows from Equation (7.6) that one of the \( \eta_i \) is a zero of a quadratic equation over \( \mathbb{Q}(\sqrt{2}) \). But in \( \mathbb{Q}(\sqrt{2}) \), the polynomial \( f(x) = x^2 + x^2 - 2x - 1 \) is irreducible. This means that \( \dim(\Pi') = 6 \).

So \( \dim(\Pi') \) is either six or odd. So we are done. \( \square \)

**Lemma 7.16** Let \( \Gamma \) be a distance-regular graph with eigenvalue \( \theta \) with multiplicity \( m \), where \( m \leq d + 1 \). If \( \Pi \) is an isometric \((2t + 1)\)-gon in \( \Gamma \) and \( \dim(\Pi') \geq t + 2 \), then \( b_{t+2} = 1 \) or \( c_{t+1} \geq 2 \).

**Proof.** Let \( V \Delta = \{ x_i \mid i \in \mathbb{Z}_{2t+1} \} \) in such a way that \( x_i \sim x_{i+1} \) \((i \in \mathbb{Z}_{2t+1})\). Let \( x_1 \sim x_2 \sim \ldots \sim x_{2t+1} \sim y_0 \sim y_1 \sim y_2 \sim \ldots \sim y_s \) such that \( d(y_s, x_1) = s + t \) and take \( s \) maximal such that \( \dim(\Pi' \cup \{ y_0, y_1, \ldots, y_s \}) = \dim(\Pi') + s \). (Then \( 0 \leq s \leq m - t - 2 \).) We will show that \( b_{t+1} = 1 \) or \( c_{t+1} \geq 2 \). Suppose not. Let \( z_1, z_2 \in \Gamma(y_i) \cup \Gamma_{t+1}(x_1) \).

If there is an \( x_i \) such that \( d(x_i, z_1) \neq d(x_i, z_2) \), then \( c_{t+1} \geq 2 \). If for all \( i, j \) we have \( d(x_i, z_1) = d(x_i, z_2) \) and \( d(y_j, z_1) = d(y_j, z_2) \), then \( z_1 = z_2 \) by Proposition 7.6 so that \( b_{t+1} = 1 \). \( \square \)

**Theorem 7.17** Let \( \Gamma \) be a distance-regular graph with an eigenvalue \( \theta \) of multiplicity \( m \geq 3 \). Then for the diameter \( d \) we have \( d \leq 2m - 1 \), with equality if and only if \( \Gamma \) is the dodecahedron.

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Proof. We may suppose that \( d \geq 2m - 1 \) and hence \( a_1 = 0 \) by Theorem 7.9. Since \( b_{n-1} = 1 \) (by Proposition 7.8), we have \( c_m = 1 \). Let \( g \) be the girth of \( \Gamma \). Let \( t \coloneqq \frac{g}{2} \). Since \( m \geq 3 \) we have \( k \geq 3 \). The distance-regular graphs with valency 3 have been classified by Biggs, Boskovic, & Shawe-Taylor [12] (cf. [19, Theorem 7.5.1]) and by checking them we see that \( \Gamma \) must be the dodecahedron. It has \( m = 3 \) and diameter 5. So we may assume that \( k \geq 4 \). The distance-regular graphs with \( m = 3 \) are the 5 platonic solids, cf. Godsil [46]. The distance-regular graphs with \( m = 4 \) have been classified by Zhu [14], cf. our future Theorem 7.30 (i), and all the graphs have diameter at most 6. So we may assume that \( m \geq 5 \). Suppose that \( t \geq 4 \). By Proposition 7.5 we have \( t \leq 2\log_{d-1}(m/2) + 1 \) and \( m \geq k(k - 1) \). Since \( k \geq 4 \) and by Proposition 7.13, the largest number \( i \) with \( c_i = 1 \), satisfies \( i \leq (k - 1)(t - 1) + 1 \leq 2(k - 1)\log_{d-1}(m/2) + 1 < m \). So we find \( c_m \geq 2 \), contradiction. Hence the girth of the graph is at most 7. Since \( c_m = 1 \) and \( m \geq 5 \), the girth is 5 or 7.

By Lemmas 7.14, 7.15 and 7.16 we have \( b_{m-1} = 1 \). Now it follows from [19, Theorem 5.9.9] that \( c_{m+1} \geq 2 \) and hence \( d \leq 2m - 2 \).

Proposition 7.5 says that if \( \lambda \) equals 1, then the valency is at most \( 2m - 2 \) if \( m \) is the multiplicity of an eigenvalue. For \( \lambda \) equal to 2 we can also show that the valency is not too big in comparison to \( m \).

Proposition 7.18 Let \( \Gamma \) be a distance-regular graph with \( \lambda = 2 \). Assume that \( \Gamma \) has an eigenvalue \( \theta \) with multiplicity \( m \geq 3 \). Then either

\[
k \leq \frac{3m}{2},
\]

or \( \Gamma \) is the icosahedron (and \( m = 3 \)).

Proof. For all vertices \( x \), the local graph induced by \( \Gamma(x) \) consists of a disjoint union of cycles. We may suppose that \( k > m \). Let

\[
b \coloneqq \frac{b}{\theta + 1}.
\]

Then (by Theorem 7.12 (iii)) the local graph \( \Gamma(x) \) has eigenvalue \( -b - 1 \) with multiplicity at least \( k - m \). By \( \lambda = 2 \) we have \( | - b - 1 | \leq 2 \). If \( \mu = 1 \), then by Proposition 7.5 (i) we obtain \( m \geq 1 + 2k/3 \). So we may suppose that \( \mu \geq 2 \). By Theorem 7.12 we have \( \theta = \theta_2 \) or \( \theta = \theta_1 \). But then by [19, Theorem 4.4.3] we get \( b \geq 1 \) or \( b \leq -2 \) or \( \Gamma \) is the icosahedron.

Assume that \( \Gamma \) is not the icosahedron. If \( b \) is not integral, then by Theorem 7.12 we may assume that \( b \geq 1 \) and hence \( b > 1 \). But then \( -1 - b < -2 \), contradiction. If \( -1 - b = \pm 2 \), then \( \Gamma(x) \) consists of at least \( k - m \) components with at least 3 vertices and this implies \( k \geq 3(k - m) \). The only case we have to do now is the case where \( b = -2 \). Then the local graph \( \Gamma(x) \) consists of at least \( (k - m)/2 \) components with at least 6 vertices each. This implies \( 6(k - m)/2 \leq k \) and thus \( 2k \leq 3m \).

Question. Does there exist a number \( m(\lambda) \) such that if \( m \geq m(\lambda) \), then \( k \leq \frac{3}{2}m(\lambda) \)? We have \( m(1) = 3 \), and \( m(2) = 4 \).

Now we will look at the case that the valency \( k \geq 2m \). We first need the following lemma.

Lemma 7.19 If \( \alpha \) is an algebraic integer, but not an integer, then for all integers \( a \), there is an algebraic conjugate of \( \alpha \), say \( \beta \), with \( |\beta - a| > 1 \).
Proof. The product $\prod (a_i - a)$ over all algebraic conjugates of $a$ is an integer not zero. So there must be at least one $a_i$ with $|a_i - a| \geq 1$. So we are done.

Let $\Gamma$ be a distance-regular graph with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ with corresponding multiplicities $m = m_0, m_1, \ldots, m_d$. Define

$$ b^+ = \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^- = \frac{b_1}{\theta_2 + 1}. $$

**Proposition 7.20** Let $\Gamma$ be a distance-regular graph with an eigenvalue $\theta$ of multiplicity $m \geq 5$ and $k \geq 2m$. Then $\theta$ is an integer, $c_2 \geq 2$, $\lambda \geq 3$ and one of the following holds.

(i) $m = m_d$, $b^- = -t - 1$, $t \in \mathbb{Z}, t \geq 1$, and $tk < \lambda(m - 2) + tm - 2$, and

(ii) $m = m_1$, $b^+ = t - 1$, $t \in \mathbb{Z}, t \geq 2$, and $tk < (t + \lambda)m - 2$.

**Proof.** From Proposition 7.5 and Lemma 7.18 it follows that $c_2 \geq 2$ and $\lambda \geq 3$.

It is well-known that the eigenvalues of a graph are algebraic integers and that if $\theta$ is an eigenvalue with multiplicity $f$ then all the algebraic conjugates of $\theta$ are eigenvalues with multiplicity $f$.

By Theorem 7.12 the eigenvalue $\theta$ is equal to either $\theta_1$ or $\theta_2$ and $\theta$ is an integer, because otherwise the local graph would have more than $k$ eigenvalues. Therefore $b := \frac{b_1}{\theta_1}$ is an integer.

Now we have to consider two cases. First assume that $\theta = \theta_1$. Then $b^+$ is an integer and by Theorem 4.4.3 of [19] it is at most $-2$. The local graph has $\lambda$ as eigenvalue with multiplicity at least one and $-b^+ - 1$ with multiplicity at least $k - m$. So there are $m - 1$ unknown eigenvalues, say $\pi_1, \ldots, \pi_{m-1}$. The trace of the adjacency matrix of the local graph is 0, so the sum over all eigenvalues is 0. We have to show that $\sum \pi_i > -\lambda(m - 1) + 2$. If the $\pi_i$ are not all integral then there is at least one greater than $-\lambda + 2$, by Lemma 7.19, and we are done. So we may assume that the $\pi_i$'s are all integral. If $-\lambda$ has multiplicity $c \geq 1$ then $\lambda$ has multiplicity at least $c$ and

$$\sum \pi_i \geq -\lambda + (m - 1)(-\lambda + 1) > -\lambda(m - 1) + 2$$

(since $m > 4$). Otherwise $\sum \pi_i \geq (m - 1)(-\lambda + 1)$, which is even larger.

Assume now that $\theta = \theta_2$. Then $b^+$ is an integer and by Theorem 4.4.3 of [19] it is at least 1. The local graph has $\lambda$ as eigenvalue with multiplicity at least one and $-b^+ - 1$ with multiplicity at least $k - m$. So there are $m - 1$ unknown eigenvalues, say $\pi_1, \ldots, \pi_{m-1}$ and we have to show that $\sum \pi_i < \lambda(m - 1) - 2$. If the $\pi_i$ are not all integral then there is at least one smaller than $\lambda - 2$, by Lemma 7.19, and we are done. So we may assume that the $\pi_i$'s are all integral and have to show that there are at least three $\pi_i$ not equal to $\lambda$. But if the local graph has at least $m - 2$ components, then, since $m > 4$, at least one of them only has two distinct eigenvalues $\lambda$ and $-b^+ - 1$, hence is complete, contradiction.

**Lemma 7.21** Let $\Gamma$ be a distance-regular graph such that eigenvalue $\theta_2$ has multiplicity $m$. If $k \geq 2m$ and $b^- = -2$ then one of the following holds.

(i) Each local graph is the complement of a line graph with $k - \lambda + 1 \leq 2m$,

(ii) $k = 2(\lambda - 1) \leq 28$,

(iii) $k = 3(\lambda - 1) \leq 27$,

(iv) $k = 4(\lambda - 1) \leq 16$.

**Proof.** Let $z$ be a vertex of $\Gamma$. Then the complement of the subgraph $\Gamma(z)$ has smallest eigenvalue $-2$. So we can apply Theorem 6.16 and find the four possibilities, because if
\( \lambda = 1 \), then by Proposition 7.5 we find \( 2m - 2 \geq k \).

If \( \Pi \) is the line graph of a graph \( \Delta \), then the maximal size of a clique in \( \Pi \) is at least \( \ell/2 + 1 \), where \( \ell \) is the maximal valency in \( \Pi \).

If \( \Gamma(z) \) is the complement of a line graph then the maximal size of a coclique in \( \Gamma(z) \) is at least \( (k - \lambda - 1)/2 + 1 \). So we find \( m \geq (k - \lambda - 1)/2 + 1 \) (by Proposition 7.4) and hence \( k - \lambda + 1 \leq 2m \).

Before we give a characterisation of the cubes we need the following lemma. We denote by \( C(x, y) \) the set \( \{ z \mid d(x, z) + d(y, z) = d(x, y) \} \).

**Lemma 7.22** Let \( \Gamma \) be a cocomponent distance-regular graph with eigenvalue with multiplicity \( m \geq 3 \). Let \( t = \dim(C(x, y)) \), where \( d(x, y) = i \). Then \( b_{m-1} \leq 1 \).

**Proof.** This lemma is a direct consequence of Proposition 7.6.

**Lemma 7.23** Let \( \Gamma \) be a cocomponent distance-regular graph with an induced quadrangle \( K_{2,2} \) and an induced \( K_{2,1,1} \) and let \( \theta \) be an eigenvalue of \( \Gamma \) of multiplicity \( m \), \( m \geq 3 \). Then \( b_{m-2} = 1 \).

**Proof.** Let \( G \) be the Gram matrix of a quadrangle and \( H \) the Gram matrix of \( K_{2,1,1} \) corresponding to an eigenvalue \( \theta \) distinct from \( k \). Then at least one of them has full rank. (Suppose not. Then \( u_2 = u_1 \) and hence \( \theta = -1 \) and it follows that \( k = 3 \) and \( d \geq 2 \), contradiction.) It follows now from Lemma 7.22 that \( b_{m-2} = 1 \).

**Theorem 7.24** Let \( \Gamma \) be a distance-regular graph with diameter \( d \geq 3 \). If \( \Gamma \) has an eigenvalue \( \theta \) with multiplicity \( d \) and \( c_2 \geq 2 \), then \( \Gamma \) is the icosahedron or a hypercube.

**Proof.** We may assume that \( c_2 \geq 2 \). By Proposition 7.8 (i) we obtain \( c_1 \leq b_{d-2} \leq 2 \) and so \( c_2 = 2 \).

There are now two cases, \( \Gamma \) has induced quadrangles or not. Suppose that \( \Gamma \) has an induced quadrangle. By Theorem 1.2 we have

\[
d \leq \frac{2k}{\lambda + 2}.
\]

By Lemma 7.23 we have that \( \Gamma \) has no induced \( K_{2,1,1} \). By Proposition 7.5 we have

\[
k\lambda \leq (\lambda + 1)d.
\]

Combining this two inequalities we get \( \lambda \leq 1 \). If \( \lambda = 1 \), then by Propositions 1.9.1 and 5.5.7 of [19] we have \( k > m \), by looking at \( a_{d-1} + b_{d-1} + c_{d-1} \). Because the locally graph has only eigenvalues 1 and \(-1\), we have \( b^+ = 0 \) or \( b^- = -2 \). By Theorem 4.4.3 of [19] we see that \( b^- = -2 \). So \( \theta = \frac{-2}{2} \). Now looking at the Gram matrix of a quadrangle we see that its rank equals to 4 and it follows from Lemma 7.22 that \( d \leq m - 1 \), contradiction. So \( \lambda = 0 \). But then \( k = m \) and thus \( \Gamma \) is a hypercube.

Suppose now that \( \Gamma \) has no induced quadrangles. But then by Theorem 1.16.3 of [19], the local graph is a strongly regular graph with \( \mu = 1 \). If the local graph has 3-claws, then by Lemma 7.22 we have \( \dim(C(x, y)) = 4 \) if \( d(x, y) = 2 \). So no local graph has induced 3-claws. By Theorem 1.2.3 of [19] it follows that \( \Gamma \) is locally the pentagon and so \( \Gamma \) is the icosahedron.
7.2 On distance-regular graphs with \(a_d\) equal to 0

The main goal of this section is to give better lower bounds on the multiplicity of an eigenvalue in a distance-regular graph with \(a_d = 0\).

**Lemma 7.25** Let \(\Gamma\) be a distance-regular graph and \(\theta\) an eigenvalue of \(\Gamma\) with multiplicity \(m\). Let \(x\) be a vertex of \(\Gamma\). Let \(a_d = 0\). Then \(m \geq \dim(\Gamma(x)) + \dim(\Gamma - \Gamma | u, v \in \Gamma_d(x))\).

**Proof.** Let \(u \in \Gamma(x)\) and \(v_1, v_2 \in \Gamma_d(x)\). Then \(d(u, v_1) = d(u, v_2) = d - 1\), so \(\pi\) is orthogonal to \(v_1 - v_2\). □

As a corollary of Lemma 7.25 and Theorem 7.12 we have the following theorem of Godsil & Hensel [48].

**Theorem 7.26 (Godsil and Hensel [48])** Let \(\Gamma\) be a distance-regular antipodal recover of diameter \(d \geq 2\) with eigenvalue \(\theta\) of multiplicity \(m\), and let \(u_0 \neq u_d\). Then \(m \geq k + r - 2\), or \(\theta \in \{\theta_1, \theta_2\}\) and either \(\theta + 1\) is an integer dividing \(b_1\), or \(\theta + 1\), \(\theta_1 + 1\) and \(\theta_2 + 1\) are conjugate algebraic integers dividing \(b_1\).

**Proof.** Let \(x \in \Gamma^r\). By the fact that %20 is antipodal and \(u_0 \neq u_d\), the set \(\Gamma_d(x)\) forms an \(r - 1\)-simplex. Therefore \(\dim(\Gamma - \Gamma | x \in \Gamma_d(x)) = r - 2\). So if \(m < k + r - 2\), then \(\dim(\Gamma(x)) < k\). Now the result follows directly from Theorem 7.12 and the above lemma. □

Another corollary is the following.

**Theorem 7.27** Let \(\Gamma\) be a distance-regular graph with \(a_d = 0\). Let \(\theta\) be an eigenvalue of \(\Gamma\) with multiplicity \(j\). Suppose that the standard array belonging to \(\theta\) satisfies \(v_2 \neq u_0 \neq u_2\). Then \(m \geq k + b_d - 1\), or \(\theta \in \{\theta_1, \theta_2\}\) and either \(\theta + 1\) is an integer dividing \(b_1\), or \(\theta + 1\), \(\theta_1 + 1\) and \(\theta_2 + 1\) are conjugate algebraic integers dividing \(b_1\).

**Proof.** Let \(x, y\) be vertices of \(\Gamma\) at distance \(d = 1\). Let \(\Delta := \Gamma_d(x) \cap \Gamma_d(y)\). If \(a, b \in A, a \neq b\), then \(d(a, b) = 2\). Now it is easy to see that \(\dim(\Gamma - \Delta | a, b \in A) = b_d - 1\). The result follows now directly from Lemma 7.25 and Theorem 7.12. □

This last theorem rules out the infinite series of feasible intersection arrays for primitive distance-regular graphs of diameter 4

\[
(\mu(2\mu + 1), (\mu - 1)(2\mu + 1), \mu, \mu - 1, \mu(\mu - 1), \mu(2\mu + 1)), \mu \geq 2.
\]

This intersection array has eigenvalue \(\mu(1 + \sqrt{2\mu + 2})\) of multiplicity \(\mu(2\mu + 1)\). It is straightforward to see that \(\frac{\mu(\mu - 1)(\mu + 1)}{1 + \mu + \sqrt{2\mu + 2}}\) is not an algebraic integer. Using the fact that the product and the sum of two algebraic integers are algebraic integers, the assumption that \(\frac{\mu(\mu - 1)(\mu + 1)}{1 + \mu + \sqrt{2\mu + 2}}\) is an algebraic integer implies that \(\frac{\mu}{1 + \mu + \sqrt{2\mu + 2}}\) is an algebraic integer, which is only true for \(\mu = 1\).

**Remarks.** (i) If distance-regular graphs with these arrays existed, then they would have two \(\Gamma\)-polynomial and two \(Q\)-polynomial structures.

(ii) On page 247 of Brouwer, Cohen, and Neumaier, [19], the authors conjecture that \(Q\)-polynomial distance-regular graphs have at most two \(Q\)-polynomial structures and all eigenvalues are integral if the diameter is greater than or equal to three, and not four. The reason that diameter 4 was excluded was the above series of intersection arrays.
7.3 Small multiplicities

By Theorem 7.9, for given $m \geq 3$ there are only finitely many distance-regular graphs with diameter at least 3 and with an eigenvalue of multiplicity $m$.

For $m = 3$ those are the five platonic solids. In the first subsection we will discuss the results of Zhu [113, 114] and Martin & Zhu [75]. In the second subsection we will discuss the algorithm we used for the classification of the distance-regular graphs with an eigenvalue of multiplicity 8. In the last subsection we shall give the classification of those graphs.

### 7.3.1 Multiplicities 4, 5, 6 and 7

**Theorem 7.28** For every integral value of $m \geq 3$, the following distance-regular graphs have an eigenvalue with multiplicity $m$.

(i) The complete graph $K_{m+1}$, with diameter $d = 1$, $v = m + 1$ vertices, eigenvalue $-1$ of multiplicity $m$ and intersection array $\{m + 1; 1\}$.

(ii) The complete multipartite graphs $K_{(m+1)t}$, with diameter $2$, $v = t(m + 1)$ vertices, eigenvalue $-t$ with multiplicity $m$ and intersection array $\{tm, t-1, 1; tm\}$, for $t = 2, 3, \ldots$

(iii) The complement of the $(2 \times (m + 1))$-grid. It has diameter $3$, $v = 2(m + 1)$ vertices, eigenvalues $\pm 1$ of multiplicity $m$, and intersection array $\{m, m - 1, 1, 1, m - 1, m\}$.

(iv) The Johnson graphs $J(m + 1, t)$, $1 \leq t \leq (m + 1)/2$ with diameter $t$, $v = (m + 1)$ vertices, eigenvalue $(t - 1)(m - t) - 1$ of multiplicity $m$ and intersection array $\{(m + 1 - t)t, (m - t)(t - 1), (m - t - 1)(t - 2), \ldots, (m + 2 - 2t); 1, 4, 9, \ldots, t^2\}$.

(v) The complement of the triangular graph, $\overline{T(m + 1)} = J(m + 1, 2)$. It has diameter $3$, $v = m(m + 1)/2$ vertices, eigenvalue $2 - m$ of multiplicity $m$ and intersection array $\{(m - 1)(m - 2)/2, 2m - 1, 1; (m - 3)(m - 2)/2\}$.

(vi) The $m$-cubes $Q_m$, with diameter $d = m$, $v = 2^m$ vertices, eigenvalues $\pm (m - 2)$ and intersection array $\{m, m - 1, \ldots, 1; 1, 1, \ldots, m\}$.

(vii) The halved $m$-cubes $\Delta(m)$, with diameter $d = \lfloor m/2 \rfloor$, $v = 2^{m-1}$ vertices, eigenvalue $(m^2 - m)/2$ of multiplicity $m$ and intersection array $\{(m^2 - m)/2, (m^2 - m)/2, (m^2 - m)/2, \ldots, (m^2 - m)/2, 1, 6, 15, 28, \ldots, (d^2 - 1)/2\}$.

If $m$ is an odd integer then also the following graphs have an eigenvalue of multiplicity $m$.

(viii) The folded cubes $\Box_m$, with diameter $(m - 1)/2$, $v = 2^{m-1}$ vertices, eigenvalue $2 - m$ with multiplicity $m$ and intersection array $\{m, m - 1, \ldots, m + 1 - d; 1, 2, \ldots, d\}$.

If $m$ is an even integer, then also the following graphs have an eigenvalue of multiplicity $m$.

(ix) The doubled Odd graph with valency $k = (m + 2)/2$, $DO_k$, with diameter $m + 1$, $v = 2^{(2k-1)}$ vertices, eigenvalues $\pm (k - 1)$ of multiplicity $m$ and intersection array $\{k, k - 1, k - 2, k - 3, \ldots, 1, 1, 1, 1, 1, 2, 2, 3, \ldots, k - 1, k\}$.

(x) The Odd graphs $O_k$, with valency $k = (m + 2)/2$, with diameter $d = k - 1$, $v = 2^{(k-1)}$ vertices, eigenvalue $1 - k$ with multiplicity $m$ and intersection array $\{k, k - 1, k - 2, k - 3, \ldots, 1, 1, 2, 2, 3, 3, \ldots\}$.

The graphs are uniquely determined by their intersection array, except for the array $\{12, 5; 1, 4\}$. For this array there are 4 graphs, namely the Johnson graph $J(8, 2)$ and the three Chang graphs.

**Proof.** Uniqueness is trivial for (i), (ii) and (iii), for the other cases see [19, Chapter 9]. The rest is obvious. $\square$

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Lemma 7.29 If $\Gamma$ is a connected strongly regular graph with an eigenvalue of multiplicity $m$, then also its complement has an eigenvalue of multiplicity $m$, unless it is a complete multipartite graph.

Proof. Directly from Theorem 1.4. \(\square\)

The next theorem gives a classification of the distance-regular graphs with an eigenvalue of multiplicity 4, 5, 6 or 7.

Theorem 7.30 Up to isomorphism, the only distance-regular graphs with an eigenvalue of multiplicity $m$, $4 \leq m \leq 7$ not mentioned in Theorem 7.28 are the following.

(i) For $m = 4$ (Zhu [114]): $K_{4 \times 2}$, $K_{3 \times 3}$, $(3 \times 3)$-grid $= H(2, 3)$, each of diameter 2, and

<table>
<thead>
<tr>
<th>name</th>
<th>intersection array</th>
<th>eigenvalue</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The line graph of Petersen graph</td>
<td>${4, 2, 1, 1, 1, 4}$</td>
<td>$\pm 2$</td>
<td>15</td>
</tr>
<tr>
<td>The Pappus graph</td>
<td>${3, 2, 2, 1, 1, 1, 2, 3}$</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>The dodecahedron</td>
<td>${3, 2, 1, 1, 1, 1, 1, 1, 2, 3}$</td>
<td>$0, -2$</td>
<td>20</td>
</tr>
</tbody>
</table>

Each of these graphs is uniquely determined by the intersection array.

(ii) For $m = 5$ (Martin & Zhu [75]): $K_{3 \times 2}$, $J(5, 2)$, the Petersen graph, each of diameter 2, and

<table>
<thead>
<tr>
<th>name</th>
<th>intersection array</th>
<th>eigenvalue</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The icosahedron</td>
<td>${5, 2, 1, 2, 5}$</td>
<td>$-1$</td>
<td>12</td>
</tr>
<tr>
<td>The line graph of Petersen graph</td>
<td>${4, 2, 1, 1, 1, 4}$</td>
<td>$\pm 2$</td>
<td>15</td>
</tr>
<tr>
<td>The Wells graph</td>
<td>${5, 4, 1, 1, 1, 4, 5}$</td>
<td>$-3$</td>
<td>32</td>
</tr>
<tr>
<td>The 3-cover of $GQ(2, 2)$</td>
<td>${6, 4, 2, 1, 1, 1, 4, 6}$</td>
<td>$-3$</td>
<td>45</td>
</tr>
<tr>
<td>(The halved Foster graph)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>The dodecahedron</td>
<td>${3, 2, 1, 1, 1, 1, 1, 1, 2, 3}$</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>The Desargues' graph</td>
<td>${3, 2, 2, 1, 1, 1, 2, 2, 3}$</td>
<td>$\pm 1$</td>
<td>20</td>
</tr>
</tbody>
</table>

Each of these graphs is uniquely determined by the intersection array.

(iii) For $m = 6$ (Martin & Zhu [75]): $K_{6 \times 2}$, $K_{3 \times 3}$, $K_{2 \times 4}$, $QR(13)$ (the Paley graph on $v = 13$ vertices), $II(2, 4)$ and its complement, the Shrikhande graph and its complement, $GQ(2, 4)$, the Schl"{o}fli graph all of diameter 2, and

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<th>intersection array</th>
<th>eigenvalue</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
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<td>The Heawood graph</td>
<td>${3, 2, 2, 1, 1, 3}$</td>
<td>$\pm \sqrt{2}$</td>
<td>14</td>
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<tr>
<td>The incidence graph of 2-(7,4,2)-design (distance 3 graph of the Heawood graph)</td>
<td>${4, 3, 2, 1, 2, 4}$</td>
<td>$\pm \sqrt{2}$</td>
<td>14</td>
</tr>
<tr>
<td>The line graph of the Heawood graph ($GH(2, 1)$)</td>
<td>${4, 2, 2, 1, 1, 2}$</td>
<td>$-1 \pm \sqrt{2}$</td>
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<tr>
<td>$II(3, 3)$</td>
<td>${6, 4, 2, 1, 2, 3}$</td>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>$GQ(2, 4)$ minus a spread (two graphs)</td>
<td>${8, 6, 1, 1, 3, 8}$</td>
<td>$-4$</td>
<td>27</td>
</tr>
<tr>
<td>The second subconstituent of Hoffman-Singleton graph</td>
<td>${6, 5, 1, 1, 1, 6}$</td>
<td>$-1$</td>
<td>42</td>
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Each of these graphs is uniquely determined by its intersection array, except for the intersection array \(\{8,6,1;1,3,8\}\). For this intersection array there are exactly two graphs.

(iv) For \(m = 7\) (Martin & Zhu [75]): \(K_{7+2}\), the three Chung graphs and their complements, each of diameter 2, and

<table>
<thead>
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<th>intersection array</th>
<th>eigenvalue</th>
<th>(v)</th>
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<tr>
<td>3-cover of (K_6) (cf. [19, p386])</td>
<td>({7,4,1;1,2,7})</td>
<td>(-1)</td>
<td>24</td>
</tr>
<tr>
<td>2-cover of (K_{14}) (cf. [19, §§12.5 and 12.7A])</td>
<td>({13,6,1;1,6,13})</td>
<td>(\pm\sqrt{13})</td>
<td>28</td>
</tr>
<tr>
<td>The Gossen graph</td>
<td>({27,10,1;1,10,27})</td>
<td>9</td>
<td>56</td>
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<tr>
<td>The Coxeter graph</td>
<td>({3,2,2,1,1,1,1,2})</td>
<td>(-1 \pm \sqrt{3})</td>
<td>28</td>
</tr>
</tbody>
</table>

Each of these graphs is uniquely determined by the intersection array. \(\Box\)

### 7.3.2 Computer-aided search

For a given multiplicity \(m\), diameter \(d\) and valency \(k\), our algorithm iterates through all possible intersection arrays of graphs \(\Gamma\) satisfying

(i) \(b_i + c_i \leq k\);

(ii) \(k = b_0 > b_1 \geq b_2 \geq \ldots \geq b_{d-1}\);

(iii) \(1 = c_1 \leq c_2 \leq \ldots \leq c_d\);

(iv) \(b_{m-1} = c_{d-m+1} = 1\);

(v) \(c_i \leq b_{d-i}\) (Taylor and Livingston [19, Prop. 4.1.6(ii)]);

(vi) the number \(k_i = b_{i-1}b_{i-1}/c_i\) of vertices at distance \(i\) from a given vertex must be an integer;

(vii) if \(a_i = 0\), then \(b_i \leq m - i\) for \(i \leq m - 1\);

(viii) \(k \geq 2a_1 + 3 - c_1\) ([19, Theorem 1.2.3(i)]);

(ix) if \(d, k \geq m + 1\), then \(a_1 + 1\) divides \(k\) and \(k \leq m(a_1 + 1)/a_1\).

Note that conditions (ii)-(iv) imply that at most \(2m - 3\) parameters are involved. This gives a rough upper bound of \(k^{2m-3}\) on the number of arrays generated. For each array that we generate, we impose the following restrictions:

(a) If \(k_i\) is odd, then \(a_i\) must be even;

(b) for \(i \leq d - m + 1\), we have \(a_i + 1 \leq c_{d-m+1}\);

(c) the multiplicities of the eigenvalues of \(\Gamma\) (obtainable from the intersection array – cf. [11, p143]) must be integers.
(d) at least one of the computed multiplicities must equal \( m \);

(e) (Yoshizawa [11], cf. [19, Corollary 5.6.2]) if the girth is at least five and \( k_d < k \), then the graph is antipodal;

These feasibility conditions were chosen by use of a subjective measure of effectiveness versus computation time. That is, there are quite a few other restrictions which could be coded here, but it is felt that the failure rate for these checks (i.e. the number of non-realizable arrays which fail the condition) does not justify the cost of implementing them. On this note, we point out that the imposition of condition (c) above involves the implementation of the QR algorithm for obtaining a basis of eigenvectors for a matrix (see [52]) and is therefore relatively expensive in computation time yet this restriction on potential intersection arrays is by far the most effective, being responsible for the great majority of our exclusions.

Henceforth, we will call an intersection array acceptable if it satisfies the conditions (i)-(ix) and (a)-(e) above. Note that this definition is only for the purposes of the present discussion and differs from the various definitions of "feasible array" found in the literature.

### 7.3.3 Multiplicity 8

In this subsection we will use the theory developed in Section 7.1 to make the classification of distance-regular graphs with an eigenvalue \( \theta \) of multiplicity 8 computable for a computer.

Let \( \Gamma \) be a distance-regular graph with an eigenvalue \( \theta \) with multiplicity 8. The diameter is at most 14, by Theorem 7.17.

Now we look at the case \( \lambda \neq 0 \).

**Proposition 7.31** Let \( \lambda \neq 0 \).

(i) If the diameter \( d \) of \( \Gamma \) is at least 10, then \( k = 6 \) and \( \lambda = 1 \).

(ii) If the diameter \( d \) equals 8 or 9, then \( e_3 > 1 \) unless \( k = 6 \) and \( \lambda = 1 \).

**Proof.** By Theorem 7.9, the number \( e_{d-m+4} \) equals 1. If the diameter is 8 then \( e_2 = 1 \), by Theorem 7.24.

If the geometric girth is at least 6 then by Proposition 7.5 we find

\[
(\lambda + 2)(k - \lambda - 1) \leq 7(\lambda + 1).
\]

If \( k = 2(\lambda + 1) \), then \( \Gamma \) is a line graph. The distance-regular line graphs are classified, cf. Theorem 4.2.16 of [19] and they have all diameter at most 6, unless \( \Gamma \) is a polygon. So \( k \geq 3(\lambda + 1) \), but then we find that the geometric girth is at most 5, unless \( k = 6 \) and \( \lambda = 1 \).

If the geometric girth equals 5 and \( e_3 = 1 \), then by [19, Proposition 4.3.11] there must be a strongly regular graph \( \Delta \) with \( \lambda(\Delta) = \lambda, \mu(\Delta) = 1 \) and \( k(\Delta) = e_3 + 1 \). The strongly regular graphs with \( \mu = 1 \) and \( \lambda \neq 0 \) have at least valency 21. Hence if \( e_3 = 1 \), then \( k = 6 \) and \( \lambda = 1 \). So we are done. \( \Box \)

**Proposition 7.32** If the diameter of \( \Gamma \) is at least 6, then the valency is at most 16.

**Proof.** Assume that the valency is at least 16 and diameter at least 6. It follows from Proposition 7.5 and Lemma 7.18 that \( \lambda \geq 3 \) and \( \mu \geq 2 \). By [19, Corollary 1.16.6], \( \Gamma \) has at least one induced quadrangle. Since Theorem 1.2 we have

\[
d(\lambda + 2) < 2k
\]
or \( \Gamma \) is a \( K_{m,2} \), a hypercube, a Johnson graph \( J(2d, d) \), a halved cube or the Gosset graph. By checking it follows that \( d(\lambda + 2) < 2k \) must be satisfied. Since Proposition 7.20 we have to consider two cases.

Case 1. \( \theta = \theta_d \). Then \( b^- \leq -2 \) and \( b^- \) is integral. If \( b^- \leq -3 \), then again by Proposition 7.20 we have \( k < 3\lambda + 7 \). But also we have \( k > 3(\lambda + 3) \). So we get a contradiction. So \( b^- = -2 \). By Lemma 7.21 we get \( k = 16 \) or \( k = 15 \). By use of \( k > 3(\lambda + 3) \) we find \( \lambda = 2 \), contradiction.

Case 2. \( \theta = \theta_d \). Then \( b^+ \geq 1 \) and integral. By [19, Theorem 4.4.11], \( b^+ \neq 1 \). So \( b^+ \geq 2 \). But then by Proposition 7.20 we get \( 3k < 8\lambda + 22 \). We also have \( k > 3(\lambda + 2) \). It follows that \( \lambda \leq 1 \), contradiction. \( \square \)

**Theorem 7.33** Up to isomorphism, the only distance-regular graphs with an eigenvalue of multiplicity \( m \), \( m = 8 \) not mentioned in the Theorem 7.28 are the following.

\( K_{4,3}, K_{3,3}, K_{8,2}, (5 \times 5) \)-grid = \( H(2, 5) \) and its complement, \( QR(17) \) (the Paley graph on \( \nu = 17 \) vertices), each of diameter 2, and

<table>
<thead>
<tr>
<th>name</th>
<th>intersection array</th>
<th>eigenvalue</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>The line graph of the Heawood graph ((GH(2,1)))</td>
<td>{4, 2, 2, 1, 1, 2}</td>
<td>-2</td>
<td>21</td>
</tr>
<tr>
<td>3-cover of ( K_8 ), cf. [19, p 386]</td>
<td>{7, 4, 1, 1, 2, 7}</td>
<td>( \pm \sqrt{7} )</td>
<td>24</td>
</tr>
<tr>
<td>( H(3, 3) )</td>
<td>{6, 4, 2, 1, 2, 3}</td>
<td>-3</td>
<td>27</td>
</tr>
<tr>
<td>( CQ(2, 4) ) minus a spread (two graphs)</td>
<td>{8, 6, 1, 1, 3, 8}</td>
<td>-1</td>
<td>27</td>
</tr>
<tr>
<td>( r )-cover of ( K_9 ), cf. [19, p 40]</td>
<td>{8, 6, 1, 1, 1, 8}</td>
<td>-1</td>
<td>63</td>
</tr>
<tr>
<td>The Coxeter graph</td>
<td>{3, 2, 2, 1, 1, 1, 1, 2}</td>
<td>2</td>
<td>28</td>
</tr>
<tr>
<td>The Wells graph</td>
<td>{5, 4, 1, 1, 1, 1, 4, 5}</td>
<td>( \pm \sqrt{5} )</td>
<td>32</td>
</tr>
<tr>
<td>The Hadamard graph of order 8</td>
<td>{8, 7, 4, 1, 1, 4, 7, 8}</td>
<td>( \pm \sqrt{8} )</td>
<td>32</td>
</tr>
<tr>
<td>( AG(2, 5) ) minus a parallel class of lines</td>
<td>{5, 4, 4, 1, 1, 1, 4, 5}</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>( H(4, 3) )</td>
<td>{8, 6, 4, 2, 1, 2, 3, 4}</td>
<td>5</td>
<td>81</td>
</tr>
</tbody>
</table>

Each of these graphs is uniquely determined by the intersection array, except the intersection array \( \{8, 6, 1, 1, 3, 8\} \). For this intersection array there are two graphs.

**Proof.** We checked the following pairs \((d, k)\), where \( d \) is the diameter, \( k \) is the valency, with the computer, using the conditions for acceptable intersection arrays of the previous subsection:

- \( d \leq 5 \) and \( k \leq 35 \).
- \( d = 6, 7 \) and \( k \leq 16 \).
- \( d = 8, 9 \) and \( k \leq 14 \).
- \( 10 \leq d \leq 14 \) and \( k \leq 8 \).

The computer found the following 36 acceptable arrays for multiplicity 8 and diameter at least 2. For each intersection array we will directly say why the graph does not exist, when the graph does not exist, and give all the graphs otherwise.

(i) \( \{9t, t - 1; 1, 9t\} \) belonging to \( K_{10,9} \) \( (t = 2, \ldots) \). It has eigenvalue \(-1\) of multiplicity 8.

(ii) \( \{5, 4; 1, 5\} \) belonging to \( K_{8,5} \). It has eigenvalue 0 of multiplicity 8.
(iii) \(8,4;1,2\) belonging to \((5 \times 5)\)-grid. It has eigenvalue 3 of multiplicity 8.

(iv) \(8,4;1,4\) belonging to the Paley graph for \(F(17)\). It has eigenvalues \(-1 \pm \sqrt{17}\) of multiplicity 8.

(v) \(9,2;1,9\) belonging to \(K_{4 \times 3}\). It has eigenvalue 0 of multiplicity 8.

(vi) \(14,1;1,14\) belonging to \(K_{8 \times 2}\). It has eigenvalue 0 of multiplicity 8.

(vii) \(14,6;1,4\) belonging to \(J(9,2)\). It has eigenvalue 5 of multiplicity 8.

(viii) \(16,6;1,12\) belonging to the complement of the \((5 \times 5)\)-grid. It has eigenvalue -4 of multiplicity 8.

(ix) \(21,10;1,15\) belonging to the complement of \(J(9,2)\). It has eigenvalue -6 of multiplicity 8.

(x) \(30,4;1,24\). The graph does not exist, because \(a_2 = 6 > 5 = k_2\).

(xi) \(30,11;1,10\). The graph does not exist, because \(v = 64 > 44\), the absolute bound for a spherical 2-distance set.

(xii) \(33,20;1,22\). The graph does not exist, because \(v = 64 > 44\), the absolute bound for a spherical 2-distance set.

(xiii) \(4,2;2,1,1,2\). Belonging to the line graph of the Heawood graph. It has eigenvalue -2 of multiplicity 8.

(xiv) \(6,4,2;1,2,3\), belonging to the Hamming graph \(H(3,3)\). It has eigenvalue -3 of multiplicity 8.

(xv) \(7,4,1;1,1,2\), belonging to a 3-cover of \(K_8\). It is unique, see [19, p386]. It has eigenvalues \(\pm \sqrt{7}\) of multiplicity 8.

(xvi) \(8,4,1;1,1,8\). The graph does not exist, since [19, Proposition 4.3.3].

(xvii) \(8,6,1;1,1,8\), belonging to a 7-cover of \(K_8\). Construction and uniqueness follow from [19, Propositions 1.17.2 and 1.17.3]. It has eigenvalues \(\pm \sqrt{8}\) of multiplicity 8.

(xviii) \(8,6,1;1,3,8\), belonging to a 3-cover of \(K_8\). There are two such graphs and both are constructed from \(GQ(2,4)\) minus a spread, cf. Brouwer [15]. They have eigenvalue -1 of multiplicity 8.

(xix) \(8,7,1;1,7,8\) belonging to the complement of the \((2 \times 9)\)-grid. It has eigenvalues \pm 1 of multiplicity 8.

(xx) \(10,6,6;1,4,5\). The graph does not exist, since \(2n_2 = 0 < 4 = a_1 + 1\), cf. [19, Proposition 5.5.1 (i)].
(xix) \(\{15,8,6,1,4,10\}\). The graph does not exist, since [19, Theorem 4.4.11].

(xz) \(\{18,10,4,1,4,9\}\), belonging to \(J(9,3)\). It has eigenvalue 9 of multiplicity 8.

(xzi) \(\{20,12,7,1,6,14\}\). The graph does not exist, since 8 is an eigenvalue of multiplicity 12, \(8 + 1 \text{ does not divide } b_1 = 12\) and Theorem 7.11.

(xzii) \(\{30,22,12;1,6,24\}\). The graph does not exist, since it does not satisfy the absolute bound of DELSARTE, GOETHALS & SEIDEL [37].

(xziii) \(\{32,22,9;1,11,24\}\). The graph does not exist since Lemma 7.21.

(xziv) \(\{32,27,4;1,16,24\}\). Again the graph does not exist since Lemma 7.21.

(xxv) \(\{3,2,2,1;1,1,1,1\}\), belonging to the Coxeter graph. It has eigenvalue 2 of multiplicity 8.

(xxvi) \(\{5,4,1,1;1,1,4,5\}\), belonging to the Wells graph. It has eigenvalues \(\pm \sqrt{5}\) of multiplicity 8.

(xxvii) \(\{5,4,4,1;1,1,4,5\}\), belonging to \(AG(2,5)\) minus a parallel class of lines. It has eigenvalue 0 of multiplicity 8.

(xxviii) \(\{5,4,4,3;1,1,2,2\}\), belonging to the Odd graph \(O_5\). It has eigenvalue \(-4\) of multiplicity 8.

(xxix) \(\{8,6,3,1;1,2,2,6\}\). The graph does not exist since \(2\sigma_5 < 3\sigma_2\), cf. [19, Theorem 5.4.1].

(xx) \(\{8,6,4,2;1,2,3,4\}\), belonging to the Hamming graph \(H(4,3)\). It has eigenvalue 5 of multiplicity 8.

(xxii) \(\{8,7,4,1;1,4,7,8\}\), belonging to the Hadamard graph of order 8, cf. [19, p20]. It has eigenvalues \(\pm \sqrt{5}\) of multiplicity 8.

(xxiii) \(\{20,12,6,2;1,4,9,16\}\), belonging to the Johnson graph \(J(9,4)\). It has eigenvalue 11 of multiplicity 8.

(xxiv) \(\{21,16,10,6;1,2,12,12\}\). The graph does not exist since it has eigenvalue 11 of multiplicity 18 and Theorem 7.11.

(xxv) \(\{28,15,6;1,1,6,15,28\}\), belonging to the halved 8-cube. It has eigenvalue 14 of multiplicity 8.

(xxvi) \(\{8,7,6,5,4,3,2,1;1,2,3,4,5,6,7,8\}\), belonging to the 8-cube. It has eigenvalues \(\pm 6\) of multiplicity 8.

(xxvii) \(\{5,4,4,3,2,2,1,1;1,1,2,2,3,3,4,4,5\}\), belonging to the doubled Odd graph \(DQ_4\). It has eigenvalues \(\pm 4\) of multiplicity 8.

\(\Box\)
Appendix

A.1 Graphs

A graph is a pair $\Gamma = (V\Gamma, E\Gamma)$ consisting of a finite set $V\Gamma$, the vertex set of $\Gamma$, and a set $E\Gamma$ of 2-subsets of $V\Gamma$, the edge set of $\Gamma$. This means that our graphs $\Gamma = (V\Gamma, E\Gamma)$ are finite, undirected, without loops or multiple edges. Elements of $V\Gamma$ are called vertices or points, elements of $E\Gamma$ are called edges. [Directed graphs, that is, pairs $(V\Gamma, E\Gamma)$, where $E\Gamma$ is a subset of $V\Gamma \times V\Gamma$, will only occur in Chapter 6. In this case an element of $E\Gamma$ is called an arc.] Instead of $\{x, y\}$ we often write $xy$. Two graphs $\Gamma$ and $\Delta$ are isomorphic if there is a bijection $\phi : V\Gamma \to V\Delta$ such that $xy \in E\Gamma$ if and only if $\phi(x)\phi(y) \in E\Delta$: the map $\phi$ is called an isomorphism. For $\Gamma, \Delta$ two graphs, a map $\phi : V\Gamma \to V\Delta$, usually denoted by $\phi : \Gamma \to \Delta$, is called a graph morphism if $xy \in E\Gamma$ implies that $\phi(x)\phi(y) \in E\Delta$. An automorphism of a graph $\Gamma$ is an isomorphism between $\Gamma$ and $\Gamma$. The group of all automorphisms is denoted by $\text{Aut}(\Gamma)$. A subgroup $G$ of $\text{Aut}(\Gamma)$ is called vertex-transitive (respectively edge-transitive) if $G$ acts transitively on the vertices of $\Gamma$ (respectively on the edges of $\Gamma$). In this case $\Gamma$ is also called vertex-transitive (respectively edge-transitive).

A flag of a graph is an ordered pair $(\text{vertex, edge})$, and $G$ (or $\Gamma$) is called flag-transitive when $G$ is transitive on the set of flags, or, equivalently, when $G$ is transitive on the ordered pairs of adjacent vertices. Clearly, flag transitivity implies edge transitivity, but the two are not equivalent, not even for regular graphs.

A graph $\Gamma$ is called empty if $V\Gamma = \emptyset$. The complement of $\Gamma$ (often denoted by $\overline{\Gamma}$) is the graph with vertex set $V\Gamma$ and edge set $\{xy \mid xy \notin E\Gamma\}$. A subgraph $\Delta$ of a graph $\Gamma$ is graph with $V\Delta \subseteq V\Gamma$ and $E\Delta = E\Gamma \cap (V\Delta \times V\Delta)$. An induced subgraph of $\Gamma$ (induced on $X$) is the graph with vertex set $X$ and whose edges are the edges of $\Gamma$ contained in $X$.

If $xy$ is an edge of $\Gamma$, then we will write $x \sim y$ and we will say that $x$ and $y$ are adjacent, joined, or that $x$ is a neighbour of $y$. A path of length $i$ from $x$ to $y$ is a sequence of vertices $x = x_0 \sim x_1 \sim \ldots \sim x_{i-1} \sim x_i = y$ such that $x_i \neq y_i$ if $i \neq j$. Being joined by a path is an equivalence relation on the vertices of $\Gamma$, its equivalence classes are the (connected) components of $\Gamma$. If there is only one component, then the graph is called connected. Remark that the empty graph is not connected.

Let $\Gamma$ be a connected graph. For $x, y$ two vertices of $\Gamma$, the distance between $x$ and $y$ in $\Gamma$, denoted by $d(x, y)$ (we usually write $d(x)$ if it is clear which graph we mean) is the length of a shortest path (geodesic) from $x$ to $y$. A subgraph of $\Gamma$, say $\Delta$, is isometric if for any two vertices $z, y$ of $\Delta$ the distance in $\Delta$ is the same as in $\Gamma$. A subgraph of $\Gamma$, say $\Delta$, is convex or geodesically closed if for all vertices $x$ and $y$ of $\Delta$, $\Delta$ contains all geodesics between $x$ and $y$. The diameter of $\Gamma$ is the maximal distance occurring in $\Gamma$. If there are no cycles in $\Gamma$, then $\Gamma$ is called a tree. A vertex $x$ of a tree $\Gamma$ is called an end
vertex (or leaf) if the induced graph on \( V \Gamma \setminus \{x\} \) is connected. If \( \Gamma \) is not a tree, then the girth of \( \Gamma \), denoted by \( g \), is the length of a shortest circuit (subgraph of valency 2) in \( \Gamma \).

A matching or 1-factor of \( \Gamma \) is a partition of the vertex set of \( \Gamma \) into edges. A \( \mu \)-graph in \( \Gamma \) is the subgraph induced by the set of common neighbours of two vertices at distance 2.

If \( x \) is a vertex of \( \Gamma \), we write \( \Gamma_i(x) \) for the set of vertices at distance \( i \) from \( x \). Instead of \( \Gamma_i(x) \) we write \( \Gamma(x) \). The graph induced by \( \Gamma(x) \) is called a local graph. A graph morphism \( \phi : \Gamma \to \Delta \) is called a local isomorphism if for each vertex \( x \) of \( \Gamma \), the map \( \phi \) restricted to \( \Gamma(x) \) is an isomorphism of graphs. The graph \( \Gamma \) is called locally \( X \) for some graph property or class of graphs \( X \) when the subgraph induced by \( \Gamma(x) \) has this property or belongs to this class for all \( x \in V \Gamma \) (e.g., we talk about locally connected, locally complete multipartite, locally Petersen, etc.). The valency \( k(x) \) of a vertex \( x \) is the cardinality of \( \Gamma(x) \). A graph is regular (with valency \( k \)) if each vertex has the same valency \( k \).

A graph is called complete (or a clique) when any two of its vertices are adjacent. The complete graph on \( n \) vertices is denoted by \( K_n \). A co-clique is a graph in which no two vertices are adjacent. A polygon is a connected graph of valency 2. An \( n \)-gon is a polygon on \( n \) vertices. A triangle is a 3-gon, a quadrangle a 4-gon, a pentagon a 5-gon, a hexagon a 6-gon and a heptagon a 7-gon. A graph \( \Gamma \) is the disjoint union of graphs \( \Gamma_1, \Gamma_2, \ldots, \Gamma_t \), written \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_t \), if \( \Gamma_1, \Gamma_2, \ldots, \Gamma_t \) are subgraphs of \( \Gamma \) such that the vertex sets of \( \Gamma_i \) partition that of \( \Gamma \), and the edge sets of \( \Gamma_i \) partition that of \( \Gamma \). The disjoint union of \( n \) graphs isomorphic to a graph \( \Gamma \) is written as \( n \Gamma \). A graph is called bipartite (with (colour) classes or parts \( R \) and \( G \)) if its vertex set can be partitioned into two co-cliques (\( R \) and \( G \)). We also will say that the vertices of \( R \), respectively \( G \), are coloured with red, respectively green. We write \( K_{m,n} \) for the complete bipartite graph with parts an \( m \)-set \( R \) and an \( n \)-set \( G \), where each red vertex is adjacent to each green vertex. A graph is the complete union of graphs \( \Gamma_1, \Gamma_2, \ldots, \Gamma_t \), written \( \Gamma = \Gamma_1 \circ \Gamma_2 \circ \ldots \circ \Gamma_t \), if each \( \Gamma_i \) (\( i = 1,2,\ldots,t \)) is a subgraph of \( \Gamma \), each vertex of \( \Gamma \) lies in exactly one \( \Gamma_i \) and the edges of \( \Gamma \) are precisely those of \( \Gamma_i \) and those of the form \( xy \) with \( x \in \Gamma_i \) and \( y \in \Gamma_j \) in different \( \Gamma_i \). A complete multipartite graph \( K_{m_1,m_2,\ldots,m_t} \) is the complete union of co-cliques of size \( m_1, m_2, \ldots, m_t \); we shall write \( K_{m_1,m_2,\ldots,m_t} \) for the complete union of \( t \) co-cliques of size \( m \). (Thus, \( K_{2,m} \) is the same as \( K_{m,m} \).) A bipartite graph with parts \( R \) and \( G \) is called semi-regular if there are constants \( k_R \) and \( k_G \) such that each red vertex has valency \( k_R \) and each green vertex has valency \( k_G \). A graph is called strongly regular when it is regular and there are constants \( \lambda, \mu \) such that any two adjacent (non-adjacent) vertices have \( \lambda \) (resp. \( \mu \)) common neighbours.

Let \( \Gamma \) be a graph. For vertices \( x,y \) of \( \Gamma \) at distance \( j \), we define \( k_j(x,y) = |\Gamma_j(x) \cap \Gamma_j(y)| \), \( c_j(x,y) = |\Gamma_j(x) \cap \Gamma_j(y) \cap \Gamma_j(z)| \), and \( b_j(x,y) = |\Gamma_{j+1}(x) \cap \Gamma_{j+1}(y)| \). We put \( k(x) = k_0(x) \), \( \lambda(x,y) = a_1(x,y) \) when \( x \sim y \), and \( \mu(x,y) = c_2(x,y) \) when \( d(x,y) = 2 \). In case \( k(x) \) does not depend on the vertex \( x \) (i.e. when \( \Gamma \) is regular) we write \( k = k(\Gamma) = k(x) \). In case \( c_2(x,y) \) does not depend on the pair \( x,y \) but only on the distance \( d(x,y) \), we say that the number \( c_2 \) exists. Similarly for the numbers \( a_1 \) and \( b_1 \). A connected graph \( \Gamma \) is called distance-regular when the numbers \( c_1, a_1, b_1 \) exist for all \( i \leq d \) where \( d \) is the diameter of \( \Gamma \).

The adjacency matrix of a graph \( \Gamma \) is the \((n \times n)\)-matrix \( A = A(\Gamma) \) (\( n = |V(\Gamma)| \)), indexed by the vertices of \( \Gamma \), whose entries \( A_{xy} \) are given by \( A_{xy} = 1 \) if \( x \sim y \), and \( A_{xy} = 0 \) otherwise. The matrix \( A \) is symmetric, \( A = A^T \). Consequently the eigenvalues of the
adjacency matrix are real; they are called the eigenvalues of $\Gamma$. If $\Gamma$ is regular of valency $k$, its adjacency matrix $A$ satisfies the equations $A J = k J$ and $A J = k J$; in particular $k$ is an eigenvalue of $\Gamma$. Here, and throughout this booklet, $J$ stands for the matrix all of whose entries are equal to one (its dimension are usually clear from the context or specified), and $j$ (sometimes also denoted by $I$) denotes the all-one vector.

If we join each pair of vertices at distance 2 in a connected bipartite graph $\Gamma$, we find two connected graphs, say $\Gamma^n$ and $\Gamma^o$ each on one of the two colour classes of $\Gamma$. The graphs $\Gamma^n$ and $\Gamma^o$ are called the halved graphs of $\Gamma$.

An antipodal graph is a connected graph $\Gamma$ for which the relation of having maximal distance is an equivalence relation. In this case, the folded graph of $\Gamma$ is defined as the graph $\tilde{\Gamma}$ whose vertices are the equivalence classes, adjacent if their union contains an edge in $\Gamma$. If, in addition, each vertex $x$ of $\Gamma$ has the same valency as its image under the folding (so that, in particular, $d \geq 3$), then $\Gamma$ is called an antipodal covering graph of $\tilde{\Gamma}$. (Note that the folding map will be a local isomorphism precisely when $D > 3$.) If, moreover, all equivalence classes have the same size $r$, then $\Gamma$ is also called an antipodal $r$-cover (double cover if $r = 2$, triple cover if $r = 3$) of $\tilde{\Gamma}$.

For any graph property $G$ we say that a graph $\Gamma$ has property $co-G$ when the complementary graph $\bar{\Gamma}$ satisfies $G$. Thus, $\Gamma$ is cococnected whenever $\Gamma$ is connected, the components of $\Gamma$ are the components of $\bar{\Gamma}$, a clique in $\Gamma$ is a clique in $\bar{\Gamma}$, etc.

### A.2 Designs

A design is an ordered pair $(X, \mathcal{B})$ with point set $X$ and set of blocks $\mathcal{B}$ such that $\mathcal{B}$ is set of subsets of $X$. A block $B$ is incident with a point $x$ if $x \in B$. The incidence matrix of $(X, \mathcal{B})$ is the $(0, 1)$-matrix with rows indexed by $X$ and columns by $\mathcal{B}$ and the $(x, B)$-entry equals 1 when $x \in B$ and 0 otherwise.

The point graph of the design $(X, \mathcal{B})$ is the graph whose vertex set is $X$ and in which two vertices are adjacent whenever there is a block containing both. The block graph of $(X, \mathcal{B})$ is the graph with vertex set $\mathcal{B}$ and in which two vertices are adjacent whenever there is a point which is contained in both. The incidence graph of $(X, \mathcal{B})$ is the bipartite graph with vertex set $X \cup \mathcal{B}$ and $(x, B)$ is an edge if and only if $x \in B$.

A $t$-$(v, k, \lambda)$ design is a design $(X, \mathcal{B})$ with $|X| = v$, $|B| = k$ for each $B \in \mathcal{B}$, and such that for each subset $T$ of $X$ with $|T| = t$, there are precisely $\lambda$ blocks containing $T$. Such a design is also called a $t$-design when $v, k, \lambda$ are not explicitly given.

A finite projective plane of order $n$ is a $2-(n^2 + n + 1, n + 1, 1)$-design. A finite affine plane of order $n$ is a $2-(n^2, n, 1)$-design. Note that we do not require projective and affine planes to be Desarguesian (i.e. coordinatized by a field). Arbitrary projective and affine planes of order $n$ are denoted by $PG(2, n)$ and $AG(2, n)$, respectively. For $d \geq 2$, $PG(d, q)$ will denote the $d$-dimensional projective space over $\mathbb{F}_q$, i.e., the lattice of subspaces of $(d + 1)$-dimensional vector space over the field $\mathbb{F}_q$. Similarly, $AG(d, q)$ is the $d$-dimensional affine space over $\mathbb{F}_q$.

A partial linear space is a design $(X, \mathcal{L})$ in which the blocks are called lines such that the lines have size at least 2 and two distinct points are joined by at most one line. A triangle of $(X, \mathcal{L})$ is a clique of the point graph of size 3 which is not contained in a line. The girth $g$ of $(X, \mathcal{L})$ is half of the girth of its incidence graph, i.e., $g$ equals three if there is a triangle; otherwise it is the minimal number $\geq 4$ such that the point graph
contains a \(g\)-gon. The point graph of a partial linear space is usually referred to as its \(collinearity\, graph\), and its block graph as the \(line\, graph\). Every graph \(\Gamma\) can be viewed as the collinearity graph of a partial linear space all of whose lines have size two (namely the edges of \(\Gamma\)). Thus, line graphs and incidence graph of graphs are defined. (We write \(L(\Gamma)\) for the line graph of \(\Gamma\)). The line graph of \(K_{1,n}\) is an \(n\)-clique, the line graph of an \(n\)-clique is the \(triangular\, graph\, \mathcal{T}(n)\), and the line graph of \(K_{m,n}\) is the \((m \times n)\)-grid.

If \(\Gamma\) is a graph such that \(\lambda\) exists and every edge lies in a \((\lambda + 2)\)-clique, then \(\Gamma\) is the collinearity graph of a partial linear space \((X, \mathcal{L})\) where \(X = V(\Gamma)\) and \(\mathcal{L}\) is the set of the \((\lambda + 2)\)-cliques. In this case the \(geometric\, girth\, g\) of \(\Gamma\) is the girth of the partial linear space \((X, \mathcal{L})\); in other words, \(g\) is the length of a shortest circuit not a triangle.

A \emph{generalized 2n-gon} is a regular graph of diameter \(n\) such that the number \(\lambda\) exists, every edge lies in a \((\lambda + 2)\)-clique and with geometric girth \(2n\). In stead of generalized 4-gon (respectively generalized 6-gon, generalized 8-gon) we write \(generalized\, quadrangle\, (respectively\, generalized\, hexagon,\, generalized\, octagon)\). A \(CQ(s, t)\) (respectively \(GH(s, t)\), \(GO(s, t)\)) is a \(generalized\, quadrangle\, (respectively\, generalized\, hexagon,\, generalized\, octagon)\) with \(\lambda = s - 1, k = s(t + 1)\). A \(spread\) in a generalized 2n-gon is partition of the vertex set in \((\lambda + 2)\)-cliques.

### A.3 Codes

Let \(Q\) be a set and \(n\) a natural number. A \(code\, C\) of \(length\, n\) over the \(alphabet\, Q\) is a subset of \(Q^n\). The code is called \(binary\, (ternary)\) when \(Q = \mathbb{F}_2\) (respectively \(Q = \mathbb{F}_3\)). The elements of \(C\) are called \(code\, words\) (and those of \(Q^n\) vectors). The \(Hamming\, distance\, d_H(u, v)\) between two code words (vectors) \(u\) and \(v\) is the number of coordinate positions where they differ. The \(minimum\, distance\, d(C)\, of\, C\) is defined as \(\min d_H(u, v)\) for \(u, v \in C, u \neq v\). Let now \(0\) be a distinguished element of \(Q\). The \(weight\, w_t(u)\) of a code word (vector) \(u\) is its number of nonzero coordinates. The \(support\, of\, a\, code\, word\, (vector)\, u\) is the set \(\{i | u_i \neq 0\}\). The code \(C\) is called \(c-error-correcting\) when \(d(C) \geq 2c + 1\). If \(C\) is a code of length \(n\), then a \(truncation\, of\, C\) is a code of length \(n - 1\) obtained by deleting a fixed coordinate position; a \(shortening\, of\, C\) is a code of length \(n - 1\) obtained by deleting a fixed coordinate position and only retaining the code words that were 0 at that position. (Thus, a shortening of \(C\) is a subcode of a truncation of \(C\); shortening does not diminish the minimum distance and truncating preserves cardinality when \(d(C) > 1\). Note that in coding theory one often uses 'puncture' instead of 'truncate'.) Conversely, in case of a binary code \(C\) of length \(n\), the \(extended\, code\) is the code of length \(n + 1\) obtained by adding a \(parity\, check\, bit\) to the words of \(C\), i.e., adding an extra coordinate so as to make the weight even. (Thus, \(C\) is a truncation of its extension.)

Often one takes for \(Q\) a finite field \(\mathbb{F}_q\). Then \(V := Q^n\) has a natural structure of vector space over \(\mathbb{F}_q\) (with a distinguished basis) and we can use vector space concepts. In particular, \(C\) is called \(linear\) when it is a linear subspace of \(V\). An \([n, k, d]\) code \(C\) over \(\mathbb{F}_q\) is linear subspace of \(\mathbb{F}_q^n\) of dimension \(k\) and with minimum distance \(d\).

### A.4 Miscellaneous notation

\(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\) and \(\mathbb{F}_q\) denote the set of natural numbers, the ring of integers, the field of rational numbers, the real field and the finite field with \(q\) elements (also known as \(GF(q)\)), respectively. \(S_n\) and \(A_n\) denote the symmetric and alternating group on \(n\) symbols, respectively. \(\mathbb{Z}_n\) denotes the cyclic group of order \(n\). For unexplained group

The cardinality (or size) of a set $X$ is denoted by $|X|$.
The linear span of a set $A$ of vectors is denoted by $\langle A \rangle$.
The expressions $\lfloor x \rfloor$ and $\lceil x \rceil$ (with $x \in \mathbb{R}$) denote the largest integer not larger than $x$,
and the smallest integer not smaller than $x$, respectively.
Bibliography


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Samenvatting

In dit proefschrift worden deelstructuren en Euclidische voorstellingen van afstandsreguliere grafen bestudeerd.

In het eerste hoofdstuk wordt een inleiding gegeven.

In hoofdstuk 2 worden voldoende voorwaarden gegeven voor het bestaan van afstandsreguliere deelgrafen in afstandsreguliere grafen. Met behulp hiervan wordt bewezen dat er geen afstandsreguliere graaf bestaat met doornsde getallen \{7, 6, 5, 4, 3; 1, 2, 3, 4, 7\}. Tevens wordt bewezen dat de doubled Odd grafen de enige afstandsreguliere grafen zijn waarbij gelijkheid optreedt in een diametergrens van Terwilliger.

In het derde hoofdstuk worden volledig reguliere codes onderzocht. W.J. Martin gaf in zijn proefschrift een vermoeden met betrekking tot de parameters van een volledig reguliere code in een afstandsreguliere graaf. Dit vermoeden wordt bewezen voor de meeste klassieke afstandsreguliere grafen. Ook worden enkele tegenvoorbeelden gegeven, zoals de vijfhoek in een dodecaëder. Ten derde worden de volledig reguliere codes in de Biggs-Smith graaf geklassificeerd. Tenslotte wordt er in dit hoofdstuk gekeken naar perfecte codes met twee verschillende decoderstralen.

In hoofdstuk 4 worden overwegingen van Cartesische produkten van grafen bestudeerd. Hiermee karakteriseren we de Doob grafen.

In het vijfde hoofdstuk worden twee constructies van uniform geodetische grafen gegeven. Ook wordt er een afstandsreguliere graaf met doornsde getallen \{45, 44, 36, 3; 1, 9, 40, 45\} gekonstrueerd.

In hoofdstuk 6 wordt er gekeken naar de metrische rangorde van grafen. Eerst worden de wortelgrafen met $\mu \geq 3$ bepaald. Daarna wordt er gekeken naar rijkelijk reguliere wortel grafen (rijkelijk regulier betekent dat de $\lambda$ en $\mu$ bestaan) met $\mu$ gelijk aan 2. Tenslotte worden de afstandsreguliere grafen met een afstandsmaat met precies één positieve eigenwaarde geklassificeerd.

In het laatste hoofdstuk wordt er gekeken naar de standaard voorstellingen van afstandsreguliere grafen. Met behulp hiervan wordt er van een oneindige reeks van rijtjes bewezen dat ze niet als de doornsde getallen van een afstandsreguliere graaf voorkomen. Ook worden alle afstandsreguliere grafen bepaald met een eigenwaarde met multipliciteit 8.
Curriculum Vitae

Jacobus Hendricus Koolen was born on September 30, 1966 in Bladel en Nettersiel, the Netherlands. He attended the schoolgemeenschap 'Pius X'-college in Bladel and completed the atheneum in 1984. From September 1984 to June 1990 he studied Mathematics at the Eindhoven University of Technology and graduated with the Master's thesis 'Metric properties of regular graphs' (cum laude, prof.dr. A.E. Brouwer). From February 1989 to May 1989 he visited the University of Aarhus in Denmark. Since September 1990, he worked in the group Discrete Mathematics at the Eindhoven University of Technology, with support from the Dutch Organization for Scientific Research (NWO). From January 1992 to April 1992 he visited the Queen Mary and Westfield College in London, United Kingdom and from September 1993 to December 1993 the University of Waterloo in Canada.
1. Alle afstandsreguliere grafen met valentie 4 zijn bekend.

2. Vermoedelijk zijn er maar eindig veel afstandsreguliere grafen met valentie 5.

3. Definieer voor een afstandsreguliere graaf $\Gamma$

   \[ l_i := |\{j | e_j = i\}| \]
   \[ m_i := |\{j | b_j = i\}|. \]

   Proposition 2.3 van dit proefschrift suggereert dat voor alle $i$ de ongelijkheden $m_i \leq l_i$ en $l_i \leq l_1$ gelden.

4. Een afstandsreguliere graaf met doorsnede getallen

   \[ \{l^2(l + 2), (l + 1)^2(l - 1), 2l^2 - r, 1, l, (l + 1)^2(l - 1), l^2(l + 2)\} \]

   met $l \geq 2$ en $1 \leq r \leq 2l^2$, is lokaal een sterk reguliere graaf met parameters \( \{l^2(l+2), l(l+1), 1, l\} \). In het bijzonder geldt: $r \geq l + 1$.

5. Vermoedelijk kan met stelling 4 bewezen worden dat er precies één afstandsreguliere graaf bestaat met doorsnede getallen \{45, 32, 12, 1; 1, 6, 32, 45\}.

6. Het feit dat in sommige landen een helm dragen op een fiets verplicht is of zal worden, terwijl het dragen van licht niet verplicht is, getuigt van een negatieve houding ten opzichte van de fietser.

7. Het niet uitspreken van de "oi" als een "ou" in bijvoorbeeld Oisterwijk duidt op onvolledige kennis van de oud-Nederlandse woorden in de geschreven taal.

8. Omdat sinds 1900 de gemiddelde lengte van de mens met ongeveer 10 procent is toegenomen, zou men de maten, zoals de hoogte van het net en de grootte van een doel, met 10 procent moeten vergroten in sporten als tennis en voetbal.

9. Onderkoeling is een onderschat risico bij het beoefenen van watersporten, zoals wild-waterkanoëën.

10. Iedereen kan leren skimoteren.

11. Een bewijs met een computer is meestal betrouwbareder dan een lang en ingewikkeld bewijs zonder computer.

12. De huidige\(^1\) problemen om een kabinet te forseren geven aan dat het in Nederland gehanteerde kieselsysteem niet bestaat en tegen grote verschuivingen in het stemgedrag.

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\(^1\)Juli 1994