ERASMUS MUNDUS MASTER OF INDUSTRIAL MATHEMATICS

Numerical Simulation of Herding and Flocking Models

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To
my Heavenly Father
and my family
Abstract

In this study, a time continuous version of the discrete Couzin-Vicsek algorithm, which describes the interactions of individual among animal societies such as flocks of birds, is used to simulate the velocities and positions of the self-driven particles (SDPs). This algorithm (the time continuous Couzin-Vicsek algorithm) is based on the discrete Couzin-Vicsek algorithm (CVA) which assumes that at each time step, the velocities and positions of the SDPs are updated, and the velocity director of each particle is changed to the director of the average velocity of the neighboring particles with addition of noise. We aim to study the behavior of SDPs subjected to different model parameters such as the interaction radius both in noise and noise free environments, analyzing the stability of the time continuous discrete CVA-model in two and three dimensions. Finally, we propose an improvement in the time continuous discrete CVA-model using the Brownian motion on the unit circle or sphere case which will always yield stable numerical solutions.
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Chapter 1

Introduction

1.1 Motivation

The interaction of individuals among animal societies such as schools of fishes, flocks of birds, herds of quadrupeds, swarms of ants, swarms of locusts, groups of bacteria, and many other biological organisms has been an interesting area of research for quite a number decades. Researchers are eager to understand the behavior of these animal societies under certain parameters in order to be able to control their mobility, as some of these organisms such as locusts have got a devastating socio-economic impact on humans and nature. Most of the time, these organisms move in large groups, some times in one direction following the leader particle (alignment) or at times forming small groups which moves in the same selected direction or choose to move in different directions. Due to the fact that these organisms or animal societies move in large numbers, such kind of motion is known as flocking or Herding depending on the type of organisms or animal societies under consideration.

A number of models have been put across to describe the behavior of such organisms or animal societies. The most outstanding model is the discrete Couzin-Vicsek algorithm...
1.1 Motivation

In this model, the individuals move with a velocity of constant magnitude and it describes in a discrete way the time evolution of the positions of the individuals and of the velocity angles measured from a reference direction. The velocities of the particles are determined by a simple rule and random fluctuations [3]. The only rule of the model is at each time step a given particle driven with a constant velocity assumes the average direction of motion of the particles in its neighborhood of radius $r$ with some random perturbation (noise) added. At each time step, the angle is updated to a new value given by the director of the average velocity of the neighboring particles, with addition of noise. The positions of the particles are also updated by adding the distance traveled during the time step in the direction specified by its velocity angle.

In some situations we are faced with problems which require modeling of larger numbers of organisms or animal societies like large schools of fish, large flocks of birds and large swarms of ants (like several millions), in such cases it may be more efficient to look for continuum like models, which describe such animal society by microscopic variables such as mean density, mean velocity, etc. A number of such models exist but in a phenomenological way (see [8, 10]).

A number of continuum models have been derived from the CVA model, but the setback in most of these models is that the derivation and mathematical qualities of the resulting models have not been fully analyzed. Of recent a model has been described in [1] in which a continuum model was derived from the kinetic version of the CVA algorithm. In this model, the CVA model was first rephrased as a time continuous dynamical system and then after passed to a mean-field version of the dynamical system. This mean-field model consists of a kinetic equation of Fokker-Planck type with a force term resulting from the alignment interactions between the particles. The mean-field model is written as:

$$\varepsilon (\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) = -\nabla_\omega \cdot (F_0^\varepsilon f^\varepsilon) + d\Delta_\omega f^\varepsilon + O(\varepsilon^2),$$  \hspace{1cm} (1.1)

$$F_0^\varepsilon(x,\omega,t) = \nu(\text{Id} - \omega \otimes \omega)\Omega^\varepsilon(x,t),$$  \hspace{1cm} (1.2)

$$\Omega^\varepsilon(x,t) = \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|} \text{ and } j^\varepsilon(x,t) = \int_{v \in S^2} v f^\varepsilon(x,v,t) dv.$$  \hspace{1cm} (1.3)

Where $f^\varepsilon(x,\omega,t)$ is the particle distribution function which depends on the space variable $x \in \mathbb{R}^3$, the velocity direction $\omega \in S^2$ and the time $t$. $d$ is a scaled diffusion constant and $F_0^\varepsilon(x,\omega,t)$ is the mean-field interaction force between the particles which depends on an interaction frequency $\nu$ (note: $F_0^\varepsilon(x,\omega,t)$ tends to align the particles to the direction $\Omega^\varepsilon$ which is the director of the particle flux $j^\varepsilon$).

The operators $\nabla_\omega$ and $\Delta_\omega$ are the gradient and the Laplace-Beltrami operators on the sphere. The matrix $(\text{Id} - \omega \otimes \omega)$ is the projection matrix onto the normal plane to the velocity director $\omega$. $\varepsilon$ is a very small parameter measuring the ratio of the microscopic length scale (the distance traveled between two interactions) to the size of the observation domain. The hydrodynamic limit $\varepsilon \to 0$ provides the large-scale dynamics of the CVA model [1].

This mean field model will be of great use in the description of the hydrodynamic limits and the derivation of the macroscopic model of the Couzin-Vicsek algorithm in chapter 3.
1.2 Thesis Objectives

The goals of this project are defined as follows:

- We will analyze numerically how the alignment of self-driven particles is affected by the model parameters such as the interaction radius, interaction frequency, time interval and noise intensity.

- We will analyze the behaviors of the self-driven particles subjected to different magnitude of noise and interaction radius in two and three dimensions.

- We will test numerically the stability of the time continuous discrete Couzin-Vicsek algorithm subjected to different model parameters.

- We will apply similar numerical simulations to the special case of Brownian motion on the unit circle (sphere) and analyze the behavior of the self-driven particles.

- We will compare the numerical results from the time continuous discrete Couzin-Vicsek algorithm with results from the case of Brownian motion on the unit circle or sphere and then suggest an improved time continuous discrete CVA-model which will always yield stable numerical solutions (solutions which satisfies the model assumptions for the velocity of the particles).

1.3 Thesis Outline

This project is organized as follows: This chapter contains the introduction. In chapter 2, we introduce the important definitions which are of great use throughout this project. These definitions will cut across a number of mathematical areas such as Vector calculus, Numerical analysis and Stochastic differential equations (SDE).

Chapter 3 is dedicated to self-driven particles models (SDPMs). First we present a brief introduction of hydrodynamic limits in relation to self-driven particles. We include a proof the macroscopic model of the Couzin-Vicsek algorithm. Next we give a derivation of a time continuous version of the discrete Couzin-Vicsek algorithm model.

In chapter 4, first we present the numerical results from the numerical simulations of the time continuous discrete Couzin-Vicsek algorithm in two and three dimensions, for a case with noise and without noise. Then using the numerical results, we discuss the how the different model parameters affect the alignment of the self-driven particles. Lastly, we analyze the stability of the model by comparing the numerical results from the model with results from the numerical simulations of the Brownian motion on the unit circle and sphere case.

In chapter 5, we give some conclusive remarks and some recommendations for the extension of this work.

All simulations in this project are carried out using MATLAB.
Chapter 2

Fundamental Theory

In this chapter, we describe in details some basic theories and definitions which used in this project. The first section is dedicated to theories on vector and tensor product of vectors and some spherical nomenclature (Gradient and Beltrami operators on the sphere). In the second section, we give a brief introduction to stochastic differential equations (SDE) and some numerical solution methods for SDE.

2.1 Tensor product of Vectors

Let $x$ and $y$ be elements of the three-dimensional (real) Euclidean space $\mathbb{R}^3$ [2]. In component form, we have $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$. Then the vector product of $x, y \in \mathbb{R}^3$ is defined as,

$$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)^T,$$

and the tensor product is defined as,

$$x \otimes y = xy^T = \begin{pmatrix} x_1y_1 & x_2y_1 & x_3y_1 \\ x_1y_2 & x_2y_2 & x_3y_2 \\ x_1y_3 & x_2y_3 & x_3y_3 \end{pmatrix}.$$

2.2 Spherical Nomenclature

Since in this project we are dealing with elements in a unit sphere (mostly the velocity director $\omega$), so first we define a unit sphere in three dimensions (since this will be very useful during the numerical simulations). Next we define the gradient and Laplace-Beltrami operators on the sphere.
2.2 Spherical Nomenclature

2.2.1 Unit Sphere in Three-Dimensions

In order to be able to introduce a system of triads on spheres, we define the vector function

\[ \Phi : [0, \infty) \times [0, 2\pi) \times [-1, 1] \rightarrow \mathbb{R}^3 \]

by

\[ \Phi(r, \varphi, t) = \left( \begin{array}{c} r \sqrt{1-t^2} \cos \varphi \\ r \sqrt{1-t^2} \sin \varphi \\ rt \end{array} \right) \]

Setting \( r = 1 \), we get a local coordinate system on the unit sphere \( \Omega \). This implies that instead of denoting any element \( \Omega \) by its vectorial representation \( \xi \), we may also use its coordinates \((\varphi, t)\). On calculating the derivatives of \( \Phi \) with \( r = 1 \), the corresponding set of orthonormal unit vectors in the directions \( r, \varphi, \) and \( t \) is determined to be,

\[ \varepsilon^r(\varphi, t) = \left( \begin{array}{c} \sqrt{1-t^2} \cos \varphi \\ \sqrt{1-t^2} \sin \varphi \\ t \end{array} \right) \] (2.1)

\[ \varepsilon^\varphi(\varphi, t) = \left( \begin{array}{c} -\sin \varphi \\ \cos \varphi \\ 0 \end{array} \right) \] (2.2)

\[ \varepsilon^t(\varphi, t) = \left( \begin{array}{c} -t \cos \varphi \\ -t \sin \varphi \\ \sqrt{1-t^2} \end{array} \right) \] (2.3)

Note that

\[ \varepsilon^t(\varphi, t) = \varepsilon^r(\varphi, t) \land \varepsilon^\varphi(\varphi, t). \]

where the vectors \( \varepsilon^r \) and \( \varepsilon^t \) mark the tangential directions [2]. Equations (2.1)-(2.3) are vector fields which define a moving orthonormal triad on the unit sphere \( \Omega \).

For our case, we use (2.1) for the numerical simulations whereby we take \( t = \cos \theta \) for all \( \theta \in [0, \pi] \) and \( \varphi \in [0, 2\pi) \) (where \( \theta \): latitude, \( \varphi \): longitude, \( t \): polar distance) [15].

2.2.2 Gradient and Laplace-Beltrami Operators on the Unit Sphere

By using the local coordinates, the gradient \( \nabla \) in \( \mathbb{R}^3 \) can be written as follows [15]:

\[ \nabla = \varepsilon^r \frac{\partial}{\partial r} + \frac{1}{r} \nabla^*, \] (2.4)

where \( \nabla^* \) is the surface gradient on the unit sphere \( \Omega \) and is defined as:
\(\nabla^* = \varepsilon\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi} + \varepsilon t \sqrt{1-t^2} \frac{\partial}{\partial t}\). \hfill (2.5)

The laplace operator \(\Delta\) in \(\mathbb{R}^3\) is defined as

\[\Delta_x = \nabla_x \cdot \nabla_x = \left( \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial x_2} \right)^2 + \left( \frac{\partial}{\partial x_3} \right)^2,\] \hfill (2.6)

Taking polar coordinates \(x = (x_1, x_2, x_3) = r\xi, r = |x|,\) and \(\xi \in \Omega,\) the laplace operator can be represented as [15]:

\[\Delta_x = \left( \frac{\partial}{\partial r} \right)^2 + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta^*_\xi,\] \hfill (2.7)

where the operator \(\Delta^*_\xi\) denotes the Laplace-Beltrami operator of the unit sphere \(\Omega,\) which is defined (in local coordinates) as

\[\Delta^*_\xi = \frac{\partial}{\partial t} (1-t^2) \frac{\partial}{\partial t} + \frac{1}{1-t^2} \left( \frac{\partial}{\partial \varphi} \right)^2.\] \hfill (2.8)

If we take \(t = \cos \theta,\) the (2.8) can be written as

\[\Delta^*_\xi = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta) \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta^2} \left( \frac{\partial}{\partial \varphi} \right)^2.\] \hfill (2.9)

### 2.3 Stochastic Differential Equations

Stochastic differential equations (SDE) models play a prominent role in a range of application areas, including biology, chemistry, epidemiology, mechanics, microelectronics, economics, and finance [5]. In this project, SDE are used to model the interaction of individual animal among animal societies in the presence of noise. In this section, first we give a brief introduction to Wiener process (Brownian motion) and stochastic integrals (Itô processes), then after we describe the numerical methods for simulating SDE (Euler-Maruyama Method and Milstein’s Higher Order Method). We assume that the reader is familiar with normally distributed random variables.

#### 2.3.1 Brownian Motion (Wiener Process)

A scalar standard Brownian motion, or standard Wiener process, over \([0, T]\) is a random variable \(W(t)\) that depends continuously on \(t \in [0, T]\) and satisfies the following three conditions [4, 5].

1. \(W(0) = 0\) (with probability 1).
2. For \(0 \leq s \leq T\) the random variable given by the increment \(W(t) - W(s)\) is normally distributed with mean zero and variance \(t-s;\) equivalently, \(W(t) - W(s) \sim \sqrt{t-s}N(0, 1),\) where \(N(0, 1)\) denotes a normally distributed random variable with mean zero and unit variance.
3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

For computational purposes it is useful to consider discretized Brownian motion, where $W(t)$ is specified at discrete time $t$ values. Thus we set $\delta t = T/M$ for some positive integer $M$ and let $W_j$ denote $W(t_j)$ with $t_j = j\delta t$. Condition (1) says $W_0 = 0$ with probability one. From conditions (2) and (3), we have:

$$W(t) - W(s) = \sqrt{t-s}N(0,1)$$

$$\Rightarrow W(t_j) - W(t_{j-1}) = \sqrt{t_j - t_{j-1}}N(0,1)$$

$$\Rightarrow W(t_j) - W(t_{j-1}) = dW_j$$

$$\Rightarrow W_j = W_{j-1} + dW_j, \quad j = 1, 2, \ldots, M,$$

(2.10)

where each $dW_j$ is an independent random variable of the form $\sqrt{\delta t}N(0,1)$ [5].

### 2.3.2 Stochastic Integrals

After introducing the concepts of Wiener process and discretized Brownian motion, now we can introduce the idea of stochastic integrals or Itô processes. Given a suitable function $h$, the integral $\int_0^T h(t)dt$ may be approximated by the Riemann sum

$$\sum_{j=0}^{M-1} h(t_j)(t_{j+1} - t_j),$$

(2.11)

where the discrete points $t_j = j\delta t$ as seen in the previous subsection for Brownian motion. The integral can be defined by taking the limit as $\delta t \to 0$ in (2.11) [5]. In a similar manner, we consider a sum of the form

$$\sum_{j=0}^{M-1} h(t_j)(W(t_{j+1}) - W(t_j)),$$

(2.12)

which by analogy with (2.11), may be regarded as an approximation to a stochastic integral $\int_0^T h(t)dW(t)$. Here, the integration of $h$ is carried out with respect to the Brownian motion. An alternative to (2.11) is given as

$$\sum_{j=0}^{M-1} h \left( \frac{t_j + t_{j+1}}{2} \right) (t_{j+1} - t_j),$$

(2.13)

which is also a Riemann sum approximation to $\int_0^T h(t)dt$. The corresponding alternative to (2.12) is
In the case where $h(t) \equiv W(t)$, the sum (2.14) requires $W(t)$ to be evaluated at $t = (t_j + t_{j+1})/2$. It can be shown that the addition of an independent increment $N(0, \Delta t/2)$ to $(W(t_{j+1}) + W(t_j))/2$ results into a value of $W((t_{j+1} + t_j)/2)$ that maintains the three conditions for the Wiener process listed above in section 2.3.1 [5].

The two stochastic Riemann sums (2.12) and (2.14) gives markedly different answers. Even for smaller $\delta t$ this mismatch or difference does not go away as $\delta t \to 0$. This highlights a significant difference between deterministic and stochastic integration. The 'left hand' sum (2.12) gives rise to what is known as the Itô integral, whereas the 'midpoint' sum (2.14) produces the Stratonovich integral [5].

### 2.3.3 The Euler-Maruyama Method

Using the Itô’s definition for stochastic integrals, a scalar autonomous SDE can be written in integral form as

$$X(t) = X_0 + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s), \quad 0 \leq t \leq T.$$  \hfill (2.15)

Here $f$ and $g$ are scalar functions and the initial condition $X_0$ is a random variable (which is normally distributed). The second integral on the right-hand side is to be taken with respect to the Brownian motion (or Wiener process) as discussed in the previously subsection and here we used the Itô version of integration [5]. The solution $X(t)$ is a random variable for each $t$. In order to avoid further explanation for $X(t)$ being a solution of (2.15), we will just define a numerical method for solving (2.15) and regard the solution $X(t)$ as the random variable that arises when we take the zero step size limit in the numerical method. It is usual to rewrite (2.15) as stochastic differential equation of the form

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T.$$  \hfill (2.16)

(2.16) is a compact way of saying that $X(t)$ solves (2.15). Note that, it is not correct to write $dW(t)/dt$, since Brownian motion is nowhere differentiable with probability 1.

If $g \equiv 0$ and $X_0$ is constant, then we have a deterministic problem and the expression (2.16) reduces to an ordinary differential equation [5].

To apply a numerical method to (2.16) over the time interval $[0, T]$, we first discretize the interval. Let $\Delta t = T/M$ for some positive integer $M$, and $t_j = j \Delta t$. The numerical approximation to $X(t_j)$ will be denoted by $X_j$.

The Euler-Maruyama (E-M) method takes the form,

$$X_j = X_{j-1} + f(X_{j-1}) \Delta t + g(X_{j-1})(W(t_j) - W(t_{j-1})), \quad j = 1, 2, \ldots, M.$$  \hfill (2.17)

To understand where (2.17) comes from, notice from the integral form (2.15) that

$$X(t_j) = X_{j-1} + \int_{t_{j-1}}^{t_j} f(X(s))ds + \int_{t_{j-1}}^{t_j} g(X(s))dW(s),$$  \hfill (2.18)
2.3 Stochastic Differential Equations

Each of the three terms on the right-hand side of (2.17) approximates the corresponding term on the right-hand side of (2.18). For the deterministic case \( g \equiv 0 \) and \( X_0 \) constant, (2.17) reduces to forward Euler method \([5]\). Note that E-M method has a strong order of convergence \( \gamma = 1/2 \) in comparison to the deterministic Euler method (converges with classical order 1).

The E-M method will be of great use in the numerical simulation of the time continuous discrete Couzin-Vicsek algorithm in the presence of the noise (see chapter 4).

2.3.4 Milstein’s Higher Order Method.

In order to raise the order of convergence of the E-M method say from \( \gamma = 1/2 \) to \( \gamma = 1 \), a correction is added to the stochastic increment, resulting into the Milstein’s method \([5]\).

The correction arises because the traditional Taylor expansion must be modified in the case of Itô calculus. Here a so-called Itô-Taylor expansion can be formed by applying Itô’s result, which is a fundamental tool of stochastic calculus \([5]\). Truncating the Itô’-Taylor expansion at an appropriate point produces Milstein’s method for the SDE (2.16):

\[
X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(t_j) - W(t_{j-1})) \\
+ \frac{1}{2}g(X_{j-1})g'(X_{j-1})((W(t_j) - W(t_{j-1}))^2 - \Delta t), \quad j = 1, 2, \ldots, M.
\]  

(2.19)

More about Milstein’s higher order method and stochastic differential equations, the reader is referred to \([4, 5]\).
Chapter 3
Self-Driven Particles Model (SDPM)

Self-driven particles (SDPs) are as described in [3], particles which are driven with a constant absolute velocity and at each time step they assume the average direction of motion of the particles in their neighborhood with some random perturbations or noise added. The velocities of these particles are determined by a simple rule (see section 1.1) and random fluctuations.

In this project, we refer to the term neighborhood in two and three dimensions as a circle and a sphere respectively. In other words, throughout this project, the term neighbor refers only to those particles inside the interaction neighborhood (circle or sphere of radius $R > 0$) of the particles in consideration including or excluding the particle itself.

This chapter is dedicated to building a complete continuum model which would account for all the important features for social interaction. One feature which is of great interest to us is the alignment of particles at short scales and attraction at large scales as it is addressed in the Couzin model [3].

This chapter plays a bigger role in the description of the major goals of this project, and it is organized as follows. In the first section, we give a brief introduction to hydrodynamic limits in relation to SDPs. Under this section, we also introduce a macroscopic model of the Couzin-Vicsek algorithm in terms of the mass and momentum equations [1]. The second section is devoted to derivation of a time continuous version of the discrete Couzin-Vicsek algorithm model.

3.1 Hydrodynamic Limits

Hydrodynamic limits were first developed in the framework of the Boltzmann theory of rarefied gases [1]. For a number of years, a lot of research has been carried out in this field (the reader can see [7, 11, 14, 16] for some current view points and major remarks in its mathematical theory).

Recently Hydrodynamics limits have been extended to a number of research areas like: traffic flows (traffic flow modeling), supply chain (supply chain research) and a number of other biological research areas such as; motion of locusts, schools of fish, flocks of birds and groups of migrating bacteria.
3.1 Hydrodynamic Limits

In this project, we look at hydrodynamic limits in relation to self-driven particles (SDPs). Here we base our discussion on the hydrodynamic limit $\epsilon \to 0$ which provides the large scale dynamics of the CVA model in its mean field version described by equations (1.1)-(1.3). To make a clear explanation of the hydrodynamic limits, we consider Theorem 3.1 below which is based on the mean field model given by equations (1.1)-(1.3) [1].

**Theorem 3.1** The formal limit as $\epsilon \to 0$ of $f_\epsilon$ is given by $f_0 = \rho(x,t)M_\Omega$ where $\rho(x,t) \geq 0$ is the total mass of $f_0$ and $\Omega = \Omega(x,t) \in S^2$ is the director of its flux. Here $\rho$ and $\Omega$ are defined as

\[
\rho(x,t) = \int_{\omega \in S^2} f_0(x,\omega,t) d\omega, \quad (3.1)
\]

\[
\Omega(x,t) = \frac{j(x,t)}{|j(x,t)|}, \quad j(x,t) = \int_{\omega \in S^2} f_0(x,\omega,t) \omega d\omega, \quad (3.2)
\]

$M_\Omega$ is a given function of $\omega \cdot \Omega$ which depends only on the interaction frequency $\nu$ and the scaled diffusion constant $d$. In addition $\rho(x,t)$ and $\Omega(x,t)$ satisfy the following system of first order partial differential equations:

\[
\frac{\partial \rho(x,t)}{\partial t} + \nabla_x \cdot (s_1 \rho(x,t) \Omega(x,t)) = 0, \quad (3.3)
\]

\[
\rho(x,t) \left( \frac{\partial \Omega(x,t)}{\partial t} + s_2(\Omega(x,t) \cdot \nabla)\Omega(x,t) \right) + \gamma (Id - \Omega(x,t) \otimes \Omega(x,t)) \nabla_x \rho(x,t) = 0, \quad (3.4)
\]

where $s_1$ and $s_2$ are convection speeds, $\gamma$ is the interaction constant.

Equations (3.3) and (3.4) describes the macroscopic model of the Couzin-Vicsek algorithm.

**Proof:** The proof is be based on the mean-field model given by equations (1.1)-(1.3). From the mean-field model, let the mean-field interaction force between the particles $F_\epsilon^x = F[f_\epsilon^x]$, where $f_\epsilon^x$ is the particle distribution function. This implies that

\[
F[f_\epsilon^x] = \nu(\omega \cdot \Omega^e)(Id - \omega \otimes \omega)\Omega^e(x,t), \quad (3.5)
\]

where

\[
\Omega^e(x,t) = \frac{j^e(x,t)}{|j^e(x,t)|}, \quad \text{and} \quad j^e(x,t) = \int_{v \in S^2} v f^e(x,v,t) dv. \quad (3.6)
\]

Also let

\[
Q(f) = -\nabla_\omega \cdot (F_0^e f^e) + d \Delta_\omega f^e = -\nabla_\omega \cdot (F[f]f) + d \Delta_\omega f, \quad (3.7)
\]

where $F[f]$ is defined by (3.5). The function $Q$ is known as the collision operator and the proof of the theorem is based on the study of this operator [6]. This operator $Q$ has got a number of properties, here we point out a few of these properties in summarized form. More details of the properties $Q$ can be found in [1].
Some properties of collision operator $Q$

First we look for the equilibrium solutions, in other words, we look for the function $f$ which cancels $Q$. Let $\mu = \cos \theta$ and denote by $\sigma(\mu)$ an antiderivative of $\nu(\mu)$, i.e. $(d\sigma/d\mu)(\mu) = \nu(\mu)$.

We define

$$M_\Omega(\omega) = C \exp(d^{-1}\sigma(\omega \cdot \Omega)), \quad \int M_\Omega(\omega) d\omega = 1. \quad (3.8)$$

where $C$ is a constant which is set by the normalization condition and it depends on $d$ and $\sigma$ but not $\Omega$.

**Lemma 3.1** (i) The operator $Q$ can be written as,

$$Q(f) = d\nabla \omega \cdot \left[ M_{\Omega[f]} \nabla \omega \left( \frac{f}{M_{\Omega[f]}} \right) \right], \quad (3.9)$$

and we have

$$H(f) := \int_{\omega \in S^2} Q(f) \frac{f}{M_{\Omega[f]}} d\omega = -d \int_{\omega \in S^2} M_{\Omega[f]} \left| \nabla \omega \left( \frac{f}{M_{\Omega[f]}} \right) \right|^2 d\omega \leq 0. \quad (3.10)$$

(ii) The equilibria, i.e. the functions $f(\omega)$ such that $Q(f) = 0$ form a three-dimensional manifold $\zeta$ given by

$$\zeta = \{ \rho M_\Omega(\omega) \mid \rho \in \mathbb{R}_+, \ \Omega \in S^2 \}, \quad (3.11)$$

and $\rho$ is the total mass while $\Omega$ is the director of the flux of $\rho M_\Omega$, i.e.

$$\int_{\omega \in S^2} \rho M_\Omega(\omega) d\omega = \rho, \quad (3.12)$$

$$\Omega = \frac{j[\rho M_\Omega]}{|j[\rho M_\Omega]|}, \quad j[\rho M_\Omega] = \int_{\omega \in S^2} \rho M_\Omega(\omega) \omega d\omega. \quad (3.13)$$

Further more, $H(f) = 0$ if and only if $f = \rho M_\Omega$ for arbitrary $\rho \in \mathbb{R}_+$ and $\Omega \in S^2$.

The flux $j[\rho M_\Omega]$ can be written as

$$j[\rho M_\Omega] = \int_{\omega \in S^2} \rho M_\Omega(\omega) \omega d\omega = \langle \cos \theta \rangle_M \rho \Omega, \quad (3.14)$$

where for any function $g(\cos \theta)$, the symbol $\langle g(\cos \theta) \rangle_M$ denotes the average of $g$ over the probability distribution function $M_\Omega$, i.e.

$$\langle g(\cos \theta) \rangle_M = \int M_\Omega(\omega) g(\omega \cdot \Omega) d\omega = \frac{\int_0^\pi g(\cos \theta) \exp \left( \frac{\sigma(\cos \theta)}{d} \right) \sin \theta d\theta}{\int_0^\pi \exp \left( \frac{\sigma(\cos \theta)}{d} \right) \sin \theta d\theta}. \quad (3.15)$$

Here $\langle g(\cos \theta) \rangle_M$ only depends on the scaled diffusion constant $d$ but not $\Omega$. 

3.1 Hydrodynamic Limits

The collision invariants of $Q$ are functions $\xi(\omega)$ such that

$$\int_{\omega \in S^2} Q(f) \xi d\Omega = 0, \quad \forall f. \quad (3.16)$$

These collision invariants are defined as

$$\xi(\omega) = (\Omega \times \omega) h(\Omega \cdot \omega). \quad (3.17)$$

for $h(cos\theta) = g(cos\theta)/\sin\theta$.

Here $\xi$ is the vector of generalized collisional invariant associated with the direction $\Omega$.

After looking at some properties of the collision operator $Q$, now we have got enough tools to prove the theorem, i.e. the limit as $\varepsilon \to 0$

Here we recall that $\varepsilon$ is a small parameter measuring the ratio of the microscopic length-scale to the size of the observation domain and by scaling we can define $\varepsilon$ as

$$\varepsilon = \frac{\text{microscopic time scale}}{\text{macroscopic observation time}},$$

so $\varepsilon \to 0$ implies that ($\text{microscopic time scale} \ll \text{macroscopic observation time}$).

We begin by assuming that all functions are regular as needed and that all convergences are as strong as needed [6]. Equation (1.1) can be rewritten after substituting for $Q(f)$ as

$$\varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon) + O(\varepsilon^2). \quad (3.18)$$

We assume that $\lim_{\varepsilon \to 0} f^\varepsilon = f$. From equation (3.18), we see that $Q(f^\varepsilon) = O(\varepsilon)$ and we deduce that $Q(f) = -\nabla_{\omega}. (F[f]f) + d\Delta_{\omega}f = 0$. Then $f = \rho M_{\Omega}$ (thanks to Lemma 3.1), with $\rho \geq 0$ and $\Omega \in S^2$. To find how $\rho$ and $\Omega$ depends on $(x, t)$, the generalized collision invariants are used. Here we first consider the constant collisional invariants, this simply means that we integrate equation (3.18) with respect to $\omega$.

$$\int_{\omega \in S^2} \varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon)d\omega = \int_{\omega \in S^2} (Q(f^\varepsilon) + O(\varepsilon^2))d\omega. \quad (3.19)$$

$$\Rightarrow \int_{\omega \in S^2} \varepsilon \partial_t f^\varepsilon d\omega + \int_{\omega \in S^2} \varepsilon \omega \cdot \nabla_x f^\varepsilon d\omega = 0$$

$$\Rightarrow \int_{\omega \in S^2} \partial_t f^\varepsilon d\omega + \int_{\omega \in S^2} \omega \cdot \nabla_x f^\varepsilon d\omega = 0$$

$$\Rightarrow \partial_t \int_{\omega \in S^2} f^\varepsilon d\omega + \nabla_x \int_{\omega \in S^2} f^\varepsilon d\omega = 0$$

$$\Rightarrow \int_{\omega \in S^2} f^\varepsilon d\omega + \int_{\omega \in S^2} f^\varepsilon \omega d\omega = 0$$

$$\Rightarrow \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0. \quad (3.20)$$

Which is a continuity equation with $\rho^\varepsilon$ as the density and $j^\varepsilon$ the flux. In the limit as $\varepsilon \to 0$, $\rho^\varepsilon \to \rho = \int_{\omega \in S^2} \rho M_{\Omega} d\omega$ and $j^\varepsilon \to j = \int_{\omega \in S^2} \rho M_{\Omega} \omega d\omega = \langle \cos \theta \rangle M_{\rho} \Omega = s_1 \rho \Omega$

where
\[ s_1 = \langle \cos \theta \rangle_M, \] 

so we end up getting the mass equation as:

\[ \partial_t \rho + \nabla_x \cdot (s_1 \rho \Omega) = 0. \] 

This proves equation (3.3).

Now we multiply equation (3.18) by \( \xi = h(\omega \cdot \Omega[f^\epsilon])(\Omega[f^\epsilon] \times \omega) \), and integrate with respect to \( \omega \)

\[
\int_{\omega \in S^2} \varepsilon(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon) h(\omega \cdot \Omega[f^\epsilon])(\Omega[f^\epsilon] \times \omega) d\omega \\
= \int_{\omega \in S^2} (Q(f^\epsilon) + O(\varepsilon^2)) h(\omega \cdot \Omega[f^\epsilon])(\Omega[f^\epsilon] \times \omega) d\omega,
\]

Taking the limit as \( \varepsilon \to 0 \), here we note that \( \Omega[f^\epsilon] \to \Omega \) and \( \xi^\epsilon \to \xi = h(\omega \cdot \Omega)(\Omega \times \omega) \) (since \( \xi \) is smooth enough).

In the limit as \( \varepsilon \to 0 \), the right-hand side of equation (3.23) equals zero. Then (3.23) can be written as:

\[
\int_{\omega \in S^2} (\partial_t f + \omega \cdot \nabla_x f) h(\omega \cdot \Omega)(\Omega \times \omega) d\omega = 0,
\]

On substituting for \( f = \rho M_\Omega \), we get

\[
\Omega \times \int_{\omega \in S^2} (\partial_t \rho M_\Omega + \omega \cdot \nabla_x \rho M_\Omega) h(\omega \cdot \Omega) \omega d\omega = 0,
\]

which can be written as:

\[
\Omega \times W = 0,
\]

where

\[
W = \int_{\omega \in S^2} (\partial_t \rho M_\Omega + \omega \cdot \nabla_x \rho M_\Omega) h(\omega \cdot \Omega) \omega d\omega.
\]

Equation (3.26) implies that the projection of \( W \) onto the plane normal to \( \Omega \) vanishes, i.e.

\[
(Id - \Omega \otimes \Omega) W = 0.
\]

Making equation (3.28) explicit, will lead us to the required equation (3.4). To achieve this, we start by looking at derivative of \( M_\Omega \) with respect \( \Omega \) acting on a tangent vector \( d\Omega \) to the sphere, i.e.
\[ \frac{\partial M}{\partial \Omega}(d\Omega) = \frac{\partial}{\partial \Omega} \exp \left( d^{-1} \sigma(\omega \cdot \Omega) \right)(d\Omega) \]
\[ = d^{-1} \frac{\partial}{\partial \Omega} \left( \sigma(\omega \cdot \Omega) \right)(d\Omega) \exp(d^{-1} \sigma(\omega \cdot \Omega)) \]
\[ = d^{-1} \frac{\partial}{\partial \Omega} \left( \sigma(\omega \cdot \Omega) \right)(\omega \cdot d\Omega) \exp \left( d^{-1} \sigma(\omega \cdot \Omega) \right) \]
\[ = d^{-1} \nu(\omega \cdot \Omega)(\omega \cdot d\Omega)M_{\Omega}. \]  

(3.29)

Then we deduce that
\[ \partial_t(\rho M_{\Omega}) = \partial_t(\rho C \exp(d^{-1} \sigma(\omega \cdot \Omega))) \]
\[ = M_{\Omega} \partial_t \rho + \rho C \partial_t(\exp(d^{-1} \sigma(\omega \cdot \Omega))) \]
\[ = M_{\Omega} \partial_t \rho + d^{-1} \rho \partial_t(\sigma(\omega \cdot \Omega))M_{\Omega} \]
\[ = M_{\Omega} (\partial_t \rho + d^{-1} \nu(\omega \cdot \Omega) \rho(\omega \cdot \partial_t \Omega)). \]  

(3.30)

and
\[ (\omega \cdot \nabla_x)(\rho M_{\Omega}) = (\omega \cdot \nabla_x)(\rho C \exp(d^{-1} \sigma(\omega \cdot \Omega))) \]
\[ = M_{\Omega} (\omega \cdot \nabla_x) \rho + \rho d^{-1} (\omega \cdot \nabla_x)(\sigma(\omega \cdot \Omega))M_{\Omega} \]
\[ = M_{\Omega} (\omega \cdot \nabla_x) \rho + \rho d^{-1} \nu(\omega \cdot \Omega) \omega \cdot ((\omega \cdot \nabla_x) \Omega)). \]  

(3.31)

Addition of equations (3.30) and (3.31) yields:
\[ \partial_t(\rho M_{\Omega}) + (\omega \cdot \nabla_x)(\rho M_{\Omega}) = M_{\Omega} \left[ \partial_t \rho + \omega \cdot \nabla_x \rho + d^{-1} \nu \rho(\omega \cdot \partial_t \Omega + \omega \cdot ((\omega \cdot \nabla_x) \Omega)) \right]. \]  

(3.32)

Now substituting equation (3.32) into (3.27), gives vector \( W \) as:
\[ W = \int_{\omega \in \mathbb{S}^2} \left[ \partial_t \rho + \omega \cdot \nabla_x \rho + d^{-1} \nu \rho(\omega \cdot \partial_t \Omega + \omega \cdot ((\omega \cdot \nabla_x) \Omega)) \right] \omega h M_{\Omega} d\omega. \]  

(3.33)

Which can be rewritten as:
\[ W = W_1 + W_2 + W_3 + W_4. \]  

(3.34)

where
\[ W_1 = \int_{\omega \in S^2} \partial_t \rho \omega h M_{\Omega} d\omega, \quad \text{(3.35)} \]
\[ W_2 = \int_{\omega \in S^2} (\omega \otimes \omega) \nabla_x \rho h M_{\Omega} d\omega, \quad \text{(3.36)} \]
\[ W_3 = d^{-1} \rho \int_{\omega \in S^2} ((\omega \otimes \omega) \partial_t \Omega) \nu h M_{\Omega} d\omega, \quad \text{(3.37)} \]
\[ W_4 = d^{-1} \rho \int_{\omega \in S^2} (\omega \otimes \omega \otimes \omega) (\nabla_x \Omega) \nu h M_{\Omega} d\omega. \quad \text{(3.38)} \]

After some computations using spherical coordinates \((\theta, \phi)\) associated with a Cartesian basis \((e_1, e_2, \Omega)\) where the vectors \(e_1\) and \(e_2\) are normal to \(\Omega\). We end up getting

\[ W_1 = 2\pi \partial_t \rho \int_0^\pi \cos \theta h M_{\Omega} \sin \theta d\theta d\Omega. \quad \text{(3.39)} \]
\[ W_2 = \pi \int_0^\pi \sin^2 \theta h M_{\Omega} \sin \theta d\theta (Id - \Omega \otimes \Omega) \nabla_x \rho \]
\[ + 2\pi \int_0^\pi \cos^2 \theta h M_{\Omega} \sin \theta d\theta (\Omega \cdot \nabla \rho) \Omega. \quad \text{(3.40)} \]
\[ W_3 = \pi d^{-1} \rho \int_0^\pi \sin^2 \theta \nu h M_{\Omega} \sin \theta d\theta (Id - \Omega \otimes \Omega) \partial_t \Omega \]
\[ + 2\pi d^{-1} \rho \int_0^\pi \cos^2 \theta \nu h M_{\Omega} \sin \theta d\theta (\Omega \cdot \partial_t \Omega) \Omega. \quad \text{(3.41)} \]
\[ W_4 = \pi d^{-1} \rho \int_0^\pi \sin^2 \theta \cos \theta \nu h M_{\Omega} \sin \theta d\theta ((Id - \Omega \otimes \Omega) : (\nabla_x \Omega)) \Omega \]
\[ + \pi d^{-1} \rho \int_0^\pi \sin^2 \theta \cos \theta \nu h M_{\Omega} \sin \theta d\theta (Id - \Omega \otimes \Omega)((\Omega \cdot \nabla) \Omega). \quad \text{(3.42)} \]

After computing the explicit integrals of \(W\), now we can rewrite equation (3.28) as

\[ (Id - \Omega \otimes \Omega) W = (Id - \Omega \otimes \Omega) (W_1 + W_2 + W_3 + W_4) = 0. \quad \text{(3.43)} \]

where

\[ (Id - \Omega \otimes \Omega) W_1 = 0. \quad \text{(3.44)} \]
\[ (Id - \Omega \otimes \Omega) W_2 = \pi \int_0^\pi \sin^2 \theta h M_{\Omega} \sin \theta d\theta (Id - \Omega \otimes \Omega) \nabla_x \rho. \quad \text{(3.45)} \]
\[ (Id - \Omega \otimes \Omega) W_3 = \pi d^{-1} \rho \int_0^\pi \sin^2 \theta \nu h M_{\Omega} \sin \theta d\theta \partial_t \Omega. \quad \text{(3.46)} \]
\[ (Id - \Omega \otimes \Omega) W_4 = \pi d^{-1} \rho \int_0^\pi \sin^2 \theta \cos \theta \nu h M_{\Omega} \sin \theta d\theta (\Omega \cdot \nabla) \Omega. \quad \text{(3.47)} \]
Using the notation in equation (3.15) of the average of a function over the probability distribution \( M_\Omega \), then we can rewrite equation (3.43) as:

\[
\langle \sin^2 \theta h \rangle_M (I_\lambda - \Omega \otimes \Omega) \nabla_x \rho + d^{-1} \rho \langle \sin^2 \theta \nu h \rangle_M \partial_t \Omega + d^{-1} \rho \langle \sin^2 \theta \cos \theta \nu h \rangle_M (\Omega \cdot \nabla) \Omega = 0.
\]  

(3.48)

Since the function \( h = h(\cos \theta) = g(\cos \theta)/\sin \theta \) is negative (by maximum principle), then we can define similar averages as in equation (3.15), substituting \( M_\Omega \) with \( \sin^2 \theta \nu h M_\Omega \) and denoting such averages as \( \langle g \rangle_{\sin^2 \theta \nu h M} \). Using these notations, we can rewrite equation (3.48) as:

\[
\langle \nu^{-1} \rangle_{\sin^2 \theta \nu h M} (I_\lambda - \Omega \otimes \Omega) \nabla_x \rho + d^{-1} \rho \partial_t \Omega + d^{-1} \rho \langle \cos \rangle_{\sin^2 \theta \nu h M} (\Omega \cdot \nabla) \Omega = 0. 
\]  

(3.49)

On rearranging the above expression, we get the momentum equation as:

\[
\rho (\partial_t \Omega + s_2 (\Omega \cdot \nabla) \Omega) + \gamma (I_\lambda - \Omega \otimes \Omega) \nabla_x \rho = 0,
\]  

(3.50)

where

\[
s_2 = \langle \cos \rangle_{\sin^2 \theta \nu h M},
\]  

(3.51)

\[
\gamma = d \langle \nu^{-1} \rangle_{\sin^2 \theta \nu h M}
\]  

(3.52)

On combining equations (3.22) and (3.50), we get the required macroscopic model of the Couzin-Vicsek algorithm as:

\[
\partial_t \rho + \nabla_x \cdot (s_1 \rho \Omega) = 0.
\]  

(3.53)

\[
\rho (\partial_t \Omega + s_2 (\Omega \cdot \nabla) \Omega) + \gamma (I_\lambda - \Omega \otimes \Omega) \nabla_x \rho = 0,
\]  

(3.54)

with convection speeds \( s_1 \) and \( s_2 \) given by equations (3.21) and (3.51) respectively, interaction constant \( \gamma \) given by equation (3.52). And this concludes the proof of Theorem 3.1.

The above theorem comes from the fact that the collision operator (right hand side of (1.1)) has got a three dimensional manifold of equilibria which is parameterized by the density \( \rho \) and the velocity director \( \Omega \) but has got only a one-dimensional set of collisional invariants which corresponding to mass conservation. In fact the interaction does no conserve momentum, hence no collisional invariant is related to this conservation [1]. To solve this problem, a broader class of collisional invariants is introduced, such that their integral (with respect to \( \omega \)) against the collision operator cancels only when the collision operator is applied to a subclass of functions [1]. In this case, a generalized class of collisional invariants is associated with each direction of \( \Omega \) on the sphere and the corresponding subclass of functions have their flux in the direction of \( \Omega \). Such a generalized collision invariants exists and they lead to (3.4). An important consequence of this result is that the large-scale dynamics of the CVA model does not present any phase transition, in fact the equilibrium is unique for a given density and velocity. Thus the model cannot exhibit any bi-stable behavior where shifts between competing equilibria would cause abrupt phase transitions. Instead the equilibrium gradually shifts from a collective one where all particles point in the same direction to an isotropic
one as the diffusion constant $\sigma$ increases from 0 to infinity. In addition the hyperbolicity of the model do not allow lines of faults across the unstable elliptic regions, as in the case of multiple-phase mixtures or phase transition in fluids or solids [1].

3.2 The Time Continuous Discrete Couzin-Vicsek Algorithm

The Couzin-Vicsek algorithm considers $N$ point particles in $\mathbb{R}^3$ labeled by $k \in 1, \ldots, N$ with positions $X^n_k$ at the discrete times $t^n = n\Delta t$ [3]. All particles have the same absolute velocity $c > 0$ and is constant in time. The velocity vector of the particles is written as $c\omega^n_k$ for $\omega^n_k$ an element in the unit sphere $S^2 = \{\omega \in \mathbb{R}^3, \text{ such that } |\omega|^2 = 1\}$ [1].

The Couzin-Vicsek algorithm (CVA) is a time-discrete algorithm were the velocities $\omega^n_k$ and the positions $X^n_k$ of the $k$-th particle are determined simultaneously at each time step $\Delta t$ [3]. These velocities and positions of the particles are updated according to the following rules.

- The position of the $k$-th particle at time $n$ is updated according to:

$$X^{n+1}_k = X^n_k + c\omega^n_k \Delta t \quad (3.55)$$

- The velocity director of the $k$-th particle, $\omega^n_k$, is changed to the director of the average velocity $\overline{\omega}_n^k$ of the neighboring particles with addition of noise (noise will account for the inaccuracies of the animal perception and cognitive systems). Note that this algorithm mimics the behavior of some animal species like fish, locust and birds, which tend to align with their neighbors. For the $k$-th particle, its neighborhood is a ball of positive radius $R$ centered at $X^n_k$ [1]. In our case, this neighborhood is a circle or a sphere of radius $R$.

The director of the average velocity $\overline{\omega}_n^k$ is given by:

$$\overline{\omega}_n^k = \frac{J^n_k}{|J^n_k|}, \quad J^n_k = \sum_{j \in \mathcal{N}_X^n_k} \omega^n_j, \quad |X^n_j - X^n_k| \leq R \quad (3.56)$$

The original Couzin-Vicsek algorithm considers a 2-dimensional space [3], here the orientations are vectors belonging to a unit sphere in $\mathbb{R}^2$. So we can write the velocity director of the particles as $w^n_k = \cos(\theta^n_k) + i \sin(\theta^n_k)$, where $\theta^n_k$ is defined modulo $2\pi$, in a similar manner, we can write the average velocity $\overline{\omega}_n^k = \cos(\overline{\theta}_n^k) + i \sin(\overline{\theta}_n^k)$ [1]. These phases can be updated by addition of a uniform noise on the small interval of angles $[-\eta, \eta]$ (where $\eta$ is a measure of the intensity of the noise), the phases takes the form

$$\theta^{n+1}_k = \overline{\theta}_n^k + \Delta \theta \quad (3.57)$$

where $\Delta \theta$ are independent identically distributed (iid) random variables with uniform distribution in $[-\eta, \eta]$. Thus the updated velocity director is given by, $\omega^{n+1}_k = \cos(\theta^{n+1}_k) + i \sin(\theta^{n+1}_k)$ [1].
A three dimensional version of the Couzin-Vicsek algorithm is considered (in this case the two-dimensional original version is taken as a particular case), and average velocity director $\bar{\omega}_k^n$ is still defined as in (3.56). In order to make the computation more simpler, a Gaussian noise (normally distributed noise) is considered instead of the uniformly distributed noise as in the original version of the CVA. In this case, the algorithm updates the velocity directors according to:

$$\omega_{k}^{n+1} = \hat{\omega}_k^n,$$  \hspace{1cm} (3.58)

where $\hat{\omega}_k^n$ are normally distributed random variables on the sphere centered at $\bar{\omega}_k^n$ with variance $\sqrt{2D\Delta t}$ where $D$ (the noise intensity) is a given coefficient [1].

If the normally distributed noise is discarded, the evolution of the orientations is given by

$$\omega_{k}^{n+1} = \bar{\omega}_k^n,$$  \hspace{1cm} (3.59)

where $\bar{\omega}_k^n$ is the average velocity director defined by (3.56).

To find a time continuous dynamics, we take the limit as $\Delta t$ tends to zero. First we consider the deterministic algorithm (3.55) and (3.59). Since each $\omega \in \mathbb{R}^3$ is an element in the unit sphere, it implies that $|\omega_k^n| = |\omega_k^{n+1}|$, from this fact, we have

$$(\omega_{k}^{n+1} - \omega_{k}^n)^2 = (\omega_{k}^{n+1} - \omega_{k}^n)(\omega_{k}^{n+1} + \omega_{k}^n) = 0.$$  \hspace{1cm} (3.60)

Defining $\omega_{k}^{n+1/2} = (\omega_{k}^{n+1} + \omega_{k}^n)/2$ and using equation (3.59), we get the relation:

$$\frac{\omega_{k}^{n+1} - \omega_{k}^n}{\Delta t} = \frac{1}{\Delta t}(\mathrm{Id} - \omega_{k}^{n+1/2} \otimes \omega_{k}^{n+1/2})(\bar{\omega}_k^n - \omega_{k}^n),$$  \hspace{1cm} (3.61)

where $\mathrm{Id} \in \mathbb{R}^{n \times n}$ denotes the identity matrix and the matrix $(\mathrm{Id} - \omega_{k}^{n+1/2} \otimes \omega_{k}^{n+1/2})$ is the orthogonal projector onto the plane orthogonal to $\omega_{k}^{n+1/2}$ [1].

By taking the limit as $\Delta t$ goes to zero, the orientations and the positions of the particles become continuous functions of time. Letting $\Delta t \rightarrow 0$ in (3.60) (using the formal definition of limits), the left-hand side tends to $\frac{\partial \omega_k}{\partial t}$. But the right hand side does not have an obvious limit. This is as a result of the improper choice of the time scale [1]. In fact taking the limit $\Delta t \rightarrow 0$ for the right-hand side of (3.60), we notice that the number of interactions per unit of time is infinite, so we should not expect to find anything meaningful if time is not rescaled. To be able to have a proper time scale for the model, we replace the time step $\Delta t$ with a typical interaction frequency $\nu$ of particles under consideration. For example in the case of fish, the time scale is $1/\nu$.

Thus we redefine (3.60) as:

$$\frac{\omega_{k}^{n+1} - \omega_{k}^n}{\Delta t} = \nu(\mathrm{Id} - \omega_{k}^{n+1/2} \otimes \omega_{k}^{n+1/2})(\bar{\omega}_k^n - \omega_{k}^n),$$  \hspace{1cm} (3.61)

where $\nu = \frac{1}{\Delta t}$.

Starting from a discrete algorithm defined by (3.55) and (3.61), then taking the limit as $\Delta t \rightarrow 0$, we obtain the following continuous dynamical system [1]:
3.2 The Time Continuous Discrete Couzin-Vicsek Algorithm

\[ \frac{dX_k}{dt} = c\omega_k, \quad (3.62) \]
\[ \frac{d\omega_k}{dt} = \nu(I - \omega_k \otimes \omega_k)\bar{\omega}_k, \quad (3.63) \]

where \((I - \omega_k \otimes \omega_k)\omega_k = 0\).

**Remark 3.1** If the Gaussian noise is retained, then we use equation (3.58) instead of (3.59), following the same steps, we obtain in the limit as \(\Delta t \to 0\) of the discrete algorithm the following Stochastic Differential Equation (SDE):

\[ \frac{dX_k}{dt} = c\omega_k, \quad (3.64) \]
\[ d\omega_k = (I - \omega_k \otimes \omega_k)(\nu\bar{\omega}_k dt + \sqrt{2D}dB_t), \quad (3.65) \]

where \(dB_t\) is Brownian motion with intensity \(\sqrt{2D}\).

Equations (3.62)-(3.65) describes the time continuous version of the discrete Couzin-Vicsek algorithm (time continuous Vicsek algorithm) model. This model is used for the numerical simulations in the next chapter. Since in the following chapter we will be looking at two different cases, i.e. with noise and without noise, so equations (3.62) and (3.63) will be used for the numerical simulations of the case without noise and equations (4.4) and (3.65), will be used for the numerical simulations of the noise case.
Chapter 4

Numerical Simulations of the Time Continuous Discrete CVA-Model

In this chapter, we present the results from the numerical simulations of the time continuous version of the discrete Couzin-Vicsek algorithm model described in the previous chapter (see chapter 3). All the numerical simulations in this chapter are based on equations (3.62)-(3.65) presented in the previous chapter.

In the first section, we present numerical results from the 2-dimensional case of the time continuous discrete CVA-model, here we look at three different cases. The first case is a two dimensional case without noise, under this, we analyze the behavior (like interaction and alignment) of the self-driven particles (SDPs) subjected to different model parameters such as the interaction radius $R$, interaction frequency $\nu$ and time interval $T$. The analysis of these particle behaviors, is based on the results from the numerical simulations of the velocities and positions of the particles subjected to different model parameters. The second case is a special case of a Wiener process (Brownian motion) on the unit circle. Numerical results from this case are used to analyze the stability of the time continuous discrete CVA-model. The third case, is the two dimensional case with noise, same steps are followed as in the case without noise. We also look at the stability of the time continuous discrete CVA-model, by comparing the numerical results from the model with numerical results from the special case of a Wiener process on the unit circle.

The second section is dedicated to the numerical simulations for the three dimensional case of the time continuous version of the discrete CVA-model. In this section we extend the numerical simulations from the three different cases in two dimensions to three dimensions. Except that in this section, the special case is that of Brownian motion on the unit sphere. Same analysis is carried out as in the two dimensional case.

4.1 Two-Dimensional Case

In this section, we look at three different cases, and in all these cases, the neighborhood of the $k$-th particle is a circle of radius $R > 0$ centered at $X^n_k$. When the noise is discarded, the velocity director of the k-th particle $\omega^n_k$ is taken to be uniformly distributed and the orientations are vectors belonging to the unit sphere $\mathbb{S}^1$ in $\mathbb{R}^2$. For the case of the white
noise, the velocity director of the k-th particle is taken to be normally distributed and we apply the Euler-Maruyama numerical method in the simulations of the noise (Gaussian noise) term.

### 4.1 Two-Dimensional Case

#### 4.1.1 Two-Dimensional Case without Noise

The equations (3.62) and (3.63) can be written in a discrete form as

\[
X_{k}^{n+1} = X_{k}^{n} + c\omega_{k}^{n} \Delta t, \tag{4.1}
\]

\[
\omega_{k}^{n+1} = \omega_{k}^{n} + \nu(Id - \omega_{k}^{n} \otimes \omega_{k}^{n})\bar{\omega}_{k}^{n} \Delta t. \tag{4.2}
\]

for \(k = 1, \ldots, N\), where \(\bar{\omega}_{k}^{n}\) is defined as in (3.56) and the symbol \(\otimes\) denotes the tensor product of vectors (note that \(\omega_{k}^{n} \otimes \omega_{k}^{n}\) can also be written as \((\omega_{k}^{n}) (\omega_{k}^{n})^T\)). We carried out the numerical simulations using equations (4.1) and (4.2), in all the simulations, we took the time step \(\Delta t = 0.005\) and the magnitude of velocity \(c = 0.03\) as in [3]. First we carried out the numerical simulations for the velocity of the self-driven particles (SDPs) subject to different model parameters. We started by fixing all the other model parameters like the interaction frequency \(\nu\) and the time interval \(T\), and varied only the interaction radius \(R\). In this case, we were able to observe clearly the effect of the interaction radius on the velocity and alignment or behavior of the self-driven particles. In Figure 4.1, we observe that in the absence of noise, the interaction radius \(R\) plays a big role in the alignment of the SDPs. For small interaction radius, we observe that the particles moves randomly in different directions, as a result the particles do not align with their neighbors. But as \(R\) is made large or increased we observe that the behavior of particles keeps on changing towards total alignment. A certain value of \(R\) (\(R_{align}\)) is attained where total alignment of SDPs is observed, in this case all the particles moves in one direction of the leader particle. Once the interaction radius \(R_{align}\) is attained, any value of \(R \geq R_{align}\), the alignment of SDPs is always observed. In other words for \(R \geq R_{align}\), the SDPs moves in the same direction of the leader particle. The reasons for the different observations can be explained as follows: 

\footnote{All numerical simulations for two-dimensional case without noise were carried out using the Matlab code nonoise2D.m.}
4.1 Two-Dimensional Case

If the interaction radius $R > 0$ is small, few particles are enclosed within this given neighborhood or circle of radius $R$. This implies that only the few particles which are enclosed within the interaction neighborhood have a chance of interacting among themselves. But since we are considering a big number of particles, it implies that the rest of the particles outside the interaction neighborhood, moves in different random directions. As a result the particles are not able to align (see Figures 4.1(a)-4.1(c)).

For large interaction radius we observe that the alignment of particles is achieved since all particles are enclosed in the given interaction neighborhood or circle of radius $R$. After some time all the particles moves in one direction following the leader particle (see Figures 4.1(d)-4.1(f)).

**Remark 4.1** Care has to be taken when selecting the time interval $T$, since the above observation of the alignment of SDPs for large $R$ can at times fail if a small time interval $T$ is consider. This effect will be addressed clearly in the next few paragraphs.
The interaction frequency $\nu$ was kept constant throughout the numerical simulations presented in Figure 4.1 (i.e. $\nu = 3$), normally this is not always the case. Due to this fact, next we analyze the effect $\nu$ on the velocity and alignment of the self-driven particles when all other model parameters are fixed. In Figure 4.2, one can see that if $\nu$ is small "say $\nu < 1$", the particles under consideration do not fully interact with their neighbors (thus the do not align) however much the interaction radius $R$ is made large (see Figures 4.2(a), 4.2(b) and 4.2(d)). This agrees with the theoretical findings since for $\nu$ small it implies that the particles do not frequently interact. As $\nu$ is made large, the alignment of SDPs is achieved, since for $\nu$ large means that the particles can frequently interact with other particles within a given neighborhood, thus making the alignment of these particles possible (see Figure 4.2(c)).

The time interval $T$ is another model parameter which affects the velocity and alignment of self-driven particles. For $T$ large, the particles tends to move in the same direction of the leader particle and they are able to align with their neighbors. But when $T$ is small 'say $T \leq 1$', the particles under consideration tends to move
4.1 Two-Dimensional Case

Figure 4.3: Velocities of SDPs for various values of the time interval $T$. Figures 4.3(a), 4.3(b) and 4.3(c), we observe that for small time interval, the particles do not align, but as $T$ is made large the alignment of SDPs is achieved (we used $R = 10$). In Figure 4.3(d), we observe that the particles do not align for smaller time intervals $T$ however much the interaction radius $R$ is made large (we used $T = 1$ and $R = 100$). In all simulations we took $\nu = 2.0$.

in different random directions (i.e. they do not align) irrespective of how large we make the other model parameters like the interaction radius (see Figure 4.3). This behavior is as expected because initially at $t = 0$, the directions of the velocities of the particles are randomly distributed. This implies that if the time interval is small, then by the time the observation of the behavior of the particles (SDPs) is taken, the particles are still in the state of random motion, so their is no possibility of aligning with their neighbors. But if the time interval is large, then chances are high that by the time the observation of the behavior of the particles is carried out, the particles have already attained an ordered kind of motion, so they are able to align with their neighbors. From these observations, we draw a conclusion that in the absence of noise, alignment of SDPs can be achieved by considering a big time interval $T$ of interaction of the particles. Note that this may not be the case in the presence of the noise as we will see in the next section.

**General observation**

In all the numerical simulations for the velocities of the SDPs, we observed that the velocities of the SDPs are always enclosed in the unit circle for all the different model parameters. This implies that in the absence of noise, the model assumptions for velocity of the particles
are always satisfied, i.e. if the noise is discarded, the time continuous discrete CVA-model yields stable numerical solutions.

Having discussed the concept of variation of velocity and alignment of self-driven particle with model parameters, now we focus on the numerical results for the positions of the particles (SDPs). As we are looking at a case without noise, the positions of the particles at any time will depend on the position of the leader particle (which of course we expect to be always ahead of the other particles). We observe from Figure 4.4 that all the particles start initially at the same point with zero displacement, thereafter, they move in different directions but their movement tend with increasing time to the direction of the leader particle. This behavior can be well observed in situations like a flock of birds moving from a given common point say a tree, at first they all fly in different directions, and after some time they move in one direction of the leader particle (leader bird in this case).

From the plot, we also observe that at any instant, the particles moves in an orderly manner, the displacements of the particles are always parallel to each other, this is as a result of the noise free environment.

Figure 4.4: Positions of the particles at different time steps over the time interval $T$, the positions of the particles in $y$ direction are plotted against the positions in $x$-direction. Here we took $N = 20$ since this would make the behavior of the particles easily observable. From the figure, we see that after some time all the particles moves in one direction of the leader particle.

All the cases (for velocity and position of SDPs) we have considered sofar, the white noise was discarded or ignored, but as we mentioned earlier in this section that we will look at a case with and without noise. The case without noise has been discussed above, next we look at the two different 2-dimensional cases with white noise. For simplicity we considered a normally distributed noise rather than a uniformly distributed noise as in the original version.
of the Couzin-Vicsek algorithm [3]. Numerical simulations are carried out using equations (3.64) and (3.65), Euler-Maruyama numerical method (see section (2.3.3)) is used for the simulation of the solution with Gaussian noise. To check the behavior of the particles and the stability of our model (the time continuous discrete CVA-model) in the presence of noise, we first consider a special case of a Wiener process on a unit circle presented in the next section.

4.1.2 Brownian Motion on the Unit Circle Case

In this section, we consider a special case of the Wiener process or Brownian motion on the unit circle (stated in definition 4.1). The numerical results obtained from the simulation of the velocities and positions of the particles, in this section are used to analyze the stability of the time continuous discrete CVA-model, by comparing them with the numerical results from the 2-dimensional cases with and without noise.

Definition 4.1 (Wiener process (Brownian motion) on the unit circle)

Let $B_t$ be a 1-dimensional Brownian motion and $g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, t) = e^{ix} = (\cos x, \sin x) \in \mathbb{R}^2$, for $x \in \mathbb{R}$. Also let $X_t = B_t$, then

$$Y(t) = g(t, X_t) = e^{iB_t} = (\cos B_t, \sin B_t) = (Y_1(t), Y_2(t))$$

is by Itô’s formula again an Itô process.

And the coordinates $Y_1$, $Y_2$ satisfy the system of equations

$$dY_1(t) = -\frac{1}{2} \cos(B_t)dt - \sin(B_t)dB_t$$

$$dY_2(t) = -\frac{1}{2} \sin(B_t)dt + \cos(B_t)dB_t$$

The process $Y(t) = (Y_1(t), Y_2(t))$ is a Wiener process (Brownian motion) on the unit circle and is the solution of the system of stochastic differential equations

$$dY_1(t) = -\frac{1}{2} Y_1dt - Y_2dB_t$$

$$dY_2(t) = -\frac{1}{2} Y_2dt + Y_1dB_t$$

or equivalently, of the vector stochastic differential equation

$$dY(t) = -\frac{1}{2} Y(t)dt + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y(t)dB_t$$

(4.3)

with a constraint that

$$|Y(t)|^2 = (Y_1(t))^2 + (Y_2(t))^2 = 1.$$  

An example of the Brownian paths for the Brownian motion on the unit circle are plotted in Figure 4.5 below;
4.1 Two-Dimensional Case

From the definition above, we consider $Y(t) = (Y_1(t), Y_2(t))$ as the initial velocity director in our numerical simulations.

In other words, we consider a case where $\omega_1 \sim Y_1$ and $\omega_2 \sim Y_2$, and then carry out the numerical simulations for for the velocities and positions of SDPs subjected to different model parameters (as we did for the case without noise).

In all the simulations, we use the stochastic differential equation;

$$\frac{dX_k}{dt} = c\omega_k, \quad (4.4)$$
$$d\omega_k = (Id - \omega_k \otimes \omega_k)(\nu\bar{\omega}_k dt) + \sqrt{2D}dY(t), \quad (4.5)$$

where $dY(t)$ is defined by equation (4.3).

First we investigate the effects of noise and interaction radius $R$ on the velocity and alignment of particles (SPDs). In order to have a clear analysis, we fixed first the noise intensity $D$ and carried out the numerical simulations for the velocity $\omega$ of SDPs for various values of the interaction radius $R > 0$. Thereafter, we varied both the noise intensity and the interaction radius.

For a fixed noise intensity $D$ (to remind the reader, here we consider the Gaussian noise), the behavior of particles (SDPs) is less affected by the magnitude of the interaction radius $R$, since for both small and large $R$, the particles tends to move in the same selected direction (see Figure 4.6), but still they do not align with their neighbors as in the case without noise (see Figure 4.1). This corresponds the theoretical finding, as it is well known that the presence of the noise affects the interaction and alignment of the self-driven particles. In all simulations, we observe that the behavior of the particles is the same for all different values of the interaction radius $R$. This implies that in the presence of noise, the interaction radius has got small effect on the behavior of the SDPs. Therefore we do not expect to achieve the alignment of SDPs by only varying the interaction radius $R$. From the plots, we also observe that the particles have got a V-shaped kind of motion (particles start at one point and they keep on moving in the same direction while spreading out with time). With this kind of motion, we can predict that after a long time the particles splits into small groups which
finally moves in different directions. In all cases, the velocity of the particles is always enclosed in the unit circle for all different values of \( R \) (stable system) and the magnitude of velocity of the particles is the same. This implies that the special case of Brownian motion on the unit circle always satisfies the model assumptions for velocity of the particles (i.e. \( \omega^n_k \) is an element in the unit sphere \( S^1 = \{ \omega \in \mathbb{R}^2, \text{ such that } |\omega|^2 = 1 \} \) and the magnitude of the velocity is the same for all particles and is constant in time). Here we also note that although the particles are traveling in small groups, the directions of the individual particles in general is not the same as that of the leader particle.

![Velocity vector for different values of \( R \)](image)

**Figure 4.6**: Velocities of SDPs for various values of the interaction radius \( R \). We observe that for different values of \( R \), the particles have got the same kind of behavior, they will always move in one selected direction but they do not align. We used \( T = 1.0 \), \( \nu = 2.0 \) and \( D = 0.1 \).

\(^2\)We carried out all the numerical simulations for Brownian motion on the unit circle case using the Matlab code Brownian2D.m.
On the other hand, variation of both the $D$ and $R$ yields some intersecting results. In Figure 4.7, we observe that for both noise and interaction radius chosen small, the particles tend to move in one selected direction, but they do not align. Although the particles moves in the same direction, but due to the presence of noise they keep on spreading out with time (i.e. moves in a V-shaped form). This predicts that after a some time, the particles forms small groups which ends up moving in different directions. The same behavior is observed even when the interaction radius $R$ is made large (see Figure 4.7(c)).

For large noise, the same behavior of particles is observed for both small and large interaction radius $R$. Compared to the case of small noise, here we see that the particles move more randomly but still in the V-shaped form (see Figures 4.7(b) and 4.7(d)). In general, we note that the interaction radius $R$ has a very small effect on the alignment or behavior of particles, since similar results are observed for a fixed noise intensity $D$ but varying interaction radius $R$.

![Figures 4.7: Velocities of SDPs for different values of $R$ and $D$. Figures 4.7(a) and 4.7(b), we fixed $R = 1$ and varied the noise intensity $D = 0.2$ and $D = 2$ respectively. Similarly for Figures 4.7(c) and 4.7(d) except that for this case we considered $R = 10$. We observe that the variation of the interaction radius $R$ has got a small effect on the behavior of particles. In all simulations we used $dt = 0.005$ and $\nu = 2.0$.](image)

The presence of the noise also affects the positions of the particles. In Figure 4.8, we observe that in the presence of noise, the particles tend to move in small groups, which implies that the particles do not follow the leader particle. These small groups moves in the same spontaneously selected direction, but they do not align as in the case without noise (see
4.1 Two-Dimensional Case

Figure 4.4). Since at each instant, the particles or small groups keeps on spreading out from each other (moving in a V-shaped form). This mimics the behavior of some animal species like locusts, and such kind of motion makes them more destructive, as it becomes more difficult to control their motion.

In general, similar conclusions can be drawn from the numerical simulations of the velocities and positions of the self-driven particles such as the V-shaped kind of motion.

![Figure 4.8](image)

(a) $R = 10$, $D = 0.1$

(b) $R = 10$, $D = 2.0$

Figure 4.8: *In this figure the positions of the particles are displayed for various values of noise. The number of particles $N = 100$ and the interaction radius $R = 10$ in each case. 4.8(a), for small noise, we observe that the particles tend to form small groups moving coherently in the same direction, here $D = 0.1$. 4.8(b), for higher noise, the particle behave in a similar way as in 4.8(a) but now with more randomness and with some correlation, here $D = 2.0$.*

4.1.3 Two-Dimensional Case with Noise

After looking at a special case of a Wiener process on the unit circle and observing the behavior of the SDPs subjected to different model parameters, we now return to the two dimensional case of the time continuous version of the discrete CVA-model with the Gaussian noise. The model has been solved numerically by implementing Euler-Maruyama numerical method that has been discussed in section 2.3.3. Still we focused on how the alignment of the SDPs (i.e measure of the average direction of motion of all the particles) is affected by the interaction radius $R$ and the magnitude of noise $D$.

First we look at the effect of the interaction radius $R$ on the velocity and alignment of particles (SDPs). From Figure 4.9, one can see that the presence of the Gaussian noise makes the alignment of the SDPs impossible. Here we see that for different values of the interaction radius, the behavior of the SDPs is almost the same, which implies that in the presence of the noise, $R$ has got a small effect of the alignment of particles. From these observations, we can conclude that, one can not achieve total alignment of SDPs by only increasing the interaction radius $R$.

In order to get some good results and to make sure that all particles have got the same magnitude of velocity, we took a large value of the interaction frequency ($\nu = 9.0$), and a
3 small noise intensity \((D = 0.1)\), but still with this small amount of noise, we never achieved the alignment of particles for all values of \(R\). This shows that the model is so sensitive to noise. Also we noted that taking \(\nu < 9.0\), the velocity of some SDPs was no longer always enclosed in the unit circle and this contradicts with the model assumptions for velocity of the SDPs. But this was not observed in the case of a Wiener process on the unit circle, since in this case, however much we varied all the model parameters, the velocity of the SDPs was always enclosed in the unit circle (see Figure 4.6). This leads us to impose a question on the stability of the time continuous discrete CVA-model.

![Image](image_url)

**Figure 4.9:** Velocities of SDPs for various values of the interaction radius \(R\). We observe that the model is so sensitive to noise, in this case even if we took a small noise, still the particles did not align however much the interaction radius was made large and in all the plots, we observe the same behavior of SDPs. In all the simulations we used \(N = 200\), \(D = 0.1\), and \(\nu = 9.0\).

---

3We carried out all the numerical simulations for the two-dimensional case with noise (Gaussian noise) using the Matlab code `noise2D.m`. 
Next we investigate what happens to the positions of the SDPs when different values of the interaction radius and noise are considered. Here we look at two cases, one with small interaction radius and the other case with large interaction radius. In both cases the magnitude of the noise was varied and we observed the behavior of the SDPs. These two cases are illustrated in Figure 4.10. From the plots, one can see that for small interaction radius and noise, the particles tend to form small groups which move randomly in different directions. And for high noise, we observe that the random motion of the SDPs increases in magnitude. For sure we do not expect the particles (SDPs) to align since the interaction radius is small and the noise is high.

Making the interaction radius large, we observe a decrease in the random motion of the particles, as we have already seen in the case without noise, that for large interaction radius $R$, the particles tend to align with their neighbors. In Figures 4.10(c) and 4.10(d), we observe the tendency of the self-driven particles to move in the same direction with time. For large $R$ and small noise, we observe the expected results of the tendency of the particles to attain an orderly motion (in this case after some time, the SDPs will tend to move in one selected direction). But here we noticed that it requires a longer time for all the particles to be able to move in the same selected direction. And making the Gaussian noise large makes this whole process take a longer time than the case for small noise.

In general, we note that the interaction radius $R$ plays a significant role in the alignment of particles both for small and high noise, but promising results are more observed in the case of small noise. Thus the motion of SDPs in one selected direction (ordered motion) can be achieved by setting the interaction radius $R$ large and taking a small noise (choosing small $D$), though one has to be a little bit patient since the total ordered motion of particles may take a longer time to be observed.
4.1 Two-Dimensional Case

In this figure, the positions of the particles are displayed for various values of interaction radius and noise. 4.10(a) For small interaction radius and noise the particles tend to move in random directions, here $R = 1.0$, $D = 0.1$. 4.10(b) For small $R$ and higher noise, the particles tend to move randomly but with some correlation, here $R = 1.0$, $D = 1.0$. 4.10(c) For large $R$ and small noise, after some time the particles tend to move in one selected direction, here $R = 10$, $D = 0.1$. 4.10(d) For larger $R$ and higher noise, particles tend to form small groups which tend to move in one selected direction with time, here $R = 10$, $D = 1.0$. In all the simulations we used $\nu = 2.0$.

4.1.4 Conclusions

In this section we give a summary of the numerical results obtained in the three different cases discussed above. We compare the numerical results from the time continuous discrete CVA-model with those obtained from the Brownian motion on the unit circle case. Furthermore, we analyze the stability of the model subjected to different model parameters, here we focus on the noise case of the model and the case of Brownian motion on the unit circle. To make a clear and simple analysis, we subject both cases to the same model parameters and observe the behavior of the particles.

First we focus on the velocities of the particles, here we check whether model assumptions for the velocity of the particles (i.e. all particles have got the same absolute velocity $c > 0$ and is constant in time, the velocity vector of the particles is written as $c\omega_k^n$ for $\omega_k^n$ an element
in the unit sphere $S^1 = (\omega \in \mathbb{R}^2, \text{ such that } |\omega|^2 = 1)$, is always satisfied for different model parameters.

Figure 4.11 shows the numerical results obtained from the the simulations of the time continuous discrete CVA-model and the Brownian motion on the unit circle case subjected to the same model parameters. From the plots, one can see that for $R = 10$ and $D = 0.1$, the time continuous discrete CVA-model yields results which contradicts with the model assumptions for the velocity of SDPs (in italics). Here, the magnitude of velocity is not the same for all particles and is not constant in time, also not all the velocity directors of the particles $\omega$ are enclosed in the unit sphere $S^1$ of $\mathbb{R}^2$ (see Figure 4.11(a)). But looking at the plot for the case of Brownian motion on the unit circle, we observe the model velocity assumptions are all well fulfilled (see Figure 4.11(b)).

Unfortunately even making the interaction radius $R$ large (from 10 to 100), did not yield better results for the time continuous discrete CVA-model, actually we observed the same behavior of the particles in both cases (for small and large $R$) as it can be seen in Figure 4.12. After observing that changing the interaction radius does not improve the numerical results or change the behavior of the SDPs for the time continuous discrete CVA-model, we fixed $R$ and varied other
4.1 Two-Dimensional Case

Figure 4.12: Velocities of the SDPs for the time continuous discrete CVA-model case and Brownian motion on the unit sphere case. In all plots, we used $R = 100$, $D = 0.1$, $N = 200$, $dt = 0.005$, and $\nu = 2.0$.

Figure 4.13: Velocities of the SDPs for the time continuous discrete CVA-model case and Brownian motion on the unit sphere case. In all plots, we used $R = 10$, $D = 0.1$, $N = 200$, $dt = 0.005$, and $\nu = 9.0$.

model parameters. One model parameter which yielded positive results was the interaction frequency $\nu$. By fixing $R = 10$, and $D = 0.1$, we observed that making $\nu$ large (for this case $\nu = 9.0$) yielded numerical results which satisfied the model assumptions for the velocity of the particles as it can be seen in Figure 4.13. These results where stable for different model parameters. Similar to the previous two cases, the case of Brownian motion on the unit circle yielded numerical results which fulfilled the model assumptions for the velocity of the particles. This implies that the model for the case of the Brownian motion on the unit circle is a stable model and can be used to check the cause of such unexpected behavior in the time continuous discrete CVA-model. After analyzing the two different model cases, i.e. the time continuous discrete CVA-model and the Brownian motion on a unit circle case, we noted that one of the causes of such differences in the numerical results, is the difference in the Brownian motion considered in both cases of the model. To make our explanation clear, we recall the stochastic differential equation given by Remark 3.1
\[ \frac{dX_k}{dt} = c\omega_k, \quad (4.6) \]
\[ d\omega_k = (Id - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt + \sqrt{2D}dB_t), \quad (4.7) \]

where \( dB_t \) is Brownian motion with intensity \( \sqrt{2D} \), and the stochastic differential equation stated in definition 4.1.

The process \( Y(t) = (\cos B_t, \sin B_t) = (Y_1(t), Y_2(t)) \) is a Wiener process (or Brownian motion) on a unit circle and is the solution of the system of stochastic differential equations

\[ dY_1(t) = -\frac{1}{2}Y_1 dt - Y_2 dB_t \]
\[ dY_2(t) = -\frac{1}{2}Y_2 dt + Y_1 dB_t \]

or equivalently, of the vector stochastic differential equation

\[ dY(t) = -\frac{1}{2}Y(t)dt + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y(t)dB_t \quad (4.8) \]

On comparing the stochastic differential equations (4.7) from the discrete CVA-model and (4.8) from the Brownian motion on the unit sphere, we see that in considering the Brownian motion in the time continuous discrete CVA-model, the first part of equation (4.8) was left out, only the second part was considered in the model. So since the solutions of equation (4.8) describe the Brownian motion on the unit circle, then it would be important to consider the whole of equation (4.8) in describing the Brownian motion in the time continuous discrete CVA-model rather than only considering the second part of equation (4.8). This would be a great improvement in the time continuous discrete CVA-model and it would result into numerical solutions which would satisfy the velocity assumptions for the model and thus leading to a stable time continuous discrete CVA-model. After this improvement in the model, the stable time continuous discrete CVA-model is given by the stochastic differential equation:

\[ \frac{dX_k}{dt} = c\omega_k, \quad (4.9) \]
\[ d\omega_k = (Id - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt + \sqrt{2D}dY(t)), \quad (4.10) \]

where \( dY(t) \) is defined by the vector stochastic differential equation defined by equation (4.8).

**Remark 4.2** For the case of Brownian motion on the unit circle or the improved time continuous discrete CVA-model (equations 4.9 and 4.10),

\[ |Y(t)|^2 = 1, \]

and

\[ |\omega|^2 = 1. \]
Proof of the remark
The proof of this remark, is based on Itô’s formula stated in the theorem below.

**Theorem 4.1 (Itô formula)**

Let \( X_t \) be a stochastic integral

\[
dX_t = u dt + v dB_t, \tag{4.11}
\]

and \( g(t, x) \) be a twice continuously differentiable function on \([0, T] \times \mathbb{R}\). Then the stochastic process

\[
Y_t = g(t, X_t) \tag{4.12}
\]

is a stochastic integral with

\[
dY_t = \frac{\partial g}{\partial t}(t, X_t) dt + \frac{\partial g}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2, \tag{4.13}
\]

where for computing \((dX_t)^2\) we use the following formula rules

\[
dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \text{ and } (dB_t)^2 = dB_t \cdot dB_t = dt. \tag{4.14}
\]

The proof of this theorem can be found in [4].

Using the above theorem (Itô formula), we start by proving that

\[
|Y(t)|^2 = 1. \tag{4.15}
\]

To prove this constraint \((|Y(t)|^2 = 1)\), its enough to prove that

\[
d(|Y(t)|^2) = 0.
\]

This can be proved as follow.

Let \( g(t, x) = x^2 \) and \( X_t = Y_t \), where \( Y_t = Y(t) \). Then

\[
F(t, Y_t) = g(t, Y_t) = Y_t^2.
\]

Then by Itô’s formula,

\[
dF(t, Y_t) = \frac{\partial g}{\partial t}(t, X_t) dt + \frac{\partial g}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2
\]

\[
= 0 + 2X_t dX_t + \frac{1}{2}(dX_t)^2
\]

\[
= 2Y_t dY_t + (dY_t)^2
\]

\[
= 0 + v^2 dt
\]

\[
= 2Y_t dY_t + v^2 dt.
\]
Hence
\[ d(Y_t^2) = 2Y_t dY_t + v^2 dt. \] (4.16)

We recall that
\[ dY(t) = \frac{1}{2} Y_t dt + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y_t dB_t. \]

Now substituting for \( dY_t = dY(t) \) into equation 4.16, we have,
\[
d(Y_t^2) = 2Y_t\left(-\frac{1}{2} Y(t) dt + K Y_t dB_t \right) + (KY_t)^2 dt, \text{ where } K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -Y_t \cdot Y_t & Y_t \cdot KY_t \\ 0 & 0 \end{bmatrix} = 0.
\]

Thus \( d(Y_t^2) = d(Y(t)^2) = 0 \), which implies that \( |Y(t)|^2 = 1 \).

Next we prove that \( |\omega|^2 = 1 \).

To prove that \( |\omega|^2 = 1 \), its enough to prove that \( d(\omega^2) = 0 \).

We recall that
\[
d \omega_k = (Id - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt) + \sqrt{2D} dY(t). \] (4.17)

Here we will drop the subscript \( k \) and just write \( \omega \) instead of \( \omega_k \).

We also recall that in the numerical simulations, we considered \( \omega_1 \sim Y_1(t) \) and \( \omega_2 \sim Y_2(t) \), this implies that \( \omega \sim Y(t) = Y_t \).

To prove that \( d(\omega^2) = 0 \), we follow the same steps as those we used when proving that \( d(Y_t^2) = 0 \).

Choose \( g(t, x) = x^2 \) and \( X_t = Y_t = \omega \), where \( Y_t = Y(t) \). Then
\[ F(t, \omega) = g(t, \omega) = \omega^2. \]

Then by Itô’s formula, we have
\[
d(\omega^2) = 2\omega d\omega + \left( \frac{d\omega}{\sqrt{2D Y_t}} \right)^2 dt. \] (4.18)

To avoid writing \( \sqrt{2D} \) ever time, we set \( D = 1/2 \), so that \( \sqrt{2D} = 1 \). So with this setting, then equation (4.18) can be rewritten as:
\[
d(\omega^2) = 2\omega d\omega + (KY_t)^2 dt. \] (4.19)
Now substituting for $d\omega$ (given by equation (4.17)) into equation (4.19), we have
\[
\begin{align*}
    d(\omega^2) &= 2\omega((Id - \omega \otimes \omega)(\nu\tilde{\omega}dt) + dY(t)) + (KY_t)^2 dt \\
    &= 2(Id - \omega \otimes \omega)\omega(\nu\tilde{\omega}dt) + 2\omega dY(t) + dt \\
    &= 2\omega dY(t) + dt \\
    &= 2\omega(-\frac{1}{2}Y(t)dt + KY(t)dB_t) + dt, \text{ where } Y(t) = \omega \\
    &= 2\omega(-\frac{1}{2}\omega dt + K\omega dB_t) + dt \\
    &= (-\omega \cdot \omega + 1)dt + 2\omega \cdot K\omega dB_t \\
    &= (-1 + 1)dt + 0dB_t \\
    &= 0.
\end{align*}
\]
Thus $d(\omega^2) = 0$, which implies that $|\omega|^2 = 1$. And this concludes the proof of Remark 4.2.

This analytic proof shows that $\omega_n^k$ will always belong to the unit sphere $S^1 = \{\omega \text{ such that } |\omega|^2 = 1\}$ of $\mathbb{R}^2$ or unit circle $S^1$ of $\mathbb{R}^2$. This is a nice condition for the improved time continuous discrete Couzin-Vicsek algorithm (Brownian motion on the unit circle case, defined by equations (4.9) and (4.10)), since it assures that the model always yields stable numerical solutions (solutions which always satisfies the model assumptions for velocity).
4.2 Three-Dimensional Case

The model we used to analyze the velocity and position of the self-driven particles in 3-dimensions has already been described in details in chapter 3. Just as a recall, the equations we used for the numerical simulations are:

\[
\frac{dX_k}{dt} = c\omega_k,
\]

\[
\frac{d\omega_k}{dt} = \nu(Id - \omega_k \otimes \omega_k)\bar{\omega}_k,
\]

in the case without noise and in the case of the Gaussian noise, we used the Stochastic Differential Equation (SDE):

\[
\frac{dX_k}{dt} = c\omega_k,
\]

\[
d\omega_k = (Id - \omega_k \otimes \omega_k)(\nu\bar{\omega}_k dt + \sqrt{2D}dB_t),
\]

where \(dB_t\) is Brownian motion with intensity \(\sqrt{2D}\) (see Remark 3.1).

The Euler-Maruyama numerical method is used for the simulation of the solution with Gaussian noise in this SDE.

As in the two-dimensional case, we first carried out the numerical simulations for the case without noise and then after look at what happens when the Gaussian noise is added to the system, thus making the analysis more easier.

In all cases, the neighborhood of the k-th particle is a sphere centered at \(X^n_k\) with positive radius \(R\), this kind of interaction neighborhood is a complicated shape, which makes the simulation a little bit challenging.

Before we carry out the numerical simulations, we point out that in all simulations the unit sphere is represented as in Figure 4.14.

![Figure 4.14](image)

(a) For time continuous CVA-model (b) For Brownian motion on the unit sphere

Figure 4.14: Plots for unit sphere which will be used in our numerical simulations for the velocities of the self-driven particles.
Figures as in Figure 4.14(a) are used for the numerical simulations for velocity of the SDPs for the time continuous discrete CVA-model, while figures as in Figure 4.14(b) for velocity simulations for the Brownian motion on the unit sphere case. These kind of representations helps us observe clearly the behavior of the velocities of SDPs inside the unit sphere. The green and pink dots should not be confused as being points representing the self-driven particles.

### 4.2.1 Three-Dimensional Case without Noise

In this section, we present the results from the numerical simulations of the three-dimensional case of the time continuous version of the discrete Couzin-Vicsek algorithm model without white noise (Gaussian noise). First, we carried out the numerical simulations for the velocity of the particles (SDPs) for different values of the interaction radius $R$ when other model parameters are kept constant (i.e. $\nu$, $T$, and $c$ are fixed). The velocities of the particles should always belong to a unit sphere $S^2 = (\omega, \text{ such that } |\omega|^2 = 1)$ of $\mathbb{R}^3$. Similar to the two dimensional case without noise, here we expect that for small interaction radius, there is no alignment of particles, but as the radius is increased to a certain value, the particles are able to interact with each other resulting into alignment, this implies that the particles tend to move in the same direction following the leader particle. Figure 4.15 illustrates the 3-dimensional plots for the velocities of the particles for different values of interaction radius $R$. In the plots, the different axes represents the components of velocity of the particles in x, y, and z directions. As we pointed out a few lines back, in Figure 4.15(a), one can see that for small interaction radius, the particles move randomly in different directions, so they do not align with each other since there is no full interaction of the particles due to the small neighborhood. In this case the particles do not follow the leader particle and even changing the other parameters does not improve the results. In Figures 4.15(b)-4.15(f), we observe that as the interaction radius $R$ is made large, the particles are able to align with their neighbors. From these figures, we also note that for $R \geq 1$, the interaction radius has got a small effect on the alignment of the particles, even when $R$ is squared, the alignment of SDPs is still preserved. Based on the above observations, we draw the conclusion that there exists some $R_{\text{min}}$ (minimum interaction radius) where the alignment of particles is achieved and any values of $R \geq R_{\text{min}}$ alignment of self-driven particles is always observed.

\footnote{We carried out all the numerical simulations for the three-dimensional case without noise using the Matlab code \texttt{Nonoise3D.m}.}
Figure 4.15: Velocities of the SDPs for different values of the interaction radius $R$. In all the simulations we used $N = 300$, $dt = 0.005$, and $\nu = 2$ (noise is discarded in this case).
After observing the behavior of the SDPs for different values of the interaction radius \( R \), we now consider a different case where we analyze the effect of the interaction frequency \( \nu \) on the velocity and the alignment of the SDPs. In this case we fix all the other model parameters including the interaction radius and vary only \( \nu \). It’s well known that to achieve total alignment of the SDPs, the particles should be able to interact with their neighbors. The more the particles are able to interact with their neighbors, the higher are the chances of alignment of these particles. The ability of these particles to interact, is equivalent to how frequent can the particles interact with their neighbors. In Figure 4.16 we observe that for small interaction frequency, the particles do not align with their neighbors even when the interaction radius \( R \) is large, this is as a result of the particles not being able to interact frequently with their neighbors. However if the interaction frequency is increased, we are able to achieve total alignment of the SDPs as particles can frequently interact with their neighbors.

In general, the alignment of SDPs is achieved by taking a large value of the interaction frequency \( \nu \), but if \( \nu \) is small, then the alignment of SDPs can’t be achieved even when a large value of \( R \) is chosen.

Now that the effect of the interaction radius and interaction frequency on the velocities and alignment of the particles (SDPs) is clear, we extend this idea to the positions of the particles, as it’s well known that the positions and velocities of particles move hand in hand according to the Couzin-Vicsek algorithm (CVA). Figure 4.17 illustrates the 3-dimensional plots for the positions of the particles in x, y, and z directions. The plots were carried out for different values of the interaction radius since our goal is to analyze the effect of the interaction radius on the positions and alignment of the particles. From the plots, we observe that for small interaction radius \( R \), the particles move in different random directions. This results from the fact that at any time \( t \) the neighborhood of the k-th particle (a sphere of radius \( R \)) is small, which implies that very few particles are enclosed in this neighborhood at any instant \( t \). So only those particles which are enclosed in the given neighborhood are able to interact with their neighbors, and since \( R \) is very small, we expect only a few particles to
be enclosed in this neighborhood. But since we are looking at a big number of particles, this implies that the other particles which are not enclosed in the given neighborhood are not be able to interact with the other particles in the same group. As a result, the particles end up moving in different random directions, and thus not being able to align with other particles (see Figure 4.17(a)). AS $R$ is increased, we observe similar behavior of the SDPs as those observed in the in the 2-dimensional case without noise, i.e. after some time the particles move in one direction following the leader particle, and the particles never cross into one another, their displacements are always parallel. This shows an orderly movement of the particles, which is as expected since the noise was discarded from the numerical simulations of the model.

![Figure 4.17](image)

Figure 4.17: In this figure, the positions of the particles are displayed for different values of interaction radius $R$. In all the simulations we used $\nu = 2.0$, $N = 20$ and $dt = 0.005$.

### 4.2.2 Three-Dimensional Case with Noise

In this section, we carry out the numerical simulations of the time continuous version of the discrete CVA-model with the Gaussian noise, for different model parameters. As we pointed out earlier, the Gaussian noise is considered in this case for simplicity, and this differs from the original version of the Couzin-Vicsek algorithm [3] where a uniformly distributed noise is considered. The Euler-Maruyama numerical method is used in this case for computing
the numerical solutions of the model. We are interested in the interaction and alignment of particles (SDPs) for different model parameters. First we analyze the velocity of the particles for different values of the interaction radius $R$ while other model parameters like noise are fixed. Afterwards, we look at the effect of the interaction radius and noise on the position and alignment of the self-driven particles.

In Figure 4.18, we present the numerical results for the velocities of the particles for varying values of the interaction radius $R$. The simulations are carried out with magnitude of velocity $c = 0.03$, time step $dt = 0.005$, number of particles $N = 300$, interaction frequency $\nu = 3.0$ and the Gaussian noise intensity $D = 0.01$. From the plots, one can see that the self-driven particles have got the same behavior for small and large values of the interaction radius $R$. This kind of behavior is similar to the one we saw in the 2-dimensional case of a Wiener process on the unit circle (see Figure 4.6). Here the particles tend to move always in the same selected direction, but they keep on spreading out with time (V-shaped kind of motion). This implies that after some time, the particles forms small groups which as time goes on are likely to separate from each other and end up moving in different directions. Even when the interaction radius $R$ is made large, we observe the same kind of behavior of particles. This implies that by only making $R$ large, we cannot achieve total alignment of the self-driven particles. One should keep in mind that here we took a small Gaussian noise, but still it had a great effect on the velocity and alignment of the particles. Therefore, here we draw a conclusion that the time continuous discrete CVA model is so sensitive to noise.

---

5 We carried out all the numerical simulations for the three-dimensional case with noise (Gaussian noise) using the Matlab code *Noise3D.m*. 
4.2 Three-Dimensional Case

Figure 4.18: Velocities of the SDPs for various values of the interaction radius $R$ (here the Gaussian noise is considered). From the plots, we observe that particles have got the same behavior ($V$-shaped kind of motion) for different values of $R$. In all the simulations we used $\nu = 3.0$ and $T = 5.0$. 
The effect of the Gaussian noise on the alignment of the SDPs can also be observed by looking at the positions of the particles in a given time interval $T$. It is well known that the introduction of the noise in a system may create some weird behavior, for our case we can observe this kind of behavior from the plots presented in Figure 4.19. From the plots we observe that for fixed interaction radius $R$, increasing the noise intensity completely makes the alignment of particles impossible.

For both $R$ and $D$ small, we observe that some particles tend to move in the same direction and others move randomly in different directions. But since the noise is small, after some time all the particles will tend to move in one selected direction (see Figure 4.19(a)). Increasing the magnitude of the noise for a fixed interaction radius $R$, gives the expected results of a more random kind of motion of the particles. In Figures 4.19(b) and 4.19(d), one can see that when the noise is increased, the particles (SDPs) tend to move in random directions but with some correlation. For example, in Figure 4.19(d), we see that for a large interaction radius, the particles tend...

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.19.png}
\caption{In this figure, the positions of the particles (SDPs) are displayed for different values of interaction radius $R$ and the noise intensity $D$. 4.19(a) For small interaction radius and noise, the particles tend to move in random directions, here $R = 1.0, D = 0.1$. 4.19(b) For small $R$ and higher noise, the particles moves in random directions but with some correlation, here $R = 1.0, D = 1.0$. 4.19(c) For large $R$ and small noise, after some time some of the particles tend to move in one selected direction, here $R = 100, D = 0.1$. 4.19(d) For larger $R$ and higher noise, the particles tend to move randomly but with some correlation and they will never follow the leader particle, here $R = 100, D = 1.0$. In all the simulations we used $\nu = 3.0$.}
\end{figure}
to move in the same direction, but due to the presence of a high noise \( D = 1.0 \), some of the particles tend to move randomly in different directions but with some correlation. The most interesting case is when the interaction radius \( R \) is large and the noise is small; in this case, most of the particles tend to move in one selected direction, but it takes a longer time for all particles to move in one direction. In other words, initially at \( t = 0 \), the particles moves randomly in different directions, but after some time (longer time) they will all move in one selected direction (see Figure 4.19(c)). This is as expected since there is a great influence of the large interaction radius, so the presence of the small noise creates an effect, but its magnitude is reduced with time. Thus resulting into all the particles moving in one spontaneously selected direction after some time.

### 4.2.3 Brownian Motion on the Unit Sphere Case

Similar to the two dimensional case where we considered the special case of the Brownian motion on the unit circle, in this section we extend the same idea to three dimensions, where we consider a special case of the Brownian motion on the unit sphere. This extension is not obvious as it seems, since the steps followed in this case are different from those considered in the two dimensional case. We start by defining the Brownian motion or Wiener process on the unit sphere.

**Definition 4.2 (Brownian motion on the unit sphere in \( \mathbb{R}^3 \))**

Let \( \psi: \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2 \) (where \( \mathbb{S}^2 \) is the unit sphere in \( \mathbb{R}^3 \)), be a function defined by

\[
\psi(x) = x \cdot |x|^{-1}; \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}.
\]

Applying the function \( \psi \) to a 3-dimensional Brownian motion \( B = (B_1, B_2, B_3) \), results into a stochastic integral \( Y = (Y_1, Y_2, Y_3) = \psi(B) = \psi(B_1, B_2, B_3) \) which by Itô’s formula is given by

\[
dY_i = \frac{|B|^2}{|B|^3} - \sum_{j \neq i} \frac{B_j B_i}{|B|^3} dB_j - \frac{B_i}{|B|^3} dt; \quad i = 1, 2, 3.
\]

Hence

\[
dY = \frac{1}{|B|} \cdot \sigma(Y) dB + b(Y) \frac{1}{|B|^2} dt,
\]

where

\[
\sigma(Y) = \frac{1}{|B|^2} \begin{bmatrix}
B_1^2 - |B|^2 & B_2 B_1 & B_3 B_1 \\
B_1 B_2 & B_2^2 - |B|^2 & B_3 B_2 \\
B_1 B_3 & B_2 B_3 & B_3^2 - |B|^2
\end{bmatrix} \in \mathbb{R}^{3 \times 3}
\]

and

\[
b(Y) = -\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = -\frac{1}{|B|} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \in \mathbb{R}^3
\]

Performing the following time change: We define

\[
Z_t(\eta) = Y_{\alpha(t, \eta)}(\eta)
\]
where
\[ \alpha_t = \alpha(t, \eta) = \beta_t^{-1}, \quad \beta(t, \eta) = \int_0^t \frac{1}{|B|^2} ds. \]

Then \( Z \) is an Itô process and we can rewrite equation (4.20) in terms of \( Z \) as
\[ dZ = \sigma(Z) d\tilde{B} + b(Z) dt, \]
where
\[ d\tilde{B} = \int_0^{\alpha_t} \sqrt{\frac{1}{|B|^2}} dB_s, \quad B_s \text{ is a 3-dimensional Brownian motion}. \]

This implies that \( Z \) is a diffusion with a characteristic operator
\[ \mathcal{A} f(y) = \frac{1}{2} \left( \Delta f(y) - \sum y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} - \sum_i y_i \frac{\partial f}{\partial y_i} ; |y| = 1. \]

Therefore \( \psi(B_{\alpha_t}) = B_{\alpha_t} \cdot |B_{\alpha_t}|^{-1} \) equals to a diffusion \( Z \) living on the unit sphere \( \mathbb{S}^2 \) of \( \mathbb{R}^3 \).

Since \( Z \) is also invariant under the orthogonal transformations in \( \mathbb{R}^3 \), then \( Z \) is called the Brownian motion on the unit sphere \( \mathbb{S}^2 \) in \( \mathbb{R}^3 \) [4].

An example of the discretized Brownian paths \((Y_{\alpha_t}^{(1)}, Y_{\alpha_t}^{(2)}, Y_{\alpha_t}^{(3)})\) for the Brownian motion on the unit sphere are plotted in Figure 4.20.

![Figure 4.20: An example of Brownian paths for Brownian motion on the unit sphere with \( dt = 0.005 \).](image)

For the numerical simulations, we consider a case where \( \omega_1 \sim Y_{\alpha_t}^{(1)}, \omega_2 \sim Y_{\alpha_t}^{(2)} \) and \( \omega_3 \sim Y_{\alpha_t}^{(3)} \).

The numerical results from this case, are be compared with the numerical results we obtained
from the three-dimensional case with Gaussian noise of the time continuous discrete CVA-model. One property which is of interest to us is the stability of the self-driven particles model (Brownian motion on the unit sphere case). We check this property by looking at the numerical simulations for the velocity of the particles for varying model parameters (i.e., we check the condition that \( \omega^n_k \) always belong to a unit sphere \( S^2 \) in \( \mathbb{R}^3 \)).

In the numerical simulations, we started by looking at the velocity of the SDPs for varying values of the interaction radius \( R \). Here we fixed all the other model parameters like the noise intensity \( D = 0.1 \), the interaction frequency \( \nu = 3.0 \), and varied only the interaction radius \( R \). By doing this we were able to analyze clearly the effects of the interaction radius on the velocity and the alignment of the particles.

From the plots in Figure 4.21, we observe almost the same behavior of the particles for different values of the interaction radius. In all cases, the particles tend to move in one selected direction but they do not align. This implies that the particles never follow the leader particle, instead they tend to form small groups which moves in the same selected direction. This kind of behavior is as expected since we are looking at the case with noise, so we do not expect to see total alignment of the self-driven particles. Therefore we cannot achieve total alignment of the particles by only varying the interaction radius \( R \), as similar behaviors are observed for both small and large values of \( R \).

The same behaviors are also observed in the 3-dimensional case of the time continuous discrete CVA-model with Gaussian noise \( (D = 0.01, \text{see Figure 4.15}) \). The difference comes in when we look at the stability of the two models. Here we noticed that for the case of the Brownian motion on the unit sphere, the velocities of the particles always belong to the unit sphere for different values of the model parameters. The magnitude of the velocity of the particles is always constant. This implies that the case of Brownian motion on the \( \text{6} \)

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\( ^6 \)We carried out all the numerical simulations for the special case of Brownian motion on the unit sphere using the Matlab code \texttt{Brown3D.m}. 

Figure 4.21: Velocities of the SDPs for various values of the interaction radius $R$. In all the simulations we used $N = 200$, $dt = 0.005$, $D = 0.1$, and $\nu = 3.0$. 
sphere is a stable model. But this is not always the case for the time continuous discrete CVA-model, since in some cases we experience situations were the particles have got different magnitudes of the velocity (the velocity of the particles is not always enclosed in the unit sphere). This contradicts with the model assumption that the velocity of the particles always belonging to the unit sphere. Here we draw a conclusion that the time continuous discrete CVA-model is not always stable.

The difference in stability of the two models can be observed in Figure 4.22, from the plots, one can see that for same magnitude of noise and interaction radius, the velocities of the SDPs for the time continuous discrete CVA-model are not always enclosed in the unit sphere as it is stated in the model assumptions (see Figure 4.22(a)). And this is not the case for Brownian motion on the unit sphere case (see Figure 4.22(b)). The reason for this kind of behavior of the time continuous discrete CVA-model is the same as in the 2-dimensional case (see section 4.1.4).

![Figure 4.22](image.png)

(a) Time continuous discrete CVA-model  
(b) Brownian motion on a unit sphere case

Figure 4.22: In this figure, the velocities of the particles are displayed for two different models but same model parameters. In both figures we used $R = 10$ and $D = 0.1$.

Finally we look at the numerical simulations for the positions of the self-driven particles for different values of the interaction radius $R$ and noise intensity $D$. We carried out the simulations in such a way that we are able to analyze the effects of noise on the behavior of the particles for a fixed values of the interaction radius $R$ (small $R$ and large $R$, in both cases we considered small and high noise). In Figure 4.23, we observe similar behavior of the SDPs as in the previous case for the velocities of the particles. One direct observation from the plots is that in the presence of the noise, the interaction radius $R$ has got a small effect on the alignment of particles, this agrees with velocity case seen previously. The difference comes in when the noise of different magnitudes is considered, we observe that for higher noise, the particles tend to move more randomly than the case with small noise. But in both cases, we note that the particles never align completely and this is as expected since noise is being considered in our simulations. Generally the behaviors of the SDPs are quite similar to those observed in the case of the time continuous version of the time continuous discrete CVA-model (see Figure 4.19). From these observations, we conclude that in the presence of the noise (of any magnitude), we cannot achieve the total alignment of SDPs by only making
the interaction radius $R$ large, we need to think of varying other model parameters such as the interaction frequency $\nu$ and time interval $T$ if our goal is to achieve total alignment of particles (SDPS).

Figure 4.23: In this figure, the positions of the particles are displayed for different values of interaction radius and noise. In all simulations we used $\nu = 3.0$.

This concludes our discussion for the numerical simulations of the time continuous discrete CVA-model. The improved model discussed in this chapter (see section 4.1.4) will always yield stable numerical solutions compared to the original time continuous discrete CVA-model presented in [1].
Chapter 5

Conclusions

In this chapter, we look back on the accomplishments of this project and evaluate which part of goals have been achieved. Furthermore, we will discuss about the possible directions of the future work. Before we present the achievements, lets first review the project goals, which can be simplified as follows:

- analyze numerically how the alignment of self-driven particles is affected by the model parameters.
- analyze the behaviors of the self-driven particles subjected to different magnitude of noise and interaction radius in two and three dimensions.
- test numerically the stability of the time continuous discrete Couzin-Vicsek algorithm subjected to different model parameters.
- compare the numerical results from the time continuous discrete Couzin-Vicsek algorithm with results from the case of Brownian motion on the unit circle or sphere.
- suggest an improved time continuous discrete CVA-model.

5.1 Achievements

In Chapter 4, we carried out the numerical simulations for velocities and positions of the self-driven particles using the time continuous discrete Couzin-Vicsek algorithm both in 2-dimensions and 3-dimensions. In both cases (2-dimensions and 3-dimensions cases), we carried out the numerical simulations for the case with and without noise for different model parameters.

For the case without noise, from the numerical simulations, we discovered that the magnitude of the interaction radius $R$ greatly affects the alignment of particles. Here we found out that for small interaction radius, the alignment of the self-driven particles is not possible, but as $R$ is made large, the alignment of particles is achieved. This implies that there exists a minimum value of the interaction radius ($R_{min}$) of which the particles are able to align, and any values of $R \geq R_{min}$ the alignment of the self-driven particles is observed. Similar behavior was observed for various values of the interaction frequency $\nu$ and the time interval.
In all the simulations for the no noise case, the model yielded numerical solutions which fulfilled the model assumptions for the velocity of the particles ($\omega_n^a$ was always enclosed in the unit circle or sphere and the magnitude of velocity was always constant in time).

In the presence of the noise (Gaussian noise), we looked at two different cases. First, we looked at the simulations of the time continuous discrete CVA-model for different model parameters. From the numerical simulations, we observed that in the presence of the Gaussian noise, the interaction radius has got a small effect on the behavior of the self-driven particles. In this case, the alignment of particles cannot be achieved by only varying the interaction radius, as both small and large values of $R$ yielded the same particle behavior (for all different values of $R$, the particles had a V-shaped kind of motion). The variation of $R$ did not always yield stable or consistent numerical solutions, at times we were faced with situations where the numerical solutions for velocities of the particles were all not enclosed in the unit circle or sphere, this violated the model assumptions for velocity. This problem was solved by taking a small noise and a large value of the interaction frequency $\nu$. We noted that, even though the velocity assumption was always fulfilled with these settings, the alignment of particles was not achieved, as particles moved in a V-shaped form. In order to achieve the alignment of particles, we had to consider a small noise intensity $D$, a large interaction frequency $\nu$, a large interaction radius $R$ and a large time interval $T$.

Then we also looked at the special case of Brownian motion on the unit circle (for 2-dimensional case) and unit sphere (for 3-dimensional case), the numerical solutions from this case always yielded solutions which fulfills the model assumptions for velocity of the particles. As in the noise case of the time continuous discrete CVA-model, the interaction radius $R$ had a small effect on the alignment or behavior of particles, since the particles always moved in the V-shaped form.

We compared the numerical results from the model (time continuous discrete Couzin-Vicsek algorithm) with results from the case of Brownian motion on the unit circle (sphere) subject to the same model parameters. From the comparison, we conclude that the time continuous discrete Couzin-Vicsek algorithm does not always yield stable numerical solutions (solutions which satisfy the model assumptions for velocity), stable numerical solutions were only observed under some conditions on the model parameters such as taking a small noise and a large interaction frequency. This was not the case for the Brownian motion on the unit sphere case, as it always yielded stable numerical solutions for different values of the model parameters.

After the above conclusions, we suggest the model for Brownian motion on the unit circle (sphere) case as an improved model for the time continuous discrete Couzin-Vicsek algorithm, which will always yield stable numerical solutions or solutions which always satisfy the model assumptions for the velocity of the self-driven particles.

## 5.2 Future Work

Throughout the project, we carried out the numerical simulations of the time continuous discrete CVA-model, by only considering a few model parameters such as the interaction radius, interaction frequency, time interval and noise intensity. But basing on the literature, we left
out a number of parameters which affect the alignment of self-driven particles. For example, one major parameter which was left out is the density $\rho$ of the particles, it is predicted in theoretical physics that as the density of animals or self-driven particles increases, transition occurs from disordered movement of individuals in a group to highly aligned collective motion. Including the density into the model would mean that we need to switch from solving the time continuous discrete Couzin-Vicsek algorithm to solving the macroscopic model of the Couzin-Vicsek algorithm (see chapter 3).

Investigations on how to solve the macroscopic model of the CVA either analytically or numerically can be done in the future and then a comparison of the solutions from the two models to see whether they have some correlation.

In this project we also pointed out that their exists a minimum interaction radius ($R_{\text{min}}$) where the alignment of particles is observed, but we were not specific on the actual value of $R_{\text{min}}$. The same thing also applies to the intensity of the noise, because in [3], Vicsek et al demonstrated experimentally that their exists a threshold value $\eta^*$ ($\eta$ being the measure of the intensity of the noise), such that for $\eta \leq \eta^*$, a coherent dynamics appears after some time where all the particles are nearly aligned and if $\eta > \eta^*$, then disorder occurs at all times. In future it would be more interesting to specify the exact values of $R_{\text{min}}$ and $\eta^*$ especially for the improved time continuous discrete Couzin-Vicsek algorithm (Brownian motion on the unit circle (sphere) case). These values can be computed either experimentally or numerically using the model.
Bibliography


Declaration

I hereby declare that I have done this thesis on my own and the no other sources than those listed in the bibliography have been used.

August 24, 2009, Kaiserslautern, Germany

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