Analysis of a Stochastic Inventory Policy with Limited Transportation Capacity

By

Said Dabia

Student identity number 0585422

In partial fulfillment of the requirements for the degree of

Master of Science in Operations Management and Logistics

Supervisors:
Dr. G.P. Kiesmüller, TU/e, OPAC
Dr. Ir. N.P. Dellaert, TU/e, OPAC
Preface

This report is the result of my graduation project which is the final stage of the two years master program Operations Management and Logistics at the Technische Universiteit Eindhoven, Eindhoven, The Netherlands. The project took place during the period from October 2006 to March 2007 within the Operations, Accounting, Control and Planning (OPAC) group of the Technology Management department at the Technische Universiteit Eindhoven. The OPAC capacity group offered me an interesting and challenging environment. Things that have always been fuelling my motivation.

The completion of this project would not have been successful without the help of many others. I would like to thank Gudrun Kiesmüller for accepting to be my first and daily supervisor. Your precious advices and valuable ideas have been very crucial to this project. You have always had time when I knocked at your door. You have been a very good listener, our discussions have been very fruitful. I would like to express my special thank to Nico Dellaert for willing to be my second supervisor. Your remarks and critics have been very useful. I enjoyed and learned a lot from our discussions. I also would like to thank my future colleagues within the OPAC group for creating such a friendly and cheerful environment.

Finally, I would like to thank my family and friends for continuously supporting me and believing in me. My special thanks to my parents and my wife Souhaila (Mshisha!) for loving me.

Said Dabia

Eindhoven, March 2007
Abstract
In this paper we consider a stochastic single-item inventory problem. A retailer keeps a
single product on stock to satisfy customers stochastic demand. The retailer is replen-
ished periodically from a supplier with ample stock. For the delivery of the product, a
truck with finite capacity is available. The lead time is considered to be equal to zero
and a fixed shipping cost is charged whenever a truck is dispatched regardless of its load.
Furthermore, linear holding and backorder costs are considered at the end of a review
period. A policy is proposed to determine replenishment quantities taking into account
transportation capacity. In fact, at each order instant it is decided on how much to order,
then the initial order size might be reduced as well as enlarged to create full truckloads.
We illustrate that the proposed policy is close to the optimal policy. Moreover, we show
how to compute the policy parameters exactly and by relying on heuristics. In a detailed
numerical study, we compare the results obtained by the heuristics with those given by
the exact analysis and the order-up-to level policy. A very good cost performance of the
proposed heuristics is observed.

Keywords
Stochastic inventory control • Transportation capacity • Full truckloads • Markov chain
Summary

In many practical situations inventory and transportation decisions are significantly correlated. On the one hand, transportation managers aim at low transportation costs. Therefore, few large shipments with highly utilized trucks are required to benefit from economics of scale. In order to obtain highly utilized trucks, products can be shipped earlier than required, leading to an increased inventory level and higher inventory holding costs. Alternatively, orders may be delayed until a certain truck utilization is reached resulting in more backorders or higher safety stocks. On the other hand, inventory managers aim at low inventory costs which may require many small shipments with probably low truck utilization, and hence high transportation cost. Transportation (ordering) capacity is an important issue that has been scarcely considered. In fact, it can happen that the optimal order quantity is much larger than the capacity of a truck, and that several trucks are needed for the shipment of an entire order. Hence, transportation costs may be much larger than assumed. Therefore, it is important to model the above trade-off carefully in order to minimize total cost (inventory and transportation costs).

In case of periodic inventory systems with unlimited ordering capacity and a fixed ordering cost, Scarf (1960) has proven that the optimal policy is an \((s, S)\)-type policy; at the beginning of each period, when the inventory position (stock on-hand minus backorders plus outstanding orders) drops to or below the level \(s\), enough is ordered to raise the inventory position up to the level \(S\). Federgruen and Zipkin (1986) consider a capacitated inventory system with no ordering costs. They have proven the optimality of the modified base stock policy; if there is enough capacity, order up to \(S\), otherwise, order as much as possible. In the case of capacitated inventory systems with fixed ordering costs, the optimal policy is not that straightforward. Shaoxiang et al. (1994) consider a single-item, periodic review inventory system with a limited ordering capacity and a fixed ordering cost. They have shown that the optimal policy, for the finite horizon case, has a system-
atic pattern, they call it the $X - Y$ band structure. The $X - Y$ band structure works as follows. Whenever the inventory position drops below level $X$, an order up to capacity takes place; when the inventory position exceeds level $Y$, no action is taken. When the inventory position is between $X$ and $Y$, the order quantity is different from state to state and no specific structure seems to be optimal. However, Gallego et al. (1998) have shown that the $X - Y$ band structure can be characterized with a four regions structure. Nevertheless, when the inventory position falls between $X$ and $Y$, the optimal decisions can differ from case to case. Chan et al. (2003) focus on the region between $X$ and $Y$. They provide some properties of this region and develop an efficient algorithm that allows the computation of the optimal ordering quantities. However, none of the above mentioned papers provides easy formulas to compute the values of $X$ and $Y$.

In this paper, similarly to Shaoxiang et al. (1994), a stochastic single-item, periodic review inventory system is considered, where a truck with fixed and finite capacity is used to ship the orders. A fixed transportation cost is charged as well as linear holding and backorder costs at the end of a period. We present a simpler and similar policy determined by the parameters $(S, Q_1, Q_2)$. The policy is similar in the sense that it has two thresholds $Q_1$ (the waiting threshold) and $Q_2$ (the full truckload threshold), and simpler in the sense that the region between $Q_1$ and $Q_2$ is a simple order-up-to level policy with order-up-to level $S$. A more detailed explanation of this policy is given in section 3. Furthermore, we provide relatively simple methods and heuristics to compute optimal and near optimal policy parameters and we show that the performance of this simple policy is optimal in many cases, and very close to the optimal in some other cases.

In section 1, a general introduction is presented as a to motivate this research. In section 2, we describe the problem. The assumptions are cited and the variables to be used are defined. Section 3 is devoted to the explanation of the proposed policy and the policy parameters are defined. In section 4, the formulation of the mathematical model enabling the computation of the optimal policy parameters is developed. The system is
modelled as discrete time Markov chain. In section 5, the optimal decisions are computed by solving a dynamic programming problem. The optimal decisions are compared with the proposed policy. The numerical study shows that the proposed policy is optimal in many cases. In section 6, two heuristics are presented as an alternative for computing the policy parameters. Relatively easy methods are developed to compute the policy parameters. In section 7, the results obtained from an extensive numerical study are shown. Finally, section 8 concludes this paper with a summary of the main results.
Contents

List of Figures 8

List of Tables 8

1 Introduction 9

2 Problem description 11

3 The \((S, Q_1, Q_2)\) replenishment policy 13

4 Model 15

4.1 Model description . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15

4.2 Model analysis: the exact analysis . . . . . . . . . . . . . . . . . . . . . . . 15

4.2.1 The decisions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

5 The optimal decisions 18

6 Heuristics 21

6.1 The S-Heuristic . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22

6.2 The SQ-Heuristic . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24

7 Numerical study 29

8 Summary and Conclusions 32
List of Figures

1  The $(S, Q_1, Q_2)$ policy. ......................................................... 13
2  The demand distributions .......................................................... 20
3  $X^*$ as a function of $A$, $h$, $p$, and $V$. ................................. 27
4  Sensitivity analysis for the optimal policy. ................................. 28

List of Tables

1  The ordered quantities at the beginning of a period ...................... 19
2  Parameter set for the numerical example in Table 1 .................... 20
3  Comparison of the $(S, Q_1, Q_2)$ policy and the optimal decisions .. 21
4  Results for the uniform distribution ........................................... 30
5  Results for the linear positive distribution .................................. 31
6  Results for the linear negative distribution .................................. 32
1 Introduction

In many practical situations inventory and transportation decisions are significantly correlated. On the one hand, transportation managers aim at low transportation costs. Therefore, few large shipments with highly utilized trucks are required to benefit from economies of scale. In order to obtain highly utilized trucks, products can be shipped earlier than required, leading to an increased inventory level and higher inventory holding costs. Alternatively, orders may be delayed until a certain truck utilization is reached resulting in more backorders or higher safety stocks. On the other hand, inventory managers aim at low inventory costs which may require many small shipments with probably low truck utilization, and hence high transportation cost. Transportation (ordering) capacity is an important issue that has been scarcely considered. In fact, it can happen that the optimal order quantity is much larger than the capacity of a truck, and that several trucks are needed for the shipment of an entire order. Hence, transportation costs may be much larger than assumed. Therefore, it is important to model the above trade-off carefully in order to minimize total cost (inventory and transportation costs).

In case of periodic inventory systems with unlimited ordering capacity and a fixed ordering cost, Scarf (1960) has proven that the optimal policy is an \((s, S)\)-type policy; at the beginning of each period, when the inventory position (stock on-hand minus backorders plus outstanding orders) drops to or below the level \(s\), enough is ordered to raise the inventory position up to the level \(S\). Federgruen and Zipkin (1986) consider a capacitated inventory system with no ordering costs. They have proven the optimality of the modified base stock policy; if there is enough capacity, order up to \(S\), otherwise, order as much as possible. In the case of capacitated inventory systems with fixed ordering costs, the optimal policy is not that straightforward. Shaoxiang et al. (1994) consider a single-item, periodic review inventory system with a limited ordering capacity and a fixed ordering cost. They have shown that the optimal policy, for the finite horizon case, has a system-
atic pattern, they call it the $X - Y$ band structure. The $X - Y$ band structure works as follows. Whenever the inventory position drops below level $X$, an order up to capacity takes place; when the inventory position exceeds level $Y$, no action is taken. When the inventory position is between $X$ and $Y$, the order quantity is different from state to state and no specific structure seems to be optimal. However, Gallego et al. (1998) have shown that the $X - Y$ band structure can be characterized with a four regions structure. Nevertheless, when the inventory position falls between $X$ and $Y$, the optimal decisions can differ from case to case. Chan et al. (2003) focus on the region between $X$ and $Y$. They provide some properties of this region and develop an efficient algorithm that allows the computation of the optimal ordering quantities. However, none of the above mentioned papers provides easy formulas to compute the values of $X$ and $Y$.

In this paper, similarly to Shaoxiang et al. (1994), a stochastic single-item, periodic review inventory system is considered, where a truck with fixed and finite capacity is used to ship the orders. A fixed transportation cost is charged as well as linear holding and backorder costs at the end of a period. We present a simpler and similar policy determined by the parameters $(S, Q_1, Q_2)$. The policy is similar in the sense that it has two thresholds $Q_1$ (the waiting threshold) and $Q_2$ (the full truckload threshold), and simpler in the sense that the region between $Q_1$ and $Q_2$ is a simple order-up-to level policy with order-up-to level $S$. A more detailed explanation of this policy is given in section 3. Furthermore, we provide relatively simple methods and heuristics to compute optimal and near optimal policy parameters and we show that the performance of this simple policy is optimal in many cases, and very close to the optimal in some other cases.

In section 2, we describe the problem. In section 3, the proposed policy is explained. Section 4 is devoted to the formulation of the mathematical model enabling the computation of the optimal policy parameters. In section 5, the optimal decisions are compared with the proposed policy. In section 6, two heuristics are presented as an alternative for computing the policy parameters. In section 7, the results obtained from an extensive
numerical study are shown. Finally, section 8 concludes this paper with a summary of the main results.

2 Problem description

A single stock location (a retailer) is considered, where a single item is stored to fulfill customers stochastic demand. Time is divided into periods of fixed length (e.g. weeks). Demand in period \( n \), \( D_n \), is a discrete random variable and the distribution \( P(D_n = k) = p_k \) is supposed to be known. Demand in subsequent periods are assumed to be independent and identically distributed. Furthermore, we assume that demand can not exceed a truck capacity and that, without loss of generality, only one truck is used to ship the orders (think about the situation where the retailer has contracts that do not allow customers to order more than a certain quantity).

The retailer is supplied from an external supplier with ample stock. This means that there is no delivery delay due to a lack of stock. The retailer is replenished by means of a truck with a finite and fixed capacity \( V \). A truck is assumed to be always available. A fixed shipping cost \( A \) is charged whenever a truck is dispatched, regardless of its load. It is assumed, without loss of generality, that deliveries are instantaneous. In other words, the lead time is assumed to be equal to zero, which is often the case in the retail environment where the orders are shipped during the weekend, or during the night, when no demand occurs. Inventories are periodically (e.g. at the beginning of each week) reviewed. The review period is given, and hence is considered to be an exogenous variable. We assume that unsatisfied demand is backordered. Moreover, at the end of each period, holding costs are charged per unit inventory on-hand and penalty costs are charged per unit backordered. Holding and penalty costs are assumed to be linear.

At the retailer the following replenishment policy is used. At the beginning of each
review period the inventory position is reviewed and an order may be placed to raise
the inventory to a certain level. The objective is to come up with a decision tool that
helps managers to smartly use their transportation capacity and benefit from economies
of scale, and hence minimize the long-run average cost consisting of inventory costs as well
as transportation costs. However, in this paper we aim to answer the following research
questions:

- What is a ”simple” and good policy to deal with the trade-off between transportation
  and inventory decisions ?

- How can the policy parameters be computed?

In the remainder of this paper, the following notation will be used. If other variables
are used, these will separately be defined:

\( V \) : Capacity of a truck
\( A \) : Fixed cost for dispatching a truck
\( D_n \) : Demand during the \( n^{th} \) period
\( X_n \) : The inventory position, before ordering, at the beginning of the \( n^{th} \) period
\( q_n \) : The quantity shipped at the beginning of the \( n^{th} \) period
\( T \) : Time between two successive shipments
\( D_T \) : Demand during \( T \)
\( h \) : Holding cost at the end of a period per item per time unit
\( p \) : Backorder cost at the end of a period per item per time unit
\( E[X] \) : Expectation of a random variable \( X \)
\( f_X \) : The probability distribution function of a continuous random variable \( X \)
\( F_X \) : The cumulative distribution function of a random variable \( X \)
\( X^+ \) : \( \max(0,X) \)
\( X^- \) : \( \max(0,-X) \)
\( \lfloor x \rfloor \) : The whole part of the real number \( x \)
\( [a,b] \) : The interval of integer number between \( a \) and \( b \) (\( a \) and \( b \) are also integers).
3 The \((S, Q_1, Q_2)\) replenishment policy

In Figure 1 we illustrate the \((S, Q_1, Q_2)\) policy. \(S\) is the order-up-to level value. The initial order size, \(O_n = \max\{0, S - X_n\}\), at the beginning of a review period, is enlarged whenever it is equal or larger than the full truckload threshold \(Q_2\) (e.g. beginning of period 4) such that a full truck is dispatched. Furthermore, the initial order size is reduced (the orders are delayed) whenever the initial order quantity is at or below the waiting threshold \(Q_1\) (e.g. beginning of periods 2 and 3). However, when the initial order size falls between \(Q_1\) and \(Q_2\), a shipment takes place to raise inventory to the level \(S\) (e.g. beginning of period 6). The circles in Figure 1 represent the quantities to be shipped at the beginning of a review period.

![Diagram](image.png)

Figure 1: The \((S, Q_1, Q_2)\) policy.

It is trivial that \(Q_1\) and \(Q_2\) should be in the interval \([0, V]\), because it does not make
sense to send an empty truck \((Q_1 < 0)\) or delay the shipment of a full truck \((Q_2 > V)\). Furthermore, it does not make sense to have \(Q_2 < Q_1\), because otherwise the target utilization \(\frac{Q_1}{Q_2}\) will never be reached.

It is very important to emphasize that because of enlargements, the inventory position \(X_n\), at the beginning of a period \(n\) before ordering can exceed the order-up-to level \(S\) (e.g. beginning of period 5). Furthermore, if the inventory position at the beginning of a period \(n\) drops below the level \(S - V\), a shipment will of course take place, but we will not be able to raise the inventory position to the value \(S\) (e.g. beginning of period 9), because we assume only one truck is used to ship the orders. Shortages are delivered in the next shipment moment. In the case of multiple-item inventory systems, Van Eijs (1994), Cachon (2001) and Kiesmüller (2006) proposed policies that combines transportation and inventory decisions. If we would relate their ideas to the \((S, Q_1, Q_2)\) policy, we observe that Kiesmüller’s idea is the special case when \(Q_1 = Q_2 = \frac{V}{2}\). Moreover, Van Eijs’s idea is the special case with \(Q_1 = 0\) and Cachon’s idea is the special case when \(Q_2 = V\). The order-up-to-level policy is the special case such that \(Q_1 = 0\) and \(Q_2 = V\).

At the beginning of a review period the quantity to be shipped is:

on the one hand, if \(Q_1 = Q_2\):

\[
q_n = \begin{cases} 
0 & , 
S - X_n \leq Q_1 \\
V & , 
S - X_n > Q_1 
\end{cases}
\]  \(\text{(1)}\)

On the other hand, if \(Q_1 \neq Q_2\):

\[
q_n = \begin{cases} 
0 & , 
S - X_n \leq Q_1 \\
S - X_n & , 
Q_1 < S - X_n < Q_2 \\
V & , 
S - X_n \geq Q_2 
\end{cases}
\]  \(\text{(2)}\)
4 Model

4.1 Model description

In this section, the mathematical model is formulated to analyze the policy. The inventory position is modelled as a discrete time Markov chain. The optimal parameters of the \((S, Q_1, Q_2)\) policy, minimizing the total cost consisting of transportation as well as inventory costs, are exactly computed. The exact analysis is compared with the optimal decisions obtained by solving a dynamic programming problem. In a numerical study, we show that, in many cases, the performance of the \((S, Q_1, Q_2)\) is optimal. In other cases, it is very close to optimal.

4.2 Model analysis: the exact analysis

Because of demand’s independency assumption, \(\{X_n, n \geq 0\}\) can be considered as a stochastic process such that the inventory position at the beginning of the \((n + 1)th\) period depends only on the inventory position at the beginning of the \(n\)th period. Hence, \(\{X_n, n \geq 0\}\) can be modelled using a Discrete Time Markov Chain. It is very important to determine the state space of the Markov chain. \(X_n\) takes its maximum value when \(S - X_{n-1} = Q_2\) and \(D_n = 0\). Hence, the maximum value \(X_n\) can take is \(S + V - Q_2\). Moreover, \(X_n\) takes its minimum value when \(S - X_{n-1} = Q_1\) and \(D_n = V\). Hence, the minimum value \(X_n\) can take is \(S - V - Q_1\). Hence, the state space of the stochastic process \(\{X_n, n \geq 0\}\) is \(SS = [S - V - Q_1, S + V - Q_2]\).

From the inventory position balance equation we have:

\[
X_{n+1} = X_n + q_n - D_n
\]
Hence, by replacing (1) and (2) in (3) we get:

If $Q_1 = Q_2$:

$$X_{n+1} = \begin{cases} X_n + V - D_n, & S - V - Q_1 \leq X_n < S - Q_1 \\ X_n - D_n, & S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases}$$ (4)

Otherwise, if $Q_1 \neq Q_2$:

$$X_{n+1} = \begin{cases} X_n + V - D_n, & S - V - Q_1 \leq X_n \leq S - Q_2 \\ S - D_n, & S - Q_2 < X_n < S - Q_1 \\ X_n - D_n, & S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases}$$ (5)

The transient probabilities matrix $P = (p_{i,j})_{(i,j) \in SS^2}$, such that for all $(i, j) \in SS^2$, $p_{i,j} = P(X_{n+1} = j | X_n = i)$, can easily be computed. For all $(i, j) \in SS^2$ we have:

If $Q_1 = Q_2$:

$$p_{i,j} = \begin{cases} P(D_n = i + V - j), & S - V - Q_1 \leq X_n < S - Q_1 \\ P(D_n = i - j), & S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases}$$ (6)

Otherwise, if $Q_1 \neq Q_2$:

$$p_{i,j} = \begin{cases} P(D_n = i + V - j), & S - V - Q_1 \leq X_n \leq S - Q_2 \\ P(D_n = S - j), & S - Q_2 < X_n < S - Q_1 \\ P(D_n = i - j), & S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases}$$ (7)

The steady state probabilities are defined as $\pi_i = \lim_{n \to \infty} P(X_n = i | X_0 = j)$, they can be computed by solving the following linear equations system:

$$\begin{cases} \sum_{i \in SS} \pi_i = 1, & 0 \leq \pi_i \leq 1 \\ \pi = \pi \times P, & \pi = (\pi_i)_{i \in SS} \end{cases}$$ (8)
4.2.1 The decisions

So far we have derived everything we need to compute the long-run cost function $C(S, Q_1, Q_2)$. We have:

$$C(S, Q_1, Q_2) = \sum_{i \in S} \pi_i c(i)$$ (9)

where $c(i)$ is the involved cost (transportation, holding and penalty costs) when the system is in the state $i$. We have:

$$c(i) = A_i + h_i + p_i$$ (10)

Moreover, we know that we only ship if the initial order size is larger than $Q_1$, hence at the end of a period we have:

$$A_i = \begin{cases} A & S - i > Q_1 \\ 0 & \text{otherwise} \end{cases}$$ (11)

Unfortunately, the cost function is not an explicit function of $S$, $Q_1$ and $Q_2$. As a consequence, it is not straightforward to prove the convexity of the cost function in the policy parameters and compute the optimal values. However, we have so far computed every thing needed to implement the model in a program that will search for the optimal values of $S$, $Q_1$ and $Q_2$. For each combination of $S$, $Q_1$ and $Q_2$ the program computes the corresponding cost, so a three dimensional cost array is constructed. The program locates the minimum cost in this array and gives the corresponding optimal values $S^*$, $Q_1^*$ and $Q_2^*$.

The exact analysis can still deliver results for small truck capacities. Unfortunately, for large truck capacities (e.g. $V = 50$) the computations become huge in the sense that it takes very long before obtaining the results. For $V = 100$, it becomes impossible to apply the exact analysis. In the next section, the $(S, Q_1, Q_2)$ policy is compared with the optimal decisions.
5 The optimal decisions

In order to compare the performance of the optimal \((S^*, Q_1^*, Q_2^*)\) policy with the overall optimal decisions, we formulate the single item ordering problem with a capacity constraint and fixed ordering cost as a dynamic programming problem. The inventory position is again used to describe the state of the system and we denote the set of all possible states by \(I\). The number of items to be ordered is non-negative and not larger than the truck capacity. Therefore, the set of possible actions is given as \(A_c = [0, V]\). If \(c_i(a)\) is defined as the one period cost when the system is in state \(i\) and action \(a\) is taken, then we can compute \(c_i(a)\) as follows:

\[
c_i(a) = A\delta(a) + h\sum_{k=0}^{i+a}(i + a - k)p_k + p\sum_{k=1+a+1}^{V}(k - i - a)p_k, \quad (i, a) \in I \times A_c
\]

(12)

where \(\delta(a)\) is defined as follows:

\[
\delta(a) := \begin{cases} 
0 & \text{if } a = 0 \\
1 & \text{if } a \neq 0
\end{cases}
\]

(13)

Finally, let \(V_n(i)\) be the expected minimal cost for an \(n\) horizon problem, if the beginning inventory level is \(i\). Then we can formulate the following dynamic programming recursion.

\[
V_n(i) = \min_{a \in A_c} \left\{ c_i(a) + \sum_{j \in I} p_{ij}(a)V_{n-1}(j) \right\}
\]

(14)

Chan et al (2001) have shown that solving equation (14) reveals that there is no simple optimal policy structure. However, the optimal ordering pattern is different from state to state. We will use (14) in order to determine the optimal decisions and the minimal costs numerically by value iteration (Tijms (1994)) and we will compare the results with the optimal \((S^*, Q_1^*, Q_2^*)\) policy (Table 1 and 3).
The results in Table 1 are based on the set of parameters presented in Table 2.

Table 1: The ordered quantities at the beginning of a period

<table>
<thead>
<tr>
<th>States</th>
<th>Case1</th>
<th>Case2</th>
<th>Case3</th>
</tr>
</thead>
<tbody>
<tr>
<td>. . .</td>
<td>. . .</td>
<td>. . .</td>
<td>. . .</td>
</tr>
<tr>
<td>-3</td>
<td>20 20</td>
<td>20 20</td>
<td>19 20</td>
</tr>
<tr>
<td>-2</td>
<td>19 20</td>
<td>20 20</td>
<td>18 20</td>
</tr>
<tr>
<td>-1</td>
<td>18 20</td>
<td>19 20</td>
<td>17 17</td>
</tr>
<tr>
<td>0</td>
<td>17 20</td>
<td>20 20</td>
<td>16 16</td>
</tr>
<tr>
<td>1</td>
<td>20 20</td>
<td>20 20</td>
<td>15 15</td>
</tr>
<tr>
<td>2</td>
<td>19 20</td>
<td>20 20</td>
<td>14 14</td>
</tr>
<tr>
<td>3</td>
<td>18 20</td>
<td>19 20</td>
<td>13 13</td>
</tr>
<tr>
<td>4</td>
<td>20 20</td>
<td>20 20</td>
<td>12 12</td>
</tr>
<tr>
<td>5</td>
<td>20 20</td>
<td>20 20</td>
<td>11 11</td>
</tr>
<tr>
<td>6</td>
<td>19 20</td>
<td>20 20</td>
<td>10 10</td>
</tr>
<tr>
<td>7</td>
<td>18 20</td>
<td>19 20</td>
<td>9  9</td>
</tr>
<tr>
<td>8</td>
<td>20 20</td>
<td>20 20</td>
<td>8  8</td>
</tr>
<tr>
<td>9</td>
<td>20 20</td>
<td>20 20</td>
<td>7  7</td>
</tr>
<tr>
<td>10</td>
<td>19 20</td>
<td>20 20</td>
<td>6  6</td>
</tr>
<tr>
<td>11</td>
<td>18 20</td>
<td>19 20</td>
<td>20  0</td>
</tr>
<tr>
<td>12</td>
<td>20 20</td>
<td>20 20</td>
<td>20  0</td>
</tr>
<tr>
<td>13</td>
<td>20 20</td>
<td>20 20</td>
<td>19  0</td>
</tr>
<tr>
<td>14</td>
<td>19 19</td>
<td>20 20</td>
<td>18  0</td>
</tr>
<tr>
<td>15</td>
<td>18 18</td>
<td>19 19</td>
<td>0  0</td>
</tr>
<tr>
<td>16</td>
<td>0  0</td>
<td>0  0</td>
<td>0  0</td>
</tr>
<tr>
<td>. . .</td>
<td>. . .</td>
<td>. . .</td>
<td>. . .</td>
</tr>
</tbody>
</table>

The two points demand distribution is such that \( P(D_n = 16) = 0.95, P(D_n = 17) = 0.05 \) and 0 otherwise. \( q_n^{opt} \) is the quantity to be ordered at the beginning of each period in case of the optimal decisions, based on the inventory position state (column 1-Table 1). \( q_n^{opt} \) is obtained by solving (14) using value iteration. \( q_n \) is the ordering quantity obtained by
using the exact analysis of the \((S, Q_1, Q_2)\) policy. We observe that, while the \((S, Q_1, Q_2)\) policy has three regions, the optimal decisions has no obvious structure. However, in Table 3 we compare the cost given by the \((S^*, Q^*_1, Q^*_2)\) policy with the cost obtained by the optimal decisions. The costs are computed for the uniform distribution, the linear positive distribution (Figure 2) and the two points distribution. As can be observe, differences, if any, are negligible (< 0.4%). Differences are more visible in case of the two points distribution, while for the other distributions, the optimal decisions are identical to the \((S^*, Q^*_1, Q^*_2)\) policy.

<table>
<thead>
<tr>
<th>Demand distribution</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Uniform</td>
<td>Linear positive</td>
<td>Two points</td>
</tr>
<tr>
<td>V</td>
<td>50</td>
<td>250</td>
<td>250</td>
</tr>
<tr>
<td>p</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>h</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2: Parameter set for the numerical example in Table 1

![Figure 2: The demand distributions](image)
<table>
<thead>
<tr>
<th></th>
<th>Uniform</th>
<th>Linear positive</th>
<th>2 pts distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_{opt}$</td>
<td>$C_{SQ1Q2}$</td>
<td>$C_{opt}$</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>43,46</td>
<td>43,46</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>60,43</td>
<td>60,43</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>91,79</td>
<td>91,79</td>
</tr>
<tr>
<td>250</td>
<td>1</td>
<td>143,46</td>
<td>143,46</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>160,43</td>
<td>160,43</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>206,25</td>
<td>206,25</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the $(S, Q_1, Q_2)$ policy and the optimal decisions

Based on the results from Table 3, it is attractive to think about developing heuristics for the $(S, Q_1, Q_2)$ policy. In the next section two heuristics are proposed.

### 6 Heuristics

In section 5, the exact analysis for the $(S, Q_1, Q_2)$ is developed. By means of a numerical study, we have shown that, in many cases, the $(S, Q_1, Q_2)$ policy is optimal. The optimal policy parameters are obtained by a search procedure that requires a lot of time in case of high truck capacity. Furthermore, no simple explicit formulas have been developed to compute the cost function and the policy parameters. Therefore, relatively easy heuristics that may give good results in a reasonable time are desirable. In this section, we propose two heuristics, we call the S-Heuristic and the SQ-Heuristic. The S-Heuristic is based on an approximated value of $S^*$. $Q_1^*$ and $Q_2^*$ are again obtained by a search procedure. In case of the SQ-Heuristic, formulas are developed to describe the region between the waiting threshold $Q_1$ and the full truckload threshold $Q_2$. $S^*$ is again estimated. The proposed heuristics are expected to perform well and decrease the computation time. In the remainder of this paper, $D_1$ denotes the demand during one period.
6.1 The S-Heuristic

The idea behind the S-Heuristic is to use an approximation to estimate the optimal value of the order-up-to level $S$. Afterwards, the Markov chain model is used to search for the optimal values of the thresholds $Q_1$ and $Q_2$. We know that in case of the order-up-to level policy, the Newsboy’s formula can be used to compute the optimal $S$. But in this case, the inventory position at the beginning of a review period, after ordering, is known to be exactly equal to $S$. Moreover, it is known that $S$ will cover the future demand occurring during the lead time plus the review period. In the case of the $(S, Q_1, Q_2)$ policy, the inventory position at the beginning of a review period after ordering, $X_{a_n}$, is not necessarily equal to $S$. Because of the enlargement and the reduction of the initial order size, the value of $X_{a_n}$ is in the interval $[S - Q_1, S + V - Q_2]$. Furthermore, the time between two successive shipments $T$ is, opposite to the case of the order-up-to level policy, not deterministic. In fact, $T$ increases with the waiting threshold $Q_1$ (the larger is $Q_1$, the longer we have to wait before a shipment takes place) and decreases with the full truckload threshold $Q_2$ (the smaller is $Q_2$, the larger is the enlargement, the longer we have to wait because of the excess of inventory). Therefore, we will estimate $X_{a_n}$ and $T$ before using the Newsboy’s equation.

Let’s first consider the simple situation where demand during one period is $a$ with probability $p_a$ and $b$ with probability $1-p_a$. The expected time between two shipments can be approximated by:

$$E[T] = 1 \frac{1}{V + Q_1 - Q_2 + 1} \sum_{k=-Q_1}^{V-Q_2} \frac{Q_1 + k}{E[D_1]}$$  \hspace{1cm} (15)$$

Because the inventory is reviewed every period we take the expected number of periods between two successive shipments equal to:

$$E[T] = 1 + \left[\frac{V + Q_1 - Q_2}{2E[D_1]}\right]$$ \hspace{1cm} (16)$$
We can see that $E[T]$ behaves as expected on $Q_1$ and $Q_2$. Furthermore, if, $Q_1 = 0$ and $Q_2 = V$, which corresponds to the case of the order-up-to level policy, $E[T] = 1$. Hence, the approximation sounds reasonable.

Based on the same reasoning, $X_n^a$ can be approximated by:

$$X_n^a = \frac{1}{V + Q_1 - Q_2 + 1} \sum_{k=-Q_1}^{V-Q_2} (S + k)$$

$$= S + \frac{V - Q_1 - Q_2}{2}$$  \hspace{1cm} (17)

For the special case where demand during one period is $a$ with probability $p_a$ and $b$ with probability $1-p_a$, $D_T$, demand during the periods between two shipments, takes values in $\bigcup_{k=0}^{V} \{ka + (n-k)b\}$ such that $k$ is the number of times the demand during one period is $a$. Hence, the probability that $D_T$ equals $m$, $P(D_T = m)$ (note that $m$ can be uniquely written as $m = ka + (n-k)b$), is equal to the probability that $D_1$ takes the value $a$ in $k$ periods from $n$ periods. Hence, $D_T$ has a binomial distribution:

$$P(D_T = m) = \left( \frac{n}{m-nb} \right)^{m-nb} p_a^{m-nb} (1-p_a)^{na-m}$$  \hspace{1cm} (18)

As a consequence, by using the approximations (16) and (17) and the probability distribution (18), $S^*$ can be easily computed by using Newsboy’s formula:

$$F_{D_T}(X_n^a) \geq \frac{p}{h + p}$$  \hspace{1cm} (19)

In the general case when demand can take any value $k$ between 0 and $V$ with probability $p_k$, the probability distribution of demand between two shipments takes the form:

$$P(D_T = m) = \sum_{(k_0, \ldots, k_V) \in C_m} \frac{E[T]!}{k_0! k_1! \ldots k_V!} p_0^{k_0} p_1^{k_1} \ldots p_V^{k_V}$$  \hspace{1cm} (20)

where $k_i$ is the number of times demand during one period is $i$, and $C_m$ is the set of combinations of $k_i$’s such that $\sum_{i=0}^{V} k_i = E[T]$ and $\sum_{i=0}^{V} ik_i = m$. 
Furthermore the following recursive relation is also useful:

\[ P(D_T = m) = \sum_{k=0}^{m} P(D_{T-1} = m - k)P(D_1 = k) \] (21)

Based on these approximations we can estimate the optimal \( S \), we denote it \( S^*_S \), which can be implemented in the Markov model to search for the estimates \( Q^*_1,S \) and \( Q^*_2,S \) of \( Q^*_1 \) and \( Q^*_2 \) corresponding to the minimum cost \( C_S \) obtained by the S-Heuristic.

6.2 The SQ-Heuristic

In this section, we develop a heuristic, we call it the SQ-Heuristic, that allows us to explicitly express the estimated cost function as a function of the policy parameters \( S, Q_1 \) and \( Q_2 \). Hence, opposite to the exacte analysis and the S-Heuristic, the policy parameters will be expressed as a function of the variables involved (e.g. \( h, p, A, V \)). As a consequence, the policy parameters are easily estimated.

The demand is assumed to be continuous and has values in the interval \([0, V]\). We develop the SQ-Heuristic for three different demand distributions (Figure 2) namely the uniform distribution (a), a positive linear probability demand distribution (b) and a negative linear probability demand distribution (c).

- a. The uniform demand distribution

The demand is assumed to have a continuous uniform distribution on \([0, V]\). Furthermore, we assume that the inventory position after ordering at the beginning of a review period is uniformly distributed in \([S - Q_1, S + V - Q_2]\). We have \( E[D_n] = \frac{V}{2} \). Based on this, the time between two shipments is estimated by:

\[ E[T] = 1 + \int_{-Q_1}^{V-Q_2} \frac{Q_1 + x}{E[D_n]} \, dx = \frac{2V + Q_1 - Q_2}{V} \] (22)
The expected inventory on-hand at the end of a period can be calculated by the following double integral:

\[
E[OH] = \int_{-Q_1}^{V-Q_2} \int_{0}^{S+\theta} (S + \theta - x) f_x f_\theta dx d\theta \\
= \frac{(S + V - Q_2)^3 - (S - Q_1)^3}{6V(V + Q_1 - Q_2)}
\] (23)

On the other hand, the expected backorders at the end of a period is:

\[
E[BO] = \int_{-Q_1}^{V-Q_2} \int_{S+\theta}^{V} (x - S - \theta) f_x f_\theta dx d\theta \\
= \frac{(S - Q_2)^3 - (S - V - Q_1)^3}{6V(V + Q_1 - Q_2)}
\] (24)

The estimated long-run cost related to the \((S, Q_1, Q_2)\) policy can be expressed as follows:

\[
C_{SQ} = \frac{A}{E[T]} + hE[OH] + pE[BO]
\] (25)

Hence, by replacing (22), (23) and (24) in (25) we get:

\[
C_{SQ} = \frac{AV}{2V + Q_1 - Q_2} + h\frac{(S + V - Q_2)^3 - (S - Q_1)^3}{6V(V + Q_1 - Q_2)} + \frac{p(S - Q_2)^3 - (S - V - Q_1)^3}{6V(V + Q_1 - Q_2)}
\] (26)

We have managed to express the estimated long-run cost as an explicit function of the policy parameters \(S, Q_1\) and \(Q_2\). We can prove that this estimated cost is convex in \(S\) and that there is an optimal \(S_{SQ}^*\). In fact, we have:

\[
\frac{\partial^2 C_{SQ}}{\partial S^2} = \frac{p + h}{V} > 0
\] (27)

We can search for the optimal value of \(S\) by setting the first partial derivative equal to zero. After some basic algebra, we find that the optimal \(S\) is such that:

\[
S_{SQ}^* = \frac{Q_1 + Q_2}{2} + \frac{(p - h)V}{p + h}
\] (28)
In the case of the order-up-to level policy, that is, \( Q_1 = 0 \) and \( Q_2 = V \) we have:

\[
S^*_{SQ} = \frac{p}{p + h} V
\]  

(29)

This is exactly what we obtain by applying the Newsboy formula to the order-up-to level policy. We can also observe that \( S^*_{SQ} \) behaves as expected on \( h \) and \( p \).

In the remainder of the paper we define the order-up-to level region as the region between \( Q_1 \) and \( Q_2 \), where the order-up-to level policy is applied. We characterize this region by the variable \( X = Q_2 - Q_1 \) such that, \( X = 0 \) means the full truckload policy, and \( X = V \) corresponds to the order-up-to level policy.

By replacing (28) in (26), we can express the estimated long-run cost as a function of \( X \). We can also prove that \( C_{SQ} \) is convex in \( X \). However, by solving the equation:

\[
\frac{dC_{SQ}(X)}{dX} = 0
\]  

(30)

we find that the optimal \( X, X^* \), such that \( 0 \leq X \leq V \), is the solution of the following important relation:

\[
(2V - X^*)^2 = \frac{12AV^2}{(p + h)(V - X^*)}
\]  

(31)

This is a very important result in the sense that simple and useful conclusions can be drawn out of it. For instance, we can observe that if \( A \approx 0 \) (negligible transportation costs), \( X^* = V \) which means we always use the order-up-to level policy. Moreover, the full truckload policy \( (X^* = 0) \) is used when:

\[
\frac{A}{p + h} \geq \frac{V}{3}
\]  

(32)

In Figure 3, we can see how the order-up-to level region varies as a function of the cost parameters and the truck capacity. We observe that, for a given \( V \), the smaller is the ratio \( \frac{A}{p + h} \), the more \( X^* \) tends to \( V \) (the order-up-to level policy), which is expected. In Figure
Figure 3: $X^*$ as a function of $A$, $h$, $p$, and $V$.

4, we illustrate which policy to use depending on the input parameters. We observe, for instance, that the larger are inventory and backorder costs ($p + h$), the less attractive is the use of the full truckload policy. The same thing can be observed when the truck capacity is increased. By a policy with a $Q_1 - Q_2$ band we mean a policy with $X > 0$. However, based on the insights gained in this section, we can develop an algorithm which we expect to allow us to quickly come up with good estimates of the optimal parameters. The algorithm is the following:

Determine $X^*$ by solving (31)
for $Q_1 := 0 \text{ to } V$

$Q_2 := \max\{Q_1 + X^*, V\}$

$S = \frac{Q_1 + Q_2}{2} + \frac{(p-h)}{p+h}V$

Compute $C(S, Q_1, Q_2)$

end

Determine $\min\{C(S, Q_1, Q_2)\}$

Give $Q_1$ and $Q_2$

$S^* = \frac{Q_1 + Q_2^*}{2} + \frac{(p-h)}{p+h}V$, $X^* = Q_2^* - Q_1^*$
We should mention that the parameters are determined using the formulas developed in this section. Afterwards, these parameters are used to compute the real cost $C(S, Q_1, Q_2)$.

Theoretically, it is possible to develop such formulas for any demand distribution. In the appendix, we show the formulas for the case of the linear positive and the linear negative distributions. But we should keep in mind that, for some distributions, derivations may become complicated. Mathematical softwares can still be used to solve the equations, but no nice formulas would be obtained.
7 Numerical study

In this section, the results obtained from a numerical study are presented. The exact analysis as well as the heuristics developed in the previous sections have been implemented in a program written in Matlab. The program uses the formulas found earlier in this paper to compute the policy parameters and cost, that is, no simulation has been relied on. The following parameter set has been used:

\[ V = 20 \; ; \; A = \{50, 250\} \; ; \; p = 100 \; ; \; h = \{1, 2, 5, 10, 20\}. \]

Additionally, the demand distributions presented in Figure 2 have been used. These three different demand distributions allow us to cover cases with high, middle and low demand (keeping in mind that demand can not exceed \( V \)). Furthermore, they allow us to have different demand variabilities. In Tables 4, 5 and 6, the results are presented separately for each demand distribution. The performance of the exact analysis is compared with the heuristics performances and with the performance the order-up-to level policy denoted \((R, S)\), with optimal order-up-to level \(S^*_R,S\) and minimal cost \(C_{R,S}\).

From Tables 4, 5 and 6, we observe that, in case of the exact analysis, the optimal value of the full truckload threshold \(Q_2\) is always \(V\). This means that it is not suitable to enlarge the initial order size. This is a very important result, in the sense that we can conclude that the optimal parameters are such that our policy always takes the form \((S, Q_1, V)\). Intuitively, this result makes sense. In fact, we have seen that enlargements cause the inventory position after ordering to exceed the value \(S\). Since the exact analysis aims to use the optimal value of \(S\), it does not make sense to take any action that may lead to any deviation from this optimum. Furthermore, the added value that may be obtained by enlarging initial order size (high truck utilization) could also be obtained by reducing them.
However, we can easily prove that when the optimal policy parameters are such that
\( Q^*_1 = Q^*_2 = Q^* \), the optimal policy is also a policy such that \( Q^*_2 = V \). In fact, we can prove that if \((S^*, Q^*, Q^*)\) are the optimal policy, all the policies in the range \((S^* + i, Q^* + i, Q^* + i)\), \(i \in [-Q^*, V - Q^*]\) are optimal. In fact, if the optimal policy takes the form \((S^*, Q^*, Q^*)\), it is characterized by the set of equations (6) and (11). Now, for each integer \(i \in [-Q^*, V - Q^*]\), the policy \((S^* + i, Q^* + i, Q^* + i)\) is characterized by the same set of equations, which means that the policy \((S^* + V - Q^*, V, V)\) is also an optimal one.

We also observe that, sometimes, the S-heuristic underestimates the value of \(S^*\), and that as a correction to this damage caused by this underestimation, the value of \(Q^*_1\) is decreased in the sense that time between two shipments is shorter and hence the backorders that would have occurred (because of the underestimation of \(S^*\)) are avoided. In most of the cases, the SQ-heuristic gives a very good estimation of \(S^*\).

In general, the S-heuristic as well as the SQ-heuristic perform quite good. To get an idea on how much the costs obtained by the heuristics deviate from the cost obtained by

<table>
<thead>
<tr>
<th>A</th>
<th>h</th>
<th>(Q^*_1)</th>
<th>(Q^*_2)</th>
<th>(S^*)</th>
<th>(C_{SQ1,Q2})</th>
<th>(Q^*_{1,S})</th>
<th>(Q^*_{2,S})</th>
<th>(S^*_S)</th>
<th>(C_S)</th>
<th>(X^*)</th>
<th>(S^*_{SQ})</th>
<th>(C_{SQ})</th>
<th>(S^*_{R,S})</th>
<th>(C_{R,S})</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1</td>
<td>20</td>
<td>20</td>
<td>37</td>
<td>43.46</td>
<td>9</td>
<td>20</td>
<td>26</td>
<td>47.70</td>
<td>1</td>
<td>37</td>
<td>43.46</td>
<td>20</td>
<td>57.62</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>20</td>
<td>20</td>
<td>36</td>
<td>60.43</td>
<td>9</td>
<td>20</td>
<td>26</td>
<td>61.48</td>
<td>1</td>
<td>37</td>
<td>60.97</td>
<td>20</td>
<td>67.62</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4</td>
<td>20</td>
<td>20</td>
<td>91.79</td>
<td>3</td>
<td>17</td>
<td>20</td>
<td>92.98</td>
<td>16</td>
<td>20</td>
<td>92.32</td>
<td>19</td>
<td>97.62</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>4</td>
<td>20</td>
<td>19</td>
<td>137.38</td>
<td>2</td>
<td>17</td>
<td>19</td>
<td>140.29</td>
<td>16</td>
<td>19</td>
<td>137.74</td>
<td>19</td>
<td>142.85</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3</td>
<td>20</td>
<td>17</td>
<td>217.48</td>
<td>2</td>
<td>17</td>
<td>17</td>
<td>219.50</td>
<td>16</td>
<td>17</td>
<td>218.57</td>
<td>17</td>
<td>221.90</td>
</tr>
<tr>
<td>250</td>
<td>1</td>
<td>20</td>
<td>20</td>
<td>37</td>
<td>143.46</td>
<td>15</td>
<td>20</td>
<td>29</td>
<td>157.42</td>
<td>0</td>
<td>34</td>
<td>147.92</td>
<td>20</td>
<td>248.09</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>20</td>
<td>20</td>
<td>36</td>
<td>160.43</td>
<td>15</td>
<td>20</td>
<td>29</td>
<td>170.55</td>
<td>0</td>
<td>34</td>
<td>162.50</td>
<td>20</td>
<td>258.09</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>20</td>
<td>20</td>
<td>34</td>
<td>206.25</td>
<td>15</td>
<td>20</td>
<td>28</td>
<td>211.95</td>
<td>0</td>
<td>33</td>
<td>206.50</td>
<td>20</td>
<td>288.09</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>31</td>
<td>271.43</td>
<td>17</td>
<td>20</td>
<td>29</td>
<td>271.71</td>
<td>1</td>
<td>32</td>
<td>272.00</td>
<td>19</td>
<td>333.33</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>9</td>
<td>20</td>
<td>19</td>
<td>358.45</td>
<td>8</td>
<td>14</td>
<td>19</td>
<td>365.36</td>
<td>9</td>
<td>19</td>
<td>360.14</td>
<td>17</td>
<td>412.38</td>
</tr>
</tbody>
</table>

Table 4: Results for the uniform distribution
On the one hand, the maximum value $\Delta C_{S\text{-heuristic}}$ can take is 13% and the minimum value it can take is 0%. On average $\Delta C_{S\text{-heuristic}} = 3.5\%$. On the other hand, the maximum value $\Delta C_{SQ\text{-heuristic}}$ can take is 5% and the minimum value it can take is 0%. On average $\Delta C_{SQ\text{-heuristic}} = 1.3\%$. Based on these values we can conclude that the proposed heuristics perform quite good, even though the SQ-heuristic clearly outperforms the S-heuristic. Moreover, we observe that the order-up-to level policy is clearly outperformed, mainly when transportation costs are high (an increase with regard to the exact analysis reaches 143%).
8 Summary and Conclusions

In this paper, we have proposed a policy that combines both inventory and transportation decisions. Our policy adapts the initial order size in the sense that they might be enlarged as well as reduced when beneficial. However, we have seen that when the optimal order-up-to level value of $S$ is used, enlargements are not preferred. We have shown how the optimal policy parameters can be computed. In a numerical study, we illustrate that, while our policy has a quite simple structure, it performs as good as the optimal decisions whose structure is much more complex. Furthermore, two heuristics have been developed for our policy, and which have proven to perform quite good, in the sense that good results can be obtained in a very reasonable amount of time. In addition, simple and useful rules could be obtained by applying the heuristics. In fact, we have seen how a simple calculation can help managers to decide on whether always sending full trucks is the best solution. Furthermore, we have shown that our policy clearly outperforms the order-up-to level policy.
Based on the results obtained in this paper, a direction to future research could be extending our analysis to the multiple-item multiple-truck situation and consequently line-item costs are taken into account. The transportation capacity will still be a constraint, but then a decision has to be made on how to smartly allocate capacity to the items to be shipped.
Appendix

b. The positive linear distribution

In case the demand has a positive linear distribution, as described in Figure 2, \( E[D_1] = \frac{2V}{3} \). Hence in the same way as for the uniform distribution, we can have:

\[
E[T] = \frac{7V + Q_1 - Q_2}{4V}
\]  

(34)

The expected on-hand and backorders at the end of a period can be similarly expressed as follows:

\[
E[OH] = \frac{(S + V - Q_2)^4 - (S - Q_1)^4}{12V^2(V + Q_1 - Q_2)}
\]

(35)

and,

\[
E[BO] = \frac{(S + V - Q_2)^4 - (S - Q_1)^4}{12V^2(V + Q_1 - Q_2)} - S + \frac{Q_1 + Q_2}{2} - \frac{V}{6}
\]

(36)

The formulas (34), (35) and (36) can be used to express the estimated cost \( C_{SQ} \). Furthermore, we can, in the same way, find \( S_{SQ}^* \) such that:

\[
S_{SQ}^* = -\frac{V - Q_1 - Q_2}{2} + \sqrt{\frac{p}{p + h}} V^2 - \frac{(V + Q_1 - Q_2)^2}{12}
\]

(37)

Again we can verify the Newsboy’s formula in case \( Q_1 = 0 \) and \( Q_2 = V \), in fact we find:

\[
S_{SQ}^* = V \sqrt{\frac{p}{p + h}}
\]

(38)

Using Newsboy formula leads to the same result.

Again we take \( X = Q_2 - Q_1 \). We can, as in the case of the uniform distribution, express \( C_{SQ} \) as a function of \( X \). Finding the \( X^* \), is the same as taking \( Q_2 = V \) and finding \( Q_{1,SQ}^* \). \( X^* \) should be equal to \( V - Q_{1,SQ}^* \). \( Q_{1,SQ}^* \) is found by solving the following equation:

\[
\frac{dC_{SQ}(Q_1)}{dQ_1} = 0
\]

(39)
Or, after some algebra, solving:

$$Q_{1, SQ}^*(Q_{1, SQ}^* + \frac{4}{3}V)^2 \sqrt{\frac{p}{p + h}}V^2 - \frac{Q_{1, SQ}^2}{12} = \frac{8AV^3}{3(p + h)}$$ \hspace{1cm} (40)

\textbf{c. The negative linear distribution}

In case the demand has a positive linear distribution, as described in Figure 2, $ED_n = \frac{V}{3}$. Hence in the same way, we can have:

$$E[T] = \frac{4V + Q_1 - Q_2}{2V}$$ \hspace{1cm} (41)

The expected on hand and backorders at the end of a period can be similarly expressed as follows:

$$E[OH] = \frac{(S - Q_1)^4 - (S + V - Q_2)^4}{12V^2(V + Q_1 - Q_2)} + \frac{(S + V - Q_2)^3 - (S - Q_1)^3}{3V(V + Q_1 - Q_2)}$$ \hspace{1cm} (42)

and,

$$E[BO] = E[OH] - S + \frac{Q_1 + Q_2}{2} - \frac{V}{6}$$ \hspace{1cm} (43)

The formulas (41), (42) and (43) can be used to express the estimated cost $C_{SQ}$. Furthermore, we can, in the same way, find $S_{SQ}^*$ such that:

$$S_{SQ}^* = \frac{V + Q_1 + Q_2}{2} + \sqrt{\frac{h}{p + h}}V^2 - \frac{(V + Q_1 - Q_2)^2}{12}$$ \hspace{1cm} (44)

The Newsboy formula can again be verified for the case of the order-up-to level policy.

Again we take $X = Q_2 - Q_1$. We can, as in the case of the uniform distribution, express $C_{SQ}$ as a function of $X$. Finding the $X^*$, is the same as taking $Q_2 = V$ and finding $Q_{1, SQ}^*$. $X^*$ should be equal to $V - Q_{1, SQ}^*$. $Q_{1, SQ}^*$ is found by solving the following equation:

$$Q_{1, SQ}^*(Q_{1, SQ}^* + \frac{2}{3}V)^2 \sqrt{\frac{h}{p + h}}V^2 - \frac{Q_{1, SQ}^2}{12} = \frac{4AV^3}{3(p + h)}$$ \hspace{1cm} (45)
References


Integrating inventory and transportation decisions

Trade-off:

**Inventory decisions:**
- Minimize holding and backorder costs
- Many small shipments
- Low truck utilization

**Transportation decisions:**
- Minimize transportation costs
- Few large shipments
- High truck utilization

System:

- WH
- L periods
- Retailer
- D
- Stochastic demand

Solution:

- Only full trucks are dispatched
- A truck is dispatched only when a minimum utilization target is reached

Input variables:
- \( p \): Backorder costs
- \( h \): Holding costs
- \( A \): Transportation costs
- \( V \): Capacity of a truck

/ faculteit technologie management