Multiple pulsed plasma torch synchronization

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Summary

The pulsed plasma torch can be applied in chemical processes (for instance etch or deposition) when controlling the characteristics of the plasma is desirable. Treatments of large surfaces demand multiple plasma torches which can operate simultaneously.

Feeding these torches in a parallel configuration from a single high voltage pulse source proves to be difficult. An equal amount of energy has to be dissipated in each torch to ensure a uniform surface treatment. The negative VI characteristic of thermal plasma causes an unstable behavior which results in an unequal energy distribution among the torches.

A circuit is developed which is able to control this energy distribution. A set of coupled inductors ensures that an equal current is fed through each torch. A second feature of the circuit is the ability to provide overvoltages to the torches that still have to break down. Some analytical analyses and simulations have been performed to study the behavior of this circuit.

Fairly well energy distribution results have been obtained in a three torch experimental setup. Experiments show that the energy distribution is similar for different pulse shapes. The generated overvoltages ensure breakdown of all torches within 100ns.
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1 Introduction

Today's plasma technology has a wide range of applications. Plasma torches can for instance be applied in etching and deposition processes. Traditional thermal plasmas are inductively coupled, DC transferred and non-transferred arcs. They usually generate a very high temperature or heat flux. These plasmas aren't suitable for some chemical processes because controlling the temperature proves to be difficult. Pulse power technology presents a solution for this problem. The characteristics of active and thermal plasma can presumably be controlled by varying the shape and repetition rate of the pulses.

1.1 Problem description

A single plasma torch can only treat a relative small area. An industrial application demands fast treatment of large surfaces. This problem can be overcome by applying a set of torches which can cover a larger area. Each torch should dissipate an equal amount of energy to ensure a uniform treatment of the surface. Feeding multiple torches from a single pulse source proves to be difficult because of instable behaviour of the plasma. Instability is caused by the negative V-I characteristic of thermal plasma (figure 1.1a).

![Figure 1.1: Multiple torch complication](image)

A parallel configuration of two torches fed by a current source is shown in figure 1.1b. Both torches will have equal impedances in case that they have a similar V-I characteristic, the current will be equally divided and the voltage over torches will have the same value. In this case a common equilibrium exists. Exactly equal V-I characteristics of torch plasmas will not exist in practice. A small deviation in electrode distance or gas flow will result in slightly different V-I characteristic. A common equilibrium will not exist. Imagine that the current through torch1 slightly increases during the unstable operation because it has a lower impedance than torch 2. The larger current will cause a relative small decrease of voltage resulting in a larger demand of power. Less power will be delivered by the source because the current is fixed and the overall voltage has dropped. The extra power can only be delivered by reducing the power dissipation of torch 2. Reduction of power means a decrease of current trough torch 2 resulting in an increase of current through torch 1, the impedance of torch 1 will drop even further. This will lead to a snowball effect. Torch 1 will become dominant and torch 2 will extinguish.

1.2 Objectives

Stable operation and homogeneous energy distribution of the parallel configuration can be ensured if an equal current is fed through each torch. A slightly different V-I characteristic will cause a small voltage difference over the torches. The resulting energy distribution will not be exactly equal but acceptable. The objective is to develop a circuit which magnetically couples the currents through the torches. The solution has to be scalable in order to easily expand the number of torches.
2 Plasma torch

Multiple tube to tube plasma torches have been used in the experimental setup. The inner and outer tube is fabricated of stainless steel. The gap is designed to breakdown at voltages of about 17-20kV. The inner electrode is suspended in the outer tube by a plastic insulator. The high voltage is fed on top of this electrode, the outer tube is grounded. Discharges take place at random positions throughout the torch but tend to occur at the bottom. Sharp edges of the inner and outer tube cause a higher electric field in this area. Another problem is the alignment of the inner electrode. The electrode is easily misaligned because it is only fixed at the top.

![Figure 2.1. Plasma torch construction](image)

The research doesn't apply to chemical aspects of the plasma, so dried compressed air is used during the experiments. Usage of dry air has the advantage of more stable operation of the plasma torch. Humidity of the air will have a constant level. The air flushes the torch, ensuring recovery after each discharge.
3 Developed synchronization circuit

A possible solution to equalize the currents through multiple torches in a parallel configuration is presented in figure 3.1. Another problem with a parallel torch configuration beside the current synchronization issue is that only one torch will break down if the voltage over the torches is increased to a high enough value. The low impedance of the plasma in that torch will cause a significant voltage drop over the other torches preventing them from breaking down. A possibility to solve this problem which will not be further discussed is to apply a Transmission Line Transformer (TLT) to provide overvoltages on the remaining torches after each breakdown. However a second feature of the circuit in figure 3.1 beside the ability of synchronizing the currents is providing these overvoltages needed to fire all torches.

Some analytical analyses have been performed to get insight in the behavior of the circuit under different circumstances. A three torch configuration is selected to keep analytical analyses fairly simple. In figure 3.1 the circuit is fed by a voltage source. The pulse source which is used during the experiments will be presented in the next chapter.

![Figure 3.1: Synchronization circuit](image)

3.1 Analyses of different states

As earlier stated, equalizing the currents is important for a good energy distribution among the torches. Overvoltages on the second and third torch are also analyzed, because they are needed for firing the remaining torches. Full analytical analyses are described in appendix A.

The following states can occur, assuming torch 1($Z_1$) breaks down first, torch 2($Z_2$) second and torch 3($Z_3$) third:

<table>
<thead>
<tr>
<th>State</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$Z_3$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>low</td>
<td>high</td>
<td>high</td>
<td>&gt;0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>low</td>
<td>low</td>
<td>high</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>low</td>
<td>low</td>
<td>low</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>

Table 3.1: Different states of the synchronization circuit
3.1.1 Characteristics state 1

Current:

A small current will be able to flow through torch1 after the first breakdown (equation 3.1). k is the coupling factor between primary and secondary winding of the coupled inductors. The inductance in series with the torch will be $\frac{2}{3}$ of $L_1$, assuming that the coupling will be near 1. So for a large $\omega L_1$ and a relative low $Z_1$ the current will be small, resulting in low energy dissipation.

$$I_1 = \frac{3V_1}{j\omega L_1 (3-k^2) + 3Z_1}$$  \hspace{1cm} (3.1)

Equation 3.2 applies for the current through the secondary windings of the coupled inductors.

$$I_{sec} = \frac{1}{3} k \sqrt{\frac{L_2}{L_1}} I_1$$  \hspace{1cm} (3.2)

Overvoltage:

When $\omega L_1 >> Z_1$, $V_3$ and $V_{Z1}$ and $V_{Z3}$ will be $\frac{3}{2} V_c$ (equation 3.5). This can easily be explained. After Torch 1 breaks down, $V_1$ will almost be equal to $V_c$ because of the low impedance of $Z_1$ and the high impedance of $\frac{2}{3} L_1$, (Tr1). That voltage will be on the secondary winding of Tr1 if we choose equal values for $L_1$ and $L_2$. $V_2$ will equally be divided over $L_2(Tr_2)$ and $L_2(Tr_3)$ because they have identical impedances. So $V_3$ and $V_5$ will be half the voltage of $V_c$. These voltages are added to $V_c$ resulting in a voltage of $\frac{3}{2} V_c$ over torch 2 and 3.

$$V_3 = V_5 = \frac{V_1}{2} jk \omega L_2 (3-k^2) + L_2 Z_1$$  \hspace{1cm} (3.3)

$$V_3 = V_5 = \frac{V_1}{2} \text{ if } k = 1, \text{ } \omega L_1 >> Z_1$$  \hspace{1cm} (3.4)

$$V_{Z3} = V_{Z5} = \frac{3}{2} V_c$$  \hspace{1cm} (3.5)

3.1.2 Characteristics state 2

Current:

Equations 3.6 and 3.7 apply for the current through torch 1 and torch 2 after the second breakdown. The inductance in series with the torch 1 and torch 2 will be $\frac{1}{3}$ of $L_1$ if we assume that the coupling will be near 1. Note that the dl/dt of the current through torch 1 will increase.

$$I_1 = \frac{3V_c(j\omega L_1 + Z_2)}{-\omega^2 L_1 (3-2k^2) + j\omega L_1 (Z_1 + Z_2)(3-k^2) + 3Z_1 Z_2}$$  \hspace{1cm} (3.6)

$$I_2 = \frac{3V_c(j\omega L_1 + Z_2)}{-\omega^2 L_1 (3-2k^2) + j\omega L_1 (Z_1 + Z_2)(3-k^2) + 3Z_1 Z_2}$$  \hspace{1cm} (3.7)

These equations can be more generally written as:

$$I_n = \frac{3V_c}{j\omega L_1 (3-2k^2) + (3-k^2)(Z_1 + Z_2)} n = 1,2 \text{ if } \omega L_1 >> 3Z_1 Z_2$$  \hspace{1cm} (3.8)

Equation 3.9 applies for the current through the secondary windings of the coupled inductors.

$$I_{sec} = \frac{1}{3} k \sqrt{\frac{L_2}{L_1}} (I_1 + I_2)$$  \hspace{1cm} (3.9)
Overvoltage:

Voltages $V_1$ and $V_2$ will be near $V_c$ if the impedance of $\frac{1}{3}L_1$ is significantly larger than the impedance of torch 2 and 3. $V_1$ and $V_2$ are added to $V_c$ and will result in a voltage of $3V_c$ over torch 3 (equation 3.12).

$$V_s = \frac{k^2 V_c (j\omega L_3 (Z_1 + Z_2) - 2\omega^2 L_3^2)}{-\omega^2 L_3^2 (3 - 2k^2) + j\omega L_3 (Z_1 + Z_2) (3 - k^2) + 3Z_1Z_2} \quad (3.10)$$

$$V_s = 2V_c \quad \text{if} \quad k = 1, \quad \omega^2 L_3^2 \gg 3Z_1Z_2 \quad (3.11)$$

$$V_{3n} \approx 3V_c \quad (3.12)$$

3.1.3 Characteristics state 3

Current:

After breakdown of all torches the following equations are valid.

$$I_1 = \frac{3V_c (\omega^2 L_3^2 - j\omega L_3 (Z_1 + Z_2) - Z_3)}{3j\omega L_3 (1 - k^2) + \omega^2 L_3^2 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j\omega L_3 (3 - k^2)(Z_1, Z_2, Z_3) - Z_3 (Z_1, Z_2, Z_3)} \quad (3.13)$$

$$I_2 = \frac{3V_c (\omega^2 L_3^2 - j\omega L_3 (Z_1 + Z_2) - Z_3)}{3j\omega L_3 (1 - k^2) + \omega^2 L_3^2 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j\omega L_3 (3 - k^2)(Z_1, Z_2, Z_3) - Z_3 (Z_1, Z_2, Z_3)} \quad (3.14)$$

$$I_3 = \frac{3V_c (\omega^2 L_3^2 - j\omega L_3 (Z_1 + Z_2) - Z_3)}{3j\omega L_3 (1 - k^2) + \omega^2 L_3^2 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j\omega L_3 (3 - k^2)(Z_1, Z_2, Z_3) - Z_3 (Z_1, Z_2, Z_3)} \quad (3.15)$$

There will only be a stray inductance in series with each torch in case of a coupling smaller than 1 and under the condition that $\omega^2 L_1^2 \gg 3Z_1Z_2Z_3$. So the currents through torch 1 and 2 will significantly increase after the last breakdown. Notice that each branch of the circuit ‘sees’ the average of the three torch impedances. Equations 3.13-3.15 can be written in a general form as:

$$I_n = \frac{3V_c}{3j\omega L_3 (1 - k^2) + (3 - 2k^2)(Z_1 + Z_2 + Z_3)} \quad n = 1, 2, 3 \quad \text{if} \quad \omega^2 L_1^2 \gg 3Z_1Z_2Z_3 \quad (3.16)$$

It’s trivial that in the case of equal plasma impedances that condition isn’t required.

$$I_n = \frac{V_c}{j\omega L_3 (1 - k^2) + Z_n} \quad n = 1, 2, 3 \quad \text{if} \quad Z_1 = Z_2 = Z_3 \quad (3.17)$$

The following equation applies for the current through the secondary windings of the coupled inductors. The current is the average of the torch currents multiplied by the ratio of primary and secondary windings.

$$I_{sec} = \frac{1}{3} k \sqrt{\frac{L_0}{L_n}} (I_1 + I_2 + I_3) = \frac{1}{3} k \sqrt{\frac{L_0}{L_n}} (I_1 + I_2 + I_3) \quad (3.18)$$

Voltage:

The voltage over the primary windings can be calculated using the following equation. Note that the voltage will be zero in case of equal plasma impedances and $k=1$. The magneto motive forces (MMF) which drive flux around the cores, created by the currents through the primary and secondary inductors are cancelled under this condition.

$$V_{2n} = V_c - Z_n I_n \quad n = 1, 2, 3 \quad (3.19)$$
3.2 N torch configuration

Analytical analyses of a configuration which consists of more than three torches can become very complex, because large sets of equations have to be solved. Although some equations for an N torch configuration can be derived using the analyses of the three torch configuration. Using the result of equations 3.5 and 3.12 we can derive an expression for the voltages over the torches which still have to break down during synchronization. N is the total number of torches and $m$ is the number of ignited torches. The exact condition under which the equation will be valid is not known. It has to be presumed that $\omega L_1 \gg Z_n$. A recommendation would be to derive exact conditions for the following equations.

$$m=0 \quad V_{\text{torch}} = \frac{1}{3} V_c + V_i$$

$$m=1 \quad V_{\text{torch}} = \frac{1}{2} V_c + V_i$$

$$m=2 \quad V_{\text{torch}} = \frac{3}{2} V_c + V_i$$

$$V_{\text{torch}} \approx \left( \frac{m}{N-m} + 1 \right) V_c = \frac{N}{N-m} V_c \quad m=0,1,\ldots,N-1 \quad (3.20)$$

The impedance in series with the torches can be derived using the result of equations 3.1, 3.8 and 3.16.

$$m=1 \quad Z_{\text{leak}} = \frac{1}{3} j \omega L_1 (3 - k^2)$$

$$m=2 \quad Z_{\text{leak}} = \frac{1}{4} j \omega L_1 (3 - 2k^2) \quad Z_{\text{leak}} = \frac{1}{N} j \omega L_1 (N - mk^2) \quad m=0,1,\ldots,N-1 \quad (3.21)$$

The currents through the torches after all torches have been fired:

$$I_n = \frac{V'}{j \omega L_1 (1 - k^2) + \frac{1}{N} \sum_{i=1}^{N} Z_i} \quad n=1,2,\ldots,N \quad (3.22)$$

3.3 Transient mode analysis

The pulse source which will further be discussed in the next chapter partly consists of a high voltage capacitor ($C_h$) which will rapidly be discharged into the plasmas. This fast discharge is regarded as transient plasma. The capacitor will be charged to voltage $V_0$ before the first torch breaks down. Because of stray inductance in the coupled inductors the current will start oscillating after the last breakdown.

![Figure 3.2: Transient mode circuit](image)

The currents through the torches will be equal under the condition that $\omega^2 L_1^2 \gg 3Z_1Z_2Z_3$ and each branch of the circuit will see the same impedance (equation 3.16). The circuit can be reduced to figure 3.3a using this equation. We can reduce the circuit even further to a
standard RLC circuit, because large currents can only flow through each branch under the condition that all switches are closed. The current through each torch will be \( \frac{1}{3} \) of the current through the RLC circuit (figure 3.3b).

\[
L_s = L_t (1 - k^2) \quad (3.23)
\]

\[
L_{sv} = \frac{1}{3} L_t (1 - k^2) \quad (3.24)
\]

\[
Z_s = \frac{1}{3} (3 - 2k^2)(Z_1 + Z_2 + Z_3) \quad (3.25)
\]

\[
Z_v = \frac{1}{3} (3 - 2k^2)(Z_1 + Z_2 + Z_3) \quad (3.26)
\]

\[
I_n = \frac{1}{3} I_{Z_v} \quad n = 1, 2, 3 \quad (3.27)
\]

The resonance frequency can be calculated using following equation.

\[
\omega = \sqrt{\frac{1}{\frac{1}{3} L_t (1 - k^2) C_{fr}}} \quad (3.28)
\]

In practice the circuit will be under damped due to low damping coefficient. Equation 3.28 is valid under the condition that:

\[
Z_s C_{fr} < \frac{4L_{fr}}{Z_v} \quad (3.29)
\]

Note that equation is multiplied by \( \frac{1}{3} \), because as explained before the current through the torches will be \( \frac{1}{3} \) of the current through the RLC circuit. Derivations of these equations are explained in detail in [2].

\[
i_n(t) = \frac{1}{3} \frac{V_0}{L_s} \sqrt{\frac{1}{L_s C_{fr}^2 - \left(\frac{Z_v}{L_v}\right)^2}} e^{-\frac{Z_v}{2L_v}} \sin\left(\sqrt{\frac{1}{L_s C_{fr}^2 - \left(\frac{Z_v}{L_v}\right)^2}} \right) \quad n = 1, 2, 3 \quad (3.30)
\]
4 Circuit topology

The basic circuit of the pulse source is shown in figure 4.1. Snubber and trigger circuits are omitted because they are not relevant to understand the operation of the circuit.

![Figure 4.1: Multiple pulsed plasma torch circuit](image)

Power to the source is fed by three phase main supply. A rectifier provides the DC current necessary to charge the buffer capacitor $C_0$. $CL$ will be resonantly charged after $T1$ is triggered. $T1$ opens when the current through the thyristor is near zero. The voltage over $CL$ will then be double the voltage of $C_0$ under the condition that $C_0 \gg CL$. $C_{th}$ and $C_{tr}$ will be resonantly charged after $T2$ is triggered. High and low voltage capacitors are matched in order that all energy in $CL$ will be transferred to the high voltage side of the circuit. The transformer multiplies the voltage by a factor 30. The function of $T3$ will be explained later in this chapter. Thesis [1] provides a more detailed description of the pulse source.

4.1 Transient and thermal mode

The voltage on $C_{tr}$ and $C_{th}$ rises during the charge cycle to a point that the first torch breaks down. Figure 4.2 shows the actual pulse forming network connected to the synchronisation circuit.

![Figure 4.2: Multiple pulsed plasma torch circuit](image)

$C_{tr}$ will rapidly discharge in the plasma's after breakdown of all torches as discussed earlier. The voltage and current will oscillate during the discharge. This fast oscillation is regarded as the transient mode of the plasma.

The energy in $C_{th}$ isn't able to discharge fast in the torches due to the relative high inductance $L_5$. $C_{th}$ and $L_5$ form a LC circuit which is prevented from oscillating by diode $D2$. The energy in $C_{th}$ will be transported into the inductor at half the period time of the oscillating frequency. $D2$ will start conducting because the voltage over $C_{th}$ will be below the diodes threshold level. The energy in $L_5$ will subsequently discharge slowly in the torches. This slow discharge of energy is regarded as the thermal mode.
4.2 Pulse source stability problem

The pulse source is designed to charge the high voltage capacitors to 30kV. The plasma torches breakdown at voltages less than 20kV. A small amount of energy in CL is able to flow in the torches after breakdown. Nevertheless CL isn’t discharged completely every pulse, resulting in stability problems. This problem will be further clarified in this paragraph.

Figure 4.3a shows an equivalent circuit which can be regarded during charging of \( C_{th} \) and \( C_{tr} \). The pulse transformer is omitted and considered ideal. Stray inductance and winding resistance are neglected, the mutual inductance is considered very high. All components at the high voltage side are transferred to the low voltage side of the circuit. They are scaled with \( 30^2 \), the square of the ratio between primary and secondary windings of the transformer. The coupled inductors Tr1-3 and the torches can be modelled as a stray inductance \( L_{sv} \) and a resistance \( L_{sv} \).

![Figure 4.3: Charge and discharge equivalent circuits](image)

\( C_{th} \) and \( C_{tr} \) are resonantly charged after thyristor \( T_2 \) is triggered (figure 4.3a). Subsequently they will be discharged in \( Z_v \) after breakdown of the torches (switch closes) as pointed out in figure 4.3b. \( T_2 \) will still be open at this time and the low voltage part of the circuit will almost virtually be short circuited. Since \( Z_v/900 \) will be near zero. Only a small amount of the energy left in CL will be dissipated in \( Z_v \). The remaining energy will flow through the circuit back in CL because of the low damping coefficient. The current through \( T_2 \) will become zero and the thyristor closes. The voltage over CL will now be inverted. The real problem occurs in the next charging cycle. Charging CL when it has a negative charge isn’t allowed because the voltage over CL will explode. Just inverting the voltage before recharging isn’t an option because CL can only be completely charged if the voltage over CL is significantly lower than the voltage over \( C_0 \). The problem is solved by dumping the remaining energy in resistor RL. A RLC circuit (figure 4.1) via \( T_3 \)-RL-L3-L1-CL is formed after \( T_3 \) is triggered. The circuit has to be critically damped to dissipate all energy. The value of RL can be calculated using following equation:

\[
RL = \frac{4(L_1 + L_3)}{CL} \tag{4.1}
\]

CL will be completely discharged if \( C_{tr} \) and \( C_{th} \) are charged to 30kV. So the maximum amount of energy in CL will be \( \frac{1}{2} (C_{tr}+C_{th})(30e3)^2 \) before discharge if the losses are neglected. The minimum energy dissipated in the torches will be \( \frac{1}{2} (C_{tr}+C_{th})(V_{breakdown})^2 \). The difference in energy will remain in CL and has to be dissipated. The remaining energy per pulse multiplied by the repetition rate \( f \) results in a worst case estimate of the power dissipation in the resistor.

\[
P_{RL} = \frac{1}{2} (C_{tr} + C_{th})((30e3)^2 - V_{breakdown}^2) f \quad \text{if} \quad CL = 30^2(C_{tr} + C_{th}) \tag{4.2}
\]
5 Simulation results

5.1 Synchronization circuit simulations

Pspice was used to simulate the behavior of the circuit during synchronization, transient and thermal mode. The schematic in figure 5.1 is a model for the complete circuit. The high voltage transformer is omitted because it doesn’t have any significant influence on the high voltage circuit. The torches are modeled as resistors with switches. Not a very good model because the impedance of the torch will change in time due to the negative VI characteristic of the plasma. Values of the resistors are chosen unequal to investigate if the circuit behaves as expected. The component values in figure 5.1 have been used during the following simulations.

Figure 5.1: Simulation circuit

5.1.1 Synchronization

CL is charged to 30kV. C1 and C2 will be resonantly charged after switch U2 closes. U1 will be closed when the current through the switch becomes zero. Switch U3 will be closed after the voltage on torch 1 has risen to 20kV. The voltage on torch 2 and 3 will increase to \( \frac{3}{2} \times 20kV \), since \( \omega L_1 \gg Z_{\text{torch1}} \) (equation 3.5). A small leakage current will start flowing into the load. After switch U4 is closed the voltage will increase even further to 3x20kV (equation 3.12). The leakage current will slightly increase. Finally the third torch is fired and \( C_T \) will rapidly discharge. The resulting oscillation is caused by stray inductance.

Figure 5.2: Breakdown simulation result
5.1.2 Transient mode

Figure 5.3b shows a simulation of the currents in the transient mode. The currents are equal because the condition $\omega^2 L_1 r^2 >> 3Z_1 Z_2 Z_3$ is easily satisfied. Substitution of the currents resonant frequency, and resistor values will lead to: $3.9e9 >> 240$. The current through the secondary winding is in this case also equal if it is scaled with the winding ratio of the coupled inductors (equation 3.18).

\[ I_{\text{Torch1}} = I_{\text{Torch2}} = I_{\text{Torch3}} \]

![Figure 5.3: Transient mode simulation result](image)

It is trivial that the voltages in figure 5.3a aren't equal. Identical voltages are only possible in case of equal torch impedances. It is expected that due to similar VI characteristic of the plasmas in each torch, the voltages will have approximately the same values.

5.1.3 Thermal mode

Figure 5.4a and b show the voltages and currents in the thermal mode. The circuit isn’t able to equalize the currents. Frequency components in the current are relatively low. The condition $\omega^2 L_1 r^2 >> 3Z_1 Z_2 Z_3$ isn’t satisfied.

![Figure 5.4: Thermal mode simulation result, $L_{(T_{0})}=4\text{mH}$](image)
The problem can be solved by increasing the value of inductor $L_1(T_{r3})$. Figure 5.5a and b show the voltages and currents in the case that $L_1$ is increased to 400mH. Note that the pulse length stays 10ms. As stated before it isn't sure that the condition $\omega^2 L_1^2 >> 3Z_1Z_2Z_3$ has to be satisfied because the plasma impedance interacts with the value of the current in time. Without a proper plasma model we can only use trail en error to find a proper value for $L_1$.

![Figure 5.5: Transient mode simulation result, $L_1(T_{r3})=400$mH](image)

### 5.2 Transient mode model validation

In paragraph 4.2 the "transient" circuit is simplified to a simple RLC circuit. The following equations are derived earlier.

$$
L_{n,v} = \frac{1}{3} L_1 (1 - k^2) \\
Z_v = \frac{1}{3} (3 - 2k^2)(Z_1 + Z_2 + Z_3)
$$

if $\omega^2 L_1^2 >> 3Z_1Z_2Z_3$

$$
I_\nu = \frac{1}{3} I_{Z_\nu} \quad n = 1,2,3
$$

Simulations have been used to validate these equations. The schematic in figure 5.6 shows both circuits next to each other. Using the equations above the value’s for $L_{sv}$ and $Z_v$ can be calculated. $C_2$ and $C_3$ are charged to 20kV. The switch of the RLC circuit is closed simultaneously with the switch of torch 3, after torch 1 and 2 have been fired. Figure 5.7 shows the simulation result. The current through the RLC circuit is divided by 3. The two waveforms seem to be identical as expected.

![Figure 5.6: Equivalent simulation circuit](image)
Figure 5.7: Transient mode model validation simulation result

Figure 5.8 shows a bode diagram of the two circuits. A 1V voltage source was used to sweep the circuits. In this case the model will be valid for frequencies higher than 3 kHz. Notice that the stray inductance can be neglected for frequencies below 10 kHz. The equation for oscillation frequency (equation 3.28) in the transient mode is valid since the frequency will be several megahertz in practice.
6 Measurement results

6.1 Synchronization circuit measurements

The following measurement results have been obtained using the configuration in table 6.1. The coupled inductors have been constructed using ferrite toroidal cores of N30 material. The cores have a cross section of 2,7cm² and a magnetic length of 26,4cm.

<table>
<thead>
<tr>
<th>CL</th>
<th>RL</th>
<th>$C_{th}$</th>
<th>$C_{tr}$</th>
<th>$L_s$</th>
<th>$n_1(T_{r1})$</th>
<th>$n_2(T_{r2})$</th>
<th>$L_1(T_{r1})$</th>
<th>$L_2(T_{r2})$</th>
<th>$k(T_{r1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6μF</td>
<td>8Ω</td>
<td>5.3nF</td>
<td>1.3nF</td>
<td>3.5mH</td>
<td>26</td>
<td>20</td>
<td>3.85mH</td>
<td>2.37mH</td>
<td>0.9991</td>
</tr>
</tbody>
</table>

Table 6.1: Circuit configuration

Voltages have been measured using three North Star PVM-01 high voltage probes. Currents have been measured using three Pearson 6600 current monitors.

6.1.1 Synchronization

Figure 6.1a shows the voltages on the torches at breakdown. Overvoltages on the remaining torches after the first breakdown are clearly visible. The overvoltages are theoretically limited to $3/2V_0$ after the first breakdown and $3V_0$ after the second breakdown. These voltages aren’t reached because the torches break down at lower voltages. The torches are fired within less than 100ns. The high frequent oscillation is caused by parasitic capacitance in the coupled inductors. Figure 6.1b shows the currents at breakdown. Note that the voltages over the torches are measured simultaneously. The currents are also measured simultaneously but not during the voltage measurement. The order of breakdown is therefore different. A small current is flowing through the torches before the first one breaks down. This could be a capacitive current between the electrodes of the torches or maybe due to some corona discharge.

![Figure 6.1: Typical waveforms at breakdown](image)

Figure 6.1: Typical waveforms at breakdown
6.1.2 Transient mode

The currents through the torches are equal as shown in figure 6.2b. Amplitude of the current depends mainly on the value of the stray inductance and \( C_{tr} \). The current offset at the end of the transient mode is caused by the current contributed of the thermal mode circuit. The voltages over the three torches are similar (figure 6.2a). This means that the impedances of the plasmas in the torches are more or less equal. The transient mode has a length of about 10\( \mu \)s, and a peak current of 120A. Resonant frequency is 2.7MHz.

![Figure 6.2: Typical waveforms in the transient mode](image)

6.1.3 Thermal mode

Accurate voltage measurement in the thermal mode proves to be difficult because a high voltage probe has to be used to measure relative low voltages. The high voltage probe on torch 2 has relatively small deviation, resulting in a large error for measuring low voltages. Since the voltage on torch 1 and 3 are equal we can conclude that the voltage over torch 2 must also be similar. Measurements with exchanged probes confirmed that the deviation is caused by the probe. The current waveforms aren’t measured simultaneously with the voltages, but the waveforms are similar because no torches extinguish premature in both figures.

![Figure 6.3: Typical waveforms in the thermal mode](image)
The circuit behaves like it should, but are there still some problems in the thermal mode. Figure 6.4 shows some examples of typical unbalanced situations. The torch currents don’t always stay synchronized until the end of the pulse.

![Graphs showing typical unbalanced situations](image)

A non-balanced situation occurs when one torch starts to become dominant. The circuit tries to keep the currents together to a point that the impedances of the plasmas (figure 6.5) become relatively high. The circuit fails therefore usually at the end of the pulse. The plasma impedance is obtained by dividing the measured torch voltage by the torch current.

![Graph showing typical plasma impedance](image)
6.2 Energy measurements

In the figure below the energy plot of one torch is shown. The energy in $C_r$ flows rapidly into the plasma after breakdown of all torches. The remaining energy of the thermal mode is slowly dissipated afterwards. The histogram shows the energy deviation per pulse. The measurement consists of 1000 samples. All energy measurements have been recorded using a Tiepie Handyscope HS3.

![Typical energy per pulse plot](image1)

![Typical energy per pulse histogram](image2)

Figure 6.6: Typical energy plots

The stability of the energy per pulse per torch depends on the breakdown voltage and the correct operation of the synchronization circuit. Less energy will be stored in $C_r$ and $C_{th}$ if the torches breakdown at a lower voltage. Beside that the problem of premature torch extinguishment can occur. This will result in unequal distribution of energy between the torches. Averaging of energy per pulse per torch is used to determine the mean energy per pulse $\mu$ (equation 6.2) and the standard deviation $\sigma$ (equation 6.3), where $N$ is the number of measured pulses. Energy per pulse is as shown in figure 6.6a calculated using equation 6.1.

\[
E_i = \int v_i(t)j_i(t)dt
\]  

\[
\mu = \frac{1}{N} \sum_{i=1}^{N} E_i
\]  

\[
\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (E_i - \mu)^2}
\]

The total energy per pulse is measured to calculate the energy deviation due to jittering of the breakdown voltage. This result is compared to the deviation of the energy per torch to determine the energy jittering due to premature torch extinguishment. The deviation $\sigma$ is divided by the mean $\mu$. The scaled deviation ($\sigma/\mu$) simplifies comparison of measurement results.
6.2.1 Value of \( L_1(Tr_x) \)

An important parameter of the synchronization circuit is the value of \( L_1(Tr_x) \). Choosing a proper value for \( L_1(Tr_x) \) is difficult due to the unstable and non linear VI characteristic of the plasma. Three configurations are compared to find a proper value for \( L_1(Tr_x) \) (table 6.2b). These configurations are chosen because of practical reasons. The values of \( L_1 \) and \( L_2 \) are measured using a LCR meter. The \( k \) factor is computed using equation 3.28 and the measured resonance frequency. The parameters in table 6.2a are fixed in all three configurations. Which results in a peak current of about 15A in the thermal mode and a total pulse length of about 750 \( \mu \)s.

<table>
<thead>
<tr>
<th>( CL )</th>
<th>( RL )</th>
<th>( C_{in} )</th>
<th>( C_{r} )</th>
<th>( L_{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6( \mu )F</td>
<td>8( \Omega )</td>
<td>5.3nF</td>
<td>1.3nF</td>
<td>3.5mH</td>
</tr>
</tbody>
</table>

(a)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>13</td>
<td>13</td>
<td>0.96mH</td>
<td>0.96mH</td>
<td>0.9834</td>
</tr>
<tr>
<td>1.2</td>
<td>20</td>
<td>13</td>
<td>2.37mH</td>
<td>0.96mH</td>
<td>0.9944</td>
</tr>
<tr>
<td>1.3</td>
<td>26</td>
<td>20</td>
<td>3.85mH</td>
<td>2.37mH</td>
<td>0.9991</td>
</tr>
</tbody>
</table>

(b)

Table 6.2: Circuit configuration 1

Results

Table 6.3 shows the measured peak currents, resonance frequencies and the length of the transient mode in all three configurations. A higher coupling of the coupled inductors results in faster dissipation of energy in the transient mode.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Peak current</th>
<th>Frequency</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>85A</td>
<td>1.36MHz</td>
<td>( \approx )15us</td>
</tr>
<tr>
<td>1.2</td>
<td>92A</td>
<td>1.49MHz</td>
<td>( \approx )15us</td>
</tr>
<tr>
<td>1.3</td>
<td>120A</td>
<td>2.7MHz</td>
<td>( \approx )10us</td>
</tr>
</tbody>
</table>

Table 6.3: Transient mode properties

A repetition rate of 10Hz was used during the energy measurements. A small gas flow is applied for recovery of the torch after each pulse. Hundred pulses per torch have been measured to calculate the mean energies and standard deviations. The bar graphs in figures 6.7-9 (a) show an almost equal distribution of energy in both transient and thermal mode for all three configurations. In figures 6.7-9(b) the scaled deviations are shown for each configuration. The total \( \sigma/\mu \) presents the deviation of energy due to jittering of the breakdown voltage. The \( \sigma/\mu \) of the total energy should be equal to the scaled deviations of the torches in case of optimal operation of the synchronization circuit. Clearly this isn’t the case but enlarging \( L_1(Tr_x) \) has a positive effect. The value of \( L_1(Tr_x) \) is of less importance for the deviation in the transient mode because of the relative high frequency.
Figure 6.7: Configuration 1.1

Figure 6.8: Configuration 1.2

Figure 6.9: Configuration 1.3
6.2.2 Value of $L_s$

Shortening the thermal mode pulse could possibly have a positive effect on the deviation of the energy per torch. The length of the pulse can be adjusted by varying the value of $L_s$. The values of $L_1$ and $L_2$ in table 6.4a are chosen because measurements with these values gave the worst result in the previous paragraph. So improvement due to a shorter pulse should be clearly visible. Measurements with three values in table 6.4b are compared which each other.

![Configuration L_s values](image)

Table 6.4: Circuit configuration 2

<table>
<thead>
<tr>
<th>Configuration</th>
<th>$L_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>3.5mH</td>
</tr>
<tr>
<td>2.2</td>
<td>2.25mH</td>
</tr>
<tr>
<td>2.3</td>
<td>1.5mH</td>
</tr>
</tbody>
</table>

Results

Table 6.5 shows the measured peak currents of the thermal mode. The energy per pulse should be equal in each configuration, resulting in a higher peak current when $L_s$ has a smaller value.

![Table 6.5: Thermal mode properties](image)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Peak current</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>15.5A</td>
<td>750us</td>
</tr>
<tr>
<td>2.2</td>
<td>17A</td>
<td>550us</td>
</tr>
<tr>
<td>2.3</td>
<td>21A</td>
<td>400us</td>
</tr>
</tbody>
</table>

The typical arc resistances per pulse are plotted in figure 6.10a. A lower value of $L_s$ results in a lower impedance at the beginning of the pulse. In each configuration $+/- 90\%$ of the energy is dissipated at $2/3$ of the pulse length (figure 6.10b). The plasma impedances have different values at this point in time. Higher impedance will probably enlarge the chance of premature torch extinguishment at the end of the pulse, resulting in a higher deviation of the energy per pulse.

![Figure 6.10: Effect of varying L_s](image)

Figures 6.11-13 show the bar graphs of mean energy and scaled deviation. The deviation of energy in the thermal mode is slightly less in configuration 2 than in configuration 1. Configuration 2 and 3 has similar results. This can be explained due to the fact that configuration 2 and 3 have a more similar arc resistance curves than configuration 1. The effect of varying $L_s$ isn't noticeable in the other two configurations where $L_1(Tr_2)$ has a higher value. Concluded can be that a smaller value of $L_s$ does not have a real positive effect on the stability.
Figure 6.11: Configuration 2.1

Figure 6.12: Configuration 2.2

Figure 6.13: Configuration 2.3
6.2.3 Value of $C_{th}$

The third parameter considered is the amount of energy per pulse in the thermal mode. The configuration in table 6.6b shows the measurement configurations. The capacity is enlarged in two steps of 3.3nF.

<table>
<thead>
<tr>
<th>$C_{th}$</th>
<th>$L_1(T_{r1})$</th>
<th>$L_2(T_{r2})$</th>
<th>$k(T_{r2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3nF</td>
<td>3.85mH</td>
<td>2.37mH</td>
<td>0.9991</td>
</tr>
</tbody>
</table>

(a)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>CL</th>
<th>RL</th>
<th>$C_{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>3μF</td>
<td>11Ω</td>
<td>2nF</td>
</tr>
<tr>
<td>3.2</td>
<td>6μF</td>
<td>8Ω</td>
<td>5.3nF</td>
</tr>
<tr>
<td>3.3</td>
<td>9μF</td>
<td>6Ω</td>
<td>8.6nF</td>
</tr>
</tbody>
</table>

(b)

Table 6.6: Circuit configuration 3

Results

Enlargement of $C_{th}$ results in a longer pulse length and a higher peak current. However the $dl/dt$ of the thermal mode remains more or less the same in contrast to the $dl/dt$ when enlarging the pulse length by varying $L_5$.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Peak current</th>
<th>Length</th>
<th>$dl/dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>9A</td>
<td>=550us</td>
<td>=16364A/s</td>
</tr>
<tr>
<td>3.2</td>
<td>13.5A</td>
<td>=750us</td>
<td>=18000A/s</td>
</tr>
<tr>
<td>3.3</td>
<td>15.5A</td>
<td>=900us</td>
<td>=17222A/s</td>
</tr>
</tbody>
</table>

Table 6.7: Thermal mode properties

Figures 6.14-16(a) shows a more or less equal energy distribution among the torches for all configurations. Remarkable is the fact that doubling $C_{th}$ does not lead to double the energy dissipated in the torches. More likely is some kind of measuring error. A reliable energy balance is difficult to obtain. The deviations for configuration 3.1 and 3.2 are similar. The deviation of the total energy in configuration 3.3 is slightly larger which results in a larger deviation of the energy dissipated in the torches.
Figure 6.14: Configuration 3.1

Figure 6.15: Configuration 3.2

Figure 6.16: Configuration 3.3
7 Conclusions and recommendations

Conclusions

- The synchronization circuit is able to ensure a breakdown of all torches within 100ns. The energy per pulse can be homogenous distributed among the three torches with a σ/μ of less than 0.14. The deviation in energy is caused by jittering of the breakdown voltage and premature torch extinguishment.

- The critical parameter in the circuit is the value of \( L_1(T_{rx}) \). Enlarging this value will result in a lower probability of premature torch extinguishment during the thermal mode. Ensuring no premature extinguishment at all will probably prove to be very difficult because the plasma impedance rises very rapidly at the end of the pulse.

- A high value for \( L_1(T_{rx}) \) isn’t required during the transient mode because of the relative high resonance frequency. The resonance frequency and peak current are determined by the value of \( C_r \) and the stray inductance in the coupled inductors.

- The pulse shape varied by \( L_5 \) and \( C_m \) has no significant influence on the energy distribution among the torches.

Recommendations

- Creating a proper plasma model which can be used to simulate the stochastic behavior during the thermal mode. The model could be applied to fine-tune the value of \( L_1(T_{rx}) \).

- Scaling up the configuration to study the effects on energy distribution and breakdown using a larger amount of torches.

- Study the possibility of saturation of the ferrite cores of the coupled inductors during the transient mode.

- Find a proper solution for the pulse source stability problem. Efficiency of the source is very poor because more than half of the energy is dumped every pulse.

- Improve voltage measurement accuracy of the thermal mode. 12 bit resolution for measurement of voltages in the range of 0 to 40kV is very poor, and will cause a measurement error during the thermal mode. The accuracy of the high voltage probe for low voltages has to be checked.
Bibliography


The voltages of transformers can be described using the following equations. M is the mutual inductance of the primary and secondary winding. Winding resistance and core losses are neglected to simplify the analysis.

\[ V_1 = j\omega L_1 I_1 + j\omega M I_{sec} \]
\[ V_2 = j\omega L_2 I_{sec} + j\omega M I_1 \]
\[ V_3 = j\omega L_3 I_2 + j\omega M I_{sec} \]
\[ V_4 = j\omega L_1 I_{sec} + j\omega M I_2 \]
\[ V_5 = j\omega L_4 I_3 + j\omega M I_{sec} \]
\[ V_6 = j\omega L_2 I_{sec} + j\omega M I_3 \]

The following states can occur, assuming torch 1 breaks down first, torch 2 second and torch 3 third.

<table>
<thead>
<tr>
<th>State</th>
<th>( Z_1 )</th>
<th>( Z_2 )</th>
<th>( Z_3 )</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>low</td>
<td>high</td>
<td>high</td>
<td>&gt;0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>low</td>
<td>low</td>
<td>high</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>low</td>
<td>low</td>
<td>low</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>

A set of equations can be formulated for each state using Kirchhoff’s law. Solving these sets will result in equations for \( I_{1-3} \) and \( V_{1-3} \).
State 1:

\[ V_2 + V_4 + V_6 = 0 \quad I_2 = I_3 = 0 \]
\[ j \omega L_2 I_{\text{sec}} + j \omega M I_1 + j \omega L_2 I_{\text{sec}} + j \omega M I_2 + j \omega L_2 I_{\text{sec}} + j \omega M I_3 = 0 \]
\[ 3 j \omega L_2 I_{\text{sec}} = -j \omega M I_1 \quad I_{\text{sec}} = -\frac{M I_1}{3L_2} = -\frac{1}{3} k \sqrt{\frac{L_2}{I_1}} \]

\[ V_c = V_i + Z_i I_i \]
\[ V_c = j \omega L_4 I_1 + j \omega M I_{\text{sec}} + Z_i I_i \]
\[ V_c = j \omega L_4 I_1 - j \omega M \frac{M I_1}{3L_2} + Z_i I_i \quad M = k \sqrt{L_4 L_2} \]
\[ V_c = j \omega L_4 I_1 - \frac{1}{3} k^2 j \omega L_4 I_1 + Z_i I_i \]
\[ V_c = (j \omega L_4 (1 - \frac{1}{3} k^2) + Z_i) I_i \]
\[ I_i = \frac{3V_c}{j \omega L_4 (3 - k^2) + 3Z_i} \]

\[ V_3 = V_5 = j \omega M I_{\text{sec}} = -j \omega M \frac{M I_1}{3L_2} = -\frac{3V_c k^2 j \omega L_4}{3 j \omega L_4 L_2 (3 - k^2) + 3L_2 Z_i} = \frac{-V_c k^2}{3 - k^2 + \frac{Z_i}{j \omega L_4}} \]
\[ V_3 = V_5 = \frac{-V_c}{2} \quad \text{if} \quad k = 1, \quad \omega L_4 >> Z_i \]
State 2:

\[ V_2 + V_4 + V_6 = 0 \quad I_3 = 0 \]

\[ j \omega L_2 I_{\text{sec}} + j \omega M I_n + j \omega L_2 I_{\text{sec}} + j \omega M I_2 + j \omega L_2 I_{\text{sec}} + j \omega M I_3 = 0 \]

\[ 3j \omega L_2 I_{\text{sec}} = -j \omega M (I_1 + I_2) \quad I_{\text{sec}} = -\frac{M (I_1 + I_2)}{3L_2} = \begin{cases} \frac{1}{2} k \sqrt{\frac{L_1}{L_2}} (I_1 + I_2) \\ \text{if } n = 1,2 \end{cases} \]

\[ V_i = V_{2n-i} + Z_n I_n \quad n = 1,2 \]

\[ V_i = j \omega L_1 I_{\text{sec}} + j \omega M I_n + Z_n I_n \]

\[ V_i = j \omega L_1 I_{\text{sec}} - j \omega M \frac{M (I_1 + I_2)}{3L_2} + Z_n I_n \quad M = k \sqrt{L_1 L_2} \]

\[ V_c = j \omega L_1 I_{\text{sec}} - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2) + Z_n I_n \]

\[ V_i = j \omega L_1 I_1 - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2) + Z I_1 \]

\[ V_i = j \omega L_1 I_2 - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2) + Z_2 I_2 \]

\[ I_1 = \frac{3V_c (j \omega L_1 + Z_1)}{-\omega^2 L_1^2 (3-2k^2) + j \omega L_1 (Z_1 + Z_2)(3-k^2) + 3Z_1 Z_2} \]

\[ I_2 = \frac{3V_c (j \omega L_1 + Z_2)}{-\omega^2 L_1^2 (3-2k^2) + j \omega L_1 (Z_1 + Z_2)(3-k^2) + 3Z_1 Z_2} \]

\[ I_1 = \frac{3V_c (1 + \frac{Z_2}{j \omega L_1})}{j \omega L_1 (3-2k^2) + (3-k^2)(Z_1 + Z_2) + \frac{3Z_1 Z_2}{j \omega L_1}} \]

\[ I_n = \frac{3V_c}{j \omega L_1 (3-2k^2) + (3-k^2)(Z_1 + Z_2)} \quad n = 1,2 \quad \text{if } \omega L_1 >> 3Z_1 Z_2 \]

\[ V_3 = j \omega M I_{\text{sec}} = -j \omega M \frac{M (I_1 + I_2)}{3L_2} = -\frac{1}{3} k^2 j \omega L_1 (I_1 + I_2) \]

\[ V_3 = \frac{-3k^2 V_c j \omega L_1 (2 j \omega L_1 + Z_1 + Z_2)}{-3 \omega^2 L_1^2 (3-2k^2) + 3 j \omega L_1 (Z_1 + Z_2)(3-k^2) + 9Z_1 Z_2} \]

\[ V_3 = \frac{-k^2 V_c (j \omega L_1 (Z_1 + Z_2) - 2 \omega^2 L_1^2)}{-\omega^2 L_1^2 (3-2k^2) + j \omega L_1 (Z_1 + Z_2)(3-k^2) + 3Z_1 Z_2} \]

\[ V_3 = \frac{k^2 V_c (j \omega L_1 (Z_1 + Z_2) - 2 \omega^2 L_1^2)}{-3 + 2k^2 + \frac{(Z_1 + Z_2)(3-k^2)}{\omega L_1^2} + \frac{3Z_1 Z_2}{\omega L_1^2}} \]

\[ V_4 = \frac{2V_c}{-1} = -2V_c \quad \text{if } k = 1, \omega^2 L_1^2 >> 3Z_1 Z_2 \]
State 3:

\[ V_2 + V_4 + V_6 = 0 \]

\[ j \omega L_2 I_{sec} + j \omega M I_1 + j \omega L_2 I_{sec} + j \omega M I_2 + j \omega L_2 I_{sec} + j \omega M I_3 = 0 \]

\[ 3 j \omega L_2 I_{sec} = -j \omega M (I_1 + I_2 + I_3) \quad I_{sec} = \frac{M (I_1 + I_2 + I_3)}{3 L_2} = -\frac{1}{3} k \sqrt{\frac{\epsilon_n}{L_2}} (I_1 + I_2 + I_3) \]

\[ V_c = V_{2n-n} + Z_n I_n \quad n = 1, 2, 3 \]

\[ V_c = j \omega L_1 I_n + j \omega M I_{sec} + Z_n I_n \]

\[ V_c = j \omega L_1 I_n - j \omega M \frac{M (I_1 + I_2 + I_3)}{3 L_2} + Z_n I_n \quad M = k \sqrt{L_1 L_2} \]

\[ V_c = j \omega L_1 I_n - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2 + I_3) + Z_n I_n \]

\[ V_c = j \omega L_1 I_n - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2 + I_3) + Z_1 I_1 \]

\[ V_c = j \omega L_1 I_n - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2 + I_3) + Z_2 I_2 \]

\[ V_c = j \omega L_1 I_n - \frac{1}{3} k^2 j \omega L_1 (I_1 + I_2 + I_3) + Z_3 I_3 \]

\[
I_1 = \frac{3 V_c (\omega^2 L_1^3 - j \omega L_1 (Z_2 + Z_3) - Z_2 Z_3)}{3 j \omega^3 L_1^4 (1 - k^2) + \omega^2 L_1^3 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j \omega L_1 (3 - k^2)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) - 3Z_1 Z_2 Z_3}
\]

\[
I_2 = \frac{3 V_c (\omega^2 L_1^3 - j \omega L_1 (Z_1 + Z_3) - Z_1 Z_3)}{3 j \omega^3 L_1^4 (1 - k^2) + \omega^2 L_1^3 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j \omega L_1 (3 - k^2)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) - 3Z_1 Z_2 Z_3}
\]

\[
I_3 = \frac{3 V_c (\omega^2 L_1^3 - j \omega L_1 (Z_1 + Z_2) - Z_1 Z_2)}{3 j \omega^3 L_1^4 (1 - k^2) + \omega^2 L_1^3 (3 - 2k^2)(Z_1 + Z_2 + Z_3) - j \omega L_1 (3 - k^2)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) - 3Z_1 Z_2 Z_3}
\]

if \( Z_1 \neq Z_2 \neq Z_3 \)

\[
I_1 = \frac{3 V_c (1 - \frac{Z_1 Z_2}{\omega L_1}) - Z_1 Z_2}{3 j \omega L_1 (1 - k^2) + (3 - 2k^2)(Z_1 + Z_2 + Z_3) - \frac{j (3 - k^2)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{\omega L_1} - \frac{3 Z_1 Z_2 Z_3}{\omega L_1^4}}
\]

\[
I_n = \frac{3 V_c (1 - \frac{Z_1 Z_2}{\omega L_1}) + \omega^2 L_1^2 Z_3}{3 j \omega L_1 (1 - k^2) + (3 - 2k^2)(Z_1 + Z_2 + Z_3) - \frac{j (3 - k^2)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{\omega L_1} - \frac{3 Z_1 Z_2 Z_3}{\omega L_1^4}} \quad \text{if} \quad \omega^2 L_1^2 >> 3Z_1 Z_2 Z_3
\]

if \( Z_1 = Z_2 = Z_3 = Z \)

\[
I_n = \frac{3 V_c (\omega^2 L_1^3 - 2 j \omega L_1 Z - Z^2)}{3 j \omega^3 L_1^4 (1 - k^2) + 3 \omega^2 L_1^2 Z(3 - 2k^2) - 3 j \omega L_1 Z^2 (3 - k^2) - 3Z^3}
\]

\[
I_n = \frac{V_c (Z + j \omega L_1) (Z + j \omega L_1) (Z + j \omega L_1) (1 - k^2)}{(Z + j \omega L_1) (Z + j \omega L_1) (Z + j \omega L_1) (1 - k^2)}
\]

\[
I_n = \frac{V_c}{j \omega L_1 (1 - k^2) + Z_n} \quad n = 1, 2, 3
\]