Electromagnetic Excitation of a Thin Wire.
An Analytic Approach.

by

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Abstract

Chaff is the military codename for clouds of thin metal wires or strips. These clouds can be used to protect ships and aircraft against hostile missiles. Electromagnetic excitation of such a cloud by the radar of the missile causes additional reflections, degrading the effectiveness of the missile. In order to simulate the electromagnetic behaviour of chaff (statistically or stochastically), it is necessary to describe the behaviour of a single wire with an analytic formula that allows fast numerical evaluation.

After deriving the integral equation for the total current on a wire we find that the exact form of Pocklington’s integral equation is an approximation, while the reduced form describes the behaviour of the wire, except for the radial currents at the end faces, exactly. For an infinite wire (no ends) we find the current, obtained by inverting the exact integral equation to be equal to the exact solution derived by separation of variables. Another reason for considering the infinite wire is that, for thin wires, there exists similarity between the particular solution and the homogeneous solution of Hallén’s integral equation. The idea is to approximate the homogeneous solution in terms of the particular solution for delta-gaps placed at suitable chosen locations.

Since there does not exist a homogeneous solution on an infinite wire, the response of a delta-gap source (transmitting state) on an infinite wire is the particular solution.

Since plane-wave excitation (receiving state) can be mathematically modeled by placing delta-gaps, excited with the correct phase, all over the finite wire, the delta-gap excitation is investigated. Inverting the exact integral equation leads to an unsolvable integral. By using Hallén’s iterative scheme we find a first-order approximation. Inverting the "exact" form of Pocklington’s integral equation leads to an approximation which allows fast numerical evaluation.

In the case of a delta-gap excitation on a finite wire, we want to model the reflections by a distribution of delta-gap sources on an infinite wire.

We distinguish the following distributions:

First, an infinite number of delta-gaps can be distributed on the, infinitely extended, finite wire such that the current outside the interval $(-L, L)$ becomes zero. This model is valid only for exponential responses of the delta-gap excitation of an infinite wire, which is the case for the thin-wire approximation. A second condition that has to be met is that the current on the delta-gap excited
infinite wire must go to zero for great distances form the source.
Second, because of computational reasons, the delta-gap distribution is truncated. Two additional sources are added to set the current to zero at the ends. In the extreme case, only two delta gaps are located at the ends of the wire. For long wires the model produced accurate results. In the case of short wires (half wavelength long) the approximation is not accurate enough due to resonance. Therefore we introduce a reflection coefficient at the ends of the wire. This results in a better approximation at the cost of a non-zero current at the ends of the wire.

When we consider an incident plane-wave on a finite wire it is not possible to write the response in terms of the response of a plane wave on an infinite wire as we did for the delta-gap excitation. The plane-wave excitation is mathematically modeled by placing "delta-gaps", excited with correct phase, all over the finite wire. Integrating the response of these delta-gaps over the length of the wire leads to the response of the current on a finite wire. For long wires we obtain an accurate response. For short wires (half wavelength) the results contain inaccuracies due to the fact that the reflection coefficient should really depend on the location of the "delta-gap" source.
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Chapter 1

Introduction

In recent years, there has been increasing interest in electromagnetic scattering by chaff. Chaff is a general term for elementary passive reflectors, absorbers, or refractors of radar, communication and other weapons system radiators, which can be suspended in air for the purpose of confusion, screening or deflecting the performance of victim electronic systems in a negative manner. Chaff is used as an Electronic CounterMeasure (ECM) to reduce the effectiveness of radar and communication systems or to protect ships and aircraft against hostile missiles. This technique originated in the 1940s to confuse German radar and is still being used during wartime.

Chaff usually consists of a large number of metal or metallic-coated (metals commonly used are aluminum, silver and zinc) strips, wires or ropes made of nylon or silicon. Radar signals, incident on the chaff, induce currents within each element (wire) of the cloud. These currents in return produce a scattered field deranging the enemy's radar.

Computing the behaviour of a chaff cloud numerically is almost impossible due to the large number of wires interacting upon each other. However applying statistical and stochastical theories could lead to useful results. Most stochastical and statistical theories require an analytical formula describing the current on a single wire. This study concentrates on finding an analytical approximation for the current on a single thin-wire. The approximation should allow fast numerical evaluation.

Next, we list the history of methods and theories describing the electromagnetic behaviour of wires. We concentrate on (semi)-analytic solution techniques, which are nearly always approximations. We can also divide these techniques in two parts, solutions in the frequency-domain and time-domain.

Frequency-domain Techniques
The electromagnetic behaviour of thin wires was described first by Pocklington's integral [1] equation in 1897. Hallén [2], [3] derived a zero-order approxima-
tion, in terms of the ratio of the radius of the wire, $a$, and the wavelength $\lambda$, or the ratio of $a$ and $2L$, the length of the wire, for the unknown current distribution. The first-order distribution can also be written in closed form. For higher-order evaluations one can use an iterative process characterized by the constant $\Omega = \ln(2L/a)$. This development has been described by Bouwkamp [4]. Van Vleck [5] reviews solutions of King and Harrison and of King and Middleton. In both methods the current is assumed to be composed of four sinusoidal currents. Using the Wiener-Hopf technique Einarson [19] found a traveling-wave solution and a closed-form expression for the far field using asymptotic expansions. King and Harrison [7] also found an iterative solution, where only the leading terms of trigonometric functions where used. This solution is valid when the wire is surrounded by a dissipative medium. Up to that point, a great deal of effort was put into the electrically small wires, while the solutions for longer wires were still inaccurate. Chin-Lin Chen [16] used the Wiener-Hopf method to solve the Pocklington integral equation. Assuming that the vector potential at one end of the wire is 'independent' on the variation of the current at the other end, and by approximating the transformed kernel, approximate expressions were found. Due to King [8], [9] better understanding of a single wire as well as of the Yagi antenna was achieved. But also here the theory was limited to relatively short dipoles. Later Jones [12], [13] [14] has shown that a unique solution exists for certain integral equations, proving that the difficulties arising in the evaluation of the integral or in the numerical computation of the current are not due to a non-existing solution. Using the method of moments described by Harrington, Jones describes the different behaviour for different cross-sections. Similar to the method of moments is a method using entire domain Galerkin expansion functions. Medgyesi-Mitschang [26] applied this successfully on Pocklington's integral operator. Chatterjee [36] used the traveling-wave model along with the Galerkin expansion to arrive at a closed-form analytical expression. Most of these results were obtained in the frequency domain.

**Time-domain Techniques**

Gomez-Martín [30] calculated analytically the radiated far field of a transient current, traveling across the wire, using reflection theory. Since he used an average reflection coefficient at the end points of the wire, his solution is still an approximation, causing the phase of the current to deviate. However, his method enhances physical insight enormously. The singular expansion (SEM) theory was used by Hoorfar, Tesche [27] to achieve a very accurate solution. That means more accurate then the solution presented by Marin [22]. Miller [24] explains in a tutorial introduction to transient electromagnetics some advantages of time-domain analysis, like the ability to handle non-linearities and the improved physical insight. It would be outside the context of this report to give a complete, detailed review of all the research that has been conducted on the electromagnetic scattering of wires. However, for a new derivation of Pockling-
ton’s and Hallén’s integral equations and the implementation using marching on in time and marching on in frequency we refer to Tijhuis et al.[38]. In this paper additional references can be found.

In this report we obtain analytical approximations for the current on thin wires. The current on a finite wire excited by a delta-gap source can be approximated by multiple delta-gap sources distributed on the extended wire. For long wires the current can be reasonably well approximated by modeling the reflection at the end by two delta-gap sources located at the ends. In the case of short wire reflection coefficients are introduced to obtain a better response at the cost of zero-current at the ends of the wire.

The results obtained from the approximations are compared with results obtained from the NEC (Numerical Electromagnetics Code) [25]. The NEC is internationally regarded as a program producing accurate data for electromagnetic problems involving thin wires. In the next chapter (Chapter Two) we define our notation and other conventions, like the geometry of the wire. Chapter Three encloses the derivation of the relevant integro-differential equations. In Chapter Four the infinite wire is investigated where we consider plane-wave excitation as well as delta-gap excitation. Chapter Five deals with the current on a finite wire which is expressed in terms of the current on the infinite wire using a weakened form of Hallén’s integral equation. Chapter Six summarizes conclusions and recommendations.
Chapter 2

Definition of Vectors, Transformations and Geometry

In this chapter we introduce the notation of vectors and scalars, define relevant transformations, describe the geometry of the wire and the two types of excitation, the delta-gap excitation corresponding to the transmitting state and the plane-wave excitation corresponding to the receiving state of the wire.

2.1 Notation of Vectors and Scalars

In this report, quantities may depend on space, time, wavenumber and frequency. Every quantity should therefore have its own representation depending on its dependence on these quantities. Vectors depending on space and time are represented by bold-face calligraphic, like $\mathbf{E}$, vectors depending on space and frequency are printed bold-face, like $E$, vectors depending on wavenumber and time are given a hat and are printed bold-face calligraphic, like $\hat{E}$, and vectors depending on the wavenumber and on the frequency are printed bold-face and are given a hat, like $\hat{E}$. In the case of scalars we use the same notation, i.e. $\mathcal{E}$, $E$, $\hat{E}$ and $\hat{E}$, respectively.

In our three-dimensional right-handed Cartesian coordinate system the position vector $\mathbf{r}$ is defined as $\mathbf{r} \equiv xu_x + yu_y + zu_z$, where $u_x, u_y$ and $u_z$ are the unit vectors in the $x$-, $y$- and $z$-directions, respectively. In cylindrical coordinates the position vector is written as $\mathbf{r} \equiv \rho u_\rho + \phi u_\phi + zu_z$.

Further, we define the wavenumber as $\mathbf{k} \equiv k_xu_x + k_yu_y + k_zu_z$.

In the frequency domain, quantities depend on $s$, the complex frequency, with $s = i\omega + \beta$, where $\omega$ and $\beta$ are real quantities. In the time domain, quantities are a function of $t$.

The gradient operator $\nabla$ is defined in the Cartesian coordinate system as $\nabla \equiv \partial_xu_x + \partial_yu_y + \partial_zu_z$. Figure 2.1 shows our three-dimensional coordinate system.
2.2 Defining Transformations

The transformation from the time domain to the frequency domain is performed by the one-sided Laplace transformation, defined by

$$\mathcal{L}\{\mathcal{F}(r, t)\} \overset{\text{def}}{=} \int_{t_0(r)}^{\infty} \mathcal{F}(r, t) \exp(-st) dt = \mathcal{F}(r, s),$$

(2.1)

where $\mathcal{F}(r, t) = 0$ for $t < t_0(r)$. Transforming back from the frequency domain to the time domain is done by the inverse Laplace transformation

$$\mathcal{L}^{-1}\{\mathcal{F}(r, s)\} \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \mathcal{F}(r, s) \exp(st) ds = \mathcal{F}(r, t),$$

(2.2)

where the path of integration is along the Bromwich contour, in such a way that $\mathcal{F}(r, s)$ is analytical for $\text{Re}(s) > \beta$.

Next we define the three-dimensional spatial Fourier transformation as

$$\hat{F}(k, s) \overset{\text{def}}{=} \int_{\mathcal{V}} F(r, s) \exp(-ik \cdot r) dV(r),$$

(2.3)

and the inverse spatial Fourier transformation as

$$F(r, s) \overset{\text{def}}{=} \frac{1}{8\pi^3} \int_{\mathcal{K}} \hat{F}(k, s) \exp(ik \cdot r) dV(k).$$

(2.4)
For the one-dimensional case the spatial Fourier transformation with respect to \( z \) is given by

\[
\hat{F}(k, s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} F(z, s) \exp(-ikz)dz,
\]

and its corresponding inverse transformation by

\[
F(z, s) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k, s) \exp(ikz)dk.
\]

### 2.3 Geometry of the Wire

In the remainder of this report we will assume that the wire is located with its axis on the \( z \)-axis from \( z = -L \) to \( z = L \). The radius of the wire is \( a \). In the receiving state, an electromagnetic field is incident on the wire. The electric component of the incident field is described by (s-domain)

\[
E^i(r, s) = E_0^i(s) \exp\left(-\frac{s}{c} \mathbf{u}^i \cdot \mathbf{r}\right) \mathbf{u}^e,
\]

where \( \mathbf{u}^i \) describes the direction of incidence and \( \mathbf{u}^e \) describes the polarization of the incident electric-field component. \( E_0^i(s) \) is the complex amplitude of the incident wave, and describes its temporal behaviour. \( c \) is the speed of light in the homogeneous medium.

In the transmitting state, a delta-gap excitation is located at \( z = z_g \), excited by \( V_g(t) \).

Figure 2.2 shows geometry of the wire.
Figure 2.2: Geometry of the wire
Chapter 3

The Integral Equations of Pocklington and Hallén

Since the wire is part of the family of perfectly conducting scatterers we will start the investigation by deriving the electric-field integral equation for a perfectly conducting scatterer. Because this integral equation depends on all three dimensions, we prefer dealing with an equation depending only on one dimension, z. Therefore we will derive the exact and approximated form of Pocklington's integral equation, followed by Hallén version of these equations. The theory is obtained from Tijhuis et al. [38]. First, we will derive these equations for an incident wave. In the last section we will extend the equations to the more general case where we locate a delta-gap at z = zg.

3.1 The Integral Relation for the Electric Field

Let us define a bounded, three dimensional, subdomain $\mathcal{D}_1$ representing the interior of a scatterer. The boundary of $\mathcal{D}_1$ is the closed surface $\partial\mathcal{D}$. The unbounded space exterior to $\partial\mathcal{D}$ is denoted by $\mathcal{D}_2$ (see Figure 3.1). In Figure 3.1 n is the normal on the surface pointing into $\mathcal{D}_2$. In $\mathcal{D}_2$, there exist electromagnetic sources represented by the electric- and magnetic-current densities, $\mathcal{J}^e_\mathcal{D}(r, t)$ and $\mathcal{J}^m_\mathcal{D}(r, t)$, respectively. The current densities appear in Maxwell’s equations as follows:

$$\nabla \times \mathcal{H}(r, t) - \varepsilon \partial_t \mathcal{E}(r, t) = \mathcal{J}^e_\mathcal{D}(r, t),$$
$$\nabla \times \mathcal{E}(r, t) + \mu \partial_t \mathcal{H}(r, t) = -\mathcal{J}^m_\mathcal{D}(r, t),$$

(3.1)

where $\mathcal{E}(r, t)$ and $\mathcal{H}(r, t)$ are the electric and magnetic field strength, respectively. The current densities $\mathcal{J}^e_\mathcal{D}$ and $\mathcal{J}^m_\mathcal{D}$ as well as the field quantities $\mathcal{E}$ and $\mathcal{H}$ are three-dimensional vectors. The time coordinate $t$ is chosen such that the current densities as well as the field quantities are zero for $t \leq t_0$, in a finite domain $\mathcal{D}_1$ only.
After transforming (3.1) from the time-domain into the s-domain, Maxwell’s equations reduce to
\[
\nabla \times \mathbf{H}(r, s) - s\varepsilon \mathbf{E}(r, s) = \mathbf{J}_D^e(r, s),
\]
\[
\nabla \times \mathbf{E}(r, s) + s\mu \mathbf{H}(r, s) = -\mathbf{J}_B^m(r, s).
\] (3.2)
Now we define the spatial Fourier transform \( \hat{\mathbf{V}}(k) \) of a vector field \( \mathbf{V}(r) \) as
\[
\hat{\mathbf{V}}(k) \overset{\text{def}}{=} \int_{D_1} \mathbf{V}(r) \exp(-ik \cdot r) \, dV(r),
\] (3.3)
where \( k = k_x \mathbf{u}_x + k_y \mathbf{u}_y + k_z \mathbf{u}_z \). This transformation is rather unusual, since it is not carried out over the entire three-dimensional space but over subdomain \( D_1 \). In order to apply this transformation to (3.2), we need the corresponding transform of \( \nabla \times \mathbf{V}(r) \).

With the aid of Gauss’ theorem, we find
\[
(\nabla \times \mathbf{V})(k) = i\mathbf{k} \times \hat{\mathbf{V}}(k) + \oint_{\partial D} [\mathbf{n}(r) \times \mathbf{V}(r)] \exp(-i\mathbf{k} \cdot r) \, dA(r). \] (3.4)
The second term of the right-hand side of (3.4) is just the spatial Fourier transform of the vector function \( \mathbf{n}(r) \times \mathbf{V}(r) \) over its domain of definition \( \partial D \). Combining (3.2)-(3.4) we obtain the following equations for \( \hat{\mathbf{H}}(k, s) \) and \( \hat{\mathbf{E}}(k, s) \):
\[
\begin{align*}
    i\mathbf{k} \times \hat{\mathbf{H}}(k, s) - s\varepsilon \hat{\mathbf{E}}(k, s) &= \mathbf{J}^e_D(k, s) + \mathbf{J}^e_B(k, s), \\
    i\mathbf{k} \times \hat{\mathbf{E}}(k, s) + s\mu \hat{\mathbf{H}}(k, s) &= -\mathbf{J}^m_B(k, s) - \mathbf{J}^m_B(k, s),
\end{align*}
\] (3.5)
in which \( \mathbf{J}^e_B(k, s) \) and \( \mathbf{J}^m_B(k, s) \) are the transforms over the boundary \( \partial D \) of the quantities
\[
\mathbf{J}^e_B(r, s) \overset{\text{def}}{=} -\mathbf{n}(r) \times \mathbf{H}(r, s) \quad \text{when} \quad r \in \partial D,
\] (3.6)
and
\[
\mathbf{J}^m_B(r, s) \overset{\text{def}}{=} \mathbf{n}(r) \times \mathbf{E}(r, s) \quad \text{when} \quad r \in \partial D,
\] (3.7)
respectively. In the right-hand sides of (3.6) and (3.7) we have taken the limiting values of the quantities upon approaching $\partial D$ via $D_1$. $\mathbf{J}_D^s(r, s)$ and $\mathbf{J}_B^m(r, s)$ in (3.5) can be interpreted as the Laplace-transformed source densities of the electric and magnetic surface currents.

We next have an algebraic system (3.5), that we can solve in closed form

$$
\mathbf{E} = (k^2 + s^2 \varepsilon \mu)^{-1} \{-s \mu (\mathbf{J}_D^s + \mathbf{J}_B^s) + \varepsilon (k_1) \mathbf{J}_B^s + \mathbf{J}_D^s \}
$$

and

$$
\mathbf{H} = (k^2 + s^2 \varepsilon \mu)^{-1} \{-s \mu (\mathbf{J}_D^m + \mathbf{J}_B^m) + \varepsilon (k_1) \mathbf{J}_B^m + \mathbf{J}_D^m \},
$$

where $k = \frac{|\mathbf{k}|}{k^2 + s^2 \varepsilon / c^2}$.

Green's Function

Green's function $G(r, t)$ is defined as the solution of the second-order differential equation

$$
[\nabla^2 - \frac{1}{c^2} \partial_t^2] G(r, t) = -\delta(r) \delta(t),
$$

where $c$ is the speed of light in a homogeneous medium, equal to $(\mu/c)^{-\frac{1}{2}}$. By subjecting (3.10) to a Laplace transformation and spatial Fourier transformation over $D_\infty$, the following result is obtained

$$
\hat{G}(k, s) = (k^2 + \frac{s^2}{c^2})^{-1}.
$$

Note that the right-hand side of (3.11) is the multiplicative factor that can also be seen in (3.8) and (3.9). $\hat{\mathbf{E}}$ in (3.8) and $\hat{\mathbf{H}}$ in (3.9) can therefore be regarded as a space-time convolution of the Green’s function in the space-time domain and the inverse Fourier and Laplace transform of the remaining parts. In the three-dimensional space we transform $\hat{G}(k, s)$ to the space-domain. The inversion integral can be written as

$$
G(r, s) = \frac{1}{8\pi^3} \int V \hat{G}(k, s) \exp(ik \cdot r) dV(k),
$$

where spherical coordinates are used to represent the $k$-domain. Here, $\theta_k$ is the angle between $\mathbf{k}$ and $\mathbf{r}$ and $\phi_k$ the angle of rotation of $\mathbf{k}$ around $\mathbf{r}$. In (3.12),
the integrals over $\phi_k$ and $\theta_k$ are elementary and the integration over $k$ can be performed by applying Jordan's lemma and Cauchy's theorem. This leads to

$$G(r, s) = G(r, s) = \frac{\exp(-sr/c)}{4\pi r}.$$ (3.13)

To obtain the corresponding time-domain signal $G(r, t)$, (3.13) is written as the solution of the Laplace transform of the time-domain signal

$$G(r, s) = \int_{0}^{\infty} \frac{\delta(t - r/c)}{4\pi r} \exp(-st) dt.$$ (3.14)

By virtue of Lerch's theorem, $G(r, t)$ is directly identified as

$$G(r, t) = \frac{\delta(t - r/c)}{4\pi r},$$ (3.15)

which completes the derivation of Green's function in the space-time-domain.

Now we can write $\mathcal{E}(r, t)$ and $\mathcal{H}(r, t)$ as space-time convolution integrals

$$\{1, \frac{1}{2}, 0\} \mathcal{E}(r, t) = -\mu \partial_t [\mathcal{A}_D^e(r, t) + \mathcal{A}_B^e(r, t)]$$
$$+\varepsilon^{-1} \int_{-\infty}^{t} \nabla \{\nabla \cdot [\mathcal{A}_D^e(r, t') + \mathcal{A}_B^e(r, t')]\} dt'$$
$$-\nabla \times [\mathcal{A}_D^{-m}(r, t) + \mathcal{A}_B^{-m}(r, t)],$$ (3.16)

and

$$\{1, \frac{1}{2}, 0\} \mathcal{H}(r, t) = -\varepsilon \partial_t [\mathcal{A}_D^{-m}(r, t) + \mathcal{A}_B^{-m}(r, t)]$$
$$+\mu^{-1} \int_{-\infty}^{t} \nabla \{\nabla \cdot [\mathcal{A}_D^{-m}(r, t') + \mathcal{A}_B^{-m}(r, t')]\} dt'$$
$$+\nabla \times [\mathcal{A}_D^e(r, t) + \mathcal{A}_B^e(r, t)],$$ (3.17)

for $r \in \{D_1, \partial D, D_2\}$, which means that in $D_1$ the left-hand side is equal to $\mathcal{H}(r, t)$, in $D_2$ the left-hand side is equal to 0 and on $\partial D$ the left-hand side is equal to $\frac{1}{2} \mathcal{H}(r, t)$. In (3.16) and (3.17), we have introduced the vector potentials

$$\mathcal{A}_D^{e,m}(r, t) = \int_{D_2} dV(r') \int_{0}^{t-R/c} G(R, t - t') \mathcal{J}_D^{e,m}(r', t') dt',$$ (3.18)

$$\mathcal{A}_B^{e,m}(r, t) = \int_{\partial D} dA(r') \int_{0}^{t-R/c} G(R, t - t') \mathcal{J}_B^{e,m}(r', t') dt',$$ (3.19)

with $R = |r - r'|$.

(3.16) and (3.17) were obtained by systemetically translating the factors $ik$ and
s into the space and time differentiations acting on the complete space-time integrals. These factors can be attributed to time differentiations acting either on the complete space-time integrals or the current distributions $J^{m,D,B}_{D,B}(r', t')$.

Now we are searching for a representation involving $n \times E$ on $\partial D$ because this component is zero. We want to express the incident field and the scattered field in $D_1$ and $D_2$ in sourceless fields. This can be done by determining the scattered field in $D_2$ and the incident field in $D_1$. Note that in $D_1$ there are no sources of the incident field since they are located in $D_2$.

First, let us consider scattering from a perfectly conducting object. In the exterior domain $D_2$ the scattered fields comply with

$$\nabla \times H^s(r, t) - \varepsilon \partial_t E^s(r, t) = 0,$$
$$\nabla \times E^s(r, t) + \mu \partial_t H^s(r, t) = 0,$$

(3.20)

when $r \in D_2$. On account of this equation, (3.17) leads to

$$\{0, -\frac{1}{2}, -1\} H^t(r, t) = \nabla \times A^{m,s}_B(r, t) - \varepsilon \partial_t A^{m,s}_B(r, t)$$
$$+ \mu^{-1} \int_{-\infty}^{t} \nabla [\nabla \cdot A^{m,s}_B(r, t')] dt',$$

(3.21)

for $r \in \{D_1, \partial D, D_2\}$. In (3.21), the vector potentials $A$ are the ones defined in (3.19), while the additional superscript $s$ reflects the fact that these potentials now pertain to the scattered field. Obviously, the propagation velocity should be taken equal to $c = (\varepsilon \mu)^{-\frac{1}{2}}$. Finally, the minus sign in (3.21) originates from the fact that $n(r)$ points into $D_2$.

In writing down (3.21), we have applied the integral relation (3.17) to an infinite domain. Strictly speaking, this is not allowed, since this relation was derived for a finite domain only. However, we can also arrive at (3.21) by considering the domain between $\partial D$ and a spherical surface with radius $r = r_\infty$. By choosing a sufficiently large $r_\infty$, we can make the contribution from this extra boundary vanish, by using the property that the scattered field vanishes when $t \leq t_0$.

Second, we consider the incident field in the interior domain $D_1$. The incident field is by definition equal to the field present in absence of the scatterer. Thus we write with the medium properties of $D_2$

$$\nabla \times H^i(r, t) - \varepsilon \partial_t E^i(r, t) = 0,$$
$$\nabla \times E^i(r, t) + \mu \partial_t H^i(r, t) = 0,$$

(3.22)
when \( \mathbf{r} \in \mathcal{D}_1 \). In (3.22), the right-hand side does not contain sources since the sources of the incident field are located in \( \mathcal{D}_2 \). (3.17) leads to

\[
\{1, \frac{1}{2}, 0\} \mathbf{H}^i(\mathbf{r}, t) = \nabla \times \mathbf{A}^e_{\mathcal{D}}(\mathbf{r}, t) - \varepsilon \partial_t \mathbf{A}^m_{\mathcal{D}}(\mathbf{r}, t)
\]

\[
+ \frac{1}{\mu c^2} \int_{-\infty}^{t} \nabla [\nabla \cdot \mathbf{A}^m_{\mathcal{D}}(\mathbf{r}, t')] dt',
\]

(3.23)

for \( \mathbf{r} \in \{\mathcal{D}_1, \partial \mathcal{D}, \mathcal{D}_2\} \).

Third, we make use of the fact that the scatterer is electrically impenetrable. This provides us with the boundary condition

\[
\mathbf{J}^m_B(\mathbf{r}, t) + \mathbf{J}^m_S(\mathbf{r}, t) = \mathbf{J}^m_B(\mathbf{r}, t) = \mathbf{n}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}, t) = 0,
\]

(3.24)

when \( \mathbf{r} \in \partial \mathcal{D} \). Using this boundary condition we can directly write for the magnetic-field strength

\[
\{0, \frac{1}{2}, 1\} \mathbf{H}(\mathbf{r}, t) = \mathbf{H}^i(\mathbf{r}, t) - \nabla \times \mathbf{A}^e_{\mathcal{D}}(\mathbf{r}, t),
\]

(3.25)

for \( \mathbf{r} \in \{\mathcal{D}_1, \partial \mathcal{D}, \mathcal{D}_2\} \). Analogous to (3.25) we can derive an expression for the electric-field component. Doing so results in

\[
\{0, \frac{1}{2}, 1\} \mathbf{E}(\mathbf{r}, t) = \mathbf{E}^i(\mathbf{r}, t) - \mu \partial_t \mathbf{A}^e_{\mathcal{D}}(\mathbf{r}, t)
\]

\[
+ \varepsilon^{-1} \int_{-\infty}^{t} \nabla [\nabla \cdot \mathbf{A}^m_{\mathcal{D}}(\mathbf{r}, t')] dt',
\]

(3.26)

for \( \mathbf{r} \in \{\mathcal{D}_1, \partial \mathcal{D}, \mathcal{D}_2\} \). By differentiating (3.26) on both sides with respect to time, the integral is eliminated. After multiplication by \( \varepsilon \) and renaming \( \mathbf{A}^m_{\mathcal{D}} \) and \( \mathbf{J}^m_B \) to the more commonly used symbols \( \mathbf{A} \) and \( \mathbf{J} \), respectively we arrive at the integral relation for the electric field.

\[
\{0, \frac{1}{2}, 1\} \varepsilon \partial_t \mathbf{E}(\mathbf{r}, t) - \varepsilon \partial_t \mathbf{E}^i(\mathbf{r}, t) = \nabla \nabla \cdot \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \partial_t^2 \mathbf{A}(\mathbf{r}, t),
\]

(3.27)

where

\[
\mathbf{A}(\mathbf{r}, t) = \oint_{\partial \mathcal{D}} \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{4\pi R} dA(\mathbf{r}')
\]

(3.28)

and \( \mathbf{r} \in \{\mathcal{D}_1, \partial \mathcal{D}, \mathcal{D}_2\} \). We will refer to \( \mathbf{A} \) as the vector potential and to \( \mathbf{J} \) as the electric current density flowing on \( \partial \mathcal{D} \). (3.27) is valid for all \( t, \mathbf{r} \) and for perfectly conducting scatterers as long as the normal on its surface is continuous.
3.2 The Pocklington Integral Equation

The scattered far-field \(|r' - r| \gg a\) is determined only by the total current \(I(z, s)\) or \(I(z, t)\). In this section, we will give the classical derivation of the reduced and exact forms of Pocklington's equation. These equations describe the relation between the incident field and the total current. Furthermore, we will derive the exact equation which will turn out to be equal to the reduced form of Pocklington's equation.

3.2.1 Reduced Form of Pocklington's Integral Equation

Perhaps the most straightforward way to arrive at the reduced form of Pocklington's integral equation is to approximate the vector potential (3.28) by

\[ A(z', t) \approx u_z \int_{-L}^{L} \frac{I(z', t - R/c)}{4\pi R} dz', \]

where

\[ R \overset{\text{def}}{=} \sqrt{(z - z')^2 + \rho^2}. \]  

The underlying approximation is that the entire current is concentrated on the z-axis. Consequently, the current does not depend on \(\phi\). (3.29) is the so-called reduced-kernel which holds for \(r \in \mathcal{D}_2\), provided that \(R \gg a\) for all values \(-L \leq z' \leq L\). Therefore it is applicable almost everywhere in \(\mathcal{D}_2\) for determining the electric-field strength \(\mathcal{E}(r, t)\) from the known space-time behaviour of \(I(r, t)\), the total current on the wire. On \(\partial \mathcal{D}\) this approximation is still valid as long as \((z - z')^2 \gg a^2\).

In addition, we can substitute (3.29) into (3.27) with \(r \in \partial \mathcal{D}\), corresponding to \(\rho = a\). Carrying out this substitution, produces the approximate integro-differential equation

\[-\varepsilon \partial_t \mathcal{E}_z^i(zu_z + au_\rho, t) = [\partial_z^2 - \frac{1}{c^2} \partial_t^2] \int_{-L}^{L} \frac{I(z', t - Ra/c)}{4\pi Ra} dz', \]

where \(Ra \overset{\text{def}}{=} \sqrt{(z - z')^2 + a^2}\) and \(I(z, t)\) is the current flowing in the z-direction. This integro-differential equation is internally inconsistent, since the value for the total current \(I(z, t)\) depends on the direction of \(u_\rho\). To overcome this problem, we require that the electric-field component \(\mathcal{E}_z^i(zu_z + au_\rho, t)\) to be independent of \(\phi\) for \(-L \leq z \leq L\). This is done by introducing the additional approximation

\[ \mathcal{E}_z^i(zu_z + au_\rho, t) \approx \mathcal{E}_z^i(zu_z, t) \]

which is comparable in accuracy to concentrating the entire current on the axis of the wire. This leads to the one-dimensional integro-differential equation

\[-\varepsilon \partial_t \mathcal{E}_z^i(zu_z, t) = [\partial_z^2 - \frac{1}{c^2} \partial_t^2] \int_{-L}^{L} \frac{I(z', t - Ra/c)}{4\pi Ra} dz', \]
which holds for \(-L \leq z \leq L\) and for \(t_0 \leq t < \infty\). The main objective to (3.33) is that its underlying approximation breaks down for \(z' - z = \mathcal{O}(a)\), where the integrand assumes its largest magnitude. This is why it is known in literature as the reduced form of Pocklington’s integral equation.

### 3.2.2 "Exact" Form of Pocklington’s Integral Equation

A more responsible approach for obtaining a one-dimensional integral equation is to consider the wire as an open-ended perfectly conducting tube, along with the current density is given by

\[
\mathcal{J}(zu_z + au_\rho, t) \approx \frac{I(z,t)}{2\pi a} u_z. \tag{3.34}
\]

Substituting this approximation in (3.27), and going through the same process as in Section 3.2.1, leads to the following expression

\[
-\varepsilon \partial_t \mathcal{E}_z^i(zu_z, t) = \left[ \partial^2_z - \frac{1}{c^2} \partial_\eta \right] \int_{-L}^{L} \oint_{\partial \mathcal{D}} \frac{I(z', t - R_\phi/c)}{4\pi^2 R_\phi} d\phi dz', \tag{3.35}
\]

where \(R_\phi \triangleq \sqrt{(z - z')^2 + 4a^2 \sin^2 \frac{1}{2} \phi}\) is the distance between two points on \(\partial \mathcal{D}\).

The integral holds for \(-L \leq z < L\) and for \(t_0 \leq t < \infty\).

### 3.2.3 The Exact Derivation

In the derivation of (3.33) and (3.35) the \(z\)-component of the incident electric field on the boundary of the wire had to be approximated by a corresponding value on the central axis in order to obtain a consistent integro-differential equation. The question that arises is, why not consider the \(z\)-component of the integral relation (3.27) on that axis directly. This component is given by

\[
-\varepsilon \partial_t \mathcal{E}_z^i(zu_z, t) = \partial_z \nabla \cdot \mathcal{A}(zu_z, t) - \frac{1}{c^2} \partial^2_z \mathcal{A}_z(zu_z, t), \tag{3.36}
\]

with \(-L < z < L\). The essential difference between (3.33) and (3.35) is the occurrence of the transverse derivatives \(\mathcal{A}(r, t)\) with respect to \(r\) in the first term on the right-hand side. These derivatives can be shown to vanish. To this end, the gradient operator is broken up into \(\nabla = \nabla_T + \partial_z u_z\) where \(\nabla_T = \partial_\rho u_\rho + \frac{1}{\rho} \partial_\phi u_\phi = \partial_z u_z + \partial_y u_y\). By straightforward differentiation of (3.28) it follows that

\[
\nabla_T \cdot \mathcal{A}(r, t) = \oint_{\partial \mathcal{D}} \left[ \frac{1}{R} + \frac{1}{c} \partial_\phi \right] (\rho' - \rho) \cdot \frac{\mathcal{J}(r', t - R/c)}{4\pi R^2} dS(r'). \tag{3.37}
\]

On the axis of the wire \(\rho = 0\). The total radial current on the end faces is estimated to be zero. The feature that \(\rho' \cdot \mathcal{J}(r, t) = 0\) on the remainder of \(\partial \mathcal{D}\) leads to the conclusion that \(\nabla_T \cdot \mathcal{A}(zu_z, t) = 0\) for \(-\infty < z < \infty\). Concluding
that only the $z$-component of the vector potential needs to be evaluated, we arrive at

$$-\varepsilon \partial_t E_z(zu_z, t) = \left[ \partial_z^2 - \frac{1}{c^2} \partial_t^2 \right] \int_{-L}^{L} \frac{I(z', t - Ra/c)}{4\pi Ra} dz', \quad (3.38)$$

for $-\infty < z < \infty$. Surprisingly, this equation turns out to be identical to the reduced form of Pocklington's integral equation. The only approximation made is the estimation of the radial current at the end faces of the wire. We can therefore conclude that the reduced form is more exact than the "exact" form of Pocklington's equation.

### 3.3 The Hallén Integral Equation

The combination of the space and time differentiations in (3.33) can be recognized as the differential operator governing the propagation of plane waves in a homogeneous, lossless dielectric. For that operator, the Green's function $G(z, t)$ can be expressed as the causal solution of the inhomogeneous second-order differential equation

$$\left[ \partial_z^2 - \frac{c^2}{c^2} \partial_t^2 \right] G(z, t) = -\delta(z)\delta(t), \quad (3.39)$$

with its solution given by

$$G(z, t) = \frac{c}{2} U(t - \frac{|z|}{c}), \quad (3.40)$$

where $U(t)$ denotes the Heavyside unit time-step function. (3.40) can be derived in a similar fashion as is done for the three dimensional Green's function in section 3.1. Rewriting $-\varepsilon \partial_t E(zu_z, t)$ as

$$-\varepsilon \partial_t E(zu_z, t) = -\varepsilon \int_{-L}^{L} dz' \int_{-\infty}^{\infty} \delta(z - z')\delta(t - t')\partial_t E(z'u_z, t') dt' \quad (3.41)$$

and substituting (3.39) into (3.41) results in

$$A_z(z, t) = -\varepsilon \int_{-L}^{L} dz' \int_{-\infty}^{\infty} G(z', t')\partial_t E_z(z'u_z, t') dt' + A^h_z(z, t)$$

$$= \varepsilon \int_{-L}^{L} dz' \int_{-\infty}^{\infty} \frac{c}{2} U(t - t' - \frac{|z - z'|}{c})\partial_t E_z(z'u_z, t') dt'$$

$$+ A^h_z(z, t), \quad (3.42)$$

where $A^h_z(z, t)$ denotes the homogeneous solution $[\partial_z^2 - \frac{c^2}{c^2} \partial_t^2] A^h_z(z, t) = 0$. Evaluating (3.42) leads to

$$\int_{-L}^{L} \frac{I(z', t - Ra/c)}{4\pi Ra} dz = \frac{Y}{2} \int_{-L}^{L} E_z(z'u_z, t - \frac{|z - z'|}{c}) dz'$$

$$+ \mathcal{F}_{-L}(t - \frac{z - L}{c}) + \mathcal{F}_{L}(t - \frac{L - z}{c}), \quad (3.43)$$

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where \( -L \leq z \leq L \) and where \( Y \overset{\text{def}}{=} \sqrt{\varepsilon / \mu} \) denotes the wave admittance of the dielectric medium. (3.43) is known as Hallén’s integral equation. The yet unknown time signals \( F_L(t - z) \) and \( F_L(t - L - z) \) represent two independent homogeneous solutions \( A_L^0(z, t) \). The first term on the right-hand side of (3.43) represents the particular solution of the vector potential, \( A_L^0(z, t) \). These signals can be determined as well, since the total current should also satisfy the boundary condition: \( I(-L, t) = I(L, t) = 0 \).

### 3.4 Extension to a Wire with a Delta Gap

Suppose we cut the wire at \( z = z_g \) and place one of the new electrically impenetrable end faces at \( z_g - \delta \) and the other at \( z_g + \delta \), then we have created a delta-gap at \( z = z_g \). Since the end faces on both sides are electrically impenetrable, we have

\[
\mathcal{E}_{\rho,\phi}(z) \approx \mathcal{E}_z(zu_z + \rho u_\rho, t) \approx 0, \quad (3.44)
\]

for \( \rho \leq a \). Because \( \mathcal{E}(r, t) \) is continuous in \( z \) and \( \delta \ll a \), it is allowed to neglect the transverse components \( \mathcal{E}_{\rho,\phi} \). Thus we have a good approximation

\[
\mathcal{E}(zu_z + \rho u_\rho, t) \approx \mathcal{E}_z(zu_z + \rho u_\rho, t)u_z, \quad (3.45)
\]

for \( z_g - \delta < z < z_g + \delta \), so that for \( 0 \leq \rho \leq a \)

\[
\int_{z_g - \delta}^{z_g + \delta} \mathcal{E}_z(zu_z + \rho, t)dz = -\mathcal{V}_g(t). \quad (3.46)
\]

\( \mathcal{V}_g(t) \) is the known applied voltage across the gap. Since the gap itself is located in the exterior domain \( D_2 \), the electric field strength is added to the total field \( \mathcal{E}(r, t) \) in \( D_2 \). By taking the limit \( \delta \to 0 \) the electric-field strength in the delta-gap approaches \( \delta(z - z_g)\mathcal{V}_g(t) \). Incorporating the delta-gap into both forms of Pocklington’s integral equation means adding on the left-hand sides of (3.33) and (3.35) \( -\varepsilon \partial_t \mathcal{V}_g(t)\delta(z - z_g) \). Incorporating the delta-gap into Hallén’s integral equation means adding on the right-hand side of (3.43) \( \frac{\varepsilon}{\mu} \mathcal{V}_g(t - \frac{|z - z_g|}{c}) \).

So far we have considered plane-wave excitation (receiving state). The next section will deal with delta-gap excitation (transmitting state).

In the next chapter we will investigate the infinite wire for two reasons. First, the infinite wire does not have end faces and, therefore, we expect the current obtained by applying the reduced form of Pocklington’s integral equation for plane-wave excitation to be exact. This will support the conclusion that the reduced form of Pocklington’s integral equation is the exact integral equation. Second, in both cases of delta-gap excitation and plane-wave excitation we point
out that there exists similarity between the particular solution and the homogeneous solution of Hallén's integral equation. The "homogeneous solution" $A_h^p(z,t)$ can be estimated in terms of "particular solutions" $A_p^p(z,t)$ on an infinite wire. On an infinite wire the homogeneous solutions do not exist.

As a result the current on a finite wire can be estimated in terms of the current on an infinite wire for delta-gap as well as for plane-wave excitation.
Chapter 4

Response of an Infinite Wire

In this chapter, we investigate the total current on an infinite wire excited by an incident field or by a delta-gap source. In Section 4.1 this is done for the reduced form of Pocklington's equation which, except for the radial currents on the end faces, is exact and for a closed-form solution obtained by separation of variables. In Section 4.2 several solutions are presented for the case where the wire is excited by a delta-gap source.

4.1 Response to a Plane-Wave Excitation

In Section 3.2.1 we derived the reduced form of Pocklington's equation which we found to be exact, with the exception of neglecting the radial current on the end faces. Because an infinite wire does not possess end faces, the solution of the reduced integral equation must be exact.

4.1.1 The Exact Response

In this subsection, the exact response is found in a similar way as Jones found it in [14].

The total current in the z-direction, which we are interested in, can be expressed as an integral over a closed contour on the surface of the wire,

\[ I(z, s) = \oint_C \mathbf{u}_z \cdot (\mathbf{u}_\rho \times \mathbf{H}(r, s)) d\ell = \int_0^{2\pi} H_\phi|_{\rho=a} a d\phi, \tag{4.1} \]

where \( \mathbf{u}_\rho \) is the radial unit vector in the cylindrical coordinate system, whose z-axis coincides with the central axis of the wire. For \( H_\phi \) we take the limit for \( \rho \downarrow a \) of \( H_\phi \), i.e. from the outside of the wire. We must now evaluate at least \( H_\phi \) to obtain the current in the z-direction. We start again from Maxwell's equations (3.1), which describe the field outside the cylinder

\[ \nabla \times \mathbf{H}(r, t) - \varepsilon \partial_t \mathbf{E}(r, t) = 0, \]
\[ \nabla \times \mathbf{E}(r, t) + \mu \partial_t \mathbf{H}(r, t) = 0, \]  
(4.2)

Further, we assume that at each position \( r = r_0 \) an initial instant can be identified as \( t = t_0(r_0) \) such that
\[ \mathbf{E}(r_0, t) = 0, \quad \mathbf{H}(r_0, t) = 0, \]  
(4.3)

for \( t \leq t_0(r_0) \). In this case, it follows from (3.1) that
\[ \nabla \cdot \mathbf{E}(r, t) = 0, \]  
(4.4)
\[ \nabla \cdot \mathbf{H}(r, t) = 0. \]  
(4.5)

From Maxwell's equations the wave equations can easily be derived:
\[ \nabla^2 \mathbf{H}(r, t) - \frac{1}{c^2} \partial_t^2 \mathbf{H}(r, t) = 0, \]  
(4.6)
\[ \nabla^2 \mathbf{E}(r, t) - \frac{1}{c^2} \partial_t^2 \mathbf{E}(r, t) = 0, \]  
(4.7)

where \( c = \frac{1}{\sqrt{\mu}} \).

Because the tangential component of the electric field is zero at the surface of the perfectly conducting wire, the boundary condition can be written as follows
\[ \mathbf{u}_r \times \mathbf{E}(r, t) = 0 \quad (\rho = a, \quad -\infty < z < \infty, \quad -\pi < \phi < \pi). \]  
(4.8)

This boundary condition can be enforced easier in the cylindrical coordinate system than in the Cartesian coordinate system. On the other hand we would like to solve (4.7) in terms of scalar components. This means that we must take the inner product of \( \mathbf{E}(r, t) \) and a unitvector which does not depend on local coordinates, thus we must take \( \mathbf{u}_z, \mathbf{u}_y \) or \( \mathbf{u}_z \) and not \( \mathbf{u}_\rho \) or \( \mathbf{u}_\phi \). \( \mathbf{u}_r \) or \( \mathbf{u}_z \) are allowed but do not fit the cylindrical coordinate system easily. Combining these preferences, we will take the inner product with \( \mathbf{u}_z \) choosing \( \mathbf{E}_z \) and \( \mathbf{H}_z \) as the fundamental field quantities. To achieve this we split \( \{ \mathbf{E}, \mathbf{H} \} \) up into two constituents \( \{ \mathbf{E}_1, \mathbf{H}_1 \} \) and \( \{ \mathbf{E}_2, \mathbf{H}_2 \} \), where \( \mathbf{H}_1 \) has a \( z \)-component and \( \mathbf{E}_1 \) does not, and \( \mathbf{E}_2 \) has a \( z \)-component and \( \mathbf{H}_2 \) does not.

Now, we need to distinguish between the incident and the scattered field. The incident field is the field that would be present in all space in absence of the wire. The scattered field represents the influence of the wire. In \( D_1 \) it cancels the incident field. In \( D_2 \) it represents a wave that propagates or decays in the outward direction, i.e., as \( \rho \to \infty \). The total field becomes a superposition of both fields:
\[ \mathbf{E}(r, t) = \mathbf{E}^i(r, t) + \mathbf{E}^s(r, t), \]  
(4.9)
\[ \mathbf{H}(r, t) = \mathbf{H}^i(r, t) + \mathbf{H}^s(r, t), \]  
(4.10)
where the electric component of the incident field can be written as

\[ E^i(r, s) = E_0^i(s) \exp\left(-\frac{s}{c} u^i \cdot r\right) u^e. \] (4.11)

(4.11) can also be written as

\[ E^i(r_T + z u_z, s) = \exp\left(-\frac{s}{c} z u_z^i\right) E^i(r_T, s), \] (4.12)

where \( r_T = x u_x + y u_y \) and \( r = z u_z + r_T \). Since the scattering object is invariant under the translation of \( z \), the scattered field is also invariant under the translation

\[ E^s(r, s) = \exp\left(-\frac{s}{c} z u_z^i\right) E^s(\rho, s). \] (4.13)

The same procedure is applicable to \( H^s(r, s) \).

Now, we can derive the wave equations for the constituents by splitting \( \nabla^2 \) into \( \nabla_T^2 + \frac{s^2}{c^2}(u_z^i)^2 \) and obtain

\[ \{\nabla_T^2 + \frac{s^2 u_z^i}{c^2} - \frac{s^2}{c^2}\} H_1(r, s) = 0, \] (4.14)

\[ \{\nabla_T^2 + \frac{s^2 u_z^i}{c^2} - \frac{s^2}{c^2}\} E_2(r, s) = 0, \] (4.15)

where \( \nabla_T^2 = \frac{1}{\rho} \partial_{\rho}(\rho \partial_{\rho}) + \frac{1}{\rho^2} \partial_\phi^2 = \partial_\rho^2 + \partial_\phi^2 \).

**Determination of \( H_1^i \) and \( E_2^i \)**

For the incident electric field we have

\[ E^i(r, s) = E_0^i(s) \exp\left(-\frac{s}{c} u^i \cdot r\right) u^e. \] (4.16)

In the case of cylindrical coordinates the inner product \( u^i \cdot r \) is equal to

\[ u^i \cdot r = \rho \sin \theta^i (\cos \phi^i \cos \phi + \sin \phi^i \sin \phi) + \cos \theta^i z = \rho \sin \theta^i \cos(\phi - \phi^i) + u_z^i z. \] (4.17)

By substituting (4.17) in (4.12) and combining the result with (4.11) the following expression is obtained:

\[ E^i(\rho, s) = E_0^i(s) u^e \exp\left(-\frac{s}{c} \rho \sin \theta^i \cos(\phi - \phi^i)\right). \] (4.18)

We also want to write \( E^s(r, s) \) as a Fourier series. In order to do so, \( E^s(r, s) \) needs to be split up into a component in the plane of incidence and a component orthogonal to the plane of incidence. The normal to the plane of incidence is given by

\[ u^n = \frac{u_z \times u^i}{|u_z \times u^i|}. \] (4.19)
Note that \( \{ \mathbf{u}^i, \mathbf{u}^n, \mathbf{u}_z \} \) forms a right-handed system. 

\( \mathbf{u}^e \) can be split into the two components

\[ \mathbf{u}^e = (\mathbf{u}^e \cdot \mathbf{u}^n)\mathbf{u}^n - \mathbf{u}^n \times (\mathbf{u}^n \times \mathbf{u}^e), \]

resulting in

\[ \mathbf{E}^i(r, s) = \mathbf{E}^i_1(r, s) + \mathbf{E}^i_2(r, s), \]

where

\[ \mathbf{E}^i_1(r, s) = (\mathbf{u}^e \cdot \mathbf{u}^n)\mathbf{u}^n \mathbf{E}^i_0(s) \exp(-\frac{s}{c} \mathbf{u}^i \cdot \mathbf{r}), \]

\[ \mathbf{E}^i_2(r, s) = -\mathbf{u}^n \times (\mathbf{u}^n \times \mathbf{u}^e)\mathbf{E}^i_0(s) \exp(-\frac{s}{c} \mathbf{u}^i \cdot \mathbf{r}). \]

Now we have \( \mathbf{E}^i_2 \). (4.22) can be rewritten as

\[ \mathbf{E}^i_1(r, s) = \frac{(\mathbf{u}^e \cdot \mathbf{u}_z \times \mathbf{u}^i)\mathbf{u}_z \times \mathbf{u}^i}{(\mathbf{u}_z \times \mathbf{u}^i \cdot \mathbf{u}_z \times \mathbf{u}^i)} \mathbf{E}^i_0(s) \exp(-\frac{s}{c} \mathbf{u}^i \cdot \mathbf{r}). \]

Using the property that \( \mathbf{H}^i(r, s) = Y(\mathbf{u}^i \times \mathbf{E}^i(r, s)) \) and

\[ \mathbf{u}^i \times (\mathbf{u}_z \times \mathbf{u}^i) = \mathbf{u}_z - (\mathbf{u}_z \cdot \mathbf{u}^i)\mathbf{u}^i, \]

the magnetic field can be evaluated as

\[ \mathbf{H}^i_1(r, s) = \frac{(\mathbf{u}^e \cdot \mathbf{u}_z \times \mathbf{u}^i)[\mathbf{u}_z - (\mathbf{u}_z \cdot \mathbf{u}^i)\mathbf{u}^i]}{(\mathbf{u}_z \times \mathbf{u}^i \cdot \mathbf{u}_z \times \mathbf{u}^i)} Y \mathbf{E}^i_0(s) \exp(-\frac{s}{c} \mathbf{u}^i \cdot \mathbf{r}), \]

where \( Y \) is the admittance given by \( Y = \sqrt{\frac{\varepsilon}{\mu}} \).

We now have \( \mathbf{H}^i_1 \). The special case, when \( \mathbf{u}^i = \mathbf{u}_z \), is left out of consideration because the direction of propagation coincides with the direction of the wire.

**Determination of the Boundary Conditions**

Previously we have written for the boundary condition

\[ \mathbf{u}_r \times \mathbf{E}(r, t) = 0 \quad (\rho = a, \quad -\infty < z < \infty, \quad -\pi < \phi < \pi). \]

In the case of determining \( \mathbf{E}_2 \) we apply the following boundary condition

\[ E_{2,z}|_{\rho=a} = 0. \]

For \( \mathbf{H}_1 \) we first evaluate Maxwell’s equations in the \( s \)-domain. The following is obtained

\[ s \varepsilon \mathbf{E}_{1,\phi} = \partial_z \mathbf{H}_{1,\rho} - \partial_{\rho} \mathbf{H}_{1,z} = -\frac{s}{c} \cos \theta \mathbf{H}_{1,\rho} - \partial_{\rho} \mathbf{H}_{1,z}, \]
and

$$-s \mu H_{1, \rho} = \frac{1}{\rho} \partial_\rho E_{1, z} - \partial_z E_{1, \phi} = \frac{1}{\rho} \partial_\rho E_{1, z} + \frac{s}{c} \cos \theta^i E_{1, \phi}. \quad (4.30)$$

When $E_{1, z} = 0$, $H_{1, \rho}$ can be eliminated and an expression containing $E_{1, \phi}$ and $H_{1, \rho}$ is left

$$s \varepsilon \sin^2 \theta^i E_{1, \phi} = -\partial_\rho H_{1, z}. \quad (4.31)$$

This means that when $E_{1, z} = 0 \mid \rho = a$ and $E_{1, \phi} = 0 \mid \rho = a$ the following boundary condition can be used

$$\partial_\rho H_{1, z} = 0 \mid \rho = a. \quad (4.32)$$

**Determination of $H_{1, z}$ and $H_{1, \phi}$**

Considering the $z$-component of (4.14) we obtain

$$\{ \nabla_T^2 + \frac{s^2 u_z}{c^2} - \frac{s^2}{c^2} \} H_{1, z}(r, s) = 0. \quad (4.33)$$

From (4.26) we obtain for the $z$-component of the incident magnetic field

$$H_{1, z}^i = u_z \cdot (u^i \times u^c) Y E_0^i(s) \exp(-\frac{s}{c} \cos \theta^i) \exp(-\frac{s}{c} \rho \sin \theta^i \cos(\phi - \phi^i)). \quad (4.34)$$

Now we write (4.34) as a Fourier series

$$H_{1, z}^i = u_z \cdot (u^i \times u^c) Y E_0^i(s) \exp(-\frac{s}{c} \cos \theta^i) \sum_{m=-\infty}^{\infty} I_m(\frac{s}{c} \rho \sin \theta^i) \exp(\text{i}m(\phi - \phi^i)). \quad (4.35)$$

Writing $H_{1, z}$ also as a Fourier series, yields

$$H_{1, z} = \sum_{m=-\infty}^{\infty} h_{1, m} \exp((\text{i}m(\phi - \phi^i)). \quad (4.36)$$

Substituting this in wave-equation (4.33), the following differential equation is obtained

$$[\rho^2 \partial_\rho^2 + \rho \partial_\rho] h_{1, m}(\rho) - [m^2 + \frac{s^2}{c^2} \sin^2 \theta^i \rho^2] h_{1, m} = 0. \quad (4.37)$$

This modified equation of Bessel has the general solution

$$h_{1, m}(\rho) = P_m I_m(\frac{s}{c} \rho \sin \theta^i) + Q_m K_m(\frac{s}{c} \rho \cos \theta^i). \quad (4.38)$$

Imposing the boundary condition (4.32) on (4.36), via (4.38), $h_{1, m}(\rho)$ becomes

$$h_{1, m}(\rho) = P_m [I_m(\frac{s}{c} \rho \sin \theta^i) - \frac{I'_m(\frac{s}{c} a \sin \theta^i)}{K'_m(\frac{s}{c} \rho \sin \theta^i)} K_m(\frac{s}{c} \rho \sin \theta^i)] \quad (4.39)$$
Matching the Fourier series (4.36) to the z-component of (4.35), $P_m$ can be determined and $H_{1,z}$ is found as

$$H_{1,z} = u_z \cdot (u^i \times u^e)E^i_0(s)\exp(-\frac{s}{c}z \cos \theta^i) \sum_{m=-\infty}^{\infty} (-1)^m \exp(im(\phi - \phi^i))$$

$$[I_m(\frac{s}{c} \rho \sin \theta^i) - \frac{I'_m(\frac{s}{c} \rho \sin \theta^i)}{K'_m(\frac{s}{c} \rho \sin \theta^i)}K_m(\frac{s}{c} \rho \sin \theta^i)]. \tag{4.40}$$

Further, we had $E_{1,z} = 0$.

By taking the first two components of Maxwell’s equations and bearing in mind that $\partial_z \rightarrow -\frac{s}{c} \cos \theta^i$, the following set of equations is obtained

$$\frac{1}{\rho} \partial_\phi H_{1,z} + \frac{s}{c} \cos \theta^i H_{1,\phi} = s \varepsilon E_{1,\rho}, \tag{4.41}$$

$$-\frac{s}{c} \cos \theta^i H_{1,\rho} - \partial_\rho H_{1,z} = s \varepsilon E_{1,\phi}, \tag{4.42}$$

$$\frac{s}{c} \cos \theta^i H_{1,\rho} = -s \mu H_{1,\phi}, \tag{4.43}$$

$$-\frac{s}{c} \cos \theta^i E_{1,\rho} = -s \mu H_{1,\phi}. \tag{4.44}$$

By substituting $E_{1,\rho}$ from (4.44) into (4.41), the first part of $H_\phi$ becomes

$$H_{1,\phi} = \frac{\cos \theta^i}{\sin^2 \theta^i} \frac{1}{s} \frac{1}{\rho} \partial_\phi H_{1,z}. \tag{4.45}$$

**Determinant $E_{2,z}$ and $H_{2,\phi}$**

Considering the z-component of (4.15) we obtain

$$\{\nabla_\rho^2 + \frac{s^2 u_z^2}{c^2} - \frac{s^2}{c^2}\} E_{2,z}(\rho, s) = 0. \tag{4.46}$$

From (4.23) we obtain for the z-component of the incident electric field

$$E^i_{2,z} = u_z \cdot u^e E^i_0(s)\exp(-\frac{s}{c}z \cos \theta^i)\exp(-\frac{s}{c} \rho \sin \theta^i \cos(\phi - \phi^i)). \tag{4.47}$$

Now we write (4.47) as a Fourier series

$$E^i_{2,z} = u_z \cdot u^e E^i_0(s)\exp(-\frac{s}{c}z \cos \theta^i)\sum_{m=-\infty}^{\infty} I_m(-\frac{s}{c} \rho \sin \theta^i)\exp(im(\phi - \phi^i)). \tag{4.48}$$

Writing $E_{2,z}$ also as a Fourier series, yields

$$E_{2,z} = \sum_{m=-\infty}^{\infty} e_{2,m}(\rho)\exp((im(\phi - \phi^i)). \tag{4.49}$$
Substituting this in the Laplace transform of wave-equation (4.46), the following differential equation is obtained

\[ \rho^2 \partial^2 \rho + \rho \partial_\rho [e_{2,m}(\rho) - [m^2 + \frac{s^2}{c^2} \sin^2 \theta \rho^2]e_{2,m}(\rho) = 0. \] (4.50)

This modified equation of Bessel has the general solution

\[ e_{2,m}(\rho) = R_m I_m \left( \frac{s}{c} \rho \sin \theta^i \right) + S_m K_m \left( \frac{s}{c} \rho \cos \theta^i \right). \] (4.51)

Imposing the boundary condition (4.28) on (4.36), via (4.51), \( e_{2,m}(\rho) \) becomes

\[ e_{2,m} = R_m \left[ I_m \left( \frac{s}{c} \rho \sin \theta^i \right) - \frac{I_m \left( \frac{s}{c} a \sin \theta^i \right)}{K_m \left( \frac{s}{c} a \sin \theta^i \right)} K_m \left( \frac{s}{c} \rho \sin \theta^i \right) \right]. \] (4.52)

By matching the Fourier series (4.49) to the z-component of (4.48), \( R_m \) can be determined and \( E_{2,z} \) is found as

\[
E_{2,z} = (u^x \cdot u_z) E^i_0(s) \exp \left( \frac{-s}{c} z \cos \theta^i \right) \sum_{m=-\infty}^{\infty} (-1)^m \exp(\im\varphi - \phi^i)) \left[ I_m \left( \frac{s}{c} \rho \sin \theta^i \right) - \frac{I_m \left( \frac{s}{c} a \sin \theta^i \right)}{K_m \left( \frac{s}{c} a \sin \theta^i \right)} K_m \left( \frac{s}{c} \rho \sin \theta^i \right) \right].
\] (4.53)

Further, we had \( H_{2,z} = 0 \).

Again, from Maxwell’s equations we have

\[ \frac{1}{\rho} \partial_\rho E_{2,x} + \frac{s}{c} \cos \theta^i E_{2,\phi} = -s \mu H_{2,\rho}, \] (4.54)

\[ \frac{s}{c} \cos \theta^i E_{2,\rho} + \partial_\rho E_{2,z} = s \mu H_{2,\phi}, \] (4.55)

\[ \frac{s}{c} \cos \theta^i H_{2,\phi} = s \epsilon E_{2,\rho}, \] (4.56)

\[ \frac{s}{c} \cos \theta^i H_{2,\rho} = s \epsilon E_{2,\phi}. \] (4.57)

Substituting \( E_{2,\rho} \) from (4.56) into (4.55), the second part of \( H_{\phi} \) turns out to be

\[ H_{2,\phi} = \frac{1}{s \mu \sin^2 \theta^i} \partial_\rho E_{2,z}, \] (4.58)
Determination of the Current

For the total magnetic field in the $\phi$-direction we simply add the two contributions form the previous paragraphs

$$H_\phi = H_{1,\phi} + H_{2,\phi}.$$  \hfill (4.59)

To obtain the current, all we have to do now is substitute $H_\phi$ in (4.1). Due to symmetry only the component $m = 0$ of $H_\phi$ contributes to the current in the $z$-direction. The modes with $m \geq 1$ cancel out and do not contribute to the total current. After integrating with respect to $\phi$ we obtain

$$I(z, s) = \frac{2\pi}{s\mu \sin^2 \theta i} (u^z \cdot u_z) \exp(-\frac{z}{c} \cos \theta i) E_0'(s).$$  \hfill (4.60)

Considering the geometry of Figure 2.2 we determine $u^z \cdot u_z$ and arrive at

$$I(z, s) = \frac{2\pi \sin \eta}{s \mu \sin \theta i} \exp(-\frac{z}{c} \cos \theta i) E_0'(s)$$

$$= W_0(s) \exp(-\frac{s}{c} z \cos \theta i) \frac{E_0'(s)}{s},$$  \hfill (4.61)

where

$$W_0(s) = \frac{2\pi \sin \eta}{\mu \cos \theta i K_0(\frac{z}{c} a \sin \theta i)}.$$  \hfill (4.62)

Since we are also interested in the time response, all we have to do is carry out the inverse Laplace transformation. Doing so leads to

$$I(z, t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} 2\pi \sin \eta \exp(-\frac{z}{c} \cos \theta i) \frac{E_0'(s)}{s \mu \sin \theta i} K_0(\frac{z}{c} a \sin \theta i) \exp(st) ds,$$  \hfill (4.63)

where $\beta \geq 0$.

Alternatively we can write $I(z, t)$ as a convolution in the time domain

$$I(z, t) = W_0(t) * \int_0^t \exp(-\frac{t - \tau}{2\sigma^2}) E_0'(\tau)d\tau,$$  \hfill (4.64)

with $W_0(t)$ the inverse Laplace transform of $W_0(s)$.

It can be observed that the response of the infinite wire to an incident plane wave excitation is also a current wave, propagating with the same velocity.

Results

Figure 4.1 shows the response to a Gaussian-pulsed plane wave using (4.63). The time-domain representation of the excitation is

$$E_0'(t) = \exp(-\frac{(t - \tau)^2}{2\sigma^2}).$$  \hfill (4.65)
4.1.2 Response Using the Exact Integral Equation

Let us begin by transforming the reduced form of Pocklington's equation (3.33) into the s-domain. Further we extend the ends of the wire at \( z = -L \) and \( z = L \) to \( z = -\infty \) and \( z = \infty \), respectively, and arrive at

\[
-c\varepsilon E_z^i(zu_z, s) = \left[ \frac{\partial^2}{\partial z^2} - \frac{s^2}{c^2} \right] \int_{-\infty}^{\infty} I(z', s)G(z - z', s)dz',
\]

where

\[
G(z - z', s) = \frac{1}{4\pi R_a} \exp\left( -\frac{s}{c}R_a \right)
\]

with \( R_a \) previously defined by \( R_a \equiv \sqrt{(z - z')^2 + a^2} \). (4.66) can be recognized as a convolution integral. Using the one-dimensional spatial Fourier transformation

\[
\hat{I}(k, s) = \int_{-\infty}^{\infty} I(z, s) \exp(-ikz)dz
\]

and its inverse transformation

\[
I(z, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{I}(k, s) \exp(ikz)dk.
\]
Likewise, by transforming \( G(z, s) \) to
\[
\hat{G}(k, s) = \int_{-\infty}^{\infty} \frac{1}{4\pi \sqrt{z^2 + a^2}} \exp(-\frac{s}{c} \sqrt{z^2 + a^2}) \exp(-ikz) dz. \tag{4.70}
\]
(4.70) can be evaluated, which leads to
\[
\hat{G}(k, s) = \int_{0}^{\infty} \frac{1}{2\pi \sqrt{z^2 + a^2}} \exp(-\frac{s}{c} \sqrt{z^2 + a^2}) \cos(kz) dz.
\]
\[
= \frac{1}{2\pi} K_0[a \sqrt{\frac{s^2}{c^2} + k^2}], \tag{4.71}
\]
where \( K_0(z) \) is a modified Bessel function [6]. After transforming to the \( k \)-domain, the \( \partial_z^2 \) operator in (4.66) can be replaced by the factor \(-k^2\). For the incident field we can write
\[
E_i^i(zu_z, s) = E_0^i \exp(-\frac{s}{c} u^i \cdot r) u_z \cdot u^e
\]
\[
= E_0^i \exp(-\frac{s}{c} z \cos \theta^i) \sin \eta \sin \theta^i \tag{4.72}
\]
and after transforming to the \( k \)-domain
\[
\hat{E}_i^i(k, s) = 2\pi E_0^i \sin \eta \sin \theta^i \delta(k - i \frac{s}{c} \cos \theta^i). \tag{4.73}
\]
Note that in (4.73), \( s \) may not have a real part. Now we can express \( \hat{I}(k, s) \) in terms of \( \hat{E}_i^i(k, s) \) and \( \hat{G}(k, s) \).
\[
\hat{I}(k, s) = \frac{2\pi \varepsilon s \hat{E}_i^i(k, s)}{(\frac{s^2}{c^2} + k^2) K_0[a \sqrt{\frac{s^2}{c^2} + k^2}]}. \tag{4.74}
\]
Substituting (4.73) into (4.74) and transforming (4.74) back into the \( z \)-domain, which is relatively easy due to the delta function in (4.73), we obtain for the current
\[
I(z, s) = \frac{2\pi \sin \eta}{\mu \sin \theta^i K_0(\frac{s}{c} a \sin \theta^i)} \exp(-\frac{s}{c} z \cos \theta^i) \frac{E_0^i(s)}{s}, \tag{4.75}
\]
which is equal to the expression for the current derived in the previous section (4.61).

We have now determined the current distribution on an infinite wire with radius \( a \), using the reduced form of Pocklington's equation and an exact method. This result supports the conclusion drawn in the previous chapter that the reduced form of Pocklington's integral equation is exact except for the radial currents on the end faces.
4.2 Response to a Delta-Gap Excitation

In this section, we assume instead of an incident field a delta-gap excitation. Because the wire is infinite the delta-gap is located at $z_g = 0$ for convenience. We will now present some solutions to this problem, for example Hallén's iterative solution, the solution resulting from using the reduced and exact form of Pocklington's equation.

4.2.1 Response Using the Exact Integral Equation

In this section we try to compute the current using the same approach as in Section 4.1.2. We start by writing down the reduced form of Pocklington's integral equation (3.38), which is the exact equation, for a delta-gap excitation described in Section 3.4. Thus we obtain in the $s$-domain,

$$-\varepsilon s\delta(z)V_g(s) = [\partial_z - \frac{s^2}{c^2}] \int_{-\infty}^{\infty} I(z', t) \frac{1}{4\pi R_a} \exp(-\frac{s}{c} R_a) dz', \quad (4.76)$$

where $R_a \overset{\text{def}}{=} \sqrt{(z - z')^2 + a^2}$. Transforming both sides of (4.76) to the $k$-domain, we obtain the following equation \[6\]

$$s\varepsilon V_g(s) = [k^2 + \frac{s^2}{c^2}] \hat{I}(k, s) \frac{1}{2\pi} K_0(a\sqrt{k^2 + \frac{s^2}{c^2}}). \quad (4.77)$$

Solving this equation for $\hat{I}(k, s)$ and transforming $\hat{I}(k, s)$ back to the $z$-domain results in

$$I(z, s) = \varepsilon s V_g(s) \int_{-\infty}^{\infty} \frac{\exp(ikz)}{[k^2 + \frac{s^2}{c^2}] K_0(a\sqrt{k^2 + \frac{s^2}{c^2}})} dk. \quad (4.78)$$

Looking closely at the integrand and realizing that $s = i\omega + \beta$ it becomes evident that there exist singularities at $k = -\infty$ and $k = \infty$ as $K_0(a\sqrt{k^2 + \frac{s^2}{c^2}})$ behaves as $\exp(-ak)$ for $ak \to \infty$ and causes singularities. In $k = \pm i\frac{s}{c}$ the integrand also becomes infinitely in magnitude. Because of these four singularities it is impossible to compute the integral analytically as well as numerically, because the integral does not exist. The reason that a delta-gap excitation on an infinite wire does not lead to a solution is that the Fourier transform of a delta function has a magnitude equal to one for all $k$. As soon as this transform does go to zero for $k \to \pm \infty$ (for example, in the case of a magnetic ring excitation [11]) a solution is found. Because a delta-gap source is not a physically realizable source it is not strange that this problem occurs. What we can do, (or try) is to approximate the current locally. This can be done by trying to solve Hallén's iterative scheme. (This scheme shifts the singularity every iteration to a higher order.) Or we can approximate the exact integral equation by the "exact" form of Pocklington's equation. (In this approximation the current is estimated locally.)
4.2.2 Hallén's Iterative First-Order Solution

In [3], Hallén presents an iterative expression for the current on a finite wire. In this section, we will apply this theory to an infinite wire using a first-order iteration. We start with (3.43) with $E_i^z = 0$ and add the additional term derived in Section 3.4 with $z_g = 0$, for convenience. By taking $L \to \infty$,

$$
\int_{-\infty}^{\infty} \frac{I(z', t - R_a/c)}{4\pi R_a} dz = \frac{Y}{2} V_p(t - \frac{|z|}{c}),
$$

which after transforming from the time domain in the s-domain yields

$$
\frac{Y}{2} V_p(s) \exp(-\frac{s}{c} |z|) = \int_{-\infty}^{\infty} I(z', s) G(z - z', s) dz',
$$

where

$$
G(z, s) = \frac{1}{4\pi \sqrt{z^2 + a^2}} \exp(-\frac{s}{c} \sqrt{z^2 + a^2}),
$$

is referred to as the kernel in this section. In analogy with Hallén's work, we add and subtract $I(z, s) \int_{-\infty}^{\infty} G(z - z', s) dz'$ to the right-hand side of (4.80) and arrive at

$$
\frac{Y}{2} V_p(s) \exp(-\frac{s}{c} |z|) = I(z, s) \int_{-\infty}^{\infty} G(z - z', s) dz' + \int_{-\infty}^{\infty} G(z - z', s)[I(z', s) - I(z, s)] dz'.
$$

The dominant contribution of the integrand is around the point $z = z'$ where the integrand tends to become singular for the first term on the right-hand side of (4.82). The impending singularity in the second term on the right-hand side is compensated for by $I(z') - I(z)$. Therefore, the largest contribution is due to the first term. We now evaluate the integral on the left-handed side of (4.82) and write [6]

$$
\int_{-\infty}^{\infty} G(z - z', s) dz' = \frac{1}{2\pi} \int_{0}^{\infty} \frac{\exp(-\frac{s}{c} \sqrt{u^2 + a^2})}{\sqrt{u^2 + a^2}} du = \frac{1}{2\pi} K_0\left(\frac{as}{c}\right) \equiv \Omega(s).
$$

The current can now be expressed as

$$
I(z, s) = \frac{Y}{2} V_p(s) \exp(-\frac{s}{c} |z|) - \int_{-\infty}^{\infty} G(z - z', s)[I(z', s) - I(z, s)] dz' \frac{\Omega(s)}{\Omega(s)}.
$$

For a thin wire we have $a^2 \ll 1$, so that the term $\Omega(s)$ is large. Therefore, the zero-order approximation is equal to the first part of the right-hand side of equation (4.84), thus

$$
I(z, s) = \frac{Y}{2} V_p(s) \exp(-\frac{s}{c_0} |z|) \frac{\Omega(s)}{\Omega(s)}.
$$
Observe that the zero-order current propagates at speed $c$ away from the feed point in both directions without attenuation. Corrections of the orders $\Omega^{-2}, \Omega^{-3}, \ldots$ are obtained by repeatedly substituting the expression for the current of (4.84) into itself. In this fashion a series development of $I(z, s)$ into powers of $\Omega(s)$ is obtained, i.e.,

$$I(z, s) = \frac{Y V_g(s)}{2} \sum_{n=0}^{\infty} \frac{I^n(z, s)}{\Omega^n(s)},$$

(4.86)

where

$$F_n(z, s) = \int_{-\infty}^{\infty} G(z - z')[I^n(z, s) - I^n(z', s)]dz',$$

(4.87)

and

$$I^{(0)}(z, s) = \exp\left(-\frac{s}{c}|z|\right).$$

(4.88)

Using this scheme, the first-order iteration of Hallén's iterative scheme can be written as

$$I^{(1)}(z, s) = \frac{Y V_p(s)\exp\left(-\frac{s}{c}|z|\right)}{\Omega(s)} + \frac{Y V_p(s)}{\Omega^2(s)} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{s}{c}\sqrt{(z - z')^2 + a^2}\right)}{4\pi \sqrt{(z - z')^2 + a^2}} \left[\exp\left(-\frac{s}{c_0}|z|\right) - \exp\left(-\frac{s}{c_0}|z'|\right)\right]dz'.$$

(4.89)

When $a$ approaches zero, the wire becomes thinner, $K_0(a^2c)$ becomes greater thus reducing the influence of the second term on the right-hand side of (4.89). Figure 4.2 shows the current distribution for several values of the radius. It can be seen that when the wire becomes thinner, the current reduces to a current flowing away from the feed point, propagating unattenuated. The other curves, referred to as Pocklington's "exact" equation, are obtained from Section 4.2.3.

### 4.2.3 Response Using the "Exact" Form of Pocklington's Integral Equation

Hallen uses the exact form of Pocklington's equation to arrive at an expression for the current on an infinite wire excited by a delta-gap source at $z_g = 0$. In this section we will give these equations and refer to Hallén's work [3] for the complete derivation. Hallén shows that the current can be written as

$$I(z, s) = 4\pi Y V_p(s) \int_{0}^{\infty} \frac{\exp\left(i\frac{s}{c}\sqrt{u^2 - 1}|z|\right)}{u\sqrt{u^2 - 1}|H_0^{(1)}(i\frac{s}{c}au)|^2} du,$$

(4.90)
Figure 4.2: Response of a Delta-Gap Excitation at $z = 0$, $\lambda = 1m$

where we choose $s = i\omega$ and the path of integration passes the upper side of branch point $u = 1$. (4.90) can also be written as

$$I(z) = 4\pi Y\psi(-\frac{S}{c}z)\exp(-\frac{S}{c}|z|),$$  \hspace{1cm} (4.91)

where for $\psi(-\frac{S}{c}z)$ after substitution of $u = \sin \phi$ for $0 \leq u \leq 1$ and $u = \frac{1}{\sin \phi}$ for $1 \leq u < \infty$ it is found that

$$\psi(-\frac{S}{c}z) = \frac{1}{\pi} \arctan \left( \frac{\pi}{2(\log \frac{4c}{-ias\pi} - \gamma)} \right) + \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} \left[ \frac{1}{\sin \phi |H_0^{(1)}(-iaz \sin \phi)|^2} - \frac{1}{\phi(1 + \frac{4}{\pi^2}(\gamma + \log \frac{as\phi}{c^2})^2)} \right] d\phi + \frac{2}{\pi^2} \exp(-\frac{S}{c}|z|) \left[ \int_0^{\frac{\pi}{2}} \frac{\exp(-\frac{S}{c}|z| \cos \phi) - \exp(-\frac{S}{c}|z|)}{\sin \phi |H_0^{(1)}(-ai\phi \sin \phi)|^2} d\phi + \right]$$

38
\[
\int_0^{\pi} \frac{\exp(i \frac{s}{c} |z| \cot \phi)}{|H_0^{(1)}(-\frac{a}{\sin \phi})|^2} d\phi.
\] (4.92)

For thin wires \(|\frac{a}{ae}| \ll 1\) we can introduce the approximation

\[
|H_0^{(1)}(\zeta)|^2 = J_0^2(\zeta) + Y_0^2(\zeta) = 1 + \frac{4}{\pi^2} (\gamma + \log \frac{\zeta}{2})^2 + O(\zeta^2),
\] (4.93)

where \(\gamma = 0.57721566\), Euler's constant. Numerical evaluation of (4.92) has been performed and found too slow because of the evaluation of the integrals. Figure 4.2 shows the current distribution for different radii. In the next section we present an approximation of (4.92) containing no integrals and allowing fast numerical evaluation.

4.2.4 A Simple Formula for the Current on a Delta-Gap Excited Infinite Wire

Shen [17] has derived an approximation for the current on an infinite wire excited by a delta-gap at \(z_g = 0\). For the complete derivation we refer to [17]. The approximation is only valid for \(\frac{s}{c} a \ll 1\). The approximated current is written as

\[
I(z, s) = iY \exp(-\frac{s}{c} z) \log|1 - \frac{2\pi i}{-2\log(-ia^{\frac{s}{c}}) - \gamma + \log(-iz^{\frac{s}{c}} + \sqrt{\exp(-2\gamma) - z^{2\frac{s}{c}}}) + i\frac{3}{2} \pi}|
\] (4.94)

This equation has been evaluated numerically. Figure 4.3 shows the current distribution for different radii. The results from the previous section are repeated for comparison. From Figure 4.3 it is observed that the simple approximation of (4.94) corresponds very closely to the results obtained from the "exact" form of Pocklington's equation, and is more accurate than Hallén's first-order iterative solution.
Figure 4.3: Response of a Delta-Gap Excitation at \( z_g = 0, \lambda = 1m \)
Chapter 5

Response of a Finite Wire

In this chapter, we will derive approximate expressions for the current on a finite wire, in terms of currents on delta-gap excited, infinite wires. In Section 5.1, we present the theory which forms the basis for a periodic delta-gap distribution. In Section 5.2, we deal with distributions of a finite number of delta-gaps. Instead of total reflection at the ends of the wire, we can choose a more general description of the reflections. In Section 5.3, we introduce a reflection coefficient. In Section 5.4, the excitation becomes an incident plane wave, for which the current distribution is approximated.

5.1 Response to a Delta-Gap Excited Wire

The idea behind the solution presented in this section is that the current on thin wires can locally be approximated by a traveling wave. Consequently, we place delta-gap excitations on the wire in such a way that they represent the reflections of the current waves at the ends. In order to explain our approach, we first give a precise, mathematical formulation of the problem at hand. Second, we solve for the vector potential and attempt to write the current as the sum of currents generated at the feed point and beyond the two ends of the wire in such a way that the current outside \((-L, L)\) is zero. Third, the current thus obtained must comply with the vector potential resulting in the demand that the response to a delta-gap excitation must be a thin-wire approximation.

Formulation of the Problem

We are looking for a solution of the differential equation

\[
(\partial_z^2 - \frac{s^2}{c^2})A_z(s, s) = -s \varepsilon V_g(s) \delta(z - z_g)
\]  

where \(z\) and \(z_g\) lie in the open interval \((-L, L)\), such that

\[
A_z(z, s) = \int_{-L}^{L} G(z - z', s) I(z', s) dz'.
\]
with \( I(-L, s) = I(L, s) = 0 \). For \( G(z - z', s) \) we can take either the "reduced" kernel of the exact integral equation

\[
G(z, s) = \frac{\exp\left(-\frac{s}{c} \sqrt{z^2 + a^2}\right)}{4\pi \sqrt{z^2 + a^2}}
\]  
(5.3)

or we can take the "exact" kernel,

\[
G(z, s) = \int_{-\pi}^{\pi} \frac{\exp\left(-\frac{s}{c} \sqrt{z^2 + 4a^2 \sin^2 \frac{1}{2} \phi}\right)}{\sqrt{z^2 + 4a^2 \sin^2 \frac{1}{2} \phi}} d\phi.
\]  
(5.4)

In essence, this is Hallén's formulation of the problem.

**Solving \( A_z(z, s) \)**

The general solution of (5.1) is

\[
A_z(z, s) = Y \frac{V_F(s)}{2} \exp\left(-\frac{s}{c} |z - z_g|\right) +
Y \frac{V_L(s)}{2} \exp\left(-\frac{s}{c} (z + L)\right) +
Y \frac{V_L(s)}{2} \exp\left(-\frac{s}{c} (L - z)\right)
\]  
(5.5)

for \(-L < z < L\), where \( V_F(s) \) and \( V_L(s) \) are constants to be determined. The first term on the right-hand side of (5.5) is the particular solution (the infinite wire solution) while the second and third term form the homogeneous solution and make sure that the boundary conditions are complied with.

Because of the finite integration interval in (5.2) we cannot invert this equation in closed form. Therefore we rewrite (5.2) as

\[
A_z(z, s) = \int_{-\infty}^{\infty} G(z - z', s) I(z', s) dz',
\]  
(5.6)

with

\[
I(z, s) = 0, \quad \text{in } -\infty < z \leq -L \quad \text{and} \quad L \leq z < \infty.
\]  
(5.7)

Now, we attempt to write \( I(z, s) \) as:

\[
I(z, s) = I_0(|z - z_g|, s) + I_{-L}(|z + L|, s) + I_L(|L - z|, s),
\]  
(5.8)

where \( I_{-L} \) and \( I_L \) can be envisaged in two ways:

- "anti-sources" located at \( z = -L \) and \( z = L \), which compensate the delta-gap solution for the infinite wire outside the interval \(-L < z < L\).

- \( I_{-L}(|z + L|, s) \) is for \( z < -L \), the cumulative current caused by virtual sources in \( z > -L \), and, for \( z > -L \), the cumulative effect of virtual sources in \( z < -L \). A similar interpretation holds for \( I_L(|L - z|, s) \).
The Iterative Approximation

Now, the first question is whether we can find suitable expressions for $I_{-L}(\zeta)$ and $I_L(\zeta)$, that satisfy (5.7). $\zeta$ is the distance from the reflection point $z = -L$ when we consider reflections at $z = -L$. When we consider reflections at $z = L$, $\zeta$ is the distance from the reflection point $z = L$.

Let us first choose $z = -L - \zeta$ with $\zeta > 0$. In that case we have $z < -L$ which means that $I(z, s) = 0$. Using the identities $|L - z| = 2L + \zeta$, $|z - z| = L + z + \zeta$ and $|z + L| = \zeta$, in combination with (5.8) we obtain

$$I_{-L}(\zeta, s) = -I_\delta(L + z_g + \zeta, s) - I_L(2L + \zeta, s), \tag{5.9}$$

for all $\zeta > 0$. Second, we take $z = L + \zeta$ with $\zeta > 0$. Then we must have $|L + z| = L + \zeta$, $|z - z| = L - z + \zeta$ and $|z - z| = \zeta$, which results in

$$I_L(\zeta, s) = -I_\delta(L - z_g + \zeta, s) - I_{-L}(2L + \zeta, s). \tag{5.10}$$

Third, we substitute (5.10) in (5.9) and obtain:

$$I_{-L}(\zeta, s) = -I_\delta(L + z_g + \zeta, s) + I_\delta(L - z_g + \zeta + 2L, s) + I_{-L}(4L + \zeta, s). \tag{5.11}$$

Now we can start an iteration by continuously substituting (5.11) in itself. Doing so we arrive at

$$I_{-L}(\zeta, s) = -I_\delta(L + z_g + \zeta, s) + I_\delta(L - z_g + \zeta + 2L, s) + I_\delta(L - z_g + \zeta + 4L, s) + I_\delta(L - z_g + \zeta + 2L + 4L, s) + I_{-L}(8L + \zeta)$$

$$= \ldots$$

$$= -\sum_{n=0}^{\infty} I_\delta(L + z_g + \zeta + 2n \cdot 2L, s) + \sum_{n=0}^{\infty} I_\delta(L - z_g + \zeta + (2n + 1) \cdot 2L, s), \tag{5.12}$$

where, in the last step it has been assumed that

$$\lim_{\zeta \to \infty} I_{-L}(\zeta, s) = \lim_{\zeta \to \infty} I_L(\zeta, s) = 0, \tag{5.13}$$

in order to obtain a meaningful summation. In a similar manner, we obtain for $I_L(\zeta, s)$ the following expression:

$$I_L(\zeta, s) = -\sum_{n=0}^{\infty} I_\delta(L - z_g + \zeta + 2n \cdot 2L, s) + \sum_{n=0}^{\infty} I_\delta(L + z_g + \zeta + (2n + 1) \cdot 2L, s). \tag{5.14}$$
Combining (5.12), (5.14) and (5.8), results in

\[ I(z, s) = I_\delta(|z - z_g|, s) \]

\[- \sum_{n=0}^{\infty} I_\delta(L + z_g + |z + L| + 4nL, s) \]

\[ + \sum_{n=0}^{\infty} I_\delta(L - z_g + |z + L| + (4n + 2)L, s) \]

\[- \sum_{n=0}^{\infty} I_\delta(L - z_g + |z - L| + 4nL, s) \]

\[ + \sum_{n=0}^{\infty} I_\delta(L + z_g + |z - L| + (4n + 2)L, s), \] (5.15)

which is the current we are interested in.

**Verification of the expression for the current**

To verify that the expression listed in (5.15) indeed satisfies the condition (5.7), we look at three different intervals.

For \(-\infty < z < -L\) we obtain

\[ I(z, s) = I_\delta(z_g - z, s) \]

\[- \sum_{n=0}^{\infty} I_\delta(z_g - z + 4nL, s) \]

\[ + \sum_{n=0}^{\infty} I_\delta(-z_g - z + (4n + 2)L, s) \]

\[- \sum_{n=0}^{\infty} I_\delta(-z_g - z + (4n + 2) \cdot 2L, s) \]

\[ + \sum_{n=0}^{\infty} I_\delta(z_g - z + (4n + 4)2L, s). \] (5.16)

After expanding the summations it can be seen that the second term on the right-hand side is canceled by the sum of the first and the last terms, and that the third and fourth terms cancel each other.

In the interval \(-L < z < L\) the following expression is obtained:

\[ I(z, s) = I_\delta(|z - z_g|, s) \]

\[- \sum_{n=0}^{\infty} I_\delta(|L + z_g| + |L + z| + 4nL, s) \]

\[ + \sum_{n=0}^{\infty} I_\delta(|L - z_g| + |L + z| + (4n + 2)L, s) \]
\[ - \sum_{n=0}^{\infty} I_\delta([L - z_g] + [L - z] + 4nL, s) \]
\[ + \sum_{n=0}^{\infty} I_\delta([L + z_g] + [L - z] + (4n + 2)L, s). \]  \hspace{1cm} (5.17)

In this case there is no cancellation.

For \( L < z < \infty \) the current assumes the form
\[ I(z, s) = I_\delta(z - z_g, s) \]
\[ - \sum_{n=0}^{\infty} I_\delta(z + z_g + (4n + 2)L, s) \]
\[ + \sum_{n=0}^{\infty} I_\delta(z - z_g + (4n + 4)L, s) \]
\[ - \sum_{n=0}^{\infty} I_\delta(z - z_g + 4nL, s) \]
\[ + \sum_{n=0}^{\infty} I_\delta(z + z_g + (4n + 2)L, s). \]  \hspace{1cm} (5.18)

As expected, the total current is also canceled out in this interval.

**Verification of the vector potential**

Of course, (5.7) is only part of the formulation of the problem. In addition, (5.5) and (5.8) must be satisfied. We must verify the following equation
\[ F_{-L}(z + L, s) = \int_{-\infty}^{\infty} G(|z + L - z'|, s)I_{-L}(|z'|, s)dz' \]
\[ = \frac{Y}{2} V_{-L}(s) \exp(-\frac{s}{c}(z + L)), \]  \hspace{1cm} (5.19)

for \(-L < z < L\). \( F_{-L}(z + L, s) \) is the vector-field caused by the outgoing current-wave. In the case of the first reflection at \( z = -L \) we observe that
\[ F_{-L}(-z - L, s) = \int_{-\infty}^{\infty} G(|-z - L - z'|, s)I_{-L}(|z'|, s)dz' \]
\[ = \int_{-\infty}^{\infty} G(|-z - L + z''|, s)I_{-L}(|-z''|, s)dz'' \]
\[ = \int_{-\infty}^{\infty} G(|z + L - z''|, s)I_{-L}(|z''|, s)dz'' \]
\[ = F_{-L}(z + L, s), \]  \hspace{1cm} (5.20)
i.e.
\[ F_{-L}(z + L, s) = F_{-L}(|z + L|, s). \]  \hspace{1cm} (5.21)

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This means that we must have
\[ F_{-L}(z + L, s) = \int_{-\infty}^{\infty} G(|z - z'|, s)I_{-L}(|z' + L|, s)dz' \]
\[ = \frac{Y}{2} V_{-L}(s) \exp\left(-\frac{s}{c}|z + L|\right), \tag{5.22} \]
for \(-3L < z < L\). Finally, it follows from analytic continuation that (5.22) must hold for \(-\infty < z < \infty\). Since (5.22) has a unique inverse as soon as it applies for all \(z\), this means we must have
\[ I_{-L}(|z + L|, s) = C_{-L}I_\delta(|z + L|, s), \tag{5.23} \]
for all values of \(z\). \(C_{-L}\) is a constant of proportion, related to \(V_{-L}(s)\). In accordance with (5.23) we write
\[ I_{-L}(\zeta, s) = -\sum_{n=0}^{\infty} I_\delta(L + z_g + \zeta + 2n \cdot 2L, s) + \]
\[-\sum_{n=0}^{\infty} I_\delta(L - z_g + \zeta + (2n + 1) \cdot 2L, s), \]
\[ C_{-L}I_\delta(\zeta, s), \tag{5.24} \]
for any combination of \(L > 0, \zeta > 0\) and \(-L < z_g < L\).
This means we must have the translation property
\[ I_\delta(\zeta_0 + \zeta) = C_0 \cdot I_\delta(\zeta) \]
\[ = \frac{I_\delta(\zeta_0)}{I_\delta(0)}I_\delta(\zeta), \tag{5.25} \]
for all \(\zeta > 0\), which is one of the definitions of the exponential function. This means that \(I_\delta(\zeta)\) must be the thin-wire approximation.
A similar reasoning is obtained for \(F_L\) in combination with \(I_L\).

**Interpretation of the Verification**

We can use the following symmetry property
\[ I_\delta(\zeta, s) = I_\delta(-\zeta, s) \tag{5.26} \]
indicating that the current on an infinite delta-gap excited wire is symmetrical in its feeding point. Using this identity, we can rewrite (5.17) as
\[ I(z, s) = I_\delta(|z - z_g|, s) \]
\[-\sum_{n=0}^{\infty} I_\delta(|z - [-L - (z_g + L) - 4nL]|, s) \]
\[+\sum_{n=0}^{\infty} I_\delta(|z - [-L - (L - z_g) - (4n + 2)L]|, s) \]
and consider this as a current on an infinite wire. Now we can use the following identity
\[
\int_{-\infty}^{\infty} G(\zeta - \zeta', s) I_s(\zeta', s) d\zeta' = \frac{Y}{2} \exp\left(-\frac{s}{c} |\zeta| \right),
\]
(5.28)
to show that \( A_z(z, s) \) as specified in (5.2) assumes the form (5.5) in the interval \(-L < z < L\). On the other hand, now the condition (5.7) is no longer satisfied automatically. Enforcing this condition will lead to the conclusion that the thin-wire approximation is satisfied. The shifts with respect to \( z \) occurring in the summations in (5.27) must be represented in \( V_{-L}(s) \) and \( V_L(s) \) from (5.5). This means that the \( I_s(\zeta, s) \) must be exponential, which is the thin-wire approximation (Section 4.2.1).

This interpretation of (5.15) amounts to classical image theory. In Figure 5.1, the location of the imaged delta-gap sources is illustrated.

Figure 5.1: Periodic Distribution of Delta-Gap Sources

We have now found an expression for the current on a delta-gap excited finite wire in terms of the current on a delta-gap excited infinite wire, using the thin wire approximation.

In practical computations, the infinite summations in (5.15) will have to be truncated at some finite \( n \). The larger we chose \( n \), the better the approximation will satisfy either one of the equations (5.5) or (5.7). Nevertheless, the current at the ends of the wire will be exactly zero. In the next section, we propose one way of at least satisfying those end conditions with a finite sum of delta-function currents.
5.2 Distribution of a Finite Number of Delta Gaps

In this section, we will truncate all sums at some finite \( N - 1 \), and include two additional delta-gap sources at \( z = \pm(4N + 1)L \). The weighting coefficients for these sources are then adjusted such that the end conditions on the wire are met. This situation is pictured in Figure 5.2. For the total current we can then write

\[
I(z, s) = I_\delta(|z - z_g|, s) \\
- \sum_{n=0}^{N-1} I_\delta(L + z_g + z + L + 4nL, s) \\
+ \sum_{n=0}^{N-1} I_\delta(L - z_g + z + L + (4n + 2)L, s) \\
- \sum_{n=0}^{N-1} I_\delta(L - z_g + z - L + 4nL, s) \\
+ \sum_{n=0}^{N-1} I_\delta(L + z_g + z - L + (4n + 2)L, s) \\
+C_{-NL}(s)I_\delta(L - z + 4NL, s) \\
+C_{NL}(s)I_\delta(L + z + 4NL, s). \quad (5.29)
\]

for \(-L < z < L\) and \(-L < z_g < L\). For the determination of \( C_{-NL}(s) \) and \( C_{NL}(s) \) we use the boundary condition \( I(-L, s) = I(L, s) = 0 \) and obtain

\[
V_{-NL}(s) = \frac{I_\delta((4N + 2)L, s)I_\delta(L + z_g + 4NL, s) - I_\delta(4NL, s)I_\delta((L - z_g + 4NL, s)}{I_\delta^2(4NL, s) - I_\delta^2((4N + 2)L, s)} \quad (5.30)
\]

and

\[
V_{NL}(s) = \frac{I_\delta((4N + 2)L, s)I_\delta(L - z_g + 4NL, s) - I_\delta(4NL, s)I_\delta((L + z_g + 4NL, s)}{I_\delta^2(4NL, s) - I_\delta^2((4N + 2)L, s)} \quad (5.31)
\]
Note, that if we choose for the thin-wire approximation, i.e. choose an exponential that depending on the length of the wire can cause the denominator of (5.30) and (5.31) to become zero, the response will be infinitely high. Another problem arising in (5.29) and especially in (5.15) is that, generally, it will take quite some time to evaluate the summations to obtain the desired accuracy.

**Special Case N=0**

One way to overcome this and obtain an expression which can be evaluated very fast, is to substitute \( N = 0 \) in (5.29). We obtain the following equation

\[
I(z, s) = I_\delta(|z_g - z|, s) + \frac{I_\delta(2L, s)I_\delta(L + z_g, s) - I_\delta(0, s)I_\delta(L - z_g, s)}{I_\delta^2(0, s) - I_\delta^2(2L, s)}I_\delta(L - z, s)
\]

\[
+ \frac{I_\delta(2L, s)I_\delta(L - z_g, s) - I_\delta(0, s)I_\delta(L + z_g, s)}{I_\delta^2(0, s) - I_\delta^2(2L, s)}I_\delta(L + z, s)
\]

(5.32)

In this situation the delta-gap sources are placed as is shown in Figure 5.3. Note that (5.32) is in fact the same approximation found by Bouwkamp [4] and Hallén [2].

**Approximation of the Delta-Gap Current**

So far we have found three expressions approximating the current on a finite wire. In order to evaluate those we need the current on a delta-gap excited infinite wire, \( I_\delta(z, s) \). This problem has been discussed in Section 4.2 where we derived several approximations. The most suitable approximation is given in Section 4.2.4 by the following equation

\[
I_\delta(z, s) = iY \exp(-\frac{S}{c}z) \log[1 - \frac{2\pi i}{-2 \log(-ia_c^2) - \gamma + \log(-iz_c^2 + \sqrt{\exp(-2\gamma) - z_c^2} + \frac{3i}{2} \pi)}]
\]

(5.33)

We choose this approximation for the following reasons.

First, the approximated current must be finite for \( z = z_g \). The approximation using the "exact" kernel results in an infinitly high imaginary part of the current.
Second, we use the thin-wire approximation. For an infinitely thin wire the current distribution is supposed to be finite with a constant magnitude. This is also a reason why the approximation, using the "exact" kernel, is out of consideration. Third, (5.33) is considered more accurate than the formula obtained from Hallén's iterative scheme. Finally (5.33) allows fast numerical evaluation which is also an advantage.

Results
Since a fast computation of the current is of great importance we concentrate on the evaluation of (5.32). We refer to curves obtained by evaluating (5.32) as model 1, N=0. Referring as model 1, N=5 means that the curve was computed using (5.29) with N=5.

From Figures 5.4 and 5.6 we may conclude that model 1 represents the simulated data from NEC reasonably accurate. We can also conclude that it does not make any difference if we end the iteration after N = 5 or after N = 20. The figures also show that representing the reflections with two delta gaps at the ends is a better approximation than representing the reflections by more of these sources.

Figure 5.4: Current on a Delta-Gap-Excited Wire with 2L = 4m, a = 0.005m, z_g = 0m and \lambda = 1m
Comparing the curves in Figure 5.5 and Figure 5.7 shows us that for short wires, the results obtained by our model differ greatly from the results obtained by the NEC.

In Figure 5.5 and Figure 5.7 we omitted results for \( N = 5 \) and \( N = 20 \) because their accuracy is even worse than for \( N = 0 \). From this data we may conclude that our model is useful for determining currents on long wires. In the next section we will present an alternative to be able to compute the current for short wires as well.

### 5.3 Response to a Delta-Gap Excitation of a Short Wire

In (5.32) we secretly assumed a reflection coefficient equal to minus one. This means that the incoming current wave is completely reflected, causing the current to be zero at the ends.

In the case of a wire with a length equal to half the wavelength, the reflected currents interact constructively resulting in current with a magnitude much too high. By introducing reflection coefficients not equal to one we can cancel this
constructive interaction and obtain a better approximation of the current at the cost of not being able to meet the zero-current condition at the ends of the wire. Now let us more generally write the current as

\[ I(z, s) = I_\delta(z - z_g, s) + C_{-L}(s)I_\delta(L + z, s) + C_L(s)I_\delta(L - z, s), \]  

(5.34)

where \( C_{-L}(s) \) and \( C_L(s) \) are the amplitudes of the outgoing waves at \( z = -L \) and \( z = L \), respectively. The amplitude of the incoming wave at \( z = -L \) is equal to \( I_\delta(z_g + L, s) + C_L(s)I_\delta(2L, s) \). The amplitude of the reflected wave was \( C_{-L}(s) \). Now we can determine the reflection coefficient at \( z = -L \) as

\[ -\Gamma = \frac{C_{-L}(s)I_\delta(0, s)}{I_\delta(z_g + L, s) + C_L(s)I_\delta(2L, s)}, \]  

(5.35)

and at \( z = L \) as

\[ -\Gamma = \frac{C_L(s)I_\delta(0, s)}{I_\delta(L - z_g, s) + C_{-L}(s)I_\delta(2L, s)}. \]  

(5.36)
Figure 5.7: Current on a Delta-Gap-Excited Wire with $2L = 0.5m$, $a = 0.005m$, $z_g = 0.2m$ and $\lambda = 1m$

Note that $\Gamma$ is dimensionless. We will assume $\Gamma$ to be known. We will now solve (5.35) and (5.36), obtaining

$$C_L(s) = \frac{I_0(0, s)I_0(L - z_g, s) - \Gamma I_0(L, s)}{I_0^2(0, s) - \frac{\Gamma^2 I_0^2(2L, s)}{2L}}.$$  \hfill (5.37)

and

$$C_L(s) = \Gamma \left[ \frac{I_0(0, s)I_0(L - z_g, s) - \Gamma I_0(L + z_g, s)}{I_0^2(0, s) - \frac{\Gamma^2 I_0^2(2L, s)}{2L}} \right].$$  \hfill (5.38)

All we have to do now is to find an acceptable method for determining $\Gamma$. Note that if we take $\Gamma = -1$ we obtain (5.32) again. Shen [17] assumed $\Gamma$ to be a slowly varying function of $|L - z_g|$, the distance between the delta-gap source and the end of the wire, so that as $L$ goes to infinity the limiting values of $\Gamma$ can be used for all $L$. In this case for $\Gamma$ is found

$$\Gamma = \frac{1}{Y\pi} \left( - \log(ka) - \gamma + i\frac{\pi}{2} \right).$$  \hfill (5.39)

An alternative method for determining $\Gamma$ is to perform a least-squares curve fit on numerical data (for example data obtained by the NEC). Doing so for certain $L - z_g$ and $a$ makes it possible to find other reflection coefficients by interpolation. There are several more expressions for $\Gamma$ to be found in literature.
Results
Now we will present some results obtained by using (5.34).
When we use (5.34) we will refer to the obtained data by model 2, $\Gamma =$ as specified.

![Graph of current on a Delta-Gap-Excited Wire with $2L = 0.5m$, $a = 0.005m$, $z_g = 0.0m$ and $\lambda = 1m$]

Figure 5.8: Current on a Delta-Gap-Excited Wire with $2L = 0.5m$, $a = 0.005m$, $z_g = 0.0m$ and $\lambda = 1m$

We computed $\Gamma$ from (5.39) (Shen) and changed it manually to show its sensitivity of the current with respect to this parameter. From Figure 5.8 and Figure 5.9 it follows that introducing a reflection coefficient in the model improves the results in Figure 5.5 and Figure 5.7. We can also conclude that by manipulating $\Gamma$ we can achieve a better approximation. A drawback is that the current is no longer zero at the ends of the wire.

5.4 Response to a Plane-Wave Excitiation

The most obvious way to determine the response to a plane-wave incident on the finite wire is to use the same scheme as described in section 5.2, which is illustrated in Figure 5.10. The current on the finite wire is here described as the sum of currents on an infinite wire generated by an incident plane wave. The reason that this reflection scheme does not work is that we have excitation
outside \(-L < z < L\). and that (5.15) was derived for excitation inside that interval. Therefore we will use a mathematical representation.

**Mathematical Representation**

In Section 2.3 we defined the incident field as

\[
E^i(r, s) = E_0^i(s) \exp\left(-\frac{s}{c} \cdot r\right)u^e, 
\]

The \(z\)-component of this field is the quantity in which we are interested. In view of the wire geometry, this component can be written as

\[
E^i_z(z, s) = E_0^i(s) \exp\left(-\frac{s}{c} z \cos \theta^i\right) \sin \eta \cos \theta^i. 
\]

With the aid of the superposition principle, a plane-wave excitation can mathematically be envisaged as the effect of a continuous distribution of delta-gap sources on a finite wire. If we excite the sources with correct phase shifts, then the total response is the superposition of all the individual responses. Therefore
we can write the total current responding to a plane-wave incident on a finite wire as

\[ I(z, s) = E_0^i(s) \sin \eta \sin \theta^i \int_{-L}^{L} \exp\left(-\frac{s}{c} z_g \cos \theta^i\right) I(z, z_g; s) dz_g, \]

(5.42)

where \( I(z, z_g; s) \) is the current due to a delta-gap excitation at \( z = z_g \) derived in the previous section. Note that the superposition is purely mathematical and that there is no physical basis because a plane-wave and a delta-gap excitation are two completely different excitations.

**Results**

We have evaluated (5.42) numerically in combination with (5.32) to obtain the following approximations displayed in Figure 5.11 and Figure 5.12. Comparing our results with those of the NEC we conclude that our method is a reasonably accurate representation of the current on a plane-wave excited wire. Because (5.32) did not perform well for short wires (see Figure 5.5 and Figure 5.7), we will use (5.34) substituted in (5.42) with a fixed \( \Gamma \). The current distribution for a plane-wave incident under 90° and 60° is depicted in Figure 5.13 and Figure 5.14, respectively, and referred to by *model 2, \( \Gamma = as specified*.* The curves in Figure 5.13 and Figure 5.14 still show some deviation. This is mainly due to the fixed \( \Gamma \) which does depends on \( L - z_g \). Therefore if we continue this search path, more accurate closed-form expressions for \( \Gamma \) must be found first.
Figure 5.11: *Current on a Plane-Wave-Excited Wire with* $2L = 4.0m$, $a = 0.005m$, $\theta^i = 90^\circ$ and $\lambda = 1m$
Figure 5.12: Current on a Plane-Wave-Excited Wire with $2L = 4.0\text{m}$, $a = 0.005\text{m}$, $\theta^i = 60^\circ$ and $\lambda = 1\text{m}$
Figure 5.13: Current on a Plane-Wave-Excited Wire with \(2L = 0.5m\), \(a = 0.005m\), \(\theta = 90^\circ\) and \(\lambda = 1m\)
Figure 5.14: Current on a Plane-Wave-Excited Wire with $2L = 0.5\,m$, $a = 0.005\,m$, $\theta^i = 60^\circ$ and $\lambda = 1\,m$
Chapter 6

Conclusions and Recommendations

6.1 Conclusions

In this report we have searched for an analytical approximation of the current on a wire. We have found several approximations for the infinite as well as for the finite wire, excited by a plane wave or by a delta-gap source. In the case of an infinitely thin wire, excited by a plane wave, we have found an exact solution in two different ways. In one method we determined the current from the $\phi$ component of the magnetic field. In the other method we used the reduced form of Pocklington's integral equation to obtain the current which is found to be equal to the current obtained by employing the first method. This result supports the conclusion that the reduced form of Pocklington's equation is the exact integral equation for the current on an infinite wire. The "exact" form of Pocklington's equation leads to an approximation of the current. Next we investigated the current distribution on a delta-gap excited infinite wire. We have found several approximate solutions which we used to determine the current on a finite wire excited by a delta-gap source. We have found a satisfactory approximation for the current distribution on a plane-wave excited finite wire. The approximation accounts for the propagation losses (radiation) of the current on the wire. The reflection of the current at the ends of the wire is not modeled correctly. In this model no radiation is emitted. The analytical expressions found allow a fast numerical evaluation. The obtained approximations are reasonable for chaff applications.

6.2 Recommendations

To obtain a more accurate approximation for the reflection coefficient $\Gamma$, further investigation is recommended. The integral describing the mathematical relation between a delta-gap excited
wire and a plane-wave excited wire is recommended to be evaluated analytically instead of numerically.
Now that we have an approximation we can consider multiple scattering. The next step could be investigating the scattering of a small number of wires numerically as well as analytically and compare the result.
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