Contour Dynamics and Applications to 2D Vortices

Pauline Vosbeek
Contour Dynamics

and

Applications to 2D Vortices
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1

Introduction

1.1 2D flows and vortices
1.2 Contour dynamics and applications
1.3 Scope of the thesis
1.1 2D flows and vortices

Flows in the atmosphere and oceans of the Earth can be considered approximately two-dimensional. This is a consequence of both atmosphere and oceans being thin layers of only a few kilometres depth, whereas the horizontal scales of the flow inside these layers are typically of the order of hundreds or even thousands of kilometres. The two-dimensional motion is furthermore encouraged by the Earth’s rotation, which causes the fluid to move locally in horizontal planes. The third aspect which enhances the two-dimensionality of the flow, is the vertical density stratification of the fluid layers. In the oceans this is caused by a temperature and salinity gradient, whereas in the atmosphere it is mainly caused by a temperature gradient. The large-scale two-dimensional flows in the oceans and atmosphere are commonly referred to as geophysical flows.

Quasi two-dimensional vortices play an important role in geophysical flows. Examples are the Gulf Stream Rings in the Atlantic Ocean and the high and low pressure areas in the atmosphere. Vortices are significant for the transport of physical properties, like momentum and heat, and salt. Therefore it is important to understand their dynamics.

Vortices can consist of several compact patches of distributed vorticity. Based on their topological shape, they can be divided into a number of classes. The most common vortex structure is the monopole, which consists of only one circular or elliptical patch of distributed vorticity. The vorticity distribution can be singly-signed, but it is also possible that the vortex core contains vorticity of one sign surrounded by a ring of vorticity with opposite sign. Examples of monopolar vortices are the high and low pressure cells in the atmosphere, Jupiter’s Great Red Spot, which exists already more than 300 years (see Figure 1.1) and Neptune’s Great Dark Spot. In literature, the monopolar vortex has been investigated theoretically and numerically as well as in laboratory experiments.

Another class of vortices is that of the dipoles. A dipolar vortex consists of two compact patches of opposite vorticity causing the structure to have a net linear momentum. As a consequence, the dipole moves steadily in a direction defined by its axis. A symmetric dipole translates along a straight line, whereas an asymmetric dipole, for which one of the patches is stronger than the other, moves along a curved path. The dipolar vortex has both been observed in nature, for example, as a blocking event in the atmosphere, and in laboratory experiments.

A more complicated structure is the tripolar vortex. This vortex consists of three distinct compact patches of distributed vorticity: a more or less elliptical core of singly-signed vorticity and two so-called satellites of oppositely-signed vorticity. Like the monopolar vortex, the tripole contains a net angular momentum resulting in a steady rotation of the structure around the axis of the core. The tripole has been observed mainly in laboratory experiments, however Pingree and Le Cann [50] observed a tripolar structure also in nature. In their paper, an infrared sea-surface picture (see Figure 1.2) is presented showing a tripole in the Bay of Biscay (observed on January 4th, 1990). The core of that tripole measures about 50 – 70 km and rotates in anticyclonic sense, i.e. clockwise on the northern hemisphere.

Although even more complicated structures might exist, like a triangular vortex with
three satellites of opposite sign which has been observed in the laboratory \cite{6, 8}, they have not (yet) been found to occur in nature.

The behaviour of these vortices changes when they experience a gradient in background vorticity, which is the case in geophysical flows. Due to the spherical shape of the Earth, the horizontal plane in which the motion occurs is parallel to the Earth's axis of rotation near the equator, and is perpendicular to it near the poles. As a consequence, the fluid effectively experiences a spatial variation of the rotation rate, which is zero near the equator and reaches its maximum near the poles. This variation causes, for example, a cyclonic monopole on the northern hemisphere to move to the north-west \cite{1, 9, 45} whereas without this background vorticity gradient it does not translate at all. If the scales of motion are not larger than about one thousand kilometres, it is common to consider the domain flat and the planetary rotation varies approximately linearly with the latitude. This approximation is usually referred to as the $\beta$-plane approximation. Near the poles, however, the variation is approximately quadratic, and in these cases the $\gamma$-plane approximation is used. If the scales of motion are small, i.e. smaller than a few hundred kilometres, the planetary rotation is assumed to be approximately uniform ($f$-plane approximation).

### 1.2 Contour dynamics and applications

Contour dynamics is a numerical method which can be used to simulate the dynamics of two-dimensional vortices. The method has originally been developed by Zabusky, Hughes...
and Roberts [60] and has been brought to full growth by the pioneering work of Dritschel [18, 19, 20, 21]. The method is based on the observation that, in case of an inviscid, incompressible, two-dimensional fluid flow, the evolution of a patch of uniform vorticity is fully determined by the evolution of its bounding contour. To obtain an approximation of a patch of distributed vorticity (like in real-life vortices), several contours can be nested.

Although the method is well-known and widely used in the field of vortex dynamics, application of contour dynamics in other fields is not so common. However, the method has also been successfully used in, for example, studying two-dimensional chemical systems with both reactions and diffusion [24, 49]. In that case, the evolution of the interface between regions of different composition is determined using a contour dynamics representation. Another application of contour dynamics is in the field of magnetic fluids [16, 38] where a contour dynamics representation is used to compute the evolution of the boundary between two thermodynamic phases, for example, two oppositely magnetised domains. A final example of an application of the method, can be found in the field of acoustics [54]. Here, the contour dynamics method is used in combination with vortex sound theory to study the generation of sound by two interacting vortices. Although this application is strongly related to vortex dynamics (in contrast to the other examples) it is not so common applied in that context.

In the past ten or fifteen years, the method itself has been studied extensively. During this period, effort has been put mainly in improving the spatial discretization of the method (see also the review paper by Pullin [51]), i.e. the interpolation of the contours. For example, Van Buskirk and Marcus [55] presented a spectrally accurate version of contour dynamics. To this end, they used a spectral parameterisation to represent the contours of the vorticity patches. Furthermore, Dritschel [18, 19] introduced the technique of contour surgery. This technique is a topological reconnection scheme that is used to remove contour features that are smaller than a prescribed length scale. The remaining parts of that contour are subsequently reconnected. The time integration part has not been examined into detail. In most versions of the method, a standard fourth-order Runge Kutta method is used.

In the field of fluid dynamics, contour dynamics has mainly been used to study vortices and their interactions in the presence of zero or uniform background vorticity and shear and strain flows. However, in geophysical applications, vortices are also influenced by the non-uniform background vorticity caused by the planetary rotation and the spherical shape of the Earth. In the research presented in this thesis, it is shown that contour dynamics can also be used to study this kind of problems. A practical problem encountered by applying contour dynamics to such problems, is that simulations can become computationally very expensive. A more efficient algorithm is presented here as well.

1.3 Scope of the thesis

The remainder of this thesis is organised as follows. In Chapter 2, the equations of flow in incompressible and inviscid flow are discussed. In addition, equations of motion for
1.3. **Scope of the thesis**

Geophysical flows are considered, as well as the $f$-, $\beta$-, and $\gamma$-plane approximations.

In Chapter 3, the contour dynamics method as used throughout this thesis is discussed. Special attention is paid to the spatial discretization and, moreover, the time integration which is carried out using a symplectic time integration scheme. This chapter appeared in a modified version as a journal paper [58].

Chapter 4 deals with a famous problem in 2D flows, namely the interaction of three monopolar vortices with particular initial positions and strengths. In this chapter, contour dynamics is used to study the evolution of the three vortices. This chapter is also published in a modified way [57].

In Chapter 5, a way to accelerate contour dynamics simulations is presented. To this end, a so-called *hierarchical-element method* is used. With this acceleration, it becomes possible to use contour dynamics for more complicated problems like the evolution of vortices in the presence of non-uniform background vorticity.

In Chapter 6 it is illustrated how to use contour dynamics for such problems. Furthermore, the method is tested on a flow problem with relevance for large-scale geophysical flows and which has been investigated in laboratory experiments: the unsteady behaviour of a tripole on a $\gamma$-plane [56]. Numerical results are compared to results of laboratory experiments.

Finally, in Chapter 7 the main conclusions from the present research are discussed. In addition also some recommendations for future research are given.
Chapter 1. Introduction
2

Mathematical Model

2.1 Introduction

2.2 Equations of motion for an incompressible flow of an inviscid fluid
   2.2.1 Principles of continuum mechanics
   2.2.2 Conservation laws for incompressible flows of an inviscid fluid
   2.2.3 The equations for a 2D-flow in vorticity-stream function formulation

2.3 Geophysical flows
   2.3.1 Effects of rotation
   2.3.2 Equations of motion for a shallow fluid layer on a rotating sphere
   2.3.3 $f$-plane, $\beta$-plane and $\gamma$-plane approximations
2.1 Introduction

In this chapter the mathematical model for a two-dimensional incompressible flow of an inviscid, Newtonian fluid is discussed (Section 2.2). To this end, first some principles of continuum mechanics are considered. With these principles, the flow equations, in terms of the primitive variables, i.e. the velocity $u$ and the pressure $p$, can be derived. In vortex dynamics, it is convenient to transform these equations into a vorticity-stream function formulation.

The flow equations obtained are valid in a fixed frame of reference. In geophysical flows, however, position and velocity are usually measured in a coordinate system fixed to the surface of the earth. Such a coordinate system thus rotates with the earth relative to an inertial frame of reference. In Section 2.3 flow equations are derived that are valid in such a rotating frame of reference. Furthermore, since the atmosphere and oceans are thin layers, the flow is considered to be locally parallel to the surface of the sphere so that the vertical velocities can be neglected. In this case, the flow can be considered two-dimensional. Finally, upon deriving the equations for such geophysical flows have been, a number of approximations to these equations are considered, namely the $f$-plane, $\beta$-plane and $\gamma$-plane approximations.

2.2 Equations of motion for an incompressible flow of an inviscid fluid

In this section, the equations of motion for an incompressible, two-dimensional flow of an inviscid fluid are discussed. First, some principles of continuum mechanics are considered. After that, these principles are used to obtain the flow equations both in the primitive variables and in vorticity-stream function formulation.

2.2.1 Principles of continuum mechanics

A continuous body $B$ of fluid is a set whose elements can be put into a bijective correspondence with the points of a region $\mathcal{G}$ in $\mathbb{R}^3$ [11]. The elements $\mathcal{P}$, $\mathcal{Q}$, etc. of $B$ are called material points. $\mathcal{G}$ is the configuration (or material volume) of $B$, i.e. the region in $\mathbb{R}^3$ that is occupied by $B$. An element of $\mathcal{G}$ is denoted by $x = (x_1, x_2, x_3)^T$. When a body is in motion, the configuration changes with time $t$, so $\mathcal{G} = \mathcal{G}(t)$. Now introduce a reference-configuration $\mathcal{G}_r \subset \mathbb{R}^3$ which is fixed. This reference-configuration may be the configuration of $B$ at a certain reference time $t_r$, but this is not a requirement. The position of a material point $\mathcal{P} \in B$ in this reference-configuration is written as $X$ ($X = (X_1, X_2, X_3)^T \in \mathcal{G}_r \subset \mathbb{R}^3$).

The mechanical processes in $B$, in consequence of its motion and deformation, can now
be described in two ways:

\[ \mathbf{X} = \mathbf{X}(\mathbf{x}, t), \]
\[ \mathbf{x} = \mathbf{x}(\mathbf{X}, t). \]

In the first way of description (often used for fluids) the situation is considered at a fixed point \( \mathbf{x} \) in space. This is called the *Eulerian* or *spatial* description of the motion. In the second formulation, one particular material point \( \mathcal{P} \) is followed during the motion or deformation and this is called the *Lagrangian* or *material* description.

Any scalar field \( \Phi \), specifying some property of the body \( \mathcal{B} \), can now be considered a function of \( \mathbf{X} \) and \( t \), \( \Phi = \Phi(\mathbf{X}, t) \), or a function of \( \mathbf{x} \) and \( t \), \( \Phi = \Phi(\mathbf{x}, t) \). The *material time derivative* of \( \Phi \) is the rate of change with time of \( \Phi \) of a specific particle, so

\[
\frac{D\Phi}{Dt} = \frac{\partial \Phi(\mathbf{X}, t)}{\partial t}.
\]

Similarly, the *local derivative* is the rate of change of \( \Phi \) with time at a fixed point in space, meaning

\[
\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi(\mathbf{x}, t)}{\partial t}.
\]

Between the material and local time derivatives, the following relation exists

\[
\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + (\mathbf{u}, \nabla \Phi).
\] (2.1)

In this relation, the particle velocity \( \mathbf{u} \) of \( \mathcal{P} \) is given by

\[
\mathbf{u} = \dot{\mathbf{x}} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}.
\]

Now, the so-called *Reynolds Transport Theorem* [12, 36] can be derived:

\[
\frac{D}{Dt} \int_{\mathcal{V}(t)} \Phi \, d\mathbf{V} = \frac{D}{Dt} \int_{\mathcal{V}_r} \tilde{\Phi} J \, d\mathbf{V},
\]

where \( J := \det(\partial \mathbf{x} / \partial \mathbf{X}) \). By applying (2.1) to the right-hand-side and using the observation that \( DJ/Dt = J(\nabla, \mathbf{u}) \) [12] it follows that

\[
\int_{\mathcal{V}_r} \frac{D\tilde{\Phi}J}{Dt} \, d\mathbf{V}_r = \int_{\mathcal{V}_r} \left( \frac{\partial \tilde{\Phi}}{\partial t} + (\nabla, \tilde{\Phi} \mathbf{u}) \right) J \, d\mathbf{V}_r,
\]

\[
= \int_{\mathcal{V}(t)} \left( \frac{\partial \Phi}{\partial t} + (\nabla, \Phi \mathbf{u}) \right) d\mathbf{V},
\]
yielding the Reynolds Transport Theorem
\[
\frac{D}{Dt} \iiint_{\Omega(t)} \Phi \, dV = \iiint_{\Omega(t)} \left( \frac{\partial \Phi}{\partial t} + (\nabla, \Phi \mathbf{u}) \right) \, dV.
\] (2.2)

This transport theorem can now be applied to derive the basic conservation laws in differential form.

### 2.2.2 Conservation laws for incompressible flows of an inviscid fluid

Conservation of mass of any material volume \( \Omega(t) \) can be written as
\[
\frac{D}{Dt} \iiint_{\Omega(t)} \rho \, dV = 0,
\]
where \( \rho \) is the density of the fluid, or, by using (2.2),
\[
\iiint_{\Omega(t)} \left( \frac{\partial \rho}{\partial t} + (\nabla, \rho \mathbf{u}) \right) \, dV = 0.
\]

Since \( \Omega(t) \) is arbitrary this leads to the differential equation
\[
\frac{\partial \rho}{\partial t} + (\nabla, \rho \mathbf{u}) = 0,
\] (2.3)
which holds at each point of the fluid. In obtaining this equation it has been assumed that \( \rho \) and \( \mathbf{u} \) are continuous and differentiable. For an incompressible flow, i.e.
\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{u}, \nabla \rho) = 0,
\]
it follows that equation (2.3) reduces to
\[
(\nabla, \mathbf{u}) = 0,
\] (2.4)
which often is called the incompressibility condition or the continuity equation.

In the absence of external body forces and effects due to viscosity, the balance of linear momentum for a Newtonian fluid is given by
\[
\frac{D}{Dt} \iiint_{\Omega(t)} \rho \mathbf{u} \, dV = - \iint_{\partial \Omega(t)} \rho \mathbf{n} \, dS,
\]
where \( \partial \Omega(t) \) is the boundary of \( \Omega(t) \), \( p \) the pressure and \( \mathbf{n} \) the outward unit normal. Now Gauss' divergence theorem can be used to convert the surface integral in the right-hand-side into a volume integral, yielding
\[
\frac{D}{Dt} \iiint_{\Omega(t)} \rho \mathbf{u} \, dV = - \iiint_{\Omega(t)} \nabla p \, dV.
\]
2.2. Equations of motion for an incompressible flow of an inviscid fluid

Application of the transport theorem (2.2) to the three components of the left-hand-side separately yields
\[
\iiint_{\mathcal{G}(t)} \left( \frac{\partial \rho u_i}{\partial t} + (\nabla, \rho u_i u) \right) \, d\mathcal{V} = - \iiint_{\mathcal{G}(t)} \frac{\partial p}{\partial x_i} \, d\mathcal{V}, \quad i = 1, 2, 3.
\]

Now by assuming incompressibility and recalling that \( \mathcal{G}(t) \) is an arbitrary volume, it can be observed that
\[
\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p, \quad (2.5)
\]

which is known as Euler's equation for an incompressible flow of an inviscid fluid. Equations (2.4) and (2.5) are the flow equations in the primitive variables \( \mathbf{u} \) and \( p \). For a two-dimensional flow, these equations can be written in vorticity-stream function formulation which is considered in the next section.

2.2.3 The equations for a 2D-flow in vorticity-stream function formulation

Define the quantity \( b \), the Bernoulli function, as
\[
b := \frac{1}{2} (\mathbf{u}, \mathbf{u}) + p/\rho, \quad (2.6)
\]
and the vorticity vector \( \omega \) as
\[
\omega := \nabla \times \mathbf{u}, \quad (2.7)
\]
then equation (2.5) becomes
\[
\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} = -\nabla b.
\]

Since \( \nabla \times (\nabla \Phi) = 0 \) for any scalar field \( \Phi \), taking the curl of this equation results in
\[
\frac{\partial \omega}{\partial t} + \nabla \times \omega \times \mathbf{u} = 0, \quad (2.8)
\]
which is often referred to as Helmholtz' equation for the transport of vorticity.

For a two-dimensional flow field \( \mathbf{u} := u e_x + v e_y \), the vorticity vector points in vertical direction, i.e. \( \omega = \omega e_z \) (note that here the notation \((x, y, z)\) is used for the frame of reference instead of \((x_1, x_2, x_3)\) above). Furthermore, since \( \omega \) and \( \mathbf{u} \) are assumed to be independent of \( z \), then
\[
\nabla \times \omega \times \mathbf{u} = \nabla \times ( -\omega u e_z + \omega v e_y )
\]
\[
= \left( \frac{\partial}{\partial x} (\omega u) + \frac{\partial}{\partial y} (\omega v) \right) e_z
\]
\[
= (\nabla, \omega \mathbf{u}) e_z,
\]
and the Helmholtz' equation reduces to a scalar equation:

\[
\frac{\partial \omega}{\partial t} + (\nabla, \omega \mathbf{u}) = 0.
\]

With \((\nabla, \omega \mathbf{u}) = \omega(\nabla, \mathbf{u}) + (\mathbf{u}, \nabla \omega)\), and equation (2.4) it follows that

\[
\frac{D \omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{u}, \nabla \omega) = 0,
\]

i.e. conservation of vorticity of a fluid particle. Since the velocity field is divergence free (see (2.4)), a stream function \(\psi\) can be defined through

\[
\mathbf{u} := \nabla \times (\psi \mathbf{e}_z),
\]

implying that

\[
\begin{cases}
  u = \frac{\partial \psi}{\partial y}, \\
  v = -\frac{\partial \psi}{\partial x}.
\end{cases}
\]  

(2.10)

This means that the system of equations is a so-called Hamiltonian system with Hamiltonian \(-\psi\). With this definition of the stream function it follows that (2.9) can be written as

\[
\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} = 0.
\]

(2.11)

Furthermore, it follows from the definitions of the vorticity \(\omega\) and the stream function \(\psi\) that these two scalar fields are related through

\[
\nabla^2 \psi = -\omega.
\]

(2.12)

Equations (2.11) and (2.12) are the governing equations for a two-dimensional incompressible flow of an inviscid fluid in the vorticity-stream function formulation.

### 2.3 Geophysical flows

The Euler equation (2.5) as given in Section 2.2.2 is valid in an inertial or fixed frame of reference. In geophysical flows, however, position and velocity are measured with respect to a frame of reference fixed to the surface of the earth rotating relative to an inertial frame of reference. In this section, the Euler equation is derived which is valid in a rotating frame of reference. This derivation is similar to the one given in the book by Kundu [36]. Furthermore, the equations of motion for a shallow fluid layer on a rotating sphere and three approximations to these equations are discussed.
2.3. Geophysical flows

![Diagram showing a frame (x, y, z) rotating with angular velocity Ω relative to the inertial frame (x', y', z').](image)

**Figure 2.1:** The frame \((x, y, z)\) rotating with angular velocity \(Ω\) relative to the inertial frame \((x', y', z')\).

### 2.3.1 Effects of rotation

Consider a fixed or inertial frame of reference \((x', y', z')\) as given in Figure 2.1. The frame of reference \((x, y, z)\) is rotating with angular velocity \(Ω\) relative to the inertial frame. An arbitrary vector \(\mathbf{v}\) can be represented in the rotating frame of reference by \(\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3\). For an observer at a fixed position in the inertial frame of reference, the directions of the rotating unit vectors \(\mathbf{e}_i\), with \(i = 1, 2, 3\), change in time. The time derivative of \(\mathbf{v}\) in the fixed frame of reference is thus given by

\[
\left( \frac{d\mathbf{v}}{dt} \right)_{\text{f}} = \sum_{i=1}^{3} \frac{dv_i}{dt} \mathbf{e}_i + \sum_{i=1}^{3} v_i \frac{d\mathbf{e}_i}{dt}.
\]

For an observer at a fixed position in the rotating frame, the time derivative of \(\mathbf{v}\) is equal to the first summation. Furthermore it can easily be shown that \(d\mathbf{e}_i/dt = \mathbf{Ω} \times \mathbf{e}_i\) [36] and hence

\[
\left( \frac{d\mathbf{v}}{dt} \right)_{\text{f}} = \left( \frac{d\mathbf{v}}{dt} \right)_{\text{R}} + \mathbf{Ω} \times \mathbf{v}.
\]  \hspace{1cm} (2.13)

Now assume that the velocity field in the inertial frame of reference is given by \(\mathbf{u}'\) and the velocity field in the rotating frame is given by \(\mathbf{u}\). By applying (2.13) to the position vector \(\mathbf{x}\), it follows that

\[
\mathbf{u}' = \mathbf{u} + \mathbf{Ω} \times \mathbf{x}.
\]  \hspace{1cm} (2.14)
Formula (2.13) can also be applied to $u'$ and by using (2.14) it follows after some vector algebra that
\[ \frac{du'}{dt} = \frac{du}{dt} + 2\Omega \times u + \Omega \times \Omega \times x. \]
The term $2\Omega \times u$ is called the Coriolis acceleration; the term $\Omega \times \Omega \times x$ the centripetal acceleration. The centripetal acceleration can be written as
\[ \Omega \times \Omega \times x = \nabla \Phi_c, \]
where $\Phi_c = -\frac{1}{2} \Omega^2 r^2$ denotes the centrifugal potential, with $r$ the distance to the rotation axis. The Euler equation for the relative motion in the rotating frame of reference now takes the following form
\[ \frac{\partial u}{\partial t} + (u, \nabla) u + 2\Omega \times u = -\frac{1}{\rho} \nabla p - \nabla \Phi_c. \]  
(2.15)

Note that the Coriolis force is defined as $-2\rho \Omega \times u$.

The Coriolis acceleration plays an important role in geophysical flows as can be observed by taking the curl of Euler's equation (2.15). By using the definition of vorticity (2.7) and the Bernoulli function $b$, equation (2.6), Euler's equation (2.15) becomes
\[ \frac{\partial u}{\partial t} + \omega \times u + 2\Omega \times u = -\nabla b - \nabla \Phi_c. \]

Taking the curl yields
\[ \frac{\partial \omega}{\partial t} + \nabla \times \omega \times u + \nabla \times 2\Omega \times u = 0. \]  
(2.16)

So compared to Helmholtz' equation (2.8) in Section 2.2.3, this equation contains an extra term that originates from the Coriolis acceleration.

### 2.3.2 Equations of motion for a shallow fluid layer on a rotating sphere

The atmosphere and the oceans can be thought of as thin layers of only a few kilometres depth, whereas the horizontal scales of the flow are typically of the order of hundreds or even thousands of kilometres. From the continuity equation (2.4) it follows that the scale of the vertical velocity $W$ is related to that of the horizontal velocity $U$ by
\[ \frac{W}{U} \sim \frac{H}{L}, \]
where $H$ is the thickness scale and $L$ is the horizontal length scale. So, if $H \ll L$ then the vertical velocity is small compared to the horizontal velocity and the flow can be considered two-dimensional. Note that due to Coriolis effects and stratification often $W$ is even smaller than $UH/L$. 
2.3. Geophysical flows

![Diagram of a rotating sphere with labeled coordinates and vectors.]

**Figure 2.2:** Definition sketch of coordinates on a rotating sphere.

Furthermore, the thin layers are assumed to be homogeneous. Although the density may vary with height (temperature and salinity gradients cause a stratification), these effects can be neglected for many situations so that the horizontal velocities can be considered independent of the vertical coordinate $z$.

In order to describe the flow in the earth's atmosphere and oceans, the earth is modelled as a sphere of radius $R$ ($R = 6371 km$). For geophysical flows with very large horizontal scales, it is most convenient to use polar coordinates. However, if the horizontal scales are much smaller than the radius $R$ then the curvature of the surface of the earth can be ignored and a local coordinate system $(x,y,z)$ as defined in Figure 2.2 may be used. The corresponding relative velocity components are then $u = (u, v, 0)$ and the relative vorticity vector is $\omega = (0, 0, \omega)$. The rotation vector $\Omega$ has components

$$\Omega = (\Omega_x, \Omega_y, \Omega_z) = (0, \Omega \cos(\varphi), \Omega \sin(\varphi)),$$

where $\varphi, \varphi \in [-\pi/2, \pi/2]$, is the geographic latitude as defined in Figure 2.2. From this it follows that the Coriolis acceleration is given by

$$2\Omega \times u = 2\Omega(-v \sin(\varphi), u \sin(\varphi), -u \cos(\varphi)),$$

or

$$2\Omega \times u = (-fv, fu, -2\Omega u \cos(\varphi)) = (-fv, fu, -u(4\Omega^2 - f^2)\frac{1}{2})$$,
where \( f = 2\Omega \sin(\varphi) \) is the so-called Coriolis parameter or planetary vorticity. Dimension analysis [48] shows that the vertical component of the Coriolis acceleration is small compared to the pressure gradient and that it can be neglected both in the horizontal and the vertical component of (2.16). This implies that only the vertical component of the earth rotation is dynamically active, yielding the scalar equation (similar to Section 2.2.3)

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u}, \nabla \omega) + (\mathbf{u}, \nabla f) = 0,
\]

or

\[
\frac{D\omega}{Dt} = -(\mathbf{u}, \nabla f).
\]

From the last equation it is clear that in a rotating frame of reference, the vorticity \( \omega \) is not materially conserved in contrast to the situation in a fixed frame of reference (Section 2.2.3). However, the absolute vorticity \( q \), which is defined as the sum of the (relative) vorticity \( \omega \) and the planetary vorticity \( f \),

\[
q := \omega + f,
\]

is materially conserved. This simply follows from equation (2.17) and \( f \) being independent of time \( t \), so that

\[
\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + (\mathbf{u}, \nabla q) = 0.
\]

The relative vorticity \( \omega \) being not conserved in an inviscid, incompressible flow on a rotating sphere has important consequences for the dynamics of the flow: if, for example, in a fixed frame of reference a particle \( P \) has vorticity \( \omega \) at a certain time, at any time later the vorticity will still be \( \omega \), independent of the direction in which \( P \) has moved. For example, passive particles, i.e. particles with zero vorticity, will never become active. On a rotating sphere, however, if \( P \) moves to the north, the planetary vorticity \( f \) (not its magnitude \( |f| \)) increases, which means that \( \omega \) (again, not its magnitude \( |\omega| \)) has to decrease because of conservation of absolute vorticity (c.f (2.19) and (2.20)). On the other hand, if \( P \) moves to the south on the northern hemisphere, \( f \) decreases so that \( \omega \) has to increase. In this way, a passive particle becomes dynamically active when it is moved north- or southward by the flow. This obviously results in very different dynamics in a rotating frame of reference.

### 2.3.3 \( f \)-plane, \( \beta \)-plane and \( \gamma \)-plane approximations

The Coriolis parameter \( f \) depending on the latitude, complicates the equations of motion. As a simplification, \( f \) is often assumed to be a constant, i.e.

\[
f = f_0 = 2\Omega \sin(\varphi_0),
\]

where \( \varphi_0 \) is the central reference latitude of the domain considered. This approximation is usually referred to as the \( f \)-plane approximation. Obviously in this case, the term \( (\mathbf{u}, \nabla f) \)
vanishes in (2.17) and (2.18), yielding conservation of relative vorticity. So in this case, the equations of motion as derived in Section 2.2.3 are valid.

An improved representation of the latitudinal variation of $f$ can be obtained by expanding the Coriolis parameter in a Taylor series around the reference latitude $\varphi_0$, yielding

$$
f(\varphi) = f(\varphi_0 + \delta \varphi) = 2\Omega(\sin(\varphi_0) + \cos(\varphi_0) \delta \varphi + \mathcal{O}(\delta \varphi^2)),$$

where $\delta \varphi = \varphi - \varphi_0$. From Figure 2.2 it follows that $R \delta \varphi = y$, and under the assumption that the quadratic and higher-order $\delta \varphi$ terms are small compared to $\delta \varphi$, it follows that

$$f(x) \doteq 2\Omega(\sin(\varphi_0) + \cos(\varphi_0) y/R) := f_0 + \beta y,$$

where $\beta := 2\Omega \cos(\varphi_0)/R$. This approximation is commonly referred to as the $\beta$-plane approximation. In this case, equation (2.17) becomes

$$\frac{\partial \omega}{\partial t} + (u, \nabla \omega) + \beta v = 0.$$

Near the poles ($\varphi = \pm \pi/2$) the cosine vanishes so that the $\beta$-effect is absent. In this case, the $\mathcal{O}(\delta \varphi^2)$-term becomes important and

$$f(x) \doteq f_0 - \gamma(x^2 + y^2), \quad (2.21)$$

with $\gamma = \pm \Omega/R^2$. Here, the origin of the coordinate frame $(x, y, z)$ coincides with the pole and $r = R \delta \varphi$ is the radial distance to the pole, i.e. $r = \|x\|$. This approximation is called the $\gamma$-plane approximation. Now equation (2.17) becomes

$$\frac{\partial \omega}{\partial t} + (u, \nabla \omega) - 2\gamma(u, x) = 0. \quad (2.22)$$

Note that the $\beta$-plane approximation is order $\mathcal{O}(\delta \varphi^2)$ accurate, whereas the $\gamma$-plane approximation is order $\mathcal{O}(\delta \varphi^4)$ accurate.
Chapter 2. Mathematical Model
3

Contour Dynamics on an $f$-Plane

3.1 Introduction
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3.1 Introduction

In this chapter a contour dynamics method is discussed that can be used for simulating vortices in two-dimensional incompressible flows of an inviscid fluid on an \( f \)-plane (see Chapter 1 and 2). The method is based on the observation that the evolution of a patch of uniform vorticity is fully determined by the evolution of its boundary contour. The method is not limited to just one region of uniform vorticity; indeed, several contours can be nested in order to obtain an approximation of a patch of distributed vorticity \([19, 20, 60]\). In Chapter 4 this contour dynamics method is used to study a physically relevant problem: the interaction of three vortices with specific initial configuration.

In contour dynamics, there are actually two types of discretizations, viz. one in space and one in time. In Section 3.2 the spatial discretization is considered. Attention is being paid to the discretization of the vorticity, the interpolation of the contours and the node redistribution. Furthermore, it is shown that the Hamiltonian character of the equations is preserved by the spatial discretization. This is important for the choice of the time integration scheme, which is discussed in Section 3.3. In that section it is shown that a so-called symplectic integration scheme is to be preferred to conventional integration methods, because of the Hamiltonian character of the equations. Furthermore, it is outlined how such a scheme can be applied to contour dynamics. Finally, in Section 3.4 aspects of the computational method like accuracy and performance are assessed on the basis of numerical experiments.

3.2 Spatial discretization

The spatial discretization of the contour dynamics method consists basically of three parts. The first part is the discretization of the continuous vorticity profile into a piecewise-uniform profile. The second part concerns the interpolation of the bounding contours of the regions of uniform vorticity. The final part, deals with the redistribution of nodes on the contours in order to approximate the contours sufficiently accurate during a simulation. In this section, these three parts will be discussed in more detail.

3.2.1 Piecewise-uniform vorticity distribution

Two-dimensional flows of an incompressible, inviscid fluid can be described by Euler's equation (see Chapter 2), expressing balance of linear momentum, and the continuity equation, expressing conservation of mass. Regarding the latter conservation law note that, for an incompressible fluid, the velocity field is divergence free and thus a stream function \( \psi \) can be introduced in the usual way (see (2.10)). The vorticity vector \( \omega \) is defined as the curl of the velocity field \( u \). Since a two-dimensional flow in the \((x, y)\)-plane is considered, this vorticity vector points in a direction perpendicular to the \((x, y)\)-plane, implying that
\[
\omega = \omega e_z.
\]
3.2. Spatial discretization

\[ G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_M. \]

**Figure 3.1:** An arbitrary patch of piecewise-uniform vorticity distribution at a certain time \( t, t \geq 0 \). The regions \( G_m \) are nested, i.e. \( G_{m+1} \subseteq G_m \) for \( m = 0, \ldots, M \).

**Figure 3.2:** A cross-section (along the dashed line in Figure 3.1) of the piecewise-uniform vorticity profile approximating the continuous profile (dashed line).

As shown in Chapter 2, the governing equations in vorticity-stream function formulation (for an \( f \)-plane) are given by the vorticity equation, equation (2.11),

\[
\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} = 0,
\]

which expresses conservation of vorticity of a fluid particle, and the Poisson equation, equation (2.12),

\[
\nabla^2 \psi = -\omega.
\]

By solving (3.2) using Green’s function, it follows that

\[
\psi(x, t) = -\iint_{\mathbb{R}^2} G(x; x') \omega(x', t) \, dx' \, dy', \quad t \geq 0,
\]

where \( x := (x, y)^T \) and

\[
G(x; x') := \frac{1}{2\pi} \ln \|x - x'\|, \tag{3.3}
\]

is Green's function in \( \mathbb{R}^2 \). The norm \( \| \cdot \| \) is defined by \( \|x\| := \sqrt{x^2 + y^2} \), for each \( x \in \mathbb{R}^2 \).
In contour dynamics, the initial continuous vorticity distribution $\omega$ of a vortex (or vortices) is replaced by a piecewise-uniform distribution $\tilde{\omega}$

$$\tilde{\omega}(x, t) = \sum_{t=0}^{m} \omega_{t}, \quad x \in \mathcal{G}(0) \setminus \mathcal{G}_{m+1}(0), \quad m = 0, \ldots, M,$$  \hspace{1cm} (3.4)

where the regions $\mathcal{G}(0)$ are nested, $\mathcal{G}_{m+1}(0) \subset \mathcal{G}(0)$, $\mathcal{G}(0) = \mathbb{R}^2$ and $\mathcal{G}_{M+1}(0) = \varnothing$; i.e. $\mathcal{G}_{M+1}(0)$ is empty. Unless specified otherwise, $\omega_0$ is considered to be zero. The $\omega_{m+1}$, $m = 0, \ldots, M - 1$, can be thought of as the jump in vorticity when moving from region $\mathcal{G}_m(0) \setminus \mathcal{G}_{m+1}(0)$ to $\mathcal{G}_{m+1}(0) \setminus \mathcal{G}_{m+2}(0)$ (the meaning of the notation $\mathcal{G}_m(0) \setminus \mathcal{G}_{m+1}(0)$ is the region $\mathcal{G}_m(0)$ without $\mathcal{G}_{m+1}(0)$). Figure 3.1 shows an example of regions of uniform vorticity $\mathcal{G}_m$ at a certain time $t$, $t \geq 0$; Figure 3.2 shows the corresponding piecewise-uniform distribution of vorticity.

Replacing the continuous distribution by a piecewise-uniform distribution is not a severe limitation of the method, as shown by Legras and Dritschel [39], who compared a contour surgery method (piecewise-uniform distribution) with a pseudo-spectral method (continuous distribution). Legras and Dritschel studied the behaviour of an initially circular vortex subject to adverse shear in a doubly periodic domain. Their results show no spurious dynamical behaviour in the contour surgery results.

Conservation of vorticity of a fluid particle ensures that a piecewise-uniform distribution remains piecewise-uniform throughout time so that

$$\bar{\psi}(x, t) = - \sum_{m=1}^{M} \sum_{t=1}^{m} \omega_{t} \iint_{\mathcal{G}_m(t) \setminus \mathcal{G}_{m+1}(t)} G(x; x') \, dx' \, dy'$$

$$= - \sum_{m=1}^{M} \sum_{t=1}^{m} \omega_{t} \iint_{\mathcal{G}(t)} G(x; x') \, dx' \, dy' + \sum_{m=2}^{M+1} \sum_{t=1}^{m-1} \omega_{t} \iint_{\mathcal{G}_m(t)} G(x; x') \, dx' \, dy'$$

$$= - \sum_{m=1}^{M} \omega_{m} \iint_{\mathcal{G}_m(t)} G(x; x') \, dx' \, dy',$$

where $\mathcal{G}_m(t)$, $m = 1, \ldots, M$, are the regions $\mathcal{G}_m$ at time $t$, $t \geq 0$.

By taking the derivatives of $\bar{\psi}$ in $x$- and $y$-direction and by applying Stokes' theorem for a scalar field, the following expression for the velocity field can be derived:

$$\bar{u}(x, t) = - \sum_{m=1}^{M} \omega_{m} \oint_{\mathcal{C}_m(t)} G(x; x') \, dx',$$  \hspace{1cm} (3.5)

where $\mathcal{C}_m(t)$ is the boundary of region $\mathcal{G}_m(t)$.

If $\omega_0 \neq 0$, then a solid body rotation, for example, $u = -\frac{1}{2}\omega_0 y e_x + \frac{1}{2}\omega_0 x e_y$, has to be added to the velocity field $\bar{u}$ in (3.5).
3.2.2 Interpolation of the contours

From (3.5), it follows that the velocity \( \mathbf{u} \) at any point of the two-dimensional plane is determined by a sum of contour integrals. To calculate these contour integrals numerically, each contour \( C_m \) is discretized into a finite but adjustable number of nodes \( N \) (\( N \) of course depends on \( m \) and time \( t \)). Between two adjacent nodes, the contours are approximated by so-called elements. These elements can, for example, be linear (in that case, the two adjacent nodes are simply connected by a straight line segment), or quadratic, cubic, etc.

The parameterisation \( \mathbf{x}_n(\xi) \) of an element \( e_n \) with nodes \( \mathbf{x}_n \) and \( \mathbf{x}_{n+1} \), is chosen such that \( \mathbf{x}_n(-1) = \mathbf{x}_n, \mathbf{x}_n(1) = \mathbf{x}_{n+1} \) and \( \mathbf{x}_N(1) = \mathbf{x}_1(-1) \). In the case of linear elements, this parameterisation is given by

\[
\mathbf{x}_n(\xi) = \frac{1}{2} (1 - \xi) \mathbf{x}_n + \frac{1}{2} (1 + \xi) \mathbf{x}_{n+1}.
\]  

(3.6)

The interpolated version of contour \( C_m \) is called \( \hat{C}_m \).

The velocity \( \hat{\mathbf{u}} := (\hat{u}, \hat{v})' \) at a point \( \mathbf{x} \) anywhere in the flow field of the spatially discretized problem is then given by

\[
\hat{\mathbf{u}}(\mathbf{x}, t) = -\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \int_{C_m(t)} \ln \| \mathbf{x} - \mathbf{x}' \| \, d\mathbf{x}'
= -\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \ln \| \mathbf{x} - \mathbf{x}_n(\xi) \| \hat{\mathbf{x}}_n(\xi) \, d\xi.
\]

(3.7)

Note that here the overdot denotes a derivative with respect to \( \xi \). In all numerical examples, linear elements shall be used. In Example 3.4.1 of Section 3.4 it is shown that this is accurate enough for the cases considered here, although higher-order interpolation would probably give better results with respect to contour shapes [20]. The integrals in (3.7) along the elements can be determined using Gaussian quadrature. Only when \( \mathbf{x} \) is equal to (or is close to) one of the elements, an analytical evaluation of this integral is used. This is necessary because the logarithm is (almost) singular in that case.

Now the following property can be proven:

**Property 3.2.1.** The velocity field \( \hat{\mathbf{u}} \) of the spatially discretized problem is divergence free.

**Proof.** The partial derivatives of the velocity field are given by

\[
\frac{\partial \hat{u}}{\partial x} = -\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \frac{(x - x_n(\xi)) \hat{x}_n(\xi)}{\| \mathbf{x} - \mathbf{x}_n(\xi) \|^2} \, d\xi,
\]

\[
\frac{\partial \hat{v}}{\partial y} = -\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \frac{(y - y_n(\xi)) \hat{y}_n(\xi)}{\| \mathbf{x} - \mathbf{x}_n(\xi) \|^2} \, d\xi.
\]
So, for the divergence it follows

\[
(\nabla, \hat{\mathbf{u}}) = \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = - \sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \frac{(x - x_n(\xi))\dot{x}_n(\xi) + (y - y_n(\xi))\dot{y}_n(\xi)}{||x - x_n(\xi)||^2} d\xi \\
= \sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \frac{d\ln ||x - x_n(\xi)||}{d\xi} d\xi \\
= \sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \left[ \ln ||x - x_n(1)|| - \ln ||x - x_n(-1)|| \right] \\
= 0,
\]

since \( x_n(1) = x_{n+1}(-1) \) for \( n = 1, \ldots, N - 1, \) and \( x_N(1) = x_1(-1), \) irrespective of the order of the interpolant \( x_n(\xi). \) This even holds when \( x \) is equal to one of the nodes on the contour. \( \blacksquare \)

From this theorem it follows that the spatially discretized problem also has a Hamiltonian (independent from the degree of interpolation) as proven in the following property.

**Property 3.2.2.** The stream function of the velocity field \( \hat{\mathbf{u}} \) is given by

\[
\hat{\psi}(x, t) = \sum_{m=1}^{M} \frac{\omega_m}{4\pi} \sum_{n=1}^{N} \int_{-1}^{1} \left[ f(x, x_n(\xi))\dot{x}_n(\xi) + g(x, x_n(\xi))\dot{y}_n(\xi) \right] d\xi,
\]  \( (3.8) \)

where

\[
f(x, x_n(\xi)) := (x - x_n(\xi)) \arctan \left( \frac{y - y_n(\xi)}{x - x_n(\xi)} \right) - (y - y_n(\xi)) \ln ||x - x_n(\xi)||, \]

\[
g(x, x_n(\xi)) := -(y - y_n(\xi)) \arctan \left( \frac{x - x_n(\xi)}{y - y_n(\xi)} \right) + (x - x_n(\xi)) \ln ||x - x_n(\xi)||. \]

**Proof.** It has to be proven that

(i) \( \frac{\partial \hat{\psi}(x, t)}{\partial y} = \hat{u}(x, t), \)

(ii) \( \frac{\partial \hat{\psi}(x, t)}{\partial x} = -\hat{v}(x, t). \)
3.2. Spatial discretization

From the partial derivatives of \( f \) and \( g \) it follows that

\[
\begin{align*}
\nabla_x f(x, x_n(\xi)) &= -\nabla x_n f(x, x_n(\xi)), \\
\nabla_x g(x, x_n(\xi)) &= -\nabla x_n g(x, x_n(\xi)), \\
\end{align*}
\]

\[(*)
\]

\[
\frac{\partial f(x, x_n(\xi))}{\partial y_n(\xi)} - \frac{\partial g(x, x_n(\xi))}{\partial x_n(\xi)} = 2\ln ||x - x_n(\xi)||.
\]

\[(**)
\]

where \( \nabla_x \) denotes the gradient with respect to \( x \) and similarly \( \nabla x_n \) the gradient with respect to \( x_n(\xi) \). Taking the derivative of \((3.8)\) with respect to \( y \) and using \((*)\) yields

\[
\frac{\partial \hat{\psi}}{\partial y} = -\sum_{m=1}^{M} \frac{\omega_m}{4\pi} \sum_{n=1}^{N} \int_{-1}^{1} \left[ \frac{\partial f(x, x_n(\xi))}{\partial y_n(\xi)} \hat{x}_n(\xi) + \frac{\partial g(x, x_n(\xi))}{\partial y_n(\xi)} \hat{y}_n(\xi) \right] d\xi.
\]

Now, \((**)\) can be used to obtain

\[
\frac{\partial \hat{\psi}}{\partial y} = -\sum_{m=1}^{M} \frac{\omega_m}{4\pi} \sum_{n=1}^{N} \int_{-1}^{1} \left[ \frac{\partial g(x, x_n(\xi))}{\partial x_n(\xi)} \hat{x}_n(\xi) + \frac{\partial g(x, x_n(\xi))}{\partial y_n(\xi)} \hat{y}_n(\xi) \right] d\xi
\]

\[-\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \sum_{n=1}^{N} \int_{-1}^{1} \ln ||x - x_n(\xi)|| \hat{x}_n(\xi) d\xi.
\]

Here, similar reasoning as in Property 3.2.1 reveals that the first term in the right-hand-side vanishes so that

\[
\frac{\partial \hat{\psi}}{\partial y} = \hat{u}(x, t).
\]

The first part \((i)\) of the proof is now completed. The second part \((ii)\) can be proven similarly.

Thus, \( \hat{\psi} \) is the stream function belonging to the velocity field \( \hat{u} \). Of course, an arbitrary constant may be added to this stream function.

If linear elements are used for the interpolation of the contours and \( h_n \) denotes the length of element \( e_n \), the following property for the discretization error can be proven:

**Property 3.2.3.** If the interpolation is linear and \( h_{\max} \) is defined as the maximum of all \( h_n = ||x_{n+1} - x_n|| \), then

\[
||\hat{u} - \hat{u}|| = \left\| \sum_{m=1}^{M} \frac{\omega_m}{2\pi} \int_{C_{\hat{C}_m} - C_m} \ln ||x - x'|| dx' \right\| = O(h_{\max}^2).
\]

Here, the integration over the difference of the two contours \( \hat{C}_m \) and \( C_m \) is defined as the difference of the integration over \( \hat{C}_m \) and that over \( C_m \).
Chapter 3. Contour Dynamics on an f-Plane

Figure 3.3: The region $A_n$ enclosed by $e_n$ and $C_n$.

Proof. For the sake of simplicity, only the situation of one contour $C$ is considered. The more general case with more than one contour can be treated similarly. Assume that the nodes $x_n$ of contour $\hat{C}$ are located on the exact contour $C$ (this probably is not exactly the case, in general, but without this assumption a proof of this kind is not possible). Then

$$\tilde{u} - u = \frac{\omega}{2\pi} \oint_{\hat{C} - C} \ln \|x - x'\| \, dx'$$

$$= \frac{\omega}{2\pi} \sum_{n=1}^{N} \oint_{e_n - C_n} \ln \|x - x'\| \, dx',$$

where $e_n$ is the straight-line segment, with length $h_n$, connecting two adjacent nodes $x_n$ and $x_{n+1}$ and $C_n$ is the part of contour $C$ that connects these points also (see Figure 3.3). Now consider the contribution vector $\delta_n$ of one contour $e_n - C_n$, i.e. the "local geometrical error", to the discretization error. Stokes' theorem for a vector field can be used to obtain a surface integral over $A_n$ (see Figure 3.3) from the contour integral

$$\delta_n := \oint_{e_n - C_n} \ln \|x - x'\| \, dx'$$

$$= \iint_{A_n} e_z \times \nabla x \ln \|x - x'\| \, dx' \, dy'$$

$$= \iint_{A_n} \left( \frac{y - y'}{\|x - x'\|^2} e_x - \frac{x - x'}{\|x - x'\|^2} e_y \right) \, dx' \, dy'.$$

Now introduce polar coordinates in the following way
3.2. Spatial discretization

\[ x - x' = r \cos(\varphi), \]
\[ y - y' = r \sin(\varphi). \]

Then, the surface integral becomes

\[ \delta_n = \int_{R_1(\varphi)}^{R_2(\varphi)} \int_{\varphi_1}^{\varphi_2} (\sin(\varphi)e_x - \cos(\varphi)e_y) \, d\varphi \, dr. \]

Here \( \varphi_1, \varphi_2, R_1(\varphi) \) and \( R_2(\varphi) \) are as in Figure 3.4. For the \( x \)-component it thus can be found that

\[ |(\delta_n, e_x)| \leq \int_{\varphi_1}^{\varphi_2} |R_2(\varphi) - R_1(\varphi)| \sin(\varphi) \, d\varphi \]
\[ \leq (\varphi_2 - \varphi_1) \max_{\varphi_1 \leq \varphi \leq \varphi_2} |R_2(\varphi) - R_1(\varphi)|, \]

\((*)\)

and similarly for the \( y \)-component

\[ |(\delta_n, e_y)| \leq \int_{\varphi_1}^{\varphi_2} |R_2(\varphi) - R_1(\varphi)| \cos(\varphi) \, d\varphi \]
\[ \leq (\varphi_2 - \varphi_1) \max_{\varphi_1 \leq \varphi \leq \varphi_2} |R_2(\varphi) - R_1(\varphi)|. \]

\((***)\)

Because of the linear interpolation, \(|R_2(\varphi) - R_1(\varphi)| \leq C h_n^2\) for \( \varphi_1 \leq \varphi \leq \varphi_2 \).
Furthermore, an expression can be derived for the angle $\theta := \varphi_2 - \varphi_1$. If $\rho$ is defined by $\rho := \|x - \frac{1}{2}(x_n + x_{n+1})\|$, then
\[
\|x - x_n\|^2 = \rho^2 + h_n \rho^2 \cos(\theta_1) + \frac{1}{4} h_n^2,
\]
\[
\|x - x_{n+1}\|^2 = \rho^2 - h_n \rho^2 \cos(\theta_1) + \frac{1}{4} h_n^2,
\]
where $\theta_1$ is the angle between $x - \frac{1}{2}(x_n + x_{n+1})$ and $\frac{1}{2}(x_{n+1} - x_n)$. For the inner product $(x - x_n, x - x_{n+1})$ it simply follows
\[
(x - x_n, x - x_{n+1}) = \rho^2 - \frac{1}{4} h_n^2.
\]

Since
\[
\cos(\theta) = \frac{(x - x_n, x - x_{n+1})}{\|x - x_n\| \|x - x_{n+1}\|},
\]
it follows that
\[
\sin^2(\theta) = 1 - \cos^2(\theta)
\]
\[
= \frac{(h_n/\rho)^2 \sin^2(\theta_1)}{1 - \frac{1}{2}(h_n/\rho)^2 \cos(2\theta_1) + \frac{1}{16} (h_n/\rho)^4}.
\]

For $\rho > h_n$, it follows from this that $\theta = O(h_n/\rho)$.

Using this result in (*) and (**) results in
\[
\|\delta_n\| = \begin{cases} O(h_n^2/\rho), & \text{if } \rho > h_n, \\ O(h_n^2), & \text{if } \rho \leq h_n. \end{cases}
\]

Now, by combining the contributions from all regions $A_n$, and by assuming that only a few elements $e_n$ are close (i.e. closer than $h_n$) to the point $x$, it may be concluded that the overall discretization error is $O(h_{\text{max}}^2)$, where $h_{\text{max}}$ is the maximum element length on contour $\bar{C}$. This completes the proof.

Note that this type of discretization error is not necessarily produced every time step during the calculations. The first time this error is produced, is when the initial contour grid is calculated. After that, spatial errors of this kind only occur at locations where nodes are added or removed. This does not imply that the spatial errors are zero when no nodes are added or removed. Since the contour shape in general changes with time, the error caused by the calculation of the initial contour grid changes with time as well. If no nodes are added, the error can grow very fast. However, if nodes are added in time, then the error caused by the initial grid can be kept of the order of its initial magnitude.

For higher-order interpolation, the spatial error as defined in Property 3.2.3 is of higher-order, of course. In general: if the interpolation is $O(h_n^p)$ accurate, then the overall error will be of order $O(h_{\text{max}}^p)$. 

Figure 3.5: An example showing what may happen when nodes are not added nor removed from the contours during a simulation. As time proceeds nodes concentrate at some parts of the outer contour and at other parts nodes become very sparse.

3.2.3 Redistribution of the nodes

Since, in general, the shape of the contours becomes increasingly complex when time proceeds, the number of nodes initially placed on the contours are inadequate to approximate the contours sufficiently accurate at later points in time. Figure 3.5 illustrates what may happen when nodes are not added nor removed during a calculation. Obviously, the number of nodes on a contour, or at least the node distribution, should be adapted during a simulation.

Several situations may occur where one has to add nodes to a contour. In general problems may arise when a small-scale feature with a high density of nodes, encounters a larger scale feature with a lower node density (see Figure 3.6). In this case, the two parts of the contours may intersect, unless nodes are added properly to the large-scale feature.

To prevent the intersection of (parts of) two contours, the following technique is implemented. A node $\mathbf{x}_i$ is called lying opposite to an element $e_n$ with nodes $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$, if the line through $\mathbf{x}_i$, perpendicular to the line $l$ through $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ (see Figure 3.7) intersects $l$ in between $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$. When no nodes are added between $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ and the local curvature at $\mathbf{x}_i$ is higher than the curvature at $\mathbf{x}_n$ or $\mathbf{x}_{n+1}$, such a point $\mathbf{x}_i$ may cause trouble. This can be avoided by properly adding a node between $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ each time the distance between the node and the element is becoming smaller than a given critical value. This distance is defined by the length of the vector $\mathbf{v}$ (see Figure 3.7) which is given by

$$
||\mathbf{v}|| = \left( ||\mathbf{x}_n - \mathbf{x}_i||^2 - \frac{(\mathbf{x}_n - \mathbf{x}_i), (\mathbf{x}_{n+1} - \mathbf{x}_n)^2}{||\mathbf{x}_{n+1} - \mathbf{x}_n||^2} \right)^{\frac{1}{2}}.
$$

It is also possible that oppositely situated points do not cause trouble, and are even located such that the local curvature is low enough to allow some nodes to be removed (for example, on a filament). Alternatively, in order to approximate the contour well enough,
nodes may have to be added at locations where the local curvature is high. Apparently, this local curvature plays an important role in the distribution of the nodes. An approximation for it can be found by fitting a quadratic polynomial through three consecutive points. For a curve parameterised by

$$\mathbf{x} = \mathbf{x}(\xi) = x(\xi) \mathbf{e}_x + y(\xi) \mathbf{e}_y,$$

the local curvature $\hat{\kappa}(\xi)$ [27] is given by

$$\hat{\kappa}(\xi) = \frac{x_{\xi\xi} y_\xi - y_{\xi\xi} x_\xi}{(x_\xi^2 + y_\xi^2)^{3/2}}, \quad (3.9)$$

where the subscript $\xi$ indicates derivation with respect to $\xi$. For a quadratic polynomial through three consecutive points $\mathbf{x}_{n-1}, \mathbf{x}_n$ and $\mathbf{x}_{n+1}$, the parameterisation is given by

$$\mathbf{x}(\xi) = \frac{1}{2} (\xi(\xi - 1) \mathbf{x}_{n-1} + 2(1 - \xi^2) \mathbf{x}_n + \xi(\xi + 1) \mathbf{x}_{n+1}), \quad (3.10)$$

where $-1 \leq \xi \leq 1$, and together with (3.9) it follows that the local curvature at node $\mathbf{x}_n$ ($\xi = 0$) can be approximated by

$$\kappa(\mathbf{x}_n) = 8 \frac{(x_{n+1} - x_n)(y_n - y_{n-1}) - (y_{n+1} - y_n)(x_n - x_{n-1})}{||\mathbf{x}_{n+1} - \mathbf{x}_{n-1}||^3}. \quad (3.11)$$

**Figure 3.6:** A small-scale feature encountering a large-scale one (from [18, 20]).

**Figure 3.7:** Node $\mathbf{x}_i$ is lying opposite to $e_n$. 
3.2. Spatial discretization

Let $h_n$ be the length of element $e_n$. A criterion based on the local curvature for adding a node between two consecutive nodes $x_n$ and $x_{n+1}$ is now given by

$$\frac{1}{2}(|\kappa(x_n)| + |\kappa(x_{n+1})|)h_n > \delta_1,$$

and a criterion for removing node $x_n$ is given by

$$\frac{1}{2}(|\kappa(x_n)| + |\kappa(x_{n+1})|)h_n < \delta_2,$$

where $\delta_1$ and $\delta_2$ are given values with $\delta_1 > \delta_2$ (depending on the required accuracy; often $\delta_1 \approx 0.1$ and $\delta_2 \approx 0.002$ appear to work fine).

Another requirement for the distribution of nodes is that the distance between two subsequent nodes should not become smaller than a minimum $h_{\text{min}}$ and not larger than a maximum $h_{\text{max}}$. The former requirement is imposed to control the number of nodes on a contour in view of the CPU-time. The latter requirement is imposed to prevent the element length to become too large in filaments (where the curvature is very low at some locations).

An additional condition the node distribution should satisfy, is that of quasi-uniformity. This can be formulated as

$$\frac{h_{n-1}}{K} \leq h_n \leq Kh_{n-1},$$

where $K$ is a constant sufficiently larger than 1 (in practice, $K \approx 2$ appears to work fine). This condition ensures that the element length of two neighbouring elements is not changing too much. This is essential since, e.g., the local curvature cannot be calculated very accurately otherwise because of its dependence on three successive nodes (see (3.11)).

All together, four criteria are necessary for adding nodes and three for removing. It is obvious that the actual removal of nodes is very simple. For adding a node however, one has to decide where the new node has to be placed. To this end, a quadratic polynomial (as given by (3.10)) is fitted through three successive nodes and the new node is placed on this polynomial. The velocity at the new node is also determined by quadratic interpolation. Here, the quasi-uniform distribution of the nodes is also essential for an accurate approximation.

In the numerical examples no surgery [19] is applied. Surgery is the technique of automatically removing fine-scale structures. It topologically reconnects contours or distinct parts of a contour when these approach each other closer than a prescribed length scale. This technique enables complex long-time simulations. However, surgery can only be applied to contours of equal vorticity level and is thus of less importance in cases of large number of contours with different vorticity levels, like in the problems discussed in Chapter 6.
3.3 Time integration

In this section, the time integration part of the method is considered. First, it is demonstrated that standard explicit integration schemes do not conserve the area within a contour. After that, it is shown how conservation can be obtained by using a so-called symplectic integration scheme.

3.3.1 Symplectic and non-symplectic Runge Kutta methods

As mentioned in Chapter 2, the equations of motion constitute a Hamiltonian system. Furthermore, in Section 3.2.2 it has been shown that the spatial discretization did not influence the Hamiltonian character. An important property of a Hamiltonian system is the concept of preservation of area [52] which is equivalent to conservation of mass in the present incompressible flow. Operators which have this property, are called symplectic. The solution operator of a Hamiltonian system is thus a symplectic operator.

In a numerical time integration, the solution operator is replaced by an approximate operator. The approximate solution operator only retains the Hamiltonian character of the exact solution operator, if it is symplectic as well. However, most standard numerical integrators replace the solution operator by a non-symplectic mapping. This is illustrated by the following simple example.

Example 3.3.1. Consider the time evolution of a circular vorticity patch (initial radius equal to $r(0)$) of uniform vorticity $\omega$. The velocity field inside the patch and on its boundary, is given by [37]

$$\begin{cases}
    u(x) = \dot{x} = -\frac{\omega}{2} y, \\
v(x) = \dot{y} = \frac{\omega}{2} x,
\end{cases} \quad t > 0, \quad \|x\| \leq 1. \quad (3.12)$$

By applying the explicit Euler forward scheme to (3.12), it follows that

$$\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = \begin{pmatrix}
1 & -\frac{\omega \Delta t}{2} \\
\frac{\omega \Delta t}{2} & 1
\end{pmatrix} \begin{pmatrix}
x(t - \Delta t) \\
y(t - \Delta t)
\end{pmatrix},$$

where $\Delta t$ is the time step. For the length $r(t)$ of the vector $x$ after $t/\Delta t$ time steps, it thus follows that

$$r^2(t) = x^2(t) + y^2(t)$$

$$= \left(1 + \left(\frac{\omega \Delta t}{2}\right)^2\right)^{t/\Delta t} \left(x^2(t - \Delta t) + y^2(t - \Delta t)\right)$$

$$= \left(1 + \left(\frac{\omega \Delta t}{2}\right)^2\right)^{t/\Delta t} r^2(0)$$

$$= r^2(0) \exp \left(t \Delta t \left(\frac{\omega}{2}\right)^2 + O\left(t \Delta t^3 \left(\frac{\omega}{2}\right)^4\right)\right).$$
3.3. Time integration

Figure 3.8: The evolution of a circular patch (initial radius equal to 1) of uniform vorticity ($\omega = 2\pi$) using the Euler forward scheme with $\Delta t = 0.05$.

Figure 3.9: The predicted behaviour of $\Delta A$ and the numerically obtained $\Delta A$ as a function of time $t$.

The radius of the patch thus grows exponentially with time, see Figure 3.8 where $r(0) = 1.0$, $\omega = 2\pi$ and $\Delta t = 0.05$ are used. Since the patch remains circular for all time, the area $A(t)$ of the patch after $t/\Delta t$ time steps obeys

$$A(t) = \pi r^2(t) = \pi r^2(0) \exp\left( t\Delta t \left(\frac{\omega}{2}\right)^2 + \mathcal{O}\left( t\Delta t^3 \left(\frac{\omega}{2}\right)^4 \right) \right).$$

(3.13)

The interesting feature here is that

$$A(t) \sim A(0) \exp(t\alpha^2\Delta t),$$

(3.14)

where $\alpha = \omega/2$; i.e. the area becomes unbounded as $t \to \infty$, although it can be kept close to $A(0)$ on any finite interval by choosing $\Delta t$ small enough. In Figure 3.9

$$\Delta A := |A(t) - A(0)|/A(0),$$

(3.15)

is plotted versus time $t$ (where the area $A(t)$ was calculated at each time step by a contour integral expression) and also the behaviour of $\Delta A$, predicted by (3.14). Clearly, (3.14) predicts the behaviour of the area of the patch very well. Furthermore it may be clear
Figure 3.10: Similar as Figure 3.9 but for RK4 and \( \Delta A := (A(t) - A(0))/A(0) \) (not the absolute value). Note the decrease of the area in this case.

that the Euler forward scheme is not symplectic and one may be forced to take many steps to stay close to conservation. In fact, this applies to any explicit method. Note that if the Euler backward scheme would have been applied to this problem (although this would hardly be feasible in practice, since implementation would require the solution of a large linear system), shrinking patches would be found instead. For higher-order (explicit) Runge-Kutta (RK) schemes, a similar result applies, albeit with a more moderate growth rate. In particular, for the classical explicit fourth-order RK-method (RK4) [28, 29], one finds by straightforward expansion

\[
A(t) \sim A(0) \exp(-t \alpha^6 \Delta t^5 / 72). \tag{3.16}
\]

In Figure 3.10 both the predicted and the numerically obtained values of \( \Delta A \) as a function of time are plotted for the same problem as before (circular vortex patch). In this figure, \( (A(t) - A(0))/A(0) \) has been plotted rather than its magnitude in order to show the decrease of the area. Again, the numerical results agree remarkably well with the theory. Although for RK4 the variation of the area is much smaller compared to the Euler forward scheme, it still can be of significant importance, for large time intervals or problems with larger values of \( \alpha \).

In the next section, attention is paid to a second-order symplectic RK-scheme: the midpoint rule and it is demonstrated how such an implicit scheme can be implemented, without needing a Jacobian matrix.
3.3. Time integration

3.3.2 Implementation of the symplectic midpoint rule

Symplectic RK-methods are implicit in general [52]. Therefore, the implementation of a symplectic method can be rather complex. In case the problem is stiff, Newton-like iteration methods are needed to solve the linear system involved. Here, however, the system normally is not stiff. Indeed, in order to follow the contour properly, regions which are moving relatively fast at all time (probably at different locations during the evolution) have to be resolved. This means that the "smaller" time scales dictate the step size at any time. Therefore a predictor-corrector scheme \((P(EC)^2E)\) [42] can be used for the implementation of the symplectic scheme. Here, an explicit scheme is used to obtain (predict) a first approximation to the velocity field at the new time step. This approximation can be used in the implicit part of the symplectic scheme (corrector) to obtain a more accurate approximation. In addition, the latter can again be used to obtain an even more accurate approximation and so on. Using such a technique implies that the method is explicit after all, but area-conservation can be achieved practically by choosing \(I\) large enough as is shown later. Here, the midpoint rule, which is second-order accurate, is chosen as corrector, but a similar formulation can be used for higher-order symplectic methods.

As can be observed from (3.7), the velocity field \(\dot{u}\) depends on the position of every node on every contour. Let \(X(t)\) be the vector of \(x\)- and \(y\)-coordinates of all nodes at a certain time \(t\)

\[
X(t) := (x_1(t), x_2(t), \ldots, x_N(t), y_1(t), y_2(t), \ldots, y_N(t))^T.
\]

Denote the velocity in \(x\)-direction at node \(x_n\) by \(\dot{u}_n(X) := \dot{u}(x_n)\) and in \(y\)-direction by \(\dot{v}_n(X) := \dot{v}(x_n)\). Furthermore, let \(U\) be the vector of velocities in the \(x\)- and \(y\)-directions

\[
U(X) := (\dot{u}_1(X), \dot{u}_2(X), \ldots, \dot{u}_N(X), \dot{v}_1(X), \dot{v}_2(X), \ldots, \dot{v}_N(X))^T.
\]

For the time evolution of the contours, the following initial value problem has to be solved

\[
\begin{align*}
\dot{X} &= U(X), \quad t > 0, \\
X(0) &= X_0.
\end{align*}
\] (3.17)

Denote the vector of approximate \(x\)- and \(y\)-coordinates of the positions of the nodes after \(k\) time steps \(\Delta t\) by \(X_k\). Furthermore, denote the vector of approximate velocities in the \(x\)- and \(y\)-directions by \(U(X_k)\). Then the midpoint rule can be expressed as

\[
X_{k+1} = X_k + \Delta t U \left( \frac{X_k + X_{k+1}}{2} \right),
\]

or

\[
\frac{X_k + X_{k+1}}{2} = X_k + \frac{\Delta t}{2} U \left( \frac{X_k + X_{k+1}}{2} \right).
\]
Introducing the vector
\[ \overline{X} := \frac{X_k + X_{k+1}}{2}, \]
it is apparently necessary to solve
\[ \overline{X} = X_k + \frac{\Delta t}{2} U(\overline{X}), \]  
(3.18)
each time step. This is achieved by the following \((P(\mathbf{E})^{T}\mathbf{E})\) method:

\[
\begin{align*}
\text{Predict:} & \quad \overline{X}^0 = X_k + \frac{\Delta t}{2} U(X_k), \\
\text{Evaluate:} & \quad U(\overline{X}^{i-1}), \\
\text{Correct:} & \quad \overline{X}^i = X_k + \frac{\Delta t}{2} U(\overline{X}^{i-1}), \quad \text{for } i = 1, \ldots, I \\
\text{Evaluate:} & \quad U(2\overline{X}^i - X_k).
\end{align*}
\]

The predictor step of this method is equivalent to the Euler forward scheme for obtaining a first approximation to \(X_{k+1}\). At the last evaluation step (i.e. after \(I\) cycles), the velocities at the (approximate) new positions
\[ X_{k+1} := 2\overline{X}^I - X_k, \]
are calculated and used for the next time interval.

In general, the number of cycles to be performed depends on both the order of the predictor scheme and the order of the corrector scheme. If the corrector is of order \(p\) and the predictor of order \(q\) \((p \geq q)\), then the local discretization error after \(i\) cycles, \(\delta^{(i)}\), is given by
\[ \delta^{(i)} = \mathcal{O}(\Delta t^p) + \mathcal{O}(\Delta t^{i+q}). \]

As far as standard accuracy arguments are concerned, it would not be necessary to carry out more than \(p - q + 1\) cycles; so here two cycles would be enough. However, a finite number of iterations implies an explicit integration after all and loss of the symplectic properties. In order to obtain an (almost) symplectic scheme, it is necessary to carry out more cycles, basically till (3.18) is solved within machine precision (see Figure 3.11). One should realize though, that each cycle requires the calculation of the velocity field, so this is very time consuming. To overcome this problem, an extrapolation method can be used to accelerate the convergence of the iteration process. The method actually used is the so-called minimal polynomial extrapolation (MPE) method [53]. This MPE method is very suitable for the problem considered here, since it is based on differences and does not need additional information about the Jacobian matrix. Before explaining the concept of this method, the following property is proven:
3.3. Time integration

![Graph showing the behavior of ΔA for different methods.](image)

**Figure 3.11:** The behavior of ΔA in case of the \(P(EC)^{14}E\) method (solid line) for the same vortex as in Figure 3.8. The behavior of ΔA in case of the Euler forward scheme (c.f. Figure 3.9) is drawn here as well (dashed line).

**Property 3.3.2.** For \(i \geq 0\),

\[
\overline{X}^i = \overline{X} - \left(\frac{\Delta t}{2}\right)^{i+1} J^i U(\overline{X}) + O(\Delta t^{i+2}),
\]

(3.19)

where \(J\) is the Jacobian matrix defined by

\[
J := \frac{\partial U(\overline{X})}{\partial \overline{X}}.
\]

**Proof.** The proof is by induction. For \(i = 0\), the result follows immediately from (3.18) since

\[
\overline{X}^0 = X_k = \overline{X} - \frac{\Delta t}{2} U(\overline{X}),
\]

Assume the property holds for \(i\). Then with the corrector part of the \((P(EC)^{i}E)\) scheme,
(3.18) and a Taylor expansion of $U$ around $\bar{X}$ it follows that

$$
\bar{X}^{i+1} = \bar{X} + \frac{\Delta t}{2} U(\bar{X}^i)
$$

$$
= \bar{X} - \left( \frac{\Delta t}{2} \right) ^{i+1} J^i U(\bar{X}) + O(\Delta t^{i+2})
$$

$$
= \bar{X} + \left( \frac{\Delta t}{2} \right) ^{i+2} J^{i+1} U(\bar{X}) + O(\Delta t^{i+3})
$$

$$
= \bar{X} - \left( \frac{\Delta t}{2} \right) ^{i+2} J^{i+1} U(\bar{X}) + O(\Delta t^{i+3}).
$$

This completes the proof. 

From this property it follows that

$$
\bar{X}^i - \bar{X} = \frac{\Delta t}{2} J(\bar{X}^{i-1} - \bar{X}) + O(\Delta t^{i+2}), \quad \text{for } i \geq 1, \quad (3.20)
$$

$$
\bar{X}^i - \bar{X}^{i-1} = \frac{\Delta t}{2} J(\bar{X}^{i-1} - \bar{X}^{i-2}) + O(\Delta t^{i+1}), \quad \text{for } i \geq 2, \quad (3.21)
$$

and the sequence $\{\bar{X}^i\}$ is linearly convergent. Since MPE is based on differences, it is convenient to have short notations for these. Define

$$
a^i := \bar{X}^i - \bar{X}^{i-1}, \quad \text{for } i = 1, \ldots, I,
$$

and

$$
A := (a^1, a^2, \ldots, a^{I-1}).
$$

Now MPE calculates the fixed point $\bar{X}$ as a weighted average of the iterates with weights determined by the coefficients of the minimal polynomial $P(\lambda)$ of $J$ with respect to $a^1$. Let $I - 1$ be the degree of $P(\lambda)$. Then, the minimal polynomial can be written as

$$
P(\lambda) = \sum_{i=0}^{I-1} c^{(i)} \lambda^i, \quad c^{(I-1)} = 1.
$$

Let the vector $c := (c^{(0)}, c^{(1)}, \ldots, c^{(I-2)})^T$ be the vector of the unknown coefficients of the minimal polynomial. Then $c$ is the solution of the system of equations

$$
Ac = -a^{I-1}. \quad (3.22)
$$

In general, $I$ is much smaller than the number of nodes. Thus, the system (3.22) has more equations than unknowns, but consistency can be proven [53]. Calculation of $c$ requires only an LU-decomposition of $A$ and the solution of the upper triangular system, which is
3.3. Time integration

Figure 3.12: The situation at $t = 0$: the outer ring has vorticity $\omega = \pi$ and the region inside the inner circle has vorticity $\omega = 2\pi$.

Figure 3.13: The variation $\Delta A$ of the area as a function of time $t$. The time step is taken to be $\Delta t = 0.05$, the number of cycles is equal to 3 and MPE-extrapolation is performed.

cheap compared to the calculation of the velocities. Once the vector $c$ has been found, the fixed point can be calculated from

$$\left( \sum_{i=0}^{I-1} c^{(i)} \right) \overline{X} = \sum_{i=0}^{I-1} c^{(i)} \overline{X}^{i+1}. \quad (3.23)$$

Of course, the degree of the minimal polynomial is not a priori clear. But this is not a problem in practice. If $I-1$ is larger than the degree of $P(\lambda)$, then there is no problem at all. If it is smaller, then instead of achieving equality in (3.22), the least-squares solution gives coefficients of an “almost annihilating” polynomial that is the “best” monic polynomial of degree $I-1$ for eliminating the influence of $I-1$ dominant components of the error [53]. These dominant error components are generated by the absolutely largest eigenvalues.

Consider now a vortex patch consisting of two concentric circular contours, where the outer contour rotates slower than the inner (the fluid enclosed by the inner contour has a larger uniform vorticity than the fluid enclosed by the outer contour). Then the eigenvalues belonging to the outer contour are smaller (in absolute sense) than those of the inner one. So it can be expected that, in the case where $I-1$ is smaller than the degree of the minimal polynomial, the area within the inner contour is better conserved than that within the outer contour. This is exactly what can be observed in Figure 3.12 and Figure 3.13. Here, the outer ring has vorticity $\omega = \pi$ and the inner circle has vorticity $\omega = 2\pi$. From Figure 3.13 it can be observed that the area enclosed by the inner contour is better conserved than that of the outer, as expected. This suggests that it might be better to split up the extrapolation
<table>
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<th>RK4</th>
<th>Exact</th>
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<tr>
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<td>249.3</td>
<td>249.3</td>
<td>249.8</td>
<td>250</td>
</tr>
</tbody>
</table>

Table 3.1: The number of revolutions of the outer and inner contour of the vortex in Figure 3.12 during the time period $[0, 500]$.

process over the contours: instead of calculating one set of coefficients $c^{(i)}$ for the whole system, a different set of coefficients is calculated for each contour in order to obtain the dominating terms for each contour. This method (called MPREXC) is implemented, and the results for the same vortex patch as in Figure 3.12 are shown in Figure 3.14a (outer contour) and Figure 3.14b (inner contour). The results for both the outer and the inner contour have improved. This is due to the larger value of $\omega$ making the extrapolation more accurate (dominance of eigenvalue is more pronounced). In the next section some more results are shown. In Figure 3.14, also results are shown of the evolution of the same vortex patch but now with the classical fourth-order explicit RK4 method. The results are in agreement with (3.16). It might be clear, that the results of the midpoint rule are much better than those of the RK4 method. This, while the same effort was needed for both methods: $I = 3$ is used, so the velocities had to be calculated four times per time step which is the same as required for the RK4 method. In addition, Figure 3.14 contains two plots (Figures 3.14c and d) where the behaviour of the error $\Delta U$ in the velocities (which is defined as $\Delta U := |(U(t) - U(0))/U(0)|$, where $U(t) = \sum_n \sqrt{a_n^2 + \dot{a}_n^2}$) as a function of time is shown, both for RK4 and MPREXC. Apparently, MPREXC also gives better results with respect to the velocities.

Another error, the phase error, may also be investigated (although it is often less interesting in practical situations). A way to consider this, is to follow the point on a contour which initially is placed at the positive part of the $x$-axis. The solution of the problem can be determined analytically, and it turns out that in this particular case such a point of the outer contour passes the positive $x$-axis exactly 170 times and that of the inner contour 250 times during the time period $[0, 500]$. For all numerical methods, this turns out to be slightly less than these analytical values as can be seen in Table 3.1. However, the differences between the investigated methods are very small, so it may be concluded that all numerical schemes produce a phase error of comparable magnitude.

### 3.4 Numerical experiments and discussion

In the previous section, some numerical results obtained by the midpoint rule and a fourth-order explicit RK method have been compared. However, that test problem was rather simple. In this section, three more complex problems are considered. Here, not only attention is being paid to the errors as a result of the time integration, but also the errors caused by the spatial discretization.
3.4. Numerical experiments and discussion

Figure 3.14: Figures a and b show the variation of the area $\Delta A$ calculated with the midpoint rule with three cycles where the extrapolation part is split up over the contours (MPREXC), with the midpoint rule with three cycles and global extrapolation (MPREX) and with a fourth-order RK-method (RK4). Figures c and d show the variation of the velocities $\Delta U$ for MPREXC and RK4. In all cases, $\Delta t = 0.05$. 
Figure 3.15: The behaviour of errors in the area ($\Delta A$), circulation ($\Delta C$) and angular momentum ($\Delta M$) as a function of time. In the left-hand-side pictures, 100 nodes are used whereas in the right-hand-side pictures 200 nodes are used.
3.4. Numerical experiments and discussion

![Graphs showing error in area, circulation, and angular momentum as a function of time for different methods and node numbers.](image)

e) $\Delta M$ for $N = 100$

f) $\Delta M$ for $N = 200$

Figure 3.15 (continued): The behaviour of errors in the area ($\Delta A$), circulation ($\Delta C$) and angular momentum ($\Delta M$) as a function of time. In the left pictures, 100 nodes are used whereas in the right pictures 200 nodes are used.

Example 3.4.1. In this example numerical results for the evolution of a so-called Kirchhoff vortex [37] are considered. A Kirchhoff vortex is an elliptical patch of uniform vorticity $\omega$ rotating with constant angular speed $ab\omega/(a + b)^2$ where $a$ and $b$ are the semi-major and semi-minor axis of the ellipse, respectively. Here, the aspect ratio is taken equal to $a/b = 2$ and the vorticity is equal to $\omega = 2\pi$. The evolution of the vortex is calculated with a fixed number of nodes, both with $N = 100$ and $N = 200$. Calculations have been performed with cubic and linear interpolation and, furthermore, using RK4 and MPREXC for the time integration. Figure 3.15 shows the errors in area ($\Delta A$, Figures 3.15a and b), circulation ($\Delta C$, Figures 3.15c and d) and angular momentum ($\Delta M$, Figures 3.15e and f) as a function of time $t$. The errors in circulation and angular momentum are defined similar to $\Delta A$ in (3.15). From these graphs it is clear that, in general, more nodes reveal more accurate results, as expected. Furthermore, it follows that higher-order interpolation not necessarily leads to more accurate results, at least not for the number of nodes used in the present study. Circulation and angular momentum behave almost the same for cubic and linear interpolation, but the area behaves even better with linear interpolation in some cases. Exception to this is the behaviour of the area in the case of 200 nodes, calculated by RK4 (in Figure 3.15b): the error behaves the same for both cubic and linear interpolation. Moreover, in comparison to the computation with $N = 100$ (Figure 3.15a), the behaviour does not improve when doubling the number of nodes. This means, that here the error is fully determined by the error in the time integration and not by the error...
due to the spatial discretization. This stresses once again the importance of accurate time integration. A further observation that can be made from the figure, is that the area is much better conserved when using MPREXC instead of RK4. Although the differences between the results of two time integrators are much less pronounced for circulation and angular momentum, MPREXC still produces better results.

**Example 3.4.2.** Next, the numerical evolution of a so-called Kida vortex [31] is considered. Initially, an elliptical patch, with aspect ratio equal to 4/3 and of uniform vorticity $\omega = 2\pi$, is placed in the centre of a strain flow $u_s$, given by

\[
\begin{align*}
    u_s &= e x, \\
    v_s &= -ey.
\end{align*}
\]

Here, $e$ is the strain rate which is chosen to be $e = 0.5$. In this case, the motion of the vortex is periodic: the vortex rotates around its centre while it remains elliptic and the aspect ratio changes periodically with time. The evolution of the vortex is calculated in three different ways: once with the midpoint rule with three cycles (i.e. four calculations of the velocities per time step) and extrapolation (MPREXC:4), once with the fourth-order Runge Kutta method (RK4) and finally with the midpoint rule with seven cycles (i.e. eight velocity calculations per time step) and extrapolation (MPREXC:8). In Figure 3.16, $\Delta A$ is plotted as a function of time $t$ for all three calculations. From this figure it can be observed that MPREXC:4 conserves the area of the patch better than RK4, this with the same computational effort. In addition, it appears that with more effort (e.g. seven cycles), the area is even better conserved. Note that again the behaviour of the area of the patch using RK4 agrees with (3.16). Furthermore, it should be remarked that the number of nodes on the contour is forced to remain constant. Of course, the variation of
the aspect ratio of the ellipse then causes a variation of the area, but this is the same for all calculations. If node redistribution would have been applied, the effect of it on the area would not have been clear and, moreover, it would probably not have been the same for all calculations; this would have hampered our assessment. A similar problem is encountered in the next example.

**Example 3.4.3.** This example concerns the evolution of a monopolar vortex into a tripolar vortex, which is a vortex consisting of an elliptic core with two satellites with vorticity of opposite sign [40, 47]. The initial configuration consists of three concentric, slightly elliptically disturbed, contours (aspect ratio is equal to 100/95). The outer ring has negative vorticity, while the core (consisting of the area enclosed by the second contour) has positive vorticity. Due to the elliptical disturbance, the monopole deforms and becomes a tripole while the core is becoming more and more elliptical. The evolution is shown in Figure 3.17. As can be observed from this figure, the outer contour (contour 1) deforms dramatically and this will have influence on the area enclosed by the outer contour because of the continual adding and removal of nodes. In Figure 3.18 the number of nodes is plotted as a function of time for each of the three contours. Although the two inner contours (contour 2 and 3) deform much less, nodes are added here too in regions where the curvature is increasing because of the elliptical shape of those contours (somewhere between \( t = 4 \) and \( t = 6 \)), and this also affects the area enclosed. Since this node redistribution might be different for calculations with different time integrators (because of the growing or shrinking of the area caused by the integrator), it is hard to compare the various results. Nevertheless, again \( \Delta A \) is plotted for all three contours as a function of time both for RK4 and MPREXC. The results are shown in Figure 3.19. For contour 1, both methods seem to produce comparable results from time \( t = 6 \) on. However, this may be caused by the node redistribution since for \( t < 6 \), MPREXC gives better results. For the other two contours, MPREXC appears to be better on the entire time interval. This may be very important for long-time integration, since the expansion of the areas of the inner contours can affect the dynamics of the whole vortex. Note that for long-time integration, surgery is recommended here.
Chapter 3. Contour Dynamics on an f-Plane

Figure 3.18: The number of nodes as a function of time $t$ for the three contours in the tripole simulation (see Figure 3.17).

Figure 3.19: $\Delta A$ plotted as a function of time for the three contours.
4

Collapse Interactions of Finite-Sized Two-Dimensional Vortices

4.1 Introduction
4.2 Point vortex motion
4.3 Contour dynamics simulations
   4.3.1 Equally-Sized Rankine vortices
   4.3.2 Distributed vorticity
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4.4 Discussion
Chapter 4. Collapse Interactions of Finite-Sized Two-Dimensional Vortices

4.1 Introduction

In this chapter the contour dynamics method, as discussed previously, is used to study collapse interactions of three finite-sized vortices. Gröbli [26] established the existence of a special self-similar motion of three point vortices of certain strength located initially at specific positions on an infinite plane (see also the paper by Aref et al. [4] for historical and scientific background). In this special case the vortex trajectories have the form of logarithmic spirals with a common origin. Depending on the signs of the strengths of the vortices, they can either escape to infinity (as is the case for which Gröbli found a solution) or move inward and collapse in the origin in a finite time. Later studies [3, 32, 33, 46] have revealed more general conditions for point vortex collapse to occur, namely some specific relations between the strengths and the initial positions of the vortices.

Although the collapse is in itself an interesting phenomenon, one may question the physical significance of this particular type of highly idealised point vortex interaction. The point vortex model has been proven to be very powerful in describing the interaction of (finite-sized) vortices [44]. Yet, it is a priori not clear whether real vortices (with finite-sized, continuous vorticity distributions) show a collapse into a single vortex, as predicted by the point vortex model. Furthermore, it was shown by Dritschel and Zabusky [23] that under certain circumstances, collapse interactions of finite-sized vortices, as discussed here, indeed occur in two-dimensional (nearly-) inviscid turbulence. This illustrates the relevance of understanding this particular phenomenon.

In this chapter, only the influence of the size and vorticity distribution on the behaviour of the vortices is examined. Effects of viscosity are discussed in the paper by Vosbeek et al. [57] where a finite-difference method is used for this purpose.

The remainder of the chapter is organised as follows. In Section 4.2 the point vortex model is discussed and in particular the case for which Gröbli found the solution. Next, in Section 4.3, a number of simulations obtained with contour dynamics is presented. In the first set of simulations, the point vortices are replaced by initially circular and equally-sized patches of uniform vorticity (Rankine vortices) with their circulation corresponding to the strengths of the point vortices of the Gröbli case. The initial patch size, relative to the initial distances between the patch centres, is the main parameter of interest here.

In the second set of simulations, the point vortices are replaced by equally-sized vortices with distributed vorticity. For this purpose, so-called Bessel vortices are used. Six contours are used to approximate this particular vorticity distribution.

In the final set of simulations, the patch size of the three vortices is chosen differently but such that the vorticity levels are identical for each vortex, and the circulation remains equal to that of the corresponding point vortex. For the latter case, simulations are presented for both Rankine vortices and vortices with distributed vorticity (Bessel vortices).

In the final section of this chapter, the results are summarised.
4.2 Point vortex motion

The two-dimensional inviscid flow problem of \( N \) interacting point vortices, strength \( \gamma_i \), position \((x_i, y_i)\) with \( i = 1, \ldots, N \), in the unbounded \((x, y)\)-plane consists of solving the following system of nonlinear first-order differential equations

\[
\begin{aligned}
\frac{du_j}{dt} &= -\frac{1}{2\pi} \sum_{i=1, i\neq j}^{N} \frac{\gamma_i (y_j - y_i)}{r_{ij}^2}, \\
\frac{dv_j}{dt} &= \frac{1}{2\pi} \sum_{i=1, i\neq j}^{N} \frac{\gamma_i (x_j - x_i)}{r_{ij}^2},
\end{aligned}
\]

(4.1)

where \( r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 \) and the initial configuration of the vortices is given by \( x_j = x_j^0 \), \( y_j = y_j^0 \) at \( t = 0 \). The system (4.1) can be written in a Hamiltonian [5] form with the Hamiltonian

\[ H = -\frac{1}{8\pi} \sum_{j=1}^{N} \sum_{i=1, i\neq j}^{N} \gamma_i \gamma_j \log r_{ij}^2, \]

in which \( H \) is a conserved quantity. In addition to \( H \), system (4.1) has three independent invariants:

\[ P_1 := \sum_{i=1}^{N} \gamma_i x_i, \quad P_2 := \sum_{i=1}^{N} \gamma_i y_i, \quad I := \sum_{i=1}^{N} \gamma_i (x_i^2 + y_i^2), \]

(4.2)

where \( P_1 \) and \( P_2 \) are the discrete forms of the first integral moments of vorticity, i.e. \( P_1 \) and \( P_2 \) divided by the total circulation of the system are the coordinates of the centroid of the vortex system. The quantity \( I \) is the discrete form of the second integral moment of vorticity, i.e. the angular momentum of the system. A combination of the invariants (4.2) provides another invariant of motion

\[ L := \sum_{i,j=1}^{N} \gamma_i \gamma_j r_{ij}^2 = 2\Gamma I - 2(P_1^2 + P_2^2), \]

with

\[ \Gamma := \sum_{i=1}^{N} \gamma_i, \]

the total circulation of the system. It was shown by Novikov and Sedov [46], Aref [3] and Kimura [32], that if the two conditions

\[ L = 0, \quad V = \sum_{j=1}^{N} \sum_{i=1, i\neq j}^{N} \gamma_i \gamma_j = 0, \]

(4.3)
are satisfied, a situation can exist where all distances between the vortices vary according to the same time dependency,

\[ r_{ij}(t) = r_{ij}(0)(1 + At)^{\frac{1}{3}} , \tag{4.4} \]

with the constant \( A \) depending on the initial conditions. If \( A < 0 \), the vortices collapse at time \( T = -1/A \).

Here, the attention is restricted to the Gröbli case, with \( N = 3 \) and

\[
\begin{align*}
x_1(0) &= -4\alpha , & y_1(0) &= 0 , & \gamma_1 &= -3\kappa , \\
x_2(0) &= -\frac{9}{2}\alpha , & y_2(0) &= \frac{3}{2}\alpha \sqrt{3} , & \gamma_2 &= 2\kappa , \\
x_3(0) &= \frac{1}{2}\alpha , & y_3(0) &= \frac{1}{2}\alpha \sqrt{3} , & \gamma_3 &= -6\kappa ,
\end{align*} \tag{4.5}
\]

where \( \alpha \) and \( \kappa \) are positive parameters. It is easy to check that the necessary conditions (4.3) are satisfied. The analytical solution of the equations of motion (4.1) satisfying these initial conditions is given by [26]

\[
\begin{align*}
x_1(t) &= 4\alpha (1 - t/T)^{\frac{1}{3}} \cos(\pi + \vartheta(t)) , \\
y_1(t) &= 4\alpha (1 - t/T)^{\frac{1}{3}} \sin(\pi + \vartheta(t)) , \\
x_2(t) &= 3\alpha \sqrt{3}(1 - t/T)^{\frac{1}{3}} \cos(\frac{5}{3}\pi + \vartheta(t)) , \\
y_2(t) &= 3\alpha \sqrt{3}(1 - t/T)^{\frac{1}{3}} \sin(\frac{5}{3}\pi + \vartheta(t)) , \\
x_3(t) &= \alpha (1 - t/T)^{\frac{1}{3}} \cos(\frac{1}{3}\pi + \vartheta(t)) , \\
y_3(t) &= \alpha (1 - t/T)^{\frac{1}{3}} \sin(\frac{1}{3}\pi + \vartheta(t)) ,
\end{align*} \tag{4.6}
\]

where \( T \) and \( \vartheta \) are given by

\[
\begin{align*}
T &= \frac{14\pi \alpha^2}{\kappa \sqrt{3}} , \\
\vartheta(t) &= \frac{5\kappa}{2\sqrt{3}} \ln(1 - t/T) , & 0 \leq t \leq T . \tag{4.7}
\end{align*}
\]

The trajectories of the point vortices are drawn in Figure 4.1 with solid lines, for the case where \( \alpha = \kappa = 1 \) and thus \( T = 14\pi / \sqrt{3} = 25.3952 \) according to (4.7). In this figure, the three vortices at \( t = 0 \) are connected with dashed lines to emphasize the shape of the triangle formed by them. Initially, this triangle has a right angle at vortex 1, and from (4.4) one can observe that this angle remains 90 degrees during the motion. In particular, the shape of the triangle does not change during the motion, as a consequence of (4.4).
Figure 4.1: The trajectories of the point vortices with initial positions (indicated by the numbers 1, 2 and 3) and strengths according to (4.5) with $\alpha = \kappa = 1$. Symbols are placed at the vortices’ positions at times $t = 0, 5, 10, 15, 20, 25$ and $t = 25.395$.

4.3 Contour dynamics simulations

In order to study the effect of a finite vortex size on the three-vortex interaction in an infinite domain, contour dynamics as discussed in Chapter 3 is used. The point vortices are replaced by vortices of a finite size with initial strength and position according to (4.5) with $\alpha = \kappa = 1$. Simulations are presented for equally-sized vortices of uniform and distributed vorticity and also for unequally-sized vortices both with uniform and distributed vorticity.

4.3.1 Equally-Sized Rankine vortices

In the first set of simulations presented here, the point vortices are replaced by initially equally-sized circular patches with uniform vorticity distribution (Rankine vortices) of radius $R$, with their centres at the initial locations defined by (4.5), with $\alpha = \kappa = 1$ and the vorticity $\omega_i$ chosen in such a way that $\omega_i \pi R^2 = \gamma_i$ for $i = 1, 2, 3$. At first, the radius $R$ is taken $R = 0.50$.

The trajectories of the centres of the three vorticity patches are obtained by computing at several moments in time the coordinates of the vorticity centres of the patches. This can be achieved by using a contour integral representation. Denote the centre of vortex $i$, for $i =$
Figure 4.2: The trajectories of the vorticity centres (solid lines) of the initially Rankine vortices (grey disks), with radius $R = 0.50$ and initial locations and strengths according to (4.5). The point vortex trajectories are drawn with dashed lines. Symbols are placed at the vortices' positions at times $t = 0, 5, 10, 15, 20$ and 25.

1, 2, 3, at time $t$ by $X_i(t)$. Then $X_i(t)$ is given by the contour integral representation [20]

$$
X_i(t) = \frac{1}{3} \sum_{m=1}^{M_i} \omega_{im} \oint_{C_{im}(t)} x(\,dx - y\,dy) \\
\frac{1}{2} \sum_{m=1}^{M_i} \omega_{im} \oint_{C_{im}(t)} (\,dy - y\,dx)
$$

where $M_i$ is the number of contours $C_{im}$ of vortex $i$ (here $M_i=1$) and $\omega_{im}$ the vorticity value belonging to contour $C_{im}$. For the numerical calculation, the contours are interpolated as discussed in Chapter 3.

In Figure 4.2, the trajectories of the vorticity patches are drawn with solid lines. The trajectories of the corresponding point vortices, initially located at the same positions, are drawn in this figure with dashed lines. It is clear from this figure, that the trajectories of the vortices are in good agreement with the trajectories of the point vortices until $t = 20$. After $t = 20$, however, the trajectories deviate from those of the point vortices, and the angle at vortex 1 starts to increase. Close inspection of the evolving vorticity patches reveals the deviation from the point-vortex trajectories to become noticeable approximately when the mutual distances between the vorticity centres of the individual patches becomes comparable to the patch sizes. Subsequently, the shape of some of the patches changes considerably. This is shown clearly in Figure 4.3, where the boundaries of the vortices are drawn with solid black lines for six moments of time from $t = 20$ on. Before $t = 20$ (not
Figure 4.3: Position and shape (solid lines) of the initially Rankine vortices of Figure 4.2 at six moments in time from $t = 20$ on. The grey shading represents the spatial distribution of the strain rate $Q$; dark shading represents a strong strain rate.
shown) vortex 1 and vortex 2 become slightly elliptic (their aspect ratios remain much smaller than 3, i.e. they remain in the stable regime [41], see later), but this almost has no influence on their interaction behaviour. After \( t = 20 \), however, vortex 1 (which is not the weakest) deforms very rapidly and is even completely torn apart at \( t = 25 \). This deformation takes place very quickly, and can be explained by observing the evolution of the strain rate of the velocity field.

According to Weiss [59] and McWilliams [43] the strain rate \( Q \) is given by

\[
Q := \text{tr}((\nabla u)^2) + \frac{1}{2} \omega^2,
\]

where \( \nabla u \) is the velocity gradient and \( \text{tr} \) is the trace. With the incompressibility condition (2.4), it easily follows that \( \text{tr}((\nabla u)^2) = -2 \det(\nabla u) \) so that

\[
Q = -2 \det(\nabla u) + \frac{1}{2} \omega^2.
\]

Figure 4.3 shows, in addition to the boundaries of the vortex patches, also contour plots of this quantity \( Q \); the dark grey regions are regions with strong strain, the white regions have (almost) zero strain.

The first feature that draws attention, is the strong strain just outside vortex 3, the strongest vortex. At \( t = 20 \), this vortex is still almost circular, and the strain can thus be described by the analytical expression (see Appendix A) for \( Q \) of a single (circular) Rankine vortex

\[
Q = \begin{cases} 
0, & r < R, \\
\frac{1}{2} \omega^2 \frac{R^4}{r^4}, & r > R,
\end{cases}
\]

where \( R \) is the radius, \( \omega \) the vorticity and \( r \) the radial distance to the centre of the Rankine vortex. Inside the vortex, the strain is zero. Just outside the vortex, the strain is strongest and equal to \( \frac{1}{2} \omega^2 \), and it decreases as \( 1/r^4 \).

The second feature that draws attention, is the presence of a rather strong strain at the long ends of the more or less elliptical vortices 1 and 2 at \( t = 20 \) and later. This can be explained employing the analytical expression for \( Q \) for a single (elliptic) Kirchhoff vortex, which can be derived from the expression of the stream function given by Lamb [37] (see Appendix A). It follows that \( Q \) is given by

\[
Q = \begin{cases} 
\frac{1}{2} \omega^2 \frac{(a - b)^2}{(a + b)^2}, & \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \\
\frac{(2ab\omega)^2}{(a^2 - b^2)^2} \frac{e^{-2\xi}}{(\cosh(2\xi) - \cos(2\eta))}, & \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1,
\end{cases}
\]

where \( a \) and \( b \) are the semi-major and semi-minor axis of the ellipse, respectively, \( \omega \) the vorticity, and \( \xi \) and \( \eta \) are elliptic coordinates which are related to the Cartesian coordinates.
Figure 4.4: The value of $Q$ plotted for two Kirchhoff vortices with different aspect ratios, both with vorticity $\omega = 1$.

$x$ and $y$ by

\[ x = (a^2 - b^2)^{\frac{1}{2}} \cosh(\xi) \cos(\eta), \]
\[ y = (a^2 - b^2)^{\frac{1}{2}} \sinh(\xi) \sin(\eta). \]

From the expression for $Q$ outside the vortex it is clear that the strain is largest for $\eta = 0$ and $\eta = \pi$ just outside the elliptical patch, i.e. at the long ends of the ellipse and smallest for $\eta = \pi/2$ and $\eta = 3\pi/2$. Note that inside the Kirchhoff vortex the strain is uniform and non-zero (unlike the Rankine vortex), its magnitude depending on the aspect ratio of the vortex and the value of $\omega$. In particular, for aspect ratios smaller than 3, the strain rate inside the vortex is smaller than the minimum value of it just outside the vortex, while for aspect ratios larger than 3 the strain rate inside the vortex is larger than the minimum value just outside it. To clarify this, the value of $Q$ is plotted in a contour plot in Figure 4.4, both for a vortex with aspect ratio 2 (Figure 4.4a) and for a vortex with aspect ratio 4 (Figure 4.4b). As in Figure 4.3, dark grey regions are regions with high values of $Q$. Since a Kirchhoff vortex is not stable for aspect ratios larger than 3 [17, 41], this suggests that the internal strain can play a role in the deformation of the vortex. This leads to the third and probably most important feature that can be observed from Figure 4.3: the presence of strain inside vortex 1 at $t = 20$, which increases rapidly when time evolves. The increasing strain results in a strong deformation of vortex 1 while it is gradually wrapped around vortex 3. Note that during this process also vortex 2 experiences substantial strain while it is becoming more elliptical. At $t = 25$ this internal strain has disappeared almost completely and the patch is less elongated.
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From the analytical expression (4.12) for the strain rate $Q$ of the Kirchhoff vortex, it follows that the internal strain is partially due to the elongated vortex shape but the strain induced by the neighbouring vortices (external strain), which come quite close to vortex 1 from $t = 20$ on, will of course also contribute significantly. In fact, this external strain causes vortex 1 to become elongated, as can be observed from Figure 4.5, where the elliptical vortex patch model described by Dritschel and de la Torre Juárez [22] is used to obtain an approximation of the external strain. In this model vortices are represented by elliptical patches of uniform vorticity and they are forced to remain elliptical during the evolution. The initial configuration used in Figure 4.5 is the same collapse configuration as in Figure 4.3 (i.e. the same initially circular patches of uniform vorticity at the same initial positions). The solid lines represent the edges of the (elliptical) vortices; the cross in the centre of the vortices displays both the magnitude and the orientation of the external strain (which is not equal to $Q$) with extension along the solid axis and compression along the short-dashed axis. The strong externally induced strain in vortex 1 from $t = 20$ up to $t = 23$ causes the elongation of the vortex. Note that comparison of the contour dynamics simulations of Figure 4.3 with this elliptical model shows hardly any difference until $t = 21$. From that moment on, however, the trajectories only slightly deviate from those of the contour dynamics simulations while the deformations of the vortices show much larger differences.

It is obvious from Figure 4.3 that a “collapse” of the patches into one single vortex patch does not take place: after $t = 25$ the two surviving vortices move along a curved path as an asymmetric dipolar structure, carrying the dynamically insignificant filament of the original vortex 1 with it.

The evolution of the vortices has also been computed for initial patch radii $R = 0.25$ and $R = 0.75$. Figure 4.6 shows the last part of the trajectories for $R = 0.25$ (a) and $R = 0.75$ (b). For the first case the trajectories of the vorticity centres agree very well with the trajectories of the point vortices beyond $t = 24$, while for the latter the trajectories are seen to deviate already at $t = 15$. This confirms the observation that the vortices no longer behave as point vortices once their mutual distance becomes of the order of their size ($2R$). The initial radius apparently only influences the time scale: the deformation of the vortices is in all cases very similar, but starts at different moments in time.

### 4.3.2 Distributed vorticity

Instead of patches of uniform vorticity, now the point vortices are replaced by vortices with distributed vorticity. To this end, so-called Bessel vortices are used. The vorticity distribution of such vortices is given by

\[
\omega(r) = \begin{cases} 
\frac{k \Gamma}{2 \pi R J_1(kR)} J_0(kr), & r \leq R, \\
0, & r \geq R,
\end{cases}
\]  

(4.13)

with $r$ the radial distance to the centre of the vortex, $R$ its radius, and $\Gamma$ its strength or circulation. $J_0$ and $J_1$ are Bessel functions of the first kind and $kR \approx 2.4048$ is the first
Figure 4.5: Six stages in the flow evolution modelled by the elliptical vortex patch model with the same initial configuration as in Figure 4.3. The cross in the centre of the vortices indicates both magnitude and orientation of the external strain.
Figure 4.6: Trajectory parts of (initially) Rankine vortices with radius $R = 0.25$ (a) and $R = 0.75$ (b) (solid lines) compared with the corresponding point vortex trajectories (dashed lines). The numbers indicate the trajectories of the corresponding vortices. Note the difference in domain size of the two pictures; the grey disk indicates the initial size of the vortices.

root of $J_0$. The maximum of vorticity is located at the centre of the vortex, where $J_0$ equals unity. The vortex given by (4.13) is an exact, stationary solution of the inviscid equations (2.11) and (2.12) in an infinite domain, which satisfies the linear relationship $\omega = k^2 \psi$.

In this simulation, Bessel vortices with radii $R = 0.5$ are used. The continuous distributions of the three vortices are replaced by piecewise-uniform distributions with six levels. The vorticity jumps inside one particular vortex are identical and depend on the maximum vorticity value of that vortex. Furthermore, the initial radii of the contours for each vortex are chosen the same and such that the piecewise-uniform distribution approximates the continuous profile. The total circulation of the piecewise-uniform distribution is equal to that of the continuous distribution. Figure 4.7 shows the trajectories of the vortices and Figure 4.8 shows the position and shape of the contours at six moments in time as well as the strain rate $Q$.

Compared to the trajectories of the Rankine vortices of the same initial size (Figure 4.2), it now appears that the trajectories coincide with those of the point vortices somewhat longer (approximately until $t = 22$) and although qualitatively the vortices behave similar to the Rankine vortices (vortex 1 is torn apart here as well), the large deformations start later in time as well. This can be observed by comparing Figure 4.8 and Figure 4.3.
A possible explanation is that for a Bessel vortex the vorticity is more concentrated in the centre of the vortex compared to a Rankine vortex of the same size and strength (the maximum of vorticity of the Bessel vortex is larger compared to that of the Rankine vortex). Therefore, a Bessel vortex approximates a point vortex better. From this observation, it may be expected that the larger the maximum vorticity value of the vortex is for a given radius and strength, i.e. the more concentrated the vorticity in the centre of the vortex, the later the large deformations start. It should be remarked, however, that this effect is smaller than the influence of the size of the vortices.

Another aspect that can be observed from a comparison between Figure 4.8 and Figure 4.3, is that the strain rate for the Bessel vortices is much higher than for the patches of uniform vorticity (note the scale difference). Nevertheless, the distribution in space of the strain rate is quite similar and the explanation provided in Section 4.3.1 still holds.

### 4.3.3 Unequally-sized vortices

In the final set of simulations, instead of using the same initial radius for each vortex and adapting the vorticity levels to achieve the required circulation, now the vorticity levels are identical for each vortex. The radii of the contours are adapted to achieve the required circulation. This implies that the strongest vortex has the largest initial size. Two simulations are presented here.

In the first simulation, Rankine vortices (uniform vorticity distributions) are used again. The radius of vortex 1 ($\gamma_1 = -3$) is $R_1 = 0.5$ and thus the vorticity level is given by
Figure 4.8: Similar to Figure 4.3 but now for the initially Bessel vortices of initial radius \( R = 0.5 \) approximated by six vorticity levels. Note that the darkest regions have much higher level of strain than in Figure 4.3.
Figure 4.9: Position and shape (solid lines) of the initially Rankine vortices, of the same vorticity but different size, at six stages in the flow from \( t = 20 \) on. The grey shading represents the spatial distribution of the strain rate \( Q \); dark regions represent strong strain.
\[ \omega = \frac{\gamma_1}{(\pi R^2_1)} = -3.81972. \] Now it easily follows from (4.5) that the radii of vortex 2 and 3 are given by \( R_2 = \sqrt{\frac{\gamma_2}{(\pi \omega_1)}} = 0.40825 \) and \( R_3 = \sqrt{\frac{\gamma_3}{(\pi \omega_1)}} = 0.707107. \) Figure 4.9 shows the shape (solid lines) and position of the vortices at six moments in time from \( t = 20 \) on. Again, the grey shading represents the distribution of the strain. Compared to the vortices with equal size of Figure 4.3 the differences are very small. Vortices 1 and 2 behave almost completely similar. The only substantial difference is the deformation of vortex 3 (and related to that the increase of the strain rate at a particular region inside the vortex) which starts at approximately \( t = 21. \) This vortex even develops a filament near the region where the strain is increasing. Apparently, vortex 3 is more affected by vortex 1 now, because it approaches the outermost contour of vortex 3 more closely than in previous simulations.

The trajectories of the vortices are not shown here since they do not differ much from those shown in Figure 4.2. Like the patches of equal size, the trajectories coincide with those of the point vortices up to \( t = 20. \)

In the second simulation Bessel vortices are used. Now, all vorticity jumps are identical. The radii of the initial contours are adapted according to the required strength. Results of this simulation are presented in Figure 4.10, where the contours (solid lines) and position of the vortices are shown, with the grey shading indicating the strain rate again. Like in the previous case, there are not many differences compared to the results of the simulations with equally-sized Bessel vortices (c.f. Figure 4.8). Again, the only substantial difference is the deformation of vortex 3 and the development of a filament similar to that in Figure 4.9. The difference with the previous simulation is that also a second filament starts to develop at the other end of vortex 3. This is clearly visible in Figure 4.11 where the three vortices are plotted at time \( t = 26. \) Of course, the deformations also start at a later stage in the evolution (similar to what occurred during the simulation discussed in Section 4.3.2). Note that in Figure 4.11 the outer contours of vortex 1 show some minor cusps at the filament tips of the inner contours. These cusps are caused by the discretization of the vorticity, as shown by Legras and Dritschel [39] in their comparison of results of simulations with piecewise-uniform (contour surgery) and continuous (pseudo-spectral method) distributions.

### 4.4 Discussion

In this chapter numerical simulations have been presented to demonstrate the influence of the finite size of a vortex on vortex interactions in a situation which leads to a vortex collapse in the limiting case of the point vortex. When the mutual distances of the vortices become of the order of their sizes, the trajectories are seen to differ from those of the corresponding point vortices and the vortices start to deform. In the three-vortex configuration considered here, one vortex in particular deforms dramatically, due to the growing internal and external strain. It is believed that this deformation into a long filament, which is gradually wrapped around the remaining vortices, prevents a collapse of the three vortices.

The larger the initial size of the vortices, the earlier they start to deform. In fact, only the time scale of the deformations is influenced by the size of the vortices; the deformations
Figure 4.10: Position and shape (solid lines) of the initially Bessel vortices, each with six contours of equal vorticity jump, at six moments in time from \( t = 20 \) on. The grey shading represents the spatial distribution of the strain rate \( Q \); dark regions represent strong strain.
Figure 4.11: The three vortices of Figure 4.10 at time $t = 26$ showing the filaments at both long ends of vortex 3.

themselves are very similar for different initial sizes of the vortices. It has been shown that the distribution of the vorticity inside the vortices does not affect the dynamics of the vortices very much either, only some minor differences in the shape of the deforming vortices can be observed. As a consequence, the vorticity distribution of the vortices has a small influence on the time scale.

In the paper by Vosbeek et al. [57], effects of viscosity are examined. Since viscosity causes the vortex to grow as time evolves, the deformations start earlier than in the inviscid case. The behaviour and deformations of the vortices, however, show good agreement with the results presented here.
5

Acceleration with a Hierarchical-Element Method

5.1 Introduction
5.2 Hierarchical-element methods
5.3 The Poisson integrals
   5.3.1 Theoretical derivation
   5.3.2 Numerical approach
5.4 Construction of finest level outer-rings
5.5 Numerical experiments and discussion
5.1 Introduction

In Chapter 3 it has been explained that in standard contour dynamics simulations nodes have to be added to the contours in order to approximate the contours satisfactorily. Example 3.4.3 of Chapter 3 illustrates this: the number of nodes on the outer contour of the tripole grows rapidly. As a consequence, the CPU-time needed per time step also increases dramatically, since the calculation of the velocity requires order \( O(N^2) \) operations where \( N \) is the total number of nodes. In the tripole example only three contours have been used during the simulation and one can imagine that in simulations where larger numbers of contours are necessary, the CPU-time per time step can become very high.

In order to be able to perform such computationally expensive calculations, it is necessary to accelerate the part of the method where the velocities are computed. To this end, a so-called hierarchical-element method has been developed. In the next section such methods are briefly reviewed and in particular the approach by Anderson [2]. Subsequently, Sections 5.3, 5.4 and 5.5 give a discussion of the necessary adaptations to Anderson's method that make the hierarchical-element method suitable for application to contour dynamics simulations. Finally, in Section 5.6, the accuracy and computational efficiency of the new method is illustrated with some numerical examples.

5.2 Hierarchical-element methods

In this section a hierarchical-element method (HEM) as described by Anderson [2] is discussed. The method is based on the fast multipole technique developed by Greengard and Rokhlin [25], but does not employ multipoles themselves. Instead, approximations based on Poisson's formula are used.

The fast multipole method (FMM) itself has been developed in order to accelerate computations of \( N \)-body interactions. For example, given \( N \) charged particles at positions \( \mathbf{x}_n \) with strength \( \kappa_n \), \( n = 1, \ldots, N \), the potential \( \Phi \) at every particle has to be calculated. Here, \( \Phi \) is a solution of

\[
\nabla^2 \Phi = \sum_{n=1}^{N} \kappa_n \delta(\mathbf{x} - \mathbf{x}_n),
\]

where \( \delta(\mathbf{x}) \) is Dirac's delta function. The solution is given by

\[
\Phi(\mathbf{x}) = \sum_{n=1}^{N} \frac{\kappa_n}{2\pi} \ln \| \mathbf{x} - \mathbf{x}_n \|. \quad (5.1)
\]

Clearly, the evaluation of \( \Phi \) at every particle requires \( O(N^2) \) operations. The FMM reduces the operation count to \( O(N \ln(N)) \) or even \( O(N) \). Note that there is a strong resemblance with point vortices: the charged particles can be replaced by point vortices and in that case, \( \Phi \) should be replaced by the stream function \( \psi \). The FMM is thus very suitable for accelerating (many) point vortex interactions as well.
The method basically consists of two parts. The first part is based on the concept of combining a large number of particles into a single computational element. When a cluster of particles is far away from a certain point at which the potential has to be calculated, the potential of the cluster is approximated by the potential induced by a single computational element inside the cluster. To this end a multipole expansion around the centre of a disk containing the cluster of particles is used. A multipole expansion [25] is an approximation of the kind

$$\Phi(z) = \text{Re} \left( \sum_{n=n_1}^{n_2} \frac{\kappa_n}{2\pi} \ln |z - z_n| \right) \approx \text{Re} \left( a_0 \ln |z - z_0| + \sum_{k=1}^{K} \frac{a_k}{(z - z_0)^k} \right),$$

where the $\mathbb{R}^2$-plane is identified with the complex plane, $z_0$ is the centre of the disk containing the particles $z_n$ with $n = n_1, \ldots, n_2$, $K$ is the order of the multipole (which depends on the desired accuracy of the method: the higher $K$ the more accurate the method) and $a_k$ are coefficients chosen so that the multipole is an accurate approximation of the potential. Clearly, the evaluation of this multipole expansion requires $O(K)$ operations, which is typically much less than the number of operations required for computing the potential of the whole collection of particles (which is $O(n_2 - n_1)$).

The second part of the method concerns the organisation of the computations in such a way that the technique of combining particles is efficient and does not lead to inaccuracies. For example, when combining particles into single elements, the more widely distributed in space, the particles of a given cluster are, the more inaccurate the multipole expansion becomes for a fixed value of $K$ and a fixed point $z$ of evaluation. However, if the evaluation point $z$ is moved away from $z_0$, then the accuracy of the approximation improves. So, if a certain degree of accuracy is desired, the potential should be approximated by a hierarchy of multipole expansions. Far away from the evaluation point, particles are combined over large regions; particles closer to the evaluation point are combined over smaller regions as indicated in Figure 5.1. This figure shows an example of a hierarchical clustering of particles which is used to create a multipole approximation to the potential at a point in the dark grey box. The potential induced by particles in the white boxes is combined into multipole approximations; the potential induced by particles in the light grey and dark grey boxes is computed using the direct interaction formula.

In the method described by Anderson, instead of multipole expansions, Poisson’s formula is used. According to Poisson’s integral the potential outside a disk with radius $a$, containing the particles $x_n$ for $n = n_1, \ldots, n_2$, is given by

$$\Phi(r, \varphi) = \kappa \ln(r) + \frac{1}{2\pi} \int_0^{2\pi} \left( \Phi(a, \vartheta) - \kappa \ln(a) \right) g\left( \frac{a}{r}, \varphi, \vartheta \right) d\vartheta, \quad \text{for } r > a, \quad \text{(5.2)}$$

where

$$g(\vartheta, \varphi, \vartheta) := \frac{1 - \varrho^2}{1 - 2\varrho \cos(\varphi - \vartheta) + \varrho^2},$$
Figure 5.1: A hierarchical clustering of particles which is used to create a multipole approximation to the potential at a point in the dark grey box (adapted from [2]).

and

\[ \kappa := \frac{1}{2\pi} \sum_{n=n_1}^{n_2} \kappa_n. \]

The coordinates \((r, \varphi)\) indicate the position in polar coordinates of the evaluation point with respect to the centre of the disk [35].

An advantage of this approach is that also collections of sources which are more general than point charges or point vortices, such as given areas of certain charge distributions or vorticity distributions, can be treated. The application of multipole expansions might be very difficult or even impossible for these particular problems.

The integral in (5.2) can be determined numerically, although problems arise when integrating it straightforwardly by, e.g., the trapezoidal rule (see Section 5.3.2). Let \(K = 2L+1\), with \(L\) a natural number, denote the number of equidistant integration points on the circle. To obtain numerically accurate results, Anderson [2] replaces the function \(g\) by a function \(g_L\) defined as

\[ g_L(\varrho, \varphi, \vartheta) := \frac{1 - \varrho^2 - 2\varrho^{L+1} \cos((L + 1)(\varphi - \vartheta)) + 2\varrho^{L+2} \cos(L(\varphi - \vartheta))}{1 - 2\varrho \cos(\varphi - \vartheta) + \varrho^2}. \]

Applying the trapezoidal rule then yields

\[ \Phi(r, \varphi) \doteq \kappa \ln(r) + \frac{1}{K} \sum_{k=1}^{K} (\Phi(\varrho_k, \varphi_k) - \kappa \ln(\varrho_k)) g_L(\frac{\varrho}{\varrho_k}, \varphi, \vartheta_k), \quad \text{for } r > a, \quad (5.3) \]
where \( \vartheta_k = 2\pi k/K \). In Section 5.3.2 a similar procedure is followed although for slightly different definitions of \( g \) and \( g_L \). The reason why more accurate results can be obtained by replacing \( g \) by \( g_L \) is also explained in more detail in that section.

In the paper by Anderson [2] the above approximation (5.3) is referred to as an outer-ring approximation. In a similar way a so-called inner-ring approximation can be defined:

\[
\Phi(r, \varphi) = \frac{1}{K} \sum_{k=1}^{K} \Phi(a, \vartheta_k) g_L\left(\frac{r}{a}, \varphi, \vartheta_k\right), \quad \text{for } r < a.
\] (5.4)

This inner-ring approximation represents the potential inside a ring with radius \( a \).

In the following, the integration points of the ring in (5.3) and (5.4) are referred to as belonging to the outer-ring and inner-ring, respectively. Furthermore, the evaluation of the potential at the outer-ring by means of (5.3) or by direct summation of the appropriate terms in (5.1) is referred to as the construction of the outer-ring and likewise for the inner-ring.

Now, the method proceeds as follows. First, a square domain is chosen which encloses all particles. Furthermore, a finest level of refinement \( l_f \) is chosen (a way to do this is discussed later). At this finest level \( l_f \), the domain is divided into \( 2^{l_f} \times 2^{l_f} \) square boxes. Similarly, at the coarser levels the domain contains of \( 2^l \times 2^l \) boxes, \( l = 1, \ldots, l_f \). Like the FMM, this method also consists of two parts.
Figure 5.4: Two sources contribute to the inner-rings: the inner-rings of the parent box at the previous coarser level (left) and outer-rings of boxes which are well separated from the current box at the current level but the parents of those boxes are not well separated from the parent box of the current box (right) (adapted from [2]).

In the first part, outer-rings are constructed at each level, starting with the finest level. Here, a ring of radius equal to the box width is chosen around each box. The centre of the ring is located at the centre of the box. Then, at the outer-ring, the potential due to the particles inside the box is determined (see also Figure 5.2) by means of direct summation of the appropriate terms in (5.1), i.e. only the terms concerning the particles inside the box are taken into account. After finishing the finest level, one proceeds to the coarser level where the outer-rings (again with radius equal to the box size and with centre located at the centre of the box) are constructed from the finer level by combining the contributions of the four “child” boxes inside the coarser box by means of (5.3). In Figure 5.3 it is shown how the contribution to the outer-ring of a “child” box (solid lines) is determined; the other three boxes (dashed lines) contribute similarly. This procedure can be repeated at each coarser level. At the end of the first part, outer-rings have been constructed for each box at every level.

In the second part of the algorithm, the contributions of the outer-rings are organised in a rather smart way. To this end, the concept of being well-separated [25] is used. If the boxes at a certain level $l$ are identified by a pair $(i, j)$, $i, j = 1, \ldots, 2^l$, with box $(1, 1)$ being the box at the bottom left and box $(2^l, 2^l)$ the box at the top right of the domain, a box $(i_1, j_1)$ is well-separated from box $(i_2, j_2)$ with distance $D$, if the maximum of the difference between their indices $\max(|i_1 - i_2|, |j_1 - j_2|)$ is larger than or equal to $D$. At a certain level, this second part consists of constructing inner-rings for each box which are used to represent the contributions to the potential from two sources. The first source is the inner-ring associated with the parent box at the previous coarser level (the left part
of Figure 5.4). The second source is the contribution from all the outer-rings of the boxes that are well separated from the given box (with distance \( D = 1 \)) and are contained within boxes at the previous coarser level that are not well separated from the parent box (the right part of Figure 5.4). The radius of the ring in the inner-ring approximations is taken equal to half the box width. This procedure is carried out for all levels, except the finest. For this level, the inner rings are only constructed from the inner-rings of the parents. At the end of this part, the potential at any given evaluation point is obtained by computing the potential of the inner-ring approximation associated with the finest level box where the point is residing. This potential is then added to the potential induced by the particles in those neighbouring finest level boxes that are not well separated from the given evaluation point box and which is obtained by using the direct interaction formula.

In pseudo-code, the algorithm can be summarised as follows:

```plaintext
begin
{Initialisation}
Choose finest level of refinement \( l_f \)
Choose number of integration points \( K \)
{Construct outer-rings at level \( l_f \)}
for each particle \( x \) do
  Determine in which box at level \( l_f \) particle \( x \) resides in
od
for each box \( b \) at level \( l_f \) do
  for each particle \( x \) in box \( b \) do
    Calculate potential due to particle \( x \) at all \( K \) integration
    points of the outer-ring of box \( b \) (see Fig 5.2)
  od
od
{Construct outer-rings at coarser levels. Note that, in the case of}
{no boundary conditions, construction of outer-rings at level 1 is}
{not necessary because of the absence of well separated boxes for}
{each box at that level}
for \( l = l_f - 1 \) downto 2 do
  for each box \( b \) at level \( l \) do
    Combine outer-rings of the four child boxes of box \( b \) using
    the outer-ring approximation (5.3) (see Fig 5.3)
  od
od
{Construct inner-rings up to level \( l_f - 1 \). Note that, in case of no}
{boundary conditions, there is no need for inner-rings to be}
{constructed at level 1}
for \( l = 2 \) to \( l_f - 1 \) do
  for each box \( b \) at level \( l \) do
    Construct inner-ring of box \( b \) from parent inner-ring (left
    part of Fig 5.4) and from boxes that are well separated
```
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from box \( b \) at this level, but their parent boxes are not
well separated from the parent box of box \( b \) at level \( l - 1 \)
(right part of Fig 5.4)

\[
\text{od}
\]

\[
\text{od}
\] {Construct inner-rings at finest level \( l_f \)}

\[
\text{for each box } b \text{ at level } l_f \text{ do}
\]

\[
\text{Construct inner-ring of box } b \text{ from parent inner-ring at}
\]

\[
\text{level } l_f - 1
\]

\[
\text{od}
\] {Evaluation of potential at the particles}

\[
\text{for each box } b \text{ at level } l_f \text{ do}
\]

\[
\text{for each particle } x \text{ in box } b \text{ do}
\]

\[
\text{Calculate the contribution of the inner-ring of box } b \text{ to the}
\]

\[
\text{potential at particle } x \text{ using the inner-ring approximation}
\]

\[
(5.4) \text{ and the contributions of all particles in boxes that}
\]

\[
\text{are not well separated from box } b \text{ by direct summation}
\]

\[
\text{od}
\]

\[
\text{od}
\]

\[
\text{end}
\]

In the foregoing, the number of refinements \( l_f \) was fixed. However, it should of course be chosen in such a way that the method is the most efficient. It may be clear, that the value of \( l_f \) depends on the number of particles in the domain: the more particles there are, the more levels of refinement should be used. However, not only the amount of particles is important, but also the way they are distributed over the domain. If there are many boxes with only a few particles (compared to the number of integration points on the rings) inside, the method can become even less efficient than direct evaluation. Obviously, in that case the number of refinements should have been taken smaller. On the other hand, if there are many boxes with very many particles inside, too many direct evaluations have to be performed, and it might be more efficient to choose more levels of refinement. In the case of uniformly distributed particles, the amount of work for a certain value of \( l_f \) can be estimated analytically. In this way, the optimal choice (the one for which the work required is minimal) of \( l_f \) can be found. However, the particles are in practice not uniformly distributed, but it is still possible to estimate the amount of work a priori. Since this amount of work at each level is completely determined by the number of particles in any given box, it can be estimated by counting the operations required: execute the hierarchical method but instead of performing all operations required, just update some counters. The counter increments are based on the density of particles in each box, and are a measure for the computational time necessary to carry out the specific computational tasks. Then the time, necessary for the work required for different levels of refinement, is estimated, and the level with the least amount of anticipated time is selected.

As mentioned earlier, this variant of the HEM can also be used for problems with given areas of charge distribution or vorticity distribution instead of point charges or point
vortices, respectively. This makes the method suitable for accelerating contour dynamics simulations dealing with piecewise-uniform vorticity distributions. In the next two sections, the necessary adaptations of Anderson's method are discussed.

### 5.3 The Poisson integrals

To accelerate the contour dynamics method with the HEM discussed in the previous section, two aspects related with the implementation of the HEM, need to be changed. The first change concerns the Poisson integrals. For contour dynamics, not the stream function is of interest, but the velocity. As a consequence, Poisson integrals as in (5.2) have to be derived both for the radial component of the velocity and the azimuthal component of the velocity. The second modification concerns the piecewise-uniform vorticity distribution instead of point vortices, which affects both the Poisson integrals and the construction of the outer-rings at the finest levels. In Section 5.3.1, the derivation of the Poisson integrals is discussed and its numerical calculation is discussed in Section 5.3.2. The construction of outer-rings at the finest level is discussed briefly in Section 5.4.

#### 5.3.1 Theoretical derivation

Consider a stream function \( \psi \) for which

\[
\nabla^2 \psi = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} = 0,
\]

(5.5)

outside a disk of radius \( a \) containing an area of piecewise-uniform vorticity \( \omega(r, \varphi) = \omega_m, (r, \varphi) \in G_m, m = m_1, \ldots, m_2 \). The regions \( G_m, m = m_1, \ldots, m_2 \) are nested such that \( G_{m_2} \subset G_{m_2-1} \subset \cdots \subset G_{m_1} \) (see also Chapter 3). The radial and azimuthal velocity \( u_r \) and \( u_\varphi \) are related to \( \psi \) by

\[
\begin{align*}
    u_r(r, \varphi) &= \frac{1}{r} \frac{\partial \psi}{\partial \varphi}, \\
    u_\varphi(r, \varphi) &= -\frac{\partial \psi}{\partial r}.
\end{align*}
\]

Before the Poisson integrals for both the radial and azimuthal velocity component can be derived, the following three properties have to be proven.

**Property 5.3.1.** For \( r \to \infty \)

\[
\begin{align*}
    ru_r &= O(1/r), \\
    ru_\varphi &= \kappa + O(1/r),
\end{align*}
\]

(5.6) (5.7)

where \( \kappa = \sum_{m=m_1}^{m_2} (\omega_m A_m)/(2\pi) \) with \( A_m \) the area of \( G_m \), i.e. \( \kappa \) is equal to the circulation divided by \( 2\pi \).
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Proof. Since $\nabla^2 \psi = -\omega_m, (r, \varphi) \in \mathcal{G}_m, m = m_1, \ldots, m_2$ inside the disk,

$$ru_r = \frac{\partial \psi}{\partial \varphi}$$

$$= - \sum_{m=m_1}^{m_2} \frac{\omega_m}{4\pi} \int \int_{\mathcal{G}_m} \frac{\partial}{\partial \rho} \left[ \ln(r^2 - 2\rho \cos(\varphi - \vartheta)) + \rho^2 \right] \rho \, d\rho \, d\vartheta$$

$$= - \sum_{m=m_1}^{m_2} \frac{\omega_m}{4\pi} \int \int_{\mathcal{G}_m} \frac{2\rho \sin(\varphi - \vartheta)}{r^2 - 2\rho \cos(\varphi - \vartheta) + \rho^2} \rho \, d\rho \, d\vartheta$$

$$= \mathcal{O} \left( \frac{1}{r} \right), \quad (r \to \infty).$$

Similarly

$$ru_\varphi = -r \frac{\partial \psi}{\partial r}$$

$$= \sum_{m=m_1}^{m_2} \frac{\omega_m}{4\pi} \int \int_{\mathcal{G}_m} \frac{\partial}{\partial r} \left[ \ln(r^2 - 2\rho \cos(\varphi - \vartheta)) + \rho^2 \right] \rho \, d\rho \, d\vartheta$$

$$= \sum_{m=m_1}^{m_2} \frac{\omega_m}{4\pi} \int \int_{\mathcal{G}_m} \frac{2r^2 - 2\rho \cos(\varphi - \vartheta)}{r^2 - 2\rho \cos(\varphi - \vartheta) + \rho^2} \rho \, d\rho \, d\vartheta$$

$$= \sum_{m=m_1}^{m_2} \frac{\omega_m}{4\pi} \int \int_{\mathcal{G}_m} \left[ 2 + \frac{2\rho \cos(\varphi - \vartheta) - 2\rho^2}{r^2 - 2\rho \cos(\varphi - \vartheta) + \rho^2} \right] \rho \, d\rho \, d\vartheta$$

$$= \kappa + \mathcal{O} \left( \frac{1}{r} \right), \quad (r \to \infty),$$

where $\kappa = \sum_{m=m_1}^{m_2} (\omega_m A_m)/(2\pi)$, with $A_m$ the area of $\mathcal{G}_m$.

Note that this property is also valid for a continuous vorticity distribution.

**Property 5.3.2.** For $r > a$, $\nabla^2 (ru_r) = \nabla^2 (ru_\varphi) = 0$.

Proof. The proof easily follows by using (5.5).

$$\nabla^2 (ru_r) = \nabla^2 \left( \frac{\partial \psi}{\partial \varphi} \right) = \frac{\partial}{\partial \varphi} (\nabla^2 \psi) = 0,$$

outside the disk and

$$\nabla^2 (ru_\varphi) = -\nabla^2 \left( r \frac{\partial \psi}{\partial r} \right) = -\frac{\partial}{\partial r} (r \nabla^2 \psi) = 0,$$

outside the disk as well.
5.3. The Poisson integrals

Property 5.3.3. The solution of the following exterior boundary-value problem

\[
\begin{align*}
\nabla^2 f(r, \varphi) &= 0, \quad r > a, 0 \leq \varphi < 2\pi, \\
f(r, \varphi) &= f(a, \varphi), \quad r = a, 0 \leq \varphi < 2\pi, \\
f(r, 0) &= f(r, 2\pi), \quad r \geq a, \\
f(r, \varphi) &= \mathcal{O} \left( \frac{1}{r} \right), \quad r \to \infty, 0 \leq \varphi < 2\pi,
\end{align*}
\]  

(5.8)

is given by

\[
f(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(a, \theta) g\left( \frac{a}{r}, \varphi, \theta \right) d\theta, \quad \text{for } r > a,
\]  

(5.9)

where

\[
g\left( \frac{a}{r}, \varphi, \theta \right) := 2a \left( \frac{\cos(\varphi - \theta) - \varphi}{1 - 2a \cos(\varphi - \theta) + \varphi^2} \right).
\]  

(5.10)

Proof. The above boundary-value problem can be solved by separation of variables. Substitution of \( f(r, \varphi) := R(r)\Phi(\varphi) \) in the Laplace equation yields the following two ordinary differential equations

\[
\frac{\tau R'(\tau)}{R(\tau)} + \frac{\tau^2 R''(\tau)}{R(\tau)} = \lambda^2,
\]

(\*)

\[
- \frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda^2,
\]

(**)

with \( \lambda \) constant. The boundary conditions yield conditions for \( R \) and \( \Phi \)

\[
R(\tau) = \mathcal{O} \left( \frac{1}{\tau} \right), \quad \tau \to \infty,
\]

(\(\circ\))

\[
\Phi(0) = \Phi(2\pi).
\]

(\(\circ\circ\))

The solution of (**) is given by

\[
\Phi(\varphi) = Ae^{i\lambda \varphi} + Be^{-i\lambda \varphi},
\]

where \( A \) and \( B \) are some arbitrary constants. From condition (\(\circ\circ\)) now follows that \( \lambda = k \in \mathbb{Z} \). The solution of (\*) is given by

\[
R(\tau) = Cr^k + Dr^{-k},
\]

with \( C \) and \( D \) constants. Condition (\(\circ\)) yields \( C = 0 \) if \( k > 0 \), \( D = 0 \) if \( k < 0 \) and \( C = D = 0 \) if \( k = 0 \). The solution of the boundary value problem (5.8) can now be written as the Fourier series

\[
f(r, \varphi) = \sum_{k=-\infty, k\neq 0}^{\infty} c_k r^{-|k|} e^{ik\varphi},
\]

where

\[
c_k = \frac{1}{2\pi} \int_0^{2\pi} f(a, \theta) g\left( \frac{a}{r}, \varphi, \theta \right) e^{-ik\theta} d\theta.
\]
where the Fourier coefficients $c_k$ can be determined from the values of $f$ on the circle $r = a$:

$$
c_k = \frac{a^{\lvert k \rvert}}{2\pi} \int_0^{2\pi} f(a, \vartheta) e^{-ik\vartheta} \, d\vartheta.
$$

Interchanging integration and summation is allowed if the series is uniformly convergent, which is the case here if $\frac{a}{\rho} < \rho$ for $\rho < 1$. Then it follows that

$$
f(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(a, \vartheta) \sum_{n=1}^{\infty} \left[ \left( \frac{ae^{i(\varphi-\vartheta)}}{r} \right)^n + \left( \frac{ae^{-i(\varphi-\vartheta)}}{r} \right)^n \right] \, d\vartheta.
$$

The outcome of the series can now easily be determined, yielding

$$
f(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(a, \vartheta) \left[ \frac{a}{r} e^{i(\varphi-\vartheta)} \left( \frac{a}{r} e^{i(\varphi-\vartheta)} + \frac{a}{r} e^{-i(\varphi-\vartheta)} \right) \right] \, d\vartheta,
$$

which can be simplified further with result (5.9).

The Poisson integrals for the radial and azimuthal velocity components now easily follow from the above properties.

**Lemma 5.3.4.**

$$
u_r(r, \varphi) = \frac{a}{2\pi r} \int_0^{2\pi} u_r(a, \vartheta) g \left( \frac{a}{r}, \varphi, \vartheta \right) \, d\vartheta, \quad (5.11)
$$

$$
u_\varphi(r, \varphi) = \frac{\kappa}{r} + \frac{a}{2\pi r} \int_0^{2\pi} \left( \nu_r(a, \vartheta) - \frac{\kappa}{a} \right) g \left( \frac{a}{r}, \varphi, \vartheta \right) \, d\vartheta, \quad (5.12)
$$

with $\kappa = \sum_{m=m_1}^{m_2} (\omega_m A_m) / (2\pi)$ with $A_m$ the area of $G_m$ and $g$ as defined (5.10).

**Proof.** Because of properties 5.3.1 and 5.3.2, the boundary value problem in property 5.3.3 is valid for both $ru_r$ and $ru_\varphi - \kappa$. Dividing the solution by $r$ yields the above Poisson integrals.

Note that the term $(\kappa/a)g(\frac{a}{r}, \varphi, \vartheta)$ in (5.12) does not contribute to the integral since

$$
\int_0^{2\pi} g \left( \frac{a}{r}, \varphi, \vartheta \right) \, d\vartheta = 0.
$$
5.3.2 Numerical approach

The numerical integration of the Poisson integrals (5.11) and (5.12) appears to be inaccurate when integrating straightforwardly by a trapezoidal rule, similar to the Poisson integral (5.2). In this section, the source of this inaccuracy is discussed and a solution (similar to that of Anderson [2]) to the problem is presented. For the sake of simplicity, the analysis is performed for a function $f$ being the solution of the boundary-value problem (5.8) in Property 5.3.3.

The function $g$ as defined in (5.10) can be expanded in a Fourier series (see also the proof of Property 5.3.3)

$$g(\theta, \varphi, \theta) = 2g \frac{\cos(\varphi - \theta) - \theta}{1 - 2g \cos(\varphi - \theta) + \theta^2} = \sum_{n=1}^{\infty} \theta^n e^{in(\varphi - \theta)} + \sum_{n=1}^{\infty} \theta^n e^{-in(\varphi - \theta)}, \text{ for } \theta < 1. \quad (5.13)$$

Furthermore, applying a $K$-point trapezoidal rule yields

$$f(r, \varphi) = \frac{1}{K} \sum_{k=1}^{K} f(a, \theta_k) g(\frac{a}{r}, \varphi, \theta_k) - \frac{1}{12} \frac{2\pi}{K^3} \sum_{k=1}^{K} \frac{\partial^2}{\partial \theta^2} \left[ f(a, \theta) g(\frac{a}{r}, \varphi, \theta) \right]_{\theta = \theta_k} =: T_K(fg) + R_K(fg), \quad (5.15)$$

where $\theta_k = 2\pi k/K$ and $\theta_k \in (\theta_{k-1}, \theta_k)$ such that the equality holds.

Since the function $f(a, \theta) g(\theta, \varphi, \theta)$ is periodic in $\theta$, it can be proven [15] that the trapezoidal rule is super-convergent. More specifically: if $fg \in C^{(2n+1)}[0, 2\pi]$, i.e. $fg$ is $2n + 1$ times continuously differentiable, $\frac{\partial^{2k-1}fg}{\partial \theta^{2k-1}}|_{\theta=0} = \frac{\partial^{2k-1}fg}{\partial \theta^{2k-1}}|_{\theta=2\pi}$ for $1 \leq k \leq n$ and $\frac{\partial^{2n+1}fg}{\partial \theta^{2n+1}}|_{\theta=0} \leq M$ for $\theta \in [0, 2\pi]$, then

$$|R_K(fg)| \leq \frac{CM}{K^{2n+1}},$$

where $C$ does not depend on $K$. This implies that application of the trapezoidal rule reveals spectrally accurate results if $fg$ is a smooth function of $\theta$.

In the particular case discussed here, $M = C\varrho$ with $\varrho = \frac{a}{r}$ and $C$ a constant such that

$$|R_K(fg)| \leq \frac{C\varrho}{K^{2n+1}}. \quad (5.16)$$

The remainder term $R_K$ is not the only error made when applying the trapezoidal rule. This can be observed by studying the Fourier series in (5.13) more in detail. The following lemma holds.
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Lemma 5.3.5. Let $K$ be an odd number, i.e. $K = 2L + 1$ for $L \in \mathbb{N}$ and let $\vartheta_k = 2\pi k / K$ for $k = 1, \ldots , K$. Then the function $g$ represented by the Fourier series (5.13), and evaluated at the discrete points $\vartheta_k$, takes the following form

$$
g(\varrho, \varphi, \vartheta_k) = \sum_{l=1}^{L} \varrho^l e^{i(\varphi - \vartheta_k)} + \sum_{l=1}^{L} \varrho^l e^{-i(\varphi - \vartheta_k)} + \frac{(oe^{i\varphi})^{2L+1}}{1 - (oe^{i\varphi})^{2L+1}} \sum_{l=-L}^{L} \varrho^l e^{i(\varphi - \vartheta_k)} + \frac{(oe^{-i\varphi})^{2L+1}}{1 - (oe^{-i\varphi})^{2L+1}} \sum_{l=-L}^{L} \varrho^l e^{-i(\varphi - \vartheta_k)}. \tag{5.17}$$

Proof. In the Fourier series in (5.13), $n$ can be replaced by $n = mK + l$ with $m \geq 0$ and $0 \leq l \leq K - 1$. Furthermore, since

$$e^{in\vartheta_k} = e^{il2\pi k / K} e^{im2\pi k} = e^{il\vartheta_k},$$

and similarly

$$e^{-in\vartheta_k} = e^{-il\vartheta_k},$$

the function $g$ can be written as

$$g(\varrho, \varphi, \vartheta_k) = \sum_{l=1}^{K-1} \varrho^l e^{i(\varphi - \vartheta_k)} + \sum_{m=1}^{\infty} (\varrho^K e^{iK\varphi})^m \sum_{l=0}^{K-1} \varrho^l e^{i(\varphi - \vartheta_k)} + \sum_{m=1}^{\infty} (\varrho^K e^{-iK\varphi})^m \sum_{l=0}^{K-1} \varrho^l e^{-i(\varphi - \vartheta_k)} + \sum_{l=1}^{K-1} \varrho^l e^{i(\varphi - \vartheta_k)} + \sum_{l=1}^{K-1} \varrho^l e^{-i(\varphi - \vartheta_k)} + \frac{(oe^{i\varphi})^{K}}{1 - (oe^{i\varphi})^{K}} \sum_{l=0}^{K-1} \varrho^l e^{i(\varphi - \vartheta_k)} + \frac{(oe^{-i\varphi})^{K}}{1 - (oe^{-i\varphi})^{K}} \sum_{l=0}^{K-1} \varrho^l e^{-i(\varphi - \vartheta_k)}.$$

If $K = 2L + 1$ for $L \in \mathbb{N}$, then

$$g(\varrho, \varphi, \vartheta_k) = \sum_{l=1}^{2L} \varrho^l e^{i(\varphi - \vartheta_k)} + \sum_{l=1}^{2L} \varrho^l e^{-i(\varphi - \vartheta_k)} + \frac{(oe^{i\varphi})^{2L+1}}{1 - (oe^{i\varphi})^{2L+1}} \sum_{l=0}^{2L} \varrho^l e^{i(\varphi - \vartheta_k)} + \frac{(oe^{-i\varphi})^{2L+1}}{1 - (oe^{-i\varphi})^{2L+1}} \sum_{l=0}^{2L} \varrho^l e^{-i(\varphi - \vartheta_k)} + \frac{1}{1 - (oe^{i\varphi})^{2L+1}} \sum_{l=1}^{2L} \varrho^l e^{i(\varphi - \vartheta_k)} + \frac{1}{1 - (oe^{-i\varphi})^{2L+1}} \sum_{l=1}^{2L} \varrho^l e^{-i(\varphi - \vartheta_k)}.$$
The summations can be split up in a part with \( l \leq L \) and a part with \( l \geq L + 1 \). The summation in the latter parts (\( l \geq L + 1 \)) can be reversed (i.e. \( l = 2L, 2L - 1, \ldots, L + 1 \) instead of \( l = L + 1, L + 2, \ldots, 2L \)). Substituting \( \hat{l} = 2L + 1 - l \) and using \( e^{\pm i(2L+1)\theta_k} = 1 \) yields, after some re-arrangement and replacement of \( \hat{l} \) by \( l \), expression (5.17).

The set of functions \( e^{il\theta_k} = e^{2\pi ik/lK} \) for \( l = -L, \ldots, L \), as well as their real-valued components \( \cos(l\theta_k) \) and \( \sin(l\theta_k) \) for \( l = 0, \ldots, 2L \), are called the basic modes. The lemma shows that the infinite number of modes of the continuous function \( g \) reduces to only \( L + 1 \) basic modes when evaluating \( g \) at the \( 2L + 1 \) discrete points \( \theta_k, k = 1, \ldots, 2L + 1 \). This effect is called aliasing [7] and can be illustrated by the following example.

**Example 5.3.6.** Figure 5.5 shows the aliasing effect. There are nine equidistant evaluation points. The function \( \sin(8x) \) is plotted versus \( x \) on the interval \([0, 2\pi]\) with a solid line and its function values at the evaluation points are indicated with symbols. At the discrete evaluation points the function \( -\sin(x) \) (dashed line) takes the same function values. As a consequence, the 8th mode \( (l = 8) \) is interpreted as the 1st mode \( (l = 1) \).

When applying the trapezoidal rule, the aliasing effect leads to additional errors in the first \( L + 1 \) modes as can be found by substituting expression (5.17) in the trapezoidal rule.
in (5.14). Then (with $K = 2L + 1$)

$$T_K(fg) = \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k) \left[ \sum_{l=1}^{L} \varphi^l e^{i(l\varphi - \vartheta_k)} + \sum_{l=-L}^{0} \varphi^l e^{-i(l\varphi - \vartheta_k)} \right] +$$

$$\frac{(qe^{i\varphi})^{2L+1}}{1 - (qe^{i\varphi})^{2L+1}} \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k) \left[ \sum_{l=-L}^{0} \varphi^l e^{i(l\varphi - \vartheta_k)} \right] +$$

$$\frac{(qe^{-i\varphi})^{2L+1}}{1 - (qe^{-i\varphi})^{2L+1}} \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k) \left[ \sum_{l=-L}^{0} \varphi^l e^{-i(l\varphi - \vartheta_k)} \right],$$

(5.18)

with $q = a/r$. The first term in this expression contains the correct first $L + 1$ modes (actually the first $L$ modes, since mode 0 is absent) and the last two terms contain the errors in the first $L + 1$ modes. This leads to the following property.

**Property 5.3.7.** A sharp bound for the error in the Poisson integral (5.9) when applying a $K = 2L + 1$ point trapezoidal rule is given by

$$|f(r, \varphi) - T_K(fg)| \leq C_1 \frac{q}{K^{2n+1}} + C_2 \frac{q^K}{|1 - q^K|},$$

(5.19)

where the first term in the right-hand-side is the error caused by the trapezoidal rule and the second is the aliasing error.

**Proof.** From (5.18) it follows that the error caused by aliasing in mode 0 is given by

$$E_0 = \frac{(qe^{i\varphi})^{2L+1}}{1 - (qe^{i\varphi})^{2L+1}} \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k) + \frac{(qe^{-i\varphi})^{2L+1}}{1 - (qe^{-i\varphi})^{2L+1}} \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k)$$

$$= 2q^{2L+1} \frac{\cos(2L + 1)\varphi - q^{2L+1}}{1 - 2q^{2L+1} \cos(2L + 1)\varphi + q^{2L+1}} \frac{1}{2L + 1} \sum_{k=1}^{2L+1} f(a, \vartheta_k).$$

If $q < 1$ then the errors in the higher modes are of higher-order than $E_0$: the error in mode $m, m > 0$ is

$$E_m = O\left( \frac{q^{2L+1+m}}{1 - q^{2L+1}} \right).$$

So $E_0$ is the leading term of the error caused by aliasing. Together with the estimate (5.16) of $R_K(fg)$ it follows that the overall error in the Poisson integral is best bounded by (5.19) where $C_1$ and $C_2$ are constants.

When $q \ll 1$, the error caused by the aliasing effect (second term) is much smaller than $|R_K|$ (first term) and it causes no trouble. However, when $q \approx 1$ the error becomes very large (much larger than $|R_K|$) and so the approximation of the Poisson integral is not accurate. Fortunately there is an easy way to solve this problem.
5.3. The Poisson integrals

Relation (5.17) suggests that the solution to the problem lies in replacing the function \( g \) by a function \( g_L \) defined as

\[
g_L(\vartheta, \varphi, \vartheta_k) := \sum_{l=1}^{L} g_l e^{il(\varphi - \vartheta_k)} + \sum_{l=1}^{L} g_l e^{-il(\varphi - \vartheta_k)} = \frac{\cos(\varphi - \vartheta_k) - \vartheta}{1 - 2\vartheta \cos(\varphi - \vartheta_k) + \vartheta^2} - \frac{\vartheta^L \cos((L + 1)(\varphi - \vartheta_k)) - \vartheta^{L+1} \cos(L(\varphi - \vartheta_k))}{1 - 2\vartheta \cos(\varphi - \vartheta_k) + \vartheta^2} ,
\]

(5.20)
i.e. \( g_L \) is equal to the first \( L + 1 \) (or actually \( L \) modes because of the absence of mode 0) modes of function \( g \). With this definition, the following property can be proven.

**Property 5.3.8.** If \( g_L \) is defined as in (5.20) then

\[
|f(r, \varphi) - T_K(fg_L)| \leq C_1 \frac{\vartheta}{K^{2m+1}} + C_3 \frac{\vartheta^{L+1}}{(L + 1)^p} .
\]

(5.21)

**Proof.** From the definition of \( g_L \) in (5.20) it follows that

\[
g_L(\vartheta, \varphi, \vartheta_k) = g(\vartheta, \varphi, \vartheta_k) - \vartheta^{L+1} (e^{i(L+1)(\varphi - \vartheta_k)} + e^{-i(L+1)(\varphi - \vartheta_k)}) + O(\vartheta^{L+2}).
\]

Applying the trapezoidal rule to \( fg_L \) yields

\[
f(r, \varphi) - T_K(fg_L) = \frac{(\vartheta e^{i\varphi})^{L+1}}{K} \sum_{k=1}^{K} f(a, \vartheta_k) e^{-i(L+1)\vartheta_k} + \frac{(\vartheta e^{-i\varphi})^{L+1}}{K} \sum_{k=1}^{K} f(a, \vartheta_k) e^{i(L+1)\vartheta_k} + O(\vartheta^{L+2}) + R_K .
\]

The sums in the first two terms can be interpreted as an approximation to the \((L+1)\)st and \(- (L + 1)\)st Fourier coefficients of the function \( f, c_{L+1} \) and \( c_{-L-1} \), respectively. According to Briggs and Henson [7], it follows that if \( f \) and its first \( p - 1 \) derivatives with respect to \( \vartheta \) are 2-\( \pi \)-periodic and continuous on \([0, 2\pi]\) and if the \( p \)th \( \vartheta \)-derivative is bounded, then

\[
|c_k| \leq \frac{C}{|k|^p} .
\]

Together with the estimate (5.16) of \( R_K(fg) \) it follows that the overall error in the Poisson integral is best bounded by (5.21) where \( C_1 \) and \( C_3 \) are constants.

For the velocity components now simply follows

\[
u_r(r, \varphi) = \frac{a}{K^r} \sum_{k=1}^{K} u_r(a, \vartheta_k) g_L(\frac{a}{r}, \varphi, \vartheta_k) + E_r ,
\]

\[
u_\varphi(r, \varphi) = \frac{\kappa}{r} + \frac{a}{K^r} \sum_{k=1}^{K} \left( u_\varphi(a, \vartheta_k) - \frac{\kappa}{a} \right) g_L(\frac{a}{r}, \varphi, \vartheta_k) + E_\varphi ,
\]

For the velocity components now simply follows
Figure 5.6: The magnitude of the error $E_r$ (a) and $E_\varphi$ (b). The dashed lines indicate the magnitude of the errors in the Poisson integrals when applying the trapezoidal rule to the original kernel (i.e. $g$); the solid lines indicate the magnitude of the errors when applying the trapezoidal rule to the modified kernel (i.e. $g_\ell$).

where

$$|E_r| \leq A_1 \frac{\varrho^2}{K^{2n+1}} + A_2 \frac{\varrho^{L+2}}{(L+1)^{p}},$$

$$|E_\varphi| \leq A_3 \frac{\varrho^2}{K^{2n+1}} + A_4,$$

with $A_i, i = 1, \ldots, 4$, constant. In Figure 5.6 the magnitude of the errors $E_r$ (left) and $E_\varphi$ (right) are plotted both for an unmodified kernel (dashed lines) and for a modified kernel (solid lines). The test problem is the same problem Anderson used [2], i.e. a ring of radius $a = 2$ centred around $(r, \varphi) = (0, 0)$ and a particle of strength $\kappa = 1$ located at $(r, \varphi) = (\frac{1}{2} \sqrt{2}, \pi/3)$. In the figure, the magnitude of the errors are plotted as a function of the distance to the centre of the ring. The evaluation points are all located at the positive $x$-axis ($\varphi = 0$). Indeed, at evaluation points close to the ring the modified kernels give better results than the unmodified kernels. The behaviour of the error agrees very well with the predicted behaviour. The cusps that are present in the figure, are caused a change in sign of the error.
5.4 Construction of finest level outer-rings

The last part of the method consists of constructing outer-rings at the finest level. At this level, the contributions of the patches of uniform vorticity in each box have to be determined at the outer-rings. Consider the piecewise-uniform vorticity distribution as depicted in Figure 5.7. In the left part of the figure, some regions \( \mathcal{G}_m \), \( m = 0, \ldots, M \) of uniform vorticity \( \omega_m \) are shown. These regions are again nested such that \( \mathcal{G}_M \subset \mathcal{G}_{M-1} \subset \ldots \subset \mathcal{G}_0 \). \( \mathcal{G}_0 \) is equal to the domain on which the HEM is applied (containing all regions of vorticity) and as in Chapter 3, \( \omega_0 \) is assumed to be zero. In the left part of the figure, the grid lines of the finest level are indicated with dashed lines. The contribution of the vorticity distribution inside the grey box \( b \) to the velocity at the integration points on the outer-ring of that box are now given by

\[
\mathbf{u}(\mathbf{x}) = -\sum_{m=1}^{M} \frac{\omega_m}{2\pi} \int_{\partial(\mathcal{G}_m \mid b)} \ln \| \mathbf{x} - \mathbf{x}' \| \, d\mathbf{x}',
\]

(5.22)
where $G_m|_b$ is the region of $G_m$ restricted to box $b$. In the example as shown in Figure 5.7, this results in

$$u(x) = -\frac{1}{2\pi} \left[ \omega_{M-3} \int_{C_0C_1C_2C_3C_0} + \omega_{M-2} \int_{P_5C_0C_1C_2P_3} + \omega_{M-1} \int_{P_3C_0C_1P_3P_5} + \omega_M \int_{P_1C_1P_3P_1} \right] \ln \|x - x'\| \, dx',$$

with the points $C_0, \ldots, C_3$ and $P_1, \ldots, P_5$ as depicted in the right part of the figure. Thus, for the construction of the finest level outer-ring of a certain box it is necessary to determine the contributions from two different sources: the first source being the parts of the contours inside the box and the second source being the correct parts of the box boundaries. In practice, it turns out to be convenient to carry out this procedure for all boxes per contour. The actual algorithm is rather involved and can be found in Appendix A.

### 5.5 Numerical experiments and discussion

In this section some numerical experiments are discussed in order to demonstrate the accuracy and the speed-up of the hierarchical-element method.

The accuracy is tested using the same initial configuration as in Example 3.4.3 in Chapter 3, i.e. the evolution of a monopole into a tripole. In this example, three contours are used. The initial elliptical disturbance of the shape of the monopolar vortex causes the monopole to deform, resulting in a tripolar structure. With the new method, four simulations have been performed for various choices of the number of integration points $K$ on the rings, namely $K = 9, 17, 25,$ and $33$. Obviously, the higher the value of $K$, the more accurate the results should become. During the simulations, the number of levels is automatically adapted (see Section 5.2) and $l_f$ increases from $l_f = 1$ at the beginning up to $l_f = 5$ at later stages. In Figure 5.8 the (relative) difference in area, enclosed by the contours, relative to the conventional method, is plotted as a function of time $t$ for the three different contours. It is clear from this figure, that $K = 9$ yields the largest difference. In fact, in that case, the differences are larger than the errors in the area caused by the time integration (see Figure 3.19) and it can be concluded that the method is not accurate enough with this value of $K$. Note that the errors for $K = 33$ tend to become larger than those for $K = 17$ and $K = 25$. The reason for this is not quite clear.

It is interesting to compare the shapes of the contours of the present simulation with those produced by the conventional method. In Figure 5.9a, the contours at time $t = 12$ for both simulations are plotted together in the same graph. The contours of the conventional method are plotted with thick grey lines, whereas the contours of the new code with $K = 9$ are plotted with thin black lines. To make the difference more clear, an enlarged view of a part of the tripole is given in Figure 5.9b. Obviously, the contours are less smooth for the case $K = 9$ than for $K = 17$ and there are substantial differences with the contours produced by the conventional method. Using these results in further calculations will yield
even larger differences at later stages. These results confirm that the results for $K = 9$ are not reliable.

For $K = 17$, however, the calculations are much better, as can be observed from both Figure 5.8 and the bottom pictures of Figure 5.9. For this value of $K$, the differences in the area remain smaller than the errors caused by the time integration. Figure 5.9c, shows again both the contours obtained with the conventional method and the ones obtained by HEM. Figure 5.9d shows an enlarged view of the same part of the tripole as for $K = 9$. Now, there is no visible difference in the shape of the contours and also further calculations did not reveal any; so for $K = 17$, the method is much more reliable.

For higher values of $K$ ($K = 25$ and $K = 33$), the differences in the area are smaller than or comparable to the $K = 17$ case. Although not shown here, it may be clear that the contours again show no visible differences with the ones produced by the conventional method.

The higher accuracy of the method for larger values of $K$ has a price, of course. The higher $K$, the more computationally expensive the method becomes. However, the speed-up is still significant as can be observed from Figure 5.10 and Figure 5.11. In Figure 5.10, the CPU-time (in seconds on a Silicon Graphics Power Challenge with 14 processors R8000, but only one of them was used) for one calculation of the velocity field, is plotted as a function of the total number of nodes. For these simulations, a circular vortex patch is used with five contours. The radii of the contours are $1.0, 2.0, 3.0, 4.0$, and $5.0$. The domain of the multipole method is chosen $[-6.0, 6.0] \times [-6.0, 6.0]$. The five contours all have the same fixed number of nodes in each calculation. Computations are performed for $100, 200, \ldots, 800, 1000, 1200, \ldots, 4000, 5000, 6000$ nodes per contour with number of levels $l_f = 1, 2, \ldots, 6$ and $K = 25$. The conventional method was tested for the same numbers
Figure 5.9: The shape of the tripole at $t = 12$. In all pictures, the contours of the tripole computed with the conventional method (thick grey line) and with the HEM method (thin black line) are drawn in the same graph. In the top pictures, $K = 9$ is used in the HEM method; in the bottom pictures $K = 17$ is used. In the pictures on the right, an enlarged view of a part of the tripole is shown.
5.5. Numerical experiments and discussion

Figure 5.10: The CPU-time, for calculating the velocities for a patch with 5 contours, as a function of the number of nodes $N$ for several values of $l_f$ (dashed lines), and the conventional method (solid line).

Figure 5.11: The speed-up factor as a function of the total number of nodes $N$ for the most optimal choice of $l_f$.

of nodes per contour. Note that in Figure 5.10 both the horizontal and vertical axis have logarithmic scales. The curve corresponding to the conventional method is drawn with a solid line, the curves for the HEM method are all drawn with differently dashed lines.

The curve corresponding to the conventional is a straight line and closer inspection reveals that this agrees with the $O(N^2)$ behaviour. Furthermore, it can be observed that the larger the value of $l_f$ is, the slower the CPU-time increases with $N$. But also: the larger $l_f$, the higher the CPU-time is for small values of $N$. From these two observations it follows that the value of $l_f$ that should be chosen in order to have maximum speed-up, depends on the number of nodes (as expected, see Section 5.2): the larger the number of nodes, the higher $l_f$. In Figure 5.11, the maximum speed-up factor is plotted as a function of $N$. This factor was determined by dividing the CPU-time needed by the conventional method by the CPU-time needed by the HEM, this for the most optimal choice of $l_f$ for $N$ nodes. Obviously, the speed-up of the HEM method is much larger for high values of $N$ (speed-up factor $\approx 25$ for $3 \cdot 10^4$ nodes!) than for small values of $N$ (speed-up factor $\approx 5$ for $5 \cdot 10^3$ nodes). The speed-up factor increases almost linear with the number of nodes, which means that if $l_f$ is chosen optimal in the computations, the method is almost of order $O(N)$. This is a substantial improvement compared to the conventional method.
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6

Contour Dynamics with Non-Uniform Background Vorticity

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6.4 Discussion
6.1 Introduction

In this chapter it is shown how contour dynamics can be used for the evolution of vortices in the presence of non-uniform background vorticity, like the case of a $\beta$-plane or a $\gamma$-plane. As pointed out in Chapter 2, the relative vorticity $\omega$ is not a conserved quantity in these cases. This has severe consequences for applying contour dynamics since an initially piecewise-uniform distribution of vorticity will not remain piecewise-uniform throughout time, which is a prerequisite for applying contour dynamics (see Chapter 3). Fortunately, the sum of the relative vorticity and the planetary or background vorticity, i.e. the absolute vorticity (see Chapter 2), is conserved so that contour dynamics can be used to deal with this kind of flow problems. For example, to use the method for a numerical simulation of the evolution of a monopole on a $\gamma$-plane, the initial vorticity distribution of the monopole should be added to the background vorticity of the $\gamma$-plane and the sum of these distributions is then replaced by a piecewise-uniform distribution. Contour integral representations can be used to determine the absolute velocity at points on the contours (like in standard contour dynamics simulations). To obtain the relative velocities, the velocity field induced by the background vorticity field has to be subtracted. In Section 6.2, the method is explained in more detail. In Section 6.3, the method is applied to a physically relevant problem: the evolution of a tripole on a $\gamma$-plane. The results obtained are compared to results obtained from laboratory experiments.

6.2 Piecewise-Uniform absolute vorticity distribution

In this section it is demonstrated how contour dynamics can be employed to numerically simulate the evolution of vortices on the $\gamma$-plane. Although attention is restricted to the $\gamma$-plane here, similar procedures can be followed for simulating flows on a $\beta$-plane or even on a rotating sphere.

6.2.1 The contour dynamics model for a $\gamma$-plane

In Chapter 3 it has been demonstrated how standard contour dynamics works. As was pointed out there, conservation of vorticity is essential. On the $\gamma$-plane, and in general for flows with non-uniform background vorticity, it is more difficult to use contour dynamics since the relative vorticity is not conserved (see Chapter 2). Fortunately, the absolute vorticity $q$, as defined in (2.19) is conserved in these cases, ensuring that an initially piecewise-uniform distribution, say $\tilde{q}$, will remain piecewise-uniform during time.

Consider now, for example, the initial absolute vorticity distribution of a circular cyclonic monopole on a $\gamma$-plane as depicted in Figure 6.1a or Figure 6.1b. The relative vorticity $\omega$ is equal to zero outside the vortex, yielding that the absolute vorticity is quadratic in $r := \|x\|$. As in the case of the $f$-plane, the relative motion depends on the relative vorticity $\omega(x,t)$:

$$\nabla^2 \psi(x,t) = -\omega(x,t) = -q(x,t) + f(x),$$  \hspace{1cm} (6.1)
where \( f(x) = f_0 - \gamma(x^2 + y^2) \) (see (2.21)) in the \( \gamma \)-plane approximation (note that the constant \( f_0 \) is not relevant for the dynamics of the flow as can be observed from equation (2.22)). Consider now the circular domain

\[
\mathcal{G} := \{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq R_0^2 \},
\]

where \( R_0 \) is assumed to be large enough (the meaning of this is explained later in Section 6.2.2). The problems considered here are assumed to have a non-zero initial relative vorticity distribution in a bounded region \( \mathcal{B} \) inside \( \mathcal{G} \). Outside \( \mathcal{B} \) the initial relative vorticity is assumed to be zero to ensure that initially no relative vorticity is present near the boundary of \( \mathcal{G} \). So in the case of cyclonic monopoles on a \( \gamma \)-plane as depicted in Figure 6.1, \( R_0 \) should be chosen such that the monopole is located completely inside \( \mathcal{G} \) and sufficiently far from its boundary to ensure that the monopole and the relative vorticity created by it at later moments in time will not influence the boundary of \( \mathcal{G} \) too much (see Section 6.2.2).

Then it follows that \( \omega(x, t = 0) = 0 \) for \( x \) outside \( \mathcal{G} \). Now assume that this relative vorticity remains small or even zero outside \( \mathcal{G} \) for a finite time \( T \) (see Section 6.2.2), then from (6.1) it follows

\[
\psi(x, t) = -\iint_{\mathcal{G}} \omega(x', t)G(x; x') \, dx' \, dy'.
\]  

(6.2)

Here, \( G(x; x') = \frac{1}{2\pi} \ln \|x - x'\| \) is Green's function belonging to the infinite two-dimensional
plane as defined in (3.3). Using $\omega = q - f$ and taking the $x$- and $y$-derivatives of $\psi$ yields

\[
\begin{align*}
\frac{\partial G}{\partial y} dx' dy' + \int_G f(x') \frac{\partial G}{\partial y} dx' dy', \\
\frac{\partial G}{\partial x} dx' dy' \end{align*}
\]

\[
\begin{align*}
\frac{\partial G}{\partial x} dx' dy' - \int_G f(x') \frac{\partial G}{\partial x} dx' dy', \\
\frac{\partial G}{\partial x} dx' dy'.
\end{align*}
\]

Since $\frac{\partial G}{\partial x} = -\frac{\partial G}{\partial y}$ and $\frac{\partial G}{\partial y} = -\frac{\partial G}{\partial y}$, it follows that

\[
\begin{align*}
u = & + \int_G q(x', t) \frac{\partial G}{\partial y} dx' dy' - \int_G f(x') \frac{\partial G}{\partial y} dx' dy' =: u_q - u_f, \\
u = & - \int_G q(x', t) \frac{\partial G}{\partial x} dx' dy' + \int_G f(x') \frac{\partial G}{\partial x} dx' dy' =: v_q - v_f.
\end{align*}
\]

The contributions $u_f$ and $v_f$ from the background vorticity $f(x) = f_0 - \gamma(x^2 + y^2)$ to the velocities $u$ and $v$ can be determined analytically.

**Property 6.2.1.** The contributions $u_f$ and $v_f$ are given by

\[
\begin{align*}
u_f = & \begin{cases}
- \left( \frac{f_0 R_0^2}{2} - \frac{\gamma R_0^4}{4} \right) \frac{y}{x^2 + y^2}, & x \notin \mathcal{G}, \\
- \frac{f_0}{2} y + \frac{\gamma}{4} y(x^2 + y^2), & x \in \mathcal{G}
\end{cases}, \\
u_f = & \begin{cases}
\frac{f_0 R_0^2}{2} - \frac{\gamma R_0^4}{4} \frac{x}{x^2 + y^2}, & x \notin \mathcal{G}, \\
\frac{f_0}{2} x - \frac{\gamma}{4} x(x^2 + y^2), & x \in \mathcal{G}
\end{cases}.
\end{align*}
\]

**Proof.** It is convenient to introduce polar coordinates. Then $f(r) = f_0 - \gamma r^2$ in the $\gamma$-plane approximation. Denote by $u_f$ and $u_{f_r}$ for the moment the radial and azimuthal velocity caused by the background vorticity $f$, respectively. Then it easily follows that $u_{f_r} = 0$. The azimuthal velocity $u_{f_r}$ is given by

\[
\begin{align*}
u_{f_r}(r) = & - \frac{\partial \psi}{\partial r} \\
= & \frac{1}{4\pi} \int_G f(\rho) \frac{\partial}{\partial r} \left[ \ln(r^2 - 2r \rho \cos(\varphi - \vartheta) + \rho^2) \right] r \, d\rho \, d\vartheta \\
= & \frac{1}{2\pi} \int_G f(\rho) \frac{r - \rho \cos(\varphi - \vartheta)}{r^2 - 2r \rho \cos(\varphi - \vartheta) + \rho^2} r \, d\rho \, d\vartheta \\
= & \frac{1}{2\pi} \int_0^{2\pi} f(\rho) \rho \int_0^R \frac{r - \rho \cos(\vartheta)}{r^2 - 2r \rho \cos(\varphi - \vartheta) + \rho^2} d\vartheta \, d\rho.
\end{align*}
\]
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The integral

\[ I(r, \rho) := \int_0^{2\pi} \frac{r - \rho \cos(\theta)}{r^2 - 2r \rho \cos(\theta) + \rho^2} d\theta, \]

can be determined by use of complex contour integration. For this purpose, the cosine is written in complex notation

\[ \cos(\theta) = \frac{1}{2} (e^{i\theta} + e^{-i\theta}), \]

and both the numerator and the denominator of the integrand are multiplied by \( e^{i\theta} \). Then

\[
I(r, \rho) = \frac{1}{2r} \int_0^{2\pi} \frac{e^{2i\theta} - 2(\rho/r) e^{i\theta} + 1}{e^{2i\theta} - (\rho/r + \rho/r) e^{i\theta} + 1} \frac{de^{i\theta}}{ie^{i\theta}}
= \frac{1}{2ir} \int_{|z|=1} \frac{z^2 - 2(\rho/r)z + 1}{z(z - \rho/r)(z - \rho/r)} dz
= \frac{1}{2ir} \int_{|z|=1} \frac{dz}{z} - \frac{1}{2ir} \int_{|z|=1} \frac{dz}{z - \rho/r} + \frac{1}{2ir} \int_{|z|=1} \frac{dz}{z - \rho/r}.
\]

These integrals can be determined by calculating the residuals associated with the poles. Here, three cases have to be distinguished: \( \rho < r, \rho = r \) and \( \rho > r \). By carefully determining these residuals it follows that

\[
I(r, \rho) = \begin{cases} \frac{2\pi}{r}, & \rho < r, \\ \frac{\pi}{r}, & \rho = r, \\ 0, & \rho > r. \end{cases}
\]

For the azimuthal velocity \( u_{f\phi} \) it now simply follows by straightforward integration

\[
u_{f\phi} = \begin{cases} \frac{f_0 R_0^3}{2r} - \frac{\Lambda R_0^4}{4r}, & r > R_0, \\ \frac{f \rho}{2} - \frac{\Lambda r^3}{4}, & r \leq R_0. \end{cases}
\]  

This results in the expressions for \( u_f \) and \( v_f \) as given in (6.3).

For the determination of \( u_{\psi} \), contour dynamics can be used. For this purpose the continuous distribution \( q \) is replaced by a piecewise-uniform distribution \( \tilde{q} \)

\[
\tilde{q}(x, t) = \sum_{l=0}^{m} q_l, \quad x \in G_m(t) \setminus G_{m+1}(t), \quad m = 0, \ldots, M, \]

(6.5)
Figure 6.2: Piecewise-uniform distribution of the absolute vorticity in the case of an isolated (a) and a non-isolated (b) cyclonic monopole on the $\gamma$-plane. $\mathcal{G}$ is the grey area in this figure.

where the regions $\mathcal{G}_m(t)$ are nested such that $\mathcal{G}_{m+1}(t) \subset \mathcal{G}_m(t)$, $\mathcal{G}_0 = \mathbb{R}^2$ and $\mathcal{G}_{M+1}(t) = \emptyset$, i.e. $\mathcal{G}_{M+1}(t)$ is empty and $q_0 = 0$ (see also Chapter 3). Figure 6.2 shows two examples of a piecewise-uniform distribution of the absolute vorticity in case of cyclonic monopoles on the $\gamma$-plane. The grey area in this figure is the circular domain $\mathcal{G}$.

Similar to (3.5), the velocity $\bar{u}_q$ is given by

$$\bar{u}_q(x, t) = - \sum_{m=0}^{M} q_m \oint_{\mathcal{C}_m(t)} G(x; x') dx'$$

The relative velocity field now follows from $\bar{u} = \bar{u}_q - u_f$. Note that the contour $\mathcal{C}_0$ should be 'far enough' from the monopole, i.e. such that $\mathcal{C}_0$ will not deform very much (i.e. relative to the motion of the vortex) during the simulations in order to keep the results accurate (see next section).

6.2.2 Accuracy aspects

In this section, some accuracy aspects of the method described in the previous section are discussed.
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Figure 6.3: The piecewise-uniform distribution $\tilde{f}$ (solid line) and the continuous profile $f$ (dashed line) in the case of $M = 20$ contours, $\gamma = 0.02$, $R_0 = 10$ and $f = \gamma R_0^2 - \gamma r^2$ (a) and their difference $\tilde{f} - f$ (b).

The first aspect concerns the accuracy of the piecewise-uniform distribution of $q$ or, more specifically, the piecewise-uniform distribution of the region around the vortex or vortices as an approximation to the background vorticity. Consider for this purpose a $\gamma$-plane, initially without any relative vorticity, i.e. $\omega(x, 0) \equiv 0$ for all $x \in \mathbb{R}^2$. In the exact case without any numerical approximation, nothing will happen so the relative vorticity will remain zero throughout time. If this problem is simulated by the contour dynamics method, then the continuous profile of the background $f$ (which is equal to $q$ in this case) is replaced by a piecewise-uniform distribution $\tilde{f}$ as shown in Figure 6.3a. In this figure, $f = f_0 - \gamma r^2$, with $\gamma = 0.02$, $R_0 = 10$ and $f_0 = \gamma R_0^2$. The value of $f_0$ is chosen so that $f = 0$ on the boundary of the domain, but this is not relevant for the computations. The jumps are chosen uniform, and equal to $\Delta f = \gamma R_0^2 / M$ with $M$ the number of jumps ($M = 20$ in Figure 6.3).

The (approximate) absolute velocities at the nodes on the contours are found by the contour dynamics procedure, and the relative velocities are found by subtracting the analytical solution (6.3) from the absolute velocities. This procedure causes an error in the relative vorticity $\omega$, and consequently $\omega$ is not exactly zero inside the computational domain. In fact, this error in relative vorticity is given by the solid lines in Figure 6.3b, when the errors due to interpolation of the contours are neglected. In this figure, the difference
between the continuous profile of $f$ (dashed line in Figure 6.3a) and the piecewise-uniform distribution $\tilde{f}$ (solid line in Figure 6.3a) is plotted as a function of $r$. It is clear that $|\tilde{f} - f| \leq \Delta f/2$. This error in the relative vorticity will, obviously, generate an error in the relative velocity field. Since the flow problem is axisymmetric, this error has a azimuthal component only. In the following, an analytic expression for this azimuthal component is derived, but first the following property is proven.

**Property 6.2.2.** Let the piecewise-uniform distribution

$$\tilde{f} = \sum_{l=1}^{m} f_l, \quad x \in \mathcal{G}_m \setminus \mathcal{G}_{m+1}, \quad m = 0, \ldots, M,$$

approximate the continuous distribution $f = \gamma R_0^2 - \gamma r^2$, for $r \leq R_0$ where the regions $\mathcal{G}_m$ are given by $\mathcal{G}_m = \{(r, \varphi) \in \mathbb{R}^2 | r \leq r_m\}$ with $R_0 \geq r_1 > r_2 > r_3 > \ldots > r_M$, $\mathcal{G}_0 = \mathbb{R}^2$ and $\mathcal{G}_{M+1} = \emptyset$. Then the absolute azimuthal velocity $\tilde{u}_{\varphi}$ is given by

$$\tilde{u}_{\varphi} = \frac{1}{2} \left( \sum_{l=1}^{m} f_l r + \sum_{l=m+1}^{M} f_l r_l^2 \right), \quad r_{m+1} \leq r \leq r_m. \quad (6.6)$$

**Proof.** Each region $\mathcal{G}_m$, $m = 1, \ldots, M$ can be considered a Rankine vortex, i.e. a circular region of uniform vorticity. Therefore, the contribution of region $\mathcal{G}_m$ is given by

$$\tilde{u}_{\varphi}^m = \begin{cases} \frac{f_m}{2} r, & r \leq r_m, \\ \frac{f_m r_m^2}{2r}, & r \geq r_m. \end{cases}$$

The total absolute azimuthal velocity $\tilde{u}_{\varphi}$, now simply follows by summing the appropriate contributions from all regions $\mathcal{G}_m$. Note that this property also follows from Property 6.2.1. 

Like in Figure 6.3, the jumps $f_m$ are taken uniform, i.e. $f_m = \Delta f = \gamma R_0^2/M$. The radii $r_m$ are chosen such that the circulation of region $\mathcal{G}_m$ is equal to the circulation of the layer of the continuous distribution between the values $f = (m - 1)\Delta f$ and $f = m\Delta f$ (see also Figure 6.4). The radii $r_m$ are then given by

$$r_m = \sqrt{\frac{M - m + \frac{1}{2}}{M} R_0}. \quad (6.7)$$

Then it follows from Property 6.2.2 that

$$\tilde{u}_{\varphi} = \frac{\Delta f}{2} \left( \sum_{l=1}^{m} r + \sum_{l=m+1}^{M} \frac{r_l^2}{r} \right) = \frac{\Delta f}{2} \left( m r + \frac{(M - m)^2 R_0^2}{2Mr} \right), \quad r_{m+1} \leq r \leq r_m. \quad (6.8)$$
If \( r_{m+1} \leq r \leq r_m \), then \( r \) can be written as

\[
r(\xi) = \sqrt{\frac{M - m + \frac{1}{2} - \xi}{M}} R_0, \quad 0 \leq \xi \leq 1.
\]

(6.9)

Note that \( r_m = r(0) \) and \( r_{m+1} = r(1) \). With this expression for \( r \), the following lemma can be proven.

**Lemma 6.2.3.** The error in the azimuthal velocity component, due to the piecewise-uniform distribution of \( f \) is given by

\[
E := u_{f,\varphi} - \tilde{u}_{f,\varphi} = \frac{\Delta f^2 (\xi - \frac{1}{2})^2}{4 \gamma} \frac{1}{r(\xi)} , \quad r_{m+1} \leq r(\xi) \leq r_m.
\]

(6.10)

Proof. With the definition (6.9) of \( r(\xi) \) and the analytical expression (6.4) it follows that the exact azimuthal component \( u_{f,\varphi} \) of \( u_f \) is given by

\[
u_{f,\varphi} = \frac{\gamma R_0^3}{4} \left( \frac{M - m + \frac{1}{2} - \xi}{M} \right)^{\frac{1}{2}} \left( \frac{M + m - \frac{1}{2} + \xi}{M} \right).
\]

Furthermore, after substitution of (6.9) in (6.8) and some rearrangement, it follows that

\[
\tilde{u}_{f,\varphi} = \frac{\gamma R_0^3}{4} \left( \frac{M - m + \frac{1}{2} - \xi}{M} \right)^{\frac{1}{2}} \left( \frac{M + m - \frac{1}{2} + \xi}{M} \right) + \frac{(\xi - \frac{1}{2})^2}{M(M - m + \frac{1}{2} - \xi)}.
\]
Figure 6.5: The error $E$ in the azimuthal component of the relative velocity as a function of $r$ on both a linear (a) and a logarithmic scale (b) for several values of $M$ and the same values of $f_0$, $\gamma$ and $R_0$ as in Figure 6.3.

so that

\[ \hat{u}_r - u_r = \frac{\gamma R_0^3}{4M\sqrt{M}} \frac{(\xi - \frac{1}{2})^2}{\sqrt{M-m+\frac{1}{2} - \xi}}. \]

Finally, the definitions of $r(\xi)$ and $\Delta f$ can be used to obtain (6.10).

Apparently, the error in the azimuthal velocity component is of order $O(\Delta f^2/r)$. This may suggest that the error near $r = 0$ is unbounded. Fortunately, it follows from the definition of $r(\xi)$ that $\xi = \frac{1}{2}$ and $m = M$ when $r(\xi) = 0$, so the error vanishes for $r = 0$. In Figure 6.5, the error is plotted as a function of $r$ for several values of $M$ on both a linear scale (a) and a logarithmic scale (b). The $O(\Delta f^2/r)$ is clearly visible as well as the quadratic behaviour between two subsequent contours. Furthermore, it is also clear that by doubling the number of contours $M$, the (maximum) error becomes four times as small, since $\Delta f$ becomes twice as small.

Although the error vanishes at $r = 0$, the error near the innermost contour is bigger than for larger values of $r$. Therefore, in the numerical simulations the density of contours is chosen higher in the centre of the computational domain.
Note that in the case where no vortices are present on the $\gamma$-plane, this velocity error (6.10) can be avoided. Indeed, by subtracting (6.6) from the absolute velocities instead of the analytical expression (6.3), the error vanishes, yielding zero relative velocities everywhere in the computational domain. However, $\tilde{u}_{f\varphi}$ is not a very smooth function since the $r$-derivative of it is discontinuous. In problems where the contours deform, for example, due to the presence of vortices, the contours do not coincide with the circles where the $r$-derivative of $\tilde{u}_{f\varphi}$ is discontinuous. Because of this, $\tilde{u}_{f\varphi}$ induces small disturbances in the shape of the contours so that the expression (6.3) is to be preferred to (6.6).

The second accuracy aspect, considered here, concerns the deformation during the simulation of the outermost contour $C_1$. This contour is initially a circle of radius $r_1$ given by (6.7). It can be expected that, the larger $R_0$ is, the smaller the deformations of this contour are and consequently the more accurate the results become. However, for larger values of $R_0$, the number of contours increases, thus requiring much more computational time. So, the question is for which minimum value of $R_0$, the results are accurate enough, i.e. the errors caused by the deformation of the outermost contour are sufficiently smaller than the overall error caused by the spatial discretization.

This question is very hard to answer, since the influence of the vortices within $\mathcal{G}$ and, moreover, the influence of the relative vorticity created by them, on the outermost contour can hardly be estimated in advance. It is, however, possible to give an a priori bound for the relative deformations, so that during the simulations it can be checked whether $R_0$ was chosen large enough. It might even be possible to enlarge the domain by adding one or more contours during a computation if that would be necessary. However, this option is currently not implemented in the present program.

Assume that during a computation, the outermost contour of the computational domain is deformed in such a way that it can be described by

$$r(t, \varphi) = r_1(1 + \varepsilon(\varphi, t)), \quad 0 \leq \varphi < 2\pi,$$

where $r$ is the radial distance to the centre of the domain, $\varphi$ is the azimuthal angle with respect to the positive $x$-axis and $\varepsilon$ is a function of $\varphi$ and time $t$ and is assumed to be small, i.e. $|\varepsilon(\varphi, t)| \ll 1$. The following lemma shows how small $|\varepsilon|$ and thus the relative deformations of the outermost contour actually should be in order to ensure sufficiently accurate results.

**Lemma 6.2.4.** If the disturbance of the outer contour can be described by (6.11) and

$$|\varepsilon(\varphi, t)| \leq -1 + \sqrt{1 + \frac{\Delta q}{2|\gamma|r_1^2}} = -1 + \sqrt{1 + \frac{\Delta q M}{2|\gamma|(M - \frac{1}{2})R_0^2}} = \frac{\Delta q M}{4|\gamma|(M - \frac{1}{2})R_0^2} + O \left( \frac{\Delta q}{|\gamma| R_0^2} \right)^2,$$

then

$$|\omega(r(t, \varphi), \varphi, t)| \leq \frac{\Delta q}{2},$$

(6.12)

(6.13)
where $\Delta q$ is the jump in absolute vorticity near the boundary of the computational domain.

Proof. The proof simply follows from conservation of absolute vorticity. Since $\omega$ is assumed to be zero on the outer contour at $t = 0$, conservation of absolute vorticity yields

$$f_0 - \gamma r_1^2 = \omega(r(t, \varphi), \varphi, t) + f_0 - \gamma r_1^2(1 + \varepsilon(\varphi, t))^2.$$

From this it follows that

$$\omega(r(t, \varphi), \varphi, t) = \gamma r_1^2(2\varepsilon(\varphi, t) + \varepsilon^2(\varphi, t)).$$

Using (6.12) simply yields

$$|\omega(r(t, \varphi), \varphi, t)| \leq |\gamma r_1^2(2\varepsilon(\varphi, t) + \varepsilon^2(\varphi, t))|$$

$$\leq |\gamma r_1^2 \frac{\Delta q}{2\gamma r_1^2 r_1^2}$$

$$= \frac{\Delta q}{2}.$$

This lemma states, that if $\varepsilon$ satisfies (6.12) during a simulation, then the relative vorticity generated by the perturbation of the outer contour remains smaller than the error in the relative vorticity caused by the piecewise-uniform distribution discussed previously. In case of uniform jumps in the absolute vorticity, $\Delta q$ is given by $\Delta q = \gamma R_0^2 / M$ with $M$ the number of jumps necessary to approximate the background $f$ accurately.

### 6.2.3 A numerical experiment

In this section, the method is tested for the following test problem. A circular monopole with relative vorticity distribution

$$\omega(r) = \frac{\Gamma}{\pi R^2} \exp \left( -\frac{r^2}{R^2} \right),$$

is placed on a $\gamma$-plane. A vortex with such vorticity distribution is often called a Lamb vortex. The azimuthal component of the velocity field induced by a Lamb vortex is given by

$$u_\varphi(r) = \frac{VR}{r} \left( 1 - \exp \left( -\frac{r^2}{R^2} \right) \right),$$

where $V = \Gamma / (2\pi R)$. In the numerical simulation the length scale $R$ is chosen to be $R = 0.466$. With this particular value of $R$, $\omega$ is approximately zero, i.e. smaller than $1\%$ of its maximum, for $r > 1$. The strength $\Gamma$ of the vortex is chosen $\Gamma = 3\pi R^2 / 2$. Furthermore, the initial position of the vortex is given by $x = (2.5, 2.5)$. The value of
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Figure 6.6: Three plots showing the initial situation of the Lamb monopole on the $\gamma$-plane. Plot (a) shows the absolute vorticity contours in the whole computational domain, while (b) and (c) show the absolute vorticity distribution and the relative vorticity distribution, respectively, in only a part of the computational domain.

$\gamma$ is taken as $\gamma = 0.02$ and the radius of the computational domain is $R_0 = 12.0$ so that $f_0 = \gamma R_0^2 = 2.88$. The total number of contours is 33, and 23 of them are used to approximate the background vorticity $f$. In Figure 6.6a the contours in the computational domain are drawn (the domain shown is $[-12,12] \times [-12,12]$). As can be observed, the density of contours is higher around the centre of the domain. Figure 6.6b shows only a part of the computational domain ($[-2,5] \times [-5,2]$) and Figure 6.6c shows a contour plot of the relative vorticity distribution $\omega$ in the same part of the domain. Positive values of $\omega$ are represented by solid lines whereas negative values are represented by dashed lines. Contours are plotted for $\omega = -0.3, -0.2, -0.1, 0.1, 0.3, \ldots, 1.5$.

The relative vorticity $\omega$ has been obtained by first computing the relative velocity field at the grid points of a uniform square grid. This relative velocity field is computed using the contour dynamics procedure, but instead of calculating the velocities at nodes on the contours, the velocities are determined at the grid points. Then the relative vorticity is obtained by numerical differentiation (using a central difference scheme). A square grid of $50 \times 50$ grid points is used for the domain $[-2,5] \times [-5,2]$. Note that the calculation of the relative vorticity is post-processing only and therefore the accuracy of it does not affect the calculation of the flow evolution.

The evolution of the absolute vorticity contours is shown in Figure 6.7; that of the relative vorticity in Figure 6.8. As can be observed from Figure 6.7, the (cyclonic) monopole moves in a north-west direction, as was to be expected. During this process, fluid in the wake of the vortex is advected in the northern direction, i.e. towards the centre of the domain which actually is the north pole, resulting in a region of negative relative vorticity (see Figure 6.8, $t = 10.0$). The negative vorticity becomes stronger as time proceeds and causes fluid behind it to move in the southern direction. As a consequence, positive relative vorticity is created ($t = 40.0$). This process is the beginning of the development of a so-
Figure 6.7: Absolute vorticity contours of the cyclonic vortex on the $\gamma$-plane at several moments in time. The domain shown is $[-2,5] \times [-2,5]$. 
Figure 6.8: Contour plots of the relative vorticity, as obtained from the relative velocity field, of the vortex of Figure 6.7 on the γ-plane at corresponding moments in time. The domain shown is \([-2, 5] \times [-2, 5]\).
Figure 6.9: The maximum value of the relative vorticity $\omega_{\text{max}}$ divided by the initial maximum value ($\hat{\omega}_{\text{max}}$) as a function of time $t$.

Figure 6.10: The path of the Lamb monopole. Symbols are placed at the position of the monopole at $t = 0, 5, 10, \ldots, 120$.

called Rossby wave which is observed to occur when a cyclonic monopole is placed on a $\beta$-plane [1, 9, 45]. When the monopole has arrived more or less at the north pole, a ring of negative vorticity is surrounding it and at $t = 90.0$ a tripolar structure is starting to develop. At $t = 120$, this tripolar structure is nicely visible in both the absolute vorticity distribution and the relative vorticity distribution.

During the evolution, the relative vorticity of the vortex decreases. This is caused by the displacement of the vortex in northern direction (since the absolute vorticity $q$ is conserved and $f$ increases when a fluid particle moves in northern direction). The decrease of the maximum of relative vorticity is shown in Figure 6.9. The path of the vortex, or rather the path of the maximum of relative vorticity, is shown in Figure 6.10. In the latter figure, it is visible that the vortex is moving in a north-west direction. It can also be observed that the velocity of the monopole changes during the evolution: it slows down when the vortex approaches the north pole.

For the present simulation, also the deformations of the outermost contour have been examined. During the evolution $|c|$ remains smaller than 0.012. The initial radius of the outermost contour is $r_1 = 11.52$ and the absolute vorticity jump at this contour is $\Delta q = 0.22357$. With these values, it follows from the theoretical considerations of the previous section that $|c|$ should remain smaller than 0.021, which apparently is the case. Moreover, it appears that the deformation of other contours remains smaller than the prescribed value as well. For example, the deformations of the eighth contour (counted from
outside and whose initial radius is equal to \( r_b = 7.39 \) are smaller than the bound required, which is 0.05, while during the simulation \(|\varepsilon| < 0.03\). Apparently, the computational domain could have been chosen smaller in this simulation.

In some pictures of Figure 6.7, for example at \( t = 80s \) and \( t = 90s \), small cusps can be observed near filament tips. This is a similar behaviour as was already observed in Figure 4.11; it is presumably caused by the piecewise-uniform discretization of the absolute vorticity.

### 6.3 Unsteady behaviour of tripoles on a \( \gamma\)-plane

The unsteady behaviour of tripolar vortices on a \( \gamma\)-plane has already been studied by Velasco Fuentes et al. [56]. In laboratory experiments carried out in a rotating fluid, they observed an asymmetric behaviour of the tripolar structure when the vortex was created at some distance (larger than zero) from the rotation axis. Detailed investigations revealed that this particular behaviour was caused by the so-called topographic \( \gamma\)-plane, i.e. the effect of certain non-uniform background vorticity. Here, this example is used to compare contour dynamics results with results from laboratory experiments, performed within the framework of this project.

#### 6.3.1 Laboratory experiments

**Experimental set-up**

The laboratory experiments are similar to those of Velasco Fuentes et al. [56] and are carried out in a rectangular tank (100cm \( \times \) 150cm) mounted on a rotating table (see Figure 6.11). The tank is filled with water up to a certain height \( H \), with \( H \) much smaller than the horizontal dimensions of the tank. The tank is rotating steadily with angular velocity \( \Omega \) yielding a (uniform) value of the Coriolis parameter \( f = f_0 = 2\Omega \). Let \( U \) and \( R \) be characteristic velocity and length scales, respectively, of the flow inside the tank. The Rossby number [36, 48] is then defined by

\[
Ro := \frac{U}{2\Omega R}.
\]

If the Rossby number is small (\(< 1\)) then the nonlinear advection terms may be neglected. As a consequence, there is a geostrophic balance in the horizontal direction and a hydrostatic balance in the vertical direction and the Taylor-Proudman theorem holds. This theorem shows that under these conditions the velocity cannot vary in vertical direction, implying that the motion is two-dimensional and will organise in vertical Taylor columns [36].

The rotation of the table causes the free surface to become parabolic as depicted in Figure 6.11. The fluid depth \( h \) as a function of the distance \( r = ||z|| \) from the rotation
axis is given by
\[ h(r) = h_0 \left(1 + \frac{\Omega^2 r^2}{2gh_0}\right), \]  
(6.15)
where \( h_0 \) is the fluid depth at the rotation axis and \( g \) the gravity acceleration. In the shallow water approximation [36, 48], conservation of potential vorticity is given by
\[ \frac{D}{Dt} \left(\frac{\omega + f}{h}\right) = 0. \]  
(6.16)
Expression (6.15) can be substituted in (6.16) and subsequently the result can be expanded in a Taylor series around \( h_0 \). If the Rossby number is assumed to be small (which is equivalent with \( \omega < f \) here) it then follows that
\[ \frac{D}{Dt} \left(\omega - \frac{f^3}{8gh_0}(x^2 + y^2)\right) = 0, \]
or
\[ \frac{\partial \omega}{\partial t} + (u, \nabla \omega) - \frac{f^3}{4gh_0} (xu + yv) = 0. \]  
(6.17)
Comparing this result with the \( \gamma \)-plane approximation (2.22) in Chapter 2, it can be observed that both equations are identical if
\[ \gamma = \frac{f^3}{8gh_0}. \]  
(6.18)
6.3. Unsteady behaviour of tripole on a $\gamma$-plane

The above approximation of the effect of the parabolic surface is referred to as the topographic $\gamma$-plane. In this approximation shallow is equivalent to north in the $\gamma$-plane approximation.

A tripolar vortex is generated by creating an unstable monopole. To this end, a bottomless cylinder of about 20 cm in diameter is placed in the tank. The liquid inside the cylinder is stirred cyclonically, i.e. in the same sense as the rotation of the table. Subsequently, the cylinder is lifted and an isolated (with net zero circulation) monopole is released. Under certain conditions, that are quite easily satisfied, the monopole becomes unstable resulting in a tripolar structure [30]. The vortex can be visualised by adding dye to the fluid inside the cylinder before lifting it. Quantitative information can be obtained by seeding the flow with small particles on the fluid's surface. It is assumed that the particles follow the flow passively without affecting it. The experiments were recorded by a video camera rotating with the table (see Figure 6.11), and the velocity of the particles could later be obtained by using the particle tracking feature of the DigImage package [14]. By interpolating these particle velocities to the points of a grid, derived quantities like vorticity $\omega$ and stream function $\psi$ can be determined.

Results

Two experiments are presented here: a dye experiment and a particle experiment. In both cases, the water depth $H$ was equal to $H = 16.5 \text{cm}$. The table was rotating with angular velocity $\Omega = 0.61 \text{s}^{-1}$ yielding $f = 2\Omega = 1.22 \text{s}^{-1}$. Furthermore, the water depth $h_0$ at the rotation axis was $h_0 = 16.0 \text{cm}$. With (6.18) it then follows that $\gamma = 0.15 \text{m}^{-2} \text{s}^{-1}$.

Figure 6.12 shows a sequence of images of the dye experiment where the vortex was created at a distance of approximately 25 cm from the rotation axis. The first image is made 25 seconds after lifting the cylinder. At that moment, the vortex still appears to be a monopolar vortex. After the vortex has become a tripole ($t = 72 \text{s}$), the $\gamma$-effect influences the behaviour of the vortex dramatically. The elliptically shaped core of the tripole moves away together with one of the satellites ($t = 85 \text{s}$). This structure is an asymmetric dipole that moves along a curved trajectory resulting in a collision with the other satellite ($t = 98 \text{s}$). Subsequently, an exchange of partners takes place ($t = 117 \text{s}$) and the process is repeated although with a longer trajectory of the new dipole (and the asymmetry becomes more pronounced). In the paper by Velasco Fuentes et al. [56], this behaviour is explained in more detail.

In the particle experiment, the vortex was also created at a distance of approximately 25 cm from the rotation axis. The purpose of the particle experiment was to obtain some quantitative information about the size and strength of the vortex compared to the $\gamma$-effect.

Figure 6.13 shows contour plots of the relative vorticity at several stages of the flow evolution. In the first two plots, contours are drawn for $\omega = -0.7, -0.5, -0.3, 0.4, 0.8, \ldots, 4.0$ while in the other images contours are drawn for $\omega = -0.6, -0.5, \ldots, -0.1, 0.2, 0.4, \ldots, 2.4$. Negative vorticity values are again represented by dashed lines, whereas positive values are represented by solid lines. As can be observed from this figure, 53 seconds after lifting the cylinder the vortex is still more or less a circular monopole. At $t = 80 \text{s}$, the ring of
Figure 6.12: A sequence of images showing the evolution of a tripolar vortex on a topographic $\gamma$-plane. The experimental parameters are $\Omega = 0.61 \text{s}^{-1}$, $h_0 = 16.0 \text{cm}$, $\gamma = 0.15 \text{m}^{-2} \text{s}^{-1}$. The vortex is created approximately 25 cm from the rotation axis.
negative vorticity has broken up and the core has become slightly elliptical. At $t = 91$, a clear tripolar vortex is visible. The asymmetric behaviour due to the parabolic surface can also be observed here, although it is less clear compared to the dye experiment.

As mentioned, the vortex is approximately circular at $t = 53s$. Closer inspection of the data reveals that the vortex remained circular up to approximately $t = 72s$. The vorticity distribution of the monopole can thus be determined up to that moment in time. Figure 6.14 shows the (scaled) relative vorticity distribution at $t = 53s$ (a) and $t = 72s$ (b) after lifting the cylinder. The vorticity profile

$$\omega(r) = \frac{U}{R} \left( 1 - \frac{\alpha}{2} \left( \frac{r}{R} \right)^2 \right) \exp \left( - \left( \frac{r}{R} \right)^2 \right),$$

(6.19)

is used to fit (solid line) the data (black dots). Here, $U$ and $R$ are some characteristic velocity and length scales and $\alpha$ is a steepness parameter controlling the shape of the profile [10] (the larger $\alpha$, the steeper the profile). This vorticity profile is, like any axisymmetric profile, a stationary solution of the two-dimensional inviscid Euler equations in an infinite domain. It can be observed that the vorticity profile at $t = 72s$ is steeper than at $t = 53s$. Moreover, at $t = 72s$ the value of $\alpha$ is larger than 2 (in contrast to $t = 53s$), which implies that the profile has become unstable [10]. Another feature that can be observed is the
Figure 6.13: Contour plots of the relative vorticity distribution at several moments in time of the particle experiment. Negative values of $\omega$ are plotted with dashed lines, positive values with solid lines. The domain shown is $[-40\text{cm}, 40\text{cm}] \times [-40\text{cm}, 40\text{cm}]$. 
6.3. Unsteady behaviour of tripole on a γ-plane

![Graphs showing the relative vorticity distribution for different times and radii.](image)

a) Fit: $\alpha = 1.8$, $R = 7.5\,\text{cm}$, $U = 37.5\,\text{cm/s}$  

b) Fit: $\alpha = 2.1$, $R = 8.3\,\text{cm}$, $U = 29.1\,\text{cm/s}$

Figure 6.14: The relative vorticity distribution of the vortex at $t = 53s$ and $t = 72s$.

The increase of the vortex size ($R = 7.5\,\text{cm}$ at $t = 53s$ and $R = 8.3\,\text{cm}$ at $t = 72s$) and a decrease of the maximum of vorticity ($\omega_{\text{max}} = U/R = 5.0s^{-1}$ at $t = 53s$ and $\omega_{\text{max}} = 3.5s^{-1}$ at $t = 72s$).

This decay of the vorticity is shown in Figure 6.15, where the maximum vorticity (scaled with $\omega_{\text{max}} = 7.0s^{-1}$) is plotted as a function of time $t$. The experimental data (black dots) are fitted (solid line) with the exponential function $\exp(-\gamma t/T_E)$ with $\gamma = 1.92$ and $T_E = 192.0$. The Ekman time $T_E$ [34, 48] is defined as $T_E := h_0/\sqrt{\nu} \Omega$ with $\nu$ the kinematic viscosity, which is $\nu = 1.14 \times 10^{-6}m^2s^{-1}$ for water at a temperature of $15^\circ C$. The maximum of vorticity decreases dramatically due to viscous effects and the spin-down mechanism associated with the recirculation flow driven by the Ekman layer at the bottom of the tank [30]. Furthermore, the displacement of the vortex in the northern direction causes the maximum vorticity to decrease as well, although this effect is rather small compared to the viscous effects.

6.3.2 A numerical experiment

From the foregoing it is clear that viscous effects and, in particular, the influence of the Ekman layer at the bottom of the tank, influences the strength of the vortex to a large extent (and also the vortex size, but less dramatically). Since such effects are not incorporated in the contour dynamics method, only a qualitative comparison is possible. In the numerical simulation, an appropriate initial vorticity field is needed to initialise the
Figure 6.15: The maximum of vorticity as a function of time $t$ plotted on (a) a linear and (b) a logarithmic scale.

simulation. To find such an initial configuration, first the $\gamma$-plane approximation (2.22) is non-dimensionalised.

If $R$ is a typical length scale and $U$ a typical velocity scale, then the dimensionless form of (2.22) is given by

$$\frac{\partial \omega}{\partial t} + (u, \nabla \omega) - 2 \gamma^* (xv + yv) = 0.$$  \hspace{1cm} (6.20)

where $\gamma^*$ is the dimensionless version of $\gamma$ and is related to it by

$$\gamma^* = \frac{R^3}{U}.$$  \hspace{1cm} (6.21)

From this expression it becomes clear that during laboratory experiments, this dimensionless number changes with time since $R$ and $U$ change with time while $\gamma$ is constant. For example, from Figure 6.14 it follows that at 53 seconds after lifting the cylinder $\gamma^*$ is given by $\gamma^* = 0.0011\gamma$ whereas at $t = 72s$ it is given by $\gamma^* = 0.0020\gamma$, which is approximately a factor of two larger. Due to the rapid decrease of the vorticity, $\gamma^*$ will presumably increase further during the experiment (an estimate of $\gamma^*$ at $t = 271s$ is given by $\gamma^* = 0.026\gamma$).

In numerical experiments, however, this $\gamma^*$ is constant during the simulation. The results can therefore not directly be compared with the results of the laboratory experiment. The following numerical experiment shows good qualitative agreement with the
dye experiment. In this simulation, a monopole with vorticity distribution according to (6.19) with \( R = 0.47, \ U = 0.71 \) is placed on a \( \gamma \)-plane with \( \gamma = 0.02 \) (resulting in a value of \( \gamma^* = 0.003 \)). The steepness parameter is chosen equal to \( \alpha = 2.5 \) to ensure that the monopole is unstable. Furthermore, the monopole has a slightly elliptical shape (aspect ratio 1.11) to enhance the instability. The initial location of the monopole is \( \mathbf{x} = (3.0,0.0) \).

The size of the computational domain is \([-8,8] \times [-8,8]\) and the number of contours used is 43. From these 43 contours, 24 are used to approximate the background vorticity. Figure 6.16 shows the evolution of the absolute vorticity contours at several moments in time. The domain shown is \([-0.2,4.0] \times [-1.85,2.35]\). Figure 6.17 shows the relative vorticity, as obtained from the relative velocity field, at the same stages in the evolution. Negative relative vorticity is again represented by dashed lines. Contours are plotted for \( \omega = -0.3, -0.2, 0.1, 0.3, \ldots , 1.5 \).

In both figures, the development of an asymmetric tripolar structure can be clearly observed. At \( t = 49s \), the core of the tripole has formed a dipolar structure together with one of the satellites, which is moving towards the other satellite. Note that compared to the simulation of the previous section (Figure 6.8) far less relative vorticity is created in this simulation. Only from \( t = 49s \) onward, some created positive relative vorticity is visible near the tail of one of the satellites.

Comparison of the two figures with the dye experiment of Figure 6.12 shows at least until \( t = 49s \) a remarkable resemblance with the images of the laboratory experiments up to \( t = 92s \). Apparently, the numerical method is able to simulate the dynamics of the vortex rather well.

### 6.4 Discussion

In this chapter it has been demonstrated how contour dynamics can be used in cases of non-uniform background vorticity like present on a \( \gamma \)-plane. Furthermore, some additional accuracy aspects (compared to standard contour dynamics) have been considered. Numerical experiments show that the present method is able to simulate the dynamics of vortices on a \( \gamma \)-plane very well.

Although only the \( \gamma \)-plane is considered here, similar procedures can be followed for problems on a \( \beta \)-plane or even on a rotating sphere. There are, however, some differences with respect to the computational domain. For the \( \beta \)-plane case, for example, the isolines of the background vorticity are parallel straight lines running in longitudinal direction. Since the contours should be closed (see Chapter 3) a singly-periodic computational domain could be chosen to achieve this. In latitudinal direction, a finite number of contours should be used to approximate the \( \beta \)-effect. Like in the \( \gamma \)-plane case considered here, the outermost contours (two straight lines now, instead of one circular contour in the \( \gamma \)-plane case) may not deform too much and similar to the \( \gamma \)-plane it is possible to monitor and control this during a computation.
Figure 6.16: A sequence of pictures showing the evolution of the absolute vorticity contours of the numerical simulation at several stages. The domain shown is $[-0.2, 4.0] \times [-1.85, 2.35]$. 
Figure 6.17: The same as Figure 6.16 but now the relative vorticity, as obtained from the relative velocity field, is shown.
Chapter 6. Contour Dynamics with Non-Uniform Background Vorticity
7

Concluding Remarks and Recommendations
This thesis reports on the contour dynamics method and a number of applications to two-dimensional vortices. Numerical aspects of the method, like spatial discretization and time integration, have been studied and also a procedure to accelerate the method. The method has been used to study collapse interactions of three vortices starting from a specific initial configuration. Furthermore, a novel application of the contour dynamics method to problems with non-uniform background vorticity has been presented.

The study of the spatial discretization and time integration revealed the importance of symplectic time integration because of the Hamiltonian character of the (spatially discretized) system of equations. Numerical experiments have shown that such symplectic Runge-Kutta time integration schemes conserve the Hamiltonian properties of the system better than standard explicit Runge-Kutta schemes.

A problem encountered, however, concerns the implementation of symplectic schemes because of their implicitness. Since the system of equations of the contour dynamics method is not stiff in general, a predictor-corrector scheme may be used for the implementation. To obtain an integrator which is effectively symplectic, sufficient corrector steps have to be carried out, i.e. basically until machine precision is reached. However, every corrector step requires the calculation of the velocities, which is rather time consuming. Therefore, an extrapolation method has been chosen to accelerate the iteration process. The minimal polynomial extrapolation method turned out to be very suitable here. It is based on the use of differences, and does not need additional information about the Jacobian matrix of the system. Performing the extrapolation for each of the contours separately, appeared to work even better in practice.

Another aspect that became clear from this study, is the observation that putting much effort into the spatial discretization does not really lead to better results with respect to conservation of area, circulation and angular momentum if the time integrator is not conservative. So if high accuracy is required, higher-order interpolation of the contours (or even spectrally accurate interpolation) is recommended to be combined with a higher-order symplectic integration scheme.

Contour dynamics has been used to examine a specific kind of interaction of three vortices, namely one that leads to a vortex collapse in the point vortex case. Several aspects (vortex sizes, vorticity distribution) that could influence the interaction behaviour have been studied. One of the main conclusions of this study is that the trajectories of the vortices, during their mutual approach, agree with those of the point vortices. However, as soon as the mutual distances of the finite-sized vortices become of the order of their sizes, the vortices start to deform and the trajectories are seen to deviate from those of the point vortices. During this deformation, one vortex in particular is torn apart and a collapse does not occur. Although the vorticity distribution with the finite-sized vortices influences the behaviour of the vortices also, this effect is small compared to the influence of vortex sizes. The vorticity distribution, and hence the number of contours used to approximate it, does not have a large influence on the interaction behaviour of the three vortices.
From the various numerical experiments presented in this thesis and also the numerous applications that can be found in literature it can be concluded that contour dynamics is a powerful method to simulate the evolution of vortices and their mutual interactions. The method is able to resolve fine-scale structures and to deal with very complex configurations. There is, however, a drawback of the method: the simulations become computationally rather expensive when the number of contours and nodes on the contours becomes large. To deal with this problem, a hierarchical-element method has been used to accelerate the velocity calculations. Numerical experiments show that this is a robust and accurate way to accelerate the calculations while preserving the accuracy, and the speed-up turns out to be substantial.

In the hierarchical-element method presented here, a square computational domain is chosen which encloses every contour. This domain is divided into a number of boxes in a uniform manner and, furthermore, several levels of uniform refinement are used. Although the speed-up is already significant, the efficiency can probably be improved further by using non-uniform refinements. For example, when the density of nodes is high in a certain area and nodes are sparse elsewhere, a local refinement in the high-density-region might improve the efficiency.

The square computational domain is very convenient in, for example, $\gamma$-plane situations, but in other cases a rectangular domain may be more suitable. It is not difficult to extend the method for a square domain to one for a rectangular domain as explained by Anderson [2]. The easiest way to do this (according to Anderson) is to choose a rectangular domain with aspect ratio a power of two. The refinements of the rectangular domain are accomplished by dividing the longer side by powers of two until the box sides are equal to the length of the short side (refinement in one direction only). Further refinements are obtained by refining in both directions.

In this thesis, the hierarchical-element method is used for flow problems in an infinite domain (although the computational domain is bounded of course). It should however also be possible to incorporate boundary conditions like in the case of a fast multipole method [25]. This would make the contour dynamics method even more generally applicable.

A final topic that has been examined within the framework of this project, is the possibility of using contour dynamics to study the behaviour of vortices in the presence of non-uniform background vorticity like in geophysical flows. It has been demonstrated how this can be carried out in the case of a $\gamma$-plane. The comparison of numerical results with laboratory experiments revealed a satisfactory qualitative agreement and it can be concluded that contour dynamics gives reliable results in this case as well.

Although not presented here, it is also possible to use the method for the evolution of vortices on a $\beta$-plane and even to simulate flows on a rotating sphere. In the $\beta$-plane case, it is convenient, or even necessary, to use a singly-periodic domain. This is a consequence of the isolines of the Coriolis parameter (i.e. lines of constant latitude) being straight lines
that are mutually parallel. Since in contour dynamics contours should be closed, these isolines have to be connected. An easy way to obtain this is to use a singly-periodic domain in longitudinal direction, either by using the appropriate Green's function \cite{20} or by incorporating the periodic boundary conditions in the hierarchical-element method. Promising numerical simulations have already been carried out.

The method as presented in this thesis, has been implemented in a FORTRAN code (called cdevol). For pre-processing (i.e. generation of initial configurations) several programs have been developed. For post-processing another program (called cdfield) has been developed. With this post-processing program quantities like strain-rate, stream function relative vorticity and velocity field can be determined on a uniform grid at selected stages in the flow evolution. Both programs cdevol and cdfield are very suitable for parallelisation, since many parts of the calculation can be carried out independently from each other. To this end, for example, PVM (Parallel Virtual Machine) may be used. In very complex cases, like $\beta$-plane and $\gamma$-plane simulations, parallelisation will surely speed-up the performance of both programs.

The simulations presented in this thesis involve only dynamically active contours, but it is also possible to study the advection of passive contours (simply by defining a contour with zero vorticity jump). Passive contours do not affect the velocity field and are therefore not incorporated in the calculation of the velocity field. This possibility of incorporating passive contours makes the method suitable to study, for example, chaotic advection of passive tracers by vortices on a $\beta$- and a $\gamma$-plane.
A

Derivation of the Strain Rate for Rankine and Kirchhoff Vortices

A.1 Rankine vortex
A.2 Kirchhoff vortex
A.1 Rankine vortex

In this section the strain rate $Q$ as defined by (4.9) in Chapter 4 is determined for a Rankine vortex (circular patch of uniform vorticity). Let $R$ be the radius of the vortex and $\omega$ the vorticity. Let furthermore the centre of the vortex coincide with the origin of the coordinate system $(x, y)$. Then the velocity field induced by this vortex is given by

$$u = \begin{cases} 
\frac{\omega}{2}(-ye_x + xe_y), & r \leq R, \\
\frac{\omega R^2}{2r^2}(-ye_x + xe_y), & r \geq R,
\end{cases}$$

where $r = \sqrt{x^2 + y^2}$, the radial distance to the centre of the vortex. From this it follows that

$$-2\text{det}(\nabla u) = \begin{cases} 
-\frac{1}{2}\omega^2, & r \leq R, \\
\frac{\omega^2 R^4}{2r^4}, & r \geq R.
\end{cases}$$

Now, expression (4.10) yields

$$Q = \begin{cases} 
0, & r < R, \\
\frac{\omega^2 R^4}{2r^4}, & r > R.
\end{cases}$$

A.2 Kirchhoff vortex

For a Kirchhoff vortex (elliptical patch of uniform vorticity), the derivation is more complex and it is therefore split up into two parts: inside and outside the patch. Assume the patch has a semi-major axis of length $a$, a semi-minor axis of length $b$ and vorticity $\omega$. Let furthermore again the centre of the vortex coincide with the origin of the coordinate system $(x, y)$.

According to Lamb [37], the stream function inside the ellipse is given by

$$\psi = -\frac{\omega}{2(a + b)}(bx^2 + ay^2).$$

The velocity field can now be found using (2.10) yielding

$$u = \frac{\omega}{(a + b)}(-aye_x + bxe_y).$$

With this expression for the flow field $\text{det}(\nabla u)$ can be determined, resulting in

$$-2\text{det}(\nabla u) = -\frac{2\omega^2 ab}{(a + b)^2} = -\frac{1}{2}\omega^2 \left(1 - \frac{(a - b)^2}{(a + b)^2}\right).$$
Expression (4.10) for $Q$ then yields

$$Q = \frac{\omega^2(a - b)^2}{2(a + b)^2}.$$

Outside the patch, the stream function is more complex [37]:

$$\psi = \frac{1}{4} ab \omega (e^{-2\xi} \cos(2\eta) + 2\xi),$$

where $\xi$ and $\eta$ are elliptic coordinates which are related to the Cartesian coordinates by

$$x = (a^2 - b^2)^{\frac{1}{2}} \cosh(\xi) \cos(\eta),$$
$$y = (a^2 - b^2)^{\frac{1}{2}} \sinh(\xi) \sin(\eta).$$

Note that the exterior of the ellipse is given by $\xi > \ln(\sqrt{a + b}/\sqrt{a - b})$ and $0 < \eta < 2\pi$ in this coordinate system. Figure A.1 shows curves $\xi = \text{constant}$ and $\eta = \text{constant}$ in the $(x, y)$ plane in the case where $a = 1.0$ and $b = 0.5$.

The velocity field can now be calculated from the stream function by using the transformation from Cartesian coordinates to elliptic coordinates, i.e.

$$u = \frac{1}{h(\xi, \eta)} \left( \frac{\partial \psi}{\partial \eta} e_\xi - \frac{\partial \psi}{\partial \xi} e_\eta \right),$$

where $h(\xi, \eta)$ is given by

$$h(\xi, \eta) = \sqrt{\left( \frac{\partial x}{\partial \xi} \right)^2 + \left( \frac{\partial x}{\partial \eta} \right)^2} = \sqrt{\left( \frac{\partial y}{\partial \xi} \right)^2 + \left( \frac{\partial y}{\partial \eta} \right)^2}$$
$$= \sqrt{(a^2 - b^2)/2} \sqrt{\cosh(2\xi) - \cos(2\eta)}. \tag{A.1}$$
With this, it follows after that the velocity field \( \mathbf{u} \) outside the vortex is given by

\[
\mathbf{u} = \frac{ab\omega}{2h(\xi, \eta)} \left( e^{-2\xi \sin(2\eta)} \mathbf{e}_\xi - (e^{-2\xi \cos(2\eta)} - 1) \mathbf{e}_\eta \right).
\]  

(A.2)

Now, the velocity gradient \( \nabla \mathbf{u} \) has the following form

\[
\nabla \mathbf{u} = \begin{pmatrix}
\frac{1}{h} \frac{\partial u_\xi}{\partial \xi} + \frac{1}{h^2} \frac{\partial h}{\partial \eta} u_\eta & \frac{1}{h} \frac{\partial u_\xi}{\partial \eta} \frac{\partial h}{\partial \xi} & \frac{1}{h} \frac{\partial u_\xi}{\partial \xi} \frac{\partial h}{\partial \eta} & \frac{1}{h^2} \frac{\partial u_\xi}{\partial \eta} \\
\frac{1}{h} \frac{\partial u_\eta}{\partial \xi} - \frac{1}{h^2} \frac{\partial h}{\partial \eta} u_\xi & \frac{1}{h} \frac{\partial u_\eta}{\partial \eta} & \frac{1}{h} \frac{\partial u_\eta}{\partial \xi} \frac{\partial h}{\partial \eta} & \frac{1}{h^2} \frac{\partial u_\eta}{\partial \eta} \\
\frac{1}{h} \frac{\partial h}{\partial \xi} u_\eta & \frac{1}{h} \frac{\partial h}{\partial \eta} u_\xi & \frac{1}{h} \frac{\partial h}{\partial \xi} & \frac{1}{h^2} \frac{\partial h}{\partial \eta} \\
\frac{1}{h} \frac{\partial h}{\partial \eta} & \frac{1}{h} \frac{\partial h}{\partial \xi} & \frac{1}{h} \frac{\partial h}{\partial \xi} & \frac{1}{h^2} \frac{\partial h}{\partial \eta}
\end{pmatrix}
\]

From the velocity field (A.2) and the expression (A.1) for \( h \), it follows that

\[
\frac{1}{h^2} \begin{pmatrix}
\frac{\partial (hu_\xi)}{\partial \xi} & \frac{\partial (hu_\xi)}{\partial \eta} \\
\frac{\partial (hu_\eta)}{\partial \xi} & \frac{\partial (hu_\eta)}{\partial \eta}
\end{pmatrix} = \frac{ab\omega e^{-2\xi}}{h^2} \begin{pmatrix}
-(\sin(2\eta) \cos(2\eta)) \\
(\cos(2\eta) \sin(2\eta))
\end{pmatrix},
\]

and, furthermore,

\[
\frac{1}{h^2} \begin{pmatrix}
\frac{\partial h}{\partial \xi} u_\eta - \frac{\partial h}{\partial \eta} u_\xi & \frac{\partial h}{\partial \xi} u_\eta + \frac{\partial h}{\partial \eta} u_\xi \\
\frac{\partial h}{\partial \xi} u_\xi + \frac{\partial h}{\partial \eta} u_\eta & \frac{\partial h}{\partial \xi} u_\xi - \frac{\partial h}{\partial \eta} u_\eta
\end{pmatrix} = \frac{ab\omega(a^2 - b^2)}{4h^4} \begin{pmatrix}
-(\sin(2\eta) \sinh(2\xi)) \\
\sinh(2\xi) \sin(2\eta)
\end{pmatrix} - e^{-2\xi} \begin{pmatrix}
-(\sin(2\eta) \sinh(2\xi) - \cos(2\eta) \sin(2\eta)) & \cos(2\eta) \sinh(2\xi) - \sin^2(2\eta) \\
\cos(2\eta) \sinh(2\xi) - \sin^2(2\eta) & \sin(2\eta) \sinh(2\xi) + \cos(2\eta) \sin(2\eta)
\end{pmatrix}
\]

Since \( \text{tr}(\nabla \mathbf{u}) = 0 \) (incompressibility) here as well, it follows that \( Q \) is given by (4.10), yielding

\[
Q = \frac{(2ab\omega)^2}{(a^2 - b^2)^2} e^{-2\xi} \frac{
}{(\cosh(2\xi) - \cos(2\eta))}.
\]

Combination of the expressions for \( Q \) inside the ellipse and outside the ellipse, yields expression (4.12) as given in Chapter 4.
The Algorithm for Constructing Outer-Rings in the HEM used for CD
In this appendix, the algorithm for the construction of the outer-rings at the finest level is discussed. In Section 5.4 it has already been mentioned that contributions from two different sources have to be taken into account, namely the contributions of the contour segments inside the box considered and the appropriate box boundary parts of that box.

The contributions of the first source can be determined by moving along the contour in positive direction, i.e. counterclockwise and determining for each node in which box it resides. The number of the box in which the node is situated is stored in such a way that for each node it is clear in which box its preceding node is situated. If the box number of the current node differs from that of its predecessor, the part of the contour between those two nodes crosses at least one grid line and the current segment contributes to the outer-ring of at least two boxes. After the intersection points with those grid lines are determined, the contributions to the various boxes can be determined. If the box number does not differ from that of the preceding node, the whole line segment is located in one box and the contribution of the segment to the outer-ring of that box (and contour nodes in neighbouring boxes) can be determined. Upon treating all nodes on the contour have been in this way, the contributions of the first source have been determined.

The determination of contributions of the second source is more complex. One problem encountered is the possibility of the contour crossing a box more than once. The intersection points of the contour with the box boundaries have already been determined in the previous part, but the sequence in which they have been found, is not of any use. This is illustrated by Figure B.1. Here, the intersection points $P_i$ are, for example, found in the order of increasing index $i$. It is clear from this figure that the parts of the box boundaries that have to be taken into account (solid lines) should always be walked along in positive direction, since the contour itself is also walked along in that direction. Thus, it is necessary to sort the array of intersection points of a given box in such a way that by traversing that array after sorting, the intersection points are encountered successively in positive direction (so, in the case of Figure B.1 the sequence of intersection points should be $P_7, P_6, P_5, P_4, P_3, P_2, P_1, P_0, P_9, P_8$, after the sorting algorithm has been applied to the original sequence). A heap sort algorithm [13] is used for this purpose.

Another feature that is clear from Figure B.1 is that if the contour moves inward into the box, it has to move outward again, since it is closed (see Chapter 3). Therefore, the number of intersection points for a given box is always an even number. Moreover, it appears that the parts of the box boundaries that have to be taken into account, are the parts between two neighbouring intersection points (not necessarily lying on the same box side) of which at the first the contour is moving outward and at the next (in positive direction) the contour is moving inward again. By giving each intersection point a flag (sside) indicating whether the contour is moving inward (sside > 0) or outward (sside < 0) at that point, it can be determined between which two subsequent intersection points a box boundary part, that should be accounted for, is situated. Since it is not a priori clear how many and which corners of the boundary are lying between such two subsequent intersection points, it is convenient to identify each side and each corner with a number (see Figure B.1). In this way, an intersection point can be assigned an integer value indicating the side it is situated on. The number of corners between two subsequent intersection points can now
be found by subtracting these values. The first corner which is encountered while moving from one intersection point to the next, is the corner with the number equal to the value of the former intersection point. For example, the values of the two subsequent intersection points $P_2$ and $P_3$ in Figure B.1 are equal to 3 and 4, respectively. Subtraction yields that there is one corner in between $P_2$ and $P_3$ and the number of that corner is equal to 3, i.e. the value of the flag of $P_3$.

This value can be combined with the flag mentioned earlier: $sside(P_i, b) = \pm k$ where $k$ is the number of the side of box $b$ where $P_i$ is lying on and the sign indicates whether the contour moves inward or outward box $b$ at $P_i$. Using this strategy, the contributions of the boundary parts of the boxes that are crossed by the contour can be determined.

Another problem that can be encountered when determining the contribution of the second source is the possibility of a box lying completely inside the interior of the contour. In this case, the whole boundary of the box contributes to the outer-ring. A way to detect this kind of boxes, is to move through the boxes from left to right, row by row. For the first box (at the left side of the domain) in a row it is clear that it is never lying completely inside the interior of the contour. Thus, it is situated completely outside the contour (when
it is not crossed by the contour) or partly outside and partly inside the contour (when the contour crosses that box). In the first case, the next box is obviously also not situated completely inside the interior of the contour. In the second case however, the next box might be situated completely inside the contour, but only if the right side of the first box is contained inside the contour. In that case, a variable, \textit{nextin}, is set to 1, otherwise this is set to 0. If \textit{nextin} = 1 and the next box is not crossed by the contour, then that box is situated completely inside the interior of the contour and \textit{nextin} is not changed, since the subsequent box might also be situated completely inside the contour. If \textit{nextin} = 1 and the next box is crossed by the contour, then \textit{nextin} is set to zero again. This procedure can be followed till the last box on the right of the row. By finishing this procedure for every row, the contributions of all the relevant box-boundary parts have been determined and the finest level outer-rings are computed.

This part of the algorithm can also be summarised in pseudo-code and it should be replacing the lines 5-14 in the pseudo-code of Section 5.2 to obtain the complete algorithm for accelerating the velocity calculations in contour dynamics simulations.

begin
\{Construct outer-rings at level $l_f$\}
for each contour $C$ do
\{Compute velocity contributions of the appropriate contour and box boundary parts to the $K$ integration points on the outer-ring of the appropriate boxes and contour nodes in their neighbour-boxes\}
for each node $x_n$ on contour $C$ do
\begin{enumerate}
\item Determine in which box at level $l_f$ node $x_n$ resides in
\item if current box $b_c = \text{previous box} b_p$ then
\item Compute the contribution of the line segment $x_{n-1} - x_n$ inside box $b_c$
\item else
\item Determine the intersection point(s) $P_i, i = 1, \ldots , I$, of the line segment between $x_{n-1}$ and $x_n$ with the grid line(s) that is (are) crossed by that line segment
\item for each $P_i$ between $x_{n-1}$ and $x_n$ do
\item Store $P_i$ according to the boxes $b_i$ and $\hat{b}_i$ it belongs to
\item {The contour is moving outward box $b_i$ at $P_i$ and it is moving inward $\hat{b}_i$ at $P_i$.}
\item Set a flag $sside$: $sside(P_i, b_i) = -k$ and $sside(P_i, \hat{b}_i) = +l$
\item {\{k and l are the numbers of the box sides, as defined in Figure B.1, $P_i$ is lying on, and the sign indicates whether the contour is moving inward (+) or outward (−) the box at $P_i$\}
\item od
\item Compute the contribution of line segment $x_{n-1} - P_i$
\item inside box $b_p = b_1$
\end{enumerate}
for each part $P_{i-1} - P_i$ of line segment $x_{n-1} - x_n$ do
  Compute the contribution of this line segment inside
  box $b_{i-1} = b_i$
  od
od
Compute the contribution of line segment $P_i - x_n$ inside
box $b_i = b_c$
fi
od
{Compute velocity contributions of the appropriate box boundary }
{parts }
for each box $b$ at level $l_f$ which is crossed by $C$ do
  Sort the intersection points of box $b$ with a heapsort algorithm in
  such a way that their sequence agrees with the sequence found
  when walking along the box boundary in positive direction
od
for each row $j$ of boxes at level $l_f$ do
  nextin = 0
for each box $b$ in row $j$ do
  if $b$ is crossed by $C$ then
    Determine the contributions of the boundary segments
    between any pair of two successive intersection points $P_{i-1}$
    and $P_i$, for which $sside(P_{i-1}, b) < 0$ (and thus
    $sside(P_i, b) > 0$)
    The numbers of corners as well as the numbers of the sides
    between $P_{i-1}$ and $P_i$ can be determined easily from the
    value of $sside(P_{i-1}, b)$ and $sside(P_i, b)$
    if the right side of box $b$ is inside $C$ then
      nextin = 1
      {The next box in this row might lie completely inside $C$}
    else
      nextin = 0
    fi
  else if nextin = 1
    {The current box $b$ is not crossed by $C$ and nextin = 1 so }
    {box $b$ lies completely inside $C$}
  fi
od
od
end
Appendix B. The Algorithm for Constructing Outer-Rings in the HEM used for CD
Bibliography


Samenvatting

Grootschalige stromingen in de atmosfeer en de oceanen, ook wel *geofysische stromingen* genoemd, kunnen bij benadering als tweedimensionaal beschouwd worden. Dit komt omdat zowel de atmosfeer als de oceanen slechts dunne schillen om de aarde zijn. Hierdoor zijn de stromingen bij benadering, lokaal gezien, voornamelijk in horizontale richting, dus parallel aan het aardoppervlak. Deze *tweedimensionaald*heid wordt bovendien nog gestimuleerd doordat de aarde om haar as roteert wat tot gevolg heeft dat verticale bewegingen d.w.z. lokaal loodrecht op het aardoppervlak, onderdrukt worden. Eenzelfde effect wordt ook veroorzaakt doordat beide media vaak geelaagd zijn in verticale richting, bijvoorbeeld door een variatie in zoutgehalte (in de oceanen) of in temperatuur.

In geofysische stromingen spelen quasi tweedimensionale wervels, d.w.z. compacte gebieden van roterende materie, een belangrijke rol in het transport van onder andere temperatuur en zout. Voorbeelden van zulke wervels zijn de hoge en lage drukgebieden in de atmosfeer en de wervels die ontstaan door de Golfstroom in de Atlantische Oceaan.

Om het gedrag van tweedimensionale wervels beter te begrijpen, worden er onder andere laboratorium experimenten uitgevoerd waarbij de wervels op kleinere schaal nagebootst worden. Verder wordt er ook veelvuldig gebruik gemaakt van numerieke programma’s om het gedrag van wervels met behulp van computers te bestuderen.

In dit proefschrift wordt één numerieke methode in het bijzonder bestudeerd, namelijk de *contour-dynamica methode*. De methode is gebaseerd op discretisatie van de vorticiteit, de locale draaiingsnelheid van een vloeistofelementje. In het algemeen is de verdeling van vorticiteit in een wervel continu. Bij contour-dynamica wordt deze nu vervangen door een stuksgewijs constante verdeling. Omdat onder bepaalde omstandigheden de vorticiteit van een vloeistofdeeltje in de tijd niet verandert, ligt de evolutie van de wervels volledig vast door de evolutie van de randen van de gebieden met constante vorticiteit. Deze randen worden *contouren* genoemd. Door de contouren te volgen in de tijd, kan een beeld verkregen worden van de evolutie van de wervels.

Een belangrijk onderdeel van de methode is de discretisatie en interpolatie van de contouren. Een contour wordt benaderd door een aantal punten welke onderling verbonden worden d.m.v. rechte lijnstukjes, zogenaamde *lineaire interpolatie*. Hiermee kunnen nu de snelheden van de punten bepaald worden, waarmee vervolgens de verplaatsingen van die punten over een kleine tijdstapje berekend kunnen worden; dit laatste wordt *tijd-integratie* genoemd. Herhaald toepassen van deze procedure levert de evolutie van de contouren in de tijd.
Een probleem is echter dat tijdens de evolutie de contouren dramatisch kunnen veranderen. Hierdoor zal het aantal punten dat initieel gekozen is, na verloop van tijd niet meer voldoende zijn om de contouren op een correcte manier te representeren, bijvoorbeeld omdat sommige punten te ver uit elkaar komen te liggen. Het is daarom noodzakelijk om tijdens de evolutie punten toe te voegen aan de contouren waar dat nodig is, of punten weg te halen waar ze overbodig worden. Tijdens het onderzoek is onder meer aandacht besteed aan deze knooppuntendistributie.

Niet minder belangrijk is ook de tijd-integratie. Een speciale manier om dit nauwkeurig te doen is uitgebreid onderzocht. In plaats van de tot nog toe gebruikelijke methoden wordt er nu een zogenaamde symplectische methode gebruikt. Deze heeft onder andere als voordeel dat het oppervlak binnen de contouren zo beter behouden blijft gedurende een berekening zodat beter voldaan wordt aan behoud van massa.

Verder is er een manier ontwikkeld om de berekeningen te versnellen. Hiertoe is een zogenaamde Héirechirsche Element Methode methode gebruikt, waarbij het rekendomein opgedeeld wordt in steeds fijne deelgebieden. Voor de bijdrage tot de snelheid in een zeker punt van een contour, worden de verder verwijderde deelgebieden tot steeds grotere deelgebieden samengevoegd. Dit levert een significante versnelling van de berekeningen op.

De versnelling is noodzakelijk voor zeer complexe berekeningen zoals bijvoorbeeld in het geval van niet-uniforme achtergrondvorticiteit waarbij ook contouren buiten de wervel(s) nodig zijn. Hierdoor kunnen de berekeningen behoorlijk reken-intensief worden. Dit soort problemen zijn echter zeker relevant met betrekking tot geofysische stromingen, omdat de bolvorm van de aarde een variatie in achtergrondvorticiteit veroorzaakt die afhankelijk is van de geografische breedtegraad. In dit proefschrift wordt getoond hoe de conventionele contour-dynamica methode aangepast dient te worden om ook dit soort problemen te kunnen simuleren.

Naast onderzoek naar de methode zelf is de methode ook toegepast op een aantal fysisch relevante problemen. Zo is onder meer de “collapse” van drie wervels onderzocht. In het geval van drie puntwervels, waarbij alle vorticiteit van de wervel geconcentreerd is in één punt, kunnen sterkte en positie zodanig gekozen worden, dat de drie wervels via spiraalvormige banen in eindige tijd naar één punt bewegen. Dit verschijnsel wordt collapse genoemd. Wanneer de puntwervels door cirkelvormige wervels met een eindig oppervlak vervangen worden, gaat de eindigheid van de wervels een rol spelen zodra de wervels dicht bij elkaar komen. Onderzoek is hoe en waarom de wervels dan gaan vervormen. Uit dit onderzoek is naar voren gekomen dat één wervel in het bijzonder, en niet de zwakste, uiteindelijk volledig uit elkaar getrokken wordt door de aanwezigheid van de andere twee. Van deze wervel blijft een dynamisch niet relevant dun filament over, terwijl de overige twee wervels als een asymmetrische dipool verder bewegen. De grootte van de wervels heeft alleen invloed op het tijdstip waarop de vervormingen beginnen; de dynamica is verder volledig gelijk.

Verder is de methode ook gebruikt om de evolutie van een tripolaire wervel op een γ-vlak te simuleren. In het geval van het γ-vlak is er sprake van niet-uniforme achtergrondvorticiteit en moest de modellering, zoals eerder opgemerkt, aangepast worden. Resultaten zijn vergeleken met laboratoriumexperimenten en tonen kwalitatief een goede overeenkomst.
Dankwoord

In de afgelopen periode heb ik met veel plezier onderzoek verricht aan de Technische Universiteit in Eindhoven. Gedurende deze tijd was ik werkzaam bij twee groepen, wat een heel bijzondere ervaring was. Alhoewel mijn naam op dit proefschrift staat, ben ik niet de enige die bijgedragen heeft aan dit onderzoek. Ik wil daarom graag iedereen bedanken die op een of andere manier een bijdrage geleverd heeft. Er zijn echter een aantal personen die ik in het bijzonder wil bedanken.

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Jean van Montfort, Wouter Graef, Roland Stolk en Veronique Mogendorff ben ik dankbaar voor hun bijdragen tijdens hun stages. Gert van der Plas en Saskia Maassen wil ik bedanken voor hun hulp bij mijn laboratorium experimenten en in het bijzonder bij het gebruik van DigImage. Furthermore, I would like to thank David Dritschel for providing his numerical code, and the elliptical model in particular. Prof.dr.ir. H.W.M. Hoeijmakers wil ik bedanken voor zijn inbreng gedurende de hele periode en in het bijzonder voor het zo zorgvuldig doorlezen van mijn proefschrift tijdens de schrijffase.

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Curriculum Vitæ

Pauline Wilhelmina Catharina Vosbeek was born in Velden on July 30, 1969. From 1981 to 1987 she followed pre-university education (VWO) at the St. Thomascollege in Venlo. Subsequently, she studied Technical Mathematics at the Eindhoven University of Technology and graduated in November 1993. Her graduation project concerned numerical simulations of two-dimensional vortices with contour dynamics, and was supervised by prof.dr. R.M.M. Mattheij and dr.ir. J.H.M. ten Thije Boonkkamp. The graduation project resulted in a continuation in the form of a trainee research assistantship (AIO) both in the Scientific Computing Group of prof.dr. R.M.M. Mattheij at the department of Mathematics and Computing Science and in the Vortex Dynamics Group of prof.dr.ir. G.J.F. van Heijst at the department of Physics, with this thesis as result.

Stellingen
behorende bij het proefschrift
“Contour Dynamics and Applications to 2D Vortices”
Pauline Vosbeek, 1 juli 1998

1. Het snelheidsveld van de ruimtelijk gediscretiseerde vergelijkingen van de contour-dynamica methode is divergentie-vrij, onafhankelijk van de gekozen graad van interpolatie.

_Dit proefschrift, Hoofdstuk 3._

2. De contour-dynamica methode biedt goede mogelijkheden voor het simuleren van twee-dimensionale wervels in de aanwezigheid van niet-uniforme achtergrond vorticiteit.

_Dit proefschrift, Hoofdstuk 6._

3. De vervorming van instabiele wervels kan goed begrepen worden aan de hand van de grootte “strain-rate” zoals onder andere gedefinieerd in Hoofdstuk 4 van dit proefschrift.

4. Het klassificatie-diagram voor de interactie van een puntwervel-dipool met een puntwervel-monopool, kan volledig analytisch bepaald worden. Het op deze manier verkregen diagram wijkt op sommige punten af van het diagram dat door Meleshko en van Heijst werd geconstrueerd op basis van numerieke simulaties.

_V.V. Meleshko & G.J.F. van Heijst, Chaos, Solitons and Fractals, 4:977, 1994_

5. Het gebruik van potentiële vorticiteit bij numerieke weersvoorspellingen kan de betrouwbaarheid gunstig beïnvloeden.

_M.H.P. Ambaum & W.T.M. Verkley, Meteorologica, 1:30, 1998_


6. Diskriminatie is net als kastruimte: hoe meer je ervan hebt, des te meer rommel je bewaart.

7. Wanneer iemands hobby zijn of haar beroep wordt, bestaat het gevaar dat deze persoon op den duur alleen nog maar werkt.

8. Het succes van (pop)musici is niet altijd evenredig met hun talent.


_www.geocities.com/Athens/Acropolis/9949_

10. De NS dreigen te ontsporen door buitensporig hoge prijzen.

11. Vrouwen maken in het algemeen weinig ophef over een mannelijke aanhef.