Sound propagation in lined ducts with parallel flow

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Sound propagation in lined ducts with parallel flow

PROEFSCHRIFT

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Notation

Coordinate systems
\[ x, r, \theta \] cylindrical coordinates
\[ x, y, z \] Cartesian coordinates
\[ u, v, w \] velocity components, cylindrical (or if mentioned explicitly: Cartesian)

Notation
\[ T_\infty, c_\infty, \ldots \] reference temperature, sound speed, etc.
\[ p_0, v_0, \ldots \] mean flow quantities
\[ p_1, v_1, \ldots \] perturbations of mean flow
\[ P, U, \ldots, \bar{P}, \bar{U}, \ldots \] modal shape functions (eigenfunctions)
\[ a \] 'physical' vector or second order tensor
\[ a \] 'numerical' vector
\[ A \] matrix

Operators / maps
\[ \frac{d}{dt} = \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \] total derivative
\[ \frac{\mathcal{D}}{\mathcal{D}t} = \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \] total derivative for parallel mean flow
\[ \nabla_\perp, \nabla_\perp \cdot \] transverse gradient / divergence
\[ \langle\langle \bar{F}, \tilde{F} \rangle\rangle \] bilinear map (arbitrary cross-section)
\[ \langle F, \tilde{F} \rangle \] bilinear map (circular cross-section)
\[ (\cdot)_y, (\cdot)_z \] \( y, z \) derivative

Latin symbols
\[ b = d + d_l \] \( \text{[m]} \) liner outer wall radius
\[ c \] \( \text{[m/s]} \) speed of sound
\[ c_p, c_v \] \( \text{[J/(kg·K)]} \) specific heats at constant pressure, volume
\[ d \] \( \text{[m]} \) duct radius
\[ d_l \] \( \text{[m]} \) liner depth
\[ \mathcal{D} \] \( \text{[J/(m^3·s)] = kg/(m·s^3)} \) perturbation energy 'dissipation'
\[ e \] \( \text{[J/kg]} \) specific internal energy
\[ E \] \( \text{[J/kg]} \) specific total energy
\[ \mathcal{E} \] \( \text{[J/m^3 = kg/(m·s^2)]} \) perturbation energy density
\[ f \] \( \text{[1/s]} \) frequency
\[ h \] \( \text{[J/kg]} \) specific enthalpy
\[ h_j, h_{\text{max}} \] continuation step size (Sec. 3.2)
\[ P^{(1)}_n, P^{(2)}_m \] Hankel functions of first and second kind
\[ I \] \( \text{[J/(m^2·s)] = kg/s^3]} \) perturbation energy flux vector
\[ J_m \] Bessel function of the first kind
\[ k \] \( \text{[1/m]} \) axial wavenumber
\[ L \] \( \text{[1/m]} \) typical length scale / duct length
Contents

\begin{itemize}
  \item \textit{M} \hspace{1cm} \text{Mach number (dim. less axial mean velocity)}
  \item \textit{m} \hspace{1cm} \text{circumferential wavenumber}
  \item \textit{p} \hspace{1cm} \text{[ N/m}^2 \text{ = Pa ] pressure}
  \item \textit{q} \hspace{1cm} \text{[ N/m}^2 \text{ = Pa ] pressure in liner region}
  \item \textit{q} \hspace{1cm} \text{[ J/(m}^2 \cdot \text{s} \text{) = W/m}^2 \text{ ] heat flux vector}
  \item \textit{R} \hspace{1cm} \text{dimensionless mean flow density}
  \item \textit{\mathcal{R}} \hspace{1cm} \text{[ J/(kg} \cdot \text{K} \text{) ] gas constant}
  \item \textit{s} \hspace{1cm} \text{[ J/(kg} \cdot \text{K} \text{) ] specific entropy}
  \item \textit{\hat{S}, \mathbf{S}, \bar{S}} \hspace{1cm} \text{interface, segment, cumulative scattering matrix}
  \item \textit{t} \hspace{1cm} \text{[ s ] time}
  \item \textit{T} \hspace{1cm} \text{[ K ] temperature}
  \item \textit{v = 1/\rho} \hspace{1cm} \text{[ m}^3 \text{/kg ] specific volume}
  \item \textit{v} \hspace{1cm} \text{[ m/s ] velocity vector}
  \item \textit{X = \varepsilon x} \hspace{1cm} \text{slow variable}
  \item \textit{Y_m} \hspace{1cm} \text{Bessel function of the second kind}
  \item \textit{Z, Z_0, Z_c} \hspace{1cm} \text{[ kg/(m}^2 \cdot \text{s} \text{) ] impedance, face sheet / characteristic (porous mat.)}
\end{itemize}

\section*{Greek symbols}
\begin{itemize}
  \item \textit{\alpha} \hspace{1cm} \text{[ m}^2 \text{/s ] thermal diffusivity (Sec. 2.1)}
  \item \textit{\alpha} \hspace{1cm} \text{[ 1/m ] radial wavenumber}
  \item \textit{\beta_s, \beta_T} \hspace{1cm} \text{[ m}^2 \text{/N = 1/Pa ] adiabatic, isothermal compressibility}
  \item \textit{\bar{\beta}, \gamma} \hspace{1cm} \text{coefficients of Pridmore-Brown eqn.}
  \item \textit{\gamma = c_p/c_v} \hspace{1cm} \text{ratio of specific heats}
  \item \textit{\delta} \hspace{1cm} \text{[ m ] boundary layer thickness}
  \item \textit{\epsilon, \epsilon} \hspace{1cm} \text{small parameter}
  \item \textit{\epsilon_j, \epsilon_{\text{tol}}} \hspace{1cm} \text{error in \kappa-prediction (Sec. 3.2)}
  \item \textit{\Theta} \hspace{1cm} \text{dimensionless mean flow temperature}
  \item \textit{\kappa} \hspace{1cm} \text{[ W/(m} \cdot \text{K} \text{) ] thermal conductivity}
  \item \textit{\lambda} \hspace{1cm} \text{[ m ] wavelength}
  \item \textit{\Lambda = \omega - \mu M} \hspace{1cm} \text{dimensionless Doppler-shifted frequency}
  \item \textit{\mu, \mu_v} \hspace{1cm} \text{[ Pa} \cdot \text{s = kg/(m} \cdot \text{s) ] dynamic / volume viscosity (Sec. 2.1)}
  \item \textit{\mu, \nu} \hspace{1cm} \text{radial mode order}
  \item \textit{\mu_p} \hspace{1cm} \text{[ 1/m ] propagation constant (porous material)}
  \item \textit{\nu} \hspace{1cm} \text{[ m}^2 \text{/s] kinematic viscosity}
  \item \textit{\Pi} \hspace{1cm} \text{dimensionless mean flow pressure}
  \item \textit{\rho} \hspace{1cm} \text{[ kg/m}^3 \text{ ] density}
  \item \textit{\rho_e} \hspace{1cm} \text{[ kg/m}^3 \text{ ] effective density (porous material)}
  \item \textit{\sigma} \hspace{1cm} \text{[ kg/(m}^3 \cdot \text{s}) ] resistivity (porous material)}
  \item \textit{\tau} \hspace{1cm} \text{[ N/m}^2 \text{ = Pa ] deviatoric stress tensor}
  \item \textit{\Phi} \hspace{1cm} \text{[ kg/(m} \cdot \text{s}^3 \text{) ] viscous dissipation term in energy equation}
  \item \textit{\omega} \hspace{1cm} \text{[ rad/s ] radial frequency}
  \item \textit{\omega := \nabla \times \mathbf{v}} \hspace{1cm} \text{vorticity vector}
  \item \textit{\Omega = \omega - ku_0} \hspace{1cm} \text{Doppler-shifted frequency}
  \item \textit{\Omega} \hspace{1cm} \text{porosity (Sec. 2.3.2)}
\end{itemize}

\section*{Dimensionless numbers}
\begin{itemize}
  \item \textit{Re} \hspace{1cm} \text{Reynolds number}
  \item \textit{Ec} \hspace{1cm} \text{Eckert number}
  \item \textit{Pr} \hspace{1cm} \text{Prandtl number}
  \item \textit{Pe} \hspace{1cm} \text{Péclet number}
\end{itemize}
Chapter 1

Introduction

1.1 Motivation: ramp noise

Have you ever had the experience that while boarding an aircraft you had to shout to your fellow travelers to make yourself audible? Sometimes passengers board a plane through a passenger boarding bridge, but boarding stairs are also frequently used, which can normally be reached by walking from the gate in open air. Especially in this last case a lot of noise can be heard coming from the aircraft while it is being prepared for departure. If you recognize this experience you have probably also noticed that the workers who are loading your luggage onto the plane are usually wearing gigantic ‘headphones’ for hearing protection.

The area where the airplane is parked so that boarding, the on- and offloading of luggage, refueling, etc. can take place, is referred to as the airport apron or ramp. The noise emitted by the aircraft in this area, while the main engines are still switched off, is consequently called ramp noise. Ramp noise is caused mainly by the Auxiliary Power Unit (APU) [62, 89].

More generally, an APU is an engine fitted on a transport vehicle that produces the power used for other purposes than propulsion. In the modern aircraft context the APU is a turbine engine in the tail of an aircraft (see Figures 1.1 and 1.2) that produces power to operate the electrical systems (e.g. for air conditioning) when the main engines are switched off. It also often produces bleed air—compressed air taken from within the engine—that is used to start the main engines, to pressurize the main cabin, for de-icing, and to operate pneumatic actuators.

In order to reduce the noise coming out of the APU exhaust duct, its inside is generally treated with an acoustically absorbing lining. Over the past decades aeroacoustic research for lined flow ducts—generally referred to as duct acoustics—was primarily aimed at reducing the noise levels in inlet and exhaust ducts of the main (turbofan) engines. Recently ramp noise has been given more attention, as regulations have become more stringent. The International Civil Aviation Organization (ICAO) has published guidelines for noise certification regarding the maximum noise levels around the aircraft during ground operations [50]. Also the European Union has defined limits to the exposure of workers to ramp noise [38]. For some airports (e.g. Copenhagen [85]) the use of the APU is even more restricted in order to reduce noise and pollution; external power supplies have to be used instead.
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As for basically any aircraft component, it is very important for an APU exhaust duct to be light-weight. Moreover, geometrical constraints exist because of the conical shape of the aircraft fuselage tail. To find an optimum for the trade-off between minimal weight and noise emissions there is a need for more insight in the main noise source propagation mechanisms and accurate design tools. This was the main motivation for the research described in this thesis. Many results, however, are equally well applicable to sound propagation in any other type of duct, like for example the turbofan inlet and exhaust ducts. We therefore proceed by sketching the wider context of aircraft duct acoustics [84].

1.2 Context: aircraft duct acoustics

The complete problem of noise radiating from an aircraft turbo-engine can be split in three parts: the noise source mechanism, the propagation of the noise through the duct, and the radiation emanating from it. This thesis concerns the second part: duct acoustic propagation. Before we turn our attention to the specifics of an APU exhaust duct, let us consider the wider context of the field of aircraft duct acoustics.

In order to be able to describe the relevant physical phenomena and some historical context here, and to introduce the problem in the next section, it is required that we first introduce some basic terminology that will be used henceforth. This introduction is very concise—we will provide more details in the following chapters.

In the field of duct acoustics we are interested in small perturbations of a steady mean flow. The acoustic problem is solved separately from the mean flow problem. For aerodynamic reasons many ducts are by and large invariant in the direction of the duct axis. Therefore, a common approach to model duct acoustics problems is to describe acoustic solutions in terms of duct modes. This approach will be used extensively throughout this thesis. These duct modes are solutions that are self-similar in the axial direction; they are characterized by a mode shape (eigenfunction), which is defined on the cross-section of the duct, and an axial wavenumber (eigenvalue), which determines the axial propagation of the mode. These mode shapes are oscillatory in nature; they can be compared to the vibrations of the membrane of a drum. In general many modes (or eigensolutions) exist. A discrete set of modes is generally ordered by the number
of oscillations, e.g. a mode of higher order is more oscillatory than a lower order mode. Some modes are more propagative than others; a mode which is (mainly) propagative in character will be referred to as cut-on, whereas a cut-off mode exponentially decays in the axial direction and does not propagate.

Classically, the field of duct acoustics was primarily concerned with musical instruments and air conditioning systems in buildings. For these applications the wavelengths are large with respect to the duct diameter, the mean flow speed is very small, and the duct walls are rigid; hence models based on plane waves (the lowest order mode) without mean flow could be used [37]. The field of duct acoustics significantly broadened with the growth of commercial aviation after World War II.

Before the advent of the turbofan engine, the turbulent exhaust jet was the primary source of aircraft noise. When the engine with bypass duct became prominent in the sixties, noise from the compressor and the fan stages inside the engine became more important. As the bypass ratio (the ratio of the mass flow rate through the annular bypass duct to that of the engine core) increased, the importance of jet noise decreased and fan noise became a relatively more important noise source. As a result, more attention was paid to sound propagating upstream through the inlet duct, and downstream through the bypass ducts.

An important early development is the so-called Tyler-Sofrin rule [117]. Most of the noise on the inlet side is due to rotor-stator interaction: rotor wakes are impinging on stator vanes (which are present to recover energy from the swirl in the mean flow), producing a specific set of interaction tones. By choosing the number of rotor blades and stator vanes in a clever way it can be ensured that the first (few) harmonics are cut-off. This greatly reduced the interaction noise.

Turbofans are very large with respect to the relevant wavelengths, so models based on plane waves were not sufficient anymore, and higher order modes had to be taken into account. Moreover, the velocity of the air flowing through the engine is high; even at take off or landing the typical Mach number (the ratio of flow speed to sound speed) is 0.7. Therefore, the Doppler-shift due to the convection of the sound waves can not be neglected, and models based on uniform flow (sometimes referred to as plug flow) were introduced [41].

The mean flow in the inlet duct is indeed almost uniform, with only a very thin boundary layer. In the annular exhaust duct however, the flow is strongly sheared, which requires models based on non-uniform mean flow velocities. For a fluid flowing along a wall, the fluid velocity can be assumed to be approximately constant away from the wall, and fall to zero in the boundary layer near the walls. As pointed out by Pridmore-Brown [97], on a ray-acoustics picture, sound propagating downstream through this fluid will be refracted towards the wall, while upstream sound rays will be bent away from the wall. Pridmore-Brown derived an ordinary differential equation for the acoustic duct modes in a parallel shear flow, which is usually referred to as the Pridmore-Brown equation. This equation plays a central role in this thesis.

It is well known that sound waves refract towards the region with the lowest sound speed, which is in stagnant flow the region with the lowest temperature. Consequently, again on a ray-acoustics picture, it can be seen that cooling the duct wall leads to refraction of the sound towards the walls for both upstream and downstream propagating waves. The effect of transverse temperature gradients can be included in the Pridmore-Brown equation [16, 56, 83]. The typical temperature gradients of the mean flow are not very large in the inlet and exhaust ducts of a turbofan, yet they are important for
Chapter 1. Introduction

Figure 1.3: Honeycomb Helmholtz resonator array liner. Copied with permission from [116].

an APU exhaust duct, as we will discuss in the next section.

To alleviate the noise problem aircraft engine ducts are treated with an acoustically absorbent lining. For turbofans these liners consists of a honeycomb structure (rectangular cavities have also been introduced) sandwiched between a solid backing plate and a facing sheet, see Figure 1.3. The face sheet generally consists of a perforated plate (for example punched or laser-drilled aluminum or mechanically-drilled carbon fiber composites) or a sheet of woven metal wires (wire mesh). This configuration is often referred to as a single degree of freedom (SDOF) type of liner, in contrast to a double degree of freedom (2DOF) liner, which adds an extra layer of honeycomb material and a porous mid sheet or septum. The 2DOF liner can be designed to attenuate a broader range of frequencies. The fact that these liners are light-weight, solid structures is an important advantage for aircraft applications [77].

Honeycomb liners can be modeled as Helmholtz resonator arrays. The depth of the individual cells of these honey-comb structures is typically a few centimeters\(^1\), and the diameter of the order of one centimeter. For the prevailing frequencies only plane waves are therefore propagating in the direction along the cell axis perpendicular to the liner wall, and no waves are permitted to propagate in the direction parallel to the liner wall. Consequently, these liners can be described by a single number which is independent of the position (the impedance, a complex-valued number at a fixed frequency); they are referred to as locally reacting.

A disadvantage of Helmholtz resonator type liners lies in their limited frequency range. When a broader range is to be attenuated also porous material may be used between the solid backplate and the face sheet, because its attenuation properties are less frequency dependent. This type of liner is referred to as a bulk absorber. For the main engines bulk absorbers are currently not implemented, since they do not have the required mechanical and thermal properties, and because of weight issues. This may change as more advanced materials, such as metal or ceramic foam or fibers become available.

As the porous material permits waves traveling in a direction parallel to the liner wall, the acoustic fields in the liner and the duct regions are inherently coupled [106]. The liner can therefore not be modeled by means of a single point value; this is why bulk absorber liners are also referred to as non-locally reacting. Also large hollow liner cavities (as can be found in mufflers), and honeycomb liners with interconnecting

\(^1\)Approximately \(\frac{1}{4} \lambda = \frac{1}{4} \frac{c}{f} \) at \(f = 1500 \text{–} 4000\) Hz.
drainage slots are non-locally reacting.

Finally, a more recent development is the zero-splice inlet liner. The inlet lining of most turbofans consists of two to three sections that are fitted together, which leaves gaps (non-treated wall) between the individual lining sections. These gaps are referred to as splices. It was always assumed that the influence of these hard-wall patches is minor because of their small surface area, so they were usually neglected during the liner design process. However, a significant noise reduction could be achieved after realizing that these splices break the circumferential symmetry of the inlet waveguide. As a result, the spinning acoustic field connected to the rotor produces an interaction field (similar to the rotor-stator interaction tones mentioned earlier) of weakly damped other modes. Because of this, the effectiveness of the liner is improved significantly by ensuring that there are no splices [11, 69].

1.3 Problem description

The problem of understanding and predicting the acoustic performance of aircraft engines can be approached in several ways: through experiments, or with the aid of analytical or numerical models. As the production of an aircraft engine is exceptionally costly, the possibilities for performing experiments on real engines are very limited, and in many cases experiments are even technically impossible to perform. During the last decades, the increase of computing power and the continuous improvement of numerical methods has enabled the study of ever more complex physical problems without resorting to experiments. However, the general importance of simplified analytical models of complex physical and technical problems can hardly be overstated [30]. A solid theoretical framework is always required for developing, performing and interpreting both computational and experimental investigations. Asymptotic results may often be used to increase numerical efficiency and accuracy. On the other hand, scientists and especially engineers are often primarily interested in the actual solution of a problem [67]. Whereas analytical models give mostly (although not always) qualitative answers and can be applied to simplified geometries, numerical methods are able to handle more complex (realistic) geometries and always give definitive numerical answers. However, the computational cost of numerical methods is often still prohibitively large for realistic problems, and the computations might produce unphysical results if not performed carefully. An analytical approach may then often be used in tandem with a numerical approach, which we loosely call ‘semi-analytical’, in order to have the best of both worlds.

As mentioned before, the research described in this thesis is motivated by the problem of sound propagation through an APU exhaust duct. We will now describe its typical properties, which are also schematically depicted in Figure 1.1, and mention a number of challenges that we wish to address.

The duct is typically straight and circular cylindrical, has a typical length of 1 m and a typical radius of 15 cm. Cool air enters the duct through the ventilation inlet on the side. This results in a shear layer between the APU turbine exhaust gas flow and the ventilation air flow with a strong thermal gradient. In reality the mean flow velocity and temperature develop along the axial direction due to the existence of thermal conduction and viscous forces. However, here we use the assumption (common for acoustics) that the gas is ideal, inviscid and non-heat conducting; in view of the fact that the boundary and shear layers develop slowly (on a length scale of the duct length,
which is typically one order of magnitude smaller than the duct radius), and because the mean flow is free of swirl (i.e. it is not rotating in the plane perpendicular to the duct axis), we assume a fully developed parallel mean flow. In other words: only the axial component of the mean flow velocity is non-zero and varies in the transverse direction, and also the mean temperature varies only in the transverse direction. Typical Mach numbers (i.e. velocity–sound speed ratios) range from 0.1 to 0.3, and the temperature difference between the hot and cold flows is typically a few hundred degrees. The uniform mean (ambient) pressure has a typical value of 1 bar. The mean density then follows from the transverse temperature profile.

Due to the conical shape of the aircraft fuselage tail, the lining inside the duct wall has an axially decreasing depth ranging from more than ten to a few centimeters. It is covered with a (mostly resistive) facing sheet consisting of a perforated plate and a wire mesh. The lining typically consists of a number of relatively large annular liner segments, which may be open or filled with porous material. The duct wall may also consist of Helmholtz resonator arrays (honeycombs) with axially varying depth.

To compute the sound inside the duct we formulate the problem in terms of acoustic duct modes, i.e. the infinitely many eigensolutions of the boundary value problem consisting of the Pridmore-Brown equation for parallel mean flow and transverse temperature gradients subject to suitable boundary conditions. We assume that all acoustic solutions can be described in terms of these modes (which might not be the case for shear flow [23]). For this purpose it is important that all relevant modes (for example with axial wavenumbers in a specific part of the complex plane) are found.

The Pridmore-Brown equation reduces to Bessel’s equation for the specific case of uniform mean flow and temperature. Consequently, the eigenfunctions are in this case Bessel functions and the eigensolutions follow from the roots of a function that is available in closed form. For boundary conditions corresponding to a hard wall these roots are even real-valued, so they are easily found numerically, since the real numbers are ordered. For the case of locally or non-locally reacting boundary conditions these roots lie in the complex plane, so there is no natural ordering. A Newton-based root-finder may be used, provided that good initial guesses are available. Depending on the lo-
cation of the initial guesses and the basins of attraction of the actual roots, Newton's method may not converge to all of the relevant roots. This problem is often mitigated somewhat in practice by using an excess number of initial guesses, but this does not fundamentally solve the problem, hence there is a risk that some modes are missed.

For the more general case of non-uniform mean flow and temperature solutions of the Pridmore-Brown equation are not known in closed form. Hence, we have to solve the boundary value problem numerically. Also in this case good initial guesses are important to ensure that all relevant modes are found.

For modes to exist it is necessary that, at least locally, the duct geometry is invariant in the axial direction. More specifically: the duct should have constant boundary conditions. For an APU exhaust duct the liner depth is axially varying even though this variation is small scale compared to the duct length. Consequently, the boundary condition varies in the axial direction. If this variation is continuous, then duct modes in a strict sense do not exist.

If the boundary conditions are step-wise continuous, i.e. if the duct is subdivided into a number of segments, each individual segment having a uniform depth, then the duct is axially invariant within each segment. At the interface between two liner segments the boundary conditions are discontinuous, and both segments have a different set of duct modes. An incoming wave from one segment is partly reflected and partly transmitted to the other segment. These scattering effects are studied by means of the mode-matching method, which relates the modal amplitudes in two adjacent segments via a scattering matrix. The entries of this matrix are integral inner products of duct eigenfunctions. For uniform flow and temperature the values of these integrals can be computed analytically since the eigenfunctions are available in closed form; for non-uniform flow and temperature we have to use the numerically computed eigenfunctions, so the integrals are not available in closed-form. Numerical quadrature can be used, but this is computationally expensive due to the oscillatory nature of the eigenfunctions, and it has only a finite numerical accuracy.

A common design criterion for liner design is the transmission loss, which is defined as the difference between the acoustic power that is radiated from the duct and the input acoustic power. Knowledge about the noise source characteristics is therefore required for its computation. Little is known about the specific characteristics of the noise coming from the APU turbine; it is typically characterized as broad-band, having frequencies ranging from 100 Hz to 10 kHz. As this thesis considers only the duct propagation, no specific knowledge of noise sources is assumed, so we choose a very general form if required. Furthermore, we assume that the duct end is reflection free, or in other words, that it has infinite length. This is in general not true, specifically for a hard-walled duct at low frequencies (wavelength comparable to the duct radius). However, for higher frequencies and for waves not too close to cut-off, the reflections are very small. In addition, when the duct wall is treated, the amplitude of the sound field is reduced, and the influence of the end reflection may be even smaller. [105, 108]

1.4 Objectives and main results

The main goal of this work is to provide semi-analytical solutions for the propagation of sound in lined flow ducts with parallel flow and strong thermal gradients, and axially varying lining. We focus primarily on the applicability of the models for an APU exhaust duct, although other applications like turbofan inlet and exhaust ducts are
equally well possible. For this purpose several ingredients are required. Firstly, we need an eigenmode solver that is numerically efficient and robust; it is important that all relevant eigensolutions are found. Secondly, we require methods to handle the axial variation of liner properties.

Before we proceed by mentioning our specific objectives and results, we first mention that the basic modeling steps that eventually lead to the Pridmore-Brown equation are presented in Chapter 2. In this chapter we also describe how sound absorbing walls can be modeled through the boundary conditions.

**Path-following to find Pridmore-Brown eigensolutions**

Our numerical approach to find Pridmore-Brown eigensolutions for non-uniform flow and temperature in an efficient and robust manner is based on a combination of the COLNEW code and a path-following (or continuation) approach (this is the topic of Sections 3.1 and 3.2). To ensure that we find all relevant eigensolutions, we start from an easy solution (for example uniform flow and temperature) and trace the solution when the relevant problem parameters are varied to the values of interest, tacitly assuming that no other solutions appear during the path. Our path-following approach is refined by using a prediction-correction scheme. The prediction is found by linear extrapolation of the previous solutions. The correction step is then an updated solution by COLNEW, with the prediction as the starting value. In addition, we take into account that for specific choices of the impedance paths some modes may appear from infinity, in which case they will be missed. We make sure that the impedance is varied such that all modes start from a finite value and move towards infinity as the impedance goes to a hard wall value. The effectiveness of the numerical method is illustrated by comparing it with some asymptotic results (which are presented in Section 2.5).

**Root finding by contour integration to find initial guesses**

Our initial guesses of the eigensolutions are found with the aid of a robust and efficient root-finding technique based on complex contour integration (this is the topic of Section 3.3). Finding the solutions for a lined duct carrying uniform flow and temperature, which serve as the initial guesses for the path-following algorithm, amounts to finding the complex-valued roots of an analytic function. To compute these roots we first construct a polynomial that has the same roots as the analytic function of interest by using contour integration (i.e. numerical quadrature). Subsequently the roots of the polynomial can be computed in a standard manner. The advantage of this method lies in the fact that it is guaranteed that all roots inside a given area of the complex plane are found (apart from finite numerical accuracy issues), whereas a Newton method only converges to all of the roots if it is started from sufficiently close initial guesses. The location of these initial guesses for the Newton method is not always self-evident, as for example in case of surface waves and modes that exist only with non-locally reacting lining.

**Mode-matching based on closed-form integrals**

One way to deal with axially varying liners is the mode-matching method, which is essentially a projection method. For non-uniform flow, in the classical approach, the necessary inner products need to be evaluated by numerical quadrature. We present
§ 1.4 Objectives and main results

in Chapter 4 a new mode-matching method, which consists of replacing the inner-products with expressions that can be evaluated in closed form. Instead of the standard inner product we use a bilinear form that resembles an inner product; it is an integral of a weighted combination of products of Pridmore-Brown modes that can be evaluated in closed form. Apart from numerical efficiency, this approach features also a higher accuracy, because it avoids the inherently inaccurate numerical quadrature of oscillating functions. Numerical results of the new and the classical approach are compared, and the agreement is excellent, with higher accuracy and greater computational efficiency for the new approach.

Once we have the mode-matching system of equations available for each interface, we compute the combined transmission and reflection effects at multiple interfaces with the aid of the numerically stable scattering-matrix formalism. In contrast, the transfer matrix formalism, which considers the propagation of cut-off modes in both directions, can potentially lead to exponentially large and small values in the matrix equations, which can result in numerically ill-posed problems. The scattering-matrix formalism only considers decaying waves; it is therefore numerically stable, also for a large number of segments.

The availability of the numerical Pridmore-Brown eigenfunctions enables us to also consider asymptotic solutions in the form of slowly varying modes of WKB type. Solutions of this type for an impedance that is continuously varying in the axial direction are presented in Chapter 5. Additionally, we compare these asymptotic results with mode-matching results. We illustrate that for a realistic APU exhaust duct configuration, for which the impedance is not slowly varying due to liner resonances, the asymptotic results are not applicable.

Application to APU exhaust duct

Finally, we aim to illustrate the practical applicability of the developed methods and to present results and insights that facilitate design studies of actual APU exhaust ducts. Based on test cases that are typical of an APU exhaust duct geometry, we describe the refraction effects due to non-uniform mean flow and temperature, and describe how this influences the sound attenuation. Furthermore, we discuss an approach to model the noise source for transmission loss calculations, when the exact source is not known. For the test cases that we present especially the temperature non-uniformity is very beneficial for the attenuation. It is also shown that modes of the lowest circumferential order are the least attenuated, and consequently the most interesting ones for design calculations.
Chapter 2

Basic equations and model

We start this chapter by formulating the equations that govern acoustics in Section 2.1. Based on the modal approach described in Section 2.2 we subsequently describe in Section 2.3 the boundary conditions that model sound absorbing walls, and derive the Pridmore-Brown equation in Section 2.4. We conclude this chapter by presenting some asymptotic approximations of solutions of this equation in Section 2.5.

2.1 Governing equations

As acoustics is often defined as small dynamic perturbations of a steady fluid flow we first recapitulate the basic equations of thermodynamics and the conservation laws of fluid mechanics. We then consider the governing equations and state our basic assumptions for both the mean flow and the acoustic problem. It will be argued that viscous and heat conductive effects can be neglected for both problems. We subsequently show that the perturbations can be described by the Linearized Euler Equations (LEE), since they are very small. Since any solution of the LEE satisfies Myers’ energy corollary, this can be used to validate numerical results.

2.1.1 Thermodynamics

In thermodynamics [109] a distinction is made between thermodynamic state properties, like for example the specific (i.e. per unit mass) internal energy $e$, and other variables, like for example the amount of specific heat $q$ that is transferred to the system and the specific work $w$ that is done by the system. Changes in thermodynamic properties are described by exact differentials $d\cdot$, since when they are integrated, the result will be independent of the thermodynamic path\(^1\). The specific heat $q$ and work $w$ however, are not state properties, so changes are described by an inexact differential $\delta \cdot$, and the value of an integral of $\delta q$ or $\delta w$ depends on the specific path.

The first law of thermodynamics, which describes the conservation of energy, can be expressed as

\[
de = \delta q - \delta w. \tag{2.1}\]

\(^1\)A differential form $dq = a(x,y)dx + b(x,y)dy$ is called exact if the value of the integral $\int dq$ is path-independent. This is true if $a$ and $b$ are the components of a conservative vector field with potential $q$, and hence $dq = \left[\frac{\partial q}{\partial x}\right] dx + \left[\frac{\partial q}{\partial y}\right] dy.$
The absolute temperature $T$, specific entropy $s$, pressure $p$, density $\rho$ and specific volume $v = 1/\rho$ are thermodynamic state functions as well. One formulation of the second law of thermodynamics, which describes the concepts of entropy and reversibility, is

$$ds \geq \frac{\delta q}{T},$$

where equality only holds for reversible processes. For a simple compressible system that can only do pressure-volume work $\delta w = pdv$ it follows that

$$de = Tds - pdv = Tds + \frac{p}{\rho^2}d\rho.$$  \hspace{1cm} (2.3)

This is the fundamental thermodynamic relation for a closed single-component system (a gas for instance). The fundamental thermodynamic relation holds for either reversible or irreversible processes, since all quantities are state functions. However, for irreversible processes, $Tds$ and $pdv$ can not be associated with heat and work any more.

As we will see later sound propagation is an isentropic (i.e. with constant $s$) process; the sound speed $c$, which is also a thermodynamic state property, is defined\footnote{We use := to denote a definition.} as

$$c^2 := \left(\frac{\partial p}{\partial \rho}\right)_s.$$  \hspace{1cm} (2.4)

Complying with the common notation in thermodynamics literature, the subscript $s$ is used here to express that $p$ is considered to be a function of $\rho$ and $s$, and that $s$ is kept constant in the partial derivative. For a simple compressible system, according to the state postulate, the thermodynamic state is completely determined by two thermodynamic state functions, so we have for example: $\rho = \rho(p, s)$. Consequently $d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds$, so we can write:

$$d\rho - \frac{1}{c^2}dp = \left(\frac{\partial \rho}{\partial s}\right)_p ds.$$  \hspace{1cm} (2.5)

The heat capacity, which describes the capacity of a material to store energy as the temperature is increased, can generally be defined as

$$c_{\text{heat}} = \frac{\delta q}{\delta T}.$$  \hspace{1cm} (2.6)

At constant pressure it follows from the fundamental thermodynamic relation that $dh = Tds$, where the specific enthalpy is defined as $h = e + pv$. For a reversible process we have $Tds = \delta q$, so $dh = \delta q$ at constant pressure. Similarly, at constant volume we have $de = \delta q$. This motivates the definition of the specific heat capacity at constant volume $c_v$ and the specific heat capacity at constant pressure $c_p$, and the ratio of heat capacities as

$$c_v := \left(\frac{\partial e}{\partial T}\right)_v, \quad c_p := \left(\frac{\partial h}{\partial T}\right)_p, \quad \gamma := \frac{c_p}{c_v}.$$  \hspace{1cm} (2.7)

For air $\gamma \approx 1.4$. Generally the state functions $c_p$ and $c_v$ depend on two of the other thermodynamic properties, e.g. pressure and temperature. For an ideal gas this is not the case, as we will show next.
Differentiating (2.3) and using one of the so-called ‘Maxwell relations’
\( \left( \frac{\partial p}{\partial T} \right)_v = \left( \frac{\partial s}{\partial v} \right)_T \)
(which follow from the fact that for exact differentials mixed partial differentials commute) yields
\[
\left( \frac{\partial e}{\partial v} \right)_T = T \left( \frac{\partial s}{\partial v} \right)_T - p
= T \left( \frac{\partial p}{\partial T} \right)_v - p = T^2 \left( \frac{\partial (p/T)}{\partial T} \right)_v.
\]
(2.8)

In this research we assume that we can use the \textit{equation of state} of an \textit{ideal gas}
\[
p = \rho R T,
\]
(2.9)
where \( R \) is the \textit{specific gas constant} (for air: \( R = 287 \text{ J/(kg·K)} \)). It follows that for an ideal gas \( \left( \frac{\partial e}{\partial v} \right)_T = T^2 \left( \frac{\partial (R/\rho)}{\partial T} \right)_v = 0 \), so \( e \) is a function of \( T \) only. It can also be shown that \( h = h(T) \) in a similar fashion. Thus for an ideal gas the relations
\[
de = c_v dT, \quad dh = c_p dT
\]
(2.10)
can be used, from which it follows that \( dh = d(e + pv) = de + d(RT) \) so \( c_p dT = c_v dT + \mathcal{R} dT \) and consequently
\[
\mathcal{R} = c_p - c_v.
\]
(2.11)
Using these relations as well as \( dh = T ds + vd p \) it can also be found that for an ideal gas
\[
ds = \frac{c_v}{p} dp - \frac{c_p}{\rho} d\rho.
\]
(2.12)
For constant entropy, i.e. \( ds = 0 \), it then follows from this relation that the sound speed of an ideal gas can be computed as
\[
c^2 = \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{\gamma p}{\rho} = \gamma \mathcal{R} T.
\]
(2.13)
For a \textit{perfect gas} \( c_p \) and \( c_v \) are constant so (2.12) can be integrated to find
\[
s = c_v \log p - c_p \log \rho + (s_{\text{init}} - c_v \log p_{\text{init}} + c_p \log \rho_{\text{init}}),
\]
(2.14)
where \( s_{\text{init}}, p_{\text{init}}, \) and \( \rho_{\text{init}} \) denote the initial entropy, pressure and density of the fluid.\(^3\)

\section*{2.1.2 Conservation laws for a Newtonian fluid}

We shall next recapitulate the conservation laws of fluid dynamics [60]. We consider a fluid with a velocity field \( \mathbf{v} \), specific density \( \rho \), pressure \( p \), deviatoric stress tensor \( \mathbf{\tau} \) (the non-isotropic part of the Cauchy stress tensor), and \textit{heat flux vector} (due to heat conduction) \( \mathbf{q} \). The \textit{specific total energy} \( E \) is the summation of the specific kinetic and

\(^3\)This particular form of the integration constant between brackets makes clear that the dimensions are correct (i.e. the arguments of \( \log \left( \frac{p}{p_{\text{init}}} \right) \) and \( \log \left( \frac{\rho}{\rho_{\text{init}}} \right) \) are dimensionless).
the specific internal energy: \( E = e + \frac{1}{2} |v|^2 \). The conservation laws in differential form describing conservation of mass, momentum and total energy are respectively

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2.15a}
\]

\[
\frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho vv) = -\nabla p + \nabla \cdot \tau, \tag{2.15b}
\]

\[
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho E v) = -\nabla \cdot q - \nabla \cdot (p v) + \nabla \cdot (\tau \cdot v). \tag{2.15c}
\]

Often these equations are referred to as the compressible Navier-Stokes equations. The fundamental thermodynamic relation (2.3) can be used, together with (2.15a) and the inner-product of (2.15b) with \( v \) to obtain a different form of the energy conservation law (2.15c), namely

\[
\rho T \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) s = -\nabla \cdot q + \tau : \nabla v, \tag{2.15d}
\]

where the double dot product (\( : \)) of two tensors can be computed as \( a : b = \sum_{i,j} a_{ij} b_{ji} \).

Constitutive equations are required to solve this system; there are more unknowns than equations in (2.15). Here we choose Fourier’s law of heat conduction and the stress relation for a Newtonian fluid. For a fluid with thermal conductivity \( \kappa \), dynamic viscosity \( \mu \) and volume viscosity \( \mu_v \) (also called second viscosity coefficient) these are

\[
q = -\kappa \nabla T, \tag{2.16a}
\]

\[
\tau = \mu \left( \nabla v + (\nabla v)^T \right) + \mu_v (\nabla \cdot v) \delta, \tag{2.16b}
\]

where \( \delta \) is the unit tensor\(^4\). For many applications the Stokes assumption \( \mu_v = -\frac{2}{3} \mu \) is found to be sufficiently accurate.

For a fluid element that is carried with the flow along a stream line we can use the convective derivative, which is defined as

\[
\frac{d}{dt} := \left( \frac{\partial}{\partial t} + v \cdot \nabla \right). \tag{2.17}
\]

We also assume that the physical parameters \( \mu, \mu_v \) and \( \kappa \) are constant. The conservation laws can now be formulated as

\[
\frac{d\rho}{dt} + \rho \nabla \cdot v = 0, \tag{2.18a}
\]

\[
\rho \frac{dv}{dt} + \nabla p = \mu \nabla^2 v + (\mu_v + \mu) \nabla (\nabla \cdot v), \tag{2.18b}
\]

\[
\rho T \frac{ds}{dt} = \kappa \nabla^2 T + \Phi, \tag{2.18c}
\]

where

\[
\Phi = \frac{1}{2} \mu \left( \nabla v + (\nabla v)^T \right) : \left( \nabla v + (\nabla v)^T \right) + \mu_v (\nabla \cdot v)^2 \tag{2.19}
\]

is the viscous dissipation term. The system of (2.18) together with the ideal gas law (2.9) and the thermodynamic relation (2.12) now form a complete system.

\(^4\delta_{ij} = 1 \text{ if } i = j \text{ and } 0 \text{ otherwise.}\)
In acoustics we are interested in small unsteady perturbations of a steady mean flow. The two separate problems (mean flow and acoustics) are very different in nature, and need to be handled separately. The results presented in this thesis are based on the assumption that the effects of viscosity and thermal conductivity can be neglected, both for the mean flow and the acoustic propagation—this will be motivated in the next sections. Consequently, the complete field can be described by the Euler equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.20a)
\]
\[
\rho \frac{d\mathbf{v}}{dt} + \nabla p = 0 \quad (2.20b)
\]
\[
\frac{ds}{dt} = 0. \quad (2.20c)
\]

An alternative for the energy equation can be found by substituting (2.20c) into (2.5):

\[
\frac{dp}{dt} = c^2 \frac{d\rho}{dt}. \quad (2.21)
\]

Note that the ideal gas law is not used in the derivation of this equation, only the assumption that the total flow is isentropic ($\frac{ds}{dt} = 0$). By using the equation for conservation of mass (2.20a) the energy equation can also be written as

\[
\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (2.22)
\]

where $\gamma p = \rho c^2$ is used, or as

\[
\frac{dT}{dt} + T(\gamma - 1) \nabla \cdot \mathbf{v} = 0, \quad (2.23)
\]

by using the ideal gas law $p = \rho R T$. Generally $p = p(\rho, s)$, but for homentropic flow we have $ds = 0$ so $s = s_0$ is uniformly constant, and consequently $p = p(\rho)$. It follows that

\[
\nabla p = \left( \frac{\partial p}{\partial \rho} \right)_s \nabla \rho, \quad \text{so we have}
\]

\[
\nabla p = c^2 \nabla \rho. \quad (2.24)
\]

This relation holds generally for homentropic gas flows ($c_p$ and $c_v$ not necessarily constant).

### 2.1.3 Mean flow equations

As mentioned above, we are interested in small unsteady perturbations of a steady mean flow, so that the field variables can be expressed as

\[
\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x}) + \mathbf{v}_1(\mathbf{x}, t), \quad p = p_0(\mathbf{x}) + p_1(\mathbf{x}, t), \quad \text{etc.} \quad (2.25)
\]

From the conservation laws (2.18) we find that the steady mean flow field satisfies

\[
\nabla \cdot (\rho_0 \mathbf{v}_0) = 0, \quad (2.26a)
\]
\[
\rho_0(\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + \nabla p_0 = \mu \nabla^2 \mathbf{v}_0 + (\mu_v + \mu) \nabla (\nabla \cdot \mathbf{v}_0), \quad (2.26b)
\]
\[
\rho_0 T_0(\mathbf{v}_0 \cdot \nabla)s_0 = \kappa \nabla^2 T_0 + \Phi_0. \quad (2.26c)
\]
In this section we will motivate why it is reasonable to assume that the mean flow is inviscid and non-heat conducting [60].

We start by considering the effect of viscosity in the momentum equation. Because the viscosity is very small the dominant balance in the flow is between the pressure and inertia forces, so we scale the pressure on $\rho_\infty v_\infty^2$, where $\rho_\infty$ and $v_\infty$ are the typical mean flow density and velocity. Thus we use

$$\tilde{v}_0 = \frac{v_0}{v_\infty}, \quad \tilde{\rho}_0 = \frac{\rho_0}{\rho_\infty}, \quad \Delta \tilde{p}_0 = \frac{\Delta p_0}{\rho_\infty v_\infty^2}, \quad \tilde{x} = \frac{x}{L},$$

where we also introduced the typical length scale $L$. The conservation laws for mass and momentum then take the form (tildes omitted)

$$\nabla \cdot (\rho_0 v_0) = 0,$$

$$\rho_0 (v_0 \cdot \nabla v_0) + \nabla p_0 = \frac{1}{Re} \left[ \nabla^2 v_0 + \frac{\mu}{\rho_\infty} \nabla (\nabla \cdot v_0) \right],$$

where the Reynolds number is defined as

$$Re := \frac{\rho_\infty v_\infty L}{\mu}.$$  

For $\rho_\infty = 1.25 \text{ kg/m}^3$ (the density of air), $v_\infty = 100 \text{ m/s}$, $L = 1 \text{ m}$, and $\mu = 1.8 \times 10^{-5} \text{ kg/(m \cdot s)}$ (the dynamic viscosity of air) this results in a Reynolds number of the order of $10^6$. Thus we conclude that for the free mean flow viscous forces can be neglected.

Close to the wall, however, a very thin boundary layer develops [60]; its typical thickness $\delta$ is much smaller than the length scale $L$. Since the inviscid equations have no solutions that satisfy the no-slip boundary condition at a surface, viscosity does play a role here. As a simple prototypical example, consider a uniform flow that impinges on a flat surface that is oriented in the direction of the flow. We use a two-dimensional Cartesian coordinate system with $x$ along the surface and $y$ normal to it. The longitudinal and transverse velocity components are $u$ and $v$. Assume that compressibility plays no role in a small region of interest along the boundary, i.e. $\nabla \cdot v_0 = 0$. The longitudinal component of the momentum equation is then

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where $v = \mu/\rho$ is the kinematic viscosity. The boundary layer changes in the $x$-direction on a length scale $L$, and in the $y$-direction on a length scale $\delta$. Furthermore it follows from $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ that $v \sim v_\infty \delta/L$, where $\sim$ means ‘of the order of’. Hence we can estimate the magnitude of the advective terms as

$$u \frac{\partial u}{\partial x} \sim \frac{u v_\infty}{L} \sim v \frac{\partial u}{\partial y}.$$  

(2.31)

A measure of the viscous term is

$$v \frac{\partial^2 u}{\partial y^2} \sim \frac{v v_\infty}{\delta^2}.$$  

(2.32)
If the viscous and advective forces are of equal importance we can estimate the development of the boundary layer thickness as

$$\delta = \frac{L}{\sqrt{Re_L}}, \quad \text{with } Re_L = \frac{Lv_\infty}{v}. \quad (2.33)$$

We see that the boundary layer develops very slowly; its thickness is typically of the order of millimeters at one meter from the edge of the surface. We therefore consider the mean flow to be parallel.

Next we consider the effects of viscosity and thermal conductivity in the energy equation. In order to make it dimensionless we first rewrite this equation as

$$\rho_0 c_p v_0 \cdot \nabla T_0 - v_0 \cdot \nabla p_0 = \kappa \nabla^2 T_0 + \Phi_0. \quad (2.34)$$

This formulation can be found by substituting $Tds = c_p dT - dp/\rho$ (the fundamental thermodynamic relation for an ideal gas) in (2.18c) and assuming steady state. Next to the scaling already mentioned in (2.27) we use

$$dT_0 = \frac{dT_0}{\Delta T}, \quad \Phi_0 = \frac{\Phi_0}{\mu v_\infty^2/L^2}, \quad (2.35)$$

where $\Delta T$ is a characteristic temperature difference. The energy equation then becomes (tildes omitted)

$$\rho_0 v_0 \cdot \nabla T_0 - Ec v_0 \cdot \nabla p_0 = \frac{1}{Pe} \nabla^2 T_0 + \frac{Ec}{Re} \Phi_0, \quad (2.36)$$

with dimensionless Eckert and Péclet numbers $Ec$ and $Pe$, which will be described next.

The *Eckert number*, which is defined as

$$Ec := \frac{v_\infty^2}{c_p \Delta T}, \quad (2.37)$$

expresses the relation between the kinetic energy and the enthalpy of a flow. Here we assume that $Ec/Re \ll 1$. Before we introduce the Péclet number we define the *Prandtl number* $Pr$ and the thermal diffusivity $\alpha$ as

$$Pr := \frac{\nu}{\alpha} = \frac{c_p \mu}{\kappa}, \quad \alpha := \frac{\kappa}{\rho_\infty c_p}. \quad (2.38)$$

The Prandtl number is a measure for the ratio of momentum diffusivity to thermal diffusivity. It is a property of a fluid; for most gases it is of order one. The *Péclet number* for the diffusion of heat is defined as

$$Pe := \frac{Lv_\infty}{\alpha} = \frac{Lv_\infty \rho_\infty c_p}{\kappa} = Pr Re. \quad (2.39)$$

It measures the ratio of the advective and the diffusive transport rate of heat, so thermal diffusion plays a small role for large Péclet numbers. Because the Prandtl number is of order one we can conclude that the Péclet number is large for gases. This leads us to conclude that the heat conduction and viscous dissipation terms in the energy equation are small so that they can be neglected. It follows that the mean flow is isentropic along stream lines, i.e. $\frac{d\delta}{dt} = 0$. 
To summarize, the steady mean flow satisfies

\[ \nabla \cdot (\rho_0 \mathbf{v}_0) = 0, \quad (2.40a) \]
\[ \rho_0 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_0 = \mathbf{0}, \quad (2.40b) \]
\[ (\mathbf{v}_0 \cdot \nabla) s_0 = 0. \quad (2.40c) \]

Moreover, from (2.21) we have as a different form of the energy equation

\[ \mathbf{v}_0 \cdot \nabla p_0 = c_0^2 \mathbf{v}_0 \cdot \nabla \rho_0. \quad (2.41) \]

We also have the ideal gas law

\[ p_0 = \rho_0 R T_0 \quad (2.42) \]

and the thermodynamic relations

\[ ds_0 = \frac{c_v}{p_0} dp_0 - \frac{c_p}{\rho_0} d\rho_0, \quad c_0^2 = \frac{\gamma p_0}{\rho_0} = \gamma R T_0. \quad (2.43) \]

This research concerns sound propagation through a straight circularly symmetric duct. Hence it is natural to use a cylindrical coordinate system \((x, r, \theta)\), with axial, radial and circumferential flow velocity components denoted by \(u, v\) and \(w\). We assume that the flow is swirl free, i.e. \(w_0 = 0\), and distinguish the following types of solutions of the steady state Euler equations:

- parallel flow with \(v_0 = u_0(r) \mathbf{e}_x\), with \(\mathbf{e}_x\) denoting the unit vector in the \(x\) direction;
- uniform flow with \(v_0 = u_0 \mathbf{e}_x\);
- no mean flow, i.e. \(v_0 = 0\).

(Note that parallel flow is not rotation free, which means that it is not possible to formulate a non-uniform flow in a straight duct in terms of a velocity potential.) For these types of flows it follows from the momentum equation (2.40b) that the mean pressure \(p_0\) is constant. The density and the entropy then follow from the temperature. We assume one of the following:

- constant mean temperature \(T_0\);
- a mean temperature that varies only in the radial coordinate: \(T_0 = T_0(r)\)

The temperature profile \(T_0(r)\) can be chosen independently of the mean flow velocity profile. If the temperature \(T_0\) (and consequently the density \(\rho_0\)) is constant, then the mean flow is homentropic, meaning that it has constant entropy \(s_0\).

### 2.1.4 Acoustics—Linearized Euler Equations

In this section we describe the equations that govern the small (acoustic) perturbations, which are denoted by the subscript \(_1\). First we will show that these perturbations are very small compared to the mean flow quantities \([108]\), and that for this reason we can neglect the nonlinear terms that arise after substitution of (2.25) into (2.18). (This
linearization matches with our common experience: we can distinguish multiple conversations in a room—each linear combination of solutions of the governing equations is a solution itself, so the individual sound fields do not influence each other [39]. Subsequently we will motivate why we neglect viscosity and thermal conductivity also for the acoustic problem [17, 94].

In the following we again use \(\infty\) to denote reference values that are used to make the equations dimensionless; these are the typical values of the mean flow variables. For simplicity we allow the reference speed \(v_\infty\) to be identical to the reference sound speed \(c_\infty\). We then have

\[
\tilde{v}_0 = \frac{v_0}{c_\infty}, \quad \tilde{\rho}_0 = \frac{\rho_0}{\rho_\infty}, \quad \tilde{p}_0 = \frac{p_0}{p_\infty}, \quad \tilde{T}_0 = \frac{T_0}{T_\infty}, \quad \tilde{s}_0 = \frac{s_0}{s_\infty},
\]

(2.44)

with the definitions

\[
\rho_\infty := \rho_\infty c_\infty^2, \quad s_\infty := \frac{c_p \Delta T}{T_\infty}.
\]

(2.45)

The definition for \(s_\infty\) is motivated by taking \(p_0\) constant in (2.43) from which we find \(d\tilde{s}_0 = c_p dT/T_0\). As before, \(\Delta T\) is the typical temperature difference of the mean flow.

In acoustics the Sound Pressure Level (SPL) is defined as

\[
\text{SPL} = 20 \log \left( \frac{p_1^{\text{rms}}}{p_{\text{ref}}} \right), \quad p_{\text{ref}} = 2 \cdot 10^{-5} \text{ Pa},
\]

(2.46)

where \(p_1^{\text{rms}}\) is the root mean square of the acoustic fluctuation. The sound pressure level corresponding to the threshold of pain is about 140 dB, which corresponds to a pressure signal with \(p_1^{\text{rms}} = 200\) Pa. Consequently, the ratio \(p_1/p_0\) is (for our purposes) of the order of \(10^{-3}\) or smaller. As this ratio is small we can linearize the thermodynamic relation for an ideal gas (2.14) to find

\[
s_1 = \frac{c_v}{p_0} p_1 - \frac{c_p}{\rho_0} \rho_1 = \frac{c_v}{p_0} (p_1 - c_0^2 \rho_1)
\]

(2.47)

if we can assume that \(\rho_1/p_0\) is of the same order as \(\rho_1/p_0\). It will be shown later that \(s_1 = 0\) for uniform temperature, in which case \(p_1 = c_0^2 \rho_1\). This relation can be used to find \(\frac{p_1}{p_0} = \gamma \frac{\tilde{p}_1}{\tilde{p}_0}\), which justifies our previous assumption. For the temperature perturbations, differentiate the logarithm of the ideal gas law and use \(\gamma p = c_p^2\) to obtain \(\frac{dp}{p} - \frac{c_p^2 dp}{\gamma p} = \frac{dT}{T}\). Based on this result we can estimate \(\frac{T_1}{T_0} = (\gamma - 1) \frac{\rho_1}{\rho_0}\). To estimate the size of the velocity perturbations we consider a plane wave traveling through a medium at rest having uniform temperature. The acoustic particle velocity is then \(u_1 = p_1/(\rho_0 c_0)\). Dividing this relation by \(c_0\) and using again \(p_1 = \rho_1 c_0^2\) yields \(\frac{u_1}{c_0} = \frac{\rho_1}{\rho_0}\). Finally, dividing (2.47) by \(s_\infty\) results in \(\frac{s_1}{s_\infty} = \frac{T_\infty}{\Delta T} \left( \frac{p_1}{\tilde{p}_0} - \frac{\rho_1}{\tilde{\rho}_0} \right)\), where \(\Delta T/T_\infty\) is of order unity.

Based on these estimates we therefore use the ratio \(\rho_1/ho_0\) as a small parameter \(\epsilon\) and use the scaled variables

\[
\tilde{p}_1 = \frac{p_1}{\epsilon p_\infty}, \quad \tilde{\rho}_1 = \frac{\rho_1}{\epsilon \rho_\infty}, \quad \tilde{v}_1 = \frac{v_1}{\epsilon c_\infty}, \quad \tilde{T}_1 = \frac{T_1}{\epsilon T_\infty}, \quad \tilde{s}_1 = \frac{s_1}{\epsilon s_\infty}.
\]

(2.48)

Furthermore, we scale time on the acoustic frequency \(f\), and distances on the acoustic wavelength \(\lambda = c_\infty/f\). Upon substituting the scaled variables in the conservation laws
(2.18), subtracting the mean flow terms that are of zeroth order in $\epsilon$, and neglecting the terms of order $\epsilon^2$, we obtain

$$\frac{\partial}{\partial t} \rho_1 + \nabla \cdot (v_0 \rho_1 + v_1 \rho_0) = 0,$$  \hspace{1cm} (2.49a)

$$\rho_0 \left( \frac{\partial}{\partial t} + v_0 \cdot \nabla \right) v_1 + \rho_0 (v_1 \cdot \nabla)v_0 + \nabla p_1 = \mu \nabla^2 v_1 + (\mu_v + \mu) \nabla(\nabla \cdot v_1),$$  \hspace{1cm} (2.49b)

$$\rho_0 T_0 \left( \frac{\partial}{\partial t} + v_0 \cdot \nabla \right) s_1 + \rho_0 T_0 v_1 \cdot \nabla s_0 = \kappa \nabla^2 T_1 + \Phi_1,$$  \hspace{1cm} (2.49c)

with the linearized viscous dissipation term $\Phi_1 = \mu(\nabla v_0 + (\nabla v_0)^T) : (\nabla v_1 + (\nabla v_1)^T) + 2\mu_v(\nabla \cdot v_0)(\nabla \cdot v_1)$. Note that gradients of mean flow and perturbation variables do not necessarily have the same length scale. Suppose that the mean flow changes over a length scale $L$ and the perturbations over the distance of one wavelength $\lambda$. Terms of the order $\epsilon \lambda / L$ then arise in the first order equations. However, even for very low frequencies $\epsilon \lambda / L \ll 1$, so the linearization is still warranted.

To illustrate why viscosity does not play a role at the frequencies of interest we consider a plane wave propagating in the $x$-direction without mean flow and at uniform mean temperature. The linearized mass and momentum equations then reduce to

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0,$$  \hspace{1cm} (2.50a)

$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = \mu N_v \frac{\partial^2 u_1}{\partial x^2},$$  \hspace{1cm} (2.50b)

where the viscosity number $N_v$ defined as

$$N_v := (\mu_v + 2\mu) / \mu$$  \hspace{1cm} (2.51)

has order of magnitude one. Together with the relation $\rho_1 = c_0^2 \rho_0$ this leads to a viscous wave equation

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 u_1}{\partial t^2} + \frac{\mu N_v}{\rho_0 c_0^2} \frac{\partial^3 u_1}{\partial x^2 \partial t} = 0.$$  \hspace{1cm} (2.52)

By substituting a plane-wave solution of the form $e^{-i\omega t + ikx}$ we find the dispersion relation

$$k^2 \left( 1 - i \left[ \frac{\omega \mu N_v}{\rho_0 c_0^2} \right] \right) = \frac{\omega^2}{c_0^2}.$$  \hspace{1cm} (2.53)

The term between square brackets, which is not present in the dispersion relation for inviscid wave propagation, can be interpreted by defining an acoustic Reynolds number

$$Re_{ac} := \frac{\rho_0 c_0 \lambda}{\mu},$$  \hspace{1cm} so that $\frac{\omega \mu N_v}{\rho_0 c_0^2} = \frac{2\pi N_v}{Re_{ac}}.$  \hspace{1cm} (2.54)

Consequently the importance of viscous effects increases with frequency. At frequencies in the audible range viscosity can be neglected: for a frequency of 10 kHz we have $Re_{ac}$ of the order of $10^9$.

We now also motivate why we neglect heat conduction for sound propagation. For simplicity we again consider plane-wave propagation in the $x$-direction and assume
no mean flow and uniform mean temperature (and consequently constant $s_0$). The linearized energy equation (2.49c) then becomes

$$\rho_0 T_0 \frac{\partial s_1}{\partial t} = \kappa \frac{\partial^2 T_1}{\partial x^2}. \quad (2.55)$$

By using the linearized versions of the fundamental thermodynamic relation (2.62) and the ideal gas law (2.61) we find

$$\rho_0 c_p \frac{\partial}{\partial t} \left( \frac{p_1}{c_0^2} - \rho_1 \right) = \kappa \frac{\partial^2}{\partial x^2} \left( \frac{\gamma p_1}{c_0^2} - \rho_1 \right). \quad (2.56)$$

From (2.50) it can easily be seen that the inviscid mass and momentum equations can be combined to find

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} = 0. \quad (2.57)$$

Together with (2.56) this leads to

$$\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p_1 = \frac{2\pi}{P_{\text{ac}}} \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial x^2} - \frac{\gamma}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p_1, \quad (2.58)$$

where the acoustic Pécelt number is defined as

$$P_{\text{ac}} := \frac{\rho_0 c_p c_0 \lambda}{\kappa}. \quad (2.59)$$

We also introduced dimensionless numbers $\tilde{t} = \omega t$ and $\tilde{x} = x\omega / c_0$ in order to compare the two terms between brackets. The bracketed term on the left hand side describes adiabatic wave propagation with sound speed $c_0^2$, whereas the bracketed term on the right describes isothermal wave propagation with sound speed $c_0^2 / \gamma$. The acoustic Pécelt number is of the same order of magnitude as the acoustic Reynolds number since they are related as $P_{\text{ac}} = Pr Re_{\text{ac}}$, where the Prandtl number $Pr$ is of the order unity for most gases. We therefore conclude that heat conduction can be neglected for sound within the audible frequency range.

Very close to walls, viscosity and thermal conductivity give rise to thin acoustic boundary layers. We assume here that the wavelength is much larger than the thickness of these boundary layers, so that they can be neglected. For more details we refer to [94] and [17].

To summarize, it follows that the perturbations satisfy the Linearized Euler equations (LEE)

$$\frac{\partial}{\partial t} \rho_1 + \nabla \cdot (v_0 \rho_1 + v_1 \rho_0) = 0, \quad (2.60a)$$

$$\rho_0 \left( \frac{\partial}{\partial t} + v_0 \cdot \nabla \right) v_1 + \rho_0 (v_1 \cdot \nabla) v_0 + \rho_1 (v_0 \cdot \nabla) v_0 + \nabla p_1 = 0, \quad (2.60b)$$

$$\left( \frac{\partial}{\partial t} + v_0 \cdot \nabla \right) s_1 + v_1 \cdot \nabla s_0 = 0. \quad (2.60c)$$

We also include here the linearized versions of the ideal gas law and the thermodynamic relation:

$$p_1 = R (\rho_0 T_1 + \rho_1 T_0), \quad (2.61)$$

$$s_1 = \frac{c_v}{p_0} p_1 - \frac{c_p}{\rho_0} \rho_1 = \frac{c_v}{p_0} \left( p_1 - c_0^2 \rho_1 \right). \quad (2.62)$$
If the mean flow is homentropic (when we have constant temperature $T_0$) it follows from (2.60c) that the entropy perturbations are constant along streamlines, which amounts to $s_1 = 0$. Only in that case the commonly used relation $p_1 = \rho_1 c_0^2$ follows from (2.62), but we emphasize here that for non-uniform temperature this relation is not valid. From (2.21) follows an alternative to the energy equation

$$\frac{\partial}{\partial t} p_1 + v_0 \cdot \nabla p_1 + v_1 \cdot \nabla p_0 = c_0^2 \left( \frac{\partial}{\partial t} \rho_1 + v_0 \cdot \nabla \rho_1 + v_1 \cdot \nabla \rho_0 \right) + c_0^2 (v_0 \cdot \nabla \rho_0) \left( \frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} \right),$$

(2.63)

which will be used later in Section 2.4. Yet another alternative to the energy equation based on (2.23) is

$$\left( \frac{\partial}{\partial t} + v_0 \cdot \nabla \right) T_1 + v_1 \cdot \nabla T_0 + T_0 (\gamma - 1) \nabla \cdot v_1 + T_1 (\gamma - 1) \nabla \cdot v_0 = 0,$$

(2.64)

which will be used in Section 5.1.

Three types of wave-like solutions to the LEE can be distinguished, namely entropy, vorticity and acoustic waves, although in general they are coupled. It has been shown [58, 114] that for uniform mean flow and temperature the three different waves are decoupled. Pure entropy and vorticity waves are convected with the mean flow. Entropy waves carry only density perturbations (no pressure and velocity perturbations), vorticity waves carry only velocity perturbations, and acoustic waves carry density, pressure and velocity perturbations. In case of non-uniform mean flow or temperature however, and in the presence of boundaries, the three waves interact [13, 14], and it is less clear how to distinguish them. Therefore, in this case we cannot strictly speak about ‘acoustic’ perturbations, as any disturbance may consist of acoustic, entropic and vortical components. In this thesis we will often refer to small perturbations as ‘acoustics’, even though they may contain entropic and vortical components.

In this thesis we investigate time-harmonic perturbations at a fixed wave frequency, and assume that turbulence does not play a role. Because we assume that the perturbations are linear it follows that time-harmonic and turbulent perturbations can be considered independently. Moreover, sound induced by a free turbulent flow is generally very quiet [31]. In other cases, such as a turbulent flow near the edge of a fan blade, turbulence is more efficient in radiating sound. However, in this thesis we are interested in sound propagation; we assume that the sound source is given, and do not investigate sound source mechanisms.

### 2.1.5 Myers’ Energy Corollary

It is often useful to evaluate the transport or radiation of acoustical energy, or to use the principle of conservation of energy as a check for the internal consistency of numerical results. For this purpose an exact corollary of the energy equation for fluid dynamics can be formulated for arbitrary steady flows, which is commonly referred to as Myers’ Energy Corollary [79, 81].

Let us assume that all quantities are expanded as a sum of perturbations of increasing order $q(x, t; \epsilon) = q_0(x) + \epsilon q_1(x, t) + \epsilon^2 q_2(x, t) + \cdots$ in a small parameter $\epsilon$. When this expansion is substituted in the Euler equations we find the zeroth order equations that govern the mean flow, the Linearized Euler equations which govern the first order
perturbations, and consecutively higher order systems of equations. In the same way the zeroth, first, second, and higher order systems of equations can be found for the energy conservation law. An energy conservation law for the linear perturbations $q_1$ is necessarily of the order $\epsilon^2$. It is shown in [81] that it is possible to express this second order part of the general energy conservation law exactly in zeroth and first order quantities only, i.e. the perturbations of second order and higher are not required to be known.

When we denote the perturbation energy density by $\mathcal{E}$, the energy flux by $\mathbf{I}$, and the dissipation term by $\mathcal{D}$, Myers’ Energy Corollary can be formulated as

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{I} = -\mathcal{D}$$ (2.65)

where

$$\mathcal{E} = \frac{p_1^2}{2\rho_0 c_0^2} + \frac{1}{2} \rho_0 |\mathbf{v}_1|^2 + \rho_1 \mathbf{v}_0 \cdot \mathbf{v}_1 + \frac{\rho_0 T_0 s_1^2}{2c_p},$$

$$\mathbf{I} = (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) \left( \frac{p_1}{\rho_0} + \mathbf{v}_0 \cdot \mathbf{v}_1 \right) + \rho_0 \mathbf{v}_0 T_1 s_1,$$

$$\mathcal{D} = -\rho_0 \mathbf{v}_0 \cdot (\omega_1 \times \mathbf{v}_1) - \rho_1 \mathbf{v}_1 \cdot (\omega_0 \times \mathbf{v}_0) + s_1 (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) \cdot \nabla T_0 - s_1 \rho_0 \mathbf{v}_0 \cdot \nabla T_1,$$ (2.66c)

and the vorticity is denoted by $\omega := \nabla \times \mathbf{v}$. This relation is valid for arbitrary steady flows, i.e. also for flows with non-uniform mean velocity and temperature. These equations show that the energy of the linear perturbations is not strictly conserved—a ‘source’-term $\mathcal{D}$ is required, which is non-zero for cases with non-zero vorticity and entropy perturbations. Note that for homentropic flows (constant temperature) $s_1 = 0$, in which case Myers’ Energy Corollary reduces to the energy relation reported by Morfey [75] and Goldstein [46]. If the fluid is quiescent ($\mathbf{v}_0 = 0$) as well, it is equivalent to Kirchhoff’s acoustic energy relation [94], the more classical result.

Taking the time average of (2.65) for time-harmonic perturbations yields

$$\nabla \cdot \mathbf{\bar{I}} = -\bar{\mathcal{D}}.$$ (2.67)

By taking the volume integral of (2.67) over a volume $\mathcal{V}$ with boundary $\partial \mathcal{V}$ and applying the divergence theorem we find

$$\int_{\partial \mathcal{V}} \mathbf{\bar{I}} \cdot \mathbf{n} \, dA + \int_{\mathcal{V}} \bar{\mathcal{D}} \, dV = 0.$$ (2.68)

This relation will be used to assess the validity and internal consistency of numerical results.

### 2.2 Duct modes

A very suitable way to approach the duct acoustic propagation problem is to describe the (acoustic) perturbation field in terms of *duct modes*. In this section we briefly discuss some basic concepts and definitions related to the modal approach, which will be used in the next sections.
It is useful to investigate linear wave propagation problems with constant coefficients in the frequency domain, i.e. we look for time-harmonic solutions of the form

$$p_1(x,y,z,t) = \text{Re}\{\hat{p}_1 e^{-i\omega t}\}. \quad (2.69)$$

with fixed radial frequency $\omega = 2\pi f$. This prevents us from investigating transient (e.g. switching) behavior and possible instabilities [33], but for our purposes this is not a restriction. More general time-dependent solutions can be constructed via Fourier synthesis if desired.

We can furthermore exploit the fact that the geometries of interest show some symmetry. The coefficients of the wave equation (which will be discussed in Section 2.4) and the boundary conditions are independent of the axial coordinate $x$ for the case of parallel flow in a cylindrical duct with uniform boundary conditions. Since the problem is invariant in the axial direction we can look for wave-like solutions that remain self-similar as they travel along the duct; it is easily shown that this amounts to duct modes of the form $P(y,z) e^{-i\omega t + ikx}$, where $P(y,z)$ is a transverse mode shape, and $k$ is a complex-valued axial wavenumber.

The mode shapes and corresponding axial wavenumbers follow as eigensolutions of a boundary value problem defined on the duct cross section, with suitable boundary conditions at the duct wall. For flow through an axisymmetric cylindrical duct having uniform mean velocity and temperature the transverse eigenvalue problem is formulated as Bessel’s equation with suitable boundary conditions. When the temperature and flow are sheared (non-uniform) and can be characterized by radial flow and temperature profiles, the transverse differential equation becomes the Pridmore-Brown equation. Hence, we are interested in the eigensolutions of this equation, which can be seen as a generalized form of Bessel’s equation. The Pridmore-Brown equation will be derived in Section 2.4.

For uniform or quiescent flow the set of axial wavenumbers is discrete, and the modes form a basis for a general solution. The general solution may then be expressed as a summation of modes,

$$p_1(x,y,z,t) = \sum_{\mu=-\infty}^{\infty} A_\mu P_\mu(y,z)e^{i k_\mu x - i \omega t}, \quad (2.70)$$

each mode having a certain amplitude $A_\mu$ such that the series converges. For non-uniform mean flow there is also continuous part of the spectrum, next to the discrete set of eigenvalues. In this thesis we assume that this contribution is small, so that it can be neglected [23, 33], and we will use (2.70) also for cases of non-uniform flow. We will comment on this again at the end of Section 2.4. Since the problem is linear, the modes can be determined separately. Once these modes are known, the amplitudes can be found from the initial conditions at a certain axial position.

If the geometry is circularly cylindrical it is most natural to use cylindrical coordinates $(x,r,\theta)$. Because of the periodicity in $\theta$ it is possible to express the solution as a Fourier series in $\theta$,

$$p_1(x,y,z,t) = \sum_{m=-\infty}^{\infty} \sum_{\mu=-\infty}^{\infty} A_{m\mu} P_{m\mu}(r)e^{ik_{m\mu}x - i \omega t + im\theta}, \quad (2.71)$$

where $m$ is the circumferential (also: azimuthal) wavenumber. This is useful if the medium and the boundary conditions are independent of $\theta$, i.e. if the problem is axisymmetric. In this case the counter $\mu$ is referred to as the radial order.
For hard walls a finite number of wavenumbers $k_\mu$ is real-valued; the rest is complex-valued. In the first case the wave is purely propagative, referred to as cut-on, while in the second case the wave decays or increases exponentially, referred to as cut-off. For frequencies below the cut-off frequency only the plane wave (for which the mode shape is constant) exists—all higher order modes are cut-off.

### 2.3 Wall boundary conditions

In this section we will describe the boundary conditions that are used to model the sound absorption in the wall. We start by defining the acoustic (normal) impedance, which measures how much the motion of a fluid particle (or a surface) is impeded when a pressure wave impinges on it, as

$$Z(\omega) = \frac{\tilde{p}_1}{\tilde{v}_1 \cdot n},$$  \hspace{1cm} (2.72)

where the normal $n$ is pointing into the wall. The impedance is a frequency-domain quantity.

As discussed in Chapter 1, the acoustic field in an acoustically absorbing wall (for example made of foam) and the acoustic field in the duct are coupled; a part of the sound waves that travel inside the wall material will propagate parallel to the wall. As a result, the impedance may vary for each position at the wall, so the wall can not be described by just a single impedance value. These walls are referred to as non-locally reacting. This is in contrast to locally reacting walls, which have an internal structure that prohibits this parallel propagation (for example through a honeycomb structure), and can consequently be described by a single impedance value $Z = Z(\omega)$ that depends on $\omega$ only. It is also possible for non-locally reacting walls to find a single impedance value that describes the entire surface if we consider only a single mode, since modal solutions retain their shape as they propagate along the wall. Hence, for non-locally reacting walls the impedance value that describes the entire surface if we consider only a single mode, since modal solutions retain their shape as they propagate along the wall. Hence, for non-locally reacting walls the impedance $Z = Z(\omega, k)$ depends on both $\omega$ and $k$. This can also be interpreted as follows: for a non-locally reacting surface the impedance depends on the angle of incidence of the sound wave, whereas for a locally reacting surface it is angle-independent.

Due to viscosity, the mean flow velocity at the wall is zero—the flow is not slipping. This gives rise to a boundary layer. Due to the very low viscosity of air this boundary layer can be very thin (as is the case for the inlet duct of an aircraft jet engine). In the limit of vanishing thickness the mean flow can be considered an inviscid slipping flow, in which case the boundary layer can be modeled as a vortex sheet. This has implications for the impedance boundary condition. Ingard [52] proposed a boundary condition for slipping flow, based on the continuity of particle displacement. This was later generalized by Myers [80] for a smooth but not flat surface. For this reason the commonly used boundary condition bears their name; the Ingard-Myers boundary condition can be formulated as

$$\tilde{v}_1 \cdot n = (-i\omega + v_0 \cdot \nabla) \frac{\tilde{p}_1}{-i\omega Z} - \frac{\tilde{p}_1}{-i\omega Z} n \cdot (n \cdot \nabla v_0).$$  \hspace{1cm} (2.73)

This condition is equivalent to continuity of particle displacement for the specific case of parallel flow along a flat surface. We will first derive the boundary condition for parallel slipping flow along a straight wall in the next section, as this is the geometry
pertinent to this research. This boundary condition is valid for both locally \( Z = Z(\omega) \) and non-locally reacting \( Z = Z(\omega, k) \) walls when we interpret \( \partial_x \) as \( ik \).

In Section 2.3.2 the propagation of sound through porous material will be discussed. Porous materials can be characterized into three different skeleton types: rigid (e.g. metal foam), limp (i.e. soft fibrous material) or elastic. Materials with an elastic skeleton are generally described by Biot theory, which governs the fluid density and the displacement of the skeleton (i.e. four degrees of freedom). In this case both compression and shear waves may exist. However, to simplify the problem often an equivalent fluid model [2, 76, 89] is used, in the same way as with rigid and limp materials. Only the compression waves are considered in an equivalent fluid model, modeled through a Helmholtz equation depending on two parameters: usually an effective density and an effective sound speed, or equivalently, a characteristic impedance and a propagation constant [110]. The various material properties like porosity, resistivity, flow resistance, etc., are modeled through these two parameters. For this purpose many different expressions exist, on both theoretical and empirical bases, such as the ones proposed by Delaney and Bazley, Allard and Champoux, and Miki [3, 35, 71, 72].

### 2.3.1 Boundary condition for parallel slipping flow

In this section we give a derivation of the Ingard-Myers boundary condition for the simple case of parallel slipping flow along a flat wall. A schematic overview of the configuration is depicted in Figure 2.1. We use a Cartesian coordinate system \((x, y, z)\) and denote the velocity components by \(u, v, w\). For \(y > \delta\) we have a uniform flow with velocity \(u = u_0\), and for \(y < \delta\) we have \(u = 0\). A vortex sheet \(S\) is located at \(y = \delta\) to represent the boundary layer. This vortex sheet is perturbed by an incoming sound wave. We use the superscripts + and − to denote variables above and below \(S\). At the wall at \(y = 0\) we have the impedance boundary condition \(p_1^− = Zv_1^−\). A vortex sheet can not support a pressure difference, so we can use \(p_1^− = p_1^+\). However, due to the fact that the mean flow velocity is discontinuous (a result of the assumption that it is inviscid), the perturbation velocity is discontinuous as well: \(v_1^− \neq v_1^+\). To model a slipping flow, we wish to find an expression for \(\tilde{p}_1^+\) and \(\tilde{v}_1^+\) as \(\delta \to 0\).

Any particle close to the surface \(S\) follows the streamlines of the interface, which can be described by

\[
x = \xi(t), \quad y = \eta(t) = \delta + s_1(\xi(t), \zeta(t), t), \quad z = \zeta(t),
\]

(2.74)
where $s_1$ is a small perturbation of the streamline due to the incoming sound wave. Instead of looking at one particle that happens to be at position $\mathbf{x} = [\xi(t), \eta(t), \zeta(t)]$ at time $t$ we can also consider a stream of particles for all $\mathbf{x}$ and $t$, so that we can take partial derivatives to space and time. By taking the time derivative of $\eta$ we then find, at $y = \delta + s_1$

$$v_1^+ = \frac{\partial s_1}{\partial t} + u_0 \frac{\partial s_1}{\partial x} + w_0 \frac{\partial s_1}{\partial z}. \tag{2.75}$$

Hence we have after linearization, above and below the streamline, at $y = \delta$

$$v_1^+ = \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) s_1, \quad v_1^- = \frac{\partial}{\partial t} s_1. \tag{2.76}$$

Eliminating $s_1$ yields

$$\left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) v_1^- = \frac{\partial}{\partial t} v_1^+. \tag{2.77}$$

Across the surface $S$ we have $p_1^+ = p_1^-$. Together with $\tilde{p}_1^- = Z \tilde{v}_1^-$ we find for time-harmonic signals $\sim e^{-i\omega t}$

$$-i\omega \tilde{v}_1^+ = \left( -i\omega + u_0 \frac{\partial}{\partial x} \right) \frac{\tilde{p}_1^+}{Z}. \tag{2.78}$$

If $s_1 \ll \delta \ll 1$ as $\delta \to 0$, i.e. if the perturbed streamline does not cross the liner surface as we take the limit, then we can use (2.78) as the boundary condition at $y = 0$ [101]. Upon using the $y$ component of (2.60b) to eliminate the velocity and assuming that $Z$ is constant in the $x$-direction we find for modes of the form $P e^{-i\omega t + ikx}$ the boundary condition, in Cartesian or cylindrical coordinates

$$\frac{i\rho_0 (\omega - ku_0)^2}{\omega Z} \frac{\partial P}{\partial y}, \quad \frac{i\rho_0 (\omega - ku_0)^2}{\omega Z} \frac{\partial P}{\partial r} \text{ at the wall.} \tag{2.79}$$

The Ingard-Myers boundary condition was shown by Brambley to be ill-posed [21]. It was shown in [33, 102] that this is the limit of a boundary layer–impedance wall instability that exists for boundary layer thicknesses less than a critical thickness (which depends on mean flow and wall properties). It was also shown that the critical thicknesses found for typical aeronautical applications are much smaller than the prevailing boundary layer thicknesses. A regularized boundary condition that includes the effect of a small but finite boundary layer thickness has been proposed, for which these instabilities are absent. Because we will stick to a fixed and real-valued $\omega$ we will not encounter this problem.

### 2.3.2 Sound propagation in porous material

In this section we summarize the basic equations for sound propagating through a porous material; for more details see [51, 76]. The basic idea is to derive a wave equation for the porous material that has an equivalent form as for air. This is called an equivalent fluid model.

We start with some definitions related to compressibility. For isentropic sound propagation we develop $\rho(p, s)$ as a Taylor series around the steady state while keeping $s$ constant:

$$\rho(p) \bigg|_{p_0} = \rho(p_0) + p_1 \left( \frac{\partial \rho}{\partial p} \right)_s \bigg|_{p_0} + O(p_1^2). \tag{2.80}$$
When we define the *adiabatic compressibility* as

\[ \beta_s := \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_s \]  

(2.81)

it follows, after neglecting the nonlinear terms, that we can write

\[ \rho_1 = \rho_0 \beta_s (p_0) p_1. \]  

(2.82)

If the sound perturbations are isothermal instead of isentropic we can replace \( \beta_s \) with the *isothermal compressibility* \( \beta_T \), which is defined as

\[ \beta_T := \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_T. \]  

(2.83)

Note that for an ideal gas with \( p = \rho R T \) we have \( \beta_s = 1/(\gamma p) \) and \( \beta_T = 1/p \).

Consider a porous material with a solid skeleton (e.g. metal foam) in which there is no mean flow and a constant temperature. We also assume that the porous material properties are uniform and isotropic (independent of orientation). Inside the porous material the local velocity \( v \) is hard to define, since it is highly dependent of the local orientation of the pores. We therefore use the averaged velocity \( u \), which is defined as the volume of fluid that crosses a unit area (which is perpendicular to the flow) per unit time. Let us also define the porosity \( \Omega \) as the fraction of the volume not occupied by the solid material, usually very close to 1. The density inside the porous material is increased slightly due to the presence of the solid; it becomes \( \rho/\Omega \). Correspondingly, since the mass flux in air must be equivalent to the mass flux in the porous material, the velocity \( u \) is slightly lower than \( v \); we can use \( u = \Omega v \).

By multiplying the linearized mass conservation equation by \( \Omega \) we find

\[ \Omega \frac{\partial}{\partial t} \rho_1 + \rho_0 \nabla \cdot u_1 = 0. \]  

(2.84)

We could use (2.82) to formulate a relation in terms of \( p_1 \) and \( u_1 \), however, the compressibility \( \beta \) is frequency dependent. For low frequencies the presence of the porous material causes the temperature to be constant—we need to use the isothermal compressibility \( \beta_T \). For high frequencies there is not enough time for heat exchange, so we need to use the adiabatic compressibility \( \beta_s \). For intermediate frequencies (typically of the order of a few kHz) the process is neither isothermal nor adiabatic. Thus, we use a compressibility \( \beta_p \) that is frequency dependent, and we formulate the mass conservation law in the frequency domain. For time-harmonic solutions \( \sim e^{-i\omega t} \) it becomes

\[ -i\omega \beta_p \Omega \dot{p}_1 + \nabla \cdot \dot{u}_1 = 0. \]  

(2.85)

In the momentum equation, a term \( \sigma u_1 \) must be included to model the retardation of the flow due to friction; the resistivity \( \sigma \) is the pressure drop required to force a unit flow through the pores. Usually \( \sigma/\rho_0 c_0 \) is between 50 and 500 \( \text{m}^{-1} \). Furthermore, the effective mass of a fluid increases when it moves through constrictions, and also due to the fact that some fibers may move with the flow. We therefore use the effective density \( \rho_p \) here, which is typically a factor of 1.5 to 5 times larger than the ambient density \( \rho_0 \). The linearized momentum equation inside the porous material thus reads

\[ \rho_p \frac{\partial}{\partial t} u_1 + \sigma u_1 + \nabla p_1 = 0. \]  

(2.86)
\( \rho_p \) can be frequency dependent as well. In the frequency domain the momentum equation can be written as
\[
- i \omega \rho_p \left( 1 + \frac{i \sigma}{\rho_p \omega} \right) \ddot{u}_1 + \nabla \ddot{p}_1 = 0.
\] (2.87)

The linearized mass and momentum conservation equations (2.85) and (2.87) can be combined into a dissipative Helmholtz equation
\[
\nabla^2 \ddot{p}_1 + \omega^2 \rho_p \beta_p \Omega \ddot{p}_1 + i \omega \beta_p \Omega \sigma \ddot{p}_1 = 0.
\] (2.88)

The wave speed can thus be defined as \( c^2_p := \frac{1}{\rho_p \beta_p \Omega} \). If we define an (again frequency dependent) effective density as
\[
\rho_e := \rho_p \left( 1 + \frac{i \sigma}{\omega \rho_p} \right),
\] (2.89)

we obtain
\[
\nabla^2 \ddot{p}_1 + \omega^2 \rho_e \beta_p \Omega \ddot{p}_1 = 0.
\] (2.90)

This motivates the definition of the effective wave speed as
\[
c^2_e := \frac{1}{\rho_e \beta_p \Omega},
\] (2.91)

which is frequency dependent as well.

In literature it is also very common to specify the bulk properties by a propagation constant and a characteristic impedance. For a plane wave satisfying (2.90) the propagation constant \( \mu_p \) and characteristic impedance \( Z_c \) can be computed as
\[
\mu_p = \frac{\omega}{c_e}, \quad Z_c = \rho_e c_e.
\] (2.92)

In that case the wave equation (2.90) and the momentum equation (2.87) have the form
\[
\nabla^2 \ddot{p}_1 + \mu^2_p \ddot{p}_1 = 0,
\] (2.93)
\[
- i \mu_p Z_c \ddot{u}_1 + \nabla \ddot{p}_1 = 0.
\] (2.94)

Note that for air we have \( \rho_p = \rho_0, \Omega = 1, \sigma = 0, \) and \( \beta_p(p_0) = \beta_s(p_0) = \frac{1}{\gamma p_0} \).

### 2.3.3 Bulk absorbing liners

We modeled the effect of slipping flow based on an impedance boundary condition at the wall \( \ddot{p}^-_1 = Z v^-_1 \). The next step is to include in the boundary condition the effect of the sound propagation inside the liner [110].

Consider the schematic overview of the geometry in Figure 2.2: we use superscripts \(+\), \(−\) and \(l\) to denote variables in the duct, boundary and liner regions. As before, the boundary and duct regions are separated by a vortex sheet, and the liner and boundary regions are separated by a facing sheet (for example a perforated plate [48] covered with a wire mesh) that causes a pressure jump, so we have \( v^+ \neq v^- = v^l \) and \( p^+ = p^- \neq p^l \). The pressure jump across the facing sheet with impedance \( Z_0 \) is related to the velocity as
\[
\ddot{p}^- - \ddot{p}^l = Z_0 \ddot{v}^l \quad \text{at } r = d.
\] (2.95)
If we use
\[ Z = Z_0 + \frac{\tilde{p}_1^l}{\tilde{v}_1^l} \] at \( r = d \)
we find again \( \tilde{p}_1^- = Z\tilde{v}_1^- \).

Here we consider a circularly cylindrical geometry, so the wave propagation in the porous material can be described in terms of modes of the form \( Q(r)e^{ikx + im\theta} \). It follows from the wave equation \( \nabla^2 \tilde{p}_1 + \mu_p^2 \tilde{p}_1 = 0 \) that \( Q \) satisfies Bessel’s equation
\[ \frac{d^2 Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} + \left( \mu_p^2 - k^2 - \frac{m^2}{r^2} \right) Q = 0 \]

Taking into account the hard-wall boundary condition \( \frac{dQ}{dr} = 0 \) at \( r = b \) with \( b := d + d_1 \), the solution to (2.97) is:
\[ Q = A [J_m(\zeta r)Y_m'(\zeta b) - Y_m(\zeta r)J_m'(\zeta b)], \] with \( \zeta^2 = \mu_p^2 - k^2 \).

Consequently, for bulk absorbing liners we finally have a \( k \)-dependent impedance of
\[ Z(k) = Z_0 + \frac{i\mu_p Z_c}{\zeta} \frac{N(\zeta)}{D(\zeta)} \]

with
\[ N(\zeta) = J_m(\zeta d)Y_m'(\zeta b) - Y_m(\zeta d)J_m'(\zeta b), \]
\[ D(\zeta) = J_m'(\zeta d)Y_m(\zeta b) - Y_m'(\zeta d)J_m(\zeta b), \] \( b = d + d_1 \).

This expression for \( Z \) can be combined with the Ingard-Myers boundary condition (2.79) to formulate the boundary condition for slipping flow along a bulk-absorbing liner.

### 2.3.4 Honeycomb liner

A honeycomb liner can be considered a special case of the more general bulk absorbing liner discussed above. It is a Helmhotz resonator array: an array of radially oriented...
cavities (tubes) covered with a facing sheet and having a cross-wise diameter that is smaller than the wavelength. This prohibits the propagation of sound in the axial and circumferential directions, so we can use \( m = k = 0 \). When the cavities are air-filled we can also use \( \mu_p = \omega/c_0, Z_c = \rho_0 c_0, \) and \( \zeta^2 = \omega^2/c_0^2 \). It then follows that (2.97) reduces to

\[
\frac{d^2Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} + \frac{\omega^2}{c_0^2} Q = 0, \tag{2.100}
\]

with the solution, analogous to (2.99)

\[
Z = Z_0 + i \rho_0 c_0 \cot \left( \frac{\omega}{c_0} d \right). \tag{2.101}
\]

If it holds that \( \frac{d_l}{d} \ll 1 \) and \( \frac{\omega d_l}{c_0} = O(1) \), then it also holds that \( \frac{\omega d}{c_0} \gg 1 \) and \( \frac{\omega b}{c_0} \gg 1 \). By using the fact that the Bessel \( J_m(z) \) and \( Y_m(z) \) functions approximate cosine and sine functions for fixed order \( m \) and large argument \( z \) (see [98], equation 10.7.8) we can find the commonly used expression [101]

\[
Z = Z_0 + i \rho_0 c_0 \cot \left( \frac{\omega}{c_0} d \right). \tag{2.102}
\]

### 2.4 Pridmore-Brown equation

#### 2.4.1 Preform for arbitrary cross-section

In this section we first derive a ‘generalized Helmholtz equation’, which governs modes (as discussed in Section 2.2) in a duct with arbitrary cross-section carrying parallel mean flow [46, 108]. This is a more general version of the so-called Pridmore-Brown equation [97] that includes the effect of temperature gradients [16, 56, 83], which will be discussed in the next section.

We assume a mean flow with only a single non-zero velocity component in the axial direction that varies only in the transverse direction. We use transverse coordinates \((y,z)\) as we first consider a cylindrical duct of arbitrary cross-sectional shape (i.e. not necessarily circular). With constant mean pressure \( p_0 \) we then have \( v_0 = u_0(y,z)e_x, \rho_0 = \rho_0(y,z) \) and \( c_0 = c_0(y,z) \). With the definition of the convective derivative for parallel flow

\[
\frac{D}{Dt} := \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right), \tag{2.103}
\]

we can then use the following identities

\[
\nabla \cdot v_0 = 0, \quad \nabla p_0 = 0, \quad (v_0 \cdot \nabla) v_0 = 0, \quad v_0 \cdot \nabla \rho_0 = 0,
\]

\[
\nabla \cdot (v_1 \cdot \nabla) v_0 = \frac{\partial v_1}{\partial x} \cdot \nabla u_0, \quad \nabla \cdot \frac{D}{Dt} v_1 = \frac{D}{Dt} \nabla \cdot v_1 + \frac{\partial v_1}{\partial x} \cdot \nabla u_0, \tag{2.104}
\]

where \( \nabla_\perp \) is used to denote a transverse gradient operator in \( y \) and \( z \).

It follows from the linearized mass (2.60a), momentum (2.60b) and energy (2.63)
equations that perturbations of a parallel mean flow are governed by

\[
\frac{D}{Dt} \rho_1 + \rho_0 \nabla \cdot v_1 + v_1 \cdot \nabla \rho_0 = 0, 
\]

\[
\frac{D}{Dt} v_1 + v_1 \cdot \nabla v_0 + \frac{1}{\rho_0} \nabla p_1 = 0, 
\]

\[
\frac{D}{Dt} \rho_1 + v_1 \cdot \nabla \rho_0 = \frac{1}{c_0^2} \frac{D}{Dt} p_1. 
\]

The last equation (2.105c) can be used to eliminate \( \rho_1 \) from the first equation (2.105a) to find

\[
\frac{1}{\rho_0 c_0^2} \frac{D}{Dt} p_1 + \nabla \cdot v_1 = 0. 
\]

When we take the convective derivative of this equation and subtract from it the divergence of the momentum equation (2.105b) we obtain (note that \( \gamma p_0 = \rho_0 c_0^2 \) is constant)

\[
\frac{1}{\rho_0 c_0^2} \frac{D^2 p_1}{Dt^2} - \nabla \cdot \left( \frac{1}{\rho_0} \nabla p_1 \right) - 2 \frac{\partial v_1}{\partial x} \cdot \nabla \perp u_0 = 0. 
\]

To eliminate \( v_1 \) we substitute the transverse components of the momentum equation (2.105b) into the convective derivative of (2.107) to arrive at

\[
\frac{D^3}{Dt^3} p_1 + 2c_0^2 \frac{\partial}{\partial x} (\nabla \perp u_0 \cdot \nabla \perp p_1) - \frac{D}{Dt} \nabla \cdot (c_0^2 \nabla p_1) = 0. 
\]

We are interested in solutions of the form \( p_1(x,y,z,t) = P(y,z) e^{ikx-\omega t} \) (as we are neglecting the continuous part of the spectrum, see the remarks at the end of the next section) that satisfy a preform of the Pridmore-Brown equation

\[
i \Omega^2 P + 2ikc_0^2 (\nabla \perp u_0 \cdot \nabla \perp P) + i \Omega (-k^2 c_0^2 P + \nabla \perp \cdot (c_0^2 \nabla \perp P)) = 0, 
\]

where \( \Omega := \omega - ku_0. \) By noting that \(-k \nabla \perp u_0 = \nabla \perp \Omega \) this equation can be further simplified to

\[
\nabla \perp \cdot \left( c_0^2 \frac{\Omega^2}{\Omega^2} \nabla \perp P \right) + \left( 1 - \frac{k^2 c_0^2}{\Omega^2} \right) P = 0. 
\]

If \( u_0 \equiv 0 \) this equation reduces to

\[
\nabla \cdot (c_0^2 \nabla P) + (\omega^2 - k^2 c_0^2) P = 0, 
\]

which is just equivalent to the Helmholtz equation if \( c_0 \) is constant.

### 2.4.2 Circularly cylindrical geometry

We can look for solutions of the form \( p_1(x,r,\theta,t) = P(r) e^{ikx-\omega t} e^{im\theta} \) if the duct cross-section is circular and the temperature and flow speed depend only on the radial coordinate \( r. \) By substituting \( P(r) e^{im\theta} \) for \( P \) in (2.110) we find

\[
\frac{\Omega^2}{c_0^2} \left[ \frac{c_0^2 r}{\Omega^2} P' \right]' + \left[ \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right] P = 0. 
\]
This is the self-adjoint form of
\[
P'' + \left[ \frac{1}{r} + \frac{T_0'}{T_0} + \frac{2ku_0'}{\omega - ku_0} \right] P' + \left[ \frac{(\omega - ku_0)^2}{\gamma RT_0} - k^2 - \frac{m^2}{r^2} \right] P = 0, \tag{2.113}
\]
which is referred to as the *Pridmore-Brown equation* in cylindrical coordinates, here written out explicitly in terms of the radial temperature and flow profiles.

The Pridmore-Brown equation together with the wall boundary condition (2.79) (possibly combined with (2.99), (2.101) or (2.102)) and a regularity condition at \( r = 0 \) forms a boundary value problem (BVP). To motivate our choice of regularity condition, we first note that (2.113) is a differential equation whose solutions can be found with the Frobenius method \[115\], as the equation can be formulated as
\[
P'' + \frac{\beta(r)}{r} P' + \gamma(r) r^2 P = 0,
\]
where \( \beta(r) \) and \( \gamma(r) \) are assumed to be analytic. At least one solution has the form
\[
P_1(r) = a_0 r^\lambda_1 + a_1 r^\lambda_1 + a_2 r^\lambda_1 + \cdots,
\]
where \( \lambda_1 = m \) is one of the two roots of the so-called *indicial equation*, which is in our case \( \lambda^2 - m^2 = 0 \). With \( \lambda_2 = -m \) the two roots differ by an integer, so according to the Frobenius method the other solution is of the form
\[
P_2(r) = cP_1(r) \log(r) + b_0 r^{-m} + b_1 r^{-m+1} + b_2 r^{-m+2} + \cdots.
\]
This second solution is singular at \( r = 0 \). To ensure that we find the non-singular solution, which behaves as \( r^m \) near \( r = 0 \), we choose the regularity condition
\[
\begin{align*}
P'(0) &= 0 \quad \text{if } m \neq 1, \\
P(0) &= 0 \quad \text{if } m = 1.
\end{align*} \tag{2.114}
\]

We are interested in finding the eigensolutions of the BVP, also referred to as (eigen)modes. Each mode is a pair of an eigenfunction \( P(r) \) and its corresponding eigenvalue \( k \). The eigenfunctions can be determined up to a multiplicative constant; we assume that \( Z \neq 0 \) and hence \( P(d) \neq 0 \) so we choose the normalization condition
\[
P(d) = 1. \tag{2.115}
\]

As mentioned in Section 2.2: the modal approach assumes that the modes form a complete set, i.e. that all (acoustic) perturbation fields can be described in terms of these modes, according to (2.71). This is in general not the case for sheared mean flow. The coefficient of the first order term of the Pridmore-Brown equation is singular when \( \omega - k u_0 = 0 \); this happens when the phase speed of the perturbation wave, which is \( \omega/k \), is equal to the local mean flow velocity. The radial location where this happens is referred to as the *critical layer*. By applying a proper forward and inverse Fourier Transform in the axial direction, it follows that the total perturbation field may consist of contributions not only from the discrete set of eigensolutions, but also from a continuous part of the spectrum, which is due to the critical layer. However, it can be shown that this last contribution is small in most cases of interest \[23, 33\]. In this thesis the continuous part of the spectrum will be neglected.

### 2.5 High-frequency approximations

Since non-trivial analytical solutions of the Pridmore-Brown equation (2.113) are unknown or at least rare it is useful to consider approximations, as these can provide some analytical insight and can be used for comparison with numerical results. Similar to the approach described in \[120\] we can find some approximate solutions for high frequency \( \omega \) by using the *WKB method*. 
2.5.1 WKB-type solutions for the radial modes

Let us first make (2.113) dimensionless based on a reference density \( \rho_\infty \) and temperature \( T_\infty \), from which follows the reference sound speed \( c_\infty = \sqrt{\gamma RT_\infty} \). We use the scaling

\[
\tilde{P} = \frac{P}{\rho_\infty c_\infty^2}, \quad \tilde{T} = \frac{T_0}{T_\infty}, \quad \tilde{r} = \frac{r}{d}, \quad M = \frac{u_0}{c_\infty}, \quad \tilde{\omega} = \frac{\omega d}{c_\infty}, \quad \tilde{k} = kd, \tag{2.116}
\]

to find a dimensionless form of the Pridmore-Brown equation (tildes omitted)

\[
P'' + \left( \frac{1}{r} + \frac{T_0'}{T_0} + \frac{2kM'}{\omega - km} \right) P' + \left[ \frac{(\omega - km)^2}{T_0^2} - k^2 - \frac{m^2}{r^2} \right] P = 0 \quad \text{on } 0 < r \leq 1. \tag{2.117}
\]

We rewrite this equation as

\[
P'' + \beta(r,k)P' + \gamma(r,k)P = 0 \quad \tag{2.118}
\]

and note that \( \beta = \mathcal{O}(1) \) and \( \gamma = \mathcal{O}(\omega^2) \) as \( \omega \to \infty \).

If \( \gamma = \omega^2 \) and \( \beta = 0 \) we have the exact solution \( P = A^+ e^{i\omega r} + A^- e^{-i\omega r} \) (with some constants \( A^+ \) and \( A^- \)), which is highly oscillatory for large \( \omega \). However, the mean flow and temperature profiles (and consequently \( \beta \) and \( \gamma \) as well) are assumed to vary only slowly compared to these oscillations. We can therefore assume a similar oscillatory solution with a slowly varying amplitude and phase of the form

\[
P = A(r)e^{i\phi(r)}. \tag{2.119}
\]

Substituting this Ansatz into (2.117) leads to

\[
\begin{aligned}
\left( \gamma - (\phi')^2 \right) A + i \left( 2A'\phi' + A(\phi')' + \beta A\phi' \right) + \left( A'' + \beta A' \right) &= 0, \\
\mathcal{O}(\omega^2) &\quad \mathcal{O}(\omega) &\quad \mathcal{O}(1)
\end{aligned} \tag{2.120}
\]

where the terms have been sorted based on order of magnitude. It then follows from the terms of order \( \omega^2 \) and \( \omega \) respectively that

\[
\phi' = \pm \sqrt{\gamma}, \quad A = C \frac{\omega - km}{\gamma^{1/4} \sqrt{rT_0}}, \tag{2.121}
\]

for some constant \( C \). The expression for the amplitude can be found by integrating \( 2 \frac{dA}{A} - \frac{1}{2} \frac{dr}{r} + \beta dr = 0 \) and using the fact that \( \beta = \frac{d}{dr} \left( \frac{\log rT_0}{(\omega - km)^2} \right) \). The final approximation can now be expressed as

\[
P(r) \simeq \frac{\omega - km}{\gamma^{1/4} \sqrt{rT_0}} \left[ A^+ \exp \left( i \int \sqrt{\gamma(s)} ds \right) + A^- \exp \left( -i \int \sqrt{\gamma(s)} ds \right) \right]. \tag{2.122}
\]

Note that the \( \exp(\cdot) \) terms are oscillatory on the interval where \( \gamma(r) > 0 \) and exponentially small otherwise. Generally \( \gamma \) is negative near \( r = 0 \) when \( m \neq 0 \) (\( \gamma \approx -m^2/r^2 \) yielding the \( P \propto r^m \) behavior near \( r = 0 \)), with a zero at say \( r_1 \). Depending on details of the velocity and temperature profiles \( \gamma \) may also be negative near \( r = 1 \), with a zero at say \( r_2 \). The zeros \( r_1 \) and \( r_2 \) of \( \gamma \) are known as turning points [103]. If \( \gamma \) is indeed negative near \( r = 1 \), in which case the solution is exponentially small near the wall,
then the effect of the boundary condition is negligible and the solution is practically independent of the wall impedance. In that case the wave numbers $k$ can be found from a quantization condition [44] that requires that an integer number of radial semi-wavelengths fits between the turning points $r_1$ and $r_2$. On each side of the interval $(r_1, r_2)$ an extra $\frac{1}{4}\pi$ is required due to the matching of the oscillatory to the decaying solution at the turning points [82]. If $m \neq 0$ (and $\gamma$ has no more zeros than $r_1$ and $r_2$) this quantization condition is

$$\int_{r_1}^{r_2} \sqrt{\gamma(r, k)} \, dr = (n - \frac{1}{2})\pi, \quad n = 1, 2, \ldots$$

(2.123)

For $m = 0$ with only one turning point we have

$$\int_{0}^{r_2} \sqrt{\gamma(r, k)} \, dr = (n - \frac{1}{4})\pi, \quad n = 1, 2, \ldots$$

(2.124)

### 2.5.2 Numerical results

Our numerical approach (which is based on the code COLNEW and will be described in Chapter 3) can be validated for test cases with uniform mean flow and temperature, as analytical solutions of the Pridmore-Brown equation are then available in terms of Bessel functions. For non-trivial cases of non-uniform mean flow it is much harder to find suitable test cases. One possible configuration is a strongly non-uniform (parabolic) mean flow with upstream running modes of sufficiently high frequency. The waves will refract to the part of the medium with the lowest (effective) sound speed [108], which is in this case the duct center. The eigenfunction becomes exponentially small near the wall, which may be challenging for numerical methods like shooting [119]. However, the WKB solution of Section (2.5.1) is very applicable and will therefore be used here for making a comparison.

Consider the case where $\omega = 25$, $m = 5$, $Z = 2 - i$, the mean flow temperature $T_0 = 1$, and the mean flow velocity is $M(r) = \frac{2}{3}(1 - \frac{1}{2}r^2)$. The first 6 upstream running eigenfunctions, decaying exponentially towards the wall, are depicted in Figure 2.3. A comparison of the axial wavenumbers found by our numerical approach and the wavenumbers determined by the quantization condition (2.123) shows an excellent agreement, see Table 2.1. As there is little or no influence of the impedance wall the wave numbers are practically real. Only for the higher order modes the damping of the wall becomes little by little effective as the imaginary parts of the wave numbers become negative.
Figure 2.3: Eigenfunctions of the first six upstream (left)-running modes in non-uniform mean flow, refracted away from the duct wall.

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<th>numerical</th>
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<tr>
<td>4</td>
<td>-46.605341</td>
<td>-46.5659 - 0.0003i</td>
</tr>
<tr>
<td>5</td>
<td>-42.195806</td>
<td>-42.1422 - 0.0212i</td>
</tr>
<tr>
<td>6</td>
<td>-37.931062</td>
<td>-37.5622 - 0.3254i</td>
</tr>
</tbody>
</table>

Table 2.1: Modal wave numbers $k$ of the first six modes, corresponding to figures 2.3, found from the WKB quantization condition and by our numerical approach.
Chapter 3

Numerical approach

In this Chapter we describe our numerical approach for finding the eigensolutions of the Pridmore-Brown equation for acoustic duct modes in parallel flow with temperature gradients.

One could ask: why would a semi-analytical approach based on modes be advantageous over a fully numerical approach, based for instance on a direct discretization of the Linearized Euler Equations (LEE) that govern the acoustic field? Computing power is ever increasing; however, computational costs for complex geometries are still prohibitively high for design calculations. This is especially the case for the high prevailing frequencies in turbofans, with Helmholtz numbers up to 100 and circumferential wavenumbers up to 50. State of the art time-domain LEE computations for a 3D exhaust analysis with a Helmholtz number of 28 take 40 hours on a cluster consisting of 12 processors [12].

To reduce computational complexity, numerical codes used in aircraft industry are therefore often based on the assumption that the mean flow and acoustic perturbation field are irrotational. In this way a formulation based on a velocity potential can be used [10], which amounts to one degree of freedom per grid point, where a full discretization of the LEE would require four. However, the modeling of swirling or parallel sheared flow, which is by definition not rotation-free, is then limited, although some efforts have been made to compute sound propagation in sheared flow based on a formulation in terms of the total fluid enthalpy (which is a scalar) [64]. The commercial code ACTRAN/TM is an example of a finite element method based on the Helmholtz equation for the acoustic velocity potential [43]. In [6] this code has been used to compute sound propagation through a 3D bellmouth intake. The CPU time is about 2 hours for a Helmholtz number of around 40. It is also noted in [6] that memory requirements prevent computations for higher Helmholtz numbers.

Our approach is based on modes, which are the eigensolutions of a boundary value problem (BVP) with a free parameter. We first describe in Section 3.1 how we solve this BVP numerically. This enables the application of a path-following approach based on a prediction-correction scheme, which will be described in Section 3.2. Finally, in Section 3.3 we describe how a good set of initial guesses for the duct modes can be obtained with the aid of a root-finding method based on complex contour-integration.
3.1 Collocation to solve the BVP

The complete boundary value problem (BVP), which consists of the ordinary differential equation (ODE) (2.117) together with the wall boundary and regularity conditions (2.79) and (2.114), made dimensionless by using the scaling of (2.116), is summarized here as

\[ P'' + \beta(r,k)P' + \gamma(r,k)P = 0 \quad \text{on} \quad 0 < r \leq 1, \quad (3.1a) \]

subject to

\[ P' + \kappa(r,k)P = 0 \quad \text{at} \quad r = 1, \quad \begin{cases} P' = 0 & \text{if} \quad m \neq 1 \\ P = 0 & \text{if} \quad m = 1 \end{cases} \quad \text{at} \quad r = 0, \quad (3.1b) \]

with

\[ \beta = \frac{1}{r} \left( T_0' + \frac{2kM'}{\omega - kM} \right), \quad \gamma = \frac{(\omega - kM)^2}{T_0} - k^2 - \frac{m^2}{r^2}, \quad \kappa = -i(\omega - kM)^2 \frac{\omega ZT_0}{\omega ZT_0}. \quad (3.1c) \]

The impedance of a bulk reacting liner is made dimensionless by using the scaling

\[ \tilde{\mu}_p = \mu_p d, \quad \tilde{\xi} = \xi d, \quad \tilde{b} = \frac{b}{d} = 1 + \frac{d_l}{d} = 1 + \tilde{d}_l, \]

which results in (tildes omitted)

\[ Z(k) = Z_0 + \frac{i\mu_p Z_c}{\xi} \frac{N(\xi)}{D(\xi)}, \quad (3.2a) \]

with

\[ N(\xi) = J_m(\xi)Y_m'(\xi b) - Y_m(\xi)J_m'(\xi b), \quad \xi^2 = \mu_p^2 - k^2, \]
\[ D(\xi) = J_m'(\xi)Y_m(\xi b) - Y_m'(\xi)J_m(\xi b), \quad b = 1 + d_l. \quad (3.2b) \]

As mentioned before, this is a boundary value problem with a free parameter \( k \), or in other words: an eigenvalue problem. Even though the Pridmore-Brown equation and the wall boundary condition are linear in \( P \), the fact that we have a free (unknown) parameter \( k \) makes the eigenvalue problem non-linear. Moreover, for non-locally reacting liners the impedance \( Z \) depends on \( k \) in a non-linear way according to (3.2).

As can be seen from (4.25), in principle it is possible to formulate the problem as an (generalized) eigenvalue system of differential equations of the form \( LF = kKF \) in terms of the vector \( F = [P,U,V,W]^T \) of separate pressure and velocity components, where \( L \) and \( K \) are differential operators in matrix form. The wall boundary condition (2.78) can be expressed in this form as well. For reasons of computational efficiency we use the problem formulation in terms of the Pridmore-Brown equation, since it is then unnecessary to store the \( U, V \) and \( W \) components.

In case of a hard wall and a mean flow with uniform velocity and temperature, the classical analytical solutions exist in the form of Bessel functions, and the axial wavenumbers \( k \) are given through the easily found (because real) zeros of the derivative of the Bessel \( J_m \) function. For most other cases solutions have to be found entirely numerically.

After its first publication [97] it took a long time before numerical solution of this problem became tractable, due to the fact that the Pridmore-Brown equation is numerically stiff for high frequencies and / or high values of \( k \) (i.e. higher order modes). One
of the first works described a *shooting* approach based on a Runge-Kutta method to find the two lowest order modes for a two-dimensional hard-walled duct with parallel shear flow [78]. More recently an efficient way to numerically find (also higher order) duct modes of an annular lined duct with parallel shear flow was proposed [119], which is also based on the shooting method.

With the shooting method the problem is considered as an initial value problem; the Pridmore-Brown equation is integrated several times for a fixed $k$ from $r = 0$ to $r = 1$, in [119] by using an implicit one-step scheme. The value of $k$ for which the boundary condition at $r = 1$ is satisfied is then found by using Newton’s method. Many initial guesses of $k$ are required if several eigensolutions are sought. Note that implicit methods can be used in the shooting method without requiring the solution of a non-linear system of equations, since the numerical integration of the ODE, which is linear in $P$, is done while the parameter $k$ is kept fixed.

Also global methods based on a finite differences or on spectral collocation have been used, for example for the study of duct modes in swirling flow [28, 29, 47, 87, 113]. These methods discretize the previously mentioned problem formulation $\Delta \mathbf{F} = k K \mathbf{F}$ on a given mesh. Together with equations that represent the boundary conditions this leads to say $N$ equations that form a generalized eigenvalue problem of the form $A \mathbf{x} = \lambda B \mathbf{x}$, which can be solved by using standard methods.

While this approach yields $N$ eigensolutions at once, only the smallest eigenvalue (which corresponds to the value for the free parameter $k$) can be determined with the accuracy of the discretization scheme, and the accuracy decreases for larger eigenvalues. (As a prototypical example, compare the analytically known eigenvalues of the operator $\frac{d^2}{dx^2}$ with homogeneous Dirichlet boundary conditions to the eigenvalues of the matrix resulting from a second order central differences discretization, [67, section 9.4.1].) Thus, to obtain a large number of sufficiently accurate eigensolutions a very fine mesh (large $N$) and a high order accurate discretization scheme are required, and it is difficult to control the accuracy of specific eigenvalues other than the smallest one.

Another potential pitfall with global methods is that in some cases spurious (not physically meaningful) modes are found due to difficulties related to the $(\omega - kM)^{-1}$ term, which can become singular [47]; however, these solutions can be ignored by recognizing that they change when the discretization is changed.

In order to control the accuracy of each of the required multiple eigensolutions we will employ a path-following strategy (which will be described in Section 3.2) combined with a separate BVP solver. This enables us to set the same error tolerance for each individual eigensolution. Clearly, there is a host of numerical methods to solve BVPs, like the above-mentioned shooting method [20, 119], finite elements, and collocation methods [19]. The advantage of a shooting method is its simplicity, but it is known to be potentially unstable. This is partly cured by employing multiple shooting. However, for more complicated problems collocation was shown to perform best in practice.

We will specifically use a code called COLNEW [15], which is also described and successfully applied to a number of practical problems in [9], and is freely available from Netlib [49]. This code, which is a modification of the code COLSYS [7, 8], is capable of solving BVPs formulated as mixed-order systems of ODEs. It is based on collocation at Gaussian points using a Runge-Kutta monomial basis representation, it uses a damped Newton solver to solve the system of non-linear equations and it has automatic meshing. The fact this code uses Gaussian points, which do not include the boundary points of each mesh subinterval, prevents potential problems with the $1/r$
singularity in $r = 0$. Because of the availability of local error estimates COLNEW has facilities to check the solution against user-prescribed tolerances, and to refine or redistribute the mesh points. This is one reason for the fact that the code is very efficient. Finally, we mention that the computed solution can be evaluated at any point inside the domain, not only at the mesh points.

We put the eigenvalue problem (3.1) in a form suitable for a standard BVP solver like COLNEW by introducing the extra equation $k' = 0$, adding a normalization condition $P(1) = 1$, and splitting the equations in real and imaginary parts. We use subscripts $R$ and $I$ to denote real and imaginary parts in order to formulate the problem as

\begin{align*}
&k'_R = 0 \\
k'_I = 0 \\
P''_R = -\beta_R P'_R + \beta_I P'_I - \gamma_R P_R + \gamma_I P_I & \quad \text{on } 0 < r \leq 1, \\
P''_I = -\beta_I P'_R - \beta_R P'_I - \gamma_I P_R - \gamma_R P_I
\end{align*}

subject to

\begin{align*}
P'_R + \kappa_R P_R - \kappa_I P_I &= 0 \quad \text{and} \quad P_R = 1 \quad \text{at } r = 1, \\
P'_I + \kappa_I P_R + \kappa_R P_I &= 0 \quad \text{and} \quad P_I = 0 \quad \text{at } r = 1, \\
\begin{cases}
P'_R - P'_I &= 0 \quad \text{if } m \neq 1 \\
P_R - P_I &= 0 \quad \text{if } m = 1
\end{cases} & \quad \text{at } r = 0.
\end{align*}

Note that the coefficients $\beta_R, \beta_I, \gamma_R, \gamma_I, \kappa_R, \kappa_I$ all depend on $r$, $k_R$ and $k_I$, which shows that the problem is non-linear. By introducing the vector $z = [k_R, k_I, P_R, P'_R, P'_I, P_I]^T$ the right-hand side of (3.3a) can be written in the form $f(r, z)$, and each of the boundary conditions in the form $g_i(z(r)) = 0$. In order to solve the non-linear system of equations by using Newton's method the Jacobian matrices of $f$ and $g$ are required, i.e. the terms $\frac{\partial f_i}{\partial z_j}$ and $\frac{\partial g_i}{\partial z_j}$. We compute these terms by using a second order central differences approximation.

### 3.2 Path-following based on linear extrapolation

As pointed out in the previous section, in order to find the relevant eigensolution it is important to have a good initial guess. This is particularly important if we want to make sure to find all (i.e. all physically relevant) modes. This is realized by a path-following (or continuation) approach, where we start from an “easy” solution (for example the analytically known solutions for a hard walled duct with uniform mean flow velocity and temperature) and trace the solution when the relevant problem parameters are varied until they reach their target values. Essentially, we embed the problem in a formulation with a continuation parameter $\lambda$. Path-following can thus be seen as an evolution problem with the problem evolving from a known solution for $\lambda = 0$ to the target solution at $\lambda = 1$.

Apart from ensuring that we find all relevant eigensolutions there are more advantages of path-following: first, we can investigate the behavior of a solution as a function of a parameter; and second, when we are interested in a series of $\lambda$-values of the same continuation, for example for the case of an axially segmented liner (where each segment has a different impedance), it is possible to save intermediate solutions when they are encountered along the way, which makes this approach very efficient.
§ 3.2 Path-following based on linear extrapolation

To compute the target solution with mean flow velocity \( M_t(r) \), temperature \( T_t(r) \) and impedance \( Z_t \) starting from the analytical solution for uniform mean flow with Mach number \( M_0 \), uniform temperature \( (T_0 = 1) \) and a given impedance \( Z_\infty \) we use the embedding

\[
M(r) = (1 - \lambda)M_0 + \lambda M_t(r),
\]
\[
T_0(r) = (1 - \lambda) + \lambda T_t(r),
\]
\[
Z = (1 - \lambda)Z_\infty + \lambda Z_t.
\]  

(Other continuation parameterizations are also possible, but this was found to work generally well.) The impedance, mean flow velocity and temperature are gradually changed to their target values in parallel, or one after another.

Path-following has been used before for the case of uniform mean flow and temperature \([24]\), and a path in the impedance plane from hard wall to the target impedance \([42]\). We will take into account, however, that not any path is suitable to find all modes, as has been noted in \([100]\). It is known that surface modes exist for certain impedance and mean flow values. Depending on the specific trajectory that is chosen for the impedance variation, some of these surface modes may appear from or move towards infinity as the impedance approaches a hard wall value. It was shown \([100]\) that the surface modes start from a finite value (and do not appear from infinity) if the impedance is varied (in the right direction) along a specific trajectory in the complex plane: starting from a nearly hard wall value we follow a straight vertical path (i.e. with \( \text{Re}(Z_t) \) fixed) upwards to the target impedance. For our sign convention \( e^{-i\omega t} \) and mean flow in the positive \( x \) direction this ensures that (the axial wavenumbers of) any possible surface waves start from a finite value and disappear to infinity.

Since each eigensolution is characterized by an eigenfunction and an axial wavenumber \( k \) we can trace \( k(\lambda) \) as a curve in the complex plane (some examples are shown in Figure 3.2 and more will be presented in Chapter 6). To determine the number of intermediate solutions and the corresponding values of \( \lambda \) we use a prediction-correction scheme \([4]\), see Figure 3.1. We use linear extrapolation of two previously computed \( k \)-values for the prediction step. In general, the prediction will not satisfy the BVP. The prediction is therefore corrected subsequently with the aid of COLNEW.

We trace the eigenvalue \( k = k(\lambda) \) and solution vector \( u = u(\lambda) \) for \( \lambda \in [0,1] \) using steps \( h_j := \lambda_j - \lambda_{j-1} \). From two previous solutions \( k_{j-2} = k(\lambda_{j-2}) \) and \( k_{j-1} = k(\lambda_{j-1}) \) we
predict a value for \( k_j \) by linear extrapolation

\[
\tilde{\kappa}_j = k_{j-1} + h_j \frac{k_{j-1} - k_{j-2}}{h_{j-1}}.
\] (3.5)

The prediction of \( u \) is performed similarly. By substituting \( \lambda_{j-1} \) and \( \lambda_{j-2} \) in the Taylor series expansion of \( k(\lambda) \) around \( \lambda = \lambda_j \) we find expressions for \( k_{j-1} \) and \( k_{j-2} \) that can be used with (3.5) to find

\[
\tilde{\kappa}_j = k_j - \frac{1}{2} h_j (h_j + h_{j-1}) k_j'' + \cdots .
\] (3.6)

Thus the error \( \epsilon_j \) between the exact and predicted value is

\[
\epsilon_j = |k_j - \tilde{\kappa}_j| = \frac{1}{2} h_j (h_j + h_{j-1}) k_j'' + \cdots = ch_j^2,
\] (3.7)

where \( k_j \) is assumed to be sufficiently smooth (such that \( c \sim k_j'' \) is not too large). Given this \( j \)-th step error, we can compute the next step size \( h_{j+1} \) such that the error remains around some tolerance \( \epsilon_{\text{tol}} = ch_{j+1}^2 \). This leads to

\[
h_{j+1} = h_j \frac{h_{j+1}}{h_j} = h_j \sqrt{\frac{ch_j^2}{ch_{j+1}^2}} = h_j \sqrt{\frac{\epsilon_{\text{tol}}}{\epsilon_j}} .
\] (3.8)

In some cases, a small change in \( \lambda \) results in a big change in the solution. It is even possible to jump to the trajectory corresponding to another mode. This is especially relevant when the minimum distance between the (a-priori unknown) trajectories of two different modes is very small, in which case we wish to set a strict error tolerance. On the other hand, if the trajectory of the mode under consideration is far away from the trajectories of the other modes (as is the case of some surface modes) we can relax \( \epsilon_{\text{tol}} \). This motivates the choice

\[
\epsilon_{\text{tol}} = \bar{\epsilon}_{\text{tol}} \Delta k_{\text{ref}},
\] (3.9)

where \( \Delta k_{\text{ref}} \) depends on the mode considered and \( \bar{\epsilon}_{\text{tol}} \) is a parameter, equal for all modes, that has to be chosen.

One option is to base \( \Delta k_{\text{ref}} \) on the distance between the axial wavenumbers corresponding to a hard wall, uniform flow with Mach number \( M_0 \) and uniform temperature. In this case the axial wavenumbers \( \hat{k} \) of circumferential order \( m \) and radial order \( \mu \) are given by

\[
\hat{k}_{m\mu} = -\omega M_0 \pm \sqrt{\omega^2 - \alpha_{m\mu}^2 (1 - M_0^2)} \frac{1 - M_0^2}{1 - M_0^2},
\] (3.10)

with \( \alpha_{m\mu} \) the \( \mu \)-th zero \( j'_{m\mu} \) of the derivative of the Bessel function \( J_m \). These zeros can be approximated by \( j'_{m\mu} \approx (\mu + \frac{1}{2} m - \frac{3}{4}) \pi \). Consequently, the difference \( \Delta k = |\hat{k}_{j+1} - \hat{k}_j| \) between two adjacent cut-off axial wavenumbers is approximately constant \( \pi/\sqrt{1 - M_0^2} \), whereas for cut-on modes \( \Delta k \) depends on \( \omega \). Another option is to base \( \Delta k_{\text{ref}} \) on the distances of a set of a-priori known axial wavenumbers, if available. For example the set of wavenumbers corresponding to a sound absorbing wall, uniform flow and uniform temperature can be used. This set can be found by using the approach that will be described in Section 3.3. Since now any possible surface waves are also known \( \epsilon_{\text{tol}} \) can be relaxed on these modes.
In order to prevent continuing on the wrong trajectory after inadvertently jumping to another mode we restart from the former solution with a halved step size when the error is too large, i.e, when $\epsilon > \epsilon_{\text{max}}$. We also add an upper limit $h_{\text{max}}$ to the step size. We finally have:

$$h_{j+1} = \min \left[ h_j \sqrt{\frac{\Delta k_{\text{ref}}}{}}, \epsilon_j, h_{\text{max}} \right].$$  \hfill (3.11)

In practice there is a trade-off in choosing the parameters $\epsilon_{\text{tol}}$, $\epsilon_{\text{max}}$, $h_{\text{max}}$ and the initial step size $h_1$; we would like to travel through the path quickly, while at the same time maintaining a certain confidence that we do not jump to another mode.

Unfortunately, it is difficult to devise a method that prevents this jumping with full confidence, since the trajectories of the modes may become arbitrarily close or they might even collapse, depending on the specific choice of the problem embedding in the continuation parameter $\lambda$. One such example is depicted in Figure 3.2. In this example we are interested in the modes corresponding to non-uniform flow and non-uniform temperature in a hard-walled duct. Figure 3.2a shows the trajectories when, starting from the solution for a hard wall, uniform flow and uniform temperature (blue), we
follow the path to the target solution (red) while keeping the wall hard. It can be seen that the trajectories of the lowest order right-running and left-running modes collapse at the point where they change from cut-off to cut-on. For some choices of the parameters $\tilde{\epsilon}_{\text{tol}}$, $\tilde{\epsilon}_{\text{max}}$, $h_{\text{max}}$ and $h_1$ this can result in jumping to the wrong mode, as depicted in Figure 3.2b. In this case it is useful to choose a different embedding. With

$$Z = 2 - \frac{2i}{4\lambda(1-\lambda)}, \quad 0 \leq \lambda \leq 1,$$

the impedance $Z$ changes from $Z = 2 - i\infty$ via $Z = 2 - 2i$ back to $Z = 2 - i\infty$ as $\lambda$ changes from 0 to 1. This specific path ensures that no surface waves are encountered (see [100]). At the same time we change the flow and temperature from uniform to non-uniform, as we did before. With this embedding the collapse of the $k$-trajectories is prevented, as shown in Figure 3.2c.

### 3.3 Finding bulk-absorber modes by using contour integration

For a duct carrying a uniform flow with uniform temperature the Pridmore-Brown equation reduces to Bessel's equation, and consequently the eigensolutions can be found as the roots of a function involving Bessel functions. For a hard-walled duct carrying uniform flow at uniform temperature the eigensolutions follow directly from the roots of the derivative of the Bessel function via the dispersion relation. These roots are easily found numerically, since they lie at approximately equidistant locations on the real line and are consequently ordered.

For a soft-walled duct this is not the case; the complex-valued roots of a function have to be found, which lie in the complex plane (consequently not ordered) at a-priori unknown distances apart. Thus it is more difficult to ensure that all eigensolutions are found as there is no way to systematically and efficiently scan the complex plane.

For locally reacting liners one option to find the modes is to start from the easily found hard-wall modes and gradually vary the impedance to the desired value along a certain trajectory. If this trajectory is chosen carefully, as discussed previously on page 41, then in principle a path-following strategy starting from the hard-wall modes can be used to find all modes for a given impedance value.

For non-locally reacting liners, however, we have to resort to other means than starting from the hard-wall modes. In this case a mode is determined through the properties of both the duct and liner regions, i.e. the eigenfunction is defined on a domain consisting of the duct and liner domains. The parts of the mode in the duct and the annular liner regions are coupled as the two regions are coupled through the facing sheet impedance. For a hard facing sheet the duct and liner regions are effectively decoupled and separate duct and liner modes exist. In that case the eigenvalue (i.e. axial wavenumber) of a liner mode is not necessarily also an eigenvalue of a duct mode. Therefore, the modes that are related to the liner region can not be reached by a path-following procedure starting from hard-walled duct modes. This motivates the use of another approach that ensures that all modes, including bulk-absorber modes, are found directly for given values of the liner properties: root-finding by contour integration, which is the topic of this section.
3.3.1 Eigenvalues as the roots of an analytic function

For uniform flow with Mach number $M_0$ and uniform temperature $T_0 = 1$ the Pridmore-Brown equation (3.1a) reduces to Bessel’s equation

$$P'' + \frac{1}{r} P' + \left( \alpha^2 - \frac{m^2}{r^2} \right) P = 0,$$

(3.13)

with the dispersion relation

$$\alpha^2 = (\omega - kM_0)^2 - k^2.$$  

(3.14)

The regular solution normalized such that $P(1) = 1$ is $P(r) = J_m(ar)/J_m(\alpha)$. Together with the wall boundary condition (3.1b) this leads to

$$Z + \frac{(\omega - kM_0)^2 J_m(\alpha)}{i\omega \alpha J_m'(\alpha)} = 0$$

(3.15)

for locally reacting liners, and

$$Z_0 + \frac{i\mu_p Z_c N(\zeta)}{\zeta D(\zeta)} + \frac{(\omega - kM_0)^2 J_m(\alpha)}{i\omega \alpha J_m'(\alpha)} = 0$$

(3.16)

for bulk absorbers. The eigensolutions that we are after can be found by searching for $k$’s that satisfy these equations.

The root-finding approach that will be described in the next section can only be applied to analytic functions. Hence, we first rewrite the eigenvalue equation of (3.16) in the form of an analytic function. First note that the Bessel functions have the following power series representations (see for example [98])

$$J_m(z) = a_0 z^m + a_1 z^{m+2} + a_2 z^{m+4} + \cdots$$

(3.17)

$$Y_m(z) = b_0 z^{-m} + b_1 z^{-m+2} + \cdots + \frac{2}{\pi} \log \left( \frac{z}{2} \right) J_m(z).$$

(3.18)

Thus, we can split the polynomial part from the logarithmic part by writing

$$Y_m(\zeta) = f(\zeta) + \frac{2}{\pi} \log \left( \frac{\zeta}{2} \right) J_m(\zeta), \quad \text{with } f(z) = b_0 z^{-m} + b_1 z^{-m+2} + \cdots,$$

(3.19)

$$Y'_m(\zeta b) = g(\zeta) + \frac{2}{\pi} \log \left( \frac{\zeta b}{2} \right) J'_m(\zeta b), \quad \text{with } g(z) = c_0 z^{-m-1} + c_1 z^{-m+1} + \cdots.$$

(3.20)

It follows that the logarithmic terms in $N(\zeta)$ and $D(\zeta)$ vanish; we can write

$$N(\zeta) = d_0 \zeta^{-1} + d_1 \zeta^1 + d_2 \zeta^3 + \cdots,$$

(3.21)

$$D(\zeta) = \epsilon_0 \zeta^{-2} + \epsilon_1 \zeta^0 + \epsilon_2 \zeta^2 + \cdots.$$  

(3.22)

The idea is to avoid difficulties related to the multi-valuedness of the square roots in $\alpha$ and $\zeta$ by making sure that we construct a function that has a series representation in even powers of $\alpha$ and $\zeta$. This can be achieved through multiplying (3.16) by $\zeta^2 D(\zeta)$ and $\alpha^{-m+1} J'_m(\alpha)$. This results in

$$F(k) = \alpha^{-m+1} \left[ \zeta^2 D(\zeta) \left( J_m(\alpha) Z_0 + \frac{(\omega - kM_0)^2 J_m(\alpha)}{i\omega \alpha} \right) + i\mu_p Z_c \zeta N(\zeta) J'_m(\alpha) \right],$$

(3.23)
of which all three terms have a series expansion of the form
\[ F(k) = (f_0 + f_1 \alpha^2 + f_2 \alpha^4 + \cdots) (\hat{f}_0 + \hat{f}_1 \xi^2 + \hat{f}_2 \xi^4 + \cdots). \] (3.24)

As a result this function is analytic. Similarly, for locally reacting liners we multiply (3.15) with \( \alpha^{-m+1} J'_m(\alpha) \) to obtain
\[ G(k) = \alpha^{-m+1} \left( J'_m(\alpha) Z_0 + \frac{(\omega - k M_0)^2}{i \omega} J_m(\alpha) \right) . \] (3.25)

Note that for \( m = 0 \) and \( Z_{\text{tot}} \to \infty \) (i.e. a hard wall) the plane wave is a valid solution. In that case (3.15) and (3.16) need to be multiplied by \( \alpha^{-m+3} J'_m(\alpha) \) instead of \( \alpha^{-m+1} J'_m(\alpha) \) in order to include the solution \( \alpha = 0 \).

### 3.3.2 Finding roots using contour integration

The original method of Delves and Lyness [36] to compute the roots \( z_i \) of an analytic function \( f(z) \) inside a contour \( \Gamma \) is based on the following.

For an analytic function \( f(z) \) the number of roots \( N \) inside a contour \( \Gamma \) can be computed by using (Cauchy’s) argument principle [1], which can be formulated as
\[ N = \frac{1}{2 \pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz . \] (3.26)

For the roots \( z_i \) of \( f(z) \) inside \( \Gamma \) the power sums \( S_n \) can be computed as
\[ S_n = \frac{1}{2 \pi i} \int_{\Gamma} z^n \frac{f'(z)}{f(z)} \, dz = \sum_i z_i^n , \] (3.27)

Note that in a quadrature implementation the function values \( f'(z)/f(z) \) that are required to compute \( N \) can be reused for \( S_n, n > 0 \). Moreover, since the integrand is periodic if the contour \( \Gamma \) is smooth we can use a trapezoidal rule very efficiently, as will be discussed in the next section.

These power sums \( S_n \), which are also sometimes referred to as Newton sums, can be used to construct a polynomial that has the same roots as \( f(z) \). Let us define this polynomial as
\[ p(z) = \prod_{i=1}^{N} (z - z_i) = \sum_{k=0}^{N} c_k z^k . \] (3.28)

We use a monic polynomial, i.e. \( c_N = 1 \). The remaining coefficients can then be computed recursively by using Newton’s identities [55], which can be written as
\[ c_k = \frac{1}{k-N} \sum_{j=1}^{N-k} S_j c_{k+j}, \quad k = N-1, N-2, \ldots, 0 . \] (3.29)

Once the coefficients of the polynomial are known, the roots can be computed using a standard polynomial root finding algorithm, like for example the following one. The companion matrix to a monic polynomial of \( N \)-th degree is the \( N \times N \) matrix
\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & -c_0 \\
1 & 0 & \cdots & 0 & -c_1 \\
0 & 1 & \cdots & 0 & -c_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -c_{N-1}
\end{bmatrix} .
\] (3.30)
The roots of the polynomial are equivalent to the eigenvalues of the companion matrix.

In this formulation the computation of the power sums requires the evaluation of the derivative $f'(z)$. In practical application $f'(z)$ may not be available or computationally very expensive to compute. A slightly different approach [34] does not require the derivative. Firstly, $N$ can be computed by considering the total change of the argument of $f(z)$ as $z$ makes one round-trip along the contour $\Gamma$:

$$N = \frac{1}{2\pi} [\Delta \arg f(z)]_{\Gamma}. \quad (3.31)$$

Secondly, the integral of (3.27) can be rewritten as [53]

$$S_n = -\frac{n}{2\pi i} \int_{\Gamma} z^{n-1} \log [(z-z_0)^{-N} f(z)] dz + Nz_0^n, \quad (3.32)$$

where $z_0$ is an arbitrary point inside $\Gamma$. Let us write the logarithmic term as $\log(g(z)) = \log(|g(z)|) + i\theta$ with $g(z) = (z-z_0)^{-N} f(z)$. Note that, since $g(z)$ has no zeros or poles, this logarithmic term is single-valued along the contour $\Gamma$: $\theta$ returns to its original value after traversing the complete contour. In a practical computer implementation, where the available subroutines for computing a logarithm usually return its principal value (for which the imaginary part is in $(-\pi, \pi)$), care must be taken that the imaginary part of the integrand in (3.32) is continuous while traversing $\Gamma$.

### 3.3.3 Implementation details

The Bessel functions $J_m$ and $Y_m$ grow exponentially for arguments with a large imaginary part. This provokes severe numerical cancelation errors in the computation of $N(\zeta)$ and $D(\zeta)$. Hankel functions of first and second kind $H^{(1)}_m$ and $H^{(2)}_m$ can be used instead of the Bessel $Y_m$ function in order to prevent this. Using $H^{(1)}_m = J_m + iY_m$ and $H^{(2)}_m = J_m - iY_m$ we can rewrite $N(\zeta)$ in (3.3b) as

$$N(\zeta) = -i \left[ J_m(\zeta) H^{(1)\prime}_m(\zeta b) - H^{(1)}_m(\zeta) J_m' (\zeta b) \right], \quad (3.33)$$

$$N(\zeta) = +i \left[ J_m(\zeta) H^{(2)\prime}_m(\zeta b) - H^{(2)}_m(\zeta) J_m' (\zeta b) \right]. \quad (3.34)$$

$H^{(1)}_m$ decreases exponentially in the upper half of the complex plane, so the two terms in $N(\zeta)$ are of order one, which avoids the cancelation problem. Similar reasoning holds for $H^{(2)}_m$, which decreases exponentially in the lower half of the complex plane. The computation of $D(\zeta)$ can be done analogously. The various types of Bessel functions are evaluated by using the subroutines developed by Amos [5].

We use circular contours centered at $z = z_0$ with radius $r_0$, which can be described as

$$\Gamma : z(t) = z_0 + r_0 e^{2\pi i t}, \quad t \in [0, 1], \quad (3.35)$$

so (3.32) can be written as

$$S_n = -n \int_0^1 (z_0 + r_0 e^{2\pi i t})^{n-1} \log [(r_0 e^{2\pi i t})^{-N} f(z_0 + r_0 e^{2\pi i t})] r_0 e^{2\pi i t} dt + Nz_0^n. \quad (3.36)$$
A suitable quadrature rule must be chosen to evaluate the integral. We choose the trapezoidal rule since it shows exponential convergence for periodic integrands. Depending on specific preliminary knowledge about the locations of the roots other contours than circles might be more suitable, but they must always be smooth to ensure exponential convergence of quadrature. The \( m \)-point trapezoidal rule approximation of (3.36) can be written as

\[
T^m_n = -\frac{n}{m} \sum_{i=1}^{m} g \left( \frac{i}{m} \right),
\]  

(3.37)

where \( g(t) \) is the integrand in (3.36). An error estimate is [34]

\[
\epsilon = |T^m_n - S_n| \approx \frac{n}{N} \varphi_{\text{max}} \exp \left( -\pi \cot \left( \frac{\varphi_{\text{max}}}{2} \right) \right),
\]  

(3.38)

where \( \varphi_{\text{max}} \) is the maximum jump in the argument of \( f(z) \) for the given \( m \)-point discretization of the contour. When evaluating the logarithmic term in the integrand of (3.36) it must be ensured that this term’s imaginary part is continuous, as mentioned above. This amounts to adding or subtracting the required multiples of \( 2\pi \) to the computed principle value of the logarithm.

In order to make the algorithm robust we implemented a number of checks. We first start the algorithm with \( m \) function evaluations to compute \( N \) using (3.31), and compute the maximum jump \( \varphi_{\text{max}} \) of \( \arg(f(z)) \) along the contour. The computed \( N \) is accepted if \( \varphi_{\text{max}} < \frac{3}{4} \pi \) and \( m \) is not too low. If \( m \) is too low for this first test to be accurate, we perform a second test [26]

\[
\frac{1}{6.1} < \left| \frac{f_k}{f_{k+1}} \right| < 6.1, \quad k = 0, \ldots, m-1.
\]  

(3.39)

We also check whether the accuracy of the quadrature rule for \( n = N \) meets a given tolerance: \( \epsilon < \text{tol} \). If one of the criteria is not met we double the number of function evaluations by adding new function evaluations for the midpoints between the old grid points.

Since the mapping from the sums \( S_n \) to the zeros becomes ill-conditioned if the number of zeros inside the contour is too large, we subdivide the circular contour into 9 smaller circular contours of radius \( r_1 = 5/12 r_0 \). One of the smaller circles is centered at \( z_0 \), the other circles are centered at

\[
z_1 = z_0 + 0.76 r_0 e^{2\pi i j/8}, \quad j = 1, \ldots, 8.
\]  

(3.40)

This subdivision is repeated recursively until the number of zeros inside each contour \( N < N_{\text{max}} \), a predetermined number for which the polynomial can be reliably constructed from the sums \( S_n \). Based on our experience we choose \( N_{\text{max}} \) around 5 to 10. Note that the smaller circles are overlapping, so some zeros might be found multiple times. To remove these multiple zeros we first group zeros that are located within a distance \( d < \text{grouptol} \) from each other and subsequently compute the center \( z_g \) of this group. Suppose that the maximum distance from the zeros in the group to the center is \( d_{\text{max}} \). We then apply the foregoing root-finding algorithm on a contour centered at \( z_g \) with radius \( r_g = \max(1.5 d_{\text{max}}, \text{quadtol}) \) to find a new set of roots that replaces the old roots in the group.
3.3.4 Numerical results

To illustrate the performance of this root-finding approach for bulk absorbing liners we consider a test case with (all parameters dimensionless) $\omega = 10$, $m = 5$, uniform flow with Mach number $M_0 = 0.3$ and uniform temperature $T_0 = 1$. The liner has a depth of $d_l = 0.33$, and is filled with a porous material for which $\mu_p/\omega = 1.3 - 0.4i$ and $Z_c = 1.3 - 0.3i$, covered with a facing sheet having an impedance $Z_0 = 1 + i$.

Figure 3.3 shows the 209 axial wavenumbers that are obtained. They are equivalent to those obtained with the NLR code BAHAMAS, which uses a root-finding approach based on the Newton-Raphson method. We required 106688 function evaluations, which is in our experience comparable to a Newton-based method with an excess number of initial guesses. For this case three different groups of modes can be distinguished, namely: the cut-on modes, which are close to the real axis; the cut-off modes with a real part around 3, which are connected to the duct; and the cut-off modes with a real part smaller than approximately 3, which are connected to the liner. Comparatively more duct modes exist than liner modes.

Figure 3.4 illustrates the behavior of our root-finding strategy. Starting with the blue circle centered at zero with radius 200, the area is subdivided into smaller circles. The area is subdivided to a larger extent in those areas that contain the largest number of zeros.
Figure 3.3: Axial wavenumbers for a bulk absorbing liner (209 modes are found using 106688 function evaluations).

Figure 3.4: Illustration of root-finding strategy; circles with many roots are subdivided into smaller ones.
Chapter 4

Mode-matching for segmented ducts

In the previous chapter we considered the numerical solution of the boundary value problem that governs the modes of an axially invariant duct. In many applications, however, the properties of the duct may suddenly change along the axial direction, for example the cross-sectional area or the wall impedance. The incoming acoustic wave is partly reflected and partly transmitted at the discontinuity. In an early work [32] the scattering of plane waves is studied by applying the equations for conservation of mass, momentum and energy to a small control volume around a sudden area discontinuity.

For higher frequencies also higher order modes must be taken into account; an incoming mode may scatter at a liner discontinuity into multiple outgoing modes in both directions. In [61] the field is expressed as a summation of modes, and the unknown modal amplitudes are found by matching the pressure and axial velocity at the axial location of the liner discontinuity—this is the mode-matching method, which is the topic of this chapter.

Mode-matching based on continuity of acoustic pressure and axial velocity has been used more recently to study a turbofan inlet duct with a wall impedance discontinuity due to a different liner depth [68]. At supersonic fan speeds the rotor-alone pressure field is well cut-on. With the aid of mode-matching the liner could be designed such that most of the acoustic energy of the fan tones is scattered into higher radial order modes, which are better absorbed by the lining. A slightly different mode-matching approach for non-axisymmetric configurations has been used to study the effect of circumferential liner splices [11]. Here, the finite element method was used to find the eigensolution at a duct cross-section.

Besides mode-matching also the Wiener-Hopf technique has been used to investigate problems with axial discontinuities [57, 74, 86]. However, this technique requires a considerable amount of complex mathematical machinery and can only be applied to very canonical geometries, which might explain the fact that its use is not widespread in engineering contexts.

In this chapter we consider mode-matching for mean flow with non-uniform velocity and temperature. As will be discussed in Section 4.2, the system of mode-matching equations results from taking inner products of the duct eigenfunctions with certain test functions. These eigenfunctions are Bessel functions for a duct with a uniform
medium. Since closed-form integrals are known for products of Bessel functions [121], the choice of a suitable set of Bessel functions as test functions allows us to determine the inner products analytically exactly. For non-uniform flow and temperature, however, the duct eigenfunctions are not Bessel functions anymore, and no exact integrals of products of these eigenfunctions are known. For the classical mode-matching approach a set of Bessel functions may still be used as test functions, although the inner products cannot (to our knowledge) be computed in closed form, so numerical quadrature is needed. We here present a new approach [91] that uses the same Pridmore-Brown modes as test functions, but instead of the standard inner product we use an associated bilinear form\(^\text{1}\) that resembles an inner product. We will show that in this way the integrals we need are available in closed form.

The outline of this chapter is as follows. We start in Section 4.1 by presenting integrals of weighted combinations of products of eigenfunctions that can be evaluated in closed form. These closed-form integrals, which will also be referred to as a bilinear maps (BLM), form the basis of a new mode-matching approach which is described in Section 4.2. First we describe classical mode-matching in Section 4.2.1 followed by our new approach based on the bilinear map in Section 4.2.2. Mode-matching yields a system of equations for the modal amplitudes of any two adjacent segments, hence, to investigate the effect of more than two segments several of these systems have to be combined. This will be done by using the scattering matrix formalism, which is described in Section 4.2.3. Finally, we compare the classical and BLM-based mode-matching approaches based on some numerical results presented in Section 4.3.

### 4.1 Exact integrals of Pridmore-Brown eigenfunctions

In order to make the exposition as clear as possible, we start with the prototypical example of the Helmholtz equation in Section 4.1.1). Next we consider the more general case of the convected Helmholtz equation for parallel flow that governs modes in a duct with an arbitrarily shaped cross-section in Section 4.1.2. We then continue in Section 4.1.3 to the circularly cylindrical case (governed by the Pridmore-Brown equation), the results of which will be used in our new mode-matching approach.

#### 4.1.1 Exact Integrals of Solutions of the Helmholtz Equation

Mode-matching is a particularly successful method for acoustic wave propagation in segmented ducts of circular or rectangular cross section with a uniform medium. The necessary integrals of the modal eigenfunctions (Bessel functions or (co)sine functions) appear to be available in closed form, which greatly simplifies the numerical evaluation. These closed-form integrals are a manifestation of a more general property of solutions of the Helmholtz equation.

Suppose that we have for parameters \(\alpha\) and \(\beta\) the solutions \(\phi\) and \(\psi\) in a region \(\mathcal{A} \in \mathbb{R}^2\) with boundary \(\Gamma\) (boundary conditions do not play a role for now) of

\[
\nabla^2 \phi + \alpha^2 \phi = 0, \\
\nabla^2 \psi + \beta^2 \psi = 0.
\]

\(^{1}\)Strictly speaking it is not an inner product. Although imprecise, we will refer to it here as inner product because of the role it plays in the mode matching procedure.
When we subtract $\phi$ times the second equation from $\psi$ times the first and integrate over $A$ we obtain
\[
(\alpha^2 - \beta^2) \iint_A \phi \psi \, dS = \iiint (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dS = \iiint \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) \, dS. \tag{4.2}
\]
After using the divergence theorem this inner product of $\phi$ and $\psi$ is given by an integral along the boundary
\[
\iiint \phi \psi \, dS = \frac{1}{\alpha^2 - \beta^2} \int_\Gamma (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell. \tag{4.3}
\]
If $\alpha = \beta$, this result can not be used. Suppose that we replace equation (4.1b) with the far more general
\[
\nabla^2 \chi + \alpha^2 \chi = f, \tag{4.4}
\]
where $f$ is an arbitrary (integrable) function. When we again cross-wise multiply and subtract as before, we find
\[
\iiint \phi f \, dS = \iiint (\phi \nabla^2 \chi - \chi \nabla^2 \phi) \, dS = \int_\Gamma (\phi \nabla \chi \cdot \mathbf{n} - \chi \nabla \phi \cdot \mathbf{n}) \, d\ell. \tag{4.5}
\]
This result was derived without specifying any boundary conditions on $\chi$, so they can be chosen arbitrarily as long as there exists a solution $\chi$. This is guaranteed if (the boundary conditions on $\chi$ are such, that) $\alpha$ is not an eigenvalue of the homogeneous version of equation (4.4). This result is related to the Fredholm alternative for linear operators [59]. Assume that $A \chi = B \nabla \chi \cdot \mathbf{n}$ on $\Gamma$ and suppose that there exists a nonzero solution $w$ of $\nabla^2 w + \alpha^2 w = 0$ with the same boundary conditions. Then we obtain, for arbitrary $f$, the contradiction $\iint_A w f \, dS = \int_\Gamma (w \nabla \chi \cdot \mathbf{n} - \chi \nabla w \cdot \mathbf{n}) \, d\ell = 0$.

The advantage of the result of (4.5) is its generality. We can substitute for $f$ any function we need, for example $f = \phi$, the solution of (4.1a). The disadvantage is of course that it requires the solution of the additional inhomogeneous equation (4.4). So in practice we will use this result only if (4.3) breaks down. Note that the equation $\nabla^2 \chi + \alpha^2 \chi = \phi$ can be found by differentiating the homogeneous equation for $\phi$ to $-\alpha^2$. In the specific case of a circular disk of radius 1 and circularly symmetric solutions $\phi = J_m(\alpha r) e^{im\theta}$ and $\psi = \hat{J}_m(\beta r) e^{-im\theta}$ (we choose opposite signs of $im\theta$ for non-trivial results later) substituted in (4.3), we obtain the well-known [121] relation for Bessel functions
\[
\int_0^1 J_m(\alpha r) J_m(\beta r) \, r \, dr = \frac{1}{\alpha^2 - \beta^2} \left[ \beta J_m(\alpha) J'_m(\beta) - \alpha J'_m(\alpha) J_m(\beta) \right]. \tag{4.6}
\]
For the case when $\alpha = \beta$ one approach is to take the limit and use l'Hôpital's rule, which yields
\[
\int_0^1 J_m(\alpha r)^2 \, r \, dr = \frac{1}{2} \left( 1 - \frac{m^2}{\alpha^2} \right) J_m(\alpha)^2 + \frac{1}{2} J'_m(\alpha)^2. \tag{4.7}
\]
A more generic approach is the one discussed above. Suppose that we have $\chi(r, \theta) = \hat{\chi}(r) e^{-im\theta}$, regular in $r = 0$, where $\hat{\chi}$ is a solution of the inhomogeneous Bessel equation
\[
\frac{1}{r} (r \hat{\chi}')' + \left( \alpha^2 - \frac{m^2}{r^2} \right) \hat{\chi} = J_m(\alpha r), \tag{4.8}
\]
(for example $\hat{\chi}(r) = -r J_m(ar)/2a$), then we have the equivalent result

$$\int_0^1 J_m(ar)^2 r dr = J_m(a) \hat{\chi}'(1) - a J'_m(a) \hat{\chi}(1). \quad (4.9)$$

Again, the boundary conditions on $\chi$ can be selected arbitrarily, except for the restriction that $A \hat{\chi}(1) + B \hat{\chi}'(1) \neq 0$ if $AJ_m(a) + aBJ'_m(a) = 0$.

The above manipulations (4.1)–(4.5) can be repeated for (2.111) (with $u_0 \equiv 0$) to obtain weighted inner product integrals of the type

$$\iint_{\mathcal{A}} c_0^2 P \tilde{P} dS, \quad (4.10)$$

but for the general case (2.110) this is not possible because $\Omega = \Omega(k)$. Indeed, no closed form expressions can be found for the standard inner products with eigenfunctions of the Pridmore-Brown equation, which are required to set up the mode-matching equations, so it seems that we have to resort to numerical quadrature to compute the integrals. With increasing radial order the eigenfunctions become more and more oscillatory, resulting in increasingly more difficult numerical computations.

Moreover, note that the integrand is based on a numerical computation, since the eigenfunctions of the Pridmore-Brown equation have to be determined numerically. This requires that either the quadrature rule has its abscissas on the same grid points as the BVP solver, or that the eigenfunctions can be interpolated on any location on or around the grid points. All this is not the case if we replace the standard inner product integrals with a dedicated integral associated to the prevailing equations, which we call (since it is not really an inner product as mentioned before) a bilinear form or bilinear map.

In the following we will construct two bilinear forms that are integrals of weighted combinations of products of eigenfunctions. One for the general case of parallel mean flow (applicable for example in distortion mode problems [111, 112]), much in the same fashion as discussed above, and one for the particular case of circular ducts with radially symmetric mean flow, the Pridmore-Brown modes. This result was inspired by [29], where a related integral was used to obtain a solvability condition for a multiple scales solution of the disturbance field for a slowly varying duct with mean swirling flow.

### 4.1.2 Exact Integrals of Parallel-Flow Modal Eigenfunctions

Analogous to (4.3) we want to construct an integral involving products of Pridmore-Brown eigenfunctions and use the divergence theorem to evaluate its value through the eigenfunction values on the boundary. Suppose that we have parallel mean flow in the $x$-direction and modes of the form

$$[\rho_1, p_1, v_1] = [R(y, z), P(y, z), U(y, z)e_x + V(y, z)e_y + W(y, z)e_z]e^{ikx - i\omega t}. \quad (4.11)$$

$U$, $V$ and $W$ are the velocity components in the $x$, $y$ and $z$ direction of a Cartesian coordinate system. For modal solutions of this form which are governed by (2.105) we
§ 4.1 Exact integrals of Pridmore-Brown eigenfunctions

have

\[
-i\Omega P + i\rho_0 c_0^2 k U + \rho_0 c_0^2 (V_y + W_z) = 0, \quad (4.12a)
\]

\[
-i\rho_0 \Omega U + \rho_0 (u_0 V + u_0 W) + ik P = 0, \quad (4.12b)
\]

\[
-i\rho_0 \Omega V + P_y = 0, \quad (4.12c)
\]

\[
-i\rho_0 \Omega W + P_z = 0, \quad (4.12d)
\]

where \(\Omega = \omega - ku_0\), and \(R\) follows directly from the other amplitudes, for example with 2.105c. (Note that the system reduces to (2.110) if \(U, V\) and \(W\) are eliminated.) Together with suitable boundary conditions this is an eigenvalue problem with eigenvalue \(k\), but this will not be used here; \(k\) will be considered as a given constant.

When the individual equations in (4.12) are multiplied by suitable combinations of other solutions of the same equations (say \(\tilde{P}, \tilde{U}, \tilde{V}\) and \(\tilde{W}\)) with constant \(\tilde{k}\) and corresponding auxiliary function \(\tilde{\Omega} = \omega - \tilde{k}u_0\), and subsequently added together, we obtain

\[
\left( -i\Omega P + i\rho_0 c_0^2 k U + \rho_0 c_0^2 V_y + \rho_0 c_0^2 W_z \right) \frac{\tilde{P}}{\rho_0 c_0^2} + \left( -i\Omega \rho_0 U + \rho_0 u_0 V + \rho_0 u_0 W + ik P \right) \frac{\tilde{k} \tilde{P}}{\rho_0 \tilde{\Omega}} - \left( -i\Omega \rho_0 V + P_y \right) \tilde{V} - \left( -i\Omega \rho_0 W + P_z \right) \tilde{W} = 0. \quad (4.13)
\]

This choice of combinations of shape functions is clearly not self-evident; it was found by first taking the products of the governing equations with arbitrary functions, and then imposing the required conditions on these functions. The resulting equations appeared to be equivalent to our original equations for \(P, U, V\) and \(W\). For more details see Appendix A. After reordering and splitting off a cross-wise divergence (4.13) is equivalent to

\[
-i \left( \frac{\Omega}{\rho_0 c_0^2} - \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P - i \frac{\Omega \tilde{k} - \tilde{\Omega} k}{\tilde{\Omega}} \tilde{P} U + i \rho_0 \Omega (\tilde{V} V + \tilde{W} W) - \tilde{V} P_y - \tilde{W} P_z + (\tilde{V}_y + \tilde{W}_z) P + \tilde{\Omega} \left( \frac{\tilde{P} V - \tilde{V} P}{\tilde{\Omega}} \right)_y + \tilde{\Omega} \left( \frac{\tilde{P} W - \tilde{W} P}{\tilde{\Omega}} \right)_z = 0. \quad (4.14)
\]

After using the defining equations (4.12) this becomes

\[
-i \left( \frac{\Omega}{\rho_0 c_0^2} - \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P - i \frac{\Omega \tilde{k} - \tilde{\Omega} k}{\tilde{\Omega}} \tilde{P} U + i \rho_0 \Omega (\tilde{V} V + \tilde{W} W) - i \tilde{\Omega} \rho_0 (\tilde{V} V + \tilde{W} W)
\]

\[
+ i \left( \frac{\tilde{\Omega}}{\rho_0 c_0^2} - \frac{\tilde{k}^2}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P + \tilde{\Omega} \left( \frac{\tilde{P} V - \tilde{V} P}{\tilde{\Omega}} \right)_y + \tilde{\Omega} \left( \frac{\tilde{P} W - \tilde{W} P}{\tilde{\Omega}} \right)_z = 0. \quad (4.15)
\]

Recombining and dividing by \(\tilde{\Omega}\) yields

\[
-i(k - \tilde{k}) \frac{1}{\tilde{\Omega}} \left[ \left( \frac{\omega}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P + \frac{\omega}{\tilde{\Omega}} \tilde{P} U - \rho_0 u_0 (\tilde{V} V + \tilde{W} W) \right] = \frac{\partial}{\partial y} \left( \frac{\tilde{P} V - \tilde{V} P}{\tilde{\Omega}} \right) + \frac{\partial}{\partial z} \left( \frac{\tilde{P} W - \tilde{W} P}{\tilde{\Omega}} \right). \quad (4.16)
\]
We can use the defining equations for $U$, $V$ and $W$ in case we want to write the left hand side in terms of $P$ only. When we integrate (4.16) over a cross section $A$ with boundary $\Gamma$ and use the divergence theorem we obtain an integral over $A$ of parallel-flow shape functions, in particular (with suitable boundary conditions and eigenvalues $k$) parallel-flow eigenfunctions, expressed as an integral along boundary $\Gamma$. We introduce the vector of shape functions $F = [P, U, V, W]$ and denote this integral by

$$\langle \langle F, \tilde{F} \rangle \rangle = \iint_A \frac{1}{\Omega} \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \Omega} \right) \tilde{P} P + \frac{\omega}{\Omega} \tilde{P} U - \rho_0 u_0 (\tilde{V} V + \tilde{W} W) \right) dS = \frac{i}{k - \tilde{k}} \int_{\Gamma} \tilde{P} (V n_y + W n_z) - (\tilde{V} n_y + \tilde{W} n_z) P \frac{1}{\Omega} d\ell, \hspace{1cm} (4.17)$$

where $n_y$ and $n_z$ denote the $y$ and $z$ components of the outward normal vector on $\Gamma$ and $k \neq \tilde{k}$. If a symmetric form is preferred, we can replace $\langle \langle F, \tilde{F} \rangle \rangle$ by $\frac{1}{2} \langle \langle F, \tilde{F} \rangle \rangle + \frac{1}{2} \langle \langle F, F \rangle \rangle$ and the right-hand side correspondingly.

If $u_0 \equiv 0$ this reduces to a regular integral inner product (with a weight function $\propto \rho_0^{-1} \propto c_0^2$)

$$\langle \langle F, \tilde{F} \rangle \rangle = \frac{k + \tilde{k}}{\omega^2} \iint_A \frac{\tilde{P} P}{\rho_0} dS = \frac{1}{(k - \tilde{k}) \omega^2} \int_{\Gamma} \tilde{P} (P_0 n_y + P_0 n_z) - (\tilde{P}_0 n_y + \tilde{P}_0 n_z) P \frac{1}{\rho_0} d\ell, \hspace{1cm} (4.18)$$

a result very similar to (4.3), which could have been obtained directly from (2.111).

For a slipping mean flow along an impedance wall at $\Gamma$ we apply Ingard-Myers conditions $V n_y + W n_z = \Omega P / \omega Z$ and $\tilde{V} n_y + \tilde{W} n_z = \tilde{\Omega} \tilde{P} / \omega \tilde{Z}$ and obtain

$$\langle \langle F, \tilde{F} \rangle \rangle = \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P} P}{\Omega \omega} \left( \frac{\Omega}{Z} - \frac{\tilde{\Omega}}{\tilde{Z}} \right) d\ell. \hspace{1cm} (4.19)$$

The integrals vanish for hard walls, i.e. when $Z = \tilde{Z} = \infty$. For a no-slip mean flow with $u_0 = 0$ along $\Gamma$ and impedance boundary conditions $P = Z (V n_y + W n_z)$ and $\tilde{P} = \tilde{Z} (\tilde{V} n_y + \tilde{W} n_z)$ we obtain

$$\langle \langle F, \tilde{F} \rangle \rangle = \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P} P}{\omega} \left( \frac{1}{Z} - \frac{1}{\tilde{Z}} \right) d\ell. \hspace{1cm} (4.20)$$

Interestingly, the integral vanishes for different modes corresponding to the same $Z$.

Although this surface integral resembles a non-standard inner product between vectors $F$ and $\tilde{F}$, it is not an inner product because it lacks positive-definiteness for $\langle \langle F, F \rangle \rangle$. We therefore refer to it as a *bilinear map*, although occasionally it may be referred to as an inner product since it plays the same role as an inner product in the classical mode matching procedure.

The above result is evidently not valid for $k = \tilde{k}$. In practice, the limit of $k = \tilde{k}$ corresponds with $F = \tilde{F}$ when we deal with modal eigenfunctions which all satisfy the same boundary condition, so we will consider that situation here.

We start with the associated inhomogeneous system of (4.12) with solution $[\hat{P}, \hat{U}, \hat{V}, \hat{W}]$, with the same $k$ as in the original system, and a solution vector $[P, U, V, W]$ satisfying
This inhomogeneous system is

\[
\begin{align*}
-i\Omega \hat{P} + i\rho_0 c_0^2 k \hat{U} + \rho_0 v_0^2 (\hat{V}_y + \hat{W}_z) &= i(u_0 P + \rho_0 c_0^2 U), \\
-i\Omega \rho_0 \hat{U} + \rho_0 u_0 \hat{V} + \rho_0 u_0 \hat{W} + ik \hat{P} &= i(\rho_0 u_0 U + P), \\
-i\Omega \rho_0 \hat{V} + \hat{P}_y &= i\rho_0 u_0 V, \\
-i\Omega \rho_0 \hat{W} + \hat{P}_z &= i\rho_0 u_0 W,
\end{align*}
\]  

(4.12)

with boundary conditions such that \( k \) is not an eigenvalue of the left-hand side in order to guarantee the existence of a solution \( \hat{P}, \hat{U}, \hat{V}, \hat{W} \). (Note that this system can be found by differentiating (4.12) with respect to \( k \).) This system is equivalent to an inhomogeneous variant of (2.110)

\[
\nabla_\perp \cdot \left( \frac{c_0^2}{\Omega^2} \nabla_\perp \hat{P} \right) + \left( 1 - \frac{k^2 c_0^2}{\Omega^2} \right) \hat{P} = 2 \frac{c_0^2}{\Omega^3} k \omega P - 2 \nabla_\perp \cdot \left( \frac{c_0^2 u_0}{\Omega^3} \nabla_\perp P \right).
\]  

(4.22)

We multiply left- and right-hand sides of (4.21) with \( P/\rho_0 c_0^2, kP/\rho_0 \Omega, -V \) and \( -W \) respectively, add and do exactly the same manipulations as before. We find that the factor \( k - k \) vanishes and obtain the final result

\[
\langle \langle F, F \rangle \rangle = \iint_A \frac{1}{\Omega} \left( \frac{u_0}{\rho_0 c_0^2} + \frac{k}{\rho_0 \Omega} \right) P^2 + \frac{\omega}{\Omega} UP - \rho_0 u_0 (V^2 + W^2) \right) \mathrm{d}S
\]

\[
= i \int_{\Gamma} \frac{\hat{P}(V n_y + W n_z) - (\hat{V} n_y + \hat{W} n_z) P}{\Omega} \mathrm{d}\ell.
\]  

(4.23)

If \( u_0 \equiv 0 \) this reduces to

\[
\langle \langle F, F \rangle \rangle = \frac{2k}{\omega^2} \iint_A \frac{P^2}{\rho_0} \mathrm{d}S = \frac{1}{\omega^2} \int_{\Gamma} \frac{\hat{P}(P y n_y + P z n_z) - (\hat{P} y n_y + \hat{P} z n_z) P}{\rho_0} \mathrm{d}\ell.
\]  

(4.24)

### 4.1.3 Exact Integrals of Radial Pridmore-Brown Modes

A special application of the above results will be for a circularly symmetric mean flow \( u_0(r), c_0(r), \rho_0(r) \) in a circular duct of radius \( d \) and cross section \( A \) (an annular duct would require only little changes) with polar coordinate system \( (x, r, \theta) \) and \( v_1 = u_1 e_x + v_1 e_r + w_1 e_\theta \). In this case the solution can be written as a sum over circumferential Fourier modes, so we can assume modal shape solutions of the form \( F(r) e^{im\theta} = [P(r), U(r), V(r), W(r)] e^{im\theta} \) satisfying

\[
\begin{align*}
-i\Omega P + i\rho_0 c_0^2 kU + \rho_0 v_0^2 \left( \frac{V'}{r} + \frac{1}{r} V + \frac{im}{r} W \right) &= 0, \\
-i\rho_0 \Omega U + \rho_0 u_0' V + ikP &= 0, \\
-i\rho_0 \Omega V + P' &= 0, \\
-i\rho_0 \Omega W + \frac{im}{r} P &= 0,
\end{align*}
\]  

(4.25a-d)

where \( \Omega = \omega - ku_0 \) and the \( ' \) denotes an \( r \)-derivative. As before we will assume \( k \) to be just a constant, but with suitable boundary conditions this system is an eigenvalue problem with eigenvalue \( k \).
Due to the symmetry it is no restriction to assume another solution of (4.25) with \( \tilde{k} \neq k \) of the form \( \hat{F}_2(\theta)e^{-im\theta} = [\hat{P}(\theta), \hat{U}(\theta), \hat{V}(\theta), -\hat{W}(\theta)]e^{-im\theta} \), such that the surface integral in (4.17) divided by \( 2\pi \) simplifies to

\[
\langle \hat{F}, \tilde{\hat{F}} \rangle = \int_0^d \frac{r}{\Omega} \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \Omega} \right) P \hat{P} + \frac{\omega}{\Omega} U \hat{P} - \rho_0 u_0 (V \hat{V} + W \hat{W}) \right| dr = \frac{id}{k - \tilde{k}} \left[ \frac{\hat{P}V - \hat{V}P}{\Omega} \right]_{r=d},
\]

where we assumed that the solutions are regular in \( r = 0 \).

If \( u_0 \equiv 0 \) we obtain

\[
\langle \hat{F}, \tilde{\hat{F}} \rangle = \frac{k + \tilde{k}}{\omega^2} \int_0^d \frac{r}{\rho_0} P \hat{P} dr = \frac{d}{(k - \tilde{k})\omega} \left[ \frac{\hat{P}P' - \hat{P}'P}{\rho_0} \right]_{r=d}.
\]

With slipping flow and impedance walls along \( r = d \) we apply Ingard-Myers boundary conditions \( V = \Omega P/\omega Z \) with \( \Omega = \omega - ku_0(d) \) for both solutions and obtain

\[
\langle \hat{F}, \tilde{\hat{F}} \rangle = \frac{id\hat{P}P}{(k - \tilde{k})\omega} \left( \frac{\Omega}{Z} - \frac{\tilde{\Omega}}{Z} \right),
\]

which vanishes if \( Z = \tilde{Z} = \infty \). With no-slip flow \( u_0(d) = 0 \) this simplifies to

\[
\langle \hat{F}, \tilde{\hat{F}} \rangle = \frac{id\hat{P}P}{(k - \tilde{k})\omega} \left( \frac{1}{Z} - \frac{1}{\tilde{Z}} \right).
\]

If \( k \) and \( \tilde{k} \) are different eigenvalues from the same impedance condition \( \tilde{Z} = Z \), then all integrals vanish in this case.

To find the degenerate case of \( \tilde{k} = k \) and \( \tilde{F} = F \) we consider with constant \( k \) the solution \( F e^{-im\theta} = [P, U, V, -W] e^{-im\theta} \) of (4.25), and the associated solution \( \tilde{F} e^{-im\theta} = [\tilde{P}, \tilde{U}, \tilde{V}, -\tilde{W}] e^{-im\theta} \) of the inhomogeneous system (with the same \( k )

\[
-i\Omega \hat{P} + i\rho_0 c_0^2 k \hat{U} + \rho_0 c_0^2 \left( \hat{V}' + \frac{1}{r} \hat{V} + \frac{im}{r} \hat{W} \right) = i(u_0 P + \rho_0 c_0^2 U),
\]

\[
-i\Omega \rho_0 \hat{U} + \rho_0 u_0' \hat{V} + ik \hat{P} = i(\rho_0 u_0 U + P),
\]

\[
-i\Omega \rho_0 \hat{V} + \hat{P}' = i \rho_0 u_0 V,
\]

\[
-i\Omega \rho_0 \hat{W} + \frac{im}{r} \hat{P} = i \rho_0 u_0 W.
\]

(Note that this system can also be found by differentiating (4.25) with respect to \( k \).

In actual practice the system (4.30) will be reduced to the following inhomogeneous Pridmore-Brown equation in \( \hat{P} \)

\[
\frac{\Omega^2}{pc_0^2} \frac{r \partial \hat{P}}{r^2} + \left( \frac{\Omega^2}{pc_0^2} - k^2 - \frac{m^2}{r^2} \right) \hat{P} = 2 \frac{\omega u_0'}{\Omega c_0^2} P' - 2 \left( \frac{u_0 \Omega}{pc_0^2} + k \right) P,
\]

which may be solved by almost the same numerical routine as is used for the Pridmore-Brown equation itself. Note, however, that it is not an eigenvalue problem in this case since \( k \) is fixed, which reduces the required computational effort. The surface integral (divided by \( 2\pi \)) of (4.23) now simplifies to

\[
\langle F, F \rangle = \int_0^d \frac{r}{\Omega} \left( \frac{u_0}{\rho_0 c_0^2} + \frac{k}{\rho_0 \Omega} \right) P^2 + \frac{\omega}{\Omega} UP - \rho_0 u_0 (V^2 + W^2) \right| dr = id \left[ \frac{\hat{P}V - \hat{V}P}{\Omega} \right]_{r=d},
\]

(4.32)
where we assumed that the solutions are regular in $r = 0$. As before, it should be noted that the inhomogeneous equation (4.31) has no solutions if the problem for $P$ is an eigenvalue problem with homogeneous boundary conditions, and the same conditions are applied to $\hat{P}$. Finally, if $u_0 \equiv 0$ we obtain

$$\langle F,F \rangle = \frac{2k}{\omega^2} \int_0^d r P^2 \, dr = \frac{d}{\omega^2} \left[ \hat{P} P' - \hat{P}' P \right]_{r=d}.$$  (4.33)

4.2 Mode-matching at an interface

In the following we consider a circularly cylindrical duct that is divided in $N$ axial segments, and assume that the wall properties are constant within each segment. We furthermore assume that the acoustic field in each segment can be expressed as a summation of eigenmodes of the Pridmore-Brown equation according to (2.71), so we ignore the contribution of the critical layer, as discussed at the end of Section 2.4.

We also note that the issue of possible instabilities due to the interaction of shear layer and impedance wall (as for example discussed in [23, 102]) will not be addressed here. Ill-posedness problems associated with a vanishing boundary layer only occur in time domain calculations, while the detection of possible unstable modes requires a causality analysis that we have not undertaken in the present context.

4.2.1 Construction of Matrix Equations (Classical Mode-Matching)

In this section we describe the classical mode-matching (CMM) approach based on continuity of pressure and axial velocity. The total field for a given circumferential wavenumber $m$ in each segment is a superposition of all right and left-running modes:

$$p'(x,r) = \sum_{\mu=1}^{\infty} \left(a_{1,\mu}^+ P_{1,\mu}^+(r) e^{ik_\mu^+(x-x_{l-1})} + a_{1,\mu}^- P_{1,\mu}^-(r) e^{ik_\mu^-(x-x_l)}\right), \quad x_{l-1} \leq x \leq x_l,$$  (4.34)

In a numerical implementation this infinite series has to be truncated; the finite number of modes $\mu_l$ to represent the field of the $l$-th segment depends on the type of liner. We use $\mu_l^{\max}$ modes to represent the field for segments with a locally reacting liner. For bulk-reacting liners we use $\mu_l = \mu_l^{\max} + \mu_l^{\text{add}}$ modes, where the number of extra modes $\mu_l^{\text{add}}$ depends on the depth of the liner (it can be different for different segments). In the following, we consider the general case of an interface between two segments with
a bulk reacting liner (for locally reacting liners $\mu_l^{\text{add}} = 0$). At the interface at $x = x_l$ we have for the pressure in segment $l$

\[
p_l(r) = \sum_{\mu=1}^{\mu_l} \left( b_{l,\mu}^+ P_{l,\mu}^+(r) + a_{l,\mu}^- P_{l,\mu}^-(r) \right),
\]

where $q$ and $Q$ are used for the pressure field in the bulk region.

Consider the hard-wall uniform flow eigenfunctions $\Psi_{v}(r) = \Phi_{l,\mu}(\eta_{l,\mu} r)$ where $\alpha_v$ are the hard-wall radial wavenumbers, which satisfy $\Psi'_{v}(d) = 0$. These functions form a complete $L_2$-basis and are at least locally, for high orders, similar in behavior as the Pridmore-Brown modes. Therefore they are suitable to serve as test functions when we set up the matrix system for the modal vectors.

Inside the duct region we impose continuity of pressure and axial velocity (approximated due to the truncation) at an interface at $x_l$ and subsequently project onto the set of test functions $\Psi_{v}$, $\nu = 1, \ldots, \nu_{\text{max}}$, which yields

\[
\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ (P_{l,\mu}^+, \Psi_{v}) + a_{l,\mu}^- (P_{l,\mu}^-, \Psi_{v}) = \sum_{\mu=1}^{\mu_{l+1}} a_{l+1,\mu}^- (P_{l+1,\mu}^+, \Psi_{v}) + b_{l+1,\mu}^- (P_{l+1,\mu}^-, \Psi_{v}),
\]

\[
\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ (U_{l,\mu}^+, \Psi_{v}) + a_{l,\mu}^- (U_{l,\mu}^-, \Psi_{v}) = \sum_{\mu=1}^{\mu_{l+1}} a_{l+1,\mu}^- (U_{l+1,\mu}^+, \Psi_{v}) + b_{l+1,\mu}^- (U_{l+1,\mu}^-, \Psi_{v}),
\]

where we use the standard inner product

\[
(f, g) = \int_0^d f(r)g(r)r \, dr.
\]

The axial velocity amplitudes follow from the pressure amplitudes via the relation

\[
U = \frac{k}{\rho_0\Omega} P - \frac{u_0'}{\rho_0\Omega^2} P',
\]

which follows from (4.25b) and (4.25c).

For the annular liner region we use (similarly to the duct region) hard-walled duct eigenfunctions as test functions, of the form (2.98). Let

\[
\Phi_{l,\nu}(r) = J_m(\eta_{l,\nu} r) Y_{m}'(\eta_{l,\nu}(d + d_l)) - Y_m(\eta_{l,\nu} r) J_m'(\eta_{l,\nu}(d + d_l)),
\]

where $\eta_{l,\nu}$ are the hard wall radial wave numbers, which satisfy $\Phi'_{l,\nu}(d) = \Phi'_{l,\nu}(d + d_l) = 0$ (note the $l$-dependency due to the fact that the depth $d_l$ of the bulk absorber can be different for each segment). Furthermore, for the axial velocity in the bulk region it follows from (2.94) that

\[
U_{\text{bulk}} = \frac{k}{\mu_p Z_c} Q.
\]

Consequently, imposing vanishing axial velocity at at the left side of the liner wall at $x = x_l$ and subsequently projecting onto the set of hard-wall bulk eigenfunctions $\Phi_{l,\nu}$, $\nu = 1, \ldots, \nu_{l}^{\text{add}}$ yields

\[
\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ (k_{l,\mu}^+ Q_{l,\mu}^+, \Phi_{l,\nu}) + a_{l,\mu}^- (k_{l,\mu}^- Q_{l,\mu}^-, \Phi_{l,\nu}) = 0.
\]
We again use the standard inner product (4.37) here, except the domain of integration is now the bulk region $d \leq r \leq d + d_1$.

All matching conditions for the duct and bulk regions together yield the following system of equations:

$$\begin{bmatrix} A^+ & A^- \\ C^+ & C^- \\ E^+ & E^- \end{bmatrix} \begin{bmatrix} b^+_l \\ a^-_l \end{bmatrix} = \begin{bmatrix} B^+ & B^- \\ D^+ & D^- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^+_l \\ b^-_{l+1} \end{bmatrix}. \tag{4.42}$$

The matrix entries corresponding to the duct region are inner products of Pridmore-Brown eigenfunctions and Bessel functions; for the matrices that correspond to the pressure equations we have

$$A^\pm_{\nu\mu} = (P^\pm_{l,\mu}, \Psi^\nu) = \int_0^d P^\pm_{l,\mu}(r)\Psi^\nu(r)r \, dr, \quad B^\pm_{\nu\mu} = (P^\pm_{l+1,\mu}, \Psi^\nu) = \int_0^d P^\pm_{l+1,\mu}(r)\Psi^\nu(r)r \, dr. \tag{4.43}$$

The matrix entries of $C^\pm$ and $D^\pm$ corresponding to the axial velocity equations are computed analogously. The matrix entries $E^\pm$ corresponding to the liner region can be computed in closed form by using the well-known [121] relation

$$\int C_m(\alpha r) \tilde{C}_m(\beta r) r \, dr = \frac{\alpha}{\alpha^2 - \beta^2} \left[ \beta C_m(\alpha r) \tilde{C}_m(\beta r) - \alpha \tilde{C}_m(\beta r) C_m(\alpha r) \right], \tag{4.44}$$

which holds for two different solutions $C_m(\alpha r)$ and $\tilde{C}_m(\beta r)$ of the Bessel equation. The term $d/(\mu p Z_c)$ cancels out so we find (superscripts $\pm$ and subscripts $l$ omitted)

$$E^\nu_{\nu\mu} = k_\mu \frac{1}{\xi_\mu^2 - \eta^2} Q'_\mu(d) \Phi(d). \tag{4.45}$$

From the boundary condition $P = Z \nu^{\text{bulk}}$ at $r = d$ (see Section 2.3.3) and the momentum equation for the liner region (2.94) we find

$$Q'_\mu(d) = \frac{i \mu p Z_c}{Z_0 + \frac{i \mu p Z_c N(\xi_\mu)}{\xi_\mu D(\xi_\mu)}} P'_{\mu}(d). \tag{4.46}$$

We can choose the normalization of $\Phi$ as $\Phi(d) = 1$ so we finally have

$$E^\nu_{\nu\mu} = k_\mu \frac{1}{\xi_\mu^2 - \eta^2} \frac{i \mu p Z_c}{Z_0 + \frac{i \mu p Z_c N(\xi_\mu)}{\xi_\mu D(\xi_\mu)}} P_{\mu}(d). \tag{4.47}$$

The matrix entries of $F^\pm$ corresponding to the liner segment to the right side of the interface can be computed analogously.

We consider the amplitudes of the modes propagating towards the interface as the unknowns. Rearranging leads to

$$\begin{bmatrix} B^+ & -A^- \\ D^+ & -C^- \\ F^+ & 0 \end{bmatrix} \begin{bmatrix} a^+_l \\ a^-_l \end{bmatrix} = \begin{bmatrix} A^+ & -B^- \\ C^+ & -D^- \\ 0 & -F^- \end{bmatrix} \begin{bmatrix} b^+_l \\ b^-_{l+1} \end{bmatrix}. \tag{4.48}$$
where the dimensions of the submatrices have been included. Thus, we have $2\nu_{max} + \nu_{l+1}^{add} + \nu_{l+1}^{add}$ equations and $\mu_l + \mu_{l+1} = 2\mu_{max} + \mu_{l+1}^{add} + \mu_{l+1}^{add}$ unknowns.

The solution to the matching problem at a liner discontinuity is not unique. This is due to the fact that at the discontinuity the boundary condition is not defined. This problem is very similar to the problem of determining the electromagnetic field around a sharp edge [18, 40, 70]. The solution can be made unique by imposing the so-called edge condition: the requirement that the stored energy in the field around the edge is finite. For the case of a duct with uniform cross-section and a locally reacting liner it can be shown that this edge condition is automatically satisfied if the number of modes in liner depth can be viewed as ducts with a discontinuity in cross-sectional area. For these cases the ratio of the total radii on both sides of the interface must be chosen equal to the ratio of the number of modes in order to satisfy the edge condition [66, 73, 88, 110]. For non-locally reacting liners with a discontinuity in depth $\mu_{l+1}^{add}$ (and $\nu_{l+1}^{add}$) could be chosen based on a similar criterion.

Finally we note that in [11] the matching conditions are formulated as a variational statement based on the time-harmonic linearized continuity and momentum equations for parallel flow and uniform temperature. For zero mean flow these conditions are equivalent to continuity of pressure and axial velocity at the interface. For non-zero mean flow, however, these conditions include an extra term: an integral over the boundary of the cross-section, which is due to the Ingard-Myers boundary condition for slipping flow.

4.2.2 Matching Conditions Based on the Bilinear Map

In this section we set up a system of equations which has the same structure as (4.42) for the classical mode-matching approach, but now we base the matching conditions on the bilinear map that was described in Section 4.1.

Let us define the vector $\mathbf{f}$ whose components are the acoustic pressure and the velocity components as

$$\mathbf{f}_l(x,r) = [p_l(x,r), u_l(x,r), v_l(x,r), w_l(x,r)].$$

The total field for a given circumferential wavenumber $m$ in each segment is a superposition of all modes

$$\mathbf{f}_l(x,r) = \sum_{\mu=1}^{\infty} \left( a_{l,\mu}^+ \mathbf{F}_{l,\mu}^+ (r) e^{ik_{l,\mu}^+(x-x_{l-1})} + a_{l,\mu}^- \mathbf{F}_{l,\mu}^- (r) e^{ik_{l,\mu}^-(x-x_l)} \right), \quad x_{l-1} \leq x \leq x_l,$$

where $\mathbf{F}$ again denotes the vector of perturbation amplitudes. At the interface at $x = x_l$ we have

$$\mathbf{f}_l(r) = \sum_{\mu=1}^{\mu_l} \left( b_{l,\mu}^+ \mathbf{F}_{l,\mu}^+ (r) + a_{l,\mu}^- \mathbf{F}_{l,\mu}^- (r) \right).$$

Inside the duct we impose continuity of $p(x,r)$, $u(x,r)$, $v(x,r)$ and $w(x,r)$ by applying the bilinear form to $\mathbf{f}_l = \mathbf{f}_{l+1}$ with the solution of the associated problem $\Psi_v$, which results in

$$\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ \langle \mathbf{F}_{l,\mu}^+, \Psi_v \rangle + a_{l,\mu}^- \langle \mathbf{F}_{l,\mu}^-, \Psi_v \rangle = \sum_{\mu=1}^{\mu_{l+1}} a_{l+1,\mu}^+ \langle \mathbf{F}_{l+1,\mu}^+, \Psi_v \rangle + b_{l+1,\mu}^- \langle \mathbf{F}_{l+1,\mu}^-, \Psi_v \rangle,$$

where $\sum_{\mu=1}^{\mu_{l+1}}$.
for $\nu = -\nu_{\text{max}}, \ldots, -1, 1, \ldots, \nu_{\text{max}}$. When we split the range of $\nu$ into left ($\nu < 0$) and right ($\nu > 0$) running parts we again obtain in matrix format

$$
\begin{bmatrix}
A^+ & A^- \\
C^+ & C^-
\end{bmatrix}
\begin{bmatrix}
b^+_l \\
a^-_l
\end{bmatrix}
= \begin{bmatrix}
B^+ & B^- \\
D^+ & D^-
\end{bmatrix}
\begin{bmatrix}
b^+_{l+1} \\
a^-_{l+1}
\end{bmatrix}.
$$

(4.53)

In order to prevent unnecessary computation of an extra set of eigenfunctions we can choose as test functions $\Psi_\nu$ the eigensolutions of, say, segment $n$. In that case the matrix entries can be computed as

$$
\begin{align}
\{A, C\}^\pm_{\nu \mu} &= \langle F_{l \mu}, F_{n \nu} \rangle, \
\{B, D\}^\pm_{\nu \mu} &= \langle F_{l+1 \mu}, F_{n \nu} \rangle,
\end{align}
$$

(4.54a, 4.54b)

where

$$
\begin{align}
\{A, B\}^+ : \mu > 0, \nu > 0, & \quad \{C, D\}^+ : \mu > 0, \nu < 0, \\
\{A, B\}^- : \mu < 0, \nu > 0, & \quad \{C, D\}^- : \mu < 0, \nu < 0.
\end{align}
$$

The values of the bilinear forms in (4.54) can be computed as follows. Suppose that we know the set of eigensolutions for two segments $l$ and $n$, with possibly different liner properties. When $Z_l \neq Z_n$ then the sets of axial wavenumbers have in general (except for a rare coincidence) no values in common, so it holds that $k_{l \mu} \neq k_{n \nu}$. Consequently we can use (4.28) to compute

$$
\langle F_{l \mu}, F_{n \nu} \rangle = \frac{id P_{l \mu} P_{n \nu}}{\Omega_{n \nu \omega(k_{l \mu} - k_{n \nu})}} \left[ \frac{\Omega_{l \mu}}{Z_{l \mu}} - \frac{\Omega_{n \nu}}{Z_{n \nu}} \right]_{r = d}, \quad \text{for} \quad Z_l \neq Z_n.
$$

(4.55)

For the case when $Z := Z_l = Z_n$ we can have $\mu \neq \nu$, which means that $k_{l \mu} \neq k_{n \nu}$, so we can compute

$$
\langle F_{l \mu}, F_{n \nu} \rangle = -\frac{id P_{l \mu} P_{n \nu} u_0}{\Omega_{n \nu \omega Z}} \bigg|_{r = d}, \quad \text{for} \quad Z_l = Z_n, \quad \mu \neq \nu,
$$

(4.56)

which is identically zero for non-slipping flow ($u_0(d) = 0$) or a hard wall ($Z \to \infty$). When $\mu = \nu$ we have $k_{l \mu} := k_{l \mu} = k_{n \nu}$ and $P_{l \mu} := P_{l \mu} = P_{n \nu}$. We require the solution $\hat{P}_{\mu}$ of the inhomogeneous Pridmore-Brown equation (4.31) to compute

$$
\langle F_{l \mu}, F_{n \nu} \rangle = \frac{d}{\rho_0 \Omega_{l \mu}} \left[ \hat{P}_{\mu} P_{l \mu}' - \frac{u_0}{\Omega_{l \mu}} P_{l \mu}' P_{l \mu} - \hat{P}_{\mu}' P_{l \mu} \right]_{r = d}, \quad \text{for} \quad Z_l = Z_n, \quad \mu = \nu.
$$

(4.57)

Note that this implies that $A^+$ and $C^-$ are diagonal matrices, and $A^-$ and $C^+$ zero matrices for non-slipping flow or hard wall.

The use of this new inner product for mode-matching thus may require the computation of an extra set of Pridmore-Brown modes to be used as test functions, or the solution of an inhomogeneous Pridmore-Brown equation. Any of these solutions have to be computed only once, regardless of the number of segments, whereas for the classical approach we need to compute all inner products at each interface. Furthermore, in some occasions the off-diagonal inner products are zero, which simplifies the calculations even more. Since the computational work to determine the extra set of (in some cases inhomogeneous) Pridmore-Brown modes are, to our experience, at worst comparable to the numerical quadrature for a single interface while the inherent numerical integration errors are avoided, we conclude that our new approach is both more accurate and cheaper than the classical mode matching methods.
4.2.3 Scattering Matrix Formalism

The effects of multiple reflections and transmissions have to be combined when liner discontinuities exist at more than one axial location. A naive coupling of the duct sections via the transmission and reflection matrices is possible, but this process is unstable in case of a large numbers of sections due to the exponentially decaying and increasing cut-off modes that are involved. In other words: the matrix that describes the combined effect of all individual segments is ill-conditioned.

An alternative approach would be an iterative one [61], where the propagation of a mode is only considered in the direction in which it decays. With this approach, starting from the amplitudes of the incident modes, the modal amplitudes in the intermediate sections are updated as more and more reflections and transmissions at different interfaces are taken into account at each new iteration, until the change in the amplitudes is below a certain threshold. However, this procedure may not converge for geometries with a large number of segments. We therefore use the scattering matrix formalism (see for example [65, 96]), which has no convergence issues and is numerically stable.

We want to express the modal amplitude vectors of the outgoing waves in terms of the amplitude vectors of the incident waves. We therefore write

\[
\begin{bmatrix}
    a_{l+1}^+ \\
    a_l^-
\end{bmatrix} = \begin{bmatrix}
    M_1^{-1} & M_2
\end{bmatrix} \begin{bmatrix}
    a_{l+1}^- \\
    a_l^+
\end{bmatrix},
\]

where we introduced the interface scattering matrix \( \hat{S} \), including the sizes of the sub-matrices. Next, we want to combine the effect of scattering at the interface and propagation through the segment. We therefore introduce the segment scattering matrix \( S_l \) according to

\[
\begin{bmatrix}
    a_{l+1}^+ \\
    a_l^-
\end{bmatrix} = \begin{bmatrix}
    \hat{S}_{11} & \hat{S}_{12} \\
    \hat{S}_{21} & \hat{S}_{22}
\end{bmatrix} \begin{bmatrix}
    x_l^- \\
    0
\end{bmatrix} \begin{bmatrix}
    a_{l+1}^- \\
    a_l^+
\end{bmatrix} = \begin{bmatrix}
    \mu_l & \mu_{l+1} \\
    \mu_{l+1} & \mu_l
\end{bmatrix} \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
    b_l^+ \\
    b_{l+1}^+
\end{bmatrix},
\]

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\]

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    a_l^-
\end{bmatrix} = \begin{bmatrix}
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    \hat{S}_{21} & \hat{S}_{22}
\end{bmatrix} \begin{bmatrix}
    x_l^- \\
    0
\end{bmatrix} \begin{bmatrix}
    a_{l+1}^- \\
    a_l^+
\end{bmatrix} = \begin{bmatrix}
    \mu_l & \mu_{l+1} \\
    \mu_{l+1} & \mu_l
\end{bmatrix} \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
    b_l^+ \\
    b_{l+1}^+
\end{bmatrix},
\]

\[
\begin{bmatrix}
    a_{l+1}^+ \\
    a_l^-
\end{bmatrix} = \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
    x_l^- \\
    0
\end{bmatrix} \begin{bmatrix}
    a_{l+1}^- \\
    a_l^+
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    \mu_{l+1} & \mu_l
\end{bmatrix} \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
    b_l^+ \\
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\]

\[
\begin{bmatrix}
    a_{l+1}^+ \\
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\end{bmatrix} = \begin{bmatrix}
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    \hat{S}_{21} & \hat{S}_{22}
\end{bmatrix} \begin{bmatrix}
    x_l^- \\
    0
\end{bmatrix} \begin{bmatrix}
    a_{l+1}^- \\
    a_l^+
\end{bmatrix} = \begin{bmatrix}
    \mu_l & \mu_{l+1} \\
    \mu_{l+1} & \mu_l
\end{bmatrix} \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
    b_l^+ \\
    b_{l+1}^+
\end{bmatrix}.
\]
where the propagation is accounted for by the diagonal matrices
\[
(X^+_l)_{\mu\nu} = e^{ih^+_l x_l}, \quad (X^-_{l+1})_{\mu\nu} = e^{-ih^-_{l+1} x_{l+1}}, \quad h_l = x_l - x_{l-1}.
\] (4.60)

Note that the exponentials are always decaying.

The effect of all layers up to layer \( l \) can be combined in the \textit{cumulative} scattering matrix \( \tilde{S}_l \) (see also Figure 4.2):
\[
\begin{bmatrix}
a^+_l \\
a^-_l
\end{bmatrix} = \begin{bmatrix} S^{11}_l & S^{12}_l \\ S^{21}_l & S^{22}_l \end{bmatrix} \begin{bmatrix} a^+_1 \\
a^-_1
\end{bmatrix}.
\] (4.61)

The cumulative scattering matrix of a certain set of segments can be computed by using the \textit{Redheffer star product}, which is defined as
\[
\begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} * \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} = \begin{bmatrix} B^{11}(I - A^{12}B^{21})^{-1} & A^{11} \\ A^{21} + A^{22}B^{21}(I - A^{12}B^{21})^{-1}A^{11} & B^{22}(I - B^{21}A^{12})^{-1}B^{22} \end{bmatrix}.
\] (4.62)

By using this definition \( \tilde{S}_l \) can be computed as
\[
\tilde{S}_l = \tilde{S}_{l-1} * S_l = S_1 * \cdots * S_l.
\] (4.63)

The Redheffer star product can be constructed as follows. We would like to construct the cumulative scattering matrix \( \tilde{S}_l \) in order to compute the effect of segment \( l \). Let us assume that we can describe the effect of all segments up to segment \( l - 1 \) via
\[
\begin{bmatrix} a^+_l \\
a^-_l
\end{bmatrix} = \begin{bmatrix} S^{11}_{l-1} & S^{12}_{l-1} \\ S^{21}_{l-1} & S^{22}_{l-1} \end{bmatrix} \begin{bmatrix} a^+_{l-1} \\
a^-_{l-1}
\end{bmatrix}.
\] (4.64)

Furthermore, we have at interface \( l \)
\[
\begin{bmatrix} a^+_l \\
a^-_l
\end{bmatrix} = \begin{bmatrix} S^{11}_l & S^{12}_l \\ S^{21}_l & S^{22}_l \end{bmatrix} \begin{bmatrix} a^+_{l-1} \\
a^-_{l-1}
\end{bmatrix}.
\] (4.65)

Substitute the second row of (4.65) into the first row of (4.64) to obtain
\[
a^+_l = \tilde{S}^{11}_{l-1}a^+_1 + \tilde{S}^{12}_{l-1}a^-_1 = \tilde{S}^{11}_{l-1}a^+_1 + \tilde{S}^{12}_{l-1}[S^{21}_l a^+_l + S^{22}_l a^-_{l+1}].
\] (4.66)

Collecting the terms gives
\[
[I - \tilde{S}^{12}_l S^{21}_l] a^+_l = \tilde{S}^{11}_{l-1}a^+_1 + \tilde{S}^{12}_{l-1}S^{22}_l a^-_{l+1},
\] (4.67)

so
\[
a^+_l = [I - \tilde{S}^{12}_l S^{21}_l]^{-1} \tilde{S}^{11}_{l-1}a^+_1 + [I - \tilde{S}^{12}_l S^{21}_l]^{-1} \tilde{S}^{12}_{l-1}S^{22}_l a^-_{l+1}.
\] (4.68)

Substituting this into the first row of (4.65) yields
\[
a^+_l = \tilde{S}^{11}_l [I - \tilde{S}^{12}_l S^{21}_l]^{-1} \tilde{S}^{11}_{l-1}a^+_1 + \left[ \tilde{S}^{11}_l [I - \tilde{S}^{12}_l S^{21}_l]^{-1} \tilde{S}^{12}_{l-1}S^{22}_l + \tilde{S}^{12}_l \right] a^-_{l+1}.
\] (4.69)
Substituting this into the second row of (4.64) yields

$$[I - S_l^{21} S_{l-1}^{12}] \mathbf{a}_l^- = S_l^{21} S_{l-1}^{11} \mathbf{a}_l^+ + S_l^{22} \mathbf{a}_{l+1}^-,$$

from which follows

$$\mathbf{a}_l^- = [I - S_l^{21} S_{l-1}^{12}]^{-1} S_l^{21} S_{l-1}^{11} \mathbf{a}_l^+ + [I - S_l^{21} S_{l-1}^{12}]^{-1} S_l^{22} \mathbf{a}_{l+1}^-.$$  (4.71)

Substituting this into the second row of (4.64) yields

$$\mathbf{a}_l^- = \left( S_l^{21} + S_l^{22} [I - S_l^{21} S_{l-1}^{12}]^{-1} S_l^{21} S_{l-1}^{11} \right) \mathbf{a}_l^+ + S_l^{22} [I - S_l^{21} S_{l-1}^{12}]^{-1} S_l^{22} \mathbf{a}_{l+1}^-.$$  (4.72)

These equations can still be used when the number of modes is different for each segment (i.e. when the four blocks of the scattering matrices are not square). To see this, consider the total number of amplitudes for the three segments numbered 1, l, and l + 1, which is 2\(\mu_l + 2\mu_l + 2\mu_{l+1}\). Of these amplitudes the \(\mu_1 + \mu_{l+1}\) amplitudes of the incident waves in segments 1 and \(l + 1\) are known. Hence, the other amplitudes can be determined by using the \(\mu_1 + 2\mu_l + \mu_{l+1}\) equations of (4.64) and (4.65). Also note that the dimensions of the terms between square brackets in (4.69) and (4.72) are \(\mu_1 \times \mu_l\), so only the solution of square systems is required.

Consequently, if all of the segment scattering matrices and the incoming amplitudes \(\mathbf{a}_1^+\) and \(\mathbf{a}_N^+\) of the outer segments are known, then the outgoing amplitudes and hence the total field in the outer segments can be computed. To compute the field inside the entire duct the amplitudes in intermediate segments are required as well. We compute these by using (4.66) and the second row of (4.65). Note that we did not invert the propagation matrices \(\mathbf{X}_i\), which would have caused growing exponentials (which might provoke numerical problems).

### 4.3 Numerical results

In order to compare the results of the classical (CMM) and the bilinear-map-based (BLM) mode-matching approaches we consider the test cases which are listed in Table 4.1. The 1m long duct with radius 0.15m is split into two segments at \(x = 0.5m\) (except for configuration IV); the left segment has a hard wall, and the right hand side segment has a locally reacting impedance wall. The incident field consists of one right-running mode, either \(\mu = 1\) or 2 for the pertinent configurations. For the BLM-based results the modes of the left (hard-wall) segment are used as test functions.

The results in this section are made dimensionless by scaling on the duct radius \(d\), a reference density \(\rho_\infty\) and a reference temperature \(T_\infty\). This implies that velocities are scaled on reference sound speed \(c_\infty = \sqrt{\gamma RT_\infty}\) and time on \(d/c_\infty\). The non-dimensional mean flow axial velocity is denoted by \(M\) (the Mach number), i.e. \(u_0 = c_\infty M\).

Configuration II has a slipping flow, so it is necessary to use the Ingard-Myers boundary condition here; this is in contrast to configurations I and III, which have non-slipping flow. The mean flow profile for configuration II has the same mass flow

\footnote{For non-uniform flow and/or temperature the numbering of the modes is not unambiguous; we number them according their similarity in eigenfunction shape and axial wavenumber value to the uniform flow modes.}
§ 4.3 Numerical results

<table>
<thead>
<tr>
<th>Configuration</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Helmholtz &amp; azi.</td>
<td>$\omega = 13.86, m = 5$</td>
<td>$\omega = 8.86, m = 5$</td>
<td>$\omega = 15, m = 5$</td>
<td>$\omega = 15, m = 5$</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T = 1$</td>
<td>$T = 1$</td>
<td>$T = 2\log(2)(1 - r^2)$</td>
<td>$T = 1$</td>
</tr>
<tr>
<td>Mean flow</td>
<td>$M = 0.5(1 - r^2)$</td>
<td>$M = 0.3 \cdot \frac{1}{4} (1 - \frac{r^2}{2})$</td>
<td>$M = 0.3 \cdot \tanh(10(1 - r))$</td>
<td>$M = 0.3 \cdot \tanh(10(1 - r))$</td>
</tr>
<tr>
<td>Soft-wall impedance</td>
<td>$Z = 1 - 1i$</td>
<td>$Z = 1 + 1i$</td>
<td>$Z = 1 - 1i$</td>
<td>$Z = 1 - 3i, ..., 1 + 3i$, $N = 20$</td>
</tr>
<tr>
<td>Incident rad. mode nr.</td>
<td>$\mu = 1$</td>
<td>$\mu = 1$</td>
<td>$\mu = 2$</td>
<td>$\mu = 1$</td>
</tr>
</tbody>
</table>

Table 4.1: Overview of test configurations.

![Classical mode-matching](image1.png)

(a) Classical mode-matching.

![Bilinear map-based mode-matching](image2.png)

(b) Bilinear map-based mode-matching.

Figure 4.3: Real part of pressure field for configuration I.

As a uniform flow with Mach number 0.3. The non-uniform temperature profile of configuration III has the same mass flow as a flow having uniform mean temperature ($T = 1$). The flow profile of configuration III is more representative of a uniform flow with a thin boundary layer. Configuration IV is a duct with $N = 20$ segments, where the impedances of the segments have an imaginary part that varies linearly between $-3$ and $3$. For all configurations we use $\mu^{\text{max}} = 50$ modes to represent the field and an equal number of test functions.

To compute the solutions of the boundary value problems we use the approach that is described in Chapter 3. It is important to note that it is not necessary to use path-following for the inhomogeneous problem, since the axial wave number $k_\mu$ is given and not part of the solution, and the inhomogeneous problem (with homogeneous Dirichlet boundary condition at $r = d$) has a unique solution. Consequently, it is much cheaper to solve the inhomogeneous problem than the original eigenvalue problem.

Figure 4.3 shows the acoustic pressure field for both the classical (CMM) and the bilinear map (BLM) based mode-matching approaches for configuration I. Figures 4.4, 4.5 and 4.6 compare the pressure, axial and radial velocity for both mode-matching methods at several radial locations. The figures show that the results of the two approaches are in very good agreement for all configuration I–III.
Figure 4.4: Comparison of classical (CMM) and bilinear-map-based (BLM) mode-matching approaches for configuration I. Pressure, axial and radial velocity at radial locations: $r = \{0.035, 0.075, 0.15\}$ m.
§ 4.3 Numerical results

Figure 4.5: Comparison of classical (CMM) and bilinear-map-based (BLM) mode-matching approaches for configuration II. Pressure, axial and radial velocity at radial locations: $r = \{0.035, 0.075, 0.15\}$ m.
Figure 4.6: Comparison of classical (CMM) and bilinear-map-based (BLM) mode-matching approaches for configuration III. Pressure, axial and radial velocity at radial locations: \( r = \{0.035, 0.075, 0.15\} \) m.
The validity of the numerical results can be assessed by checking whether they satisfy the balance of energy. For that purpose we use Myers’ Energy Corollary (see Section 2.1.5), which holds exactly for the disturbance field. Figure 4.7 shows the sum of the acoustic fluxes through the wall, the inlet and the outlet plane, and the volume integral over the source term. We use a relative numerical accuracy of $10^{-6}$ for the boundary value problem solver, and use Simpson’s rule for the numerical quadrature on a grid of 151 by 1001 grid points to compute the energy terms. Thus, this sum (which is normalized on the flux through the inlet plane), which ideally should be zero, is not expected to be bigger than $10^{-6}$. Figure 4.7 shows that the energy balance is satisfied better as the number of modes $\mu_{max}$ increases, which is to be expected. Furthermore, the BLM-based mode-matching method performs even better than the classical one for this configuration I. Incidentally, the energy integral being so accurately satisfied confirms the assumption of a negligible continuous spectrum contribution.

To verify the regular behavior of the solution at the liner discontinuities along the wall (where the edge condition has to be satisfied, which is that any volume surrounding the edge carries finite energy, as discussed on page 62) we check the uniform convergence of the modal series via the convergence rate of the found modal amplitudes $A_n$. If we assume that $A_n = \mathcal{O}(n^p)$ for $n \to \infty$ such that $\log |A_n| = p \log n + \mathcal{O}(1)$, then $p_n$ defined as

$$p_n := \frac{\log |A_n|}{\log n} \quad (4.73)$$

is expected to approach $p$ with $p_n = p + \mathcal{O}(1/\log(n))$. Anticipating a local behavior of the modal functions that is asymptotically similar to a Fourier series, a convergence rate $p < -1$ will be sufficient for uniform convergence. As shown in Figure 4.8, $p \approx -2$ for the considered configurations I–III, so we see that the condition is indeed satisfied, in particular for both mode matching methods in the same way.

Moreover, there is another interesting observation possible from these plots. The behavior of $p_n$ from the classical mode-matching method is not as smooth as from the
Figure 4.8: Convergence rate of amplitudes for classical and bilinear-map-based mode-matching, for configurations I, II and III.
BLM-based mode-matching as \(n\) becomes larger. Apparently the amplitudes from the classical method are more inaccurate for large \(n\). This is a confirmation of the fact that the inner products based on closed-form expressions are not prone to the quadrature errors for large \(n\), for which the eigenfunctions are most oscillatory and quadrature is consequently most difficult.

Finally, to illustrate that a high number of segments poses no problem for mode-matching with the scattering matrix formalism we included Figure 4.9, which depicts the pressure field for a configuration with \(N = 20\) segments. From our experience an iterative procedure often does not converge for more than 10 segments. The artifacts near \(r = 0\) are due to the fact that at that location the field is almost zero, so very close to the level curve at zero.
Chapter 5

Asymptotic solutions for slowly varying impedance

In the previous chapters we considered configurations whose properties are (at least locally, in case of a segmented liner) invariant in the axial direction; the shape of the cross-section, the mean flow and the wall impedance were independent of the axial position. This permitted us to describe the sound field in terms of duct modes. Modes in a strict sense cease to exist if one of the problem parameters varies with the axial coordinate.

Fortunately, if this variation is slow it is possible to find a small parameter that may be used for solutions of multiple scales or WKB type (a special case of multiple scales analysis). This approach is based on slowly varying modes, for which it is assumed that the fast scale of the wave motion and the slow scale of the axially varying ‘wave envelope’ (mode shape and wave number) are independent. (Note that in Section 2.5 we also employed the WKB method, but in cross-wise direction.)

One specific example is the case of sound propagation through flow ducts with a slowly varying cross-section [103, 104]. The usefulness of the approximation was shown by a numerical comparison for a geometry that is very typical of the shape of an aero-engine duct [107]. Moreover, the reflection of modes at turning points was studied, where a mode changes from cut-on to cut-off due to a decrease in cross-sectional area. WKB solutions for slowly varying cross-section have also been used to match the flow perturbations of the source region (in and near the region of rotor and stator blades, computed from fully numerical simulations) to the acoustic region in the inlet and bypass duct [92, 118].

The multiple-scales method can in principle be used for the variation of any of the problem parameters [22, 28, 29, 63, 93], but with the mean flow it is difficult since we do not have an analytic expression available for a slowly varying ducted shear flow. We will consider here a slowly varying impedance. As mentioned in Chapter 1 and depicted in Figure 1.2, the sound absorbing liner of an APU exhaust duct has an axially decreasing depth, which results in an axially varying wall impedance.

In Section 5.1 we present a solution\(^1\) in terms of slowly varying modes for a duct with an axially varying impedance and a mean flow with non-uniform velocity and

---

\(^1\)The work described in this chapter is the result of a collaboration with W. Lazeroms, who derived most of the analytical results of Section 5.1, which can also be found in [63].
temperature \([63, 90]\). The cross-wise mode shape of a slowly varying mode is given by the Pridmore-Brown equation, which is therefore solved at each axial location with the numerical approach described in Chapter 3. In Section 5.2 some numerical results obtained with the WKB approach will be compared to some mode-matching results.

### 5.1 WKB solutions

In this section we use the Linearized Euler Equations for parallel flow in the formulation of (2.105a), (2.105b) and (2.64), in which the last term vanishes for parallel flow. Together with the linearized ideal gas law (2.61) these equations are made dimensionless in the same way as the Pridmore-Brown equation in Sections 2.5 and 3.1, i.e. based on the duct radius \(d\), a reference density \(\rho_\infty\) and temperature \(T_\infty\), from which follows the reference sound speed \(c_\infty = \sqrt{\gamma R T_\infty}\). This amounts to the scaling

\[
\begin{align*}
M &= \frac{u_0}{c_\infty}, & R &= \frac{\rho_0}{\rho_\infty}, & \Theta &= \frac{T_0}{T_\infty}, & \Pi &= \frac{p_0}{\rho_\infty c_\infty^2}, & \tilde{\omega} &= \frac{\omega d}{c_\infty}.
\end{align*}
\] (5.1)

which implies that we can use \(R = 1/\Theta\) and \(\Pi = 1/\gamma\). In the following we will suppress the tildes and use dimensionless quantities unless stated otherwise. Up to this point we denoted the mean flow and perturbations by subscripts \(0\) and \(1\), but these will be used later to denote the orders in an asymptotic expansion, and we will now write the total field consisting of the steady parallel mean flow plus perturbations of the form \(e^{-i\omega t + im\theta}\) as

\[
[p, u, v, w, \rho, T](x, r, \theta, t) = [\Pi, M, 0, 0, R, \Theta](r) + \text{Re} \left\{ [P, U, V, W, D, T](x, r) e^{im\theta - i\omega t} \right\}.
\] (5.2)

The governing equations (2.105a), (2.105b), (2.64) and (2.61) then become

\[
\begin{align*}
\left( -i\omega + M \frac{\partial}{\partial x} \right) D + \frac{1}{r} \frac{\partial (r RV)}{\partial r} + R \left( \frac{i m}{r} V + \frac{\partial U}{\partial x} \right) &= 0, \\
\left( -i\omega + M \frac{\partial}{\partial x} \right) U + \frac{\partial M}{\partial r} + \frac{1}{R} \frac{\partial D}{\partial x} &= 0, \\
\left( -i\omega + M \frac{\partial}{\partial x} \right) V + \frac{1}{\rho_\infty} \frac{\partial P}{\partial r} &= 0, \\
\left( -i\omega + M \frac{\partial}{\partial x} \right) W + \frac{im}{r R} \frac{\partial V}{\partial r} &= 0, \\
\left( -i\omega + M \frac{\partial}{\partial x} \right) T + \frac{\partial \Theta}{\partial r} V + (\gamma - 1) \Theta \left( \frac{1}{r} \frac{\partial}{\partial r} (r V) + \frac{im}{r} \frac{\partial W}{\partial x} \right) &= 0, \\
\gamma P &= R T + D \Theta.
\end{align*}
\] (5.3)

Note that for modes of the form \(P(x, r) = P(r) e^{ikx}\) we obtain a system of equations for the modal amplitudes depending only on \(r\). This system can be reduced to the Pridmore-Brown equation. The Ingard-Myers boundary condition (2.78) in dimensionless form is in the present notation

\[
-i\omega V = \left( -i\omega + M \frac{\partial}{\partial x} \right) \frac{P}{Z} \quad \text{at } r = 1.
\] (5.3g)
If the impedance varies in the axial direction, the problem is not axially invariant anymore, so strictly speaking it is not possible to describe the field in terms of modes of the form $P(r)e^{ikx}$. However, if we assume that the inherent length scale $L$ of typical variations of $Z(\bar{x}/L)$ (where $\bar{x} = xd$ and $x$ are the dimensional and dimensionless axial coordinates) is large compared to the duct radius $d$, i.e.

$$Z\left(\frac{\bar{x}}{L}\right) = Z\left(\frac{d}{L}x\right) = Z(X), \quad \varepsilon = \frac{d}{L} \ll 1, \quad X := \varepsilon x,$$

(5.4)

we can use this small parameter $\varepsilon$ to construct so-called *slowly varying modes* by a variant of the WKB method. Assuming that the axial wavenumbers are typically equal or larger than $\mathcal{O}(1)$, we approximate the acoustic field by modes of which the amplitude, mode shape and axial wavenumber vary only slowly in the axial direction; we look for modal solutions of the form

$$[P, U, V, W, D, T](r, X; \varepsilon) = \left[P_0, U_0, V_0, W_0, D_0, T_0\right](r, X) \exp\left(\frac{i}{\varepsilon} \int_{0}^{X} \mu(\eta) d\eta\right),$$

(5.5)

where $\mu(X)$ is the axial wavenumber depending on the slow coordinate $X$. Substituting this WKB Ansatz (5.5) into (5.3) yields

$$-i\Lambda D + \varepsilon M \frac{\partial D}{\partial X} + \frac{1}{r} \frac{\partial}{\partial r} (r RV) + R \left(\frac{im}{r} W + \varepsilon \frac{\partial U}{\partial X} + i\mu U\right) = 0,$$

(5.6a)

$$-i\Lambda U + \varepsilon M \frac{\partial U}{\partial X} + \frac{dM}{dr} V + \frac{1}{R} \left(\varepsilon \frac{\partial P}{\partial X} + i\mu P\right) = 0,$$

(5.6b)

$$-i\Lambda V + \varepsilon M \frac{\partial V}{\partial X} + \frac{1}{R} \frac{\partial P}{\partial r} = 0,$$

(5.6c)

$$-i\Lambda W + \varepsilon M \frac{\partial W}{\partial X} + \frac{im}{r} P = 0,$$

(5.6d)

$$-i\Lambda T + \varepsilon M \frac{\partial T}{\partial X} + \frac{d\Theta}{dr} V + (\gamma - 1)\Theta \left[\frac{1}{r} \frac{\partial}{\partial r} (r V) + \frac{im}{r} W + \varepsilon \frac{\partial U}{\partial X} + i\mu U\right] = 0,$$

(5.6e)

$$\gamma P = RT + \Theta D,$$

(5.6f)

with $\Lambda := \omega - \mu M$. The corresponding boundary condition obtained from (5.3g) is

$$-i\omega V = -\frac{i\Lambda}{Z} P + \varepsilon M \frac{\partial}{\partial X} \left(\frac{P}{Z}\right) \quad \text{at } r = 1.$$

(5.6g)

Now we expand the amplitude functions in $\varepsilon$ as

$$[P, U, V, W, D, T](r, X; \varepsilon) = [P_0, U_0, V_0, W_0, D_0, T_0](r, X) + \varepsilon[P_1, U_1, V_1, W_1, D_1, T_1](r, X) + \mathcal{O}(\varepsilon^2).$$

(5.7)
The leading order equations obtained after substitution of this expansion in (5.6) are

\[-i\Lambda D_0 + \frac{1}{r} \frac{\partial}{\partial r} (rRV_0) + R \left( \frac{im}{r} W_0 + i\mu U_0 \right) = 0, \quad (5.8a)\]
\[-i\Lambda U_0 + \frac{dM}{dr} V_0 + i\mu R P_0 = 0, \quad (5.8b)\]
\[-i\Lambda V_0 + \frac{1}{R} \frac{\partial P_0}{\partial r} = 0, \quad (5.8c)\]
\[-i\Lambda W_0 + \frac{im}{rR} P_0 = 0, \quad (5.8d)\]
\[-i\Lambda T_0 + \frac{d\Theta}{dr} V_0 + (\gamma - 1)\Theta \left[ \frac{1}{r} \frac{\partial}{\partial r} (rV_0) + \frac{im}{r} W_0 + i\mu U_0 \right] = 0, \quad (5.8e)\]
\[\gamma P_0 = RT_0 + \Theta D_0, \quad (5.8f)\]

with the boundary condition

\[-i\omega V_0 = -\frac{i\Lambda}{Z} P_0 \quad \text{at} \quad r = 1. \quad (5.8g)\]

The system of equations that arises for modal solutions of the form \(P(x, r) = P(r)e^{ikx}\) has the same structure as (5.8), except that in (5.8) the modal amplitudes and the axial wavenumber depend on \(X\), which serves as a parameter. It follows that \(P_0\) satisfies the Pridmore-Brown equation (2.117), i.e.

\[LP_0 := \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{2}{\Lambda} \frac{d\Lambda}{dr} + \frac{1}{\Theta} \frac{d\Theta}{dr} \right] \frac{\partial}{\partial r} + \left( \frac{\Lambda^2}{\Theta} - \mu^2 - \frac{m^2}{r^2} \right) P_0 = 0, \quad (5.9a)\]

together with the Ingard-Myers boundary condition (3.1b), i.e.

\[\frac{\partial P_0}{\partial r} - \frac{i\Lambda^2 R}{\omega Z} P_0 = 0 \quad \text{at} \quad r = 1. \quad (5.9b)\]

For every fixed axial location \(X\) this problem (together with the regularity condition at \(r = 0\)) is exactly the same as the constant impedance problem described by (3.1). The general leading order solution will be of the form

\[P_0(r, X) = N(X)\psi_{mn}(r, X), \quad (5.10)\]

where \(\psi_{mn}\) is again an eigenfunction (for clarity considered normalized), this time parametrically depending on \(X\), and \(N(X)\) is a slowly varying amplitude function that is still unknown. As before, solving the Pridmore-Brown equation to find the eigenfunction \(\psi_{mn}(X)\) of radial order \(n\) also yields the axial wavenumber \(\mu_{mn}(X)\) for any given frequency \(\omega\) and circumferential wavenumber \(m\).

To determine \(N(X)\) the first order equations obtained after substituting the expansion (5.7) in (5.6) are needed. \(N(X)\) can be determined from a condition that the asymptotic expansion remains uniform in \(X\) and the first order problem is solvable, as we will see later—we do not have to solve this first order system explicitly. These first
order equations are
\[-i \Lambda D_1 + \frac{1}{r} \frac{\partial}{\partial r} (rRV_1) + R \left( \frac{im}{r} W_1 + i\mu U_1 \right) = - \left( M \frac{\partial D_0}{\partial X} + R \frac{\partial U_0}{\partial X} \right), \quad (5.11a)\]
\[-i \Lambda U_1 + \frac{dM}{dr} V_1 + \frac{i\mu}{R} P_1 = - \left( M \frac{\partial U_0}{\partial X} + \frac{1}{R} \frac{\partial P_0}{\partial X} \right), \quad (5.11b)\]
\[-i \Lambda V_1 + \frac{1}{R} \frac{\partial P_1}{\partial r} = - M \frac{\partial V_0}{\partial X}, \quad (5.11c)\]
\[-i \Lambda W_1 + \frac{im}{rR} P_1 = - M \frac{\partial W_0}{\partial X}, \quad (5.11d)\]
\[-i \Lambda T_1 + \frac{d\Theta}{dr} V_1 + (\gamma - 1)\Theta \left( \frac{1}{r} \frac{\partial}{\partial r} (rV_1) + \frac{im}{r} W_1 + i\mu U_1 \right) = - \left( M \frac{\partial T_0}{\partial X} + (\gamma - 1)\Theta \frac{\partial U_0}{\partial X} \right), \quad (5.11e)\]
\[\gamma P_1 = RT_1 + \Theta D_1, \quad (5.11f)\]
with the boundary condition
\[-i \omega V_1 + \frac{i\Lambda}{Z} P_1 = M \frac{\partial}{\partial X} \left( \frac{P_0}{Z} \right) \text{ at } r = 1. \quad (5.11g)\]

In a similar way as before we can find a single equation for $P_1$ with a right-hand side that only contains $P_0$. The result is
\[\mathcal{L} P_1 = \mathcal{F} := - \frac{2iM}{\Lambda} \frac{\partial}{\partial X} \left[ \frac{\partial}{\partial r} \ln \left( \frac{\Lambda}{M} \right) \frac{\partial P_0}{\partial r} \right] - \frac{iM}{P_0} \frac{\partial}{\partial X} \left[ \left( \frac{\Lambda}{\Theta} + \frac{\mu}{M} \right) P_0^2 \right], \quad (5.12a)\]
with the boundary condition
\[\frac{\partial P_1}{\partial r} - \frac{i\Lambda^2 R}{\omega Z} P_1 = - \frac{MRZ}{\omega P_0} \frac{\partial}{\partial X} \left( \frac{\Lambda P_0^2}{Z^2} \right) \text{ at } r = 1. \quad (5.12b)\]

Instead of solving for $P_1$, which would lead to yet another undetermined amplitude function analogous to $N(X)$ in (5.10), we will derive a solvability condition [82, chapter 15] for (5.12) that only contains leading order terms. If this condition holds, then the solution of the form (5.5) exists, and the leading and first order systems (5.9) and (5.12) are consistent with a regular asymptotic expansion. This solvability condition can be found as follows.

We start by multiplying (5.12a) by $r\Theta P_0/\Lambda^2$, (5.9a) by $r\Theta P_1/\Lambda^2$, subtracting the two equations and integrating the result. This yields
\[\int_0^1 \frac{\Theta}{\Lambda^2} (P_0 \mathcal{L} P_1 - P_1 \mathcal{L} P_0) \, r \, dr = \int_0^1 \frac{\Theta}{\Lambda^2} P_0 \mathcal{F} \, r \, dr. \quad (5.13)\]

Because the operator $(r\Theta/\Lambda^2)\mathcal{L}$ is the self-adjoint form of $\mathcal{L}$ (compare with (2.112)), i.e.
\[r\Theta \frac{\Lambda^2}{\Lambda^2} \mathcal{L} P = \frac{\partial}{\partial r} \left[ r\Theta \frac{\partial P}{\partial r} \right] + r\Theta \left( \frac{\Lambda^2}{\Theta} - \frac{\mu^2}{r^2} - \frac{m^2}{r^2} \right) P, \quad (5.14)\]
the left-hand side of (5.13) can be reduced to an expression containing only boundary
terms, i.e.

\[
\int_0^1 \frac{\Theta}{\Lambda^2} (P_0 \mathcal{L}P_1 - P_1 \mathcal{L}P_0) r dr = \int_0^1 \frac{\partial}{\partial r} \left[ \frac{r \Theta}{\Lambda^2} \left( \frac{P_0}{\partial r} - \frac{P_1}{\partial r} \right) \right] dr
\]

\[= \left. \left[ \frac{r \Theta}{\Lambda^2} \left( \frac{P_0}{\partial r} - \frac{P_1}{\partial r} \right) \right] \right|_{r=1}^{r=0} = \left. \frac{\Theta}{\Lambda^2} \left( \frac{P_0}{\partial r} - \frac{P_1}{\partial r} \right) \right|_{r=1}^{r=0}
\]

\[= - \frac{MZ}{\omega \Lambda^2 \partial X} \left( \frac{\Lambda P_0^2}{Z^2} \right) \right|_{r=1}^{r=1}, \tag{5.15}
\]

where we used the boundary conditions (5.9b) and (5.12b) and the relation \(R \Theta = 1\) in the last step. Using this result together with (5.12a) in (5.13) then leads to the following solvability condition for the first order problem:

\[
i \int_0^1 \left\{ \frac{2M \Theta}{\Lambda^3} P_0 \frac{\partial}{\partial X} \left[ \frac{\partial}{\partial r} \ln \left( \frac{\Lambda}{M} \right) \right] \right\} dr = \frac{MZ}{\omega \Lambda^2 \partial X} \left( \frac{\Lambda P_0^2}{Z^2} \right) \right|_{r=1}^{r=1}, \tag{5.16}
\]

The next step is to substitute the general solution of \(P_0\) shown in (5.10) into (5.16). After working out and rearranging terms we arrive at a first order equation for the amplitude function \(N\)

\[
g(X) \frac{d}{dX} [N(X)]^2 = -f(X) [N(X)]^2, \tag{5.17a}
\]

with (in principle known) functions

\[
f(X) := i \int_0^1 \left\{ \frac{2 \psi}{\Lambda^3} \frac{\partial}{\partial X} \left[ \frac{\partial}{\partial r} \ln \left( \frac{\Lambda}{M} \right) \right] \right\} r dr - \frac{MZ}{\omega \Lambda^2 \partial X} \left( \frac{\Lambda \psi^2}{Z^2} \right) \right|_{r=1}^{r=1},
\]

\[
g(X) := i \int_0^1 \left\{ \frac{\psi}{\Lambda^3} M \Theta \frac{\partial}{\partial r} \ln \left( \frac{\Lambda}{M} \right) \right\} r dr - \frac{MZ}{\omega \Lambda^2} \left( \frac{\Lambda \psi^2}{Z^2} \right) \right|_{r=1}^{r=1}. \tag{5.17b}
\]

This equation has the solution

\[
N^2 = N_0^2 \exp \left( - \int_0^X \frac{f(\eta)}{g(\eta)} d\eta \right), \tag{5.18}
\]

where \(N_0\) is a normalization constant that can be determined from the source, e.g. at the beginning of the duct. (Note that any normalization of \(\psi\) drops out in the term \(f(X)/g(X)\).)

In principle (5.10) can now be computed, provided that the derivatives \(\partial \psi/\partial X\) and \(d \mu/dX\) are known. Since \(\mu(X)\) and \(\psi(r,X)\) are only known from the solution of the Pridmore-Brown equation, which are solved numerically (unless flow and temperature are uniform), the \(X\)-derivatives have to be computed numerically as well.

The results can be simplified for some special cases. For uniform mean flow and arbitrary temperature we have

\[
f(X) = \frac{1}{a(X)} \left( \frac{\partial}{\partial X} a(X) - h(X) \right), \tag{5.19a}
\]
§ 5.2 Numerical comparison between WKB and mode-matching

with

\[
a(X) := \int_0^1 (M \Lambda + \mu \Theta) \psi^2 r \, dr + \left[ \frac{i M \Lambda}{\omega Z} \psi^2 \right]_{r=1}, \quad h(X) := \left[ \frac{i M \Lambda}{\omega Z} \psi^2 \frac{dZ}{dX} \right]_{r=1},
\]

such that (5.18) reduces to

\[
N^2(X) = \frac{N_0^2}{a(X)} \exp \left( - \int_0^X \frac{h(\eta)}{a(\eta)} \, d\eta \right).
\]

For zero mean flow and arbitrary temperature this can be reduced even further, to

\[
N^2(X) = \frac{N_0^2}{\mu} \left( \int_0^1 \Theta \psi^2 r \, dr \right)^{-1}.
\]

Note that for these special cases it is not required to compute \( \partial \psi / \partial X \) and \( d \mu / dX \).

The result in (5.18) may be compared with the one given by [28], where the duct varies in diameter and the mean flow includes swirl, but the impedance is taken constant. With the APU geometry in mind of a constant duct but varying impedance, we could obtain a simpler result, that is relatively easily applicable when numerical solutions of the Pridmore-Brown equation are available.

Finally we note that the small parameter \( \varepsilon \) only acts as a bookkeeping parameter; since (5.17a) does not depend explicitly on \( \varepsilon \) and both sides of the equality sign contain only single \( X \)-derivatives we can replace \( X \) by \( x \).

### 5.2 Numerical comparison between WKB and mode-matching

In this section we show some numerical results for test cases motivated by a realistic APU exhaust duct, having a length of 1m and a radius \( d \) of 0.15 m. We choose the reference temperature \( T_\infty = 700 \) K and density \( \rho_\infty = 0.5 \) kg/m\(^3\), corresponding to a reference sound speed of \( c_\infty = 549 \) m/s. The frequency \( \omega \), wavenumber \( m \), and the details of the velocity and temperature of the mean flow are given in the captions of the respective figures. We consider cases where \( \text{Im}(Z(x)) \) varies linearly with fixed \( \text{Re}(Z) = 1.5 \), and cases where the liner is modeled as a Helmholtz resonator (see Section 2.3.4).

In the following we will use \( \tilde{x} = x d \) and \( x \) to denote the dimensional and dimensionless axial coordinates respectively, as we did in (5.4).

To assess the applicability of the WKB method and estimate the truncation error in the WKB expansion (which is \( O(\varepsilon) \), see (5.7)) we need to estimate \( \varepsilon \). Since by assumption \( Z'(X)/Z(X) = O(1) \) this can be done by noting that if \( Z \) varies typically over a length scale \( L \) we have

\[
d \frac{d}{d \tilde{x}} \frac{Z(\tilde{x}/L)}{Z(\tilde{x}/L)} = \frac{d}{L} \frac{Z'(X)}{Z(X)} = O(\varepsilon).
\]

We plot contour lines of the pressure field, where the source at \( \tilde{x} = 0 \) m consists of a single \( n \)-th order radial mode (usually, \( n = 1 \)). For cases of uniform mean flow velocity
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and temperature we compare the (axial) WKB results with mode-matching\(^2\) results (see Chapter 4). The eigenfunctions are normalized according to

\[
\int_0^1 |\psi|^2 r \, dr = 1, \quad \psi'(1) \in \mathbb{R}^+, \quad (5.23)
\]

i.e. the phase is chosen such that \(\psi'(1)\) is real-valued and positive.

For the WKB results we first compute the eigenfunctions and axial wavenumbers for 200 equally spaced axial positions along the duct with the numerical approach described in Chapter 3. The amplitudes \(N(X)\) are computed by using (5.18) or the special cases (5.20) and (5.21). The integrals over \(\eta\) are computed with a trapezoidal rule, which has equally spaced grid-points; this enables us to use all computed eigenfunctions (for all axial positions) for the field plot. The integrals over \(r\) in (5.17b), (5.19b) and (5.21) are computed by using the QUADPACK [95] code (which is based on an adaptive Gauss-Kronrod rule). We motivate this choice by noting that the eigenfunction (and hence the integrand of the radial integral) can be oscillatory, while the integrands of the axial integrals are only slowly varying. For non-uniform mean flow we need to compute \(f(X)\). Working out \(f(X)\) in (5.17b) yields

\[
f(X) = i \int_0^1 \left\{ \frac{-2 \omega \psi \Theta}{\Lambda^2} \frac{\partial M}{\partial r} \left[ \Lambda \frac{\partial^2 \psi}{\partial r \partial X} + M \frac{\partial \mu}{\partial X} \frac{\partial \psi}{\partial r} \right] + \frac{\psi}{\Lambda^2} \left[ (\Theta - M^2) \frac{\partial \mu}{\partial X} \psi + 2(M \Lambda + \mu \Theta) \frac{\partial \psi}{\partial X} \right] \right\} r \, dr
\]

\[
- \frac{M \psi}{\omega \Lambda^2 Z^2} \left[ 2 \Lambda Z \frac{\partial \psi}{\partial X} - \left( ZM \frac{\partial \mu}{\partial X} + 2 \Lambda \frac{\partial Z}{\partial X} \right) \psi \right] r=1. \quad (5.24)
\]

The \(X\)-derivatives \(\partial \psi/\partial X\) and \(d\mu/dX\) (which are not available analytically, as mentioned before) are computed by using a second order finite differences approximation.

To test the WKB approach we first let \(Z(\bar{x})\) vary linearly from \(1.5 - i\) to \(1.5 + i\), which corresponds to an estimated \(\varepsilon = 0.2\). Figure 5.1 shows the piecewise impedance for the segments that are used in the mode-matching approach. From Figure 5.2a and Figure 5.2b it is clear that the difference between constant impedance and axially varying impedance is significant, so the \(\bar{x}\)-dependency of \(Z\) has to be taken into account. Figure 5.2b shows that the WKB and the mode-matching results agree reasonably well for this not very small choice of \(\varepsilon\). From Figure 5.2c, with the same parameter values and mean flow mass flux as before but a parabolic mean flow profile, we conclude that the effect of a non-uniform mean flow should not be neglected. The present difference is explained by the fact that downstream running sound waves are refracted towards the lined wall [108], which results in more damping.

In the next configuration we let \(Z(\bar{x})\) vary linearly from \(1.5 - 2i\) to \(1.5 + 2i\) along the same interval, which corresponds to a higher estimated \(\varepsilon = 0.4\). The truncation error in the WKB approximation is now larger and we may expect a bigger difference between the WKB and the mode-matching solutions. This is indeed the case, as is shown in Figure 5.3. Near \(\bar{x} = 0.7\) m some signs of intermodal scattering are visible, which is explicitly not taken into account by the WKB method.

To evaluate the applicability of the WKB method for a realistic APU exhaust duct geometry we now consider a locally reacting liner with a cell depth \(d_l(\bar{x})\) that is axially

\(^2\)In [90] (on which this chapter is based) we referred to the NLR-developed mode-matching code BAHAMAS, which handles uniform flow and temperature, since a mode-matching code that handles non-uniform flow and temperature was not available at the time of writing.
§5.2 Numerical comparison between WKB and mode-matching

Figure 5.1: $Z$ varies linearly from $1.5 - i$ to $1.5 + i$, $\varepsilon \approx 0.2$.

(a) $Z = 1.5 - i$, uniform mean flow velocity $M_0 = 0.3$.

(b) Same as Figure 5.2a, except $Z$ varies linearly from $1.5 - i$ to $1.5 + i$ so $\varepsilon \approx 0.2$.

(c) Same as Figure 5.2b, except mean flow velocity $M(r) = M_0 \frac{3}{2} (1 - \frac{1}{2} r^2)$ with $M_0 = 0.3$.

Figure 5.2: $\omega = 10$, $m = 2$, $n = 1$
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0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.05
0.1
0.15

Figure 5.3: \( \omega = 10, m = 2, n = 1 \), uniform mean flow velocity \( M_0 = 0.3 \), \( Z \) varies linearly from 1.5 − 2i to 1.5 + 2i so \( \varepsilon \approx 0.4 \).

0
0.2
0.4
0.6
0.8
1
−3
−2
−1
0
1

Figure 5.4: \( Z \) modeled as Helmholtz resonator with liner depth that varies linearly from 7 cm to 1 cm. Left: \( \omega = 6 \ (\varepsilon \approx 0.3) \), right: \( \omega = 10 \) (resonance occurs).

varying from 7 cm to 1 cm along the duct. The liner is modeled as a Helmholtz resonator array (see Section 2.3.4), which has an impedance given by

\[
\bar{Z} = \bar{Z}_0 + i \rho_\infty c_\infty \cot \left( \frac{\bar{\omega}}{c_\infty} \bar{d}(\bar{x}) \right),
\]

(5.25)

where the overbars are used to denote dimensional quantities, and \( \bar{Z}_0 \) is the face sheet impedance

\[
\bar{Z}_0 = R_0 - i \bar{\omega} \bar{\chi}.
\]

(5.26)

We choose a facing sheet resistance of \( R_0 = 400 \ \text{kgm}^{-2} \text{s}^{-1} \), and a mass reactance of \( \bar{\chi} = 0.001 \ \text{kg/m}^2 \). We remark that this is only a model and the reference sound speed \( c_\infty \) may be different from the sound speed at the wall for non-uniform temperatures. Figure 5.4 shows the imaginary part of the impedance as a function of \( \bar{x} \) for two different frequencies. Note that for \( \omega = 10 \) resonance occurs; close to \( \bar{x} = 0.4 \) m the impedance becomes very large so the liner behaves as a hard wall.

Figure 5.5 show the acoustic field for \( \omega = 6 \). On the interval considered there is no location where the Helmholtz resonator is in resonance; \( Z(\bar{x}) \) is slowly varying with an estimated \( \varepsilon = 0.3 \). The WKB and mode-matching results show rather good agreement, about what can be expected from this value of \( \varepsilon \).

However, for \( \omega = 10 \) we do have resonance near \( \bar{x} = 0.4 \) m so the assumption of a slowly varying \( Z(\bar{x}) \) is, at least near this point, not valid anymore. This can indeed be
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Figure 5.5: $\omega = 6$, $m = 2$, $n = 1$, uniform mean flow velocity $M_0 = 0.3$, $Z$ modeled as Helmholtz resonator with liner depth that varies linearly from 7 cm to 1 cm.

Figure 5.6: $\omega = 10$, $m = 2$, $n = 1$, uniform mean flow velocity $M_0 = 0.3$, $Z$ modeled as Helmholtz resonator with liner depth that varies linearly from 7 cm to 1 cm.

identified from Figure 5.6, where the region of resonance seems to excite the second radial mode, an effect which cannot be described by (straight-forward) application of the WKB method [92].

In a realistic APU exhaust duct cool air is let in near $\bar{x} = 0$ m along the wall (see Figure 1.4). This produces a strong radial temperature gradient. We modeled this by the tanh-type profile that is depicted in Figure 5.7. The effect of this temperature gradient is that it creates effectively two concentric ducts, each with its own propagation properties. These duct fields are not completely independent of each other. Sound waves from the center region (with the highest sound speed) will refract (by a form of Snell’s law) to the colder annular region. However, sound waves in the annular region refracts only if the angle between duct axis and their propagation direction is not too small. Otherwise the annular region will act as a duct on its own.

This is illustrated in Figure 5.8, where the fields are plotted for the first two right-running radial modes. In case of the first mode the field is virtually only existent in the colder outer region. The field of the second mode exists in both, but such that the sound waves refract from inside to outside.
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Figure 5.7: Temperature profile $\Theta(r) = \frac{1}{4} + \frac{5}{8} \left[ 1 + \tanh \left( 50 \left( \frac{3}{4} - r \right) \right) \right]$.

Figure 5.8: $\omega = 10$, $m = 2$, $n = 1$ and $n = 2$, uniform mean flow velocity $M_0 = 0.3$, strong temperature gradient (see Figure 5.7), $Z(x)$ varies linearly from $1.5 - i$ to $1.5 + i$. 
Chapter 6

Application to APU exhaust duct

In this chapter we present numerical results based on the theory that was described in the preceding chapters. The parameters have been chosen with the application of an APU exhaust duct in mind. We aim to illustrate the practical applicability of the methods developed above and to present results and insights that facilitate design studies.

For easier interpretation, all parameters and variables in this chapter are made dimensionless by scaling on the duct radius $d$, a reference density $\rho_\infty$ and a reference temperature $T_\infty$. This implies that velocities are scaled on reference sound speed $c_\infty = \sqrt{\gamma R T_\infty}$, time on $d/c_\infty$ and impedance on $\rho_\infty c_\infty$ (note that for non-uniform temperature the local sound speed at the wall might be different from $c_\infty$). The dimensionless frequency, also referred to as Helmholtz number, is computed as $\omega = 2\pi f d/c_\infty$. The dimensionless mean flow axial velocity is denoted by $M$ (the Mach number) and the dimensionless mean temperature is by $T_0$, in other words, $u_0 = c_\infty M(r)$ and $T_0 = T_\infty T(r)$.

Unless mentioned otherwise, we choose the duct length $L = 1$ m, radius $d = 0.15$ m, the reference density $\rho_\infty = 0.5$ kg/m$^3$, and reference temperature $T_\infty = 700$ K, and the gas to be a perfect gas (see Section 2.1.1) with $\gamma = 1.4$. This amounts to a reference sound speed $c_\infty = 531.7$ m/s and pressure $p_\infty = \rho_\infty R T_\infty = 100800$ Pa. We choose the kinematic viscosity $\nu = 6.8 \cdot 10^{-5}$ m$^2$/s.

As mentioned in Section 1.3, we assume that the duct end is reflection free. We justify this by noting that for high dimensionless frequencies, and for modes not too close to cut-off, the reflections are very small. In addition, when the duct is treated, the amplitude of the sound field is reduced, and the influence of the end reflection may be even smaller. Finally, we note that including the duct-end reflection and radiation could obfuscate the propagation effects that we wish to investigate.

6.1 Effect of non-uniform mean flow

First we investigate the influence of the flow profile on the axial wavenumbers. We illustrate the effect of a widening boundary layer by introducing a (simple) flow profile,
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(a) \( M(r) = M_0 \frac{b+2}{2}(1-r^b) \) with \( M_0 = 0.4 \) for various \( b \) values.

(b) Blasius-like profile of (6.2) with \( M_0 = 0.3 \) at several values of \( x = \xi d \).

Figure 6.1: Mean flow profiles with varying boundary layer thickness.

only depending on the parameter \( b \)

\[
M(r) = M_0 \frac{(b+2)}{b}(1-r^b), \quad b \in (2, 3, 5, 10, 20, 50, 100),
\]

see Figure 6.1a. This profile has the same mass flux for all \( b \), equal to that of a flow with uniform temperature and Mach number \( M_0 \).

Figure 6.2a shows the trajectories of the ten axial wavenumbers that are closest to the real axis for dimensionless frequency \( \omega = 10 \), circumferential wavenumber \( m = 5 \) and locally reacting impedance \( Z = 1 + 1i \). It is seen that the influence of the mean flow profile is relatively minor, except for the wavenumber ending at \( k = -11.3 - 3.6i \). For the highest \( b \)-values, when the mean flow profile is closest to uniform, this mode behaves as a surface wave, which is transformed into a mode which does not exhibit this surface wave behavior as the boundary layer is widened. This is also illustrated in Figure 6.2b, which depicts the modulus of the eigenfunctions. This figure shows that the eigenfunction of the third left-running mode \( (\mu = -3) \) changes its character, from surface wave to regular mode, whereas the other modes do not.

Furthermore, it can be seen that the right-running mode starting at \( k = 6.1 + 1.4i \), which is close to cut-on, becomes more cut-off as the boundary layer is widened. The reverse effect can be seen for the left-running mode starting at \( k = -11.4 - 0.8i \), which becomes more cut-on. In the direction of the flow the wave is refracted towards the wall and hence damped more strongly. In the upstream direction the wave is refracted away from the wall, which almost annihilates the damping (see also Section 2.5). This refraction effect increases when the flow becomes less uniform.

Next we consider a more physically inspired profile, which is an approximation of the Blasius solution for the boundary layer development of a flow parallel to a semi-infinite flat plate, see Appendix B. We choose the mean flow profile \( M(r; x) \) at a given (dimensionless) axial location \( \xi = x/d \) with respect to where the flow is still uniform
§ 6.1 Effect of non-uniform mean flow

(a) Trajectory of axial wavenumbers, which move from from blue to red passing the $b$-values $b = 100, 50, 20, 10, 5, 3$ (indicated by dots).

(b) Magnitude of normalized eigenfunctions of the first six left running modes as a function of the radial coordinate, for different values of $b$.

Figure 6.2: Influence of boundary layer thickness for mean flow profile (6.1).
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Figure 6.3: Trajectory of axial wavenumbers for the mean flow profile of (6.2) with $x = 1$ m, $Z = 1.5 + 0.5i$.

(corresponding to the plate edge for the Blasius solution) as

$$M(r; x) = M_0 \left[ \tanh \left( a(r; \xi) b(r; \xi) \right) \right]^{1/b(r; \xi)} , \quad a(r; \xi) = B_0 r / \sqrt{\xi} , \quad b(r; \xi) = a_0 + B_1 r / \sqrt{\xi}$$

with $\xi = x/d$, $B_0 = \lambda \sqrt{U_0 d / \nu}$, $B_1 = a_1 \sqrt{U_0 d / \nu}$, $U_0 = M_0 c_\infty$,

$$a_0 = 1.4853 , \quad a_1 = 0.0555 , \quad \lambda = 0.33206 .$$

(6.2)

where $U_0$ is the dimensional mean flow velocity far away from the wall. See Figure 6.1b for the boundary layer development for axial locations up to $x = 1$ m.

Figure 6.3 shows the axial wavenumber trajectories for a test case with $\omega = 3$, $m = 1$, and a Blasius-like boundary layer with $M_0 = 0.3$ and $x = 1$ m. The impedance of the locally reacting liner has been chosen such that two acoustic and two hydrodynamic surface waves exists (see [100], note the different sign convention). The trajectories, which move from blue to red, consist of two paths designated by the dots; first the impedance is varied from hard-wall to $Z = 1.5 + 0.5i$, after which the mean flow profile is varied from uniform flow to the boundary layer profile. As the boundary layer thickness is increased the hydrodynamic surface wave ceases to exist, which is demonstrated by the fact that the corresponding axial wavenumber disappears to infinity in the lower half of the complex $k$-plane.

To illustrate the effect of a developing boundary layer on the axial wavenumbers we plotted in Figure 6.4 the $k$-trajectories for a test case with $\omega = 10$, $m = 0$ and $Z = 1 + 1i$, as the mean flow profile is gradually changed from the Blasius-like profile corresponding to $x = 10$ cm to the profile corresponding to $x = 1$ m. It can be seen that the influence
§ 6.2 Effect of non-uniform temperature

of the mean flow profile is in this case very small, which is to be expected in view of the very thin boundary layer (99% of the centerline Mach number is reached at a distance from the wall of approximately 1.0–3.5% of the duct radius). The surface wave in the third quadrant is influenced most strongly.

6.2 Effect of non-uniform temperature

To investigate the influence of a temperature gradient we consider a (dimensionless) temperature profile of the form

\[ T(r) = 1 + \frac{1}{2} \tanh(q(r_0 - r)), \]

(6.3)

where the parameter \( q \) is used to set the magnitude of the gradient, and the parameter \( r_0 \) to set its radial location. We take a dimensionless impedance close to hard wall \((Z = 100 + 1i)\), in order to avoid the special embedding of (3.12) for non-uniform flow and temperature combined with a hard wall as described in Section 3.2. The circumferential wavenumber \( m = 0 \). With zero mean flow, the left and right axial wavenumbers are identical except for their sign, so we plot only the right-running modes.

For a hard-walled duct with uniform flow and temperature we have the dispersion relation

\[ a^2 = \frac{1}{T} \left( \frac{(\omega - kM)^2}{T} - k^2 \right). \]

(6.4)
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The radial wavenumber $\alpha$ follows from the boundary conditions; the axial wavenumber is then

$$k = -\frac{\omega M \pm \sqrt{\omega^2 - \alpha^2(T - M^2)}}{T - M^2}. \tag{6.5}$$

For cases where $M^2 \ll T$ (which is reasonable for an APU since $T$ is typically close to 1 and $M$ to 0.3) it follows that

$$k \approx \pm \sqrt{\frac{\omega^2}{T} - \alpha^2} \quad \text{for} \quad M^2 \ll T. \tag{6.6}$$

This means that the cut-on modes move towards the origin as $T$ increases. For the cut-off modes (for which holds $\alpha^2 > \omega^2/T$) that are close to cut-on the wavenumbers move away from the origin, and for the modes that are strongly cut-off ($\alpha^2 \gg \omega^2/T$) there is little influence of the temperature on $k$.

Figure 6.5a shows the trajectories of the axial wavenumbers when the temperature gradually varies from a uniform $T = 1$ to the non-uniform profile of (6.3) with $r_0 = 0.95$ and $q = 30$. The dimensionless frequency $\omega = 10$. This figure illustrates that for this case, with the location of the gradient close to the wall, the behavior of the wavenumbers can be understood like above, in other words, for the axial wavenumbers the situation is very similar to uniform temperature.

Looking at the eigenfunctions depicted in Figure 6.5b it can be seen that the first mode ($\mu = 1$) is strongly influenced by the non-uniform temperature. As waves refract towards the region with the lowest wave speed (the region with the lowest temperature, since $c_0^2 = \sqrt{\gamma R T_0}$) the energy of each mode is mostly concentrated in the region near the wall. This is also visualized in Figure 6.5c, where the time-averaged perturbation ('acoustic') energy flux vector field is plotted on top of the pressure field.

It is also clear from Figure 6.5b that the influence of the temperature gradient decreases for the higher order modes. Let us assume that a mode is made by two waves bouncing back and forth between the duct wall and the center of the duct, these waves having a propagation direction along the vector with axial component $k$ and radial component $\pm \alpha$. (This is a strongly simplified description, which is nevertheless

(a) Temperature profile of (6.3) with $q = 50$ and $r_0 = \{0.65, 0.8, 0.95\}$.

(b) Temperature profile of (6.3) with $r_0 = 0.8$ and $q = \{100, 22, 3, 5\}$.

Figure 6.5: Temperature profiles.
\( \S 6.2 \) Effect of non-uniform temperature

(a) Trajectory of axial wavenumbers, which move from blue to red as the temperature varies from \( T = 1 \) to \( T(r) \).

\[ \mu = 1, \; k = 8.7 + 0.0i \]

\[ \mu = 2, \; k = 7.4 + 0.0i \]

\[ \mu = 3, \; k = 4.4 + 0.0i \]

\[ \mu = 4, \; k = 0.0 + 6.1i \]

(b) Magnitude of normalized eigenfunctions of the first six right running modes for \( T = 1 \) (denoted by \( \infty \)) and \( T(r) \).

(c) Pressure field and time-averaged acoustic intensity vectors for the first mode.

Figure 6.6: Influence of the non-uniform temperature profile \( T(r) \) of (6.3) with \( q = 30 \) and \( r_0 = 0.95 \) (see Figure 6.5a), \( \omega = 10 \), \( m = 0 \) and \( Z = 100 + 1i \).
Figure 6.7: Trajectory of axial wavenumbers, which move from from blue to red as the temperature varies from $T = 1$ to $T(r)$ of (6.3) with $q = 50$ and $r_0 = 0.8$, with $\omega = 20$, $m = 0$ and $Z = 100 + 1i$. The inset is a vertical zoom of the area close to the real axis.

suitable for our purposes; a more precise description of the ray structure of circular duct modes can be found in [27].) The higher radial orders, for which $\alpha$ becomes larger, make a larger angle with the duct axis. Consequently, the wavefront of these modes becomes more perpendicular to, and hence is less influenced by, the radial temperature gradient.

For the next test cases we increase the frequency to $\omega = 20$. We first consider a temperature profile with $q = 50$ and $r_0 = 0.8$. Figure 6.7 shows that only the first order mode moves away from the origin, contrary to what would be expected in view of the average temperature increase. Note also that modes can change from cut-on to cut-off or vice versa.

To investigate the effect of the location $r_0$ of the temperature gradient we now consider a temperature profile with $q = 50$ and $r_0 = \{0.65, 0.8, 0.95\}$. Figure 6.8 illustrates that the steep temperature gradient can produce a tunneling effect. Considering the case for which $r_0 = 0.95$ we see that only the first radial-order mode is fully trapped inside the low temperature region (it is zero elsewhere). For $r_0 = 0.8$ the first two radial-order modes are trapped, and for $r_0 = 0.65$ the first three modes.

Finally we investigate the influence of the magnitude of the temperature gradient by considering a profile with $r_0 = 0.8$ and $q = \{100, 22.3, 5\}$. The tunneling effect is clearly the most pronounced for the steepest temperature gradients. However, it is still present for the smaller temperature gradients as well. As before, for higher radial orders the influence of the temperature profile decreases.
§ 6.2 Effect of non-uniform temperature

Figure 6.8: Magnitude of normalized eigenfunctions of the first six right running modes as a function of the radial coordinate, for $T = 1$ or the temperature profile of (6.3) with $q = 50$ and $r_0 = \{0.65, 0.8, 0.95\}$, with $\omega = 20$, $m = 0$ and $Z = 100 + i$. 
Figure 6.9: Magnitude of normalized eigenfunctions of the first six right-running modes as a function of the radial coordinate, for $T = 1$ or the temperature profile of (6.3) with $q = 50$ and $r_0 = \{0.65, 0.8, 0.95\}$, with $\omega = 20$, $m = 0$ and $Z = 100 + 1i$. 
§ 6.3 Role of source in transmission loss

The transmission loss always depends on the (assumed) distribution of the modal source amplitudes. Very often this source is not (exactly) known. A common approach is then to create a source field that consists of all cut-on modes and a few cut-off modes, where the individual modes have equal modal energy (often referred to as ‘equipartition of energy’) and random phases, in other words, they are treated as being incoherent. In this section we describe how the expected value of the transmission loss can be computed directly by considering the source modes separately.

Each source mode produces a different acoustic field in the entire duct. The combined acoustic field due to all source modes is the sum of these fields. When the source modes have different azimuthal orders, the power of the combined field is equal to the sum of the powers of the separate fields, since the azimuthal eigenfunctions \( e^{im\theta} \) are orthogonal. However, with fixed \( m \) the fields couple radially (irrespective of any possible orthogonality of the radial eigenfunctions), as we will show next. Consequently, in that case the power of the combined field is not equal to the sum of the powers of the separate fields. We will show this in the following.

Suppose that we have a certain set of source mode amplitudes \( A_\mu \) at the duct entrance. If we keep only the \( j \)-th source mode and set all others to zero this gives rise to a certain field at the duct end. Suppose also that the \( j \)-th source mode has an arbitrary phase shift \( \phi_j \). Let us denote the mean density by \( \rho \), the mean flow in the axial direction by \( M \), disturbance pressure, density, axial velocity, temperature and entropy fields respectively by \( p \), \( \rho \), \( u \), \( \tau \) and \( s \). The time average over one period \( \bar{A} = \frac{\omega}{2\pi} \int_0^{2\pi} A(t) dt \) of the product of two time-harmonic perturbations of the form of \( \bar{q} = \bar{q} e^{-i\omega t} \) can be computed as \( \bar{q}_1 \bar{q}_2 = \frac{1}{2} \text{Re}(\bar{q}_1 \bar{q}_2^*) \), where \((\cdot)^*\) is used to denote the complex conjugate. Consequently, the power flux (see Section 2.1.5) through a circular duct cross section due to a single source mode \( j \) can be defined as

\[
\mathcal{P}_j := \pi \text{Re} \int_0^d \left\{ \left[ u_j + \frac{M}{R} p_j \right] \left[ p_j + M R u_j \right]^* + M R \tau_j s_j^* \right\} r dr. \tag{6.7}
\]

The power of the combined field due to two source modes 1 and 2 with phase shifts \( e^{i\phi_1} \) and \( e^{i\phi_2} \) can then be computed as

\[
\mathcal{P}_{12} = \pi \text{Re} \int_0^d \left\{ \left[ u_1 e^{i\phi_1} + u_2 e^{i\phi_2} \right] + \frac{M}{R} \left( \rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2} \right) \right\} \left\{ \left[ p_1 e^{i\phi_1} + p_2 e^{i\phi_2} \right] + M R \left( u_1 e^{i\phi_1} + u_2 e^{i\phi_2} \right) \right\}^* r dr
\]

\[
+ \pi \text{Re} \int_0^d M R \left( \tau_1 e^{i\phi_1} + \tau_2 e^{i\phi_2} \right) \left\{ s_1 e^{i\phi_1} + s_2 e^{i\phi_2} \right\}^* r dr
\]

\[
= \mathcal{P}_1 + \mathcal{P}_2 + \pi \text{Re} \left\{ e^{i(\phi_2 - \phi_1)} \int_0^d \left\{ \left[ u_2 + \frac{M}{R} \rho_2 \right] \left[ p_1 + M R u_1 \right]^* + M R \tau_2 s_1 \right\} r dr \right\}
\]

\[
+ \pi \text{Re} \left\{ e^{i(\phi_1 - \phi_2)} \int_0^d \left\{ \left[ u_1 + \frac{M}{R} \rho_1 \right] \left[ p_2 + M R u_2 \right]^* + M R \tau_1 s_2 \right\} r dr \right\}. \tag{6.8}
\]
More generally, the power of a combined field due to multiple source modes consists of the sum of the powers of the individual fields plus their interaction terms, so generally

\[ P_{\text{tot}} \neq \sum_{j} P_j. \]  

(6.9)

One option to obtain a representative transmission loss while taking into account the random phases of the source modes would be to compute the power of the combined field for multiple realizations of a set of random input phases, and then average the results. This is computationally expensive for many realizations. However, by taking the expected value of (6.8) the cross-terms vanish, since the phases are random. More generally we have

\[ \mathbb{E}\{P_{\text{tot}}\} = \sum_{j} P_j. \]  

(6.10)

Consequently, we can compute the expected value of the transmission loss directly based on the output powers due to separate source modes.

Therefore, in the following we compute the transmission loss as

\[ TL := 10 \log_{10} \left( \frac{\sum_{m,j} P_{\text{in}}^{m,j}}{\sum_{m,j} P_{\text{out}}^{m,j}} \right), \]  

(6.11)

where the sum is over all cut-on modes for a given \((\omega, m)\) combination and add two extra cut-off modes. The extra cut-off modes are included to capture a possible near field that might be scattered into propagating modes after the first segment. We compute the power flux through the duct entrance and exit due to the separate incident modes by means of numerical quadrature. In order to produce easily reproducible results and adhere at the same time to the ‘equipartition of energy’ assumption mentioned earlier, we set the amplitudes of the individual source modes to one. This approximates equal energy per mode, since the eigenfunctions are normalized according to \(\int_{0}^{1} P^2 r dr = 1\), and the power through a hard-walled duct cross section carrying uniform flow is similar to \(\sum_{\mu} \frac{\text{Re}(k_{\mu})}{\omega} |A_{\mu}|^2\). We could have included a factor \(\omega\), but this term cancels in (6.11).

### 6.4 Liner with varying depth

As discussed in the introductory chapter, due to the geometrical constraint of the conically shaped tail of an aircraft, the APU exhaust duct typically has a liner depth which varies in axial direction. Furthermore, strong radial temperature gradients may exist due to the inflow of cold air from the sides. In this section we investigate the transmission loss for some representative test cases based on this knowledge. We consider a case (I) with uniform flow and temperature, a case (II) with non-uniform flow and constant temperature, and a case (III) with both non-uniform temperature and flow.

For these test cases we consider frequencies ranging from \(\omega = 1\) to 20 (which corresponds to dimensional frequencies of approximately 550–11000 Hz). The duct of length 1 m is divided into 12 segments; the first and last segments are hard-walled and have a width of 1 cm. The remaining 10 segments have an equal width and a locally reacting liner with a depth \(d_l\) which varies from 7 to 1 cm in the positive \(x\)-direction (i.e. downstream). The liner is modeled as a Helmholtz resonator having a dimensional impedance

\[ Z = \rho_{\infty} c_{\infty} \left( Z_{0} + i \cot \left( \frac{2\pi f d_l}{c_{\infty}} \right) \right), \]  

(6.12)
where $f$ is the (dimensional) frequency and $Z_0$ the dimensionless face sheet impedance, here $Z_0 = 1.5$. The imaginary part of the dimensionless impedance is plotted for all frequencies in Figure 6.10.

The non-uniform mean flow profile that we use here is in dimensionless form

$$M(r) = 0.3 \tanh(20(1-r)).$$

(6.13)

For the case with non-uniform temperature we use the profile (again in dimensionless form)

$$T(r) = N \left[1 + \frac{1}{2} \tanh(30(0.8-r))\right], \quad N = 1.10594,$$

(6.14)

where the normalization constant $N$ is computed such that this profile has the equivalent mass flux as a constant temperature ($T = 1$) profile.

For the mode-matching the field is expressed as a sum of $\mu_{\text{max}} = 15$ modes in both directions. Bilinear map based (BLM) mode-matching is used; classical mode-matching gave identical results, which are not included here.

The transmission loss is computed based on all modes that are cut-on in the hard-walled segments at a given frequency plus two extra cut-off modes. Figures 6.12, 6.13 and 6.14 show the number of cut-on modes for all ($\omega, m$) combinations for case I, II and III respectively (for $m > \omega$ cut-on modes are not to be expected, so they have not been tested).

Figure 6.15a depicts the transmission loss versus the frequency for case I. For very low frequencies the transmission loss is very small; this can be explained by the fact that the lower radial order modes propagate mostly in the axial direction and hence do not interact with the wall. It can be seen from Figure 6.10 that for $\omega = 7$ and $\omega = 14$ a nearly hard-wall segment (a liner resonance) is close to the source plane. This matches with the local minima at these frequencies in the transmission loss curve.

When the transmission loss is split out for individual $m$-values up to 5, as depicted in Figure 6.15b, we see that the lowest $m$-values are least attenuated, and hence are of the most interest for design calculations. The figure also shows that the transmission loss decays far stronger with frequency if $m$ is high. This trend can be explained on a simplified ray-acoustics picture (for details of the ray structure of duct modes, see [27, 99]). The wave vector of length $\omega$ has axial and radial components $k$ and $\alpha$. For the angle $\theta$ between the wave vector and the duct axis we have $\theta = \arcsin(\alpha/\omega)$. Furthermore, the radial eigenvalue of the first radial-order mode $\alpha_{m1} = j'_{m1} \approx m$, where $j'_{m1}$ is the first zero of the derivative of the Bessel $J_m$ function. Hence, $\theta$ increases with $m$ at fixed frequency, and consequently the mode is damped more strongly. Increasing $\omega$ while keeping $m$ fixed has the opposite effect.

The non-uniform flow results of case II, depicted in Figures 6.16a and 6.16b, are very similar to the uniform flow results of case I. There is a slight increase of the transmission loss, which can be explained by the fact that the acoustic field that propagates downstream is refracted towards the wall, and hence damped more strongly.

Finally, we consider case III for which both the flow and the temperature are non-uniform. Figure 6.17 shows again that the attenuation is relatively small for low frequencies. Overall the transmission loss is of the order of 10 dB higher compared to the uniform temperature cases. It appears that the refraction of the acoustic energy towards the lower temperature region near the wall is very beneficial for the transmission loss. The local minima are now not visible anymore. As before, only the lowest $m$-values are of most interest as they are the least attenuated.
Figure 6.10: Imaginary part of dimensionless impedance versus axial location (m); the liner depth $d$ varies from 7 to 1 cm for all frequencies.
§ 6.4 Liner with varying depth

Figure 6.11: Non-uniform mean flow and temperature profiles of (6.13) and (6.14).

Figure 6.12: Number of cut-on modes for all \((\omega, m)\) combinations, uniform flow and uniform temperature (case I).
Figure 6.13: Number of cut-on modes for all \((\omega, m)\) combinations, non-uniform flow and uniform temperature (case II).

Figure 6.14: Number of cut-on modes for all \((\omega, m)\) combinations, non-uniform temperature and non-uniform flow (case III).
§ 6.4 Liner with varying depth

Transmission loss versus Helmholtz number $\omega$ for case with uniform flow and uniform temperature (case I).

(a) Transmission loss based on cut-on modes for all $m$’s.

(b) Transmission loss based on cut-on modes for separate $m = 0, \ldots, 5$.  

Figure 6.15: Transmission loss versus Helmholtz number $\omega$ for case with uniform flow and uniform temperature (case I).
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(a) Transmission loss based on cut-on modes for all $m$'s.

(b) Transmission loss based on cut-on modes for separate $m = 0, \ldots, 5$.

Figure 6.16: Transmission loss versus Helmholtz number $\omega$ for case with non-uniform flow and uniform temperature (case II).
(a) Transmission loss based on cut-on modes for all $m$’s.

(b) Transmission loss based on cut-on modes for separate $m = 0, \ldots, 5$.

Figure 6.17: Transmission loss versus Helmholtz number $\omega$ for case with both non-uniform flow and temperature (case III).
Chapter 7

Concluding remarks

In this concluding chapter we formulate some open questions which could serve as the starting point of future work.

In Section 2.5 we presented an approximate high-frequency solution of the Pridmore-Brown equation with non-uniform flow in the form of a WKB approximation. As pointed out, analytical solutions of this equation are rare, which is unfortunate as they are very useful for comparison with numerical computations and can provide analytical insight. It is suggested that next to this WKB approximation a larger collection of asymptotic results for various flow and temperature profiles will be derived. One example of an ‘exact’ analytical solution that could be considered is the case of cross-wise waves in parallel flow [45]; the frequency could be chosen such that $k = 0$ is a solution (hence the mode is $x$-independent), in which case the Pridmore-Brown equation reduces to Bessel’s equation, and the other components follow directly from a system like (4.25). Furthermore, next to the exact solution for a linear flow velocity profile, also the analytical solution for an exponential boundary layer [25] could be considered.

In Section 3.1 we discussed the use of the collocation code COLNEW that is used to numerically solve the boundary value problem consisting of the Pridmore-Brown equation subject to the pertinent boundary conditions. Even though this code is very efficient, for high order modes the computations become increasingly expensive due to the highly oscillatory nature of the eigenfunctions. Specialized methods for highly oscillatory ODEs exist, like Magnus methods [54, 122], which have been successfully applied to Bessel’s and Airy’s equations. It might be interesting to investigate whether the Pridmore-Brown equation is amenable to such methods.

In order to describe the acoustic field as a sum of eigenmodes, and use this in a mode-matching approach, it is very important to find all relevant modes with certainty. For simple cases of an absorbing wall combined with uniform flow and temperature this can be achieved with the aid of a contour-integration based root-finding method, as discussed in Section 3.3. The axial wavenumbers (eigenvalues) are the roots of an analytic function that follows from substituting the solutions of the governing Bessel’s equation into the wall boundary condition, hence this function is available in closed form. For non-uniform flow and temperature we used a path-following approach, which certainly improves certainty about finding all the modes, but it is never completely fail-safe: the paths of two modes may come close to each other, which might result in the path-following algorithm jumping from one mode to another, or the path may traverse an unanticipated singular situation for a certain value of the continuation parameter.
This raises the question whether a contour-integration based root-finding approach could also be applied for the non-uniform flow case. In that case the roots are sought of a function of the axial wavenumber that follows from substituting a numerical solution of the Pridmore-Brown equation for a certain fixed $k$ into the wall boundary condition. At least two difficulties will need to be addressed: firstly, this function of $k$ (the dispersion relation) is not analytic in the whole complex plane due to the continuous part of the spectrum, so the integration contour needs to be deformed. Secondly, each function evaluation consists of the solution of an initial value problem (IVP), instead of a Bessel-function evaluation. Even though the solution of this IVP is less expensive than the solution of the BVP, it might render the whole procedure very costly.

In Chapter 4 a new mode-matching procedure for non-uniform flow was presented, based on closed-form expressions for integrals of weighted combinations of products of Pridmore-Brown modes. We noted that these integrals, which serve the same purpose as a standard inner product in the classical mode-matching method, are in fact not really inner products, so we referred to them as bilinear maps. By their construction these bilinear maps can be considered as the natural generalizations of the Bessel function product integrals. Although we did not encounter it in practice, the fact that the bilinear map is not an inner product might lead to problems in the mode-matching procedure (e.g. ill-conditioned scattering matrices, since the Pridmore-Brown modes might not be independent enough as they can not be proven to be orthogonal). It would therefore be an important step theoretically if a real inner product of Pridmore-Brown modes that can be evaluated in closed-form could be constructed. We also mention here that the bilinear maps might be useful to formulate a solvability condition for slowly varying solutions of WKB-type.

As mentioned in Chapter 4 we checked whether our numerical mode-matching solutions satisfied the edge condition (finite energy around the liner discontinuity) by investigating the convergence of the found modal amplitudes. This was for locally reacting liners. We also mentioned that for non-locally reacting liners the situation can be compared to that of a thin iris. For a thin iris it was shown [73] that the ratio of the number of modes in each of the adjacent segments has to be chosen proportionally to the ratio of the radii in order to satisfy the edge condition. For non-locally reacting liners it is expected that the number of extra modes $\nu_{\text{add}}$ has to be chosen according to a similar criterion; however, this is not completely understood and therefore requires further research.

Finally, we remark that it is of great practical interest to the aircraft industry to develop models for cases with developing (i.e. axially varying) flow. Concerning the possible application of mode-matching for such cases we note that the bilinear map in the current form can not be used to match segments with different flow profiles, since its construction was based on the fact that the governing ODE (including its coefficients, as for example the mean flow profile) is the same in both segments. Moreover, it is unclear how the mean flow should be modeled; first of all, an analytical expression (that satisfies the continuity and momentum equations) for a developing flow is required, and also the discontinuities at the interfaces between segments (i.e. the jumps of the mean flow velocity along the streamlines) have to be handled appropriately. Additionally, it is unclear how discontinuities in the mean flow should be accounted for in the matching conditions; at any rate straightaway application of continuity of acoustic pressure and velocity will be ill-advised.
Appendix A

Construction of bilinear map

For modal solutions of the form

$$\rho_1, p_1, v_1 = [R(y, z), P(y, z), U(y, z)e_x + V(y, z)e_y + W(y, z)e_z]e^{ikx-iot} \quad (A.1)$$

which are governed by (2.105) we have

$$-i\Omega P + i\rho_0 c_0^2 kU + \rho_0 c_0^2 (V_y + W_z) = 0, \quad (A.2a)$$

$$-i\rho_0 \Omega U + \rho_0 (u_{0y} V + u_{0z} W) + ikP = 0, \quad (A.2b)$$

$$-i\rho_0 \Omega V + P_y = 0, \quad (A.2c)$$

$$-i\rho_0 \Omega W + P_z = 0, \quad (A.2d)$$

where $\Omega = \omega - ku_0$, and $R$ follows directly from the other amplitudes, for example with 2.105c. When the individual equations in (A.2) are multiplied respectively by $\psi, \xi, \phi$ and $\chi$ and added together it follows that

$$(-i\rho_0 \xi + ik\rho_0 c_0^2 \psi)U + \rho_0 (-i\Omega \psi + u_{0y} \xi)V + \rho_0 (-i\Omega \chi + u_{0z} \xi)W - i(\Omega \psi - k \xi)P$$

$$+ \rho_0 c_0^2 V_y \psi + \rho_0 c_0^2 W_z \psi + P_y \phi + P_z \chi = 0. \quad (A.3)$$

A cross-wise divergence can be split off, i.e.

$$\rho_0 (-i\Omega \xi + ik c_0^2 \psi) U + \rho_0 (-i\Omega \phi + u_{0y} \xi - c_0^2 \psi_y) V$$

$$+ \rho_0 (-i\Omega \chi + u_{0z} \xi - c_0^2 \psi_z) W + (-i\Omega \psi + ik \xi - \phi_y - \chi_z) P$$

$$+(\rho_0 c_0^2 V \psi + P \phi)_y + (\rho_0 c_0^2 W \psi + P \chi)_z = 0. \quad (A.4)$$

Suppose that $\psi, \xi, \phi, \chi$ satisfy the following ‘associated’ system

$$\rho_0 (-i\hat{\Omega} \xi + ik \psi) = 0, \quad (A.5a)$$

$$\rho_0 (-i\hat{\Omega} \phi + u_{0y} \xi - c_0^2 \psi_y) = 0, \quad (A.5b)$$

$$\rho_0 (-i\hat{\Omega} \chi + u_{0z} \xi - c_0^2 \psi_z) = 0, \quad (A.5c)$$

$$-i\hat{\Omega} \psi + ik \xi - \phi_y - \chi_z = 0, \quad (A.5d)$$

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with $\tilde{k}$ and $\tilde{\Omega} = \omega - \tilde{k}u_0$. After multiplying these equations respectively by $U$, $V$, $W$ and $P$ and subtracting them from (A.4) we have

$$
(r_0c_0^2 V \psi + P \phi)_y + (r_0c_0^2 W \psi + P \chi)_z
= -i(k - \tilde{k}) \left[ (u_0r_0 U + P)\xi + (r_0c_0^2 U + u_0 P)\psi + r_0u_0(\phi V + \chi W) \right]. 
$$

(A.6)

Note that (A.5) is equivalent to (A.2) after the transformation

$$
k \equiv \tilde{k}, \quad r_0c_0^2 \tilde{\Omega} \psi \equiv \tilde{P}, \quad \tilde{\Omega} \phi \equiv \tilde{V}, \quad \tilde{\Omega} \chi \equiv \tilde{W}, \quad r_0\tilde{\Omega}^2 \xi \equiv \tilde{k}\tilde{P}. 
$$

(A.7)

This explains why in Section 4.1.2 we multiply the individual equations of (4.12) with $P/r_0c_0^2$, $kP/r_0\Omega$, $-V$ and $-W$ respectively to obtain the bilinear map.
Appendix B

Approximate Blasius solution

In this chapter we discuss the classical solution found by Blasius of the problem of the development of a boundary layer along a flat plate, see for example [60]. In the following we use a Cartesian coordinate system \((x, y)\) with velocity components \(u\) and \(v\). Consider a uniform flow \(U_0\) parallel to a semi-infinite flat plate oriented along the \(x\) direction \((y = 0, x \in [0, \infty))\), where \(y\) is the coordinate orthogonal to the plate. Assume that we have a Reynolds number \(Re_x\), a boundary layer thickness \(\delta(x)\) and a scaled \(y\) coordinate \(\eta\) as

\[
Re_x = \frac{U_0 x}{\nu}, \quad \delta(x) = \frac{x}{\sqrt{Re_x}}, \quad \eta = \frac{y}{\delta(x)},
\]

where \(\nu\) is the kinematic viscosity. It can be shown that the velocity \(u(y)\) can be written as

\[
u f'(\eta), \quad \eta = \frac{y}{\delta(x)},
\]

where the shape of boundary layer (equivalent at each \(x\)) follows from the solution \(f\) of the following boundary value problem

\[
\begin{align*}
&f''' + \frac{1}{2} f f'' = 0, \\
&f(0) = f'(0) = 0, \quad f'(\infty) = 1.
\end{align*}
\]

It appears that \(f'\) is well approximated by

\[
f'(\eta) \approx \left[\tanh\left((\lambda \eta)^{a(\eta)}\right)\right]^{1/a(\eta)}, \quad a(\eta) = a_0 + a_1 \eta, \quad a_0 = 1.4853, \quad a_1 = 0.0555, \quad \lambda = f''(0) \approx 0.33206.
\]

This is evidenced by Figure B.1, where we compare the numerical solution of (B.3) with the approximation (B.4). The numerical solution of (B.3) is found with the Matlab routine \texttt{ode45} (which is based on the explicit Runge-Kutta (4,5) formula of Dormand and Prince), where we used a relative error tolerance of \(10^{-13}\). It can be seen that the difference with the approximation (B.4) is of the order \(10^{-4}\).

We now switch to cylindrical coordinates and make everything dimensionless. Based on the approximation of B.4 we choose the mean flow profile \(M(r)\) at a given axial loca-
Appendix B. Approximate Blasius solution

Figure B.1: Numerical solution of (B.3) compared to the approximation (B.4).

ution ζ (non-dimensional) away from the plate edge as

\[ M(r; \xi) = M_0 \left[ \tanh \left( a(r; \xi)^b(r; \xi) \right) \right]^{1/b(r; \xi)}, \quad a(r; \xi) = B_0 r/\sqrt{\xi}, \quad b(r; \xi) = a_0 + B_1 r/\sqrt{\xi} \]

with \( \xi = x/d, \quad B_0 = \lambda \sqrt{U_0 d/\nu}, \quad B_1 = a_1 \sqrt{U_0 d/\nu}, \quad U_0 = M_0 c_\infty. \) (B.5)

where \( d, U_0, c_\infty \) and \( \nu \) are the duct radius, mean flow velocity, reference sound speed and kinematic viscosity (all dimensional).
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Summary

Sound propagation in lined ducts with parallel flow

Over the past decades aeroacoustic research for lined flow ducts (duct acoustics) was primarily aimed at reducing the noise levels in inlet and exhaust ducts of the main turbofan engines of aircraft. Recently the so-called ramp noise—to which workers are exposed while the aircraft is parked on the ground—has been given more attention, as regulations have become more stringent.

A large contribution of the ramp noise comes from the auxiliary power unit (APU), which is a turbine engine in the tail of the aircraft that produces power while the main engines are switched off. The APU exhaust duct, which is typically straight and circular-cylindrical, carries a non-uniform flow with strong temperature gradients due to the inflow of cold air from the sides. To reduce the noise coming out of the APU exhaust duct, its walls are generally treated with an acoustically absorbing lining, which typically is a locally reacting liner in the form of a honeycomb structure, or a non-locally reacting liner in the form of a series of hollow annular segments, each covered by a porous facing sheet. The liner depth typically varies along the axial direction due to the conical geometry of the aircraft fuselage tail.

To find an optimum for the trade-off between minimal weight and noise emissions there is a need for more insight in the main noise source propagation mechanisms and accurate design tools. This motivates the main goal of this work, which is to provide semi-analytical solutions for the propagation of sound in lined flow ducts with parallel flow and strong thermal gradients, and axially varying lining. We formulate the sound propagation problem in terms of acoustic duct modes. These modes are solutions of the Pridmore-Brown equation; together with suitable boundary conditions this equation forms a boundary value problem that describes small (acoustic) perturbations of a parallel mean flow with transverse temperature gradients. We present some asymptotic solutions of the Pridmore-Brown equation, which are used to validate our numerical approach.

For non-uniform flow and temperature we solve the Pridmore-Brown equation numerically in a robust and efficient manner with the aid of the COLNEW code, which solves boundary value problems in ordinary differential equations by collocation. To ensure that we find all relevant solutions we use a path-following (or continuation) approach, where we start from an easy solution (e.g. uniform flow and temperature) and trace the solution when the relevant problem parameters are varied to the values of interest. Our path-following approach is refined by using a prediction-correction scheme, where the prediction is found by linear extrapolation of the previous solutions. The correction step is then an updated solution by COLNEW, with the prediction as
the starting value.

For uniform flow and temperature an exact dispersion relation is available (in terms of Bessel functions), both for locally and non-locally reacting boundary conditions. This enables us to find the eigenvalues (modal axial wavenumbers) as the roots of an analytic function. To ensure that all relevant eigenvalues are found numerically we employ the root-finding method of Davies (a derivative-free adaptation of the method of Delves and Lyness), which is based on complex contour integration. This method guarantees that all roots inside a given area of the complex plane are found, whereas a Newton method only converges to all of the roots if it is started from sufficiently close initial guesses.

A suitable approach to compute sound propagation for a geometry that consists of several annular segments with different liner properties is mode-matching. Here, the acoustic field is expressed as a summation of modes, and the unknown modal amplitudes are found by imposing continuity of the acoustic pressure and axial velocity fields at the axial location of the liner discontinuity. The matching is based on the projection (by inner products) of the continuity conditions onto a suitable set of test functions. The scattering matrix formalism is used to compute the combined propagation through multiple interfaces. This approach is numerically stable because it only considers the propagation of cut-off modes in the direction of exponential decay.

For uniform flow the necessary inner products can be found in closed form if Bessel functions are used as test functions (the classical approach). For non-uniform flow however, numerical quadrature is required to compute the inner products for the classical approach. In this dissertation a new bilinear form is presented, which is an integral of a weighted combination of products of Pridmore-Brown modes that can be evaluated in closed form. As this bilinear form resembles an inner product of Pridmore-Brown functions, it can be used instead of the standard inner product to form the basis of a new mode-matching procedure. Since this new approach avoids the inherently inaccurate numerical quadrature of oscillating functions we conclude that it is both more accurate and cheaper than the classical approach, and demonstrate this by numerical experiments.

The availability of the numerical Pridmore-Brown eigenfunctions enables us to consider asymptotic solutions in the form of slowly varying modes of WKB type for the case of an impedance that is slowly varying in the axial direction due to varying liner depth. We compare these asymptotic results with mode-matching results and illustrate that for most APU exhaust duct configurations, for which the impedance is not slowly varying due to liner resonances, the asymptotic results are not applicable.

Finally, the developed methods are applied to illustrate some refraction effects due to non-uniform mean flow and temperature, and we describe how this influences the sound attenuation. We also discuss an approach to model the noise source for transmission loss calculations, when the exact modal amplitude distribution for the noise source is not known (which is generally the case in practice). Based on this approach we consider the transmission loss of three test cases that are typical of a realistic APU exhaust duct geometry. We conclude that especially the temperature non-uniformity is very beneficial for the attenuation, and that modes of the lowest circumferential orders are most relevant for design calculations since these are the least attenuated.
Samenvatting

Geluidsvoortplanting door akoestisch beklede kanalen met parallelstroming

Gedurende de afgelopen decennia was aero-akoestisch onderzoek voor stromingskanalen met wandbekleding (kanaalakoestiek) voornamelijk gericht op geluidsreductie in de in- en uitlaatkanalen van de hoofdmotoren (turbofans) van vliegtuigen. Sinds enige tijd heeft ook het zogenaamde ramp-lawaai—waaraan grondpersoneel wordt blootgesteld terwijl het vliegtuig aan de grond staat—meer aandacht gekregen, aangezien de regelgeving strenger is geworden.

Een belangrijke bijdrage aan het ramp-lawaai wordt geleverd door de Auxilliary Power Unit (APU), een turbinemotor in de staart van een vliegtuig die vermogen levert zolang de hoofdmotoren uitgeschakeld zijn. Het uitlaatkanaal van de APU, dat doorgaans recht en cirkelcylindrisch is, voert een niet-uniforme stroming met sterke temperatuurgradiënten die veroorzaakt worden door het binnenstromen van koude lucht via de zijkanten. Om het lawaai uit het APU-uitlaatkanaal te verminderen, wordt de wand over het algemeen bekleed met geluidsabsorberend materiaal: doorgaans een lokaal-reagerende bekleding in de vorm van een honingraatstructuur, of een niet lokaal-reagerende bekleding in de vorm van een reeks holle, annulaire segmenten, ieder bedekt met een poreuze afdekplaat. De diepte van de wandbekleding varieert meestal in de lengterichting vanwege de conische vorm van de staart van de vliegtuigrump.

Om een goede afweging te kunnen maken tussen minimaal gewicht en zo min mogelijk lawaai bestaat de behoefte aan meer begrip van de belangrijkste geluidsvoortplantingsmechanismen en aan nauwkeurige ontwerpgereedschappen. Dit motiveert het hoofddoel van dit onderzoek, namelijk het beschikbaar maken van semi-analytische oplossingen voor geluidsvoortplanting in akoestisch beklede kanalen met parallelstroming en sterke temperatuurgradiënten, en axiaal variërende wandeigenschappen. Wij formuleren het geluidsvoortplantingsprobleem in termen van akoestische kanaalmodes. Deze modes zijn oplossingen van de Pridmore-Brown-vergelijking; samen met geschikte randvoorwaarden vormt deze vergelijking een randwaardeprobleem dat kleine (akoestische) verstoringen van een parallelstroming met transversale temperatuurgradiënten beschrijft. We presenteren enkele asymptotische oplossingen van de Pridmore-Brown-vergelijking, die gebruikt worden om onze numerieke aanpak te valideren.

Voor niet-uniforme stroming en temperatuur bepalen we numerieke oplossingen van de Pridmore-Brown-vergelijking op een robuuste en efficiënte wijze door gebruikmaking van de COLNEW computercode, die randwaardeproblemen in termen van ge-
wone differentiaalvergelijkingen oplost met behulp van collocatie. Om te garanderen dat alle relevante oplossingen gevonden worden, gebruiken we een padvolgende (of continueringen-) aanpak, waarbij we starten met een gemakkelijke oplossing (bijvoorbeeld uniforme stroming en temperatuur) en vervolgens deze oplossing volgen terwijl de relevante parameters worden gevarieerd tot de gewenste waarden bereikt zijn. Onze padvolgingsaanpak is verfijnd door gebruikmaking van een predictie-correctie-regeling, waarbij de predictie wordt bepaald door lineaire extrapolatie van de voorwaande oplossingen. De correctiestap bestaat vervolgens uit het aanpassen van de oplossing door middel van COLNEW, met de predictie als startwaarde.

Voor uniforme stroming en temperatuur is een exacte dispersierelatie beschikbaar (door gebruik te maken van Besselfuncties), zowel voor lokaal- als voor niet-lokaal-reagerende randvoorwaarden. Dit maakt het mogelijk om de akoestische modes te vinden door hun axiale golfgetallen op te lossen als de nulpunten van een analytische functie. Om te garanderen dat alle relevante oplossingen worden gevonden, passen we voor het zoeken van nulpunten de methode van Davies toe, die gebaseerd is op contourintegratie in het complexe vlak (een aanpassing van de methode van Delves and Lyness zodanig dat afgeleiden niet benodigd zijn). Deze methode garandeert dat alle nulpunten binnen een bepaald gebied van het complexe vlak worden gevonden, dus dat er geen afhankelijkheid is van de beginschattingen, zoals bij de methode van Newton.

Een geschikte aanpak om geluidsvoortplanting te berekenen voor een configuratie die bestaat uit meerdere annulaire segmenten met verschillende wandeigenschappen is mode-matching. Hierbij wordt het akoestische veld beschreven als een sommatie van modes en worden de onbekende modale amplitudes gevonden door het opleggen van de continuïteit van akoestische druk en axiaalsnelheid op de plaats waar de wandeigenschappen discontinu zijn. Het koppelen van de akoestische velden gebeurt gebeurt op basis van de projectie van de continuïteitsvoorwaarden op een geschikte basis van testfuncties. Het scattering matrix-formalisme wordt toegepast om de geluidsvoortplanting door een reeks van opeenvolgende segmenten te berekenen. Deze methode is numeriek stabiel aangezien de voortplanting van niet-propagerende modes alleen wordt beschouwd in de richting van exponentiële afname.

Voor uniforme stroming kunnen de benodigde inproducten worden gevonden als analytische uitdrukkingen indien Besselfuncties worden gebruikt als testfuncties (de klassieke aanpak). Echter, in het geval van niet-uniforme stroming is numerieke integratie vereist om de inproducten te berekenen bij toepassing van de klassieke aanpak. In dit proefschrift wordt een nieuwe bilineaire afbeelding voorgesteld in de vorm van een integraal van een gewogen combinatie van producten van Pridmore-Brown modes waarvan de waarde kan worden berekend met behulp van een analytische uitdrukking. Aangezien deze bilineaire afbeelding lijkt op een inproduct van Pridmore-Brown modes kan deze worden toegepast in plaats van het standaard integraal-inproduct om hiermee de basis te vormen van een nieuwe mode-matching-procedure. Omdat met deze nieuwe aanpak de inherent onnauwkeurige numerieke integratie van oscillatorische functies vermeden wordt, is deze aanpak nauwkeuriger numeriek goedkoper dan de klassieke aanpak, wat aangetoond wordt met numerieke experimenten.

De beschikbaarheid van de numerieke Pridmore-Brown modes stelt ons in staat om asymptotische oplossingen in de vorm van langzaam variërende moden van het WKB-type te beschouwen, voor het geval dat de impedantie langzaam varieert in de lengterichting vanwege de variërende wanddikte. We vergelijken deze asymptotische
benaderingen met de resultaten van mode-matching en laten zien dat voor de meeste
APU-uitlaatkanalen, waarvoor geldt dat de impedantie niet langzaam varieert als ge-
volg van resonanties in de wand, deze asymptotische benaderingen niet toepasbaar
zijn.

Ten slotte worden de ontwikkelde methodes toegepast om enkele afbuigingseffecten
te illustreren die het gevolg zijn van niet-uniforme stroming en temperatuur, en bespre-
ken we hoe dit de geluidsdemping beïnvloedt. Daarnaast presenteren we een aanpak
om bij transmissieverliesberekeningen het brongeluid te modelleren als de exacte mo-
dale verdeling hiervan niet bekend is (wat meestal het geval is in de praktijk). Op
basis van deze aanpak beschouwen we het transmissieverlies voor drie testgevallen
die representatief zijn voor een realistische configuratie van een APU uitlaatkanaal.
We concluderen dat in het bijzonder de niet-uniformiteit van de temperatuur een erg
gunstige invloed heeft op de demping, en dat modes met een laag omtreksgolfgetal
het meest relevant zijn voor ontwerpberekeningen aangezien deze het minst worden
gedempt.
Curriculum Vitae

Martien Oppeneer was born on the 28th of May 1982 in Axel, The Netherlands. After finishing his pre-university education (VWO) in 2000 at the Zeldenrustcollege in Terneuzen he began his studies in Electrical Engineering at the TU Eindhoven. As part of his Master's education he did internships at the Netherlands Organisation for Applied Scientific Research (TNO) on the topic of radio wave propagation of cellular networks, and at the Electromagnetics department of the TU Eindhoven on the topic of thin-wire antennas.

In 2008 he conducted his Master's thesis research under the supervision of prof.dr. A.C. Cangellaris at the University of Illinois at Urbana-Champaign (UIUC) where he worked on numerical models for multi-terminal resistance extraction of the power distribution network of microchips, using algebraic multigrid based on element interpolation (AMGe) and stochastic collocation techniques. He graduated on this topic within the Electromagnetics (EM) group of prof.dr. A.G. Tijhuis at the Department of Electrical Engineering of the TU Eindhoven in 2010. During his studies he has been an active member of the student organization VGSEi, of which he was president from October 2003 to October 2004.

From September 2009 he was employed as a PhD student at the Dutch National Aerospace Laboratory (Nationaal Lucht- en Ruimtevaartlaboratorium—NLR), where he worked on a PhD project in cooperation with the Center for Analysis, Scientific Computing and Applications (CASA) under the supervision of prof.dr. R.M.M. Mattheij, dr. S.W. Rienstra, and dr. P. Sijtsma (NLR); the results are presented in this dissertation. As a part of this project he did a short internship in 2012 at the Acoustics and Environment Department of Airbus in Toulouse, France.

Since March 2014 he is employed as a Scientific Software Engineer at VORtech Computing in Delft.
Dankwoord / Acknowledgements

Binnen de antropologie wordt wel de term *rite de passage* gebruikt om een overgangsritueel aan te duiden dat de verandering van iemands sociale status markeert, zoals bijvoorbeeld een huwelijk. Een *rite de passage* gaat vaak gepaard met beproevingen en veel symboliek. De promotieplechtigheid kan wellicht ook als een voorbeeld hiervan worden gezien, een soort *coming of age* in wetenschappelijke zin. Het volbrengen en afronden van het promotietraject dat zijn weerslag gevonden heeft in dit proefschrift is voor mij op sommige momenten werkelijk een beproeving geweest. Hoewel een doctorsstitel een bewijs is van wetenschappelijke zelfstandigheid, is de hulp en het getoonde vertrouwen van anderen op sommige momenten voor mij onontbeerlijk geweest, zowel op inhoudelijk als op persoonlijk vlak.

Voor dit project ben ik in dienst geweest bij de afdeling AVHA van het Nationaal Lucht- en Ruimtevaartlaboratorium (NLR) en heb ik een aanzienlijk deel van mijn tijd doorgebracht bij het Centre for Analysis, Scientific computing and Applications (CASA) van de faculteit Wiskunde en Informatica van de TU Eindhoven. Allereerst dank ik het NLR voor het financieel mogelijk maken van dit project en CASA voor het bieden van een interessante multi-disciplinaire wetenschappelijke omgeving. Ik heb het waardevol gevonden om deel uit te maken van zowel de meer bedrijfsmatige omgeving van het NLR, waar een sterke link is met de experimentele wereld, als de wetenschappelijke en meer theoretische omgeving van de TU.

Ik dank mijn eerste promotor Bob Mattheij voor het geschonken vertrouwen vanaf het begin, de vele inhoudelijke discussies gedurende het project en het op vriendelijke en tegelijkertijd resolute wijze bewaken van de voortgang tijdens de afrondende fase. Ook dank ik Pieter Sijtsma en Sjoerd Rienstra, de geestelijke vaders van dit project. Ik dank Pieter, niet alleen voor het feit dat ik met ‘zijn’ BAHAMAS een uitstekend beginpunt voor mijn eigen onderzoek had en voor het trekken van mijn project binnen het NLR, maar ook voor het laten zien hoe je geavanceerde wiskunde in kunt zetten voor concrete technische toepassingen. Ik ben ook zeer veel dank verschuldigd aan Sjoerd; voor zijn fundamentele inzichten en diepgaande kennis op het gebied van aeroakoestiek, voor de inspirerende discussies en creatieve ideeën, voor het demonstreren van het belang van goed modelleren, en voor het regelmatig (soms wat uitgebreid) uitwijzen over een onderwerp, al dan niet wetenschappelijk van aard. Ik dank Barry Koren voor zijn bereidwilligheid om als tweede promotor te fungeren en voor zijn betrokkenheid gedurende de afrondende fase van het project. I would also like to thank the professors Schröder, Hoeijmakers and Slot for carefully evaluating my thesis, and professor Aarts for being the chairman during the defense ceremony.
Ik heb mij binnen de afdeling AVHA van het NLR goed thuis gevoeld; ik wil mijn huidige chef Joost Hakkaart en zijn voorganger Christophe Hermans hartelijk danken voor de ondersteuning en prettige samenwerking. Ook mijn overige (oud-)afdelingsgenoten uit de Noordoostpolder en de andere collega’s bij het NLR dank ik hartelijk voor de goede sfeer. De therapeutische boswandelingen hebben daaraan zeker bijgedragen.

Also at the CASA department I very much enjoyed the friendly and open atmosphere. I would like to thank the current and former PhD students, postdocs and staff members for that. In het bijzonder dank ik Werner Lazeroms voor de prettige samenwerking v.w.b. de WKB-oplossingen die beschreven zijn in Hoofdstuk 5. At CASA an atmosphere is fostered in which it is easy to cross the gaps between people from very different backgrounds, both scientifically as well as culturally, which is very valuable.

This PhD project was motivated by the practical problem of noise coming from an APU exhaust duct, a problem that is studied extensively at Airbus. For this reason I spent some time in Toulouse during the fall of 2012. This has been a great learning experience, and also in other respects very enjoyable due to the friendly people, the nice climate and the good food. Je remercie Maud Lavieille pour l’avoir rendu possible.

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