Non-backtracking lace expansion

The NoBLE project
Non-backtracking lace expansion

The NoBLE project

PROEFSCHRIFT

der verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
rector magnificus, prof.dr.ir. van Duijn, voor een
commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen
op dinsdag 2 juli 2013 om 16.00 uur

door

Robert Jörg Fitzner

geboren te Berlijn, Duisland
Dit proefschrift is goedgekeurd door de promotor:

prof.dr. R.W. van der Hofstad
Summary

Non-backtracking lace expansion

We study nearest-neighbor statistical mechanical models on the $d$-dimensional lattice above, but close to, their so-called upper critical dimension. Above the upper critical dimension these models are expected to behave similarly as the respective model on a tree. The corresponding model on the tree is called the mean-field model, due to the fact that in high dimensions far away vertices behave almost independently. In our analysis we rely upon the non-backtracking lace expansion (NoBLE), an adaptation of the so-called lace expansion. The lace expansion is a perturbation technique that requires a small parameter. For nearest-neighbor models this parameter equals the inverse dimension $1/d$. Thus, this technique is restricted to sufficiently high dimensions.

Since the ground breaking result of Takashi Hara and Gordon Slade in 1990-1992 the lace expansion has become an indispensable tool to prove mean-field behavior of statistical mechanical models. For dimensions above, but close to, their upper critical dimension the parameter $1/d$ is not small enough to use perturbative techniques directly.

The NoBLE perturbs around the non-backtracking random walk, a relatively simple object that is, in high dimensions, close to the original statistical mechanical model. In particular, non-backtracking walk is closer to the original model than simple random walk, which is used in the classical lace expansion. For this reason the perturbation in our method is one order $1/d$ smaller so that it applies to dimensions closer to the upper critical dimension.

We rely on explicit numerical computations, leading to a computer-assisted proof, to verify in which dimension we can prove mean-field behavior.

We derive the NoBLE for four different models: self-avoiding walk, lattice trees, lattice animals and percolation. The common feature of all these models is that they have a phase transition at a certain critical value. For percolation this critical value is the critical bond probability and for the other three models the critical value is the radius of convergence of the Green's function. Using the lace expansion Hara and Slade proved mean-field behavior in for SAW in $d > 4$. This is the optimal
result in the sense that mean-field behavior is not expected for $d \leq 4$. For lattice trees and animals Hara and Slade proved mean-field behavior in sufficiently high dimensions, but the precise dimension above which the results apply is unknown. In this thesis we prove that the mean-field behavior holds in $d \geq 20$ for lattice trees and in $d \geq 21$ for lattice animals.

The grand challenge is to prove that nearest-neighbor percolation above 6 dimensions displays critical mean-field behavior. Currently such results are only known above 18 dimensions. The computations necessary for this are so involved that they have never been published. With an analysis that is substantially simpler we prove critical mean-field behavior in $d > 37$. Moreover, we create a second analysis, based upon ideas of Hara and Slade, to prove critical mean-field behavior in $d > 14$. The documented implementation in Mathematica of the computer-assisted proof is available at the website of the author. This will make the method more transparent and, more importantly, verifiable. Further, it allows other researchers to obtain bounds for the critical values and other quantities of interest.
# CONTENTS

## 1.8 Structure of the thesis ........................................... 33

## 2 Expansions ........................................................... 35

### 2.1 Classical lace expansion vs. NoBLE .......................... 35

### 2.2 Self-avoiding walk .............................................. 44

#### 2.2.1 The algebraic derivation of the expansion ................. 44

#### 2.2.2 Completion of the derivation of the expansion ............ 47

#### 2.2.3 Interpretation of the lace-expansion coefficients ......... 48

### 2.3 Lattice trees and animals ....................................... 51

### 2.4 Percolation ......................................................... 58

#### 2.4.1 Notation ......................................................... 58

#### 2.4.2 Expansions for restricted two-point functions .......... 60

#### 2.4.3 Completion of the expansion ................................ 65

### 2.5 Discussion ......................................................... 67

## 3 Analysis .............................................................. 69

### 3.1 Structure of the analysis ....................................... 70

### 3.2 The general setting ............................................. 72

#### 3.2.1 The NoBLE relation ........................................ 72

#### 3.2.2 Assumptions .................................................. 73

### 3.3 Analysis ............................................................ 76

#### 3.3.1 The main result ............................................... 76

#### 3.3.2 Overview of the proof ....................................... 78

#### 3.3.3 The bootstrap for $f_1$ ....................................... 80

#### 3.3.4 The bootstrap for $f_2$ ....................................... 81

#### 3.3.5 The bootstrap for $f_3$ ....................................... 82

### 3.4 Proof of key inequalities in Proposition 3.3.1 .................. 85

#### 3.4.1 Computation of the Fourier inverse of $\hat{F}$ and $\hat{\Phi}$ 86

#### 3.4.2 Bound on the absolute value of $\hat{F}$ and $\hat{\Phi}$ ........ 87

#### 3.4.3 Bounds on differences ....................................... 88

### 3.5 Analysis with an alternative bound on the displacement ... 93

#### 3.5.1 Introduction of the alternative bootstrap function .... 93

#### 3.5.2 The modified analysis ...................................... 96

#### 3.5.3 Rewrite of the two-point function .......................... 98

### 3.6 The bootstrap for $\bar{f}_3$ ....................................... 99

#### 3.6.1 Continuity ..................................................... 99

#### 3.6.2 Bound for the initial point ................................ 100

#### 3.6.3 The idea of the improvement of bounds .................... 102

#### 3.6.4 Improvement of bounds .................................... 106

### 3.7 Proof of model assumptions .................................... 111

#### 3.7.1 Self-avoiding walk ......................................... 111

#### 3.7.2 Lattice trees and animals ................................... 113
## 5.2 Simple random walk integrals

### 5.2.1 Computation of $I_{n,0}$

### 5.2.2 Computation of $I_{0,m}$

### 5.2.3 Bounds on involved SRW-Integrals

## 5.3 Series involving matrices

## 5.4 Model specific features

### 5.4.1 Comments for self-avoiding walk

### 5.4.2 Comments for lattice trees and animals

### 5.4.3 Comments for percolation

## 5.5 Discussion

## 6 Alternative expansion for SAW

### 6.1 The idea

### 6.2 The classical lace expansion

### 6.3 The derivation of the new expansion

### 6.4 Bounds on $\Phi_z$

### 6.5 Bounds on $\Pi_z$

#### 6.5.1 Bounds on $\tilde{\Pi}_z(k)$

#### 6.5.2 Bounds on $\tilde{\Pi}_z(0) - \tilde{\Pi}_z(k)$

#### 6.5.3 Bounds on $\tilde{\Pi}_z^{\text{AV}}(0) - \tilde{\Pi}_z^{\text{AV}}(k)$

### 6.6 Analysis

### 6.7 Discussion

## 7 Non-backtracking random walk

### 7.1 Non-backtracking random walk on $\mathbb{Z}^d$

#### 7.1.1 Setting

#### 7.1.2 Transition matrix

#### 7.1.3 Central limit theorem

#### 7.1.4 Extension to non-nearest-neighbor settings

### 7.2 Non-backtracking random walk on tori

#### 7.2.1 Setting

#### 7.2.2 Asymptotics for NBW on the torus

#### 7.2.3 NBW on the hypercube

## Bibliography

## About the author

## Acknowledgements
Chapter 1
Introduction and results

1.1 Introduction

1.1.1 Motivation

In statistical physics we use probabilistic models to study the behavior of physical systems, especially interacting particle systems. These can be systems of particles of a gas, the movement of electrons or the behavior of water at different temperatures, the formation of crystals as well as the spatial form of large molecules. Some of these probabilistic models are created by assuming the network of particles to have simple pairwise interactions, e.g. the attractive forces between two atoms. We want to understand the global behavior of a system, that is created by these simple interactions and how variations of the interaction strength changes the global behavior.

Many statistical physics models have a phase transition. A phase transition occurs when there exists a critical value for the interaction strength such that slight changes of the strength, near the critical value, change the global behavior of the systems drastically. If such a critical value exists, then we speak of a phase transition. The prime example for such an abrupt change of the global behavior is the behavior of water below and above the boiling temperature. We call a system critical (near-critical), if the strength of interaction equals the critical value (is close to the critical value).

An important notion for phase transitions is the concept of universality. This notion refers to the insight that many features of the critical and the near-critical system are independent of many details of the system. In particular, it is believed that the behavior of critical and near-critical systems can be characterized by critical exponents, where the critical exponents are universal in the sense that their values remain unchanged under moderate changes of the definition of the systems.
For some important interacting models it is expected that these critical exponents have the same value as a corresponding model without any interaction. This means that the interaction does not influence the global behavior of the critical model. We call this behavior of the critical/ near-critical model mean-field behavior. The name mean-field behavior comes from the Ising model. In the critical Ising model the interaction of any particles to any other particles in the system is the same. Thus, the behavior of one particle is influenced by the average/mean behavior of the whole system.

Mean-field behavior occurs typically in high dimensions, where far away particles are close to being independent in the critical system. In this thesis we develop a technique that can prove mean-field behavior for nearest-neighbor percolation, self-avoiding walks, lattice trees, and lattice animals in high dimensions.

1.1.2 Structure of this chapter

We will begin this chapter by introducing the four models we consider in this thesis. All these models have a critical value and are expected to show mean-field behavior in high dimensions. In Section 1.5 we motivate the concept of mean-field behavior and introduce the critical exponents of the corresponding systems without interactions. We will prove mean-field behavior for the four models using a technique called non-backtracking lace expansion (NoBLE), which we discuss in Section 1.6. In Section 1.7 we state the result obtained using the NoBLE and discuss two related problems that we are also part of this thesis. We close this chapter by giving an outline of the proof of mean-field behavior using the NoBLE and explain the structure of the thesis.

1.1.3 Notation

We briefly introduce the notation used throughout the thesis. We use the letters $n, m, d \in \mathbb{N}$ for integers and $u, v, w, x, y \in \mathbb{Z}^d$ for points in the hypercubic lattice. We write $x_i$ for the $i$th component of the vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$. We denote by $|\cdot|$ the one-norm/absolute value and by $\|\cdot\|_2$ the Euclidian norm, i.e., $|x| = \sum_{i=1}^{d} |x_i|$ and $\|x\|_2^2 = \sum_{i=1}^{d} |x_i|^2$ respectively, for $x \in \mathbb{Z}^d$. We call two points $x, y$ neighbors if $\|x - y\| = 1$. We denote by a bond a pair of neighbors $\{x, y\}$ and by an oriented bond an order pair of neighbors $(x, y)$. For an oriented bond $b = (x, y)$ we write $b_0$ for the starting point $x$ and $b_1$ of the endpoint $y$ of the oriented bond.

We exclusively use the Greek letters $\iota$ and $\kappa$ for values in $\{-d, -d+1, \ldots, -1, 1, 2, \ldots, d\}$ and denote by $e_\iota \in \mathbb{Z}^d$ the unit vector in direction $\iota$, e.g. $(e_\iota)_i = \text{sign}(\iota)\delta_{|\iota|,i}$ (beware of the minus sign when $\iota$ is negative, which is somewhat different from the usual choice of a unit vector).

We use $p \in [0, 1]$ to denote a probability and denote by $z$ a non-negative real number. For event $E$ we define $\mathbb{I}_{|A|}$ to be the indicator that $A$ occurs. Further, we use $\delta$ for
the Kronecker delta, i.e., for $x, y \in \mathbb{Z}^d$ we define $\delta_{x,y} = \mathbb{1}_{x=y}$.

Let $f, g : \mathbb{Z} \to \mathbb{C}$. For an absolutely summable function $f$ we define the Fourier transform of $f$ by

$$
\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{i k \cdot x} \quad \text{for} \quad k \in [-\pi, \pi]^d,
$$

where $k \cdot x = \sum_{i=1}^d k_i x_i$, with inverse

$$
f(x) = \int_{[-\pi, \pi]^d} \hat{f}(k) e^{-i k \cdot x} \frac{d^d k}{(2\pi)^d}.
$$

We will use the letters $k$ and $l$ exclusively to denote values in the Fourier dual space $[-\pi, \pi]^d$. We denote by $\ast$ the convolution of two functions, i.e.,

$$
(f \ast g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x-y),
$$

and by $f^{*n}$ the $n$-times convolution:

$$
f^{*n}(x) = (f \ast f^{*(n-1)})(x) = (f \ast f \ast f \ast \cdots \ast f)(x).
$$

Further, we note that the Fourier transform of $f^{*n}(x)$ is given by $\hat{f}(k)^n$.

**Landau symbols.** The symbols $O, o$ and $\sim$ have the following meanings:

$$
\begin{align*}
    f(x) = O(g(x)) & \text{ as } x \to a \text{ means } f(x)/g(x) \text{ is bounded as } x \to a, \\
    f(x) = o(g(x)) & \text{ as } x \to a \text{ means } f(x)/g(x) \to 0 \text{ as } x \to a, \\
    f(x) \sim g(x) & \text{ as } x \to a \text{ means } f(x)/g(x) \to 1 \text{ as } x \to a.
\end{align*}
$$

**Matrix notation.** In our analysis, we use $\mathbb{C}^{2^d}$-valued and $\mathbb{C}^{2^d} \times \mathbb{C}^{2^d}$-valued functions. For a clear distinction between scalar-, vector- and matrix-valued quantities, we always write $\mathbb{C}^{2^d}$-valued functions with a vector arrow (e.g. $\vec{v}$) and matrix-valued functions with bold capital letters (e.g. $\mathbf{M}$). We do not use $\{1,2,\ldots,2d\}$ as index set for the elements of a vector or a matrix, but use $\{-d,-d+1,\ldots,-1,1,2,\ldots,d\}$ instead. Further, we define for a $k \in [-\pi, \pi]^d$ and negative index $\iota \in \{-d,-d+1,\ldots,-1\}$ the notation $k_\iota = -k_{|\iota|}$.

We denote the identity matrix by $\mathbf{I} \in \mathbb{C}^{2^d \times 2^d}$ and the all-one vector by $\vec{1} = (1,1,\ldots,1)^T \in \mathbb{C}^{2^d}$. Moreover, we define the matrix $\mathbf{J}, \hat{\mathbf{D}}(k) \in \mathbb{C}^{2^d \times 2^d}$ by $(\mathbf{J})_{i,\kappa} = \delta_{i,-\kappa}$ and $(\hat{\mathbf{D}}(k))_{i,\kappa} = \delta_{i,\iota} e^{i k \cdot x}$.
1.2 Percolation

1.2.1 A brief overview of percolation

Motivation of the problem. Percolation is a model to describe the spread of fluid through a medium. We can consider the movement of a liquid through a porous stone, the flow of electrons through an atomic lattice, or the spread of a disease in a community. For us the only source of randomness of the spread of the fluid is in the random structure of the media.

Percolation is obtained by modeling the solid medium by a graph and then independently declaring edges of the graph to be open or closed. The graph of open edges then represents all possible canals that the fluid can use. How far the fluid can spread depends sensitively on the fraction of open edges. If most of the edges are open, then the fluid reaches most of the graph. In particular, if the graph is infinite, then the fluid spreads to an infinite number of vertices. When we open only a small proportion of edges then the fluid will not spread far. For many infinite graphs, including lattices, the transition from reaching only a finite number of vertices to reaching infinitely many vertices is rather sharp. Thus, despite the simplicity of its definition, this model has a phase transition. We visualize this phase transition for the example of the square lattice in Figures 1.1 and 1.2.

General references. Percolation as a stochastic process was first introduced by Broadbent and Hammersley (1957) in [20]. The idea for this model is even older. The earliest mention of such a model can be found in the problem section of the first volume of the American Mathematical Monthly (1894) [101]. In Flory (1941) used the key concept of the percolation model to describe the reaction of small branching molecules and the formation of very large networks of molecules connected by chemical bonds. In 1954 Hammersley and Morton then reinvented the idea for such a model in [34, Example 4].

Since its introduction, percolation has attracted many researchers in mathematics and physics and found applications in numerous fields. An overview of applications of percolation processes is given in [87] and [88]. There are many beautiful results for percolation and still many interesting open problems. To mention only some keywords: first passage percolation, the incipient infinite cluster, the scaling limit of the critical percolation cluster (SLE₆ in dimension 2). We will not even try to give a complete overview of the field and refer the reader to [29] or [14] for more mathematical references and to [97] for an introduction to the physics point of view. A review of the more recent results can be found in [30] and references therein. In the following we only discuss the features that are relevant for this thesis.
1.2.2 Formal definition of percolation

We consider Bernoulli percolation on the hypercubical lattice. We use the definition of [92] Section 9: To each nearest-neighbor bond \( \{x, y\} \) we associate an independent Bernoulli random variable \( n_{\{x, y\}} \) which takes the value 1 with probability \( p \) and the value 0 with probability \( 1 - p \), where \( p \in [0, 1] \). If \( n_{\{x, y\}} = 1 \), then we say that the bond \( \{x, y\} \) is open, and otherwise we say that it is closed. A configuration is a realization of the random variables of all bonds. The joint probability distribution of the bond variables is denoted by \( P_p \) with corresponding expectation \( E_p \).

We say that \( x \) and \( y \) are connected, denoted by \( x \leftrightarrow y \), if there exists a path consisting of open bonds connecting \( x \) and \( y \), or if \( x = y \). We denote by \( C(x) \) the random set of vertices connected to \( x \) and denote its cardinality by \( |C(x)| \). The two-point function \( \tau_p(x) \) is defined to be the probability that 0 and \( x \) are connected:

\[
\tau_p(x) = P_p(0 \leftrightarrow x). \tag{1.2.1}
\]

By translation invariance \( P_p(x \leftrightarrow y) = \tau_p(x - y) \) for all \( x, y \in \mathbb{Z}^d \). We define the susceptibility, or expected cluster size, by

\[
\chi(p) = \sum_{x \in \mathbb{Z}^d} \tau_p(x) = E_p[|C(0)|]. \tag{1.2.2}
\]

We say that the system percolates if there exists a cluster \( C(x) \) such that \( |C(x)| = \infty \). We define \( \theta(p) \) as the probability that the origin is part of an infinite cluster, so that 0 is connected to infinitely many points:

\[
\theta(p) = P_p(|C(0)| = \infty). \tag{1.2.3}
\]

For \( d \geq 2 \), there exists a critical value \( p_c \in (0, 1) \) such that the probability \( \theta(p) \) is zero for \( p < p_c \) and strictly positive for \( p > p_c \):

\[
p_c(d) = \inf\{p|\theta(p) > 0\}. \tag{1.2.4}
\]

Menshikov (1986) [75], as well as Aizenmann and Barsky (1987) [2], have proven that the critical value can also be characterized as

\[
p_c(d) = \sup\{p|\chi(p) < \infty\}. \tag{1.2.5}
\]

The percolation probability \( \theta \) as a function of \( p \) is clearly continuous on \((0, p_c)\), and it is also continuous (and even infinitely differentiable) on \((p_c, 1]\) by the results of [12] (for infinite differentiability of \( p \to \theta(p) \) for \( p \in (p_c, 1] \), see [85]). Thus, continuity of \( p \to \theta(p) \) is equivalent to the statement that \( \theta(p_c(d)) = 0 \).
Figure 1.1: Percolation configuration on a box of size 160x100 sites. The biggest cluster is marked in black. The bond probability is marked in the lower right corner of the picture.
Figure 1.2: The pictures are created by an applet that can be found on the website of the author.
Critical exponents. We introduce three critical exponents for percolation. It is widely believed that the following limits exist in all dimensions

\[
\gamma = -\lim_{p \to p_c} \frac{\log \chi(p)}{\log|p - p_c|},
\]
(1.2.6)

\[
\beta = -\lim_{p \to p_c} \frac{\log \theta(p)}{\log|p - p_c|}.
\]
(1.2.7)

A strong form of (1.2.6) and (1.2.7) is that

\[
\chi(p) \sim c_{\chi}(p_c - p)^{-\gamma} \quad \text{as} \quad p \to p_c,
\]
(1.2.8)

\[
\theta(p) \sim c_{\theta}(p - p_c)^{\beta} \quad \text{as} \quad p \to p_c.
\]
(1.2.9)

and is expected to hold in all dimensions. The constants \(c_{\chi}\) and \(c_{\theta}\) depend on the dimension. Further, it is believed that there exists \(\eta\) such that

\[
\tau_{p_c}(x) \sim c_1 \frac{1}{|x|^{d-2+\eta}}, \quad \hat{\tau}_{p_c}(k) \sim c_2 \frac{1}{|k|^{2-\eta}},
\]
(1.2.10)

where \(c_1\) and \(c_2\) depend on the dimension. For percolation, the existence of many more exponents is conjectured and partially also proven. These exponents describe quantities like the correlation length, magnetization, the size distribution of the cluster size, etc. We restrict ourself to the exponents relevant to this thesis. We refer the interested reader to [29, Chapter 10].

1.2.3 Results and conjectures

In the five decades since its introduction in [20], many interesting results for percolation were obtained. In the following we give a brief review of results on the critical exponents and the percolation probability \(\theta(p)\) in high dimension. Let us first discuss the critical probability, see [29, Chapter 3] for a more extensive review.

Critical probability \(p_c\). For two infinite connected graph \(G_1\) and \(G_2\), such that \(G_2\) is a subgraph of \(G_1\) is know that the percolation probability on these graph satisfies \(p_c(G_1) \leq p_c(G_2)\). From this follows that the critical probability \(p_c(d)\) of the hypercubical lattice \(\mathbb{Z}^d\) is decreasing \(d\). A discussion about such inequalities for the percolation probability and the question in which cases strict inequality holds can be found in [29 Sections 3.2-3.3].

Kesten proven in [66] that \(p_c(d) \sim (2d)^{-1}\) as \(d \to \infty\). In [43], [57] the following asymptotic expansion for \(p_c\) has been proven using the lace expansion:

\[
p_c(d) = \frac{1}{2d} + \frac{2}{(2d)^2} + \frac{7}{(2d)^3} + O\left(\frac{1}{(2d)^4}\right).
\]
(1.2.11)
To compute the precise value of $p_c$ remains an open problem. In [28] you can find numerical estimates on $p_c$ for $d \in [4, 13]$. For $d = 2$ we can couple percolation on the lattice with bond probability $p$ to percolation with bond probability $1 - p$. This self-duality of the square lattice $\mathbb{Z}^2$ allowed Kesten [63], building upon results of Harris [46], Russo [85] and Seymour and Welsh [91], to prove that $p_c = 1/2$. Russo [86] generalized the arguments of Kesten and computed the critical bond probability for the triangular lattice and the honeycomb lattice, which are dual to each other. Further, Kesten [64] was able to prove that there exists no infinite cluster for $p_c = 1/2$, so that $\theta$ is continuous in $p$ with $\theta(1/2) = 0$.

**Critical exponents.** The critical exponents for percolation have received considerable attention in the literature. For the critical exponents introduced above it is known that $\beta \leq 1$ and $\gamma \geq 1$ for all $d \geq 2$, see Chayes and Chayes [23] and Aizenman and Newman [4].

Further, we know that if $\beta$ and $\gamma$ exists, then $\beta \in (0, 1]$ and $\gamma \in [1, \infty)$, see [29] Section 10.2 and 10.4. We lack a technique to analyze percolation in $d = 3, 4, 5$, so these bounds on the critical exponents are the best bounds known to us.

**Critical exponents for $d = 2$.** For $d = 2$ the critical exponents are conjectured to have the value $\gamma = 43/18$, $\beta = 5/36$, $\eta = 5/24$, see [59]. This conjecture is supported by [94] and [31]. In [94] some critical exponents, including $\gamma = 43/18$, are proven for site percolation on the triangular lattice. By the hyperscaling relations, that are proven to hold by Kesten [65], we conclude that $\eta = 5/24$. In [31] it is proven that bond percolation on the square lattice, the triangular lattice, and the hexagonal lattice, belong to the same universality class, meaning that they have identical critical exponents. If the conjecture that bond and site percolation are in the same universality class hold, then this would prove the conjectured values of the critical exponents also for bond percolation on the square lattice. Further, it is conjectured that the scaling limit of critical percolation equals SLE$_6$ in dimensions 2, see [30, Section 3] and references therein.

**Critical exponents for $d > 6$.** In high dimensions we expect *mean-field behavior* for percolation. Namely, we expect that the critical exponents correspond to the exponents of the regular tree: $\gamma = 1$, $\beta = 1$, and $\eta = 0$. An important step to prove mean-field behavior for percolation is the result of Aizenman and Newman [4], that the finiteness of the triangle, defined by

$$ (\tau_p \star \tau_p \star \tau_p)(0), \quad (1.2.12) $$

implies that $\gamma = 1$. This triangle condition also implies that $\beta = 1$, see [11]. In particular, this implies that $p \to \theta(p)$ is continuous.
Hara and Slade [36] use the lace expansion, which we discuss in more detail in Section 1.6, to prove $\eta = 0$ in Fourier space and the finiteness of triangle condition for $d \geq 7$ in the spread-out setting with a parameter $L$ big. In the spread-out setting all bonds $\{x, y\}$ with $|x - y| \leq L$ are independently open or closed. This is an optimal result in the sense that mean-field behavior is not expected in $d \leq 6$, see [99], where Toulouse argues that the upper critical dimensions $d_c$, above which we can expect mean-field behavior, equals 6.

For the nearest-neighbor setting Hara and Slade proved mean-field behavior in sufficiently high dimensions [36]. Later, they verified that $d = 19$ is sufficiently high. In private communication with Takashi Hara the author learned that in a recent computation the mean-field result was established in $d = 15$. This also implies that $\theta$ is continuous in $d \geq 15$. The computations for both results ($d \geq 15$ and $d \geq 19$) were never published.

1.3 Self-avoiding walk

1.3.1 A brief overview of self-avoiding walk

A self-avoiding walk (SAW) is a path on the lattice that does not visit the same site more than once. SAW is used to model the behavior of long polymer chains in a dilute solvent. On the lattice the bond could be considered to be the chemical bond between two adjacent atoms. SAW provides an interesting and difficult problem in combinatorics and computer science.

In spite of its simple definition, SAW is a difficult model to study. The problem with analyzing SAW is that it is not Markovian, indeed it is not even a stochastic process. Therefore, the standard probabilistic methods do not apply and other techniques are required to study SAW. For example, we need to compute all SAWs with $n$ steps to determine the distribution of the end-point of $n$-step SAWs. The computation of all $n$-steps walks is exponentially hard in $n$. See [24] for a recent computational approach to the problem.

We will only review the properties that are relevant for this thesis. For a review on SAW we refer the reader to the general references for SAW, [74] and [58]. A survey of recent results can be found in [93].

1.3.2 Formal definition of self-avoiding walk

A nearest-neighbor $n$-step SAW on $\mathbb{Z}^d$ is an ordered $n + 1$-tuple $\omega = (\omega_0, \omega_1, \omega_2, \ldots, \omega_n)$, with $\omega_i \in \mathbb{Z}^d$, $\|\omega_i - \omega_{i+1}\|_2 = 1$, and $\omega_i \neq \omega_m$ for all $0 \leq i < m \leq n$. Unless stated otherwise we take $\omega_0 = \vec{0} = (0, 0, \ldots, 0)$.

We denote by $c_n$ the number of all $n$-step SAWs, and for $x \in \mathbb{Z}^d$ we denote by $c_n(x)$ the number of $n$-step SAWs for which $\omega_n = x$.

We can split each $(n + m)$-step SAW into two SAWs with $n$ and $m$ steps respectively.
From this we conclude that $c_{n+m} \leq c_n c_m$. Thus, $\log c_n$ is a subadditive sequence. We use [74] Lemma 1.2.2 to conclude that $\mu = \lim_{n \to \infty} c_n^{1/n}$ exists and $c_n \geq \mu^n$. The constant $\mu$ is known as the connective constant and its exact value is unknown in general.

We define the susceptibility of SAW $\chi^{SAW}(z)$ by

$$\chi^{SAW}(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and by $z_c$ the radius of convergence of $\chi^{SAW}(z)$. Since $z_c = 1/\mu$ the susceptibility allows us to study $\mu$. We use SAW two-point function to study the asymptotic behavior of $c_n$:

$$G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n,$$

which is defined for all $z \in [0, z_c)$. We note that $\chi^{SAW}(z) = \sum_x G_z(x) = \hat{G}_z(0)$. Another quantity of interest is the mean-square displacement:

$$\frac{1}{c_n} \sum_x \|x\|^2 c_n(x).$$

**Critical exponents.** We now introduce the critical exponents for SAW that are relevant to us. For all $d \geq 2$ it is expected that there exist critical exponents $\gamma, \nu$ and $\eta$ and constants $A_i$ such that

$$c_n \sim A_1 z_c^n n^{\gamma-1},$$

$$\chi^{SAW}(z) \sim A_2 (z_c - z)^{-\gamma} \quad \text{as } z \not\to z_c,$$

$$\frac{1}{c_n} \sum_{x \in \mathbb{Z}^d} \|x\|^2 c_n(x) \sim A_3 n^{2\nu},$$

$$G_{zc}(x) \sim \frac{A_4}{\|x\|^{2-d-\eta}} \quad \text{and} \quad \hat{G}_{zc}(k) \sim A_5 \|k\|^{2-\eta},$$

where $\gamma, \nu$ and $\eta$ should fulfill Fisher’s scaling law $\gamma = (2 - \eta)\nu$. While the constants $A_i$ depend on the details of the lattice, the critical exponents are believed to be universal in the sense that they are insensitive to minor modifications of the model, like changes of the underlying graph. Thus, the values of $\gamma, \nu$ and $\eta$ are the same for the spread-out setting, in which the walk can not only take near-neighbor steps, but can jump to any vertex at distance at most $L \geq 1$.

**1.3.3 Results and conjectures**

The value of $\mu$ has been studied extensively in literature. For low dimension most results were obtained using numerical estimates, see Table 1.3.3 For an overview
about such results we refer the reader to [24]. On the website of Gordon Slade you find some values for \(c_n\) and estimates on \(\mu\) and \(\nu\) for \(d = 2, \ldots, 12\). Using the lace expansion it has been proven that \(\mu\) has an asymptotic expansion to all orders in \(1/d\) and that

\[
\mu = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{16}{(2d)^3} + O\left(\frac{1}{(2d)^4}\right),
\]

where the coefficients are computed up to and including the \((2d)^{-11}\)-term, see [24].

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
   d  & 4   & 5   & 6   & 7   & 8   & 9   & 10  \\
\hline
(2d-1)z_c(d) & 1.033 & 1.018 & 1.011 & 1.008 & 1.005 & 1.004 & 1.003 \\
\hline
\end{array}
\]

Table 1.1: Numerical values of \(\mu(d)\) taken from [24, Table 15] for \(d = 4, \ldots, 8\) and approximated for \(d = 9, 10\) using the power expansion of \(1/z_c\) in [24, Section 1.3, equation (1)].

\[
\begin{array}{|c|c|c|c|}
\hline
   d  & \gamma & \nu & \eta \\
\hline
2   & \frac{43}{32} & \frac{4}{3} & \frac{5}{24} \\
3   & \approx 1.15 & \approx 0.587 & \approx 0.041 \\
\geq 4 & 1 & \frac{1}{2} & 0 \\
\hline
\end{array}
\]

Table 1.2: Conjectured values for the critical exponents. In dimension 4 (1.3.1)-(1.3.4) are believed to hold with logarithmic corrections.

In Table 1.3.3 we note the conjectured values of the critical exponents. For \(d = 2\) the value of \(\gamma\) and \(\nu\) are predicted by Nienhuis [79]. These values are supported by numerical estimates using Monte Carlo experiments (see e.g. [72]) and computations as in [24]. The value of \(\gamma\) and \(\nu\) for \(d = 3\) are numerical estimates obtained in [24]. The values for \(\eta\) for \(d = 2, 3\) are concluded from the Fisher’s scaling relation that is believed to hold for all \(d \geq 2\). For \(d \geq 5\) Hara and Slade ([42,40]) have proven that the critical exponents \(\gamma, \nu\) and \(\eta\) take their mean-field values given in Table 1.3.3 using the lace expansion. Further, they conclude that the SAW-path properly scaled converges to Brownian motion. The result of Hara and Slade is optimal in the sense that mean-field behavior is not expected in dimension 4 and lower.

In the critical dimension \(d = 4\) (1.3.1) and (1.3.3) are predicted to hold with \((\log n)^{1/4}\) corrections [19]. A first result in this direction is announced by Brydges and Slade [21] using renormalization group techniques. Their results apply to continuous-time weakly SAW.
1.4 Lattice trees and animals

1.4.1 A brief overview of lattice trees and animals

A lattice animal (LA) is a finite, connected set of bonds. A lattice tree (LT) is a finite, connected set of bonds that contains no cycles. The following passage is taken from Whittington and Soteros [95], where a good overview of results and open problems for LA is given: “These animals and trees have been considered as models of branched polymers with excluded volume in much the same way that self-avoiding walks have been used as models of linear polymers with excluded volume, and the techniques used to handle the animal problem are closely related to techniques in the theory of self-avoiding walks (Hammersley 1957 [33]; Kesten 1963 [62]). Lattice animals are also closely related to percolation clusters although the associated weights are different in the two problems (Broadbent and Hammersley 1957[20]; Kesten 1982 [64]).” For further information of applications of LT/LA model we refer the interested reader to van Rensburg (2000) [60]. Besides being a model for randomly branched polymers in a good solvent, LT and LA proved to be an interesting combinatorial problem. In particular, the enumerations of lattice trees and animals receives considerable attention in the literature, see e.g. [61] and [26] and references therein. Moreover, series expansions for various percolation properties, such as the percolation probability or the average cluster size, can be obtained as weighted sums over the number of lattice animals, enumerated according to the number of sites \( n \) and perimeter \( m \), [29, Section 4.2] or [64, Chapter 5].

1.4.2 Formal definition of lattice trees and lattice animals

A nearest-neighbor lattice tree on \( \mathbb{Z}^d \) is a finite, connected set of nearest-neighbor bonds which contains no cycles (closed loops). A nearest-neighbor lattice animal on \( \mathbb{Z}^d \) is defined to be a finite, connected set of nearest-neighbor bonds, which may or may not contain cycles. Although a tree/animal \( A \) is defined as a set of bonds, we write \( x \in A \) for \( x \in \mathbb{Z}^d \), to denote that \( x \) is an element of a bond of \( A \). The number of bonds in \( A \) is denoted by \( |A| \). We define \( t_{n}^{(a)}(x) \) and \( t_{n}^{(t)}(x) \) to be the number of LAs and LTs respectively, that consist of exactly \( n \) bonds and contains the origin and \( x \in \mathbb{Z}^d \). We study LA and LT using the one-point function \( g_z \) and the two-point function \( G_z \)

\[
\begin{align*}
    t_{n}^{(a)}(x) & = \sum_{A: A \ni 0} t_{n}^{(a)}(x) = \sum_{A: A \ni 0} z^{|A|}, \\
    t_{n}^{(t)}(x) & = \sum_{T: T \ni 0} t_{n}^{(t)}(x) = \sum_{T: T \ni 0} z^{|T|}, \\
    g_z^{(a)}(0) & = \sum_{A: A \ni 0} z^{|A|}, \\
    g_z^{(t)}(0) & = \sum_{T: T \ni 0} z^{|T|}, \\
    \tilde{G}_z^{(a)}(x) & = \sum_{n=0}^{\infty} t_{n}^{(a)}(x) = \sum_{A: A \ni 0, x} z^{|A|}, \\
    \tilde{G}_z^{(t)}(x) & = \sum_{n=0}^{\infty} t_{n}^{(t)}(x) = \sum_{x \in \mathbb{Z}^d} \sum_{T: T \ni 0, x} z^{|T|},
\end{align*}
\]

where we sum over animals \( A \) and trees \( T \) respectively. You might wonder at this point why we denote the two-point function with \( \tilde{G}_z \), instead of \( G_z \). The
reason for this is that we will normalize $\tilde{G}_z$ by the one-point function $g_z$ and define $G_z(x) = \tilde{G}_z(x)/g_z$. This is not necessary but simplifies our analysis and improves the numerical performance of the bounds.

We define the susceptibility of LAs and LTs by

$$
\chi(z) = \tilde{G}_z(0), \quad \chi(t) = \hat{G}_z(0),
$$

(1.4.3)

and denote the radii of convergence of these sums by $z^{(a)}_c$ and $z^{(t)}_c$. As for SAW, $1/z_c$ describes the growth of the number of LT/LA as $n$ grows.

The typical length scale of a lattice tree/animal is characterized by the average radius of gyration $R_n$, that can be defined by

$$
R_n = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \|x\|_2 n^{(a)}(x),
$$

(1.4.4)

Critical exponents. The asymptotic behavior of $t_n$ and $\tilde{G}_z$ can be described using critical exponents. Now we define three of these critical exponents for LA and LT. In doing so we drop the superscripts $(a)$ and $(t)$ as the following holds for LA and LT. It is believed that there exist $\gamma, \eta, \nu$ and $A_1, A_2, A_3, A_4 > 0$ such that

$$
\chi(z) \sim A_1 \frac{1}{1 - z/z_c}^{\gamma},
$$

(1.4.5)

$$
R_n \sim A_2 \cdot n^{\nu},
$$

(1.4.6)

$$
\tilde{G}_{z_c}(x) \sim \frac{A_3}{\|x\|_2^{2-d-\eta}}, \quad \text{and} \quad \hat{G}_{z_c}(k) \sim A_4 \|k\|_2^{\eta-2},
$$

(1.4.7)

as $|x| \to \infty$ and $k \to 0$. The exponents are believed to be universal, in the sense that they do not depend on the detailed structure of the lattice. In particular, it is believed that the values of $\gamma, \eta$ and $\nu$ are the same in the nearest-neighbor setting, that we consider here, and in the spread-out setting. The constants $A_i$ depend on the structure of the lattice.

1.4.3 Results and conjectures

For LT and LA not only $1/z$ but also $g_z = \tilde{G}_z(0)$ is unknown. Recently it has been proven in [76] that

$$
z_c = e^{-\frac{1}{2d} \left[ \frac{1}{(2d)^2} + \frac{3}{2} \frac{115}{24} \frac{1}{(2d)^3} \right] + o(2d)^{-3}},
$$

(1.4.8)

$$
g_{z_c} = e \left[ \frac{1}{2d} \frac{3}{2} \frac{263}{24} \frac{1}{(2d)^2} + \frac{115}{24} \frac{1}{(2d)^3} \right] + o(2d)^{-2},
$$

(1.4.9)
where the factor $\mathbb{1}_a$ is 1 for LA and 0 for LT. The rigorous bound on the error terms was created using the lace expansion. These asymptotic expansions have also been studied extensively in the physics literature, where results were obtained up to the order of $1/(2d)^6$, but with non-rigorous estimates on the error term. We refer the reader to [76] for an overview of such results and a discussion of the $1/d$ expansions of the critical value $z_c$ of SAW, percolation, LT, and LA.

**Critical exponents.** As for SAW it is believed that the critical exponents $\gamma, \nu, \eta$ are related by the Fisher relation $\gamma = (2 - \eta)\nu$ and it has been proven in [18] that $\gamma \geq 1/2$ in all dimensions. Further, it is believed that there exists an upper critical dimension $d_c$ such that the critical exponents of LT/LA in $d > d_c$ take their mean-field values, which are $\gamma = 1/2, \nu = 1/4, \eta = 0$.

These values correspond to the mean-field model of LT and LA, studied in [16]. It is conjectured in [73] that the upper critical dimension of LT and LA is eight 8. This conjecture is supported by rigorous work in [44], where it is shown that if the “square diagram” is finite at the critical point, as is believed for $d > 8$, then the critical exponent $\gamma$ is smaller than 1/2. Hara and Slade proved in [39] that mean-field behavior holds for LT and LA in the *spread-out setting* with $L$ big enough.

For the nearest-neighbor setting that we consider, Hara and Slade give a rigorous proof of mean-field behavior for LT and LA in *sufficiently high dimensions*, see [39]. What sufficiently high dimensions means was not made precise. The author learned through private communication with Takasi Hara that it was not verified down to which dimension the classical lace expansion works, since Hara and Slade expected the lace expansion to be only successful in dimensions much bigger than $d_c = 8$. We explain the reason for this in Section 1.6.

In [32] site lattice trees and animals are simulated and the conjectured values of the critical exponents are met by these simulations. These computations are only done for site lattice trees, but as the critical exponents are expected to be universal, the values should be the same as of the bond tree/animal discussed here.

1.5 **Mean-field models and behavior**

In this section we give a heuristic explanation why mean-field behavior is expected in high dimensions for SAW, LT, LA, and percolation. Moreover, we discuss the predicted value of the upper critical dimension. We begin by introducing two models that can be used as mean-field models: simple random walk (SRW) and branching random walk (BRW). Further, we introduce non-backtracking walk (NBW), and non-backtracking branching walk (NBBW) that can also be used as mean-field models and become useful to motivate the non-backtracking lace expansion.
1.5.1 Simple random walk

Simple random walk (SRW) is one of the simplest stochastic processes imaginable and has proven to be useful in countless applications. For a review of SRW and connected models we refer the reader to [96], [58] or [70].

An n-step nearest-neighbor simple random walk on \( \mathbb{Z}^d \) is an ordered \((n + 1)\)-tuple \( \omega = (\omega_0, \omega_1, \omega_2, \ldots, \omega_n) \), with \( \omega_i \in \mathbb{Z}^d \) and \( \| \omega_i - \omega_{i+1} \|_1 = 1 \), where \( \| x \|_1 = \sum_{i=1}^d |x_i| \).

Unless stated otherwise we take \( \omega_0 = \vec{0} = (0, 0, \ldots, 0) \).

The step distribution of SRW is given by

\[
D(x) = \frac{1}{2^d} \delta_{\|x\|_1, 1} \quad \text{with} \quad \hat{D}(k) = \frac{1}{d} \sum_{i=1}^d \cos(k_i),
\]

(1.5.1)

where \( \delta \) is the Kronecker delta. We define \( p_n(x) \) as the number of \( n \)-step SRWs with \( \omega_n = x \) and \( p_n \) to be the total number of \( n \)-step SRWs. We know that \( p_n = (2d)^n \) and that, for \( n \geq 1 \),

\[
p_n(x) = \sum_{y \in \mathbb{Z}^d} 2d D(y) p_{n-1}(x-y) = 2d (D \ast p_{n-1})(x) = (2d)^n D_*^n(x). \quad (1.5.2)
\]

We define the SRW two-point function to be the generating function of the sequence \( p_n \), i.e.

\[
C_z(x) = \sum_{n=0}^{\infty} p_n(x) z^n = \sum_{n=0}^{\infty} D_*^n(x)(2d z)^n.
\]

(1.5.3)

Applying the Fourier transform, defined in (1.1.1), we see that \( \hat{p}_n(k) = (2d)^n(\hat{D}(k))^n \) and thereby

\[
\hat{C}_z(k) = \frac{1}{1 - 2d z \hat{D}(k)}. \quad (1.5.4)
\]

The SRW-susceptibility is given by

\[
\chi_{\text{SRW}}(z) = \hat{C}_z(0) = \frac{1}{1 - 2d z} = \frac{1}{2d} \left( \frac{1}{2d} - z \right)^{-1}.
\]

(1.5.5)

The critical point for SRW is the singularity \( z_c = 1/2d \) of the susceptibility. In the context of random walks the critical two-point function \( C_{1/2d} \) coincides with the Green’s function of the system.

**Critical exponents.** We now review the critical exponents \( \gamma, \nu \) and \( \eta \) for SRW to be able to compare the behavior of SRW with the behavior of the other models. In (1.5.5) we see that \( \gamma = 1 \). Since \( 1 - \cos(a) \approx a^2 \) for small \( a \in \mathbb{R} \), so that

\[
\hat{C}_{1/2d}(k) = \frac{1}{1 - \hat{D}(k)} \approx \frac{1}{\|k\|_2^2},
\]

(1.5.6)
for small \( k \) and we conclude that \( \eta = 0 \) in Fourier space. Further, it is known that the mean-square displacement of SRW is given by

\[
\frac{1}{p_n} \sum_x \|x\|_2^2 p_n(x) = n,
\]

so that \( \nu = 1/2 \). We will now prove (1.5.7) as we will use later similar arguments:

**Proof of (1.5.7).** Let \( \Delta^2_k = \sum_{j=1}^d \left( \frac{\partial}{\partial k_j} \right)^2 \). Since \( \frac{\partial^2}{\partial k_j^2} e^{i k \cdot x} = -x_j^2 e^{i k \cdot x} \) we know that \( \Delta_k e^{i k \cdot x} = -\|x\|_2^2 e^{i k \cdot x} \). Thus,

\[
-\frac{1}{p_n} \Delta^2_k \hat{p}_n(k)|_{k=0} = -\frac{1}{p_n} \sum_x \Delta^2_k p_n(x) e^{i k \cdot x} |_{k=0} = \frac{1}{p_n} \sum_x \|x\|_2^2 p_n(x).
\]

We recall the form of \( \hat{D} \) in (1.5.1) and compute

\[
\frac{\partial}{\partial k_j} \hat{D}(k)|_{k=0} = -\frac{1}{d} \sin(k_j)|_{k=0} = 0, \quad \Delta^2_k \hat{D}(k) = -\hat{D}(k).
\]

We use this and \( \hat{p}_n(k) = (2d)^n \hat{D}(k)^n \) to obtain

\[
\frac{1}{(2d)^n} \sum_{x \in \mathbb{Z}^d} \|x\|_2^2 c_n(x) = -\frac{1}{(2d)^n} \Delta^2_k \hat{c}_n(k)|_{k=0} = -\Delta^2_k \hat{D}^n(k)|_{k=0}
\]

\[
= -n \Delta^2_k \hat{D}(k) |_{k=0} \hat{D}^{n-1}(0) - n(n-1) \sum_{j=1}^d \left( \frac{\partial}{\partial k_j} \hat{D}(k) \right)^2 |_{k=0} \hat{D}^{n-2}(0)
\]

\[
= n \hat{D}^n(0) = n.
\]

\[\square\]

### 1.5.2 Non-backtracking walk

By a backtracking of a walk we denote the event that the walker retraces its steps. If you prefer, we can also speak of a reversal of its previous step. We define a non-backtracking walk (NBW) to be a SRW that does not have backtrackings. Considering NBW to be a process on vertices of the lattice, NBW is not Markovian. However, when we define NBW in terms of steps on bonds, the Markovian property is recovered.

For NBW we will assume that \( d \geq 2 \), as the problem is trivial for \( d = 1 \). As we will show now the introduction of the non-backtracking condition does not change the critical exponents or the limiting behavior. A more detailed discussion of NBW can be found in Chapter 7.
**Introduction and results**

In Fourier space, these relations translate to:

\[ \hat{b}_n(x) = |\hat{w}_n^{\text{NBW}}(0, x)| = |\{\omega : \omega \in \hat{w}_n^{\text{NBW}}(0, x), \omega_1 \neq e_i\}| = |\{\omega : \omega \in \hat{w}_n^{\text{NBW}}(0, x-e_i), \omega_1 = 0\}| = |\{\omega : \omega \in \hat{w}_n^{\text{NBW}}(0, x-e_i), \omega_1 = -e_i\}|. \]  

We use this to obtain the following characterization of \( b_n \), for \( n \geq 1 \),

\[ b_n(x) = \sum_{i \in \{\pm 1, \ldots, \pm d\}} b_{n-1}^{-i}(x-e_i), \quad \text{and} \quad b_n(x) = b_n^i(x) + b_{n-1}^{-i}(x-e_i). \]  

The first equality is obtained by summing over the direction of the first step and the second equality by conditioning whether the walk will visit \( e_i \) in its first step or not. We define the **NBW two-point functions** as

\[ B_\mu(x) = \sum_{n=0}^{\infty} b_n(x) \mu^n, \quad B_\mu^i(x) = \sum_{n=0}^{\infty} b_n^i(x) \mu^n, \]

and see that (1.5.12) implies

\[ B_\mu(x) = \delta_{0,x} + \mu \sum_{i \in \{\pm 1, \ldots, \pm d\}} B_\mu^{-i}(x-e_i) \quad \text{and} \quad B_\mu(x) = B_\mu^i(x) + \mu B_\mu^{-i}(x-e_i). \]  

In Fourier space, these relations translate to:

\[ \hat{B}_\mu(k) = 1 + \mu \sum_{i \in \{\pm 1, \ldots, \pm d\}} e^{-ik \cdot e_i} \hat{B}_\mu^i(k) \quad \text{and} \quad \hat{B}_\mu(k) = \hat{B}_\mu^i(k) + e^{ik \cdot e_i} \hat{B}_\mu^{-i}(k). \]  

In the next step, we rewrite (1.5.14) using matrix notation (see Section 1.1.3). We define the \( C^{2d} \)-valued functions \( \hat{B}_\mu(k) \) by \( \{\hat{B}_\mu(k)\} = \tilde{\hat{B}}_\mu(k) \), rewrite (1.5.14) to

\[ \hat{B}_\mu(k) = 1 + \mu \tilde{\hat{D}}(-k) \hat{B}_\mu(k), \quad \text{and} \quad \hat{B}_\mu(k) \tilde{1} = \tilde{\hat{B}}_\mu(k) + \mu \tilde{\hat{D}}(k) \mathbf{J} \hat{B}_\mu(k), \]

and conclude that

\[ \hat{B}_\mu(k) = \frac{1}{1 - \mu \tilde{\hat{D}}(-k) \mathbf{I}} \left[ I + \mu \tilde{\hat{D}}(k) \mathbf{J} \right]^{-1} \tilde{1} = \frac{1}{1 - \mu \tilde{\hat{D}}(k) + \mu \mathbf{J}} \frac{1}{1 - \mu \tilde{\hat{D}}(-k)}. \]  

**Formal definition.** If an \( n \)-step SRW \( \omega \) satisfies \( \omega_i \neq \omega_{i+2} \) for all \( i = 0, 1, 2, \ldots, n-2 \), then we call \( \omega \) **non-backtracking**. In order to analyze NBW we want to obtain an equation similar to (1.5.2). The same equation does not hold for NBW as it neglects the condition that the walk does not revisit the origin after the second step. For this reason, we introduce the condition that a walk should not go in a certain direction \( \iota \)

We define \( w_n^{\text{NBW}}(x, y) \) to be the set of all \( n \)-step NBWs starting at \( x \) and ending at \( y \) and \( w_n^{\text{NBW}}(x, y) \) to be the set of all \( \omega \in w_n^{\text{NBW}}(x, y) \) with \( \omega_1 \neq e_i + x \).

Let \( b_n(x) = |w_n^{\text{NBW}}(0, x)| \) and \( b_n^i(x) = |w_n^{\text{NBW}}(0, x)| \), where \( |\cdot| \) denotes the number of elements in the set. For \( b_n^i(x) \) we see that

\[ b_n^i(x) = |w_n^{\text{NBW}}(0, x)| = |\{\omega : \omega \in w_n^{\text{NBW}}(0, x), \omega_1 \neq e_i\}| = |\{\omega : \omega \in w_{n+1}^{\text{NBW}}(e_i, x), \omega_1 = 0\}| = |\{\omega : \omega \in w_n^{\text{NBW}}(0, x-e_i), \omega_1 = 0\}|. \]  

We can rewrite (1.5.14) to

\[ \hat{B}_\mu(k) = 1 + \mu \cdot \text{some expression} + \mu \cdot \text{some expression}. \]
1.5 Mean-field models and behavior

Since $\mathbf{D}(k) = J \mathbf{D}(-k)$ and $(\mathbf{D}(k))^{-1} = \mathbf{D}(-k)$ we can compute

$$[\mathbf{D}(k) + \mu \mathbf{J}]^{-1} = \frac{1}{1 - \mu^2} (\mathbf{D}(-k) - \mu \mathbf{J}).$$

(1.5.17)

As $\mathbf{1}^T \mathbf{D}(k) \mathbf{1} = 2d \hat{D}(k)$ and $\mathbf{1}^T \mathbf{J} \mathbf{1} = 2d$ we conclude

$$\hat{B}_\mu(k) = \frac{1}{1 + (2d - 1)\mu^2 - \frac{2d\mu}{1 - \mu^2} \hat{D}(k)} = \frac{1 - \mu^2}{1 + (2d - 1)\mu^2 - 2d\mu \hat{D}(k)}.$$  

(1.5.18)

The susceptibility of NBW is given by $\chi_{NBW}(\mu) = \hat{B}_\mu(0)$ with critical point $\mu_c = 1/(2d - 1)$. Remarkably, NBW and SRW two-point functions are linked by

$$\hat{B}_\mu(k) = \frac{1 - \mu^2}{1 + (2d - 1)\mu^2 - 2d\mu \hat{D}(k)} = \frac{1 - \mu^2}{1 + (2d - 1)\mu^2 - 2d\mu \hat{D}(k)}.$$  

(1.5.19)

This link allows us to compute values for NBW two-point function in $x$ and $k$ space using known SRW algorithms, see Section 5.2.

**Critical exponents.** As for SRW, we will review the value of the critical exponents for NBW. First, we compute

$$\chi_{NBW}(\mu) = \hat{B}_\mu(0) = \frac{1}{1 - \frac{2d\mu}{1 - \mu^2}(1 - \mu)} = \frac{1 + \mu}{2d - 1} \left( \frac{1}{2d - 1} - \mu \right)^{-1},$$  

(1.5.20)

so that $\gamma = 1$. In Chapter 7 we derive a closed form expression for $\hat{b}_n(k)$ and determine $\Delta_k^2 \hat{b}_n(k) |_{k=0}$ (analog to (1.5.10)) to compute that the average displacement is given by

$$\sum_{x \in \mathbb{Z}^d} \frac{\|x\|^2_2 b_n(x)}{2d(2d - 1)^{n-1}} = d \frac{d}{d - 1} n + 4d - 1 \frac{4d - 1}{2(d - 1)^2} + d \frac{d}{2(d - 1)(2d - 1)^{n-2}},$$  

(1.5.21)

see Lemma 7.1.7. From this follows that $\nu = 1/2$ for NBW. Further, we can conclude for (1.5.6) and (1.5.19) that $\eta = 0$ as

$$\hat{B}_{\mu_c}(k) \approx \frac{2(d - 1)}{2d - 1} \frac{1}{\|k\|^2_2}$$  

for small $k$.  

(1.5.22)

1.5.3 Branching random walk

Branching random walk is a generalization of simple random walk and branching process. Branching processes describe the evolution of a population over time. The most common formulation of a branching process is the Galton-Watson process. In the branching random walk a spatial component is added to the evolution of the population. Thus, we study the movement of particles/walkers in space. Each particle has the chance to die or to give offspring to new particles at any given time. For a review of branching processes we refer the reader to [9] or [45].
Branching random walk. We now define branching random walk (BRW). We begin with one particle. This particle arrives at time 0 at the origin and dies. While dying it sends to each of the \(2d\) neighboring points independently a new particle with probability \(p\). These new particles arrive at the corresponding neighboring points at time 1 to die there and give again birth to new particles, following the same rule as the original particle. Results about such a simple model can be found in [84, Chapter 4 and 7]. Please note that this is slightly different from the usual definition of BRW, where the newborn particle choose its self where to go. In our definition the parent particle tells the children where to go.

The branching process underlying BRW is a discrete-time branching process with a Binomial offspring distribution with parameters \((p, 2d)\). We define the BRW two-point function by

\[
G_p(x) = E_p[\text{number of particles that have died at } x] \quad (1.5.23)
\]

and use the independence of the particles to compute

\[
G_p(x) = \delta_{0,x} + \sum_i E_p[\text{number of particles that have died at } x - e_i] = \delta_{0,x} + 2d p(D \star G_p)(x). \quad (1.5.24)
\]

From this it follows that

\[
\hat{G}_p(k) = \frac{1}{1 - 2d p \hat{D}(k)}, \quad (1.5.25)
\]

so that \(\hat{G}_p(k) = \hat{C}_p(k)\). We define the susceptibility of BRW by

\[
\chi^{\text{BRW}}(p) = \hat{G}_p(0) = E[\text{number of particles that have died}] = \frac{1}{1 - 2d p} = \frac{1}{2d \left( \frac{1}{2d} - p \right)}^{-1}, \quad (1.5.26)
\]

with critical value \(z_c = 1/(2d)\). Thus, the susceptibility and the two-point function of BRW and SRW are the same and we conclude that \(\gamma = 1\) and \(\eta = 0\) for BRW. For branching random walk, the susceptibility corresponds to the expected total progeny of the branching process. We also compute the mean-square displacement in the \(n\)th generation

\[
\sum_x \|x\|^2 E_{z_c}[\text{number of particles that died at } x \text{ at time } n] = \sum_x \|x\|^2 p_n(x) z_c^n = n, \quad (1.5.27)
\]

so that we \(\nu = 1/2\) as for SRW. When we condition on the size of the total population \(T\) of the branching process we obtain

\[
\sum_{x \in \mathbb{Z}^d} \|x\|^2 E_p[\text{number of particles that have died at } x \mid T = n] = E_p[\text{average time at which a particle dies } \mid T = n] \approx n^{1/2} = n^{2\nu}, \quad (1.5.28)
\]
so that \( \nu' = 1/4 \). The relation [1.5.28] follows from standard arguments about Ulam-Harris tree and Galton-Watson tree. For branching random walks we compute an analog to critical exponent \( \beta \) of percolation, which describes the probability that the BRW dies out:

\[
\zeta(p) = \mathbb{P}[\text{number of particles that have died} < \infty] = (1 - p + 2d \rho \zeta(p))^{2d}.
\] (1.5.29)

For all \( p < 1/2d \), we can compute

\[
\mathbb{P}(|\mathcal{T}| = \infty) = 1 - \zeta(p) = c \left( p - \frac{1}{2d} \right) (1 + o(1)),
\] (1.5.30)

for a constant \( c > 0 \), see e.g. [48, Proof of Theorem 1.1]. Thus, the critical exponent \( \beta \) is given by 1.

**Non-backtracking branching random walk.** We introduce a second example of a branching walk, that we call non-backtracking branching random walk (NBBRW). As for BRW, we begin with one particle. This particle arrives at time 0 at the origin and dies. While dying the particle independently sends a new particle to each of the \( 2d \) neighboring points with probability \( p \). These new particles arrive at the neighboring points at time 1 to die there. While dying each particle sends new particles to each of the neighboring points with probability \( p \), except to the point where it was born. The following generations of particles repeat this behavior. They die at time \( n \) and possibly produce offspring which they send to the neighboring points, except for the birthplace of their parent particle.

The branching process underlying this NBBRW has a Binomial offspring distribution with parameters \( (p, 2d) \) for the first steps and \( (p, 2d - 1) \) for all the other steps. We define the *NBBRW two-point function* by

\[
G_p(x) = \mathbb{E}_p[\text{number of particles that have died at } x].
\] (1.5.31)

We see that for NBBRW recursive relations analog to [1.5.13] hold and conclude

\[
G_p(x) = B_p(x) = \frac{1 - \mu^2}{1 + (2d - 1) \mu^2 - 2d \mu \hat{D}(k)}.
\] (1.5.32)

Therefore, the *susceptibility* of NBBRW is also the same as for NBW and the *critical value* is given by \( z_c = 1/(2d - 1) \). Moreover, the critical exponents are the same as for BRW, \( \gamma = 1 \) and \( \eta = 0 \). As for BRW we know for \( p < 1/(2d - 1) \) that

\[
\mathbb{P}(|\mathcal{T}| = \infty) = 1 - \zeta(p) = c \left( p - \frac{1}{2d - 1} \right) (1 + o(1)),
\] (1.5.33)

and that the critical exponent \( \beta \) equals 1.
1.5.4 Heuristics of the mean-field conjecture

In this section we give a heuristic argument why we expect mean-field behavior for SAW, LT, LA and percolation and give an interpretation of the upper critical dimension. We start with SRW, which is the mean-field model for SAW. The two-point function characterizes the connection between two points. For SRW the critical two-point function \( C_{1/2d} \) corresponds to the Green’s function:

\[
C_{1/2d}(x) = \mathbb{E}[\text{number of visits to } x \text{ by a SRW}].
\] (1.5.34)

We define the SRW-bubble by

\[
B_{SRW}(z) = \sum_{x \in \mathbb{Z}^d} C_z(x)^2,
\] (1.5.35)

and see that the SRW-bubble at criticality also bears a probabilistic interpretation:

\[
B_{SRW}(z) = \sum_{n, m \in \mathbb{N}} \mathbb{P}(\omega_1^n = \omega_2^m) = \int_{[-\pi, \pi]^d} (\hat{C}_z(k))^2 \frac{d^d k}{(2\pi)^d},
\] (1.5.36)

where \( \omega_1 \) and \( \omega_2 \) are two independent SRWs. Thus, \( C_{1/2d}(x) \) describes the expected number of visits to \( x \) and the SRW-bubble the expected number of intersections of two independent SRW-paths. By Borel-Cantelli we know that if \( C_{1/2d}(x) < \infty \), then the walk is transient at \( x \), and that if \( B_{SRW}(1/2d) < \infty \), then two independent, infinite SRW-paths only intersect finitely often \( \mathbb{P}\)-a.s. As \( \eta = 0 \) for SRW \( \eta = 0 \), we conclude that

\[
C_{1/2d}(x) = \int_{[-\pi, \pi]^d} \hat{C}_z(k) e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d} \leq c_1 \int_{[-\pi, \pi]^d} \frac{1}{\| k \|^2} \frac{d^d k}{(2\pi)^d} = \infty \quad \text{if } d \leq 2,
\] (1.5.37)

\[
B_{SRW}(z) = \int_{[-\pi, \pi]^d} (\hat{C}_z(k))^2 \frac{d^d k}{(2\pi)^d} \leq c_1^2 \int_{[-\pi, \pi]^d} \frac{1}{\| k \|^4} \frac{d^d k}{(2\pi)^d} < \infty \quad \text{if } d > 2.
\] (1.5.38)

We saw that the critical exponents of SRW and NBW are the same for \( d \geq 2 \). Thus, the addition of the non-backtracking condition did not change the asymptotic behavior, even if an infinite-step SRW will not be a NBW \( \mathbb{P}\)-a.s. A memory-\( m \) walk is a SRW that does not have self-intersections within a finite memory \( m \), i.e. \( \omega_i \neq \omega_{i+j} \) for \( j \neq m \) and all \( i \). Hence, NBW is a memory-2 walk. It is believed that the finite interaction of the memory-\( m \) walks does not influence the asymptotic behavior of the system on the level of exponents. Thus, the critical exponents of the memory-\( m \) walk should have the same value as for SRW, because any finite interaction should be lost in the scaling limit.

When we consider a SAW, a memory-\( \infty \) walk, the interaction range is infinite and can change its asymptotic behavior. We first argue why the asymptotic mean-square
displacement differs and then give a more abstract reasoning for the difference of the behavior for \( d \leq 4 \) and \( d > 5 \). It is expected that a uniformly chosen SAW \( n \)-step walk with \( n \gg 1 \) is less likely to end close to the origin. Thus, SAW can be expected to have a bigger mean-square displacement than SRW. For \( d > 5 \) the SRW-bubble is finite, so that a SRW has only finitely many self-intersections \( \mathbb{P} \)-a.s. Therefore, the self-avoidance constraint to not hit any point near the origin does not influence the walk very much as a typical SRW will not return to the early history very often anyway. Thus, the critical exponent \( \nu \) of SAW can still be \( 1/2 \). For \( d < 4 \) the bubble is infinite, so that an infinite SRW has infinitely many self-intersections, while a SAW is not allowed to intersect even once. This forces SAW to leave regions of the lattice fast where it already walked, thus causing the system to grow faster than the diffusive SRW, \( \nu_{\text{SAW}} > \nu_{\text{SRW}} = 1/2 \). Interesting here is the upper critical dimension \( d_c = 4 \) for which the influence of the self-avoidance constraint is expected to be just strong enough to change the asymptotic behavior by adding an logarithmic term. Let us now review the value of the upper critical dimension \( d_c = 4 \) using the dimensionality of the trajectory of a walks. SRW converges, properly scaled, to Brownian motion, which is known to have Hausdorff dimension 2. Thus, we can say that the path of an infinite SRW is a two-dimensional object and we can expect that the path of all walks are two-dimensional. Let us consider an infinite step SAW that we split at some point into two infinite pieces \( \omega_1, \omega_2 \). In \( d > 2 + 2 = 4 \) dimensions the walks \( \omega_1 \) and \( \omega_2 \), that are not allowed to intersect, have enough “space” to be both two-dimensional and both of them can behave almost independently. In \( d < 2 + 2 = 4 \) the lattice simply does not give both the walks \( \omega_1, \omega_2 \) enough space to not interact. Therefore, neither of them can be two-dimensional which prohibits the walk to have the same asymptotic behavior as SRW. At the critical dimension \( d_c \) the interaction between the two pieces \( \omega_1 \) and \( \omega_2 \) causes a logarithmic correction in (1.3.1) and (1.3.3).

In an effort to make this comparison of SRW and SAW rigourous the loop-erased random walk was invented by Lawler (1980) \cite{68}. A loop-erased random walk (LERW) is created by taking a SRW and removing all loops that have been created by the self-intersections. Thus, each path of a LERW is also a SAW. It was hoped that LERW can be used to study properties of SAW. However, it turned out that LERW has a different asymptotic behavior as SAW in \( d = 2, 3, 4 \) and became an interesting model to study on itself. As for \( d \geq 5 \) self-intersections happen only finitely often LERW is expected to have in these dimensions the same asymptotic behavior as SRW and SAW. We refer the interested reader to \cite{69} or \cite{70}.

For a similar argument for LA, LT and percolation we define the SRW-triangle and
the SRW-square by

\[
T(z) = \sum_{x,y \in \mathbb{Z}^d} C_z(x)C_z(x-y)C_z(y) = \int_{[-\pi,\pi]^d} (\hat{C}_z(k))^3 \frac{d^d k}{(2\pi)^d},
\]

(1.5.39)

\[
S(z) = \sum_{x,y,u \in \mathbb{Z}^d} C_z(x)C_z(y-x)C_z(u-y)C_z(u) = \int_{[-\pi,\pi]^d} (\hat{C}_z(k))^4 \frac{d^d k}{(2\pi)^d}.
\]

(1.5.40)

As \(\hat{C}_{z_c}(k) \approx \|k\|_2^{-2}\) for small \(k\) and we can bound \(\hat{C}_{z_c}(k)\) uniformly for non-small \(k\), we conclude that the SRW-bubble condition is satisfied for \(d > 4\), the SRW-triangle condition is satisfied for \(d > 6\) and the SRW-square condition is satisfied for \(d > 8\).

BRW as a branching version of SRW is \(2^2 + 2 = 4\) dimensional and the critical square \(S(1/2d)\) is finite exactly when two critical branching random walks intersect only finitely often. BRW is the mean-field model for LT and LA. As argued for SAW, LT and LA can only be expected to show mean-field behavior in \(d > 8\) as in smaller dimensions the interaction between the pieces is too strong. We refer the interested reader to [92, Chapters 15 and 16] for a more detailed intuitive argument.

In most literature percolation on a regular tree is considered to be the mean-field model for percolation on the lattice. For our explanation we will consider BRW to be the mean-field model and make the link to percolation using an algorithm called cluster exploration. We define the cluster exploration as follows:

1.) We start by declaring all bonds of the lattice to be closed.

2.) We independently open each of the \(2d\) edges incident to the origin with probability \(p\).

3.) We define \(S\) to be the set of all vertices \(x \neq 0\) to which the origin is now connected by an open bond \(b = (0, x)\) and call the origin explored.

4.) For all vertices \(v \in S\) we now proceed as follows:

   i.) We independently open each bond \(b\) that contains \(v\) with probability \(p\), if the other point of the bond \(b\) is not already explored.

   ii.) We declare \(v\) to be explored.

5.) We redefine \(S\) to be the set of all unexplored vertices to which we have opened a bond in the last step.

6.) If \(S\) is non-empty then we repeat the procedure starting in 4.) with the updated set \(S\).

7.) If \(S\) is empty then we define \(E(d)\) to be the set of all vertices that have been explored.
If \( p \) is smaller than the critical bond probability \( p_c \) then this procedure \( \mathbb{P}_p \)-a.s. terminates after a finite number of steps. For \( p > p_c \) there is a positive probability that the algorithm never terminates. It is easy to see that \( E(d) \) has the same distribution as the percolation cluster of the origin \( C(0) \).

The cluster exploration differs from a BRW only in the sense that it does not open edges to vertices that have already been explored. If the square diagram is finite then two BRWs intersect only finitely often, so that we can expect the difference between the exploration and BRW not to be too big. Thus, it is reasonable to expect mean-field behavior for percolation in \( d > 8 \).

1.6 Lace expansion

In the preceding section we have introduced the concept of mean-field behavior and have given an intuition of why it can be expected above the model’s upper critical dimension. The lace expansion (LE) is a perturbative technique to prove mean-field behavior in high dimension. It was first introduced by Brydges and Spencer \([22]\) to show mean-field behavior for the weakly self-avoiding walk. With the subsequent extension to SAW, percolation, LT, LA, and the Ising model, the lace expansion has become an indispensable tool to prove mean-field behavior of statistical mechanical models above their upper critical dimension.

The LE can be used to prove that the two-point function of the general system \( G_z \) is a perturbation of the two-point function \( C_z \) of SRW. Being a perturbative method in nature, applications of the lace expansion typically necessitate a small parameter. This small parameter tends to be the degree of the underlying graph. There are three possible approaches to obtain a small parameter. The first is to work in a so-called spread-out model, where long but finite-range connections over a distance \( L \) are possible, and we take \( L \) large. This approach has the advantage that the results hold, for \( L \) sufficiently large, all the way down to the critical dimension of the corresponding model.

The second approach is to consider weakly-interacting models, in which intersections are not forbidden, but just penalized. This approach is not applicable to percolation. By making the interaction weak enough the mean-field result can be
shown down to the upper critical dimension, see [22], [16], [92, Section 15.3]. The third approach applies to the nearest-neighbor setting, which we discuss in this thesis. For the nearest-neighbor model, the degree of a vertex is $2d$ which then has to be taken to be large in order to prove mean-field results. Thus, we need to take the dimension large, and therefore obtain suboptimal results in terms of the dimension above which the results hold.

**Lace expansion for SAW.** The LE was first introduced to study weakly self-avoiding walk by Brydges and Spencer [22]. Then it was developed further by Hara and Slade who obtained in 1991 [42], [41] the seminal result of mean-field behavior for all dimensions above the upper critical dimension $d_c = 4$ in the nearest-neighbor setting. To achieve this they use an extensive analysis of the perturbation term and a computer assisted-proof to show that in $d \geq 5$ the perturbation is small enough to obtain mean-field behavior.

**Lace expansion for LT and LA.** Hara and Slade developed the LE for LA and LT in [39]. They prove that the square condition

$$S(z_c) = (\bar{G}_{z_c} \ast \bar{G}_{z_c} \ast \bar{G}_{z_c} \ast \bar{G}_{z_c})(0) = \bar{G}_{z_c}^4(0) < \infty \quad (1.6.1)$$

holds in sufficiently high dimensions and prove that $\nu = 1/4$. By [98] and [44] the square condition implies that $\gamma = 1/2$. Thus, they have proven mean-field behavior in sufficiently high dimension. They never verified what dimension is sufficiently high as they expected that their result would only hold in very high dimensions $d \gg 8$.

**Lace expansion for percolation.** In [38] Hara and Slade developed the LE for percolation and proved that the percolation triangle condition

$$T(p_c) = (\tau_{p_c} \ast \tau_{p_c} \ast \tau_{p_c})(0) < \infty \quad (1.6.2)$$

holds in sufficiently high dimensions. By Aizenman and Newman [4] and Aizenman and Barsky [3] the triangle condition implies that the critical exponents $\gamma, \beta$ exist with $\gamma = 1$ and $\beta = 1$. Thus, finiteness of the triangle implies mean-field behavior. Hara and Slade used a computed-assisted proof to verify for which dimensions they can show the finiteness of the triangle. Over the years they have improved their technique. Initially, they stated that $d \geq 92$ is high enough (1989) [37]. In 1990 [38] they improved this to $d \geq 48$ and in 1994 announced $d \geq 19$, a number that is still used in literature. Through private communication with Takashi Hara the author learned that in a recent rework of the analysis and implementation the result was further improved to $d \geq 15$.

Hara and Slade never published the required computation as they could not prove
the mean-field result down to the upper critical dimension $d_c = 6$. In [38] they remark: “Our current best estimate is $d_0 = 48$, obtained by a slightly more complicated analysis than that presented in this paper. This value can doubtless be improved, but a new idea will be needed to obtain the triangle condition for the nearest-neighbour model right down to the expected upper critical dimension of six. The fact that we are unable to do much better than $d_0 = 48$ suggests that we still do not have a very efficient expansion for percolation.” One possible idea to obtain an improved technique is to extract more of the interaction into explicit contributions and obtain in this way an expansion with a smaller perturbation term. This idea is the starting point of this thesis.

Non-backtracking lace expansion. We develop the non-backtracking lace expansion (NoBLE) to improve the results obtained by using the LE in the nearest-neighbor setting. The LE perturbs the two-point function of the system $G_z$ around the SRW two-point function $C_z$. With the NoBLE we show that $G_z$ is a perturbation of the critical NBW two-point function $B_{1/(2d−1)}$ in high-enough dimensions. By perturbing around a NBW we make the biggest contribution of the perturbation in the LE into an explicit term and thereby reduce the size of the perturbation. In this way we are able to reduce the dimensions above which we can show mean-field behavior of the models. We elaborate this idea of a smaller perturbation in the next paragraph.

The idea of such an expansion originated from Remco van der Hofstad and was discussed with David Brydges and Gordon Slade already years ago. At this point it was not clear how to use this idea as perturbing around a NBW requires a new, more involved analysis. The work on this technique was only started when the author began his PhD-project. In Section 2.1 we compare the LE and NoBLE at the example of SAW and explain the general idea of the NoBLE. Here we only want to give an idea of why it is better to perturb around NBW than around SRW.

The size of the perturbation of the LE and NoBLE. The LE and NoBLE have the fundamental fault that they prove mean-field behavior only if the perturbation is small. Now we will discuss the expected size of the perturbation of SAW and percolation.

In the classical lace expansion we perturb $G_z$ around $C_z$. The contribution of walks with many steps is relatively small as every additional step adds a factor of $z ≈ 1/2d$, see [1.3.5]. Therefore, the biggest contribution to perturbation for SAW is created by self-intersections that are realized in a small number of steps. Indeed, the biggest contribution is created by back and forth movements $ω = \{0, x, 0\}$, with $∥x∥_2 = 1$, which create a perturbation of the size $2dz^2 ≈ 1/(2d)$. In the NoBLE we compare SAW to NBW: The smallest NBW-loop has four steps. As there are $2d(2d−2)$ four-step NBW-loops, the perturbation of the NoBLE is at least
of the size $2d(2d - 2)z^4 \approx (2d)^{-2}$. These contributions are actually the dominant contributions to the perturbation term.

For percolation we compare the cluster exploration, defined in Section 1.5.4 with BRW. The chance that a BRW does not correspond to a cluster exploration is at least $2^d p^2$ as each of the particles dying at time 1 could try to send a particle back to the origin. As $p_c \approx 1/2d$, see (1.2.11), we see that also for percolation the LE perturbation term is of the order $O(1/2d)$. When we compare the exploration process with NBBRW, the biggest perturbation of the NoBLE for percolation is also created by four-step loops and is thus of the order $(2d)^{-2}$. For both models we see that the NoBLE has a perturbation that is by an order of $1/(2d)$ smaller as the perturbation of the LE.

**Memory-$m$ lace expansion.** The idea of extracting explicit terms from the perturbation term to create an expansion with a smaller perturbation can be continued by extracting all loops with at most $m$ steps. This would then correspond to a memory-$m$ expansion, where we would perturb around a memory-$m$ walk. An $n$-step memory-$m$ walk is a SRW $\omega$ with $\omega_i \neq \omega_{i+j}$ for all $0 \leq i < i+j \leq n$ and $j \in [1,m]$.

For all finite $m$ it is possible to create such a memory-$m$ expansion for the models discussed in the thesis. Due to time constraints when writing the thesis, the derivation of such an expansion has not been included in the thesis. The problem with the memory-$m$ expansion is that the created relations for the two-point function become very complicated and the lace expansion requires, next to the expansion itself, an elaborate analysis that uses these relations to prove mean-field behavior. For the NoBLE, a memory-2 expansion, Remco van der Hofstad and the author create the required analysis (Chapter 3). The biggest obstacle to use the memory-$m$ expansion is the creation of the necessary analysis.

**Analysis of the lace expansion.** The lace expansion technique requires a sophisticated analysis of the relations created by the expansion. We created such an analysis for the NoBLE by adapting the analysis of [47]. This analysis uses the trigonometric method developed in [17] and also used in [92], where an introduction to the LE is given. To make the document as accessible as possible we orient our notation and the structure of the proofs at the general reference in the field [92].

This analysis is not the most effective in bounding the function $G_z(x;k) = G_z(x)(1 - \cos(k \cdot x))$, that is required for the analysis. Therefore, we derive a second analysis, that is an adaptation of the technique used by Hara and Slade [41] to prove mean-field behavior for SAW down to the upper critical dimension. The second analysis is technically more demanding, but produces better bounds. The idea for this second analysis were developed in discussion with Takashi Hara in 2012. The first analysis is explained in Section 3.2-3.4. In Section 3.5-3.6 we explain the second analysis.
Other analyses of the lace expansion. For several models there exist two more ways to analyze the relations created by the LE. When the model has a time parameter we can use an inductive approach to bound the perturbation and prove mean-field behavior. This analysis was invented by van der Hofstad, van der Hollander and Slade in [49] for weakly SAW. The major advantage of this technique is that the result can be obtained without the use of generating functions, so that $c_n(x)$ (for SAW) and $t_n(x)$ (for LT) are analyzed directly.

In [54] a generalized analysis is developed that is used in [56], [55], [52], [53] to analyze SAW, oriented percolation, lattice trees and the contact process in the spread-out models. Further, Ueltschi [100] used the techniques of [54] to prove that the connective constant exists and that the system has diffusive behavior for a spread-out SAW with nearest-neighbor attractions.

Another approach to analyze the lace expansion was invented by Bolthausen and Ritzmann [15], and is based on a fixed-point argument. Until now this method has been only used for weakly SAW. Recently, this technique has been extended to analyze the continuous-space weakly SAW [10].

Lace expansion for other models. The lace expansion was also used for models not discussed in this thesis. The most prominent of these model is the Ising model, a simple ferromagnetic model, that is expected to show mean-field behavior above dimension four. Sakai [90] developed the LE for the Ising model to prove mean-field behavior in the spread-out setting, see also [47]. For the nearest-neighbor setting the finiteness of the bubble of the Ising model, which implies mean-field behavior for this model, can be obtained using reflection positivity. For the spread-out model such a relation is not available.

Further, the LE has been used to analyze oriented percolation [77], [55]. It is proven in [78] that oriented percolation shows mean-field behavior for all dimensions above the upper critical dimension $d_c = 4$ in the spread-out setting with $L$ big enough and for sufficiently high dimensions for the nearest-neighbor setting. The LE for oriented percolation can be adapted to create a LE for contact processes, see [89], [52].

1.7 Results

1.7.1 Result obtained by the NoBLE

In this thesis we develop the non-backtracking lace expansion (NoBLE) to improve mean-field results obtained by the lace expansion (LE) in the nearest-neighbor setting. The LE can only be applied if the perturbation is small enough. We create the NoBLE that has a perturbation which is one order $1/d$ smaller than the perturbation of the LE. For this reason, we can apply the technique in dimensions closer
to the upper critical dimension.

We now give the main statement of the thesis, which we state for all models at once.

**Theorem 1.7.1** (Infrared bound). *For each of the models SAW, LT, LA, and percolation there exists a dimension $d_0 > d_c$, given in Table 1.3, such that there exist $A(d) > 0$ with*

$$
\hat{G}_z(k) \leq \frac{A(d)}{\chi^{-1}(z) + [1 - \hat{D}(k)]}
$$

**(1.7.1)**

*uniformly for $z \leq z_c$.*

Theorem 1.7.1 implies that $\hat{G}_{z_c}(k) \sim \|k\|^{-2}$, see (1.5.6).

<table>
<thead>
<tr>
<th>mean-field behavior</th>
<th>SAW</th>
<th>LT</th>
<th>LA</th>
<th>percolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>expected for</td>
<td>$d \geq 5$</td>
<td>$d \geq 9$</td>
<td>$d \geq 9$</td>
<td>$d \geq 7$</td>
</tr>
<tr>
<td>reported in the literature in</td>
<td>$d \geq 5$</td>
<td>sufficiently high</td>
<td>sufficiently high</td>
<td>$d \geq 19$</td>
</tr>
<tr>
<td>proven using Section</td>
<td>$d \geq 8$</td>
<td>$d \geq 29$</td>
<td>$d \geq 49$</td>
<td>$d \geq 38$</td>
</tr>
<tr>
<td>proven using Section</td>
<td>not computed</td>
<td>$d \geq 20$</td>
<td>$d \geq 21$</td>
<td>$d \geq 15$</td>
</tr>
<tr>
<td>value of $d_0$</td>
<td>$7^*$</td>
<td>20</td>
<td>21</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 1.3: Result concerning mean-field behavior for the different models. * In Chapter 6 we prove that SAW shows mean-field behavior for $d \geq 7$.

We identify the value of $d_0$ using an computer-assisted proof. The computations of the computer-assisted proof also create bounds on various quantities of interest, see Tables 1.4 | 1.5.

The infrared bound implies that $\eta = 0$ in Fourier space and allows us to conclude whether the bubble, triangle, and square diagrams are finite. We can conclude mean-field behavior for a given model from this.

For SAW we see that the SAW-bubble is finite in $d \geq d_0 = 8$. Using the techniques of [74 Chapter 6] we can conclude that $\gamma = 1$ and $\nu = 1/2$ take their mean-field values.

For LT and LA we know by [18] that $\gamma \geq 1/2$ for all dimensions. We prove here that the LT-square is finite for $d \geq 20$ and the LA-square is finite for $d \geq 21$. This implies by [44] that $\gamma = 1/2$ for these dimensions.

For percolation Theorem 1.7.1 implies that the triangle condition holds for $d \geq 15$. This implies by [4] and [11] that $\gamma = 1$ and $\beta = 1$, see [11]. In particular, we then also know that the percolation probability $p \rightarrow \theta(p)$ is continuous for $d \geq 15$. 
## 1.7 Results

<table>
<thead>
<tr>
<th>dimension $d$</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>bound on $p_c(d)$</td>
<td>0.0360131</td>
<td>0.033563</td>
<td>0.0314314</td>
<td>0.017266</td>
</tr>
<tr>
<td>bound on $(2d-1)p_c(d)$</td>
<td>1.04438</td>
<td>1.04045</td>
<td>1.03724</td>
<td>1.01869</td>
</tr>
<tr>
<td>bound on $A(d)$</td>
<td>1.05935</td>
<td>1.03455</td>
<td>1.02501</td>
<td>1.00418</td>
</tr>
</tbody>
</table>

Table 1.4: Percolation: Rigorous numerical upper bound on the critical bond probability and the constant $A(d)$ of Theorem 1.7.1.

<table>
<thead>
<tr>
<th>dimension $d$</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>bound on $g_z(d)$</td>
<td>2.8668</td>
<td>2.85617</td>
<td>2.84722</td>
<td>2.80475</td>
</tr>
<tr>
<td>bound on $A(d)$</td>
<td>1.12417</td>
<td>1.1009</td>
<td>1.08635</td>
<td>1.0427</td>
</tr>
</tbody>
</table>

Table 1.5: Lattice trees: Rigorous numerical upper bound on the one point function and the constant $A(d)$ of Theorem 1.7.1.

### Discussion of the value of $d_0$.

The results are not optimal in the sense that we can not prove the results down to the upper critical dimension. Hara and Slade proved the optimal result for SAW. We were not able to reestablish this result with the techniques derived in this thesis. We are convinced that we could do this by using the analysis of Section 3.5 and more elaborate bounds on the perturbation. However, we did not try to improve our technique for SAW as the optimal result is already known.

For LT and LA we computed the size of the perturbation in a given dimension and were able to prove mean-field behavior for $d \geq d_0$, see Table 1.3 where $d_0 = 20$ for LT and $d_0 = 21$ for LA. With a more elaborate analysis of the perturbation the value of $d_0$ can be improved. Especially, the bound on the first terms of the perturbation can be easily improved by an explicit computations of the leading behavior. For the thesis we decided to use the bounds proven in Sections 4.3, 4.4 that are designed to be simple to state, while still being effective. For LT and LA it is remarkable that the analysis of Section 3.5 proves the results in comparable dimensions (LT 20, LA 21), while the analysis of Section 3.3 works for LA only in dimensions much high than for LT (LT 29, LA 49). We explain the reason for this at the end of Chapter 4.

For percolation we were able to improve the result of Hara and Slade ($d \geq 19$) and meet the newer result of Hara ($d \geq 15$). The NoBLE may not be as effective as we had hoped for, which exemplifies that the bounds used in the unpublished computations of Hara and Slade are quite sophisticated. With more elaborate bounds on the perturbation we might be able to extend the result to $d \geq 12$. We explain ideas of how to improve the results for LT, LA and percolation at the end of Chapter 4.
**General discussion.** With the NoBLE we give the first verification above which dimension mean-field behavior holds for LT and LA. Further, we give the first complete proof of mean-field behavior for percolation above $d \geq 15$.

We create an analysis for the matrix-valued equation created by the NoBLE and bound the perturbation terms using the structure of the perturbation. Moreover, we give in Chapter 5 an overview of how to compute the perturbation numerically, which has not been done in literature since 1992 [41] and contains many novel ideas.

While reducing the size of the perturbation of the LE we are not able to prove mean-field behavior for all dimensions above the upper critical dimensions. This is not too surprising as the NoBLE also requires the perturbation to be small. In dimensions just above the upper critical dimensions the effect of the interaction will be quite strong, even if it is supposed not to affect critical exponents.

As already indicated we could develop our technique further by either perturbing around a memory-$m$ walk and/or by the explicit computation of the biggest perturbation terms. For a memory $m$-lace expansion, it is not clear whether the benefit of a small perturbation is not canceled by the more elaborate analysis required for such an expansion. In an explicit computation we would enumerate the structures contributing to the perturbation terms. The logical way to do this would be a computed-based algorithm. Both approaches would require much time and effort and will not offer any more insight into the problem. Further, it is not clear whether these two ideas would allow us to prove the mean-field results for all dimensions above the upper critical dimension.

**Computations.** The NoBLE uses a computer assisted-proof to show mean-field behavior. The computations were done using Mathematica-notebooks, that are available on the website of the author as Mathematica notebooks and PDF’s. We use Mathematica for our computations as it allows us to combine numerical precision and readability. The Mathematica notebooks can are formatted, which makes them more readable in a printed version. Further, they can be understood by readers that have no prior knowledge of programming. On the other hand, Mathematica allows us to perform the computations up to any given precision. As we do not have very demanding computation we can omit a detailed discussion of numerical errors, see Section 5.5.

### 1.7.2 An alternative expansion based on LE

In the process of creating NoBLE we created three different lace expansion for SAW that use the non-backtracking idea. In Chapter 6 we show one of these techniques that can be used to prove mean-field behavior for SAW also in dimension $d = 7$.

The expansion of Chapter 6 is created by extracting all two-step loops from the perturbation term of LE. After a minor modification we can use the analysis of
Section 3.3 to prove mean-field behavior for $d \geq 7$. It is not clear whether we can use a similar idea for the other models as well.

1.7.3 Non-backtracking random walk results

In Chapter 7 we analyze NBW, introduced in Section 1.5.2, on the $\mathbb{Z}^d$-lattice and on tori. We use Fourier analysis to derive the $2d \times 2d$-dimensional transition matrix of NBW on the lattice. We evaluate the eigensystem of the transition matrix and use its properties to show a functional central limit theorem for NBW on $\mathbb{Z}^d$ and to obtain estimates on the convergence towards the stationary distribution for NBW on the torus.

In the beginning of the work on the NoBLE we tried to analyze the created equations using matrix perturbation theory. This required a good understanding of the NBW. In particular, we needed to understand the properties of its transition matrix around which we wanted to perturb. Unfortunately, we did not find a good way to bound the perturbation in the matrix valued equation, as the transition matrix lacks most of the properties using the classical matrix perturbation theory. We solved this problem by reformulating the NoBLE to obtain scalar-valued equations and analyzed these equations, see especially Section 3.4.

1.8 Structure of the thesis

At the end of this first chapter we want to give the reader an overview over the chapters lying ahead. In Chapters 2 to 5 we derive the NoBLE technique. We now explain the structure of the NoBLE, and in doing so the function of Chapters 2 to 5.

Classically, the lace expansion consists of three parts: expansion, bounding and analysis. First, an expansion is created that characterizes the two-point function using a recursive relation that resembles a corresponding relation of the mean-field model. This relation includes a coefficient that characterizes the perturbation. We bound the coefficient by a combination of simple diagrams, like bubbles and triangles. The analysis yields a bound on simple diagrams of the order $1/d$, and if these bounds are small enough then the analysis can be used to prove the infrared bound. The statement that the simple diagrams are of the order $1/d$ is the reason that most results are stated for sufficiently high dimensions. When we take $d$ big enough the perturbation will be small enough to succeed.

We actually want to compute the dimension above which we can prove our result. Therefore, we need to add a fourth step in our analysis, namely explicit computations to bound the perturbation numerically. In Figure 1.8 we give a small pictorial representation of the structure of the NoBLE.

In Chapter 2 we create the expansion by creating relations for the two-point functions that resemble the relations (1.5.13) of NBW. These relations characterize the
Figure 1.3: Structure of the non-backtracking lace expansion.

perturbation in terms of the lace-expansion coefficients $\Xi_z, \Xi_z^l, \Psi_{z}^k, \Pi_{z}^{\kappa}$. In Chapter 3 we explain the two analyses that we use to prove mean-field behavior. We do this in a generalized way, so that the same analysis can be used for all four models.

In Chapter 4 we prove that the coefficients can be bounded by combinations of bubbles, triangles and square diagrams.

In Chapter 5 we explain how we bound the diagrams by using simple combinatorial arguments and explicit computation of SRW-integrals. For these bounds to hold, we need to assume a priori bounds on the two-point function in terms of the SRW two-point function. These a priori bounds are established in the analysis of Chapter 3. We give an idea of the techniques used at the beginning of each chapter.

The last two chapters are not part of the NoBLE. In Chapter 6 we explain an alternative lace expansion with which we can show mean-field behavior for the SAW in $d \geq 7$. In the final chapter we analyze NBW.
Chapter 2
Expansions

This chapter is devoted to the derivation of the so-called non-backtracking lace expansion (NoBLE). The NoBLE is a modification of the classical lace expansion (LE) as introduced by Brydges and Spencer [22] for self-avoiding walk (SAW) and by Hara and Slade for lattice trees (LT), lattice animals (LA) [39] and percolation [38]. The LE is a perturbation technique that is able to show that the two-point function of a model behaves like the two-point function of simple random walk (SRW) in sufficiently high dimensions. We adapt this perturbation argument. Instead of perturbing around SRW, we perturb around non-backtracking random walk (NBW). This removes the biggest correction term of the LE and are able to use the lace expansion analysis successfully in dimension closer to the upper critical dimensions.

We begin this chapter with a comparison of the LE and the NoBLE for SAW in Section 2.1. Further, we explain the idea of the NoBLE for general models on a heuristic level in the first section. Then, we derive the expansions for the different models. First, we use a graph-based proof for SAW, LT and LA. To make this argument as accessible as possible we keep the notation and arguments as close as possible to [92]. Then, we use the inclusion-exclusion argument outlined in the end of Section 2.1 to derive the expansion for percolation, being from a technical point of view the most difficult expansion considered in this thesis.

2.1 Classical lace expansion vs. NoBLE

There are two ways to obtain the lace expansions for SAW. Either we can use a family of graphs to describe the self-avoidance constraint and then expand this graph-based formulation or we can perform an inclusion-exclusion procedure for the number of walks. The first way, using graphs, was introduced by Brydges and
Spencer [22] to study the weakly self-avoiding walk and offers a very elegant way to derive the lace expansion. For this reason we use the graph-based approach to derive the NoBLE for SAW in Section 2.2.

As the inclusion-exclusion approach gives more insight into the difference between the expansion we use it to make the comparison. Then we generalize the inclusion-exclusion idea of the NoBLE to the other models.

We define $W_n^{SAW}(x, y)$ to be the set of $n$-step self-avoiding walks with $\omega_0 = x$ and $\omega_n = y$ and recall the definition of $c_n(x)$ as the number of $n$-step SAWs starting at 0 and ending at $x$, see Section 1.3 i.e., $c_n(x) = |W_n^{SAW}(0, x)|$.

**Classical lace expansion.** In the LE we aim at reproducing the simple random walk recursion formula for $n \geq 1$:

$$p_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} 2dD(y)p_n(x-y) = \sum_{\kappa \in \{\pm 1, \pm 2, \ldots, \pm d\}} p_n(x- \kappa). \quad (2.1.1)$$

For brevity we omit the domain $\{\pm 1, \pm 2, \ldots, \pm d\}$ when summing over $\kappa$. The LE can be derived as follows: We sum over all possible directions $\kappa \in \{\pm 1, \pm 2, \ldots, \pm d\}$ of the first step and obtain for $n \geq 1$:

$$c_{n+1}(x) = \sum_{\kappa} \sum_{\omega \in W_{n+1}^{SAW}(0, x)} \delta_{\omega, \kappa} = \sum_{\kappa} \sum_{\omega \in W_n^{SAW}(\kappa, x)} \mathbb{1}_{(\omega_1 \neq 0 \text{ for } i=0, \ldots, n)}$$

$$= 2d(D \ast c_1)(x) - \sum_{\kappa} \sum_{\omega \in W_n^{SAW}(\kappa, x)} \mathbb{1}_{(\exists \kappa \in \{0, \ldots, n\} \text{ with } \omega_1=0)}. \quad (2.1.2)$$

The first term is similar to the SRW contribution (2.1.1) and the second is an error term which we expand further. To continue, we split a walk $\omega \in W_n^{SAW}(\kappa, x)$, that is contributing to the error, at the step where it visits the origin into two independent SAWs $\omega^1$ and $\omega^2$, i.e., we ignore their mutual avoidance. We overcount since now $\omega^1$ and $\omega^2$ could intersect. This overcounting produces another error term:

$$\sum_{\omega \in W_n^{SAW}(e_1, x)} \mathbb{1}_{(\exists \kappa \in \{0, \ldots, n\} \text{ with } \omega_1=0)}$$

$$= \sum_{m=1}^n \sum_{\omega \in W_m^{SAW}(e_1, 0)} c_{n-m}(x) - \sum_{m=1}^n \sum_{\omega \in W_m^{SAW}(e_1, 0)} \sum_{\omega' \in W_{n-m}^{SAW}(0, x)} \mathbb{1}_{(\omega_1 \cap \omega^2 \neq \emptyset)}, \quad (2.1.3)$$

where we denote by $\omega^1 \cap \omega^2$ the set of all points at which the two walks $\omega^1$ and $\omega^2$ intersect. We define $\pi_n^{(1)}(x) = \delta_{0,x}c_{n-1}(e_1)$, so that the first term equals $\sum_{m=2}^{n+1}(\pi_m^{(1)} \ast c_{n+1-m})(x)$. In the next step of the expansion we split the walk $\omega^2$ at the first intersection with $\omega^1$ into two independent SAWs $\omega^3$ and $\omega^4$. Thus, we overcount again as $\omega^1$ and $\omega^2$ could intersect and obtain another error term. To describe the
2.1 Classical lace expansion vs. NoBLE

walks up to the first intersection of $\omega^1$ and $\omega^2$ we define

$$\pi^{(2)}_n(x) = (1 - \delta_{0,x}) \sum_{m_1=1}^{n} \sum_{m_2=0}^{n-m_1} \sum_{\omega^1 \in \mathcal{W}^{SAW}_{m_1}(0,x)} \sum_{\omega^2 \in \mathcal{W}^{SAW}_{m_2}(x,0)} \sum_{\omega^3 \in \mathcal{W}^{SAW}_{n-m_1-m_2}(0,x)} \times \mathbb{I}_{[\omega^1 \cap \omega^2 = (\omega^1 \cap \omega^3) = (\omega^2 \cap \omega^3) = [0,x]}}$$

(2.1.5)

and see that we can rewrite (2.1.3) to

$$c_{n+1}(x) = 2d(D \star c_n)(x) - \sum_{m=2}^{n+1} \left[ (\pi^{(1)}_m \star c_{n+1-m})(x) - (\pi^{(2)}_m \star c_{n+1-m})(x) \right]$$

(2.1.6)

We see, that at each step of the expansion an error term is produced. Proceeding the inclusion-exclusion idea infinitely often, while assuming that these errors converge to zero, we obtain the following recursive relation:

$$c_{n+1}(x) = 2d(D \star c_n)(x) + \sum_{m=2}^{n+1} (\pi_m \star c_{n+1-m})(x)$$

(2.1.7)

with $\pi_n(x) = \sum_{N=1}^{\infty} (-1)^N \Pi^{(N)}_n(x)$. This finishes the inclusion-exclusion derivation of the LE.

We take the Green’s function of (2.1.7) to obtain

$$G_z(x) = \delta_{0,x} + 2dz(D \star G_z)(x) + (\Pi_z \star G_z)(x)$$

(2.1.8)

Applying the Fourier transformation yields

$$\hat{G}_z(k) = \frac{1}{1 - 2dz\hat{D}(k) - \hat{\Pi}_z(k)},$$

(2.1.9)

with $\Pi_z(x) = \sum_{n=1}^{\infty} \pi_n(x)z^n$. We see that (2.1.9) is only a perturbation of the SRW equation (1.5.4), if $\hat{\Pi}_z(k)$ is small.

**Non-backtracking lace expansion.** The NoBLE creates a relation similar to the NBW equations

$$b_n(x) = \sum_{i} b_{n-1}^{-i}(x - e_i), \quad \text{and} \quad b_n(x) = b'_n(x) + b''_{n-1}(x - e_i)$$

(1.5.12)
for SAW. For a set of walks $S$ let $|S|$ denote the number of walks in $S$. Then we see that $c_n(x) = |W_n^{SAW}(0, x)|$. Further, we define $c'_n(x)$ by

$$
c'_n(x) = |\{ \omega \in W_n^{SAW}(0, x) \mid \omega_i \neq e_i \text{ for all } i = 0, 1, \ldots, n\}| \quad (2.1.10)
$$

$$
c'_n(x) = |\{ \omega \in W_{n+1}^{SAW}(e_t, x) \mid \omega_1 = 0\}|. \quad (2.1.11)
$$

We see that the definition (2.1.11) implies that

$$
c_n(x) = \sum_{t} c'_{n-1}(x - e_t), \quad (2.1.12)
$$

which equals the first equation in (1.5.12). To obtain a relation similar to the second part of (1.5.12) we sort the walk depending on when they visit the point $e_t$, i.e.,

$$
c_n(x) = \sum_{\omega \in W_n^{SAW}(0, x)} \left( \mathbb{I}_{[\omega \cap \mathcal{F}]} + \mathbb{I}_{[\omega = \omega_1]} + \sum_{t=2}^{n} \mathbb{I}_{[\omega = \omega_t]} \right). \quad (2.1.13)
$$

Using the definition of $c'_n$, we obtain

$$
c_n(x) = c'_n(x) + c'_{n-1}(x - e_t) + \sum_{\omega \in W_n^{SAW}(0, x)} \sum_{t=2}^{n} \mathbb{I}_{[\omega = \omega_t]}. \quad (2.1.14)
$$

Equation (2.1.14) agrees with the second equation of (1.5.12), except for the last term. It is this term that we expand further. We define

$$
\xi^{(0), t}_n(x) = \delta_{x,e_t} \mathbb{I}_{[n \geq 2]} c_n(e_t), \quad (2.1.15)
$$

$$
\pi^{(0), t,K}_n(x) = \delta_{x,e_t} \mathbb{I}_{[n \geq 3]} |\{ \omega \in W_n^{SAW}(0, e_t - e_K) \mid \omega_{n-1} = e_t \}| \quad (2.1.16)
$$

and cut the walk at the step it takes after visiting $e_t$ to obtain

$$
\sum_{\omega \in W_n^{SAW}(0, x)} \sum_{t=3}^{n} \mathbb{I}_{[\omega = \omega_t]}
$$

$$
= \sum_{t=3}^{n} \sum_{y \in \mathbb{Z}^d} \pi^{(0), t,K}_t(y) c^{K}_{n-t}(x - y + e_K) + \xi^{(0), t}_n(x)
$$

$$
- \sum_{t_1=3}^{n-3} \sum_{t_2=t_1+2}^{n} \sum_{y \in \mathbb{Z}^d} \omega^1 \in W^{SAW}_{t_1}(0, y) \sum_{\omega^2 \in W^{SAW}_{n-t_2}(y, e_t)} \sum_{\omega^3 \in W^{SAW}_{t_2-s}(e_t, y)} \mathbb{I}_{[t_1 \leq t_2 \leq s]} \mathbb{I}_{[(\omega^1 \cap \omega^2) = (\omega^1 \cap \omega^3) = (\omega^2 \cap \omega^3) = (\omega^3 \cap \omega^4) = \{y\}]} \mathbb{I}_{[(\omega^2 \cap \omega^3) = \{e_t, y\}]}.
$$

(2.1.17)

The error term at this stage is created by possible intersections between walks counted in $\pi^{(0), t,K}_t(y)$ and $c^{K}_{n-t}(x - y + e_K)$. The constraint on the values of $t_1$ and $t_2$ follows naturally from the fact that the walks counted in $c^{K}_{n-t}(x - y + e_K)$ should
avoid \( y \).

To continue the inclusion-exclusion procedure we cut the first bond after the intersection between \( \omega_3 \) and \( \omega_4 \) and remember its direction. We define \( \xi_{\omega_1}^{(1)}(y) \) as the contribution of \( y = x \) and \( \pi_{\omega_1}^{(1),i,K}(y) \) for walks that follow the path \( 0 \rightarrow y \rightarrow e_i \rightarrow y \rightarrow y + e_\kappa \). We omit the formal definition at this point, as this is not very informative. In the next step we obtain a formula of the form

\[
\sum_{\omega \in \mathcal{P}_{SNW}(0,x)} \sum_{\ell=2}^{n} \mathbb{1}_{e_\ell = \omega_\ell} = \sum_{m=3}^{n} \sum_{\kappa} \sum_{y \in \mathbb{Z}^d} \left( (\pi_m^{(0),i,K}(y) - \pi_m^{(1),i,K}(y)) c_n^{\kappa}(x - y + e_\kappa) + \xi_n^{(0),i}(x) - \xi_n^{(1),i}(x) + r_n^{(1)}(x) \right).
\]

In Figure 2.1 we give a schematic representation of a walk counted in the error term \( r_n^{(1)}(x) \). We see that a walk counted in \( r_n^{(1)}(x) \) first follow the pattern of \( \pi_m^{(1),i,K}(y) \) and then intersects the walk \( \omega_3 \), at a point which we label as \( v \) in Figure 2.1. We split the walks again at \( v \) and proceed the inclusion-exclusion expansion by always splitting the walk at the first intersection with the former piece. We obtain two different lace-expansion coefficients. Loosely speaking, at each step \( \pi_{\omega_1}^{(N),i,K} \) will contain intersecting walks and the bond after the last intersection at which we cut the connection, while \( \xi_{\omega_1}^{(N),i} \) captures the contribution of walks for which the last intersection was at \( x \) (so that there is no bond after this intersection). Continuing this forever and defining

\[
\pi_n^{i,K}(x) = \sum_{N=0}^{\infty} (-1)^N \pi_n^{(N),i,K}(x) \quad \text{and} \quad \xi_n^{i}(x) = \sum_{N=0}^{\infty} (-1)^N \xi_n^{(N),i}(x),
\]

we obtain the NoBLE:

\[
c_n(x) = c_n^{i}(x) + c_{n-1}^{\ell}(x - e_\ell) \\
+ \sum_{m=3}^{n} \sum_{\kappa} \sum_{y \in \mathbb{Z}^d} \pi_m^{i,K}(y) c_{n-m}^{\kappa}(x - y + e_\kappa) + \xi_n^{i}(x).
\]
In Section 2.2 we derive the NoBLE using a graph-based algebraic representation, which gives an explicit representation of the lace-expansion coefficients $\xi_n^l$ and $\pi_n^{l,k}$. Let the generating functions of $c_n^l$, $\pi_n^{l,k}$ and $\xi_n$ be defined as

$$G_z^l(x) = \sum_{n=0}^\infty c_n^l(x)z^n, \quad \Pi_z^{l,k}(x) = \sum_{n=0}^\infty \pi_n^{l,k}(x)z^n, \quad \Xi_z^l(x) = \sum_{n=0}^\infty \xi_n^{l,k}(x)z^n.$$ (2.1.22)

Then (2.1.12) and (2.1.21) imply that

$$G_z(x) = \delta_{0,x} + z \sum_i G_z^{-i}(x - e_i), \quad \Pi_z^{l,k}(x) = \sum_i \sum_{y \in \mathbb{Z}^d} \Pi_z^{l,k}(y)G_z^l(x - y + e_k) + \Xi_z^l(x).$$ (2.1.24)

Applying the Fourier transformation we obtain

$$\hat{G}_z(k) = 1 + z \sum_i e^{ik \cdot e_i} \hat{G}_z^{-i}(k),$$ (2.1.25)

$$\hat{G}_z(k) = \hat{G}_z^l(k) + z e^{ik \cdot e_l} \hat{G}_z^{-l}(k) + \sum_k \sum_{y \in \mathbb{Z}^d} \hat{\Pi}_z^{l,k}(y)\hat{G}_z^k(k) + \hat{\Xi}_z^l(k).$$ (2.1.26)

In the next step, we rewrite these two lines using vector- and matrix-notations. Therefore, we recall the matrices $I, J, \hat{D}(k)$ defined in Section 1.5.2. We define the matrix $\hat{\Pi}_z(k)$ and the vectors $\hat{\Xi}(k), \hat{G}(k)$ by

$$\begin{pmatrix} \hat{\Pi}_z^l(k) \\ \hat{\Xi}(k) \end{pmatrix} = \begin{pmatrix} \hat{\Xi}_z^l(k) \\ \hat{G}(k) \end{pmatrix} = \hat{G}_z^l(k).$$ (2.1.27)

Then we can rewrite (2.1.25) and (2.1.26) to

$$\begin{align*}
\hat{G}_z(k) &= I^T \hat{D}(-k) \hat{G}(k), \\
\hat{G}_z(k) I &= \hat{G}(k) + z \hat{D}(k) \hat{G}(k) + \hat{\Pi}_z(k) \hat{D}(-k) \hat{G}(k) + \hat{\Xi}(k).
\end{align*}$$ (2.1.28)

Using $\hat{D}(k)J = J\hat{D}(-k)$ and $\hat{D}(k)^{-1} = \hat{D}(-k)$ we rewrite (2.1.29) as

$$\hat{G}_z(k) I - \hat{\Xi}(k) = [\hat{D}(k) + zJ + \hat{\Pi}_z(k)] \hat{D}(-k) \hat{G}(k).$$ (2.1.30)

Inverting the matrix and combining it with (2.1.28) we obtain

$$\hat{G}_z(k) = \frac{1 - zI^T [\hat{D}(k) + zJ + \hat{\Pi}_z(k)]^{-1} \hat{\Xi}(k)}{1 - zI^T [\hat{D}(k) + zJ + \hat{\Pi}_z(k)]^{-1} I}. \quad (2.1.31)$$

In high dimensions the entries of $\hat{\Pi}_z(k)$ and $\hat{\Xi}(k)$ are expected to be small, so that $\hat{G}_z(k)$ is only a small perturbation of $\hat{B}_\mu(k)$ (see (1.5.16)).
2.1 Classical lace expansion vs. NoBLE

Comparison of the two expansions. In the derivation of both expansions we cut a walk at the first intersection with the piece created in the previous inclusion-exclusion step. By cutting we mean replacing the original SAW by two independent SAWs. When we cut a walk in the NoBLE we identify the orientation of the bond cut to guarantee that when recombined the two walks remain a NBW. Thus, the biggest correction expressed by a coefficient only counts self-intersections which involve a loop consisting of at least four steps.

In the LE that does not use the information of the orientation such self-intersections can also be created by immediate reversals which consist of only two steps. This is an important remark as the dominant contributions to \( \Pi_z \) are created by short walks. Indeed, the biggest contribution to \( \Pi_z \) in the LE is created by the direct forth and back movement of the walker between the origin and a neighbor \( e_i \). As each step obtains the weight \( z_c \approx (2d)^{-1} \) and there are \( 2d \) direct neighbors of the origin, we obtain that \( \hat{\Pi}_z(0) \sim (2d)^{-1} \). In the NoBLE we remove this contribution and obtain an error being at least one order of \( 1/d \) smaller than in the LE.

The form of the two-point function produced by the NoBLE (2.1.31) is more involved than the corresponding formula of the LE (2.1.9). Therefore, also the analysis of the NoBLE is somewhat more involved. On the other hand, the NoBLE coefficients are one order \( 1/d \) smaller than the corresponding coefficients of the LE. The major objective of the “NoBLE project” was to find an analysis that is effective enough to actually gain from the fact that we have smaller coefficients, and on the other hand, is simple and less computer intensive than the techniques used by Hara and Slade in the 90’s.

Idea of the NoBLE. The rigorous derivation of the NoBLE does not convey the underlying idea. For this reason we now explain how to generalize the idea of the NoBLE for SAW to the other models. We make some simplifications and show the inclusion-exclusion approach on a purely heuristical level. Especially the expansion for percolation, in Section 2.4, involves a number of very delicate technical steps that are completely omitted here.

Our goal is to obtain an approximation of (1.5.13):

\[
B_\mu(x) = \mu \sum_{\iota \in \{\pm 1, \ldots, \pm d\}} B^{-\iota}_\mu(x - e_\iota) \quad \text{and} \quad B_\mu(x) = B^\iota_\mu(x) + \mu B^{-\iota}_\mu(x - e_\iota)
\]  

(1.5.13)

in the context of the other models. The two-point function always involves statements about possible connections between 0 and \( x \) by a path of bonds from 0 to \( x \). We call a bond pivotal for the connection from 0 to \( x \), if its removal destroys the connection. These pivotal bonds have a natural order, as every self-avoiding path from 0 to \( x \) has to traverse the pivotal bonds in the same order. We call the connected components that remain when removing all pivotal bonds sausages. Further, we consider the pivotal bonds to be directed such that all self-avoiding...
paths traverse the pivotal bonds in the same direction. For a directed bond $b$, we denote the starting point of $b$ by $b$ and its ending point by $\overline{b}$. For SAW and LT there only exists one unique self-avoiding path from 0 to $x$, thus all bonds on this path are pivotal. Further, we note that for SAW all sausages consist of a single vertex. See Figure 2.2 for a visualization of these different concepts.

We expect that the value of the two-point function $G_z(x)$, for $x \in \mathbb{Z}^d$, is dominated by connections where most of the bonds on a path between 0 and $x$ are pivotal. Take percolation as an example. It is highly unlikely that there exist two bond-disjoint paths between 0 and $x$ when the bond probability is only $p_c \approx (2d - 1)^{-1}$. Since this is especially true when the points are not direct neighbors, we expect a “typical” sausage to be small. Therefore, a “typical” connection for 0 to $x \in \mathbb{Z}^d$ with $|x| \gg 1$ should include many pivotal bonds and only small sausages.

We begin the inclusion-exclusion-based derivation in the same way as for the classical lace expansion. We denote by $b_1 = (\overline{b}_1, b_1)$ the first pivotal bond on the connection from 0 to $x$. As $b_1$ is pivotal, we know that every path from $\overline{b}_1$ to $x$ has to avoid the first sausage $A_0$. We define $\Xi^{(0)}(y)$ to count the contributions of sausages that include 0 and $y$. We cut after the pivotal bond $b_1$ and make an inclusion-exclusion step for the constraint that the connection $\overline{b}_1$ to $x$ should avoid the sausage $A_0$. The first step for the classical lace expansion produces a term of the form

$$G_z(x) = \Xi^{(0)}_2(x) + z \sum_y \sum_x \Xi^{(0)}_2(x,y)G_z(x - y - e) - \text{remainder term. (2.1.32)}$$

The remainder term contains contributions of connections intersecting the first sausage $A_0$. For all models under consideration the dominant contribution to this is due to a reversal of the bond at which we cut. To remove this contribution from the correction terms we will remember the orientation of the bond in the inclusion-
exclusion step. We define $\Psi_{z}^{(0),K}(y)$ as the contribution of a sausage that includes 0 and $y$, but does not include $y + e_\kappa$. In the first expansion step of the NoBLE we obtain

$$G_z(x) = \Xi_z^{(0)}(x) + \tilde{a}_z \sum_y \sum_\kappa \Psi_z^{(0),K}(y) G_z^{-K}(x - y - e_\kappa) + R_0(x), \quad (2.1.33)$$

where $\tilde{a}_z$ is a contribution created by cutting at the pivotal bound. The first sausage was conditioned not to include $b_1$ and the connection $\bar{b}_1$ to $x$ does not include the point $b_1$. Thus, every intersection contribution to $R_0$ requires at least four bonds. To continue the inclusion-exclusion procedure, we denote by $b_2$ the first pivotal bond of the connection $\bar{b}_1$ to $x$ after the first intersection with the first sausage. We define $\Xi_z^{(1)}(x)$ for the contribution were no second pivotal bond $b_2$ exist and $\Psi_z^{(1),K}(y)$ for the contribution were $b_2 = (y, y + e_\kappa)$. Then,

$$G_z(x) = \Xi_z^{(0)}(x) - \Xi_z^{(1)}(x) + \tilde{a}_z \sum_y \sum_\kappa (\Psi_z^{(0),K}(y) - \Psi_z^{(1),K}(y)) G_z^{-K}(x - y - e_\kappa) + R_1(x). \quad (2.1.34)$$

The error $R_1$ is created by a possible intersection of the path contributing to $G_z^{-K}(x - y - e_\kappa)$ and the sausages between $\bar{b}_1$ and $b_2$. In the next step, we denote by $b_3$ the first pivotal bond after the first intersection, and define $\Xi_z^{(2)}$ to count contributions where no $b_3$ exists, and $\Psi_z^{(2),K}$ for the case that $b_3$ does exist. Continuing in this way forever and defining

$$\Xi_z(x) = \sum_{N=0} \sum_y (-1)^N \Xi_z^{(N)}(x) \quad \text{and} \quad \Psi_z^{K}(x) = \sum_{N=0} \sum_y (-1)^N \Psi_z^{(N),K}(x), \quad (2.1.35)$$

we obtain a recursion similar to the first part of (1.5.13):

$$G_z(x) = \Xi_z(x) + \tilde{a}_z \sum_y \sum_\kappa \Psi_z^{K}(y) G_z^{-K}(x - y - e_\kappa). \quad (2.1.36)$$

To obtain an approximation of $B_\mu(x) = B_\mu^l(x) + \mu B_\mu^{-l}(x - e_\kappa)$, we condition on the function of the point $e_\kappa$ for the connection 0 to $x$, as in (2.1.12) - (2.1.13). We split between contributions where the point $e_\kappa$ is not part of the connection, where $(0, e_\kappa)$ is a pivotal bond and cases where $e_\kappa$ needs to be traversed by any connection from 0 to $x$. The first two cases produce the contributions $G_z^l(x)$ and $\alpha z G_z^{-l}(x - e_\kappa)$. To analyze the last case we identify the first pivotal bond $(y, y + e_\kappa)$ after traversing $e_\kappa$. If such a bond $(y, y + e_\kappa)$ exists then we cut after it and capture the contribution with the coefficient $\Pi_z^{(0),l,K}(y)$. If no pivotal bond exists, after $e_\kappa$, we capture the contribution with $\Xi_z^{(0),l}(x)$. This way, we obtain

$$G_z(x) = G_z^l(x) + \alpha z G_z^{-l}(x - e_\kappa) + \sum_{y,k} \Pi_z^{(0),l,K}(y) G_z^K(x - y - e_\kappa) + \Xi_z^{(0),l}(x) + R^{(0),l}(x), \quad (2.1.37)$$
where $\alpha_z$ is created by cutting after the pivotal bond $(0, e_i)$. The error term $R^{(0),l}_z$ has to compensate for the overcounting that occurs when cutting the bond. As argued before, we identify the first pivotal bond after an intersection cut after that pivotal bond, and create a smaller error. At each step we define $\Xi^{(N),l}_z(x)$ for the case that there is no pivotal bond after an intersection and $\Pi^{(N),l,K}_z(y)$ for the case that a pivotal bond exists. Continuing in this way forever, we obtain a relation similar to the second part of (1.5.13).

As for the SAW, the difference between the LE and the NoBLE is that the NoBLE keeps track of the orientation of the bond after which we cut. Keeping track of the orientation, we avoid that connections contributing to $G_\kappa(x - y - e\kappa)$ immediately intersect with $y = b_N$ and thereby remove the biggest contribution to the LE coefficient.

2.2 Self-avoiding walk

In this section we derive the NoBLE for self-avoiding walk (SAW). We first derive a recursive structure for a set of graphs using an algebraic expansion in Section 2.2.1. Then we create the NoBLE in Section 2.2.2. In Section 2.2.3 we identify the structure of the walks counted by the lace-expansion coefficients. The structure of this section and the notation is in the style of [92, Section 3.2-3.3].

2.2.1 The algebraic derivation of the expansion

In this part, we give a description of the self-avoidance constraint in terms of graphs that will be used in the next section to derive a formula for $\pi_{n,i}^\kappa$ and $\xi_n^i$.

Definition 2.2.1 (Set of non-backtracking walks). Let $\mathcal{W}_{NBW}$ be the set of all NBWs. We define the subsets of NBWs with a certain length $n$ and/or with endpoint $x$ to be $\mathcal{W}_{nbw}^n$, $\mathcal{W}_{nbw}^n(x)$ and $\mathcal{W}_{nbw}^n(x)$, respectively.

Definition 2.2.2 (Graphs and the set of all graphs). Let $a, b \in \mathbb{N}$ with $a < b$. For $s, t \in [a, b] \cap \mathbb{N}$ with $s < t$, the edge between $s$ and $t$ is the tuple $(s, t)$. We abbreviate $st$ for $(s, t)$. We call a set of edges a graph. We denote the set of all graphs on $[a, b]$ that have only edges $st$ with $t - s \geq 4$ by $\mathcal{B}[a, b]$.

Note that, by this definition, $\mathcal{B}[a, b]$ contains the empty graph.

Definition 2.2.3 (Connected graphs, minimally connected graphs and two-connected lace). We say that a graph $\Gamma \in \mathcal{B}[a, b]$ is connected, if, for all $i \in (a, b)$, there exists an edge $st \in \Gamma$ such that $s < i < t$. We call a connected graph minimally connected or lace, if the removal of any edge would disconnect the graph. We call a lace $L = \{s_1 t_1, s_2 t_2, \ldots, s_N t_N\}$, with $s_i < s_{i+1}$, two-connected, if $L$ consists only of
2.2 Self-avoiding walk

one edge or if \( t_i \leq t_{i+1} - 2 \) for all \( i = 1, \ldots, N - 1 \). We denote by \( \mathcal{L}[a, b] \) the set of all two-connected laces on \([a, b]\).

**Definition 2.2.4** (Two-connected laces and corresponding graphs). We define the function

\[
L : \mathcal{B}[a, b] \rightarrow \bigcup_{c=a+4}^b \mathcal{L}[a, c] \cup \{\phi\}
\]

in a constructive manner as follows: Let \( \Gamma \in \mathcal{B}[a, b] \). If there does not exists \( t \in (a+1, b) \) such that \( at \in \Gamma \), then we define \( L(\Gamma) = \phi \). Otherwise we define \( s_1 = a \) and \( t_1 = \sup\{t\} \) such that \( at \in \Gamma \). Further we define

\[
t_i = \sup\{t \geq t_{i-1} + 2 \mid \exists s < t_{i-1} \text{ such that } st \in \Gamma\} \quad \text{and} \quad s_i = \inf\{s\} \text{ such that } st_i \in \Gamma.
\]

This procedure ends after a finite number of steps \( N \). We denote by \( L(\Gamma) \) the resulting graph \( L = \{s_1 t_1, s_2 t_2, \ldots, s_N t_N\} \), which by construction is in \( \mathcal{L}[a, t_N] \). We define the inverse images of \( L \) by

\[
L^{-1}(L) := \{\Gamma \in \mathcal{B}[a, b] \mid L(\Gamma) = L\} \quad \text{for } L \in \mathcal{L}[a, b],
\]

\[
L^{-1}(\mathcal{L}[a, b]) := \{\Gamma \in \mathcal{B}[a, b] \mid \exists L \in \mathcal{L}[a, b] \text{ such that } L(\Gamma) = L\}.
\]

We also require the following notation to derive the NoBLE.

**Definition 2.2.5** (Restricted graph and two-connected component). For \( a, b, c, d \in \mathbb{N} \) with \( a \leq c < d \leq b \) and a graph \( \Gamma \in \mathcal{B}[a, b] \), we denote by \( \Gamma[c, d] \subset \Gamma \) the graph \( \Gamma \) restricted to \([c, d]\), i.e., \( \Gamma[c, d] = \{st \in \Gamma \mid s, t \in [c, d]\} \). We define the set of irreducible components \( \mathcal{R}[a, b] \) to be the following set of graphs:

\[
\mathcal{R}[a, b] = \left\{ \Gamma \in \mathcal{B}[a, b] \mid L(\Gamma) \in \mathcal{L}[a, b-1] \right\}.
\]

(2.2.1)

Figure 2.3: An example of a two-connected component \( \Gamma \). The edges in \( L(\Gamma) \) are drawn solid, while the others are dashed.

To understand the relevance of these sets of graphs let us define for \( 0 \leq s < t \) the indicator of an intersection \( \mathcal{H}_{st}(\omega) = -\delta_{\omega_s, \omega_t} \). Further, we define for \( a \geq 0 \) and
\[ b \geq a: \]

\[
K[a, b](\omega) = \sum_{\Gamma \in \mathcal{R}[a, b]} \prod_{s \in \Gamma} U_{st}(\omega), \tag{2.2.2}
\]

\[
C[a, b](\omega) = \sum_{\Gamma \in \mathcal{R}[a, b]} \prod_{s \in \Gamma} U_{st}(\omega), \tag{2.2.3}
\]

\[
J[a, b](\omega) = \sum_{\Gamma \in \mathcal{L}^{-1}(\mathcal{L}[a, b])} \prod_{s \in \Gamma} U_{st}(\omega). \tag{2.2.4}
\]

For abbreviation we often drop the argument \( \omega \). For a NBW \( \omega \) the following can be easily proven, e.g. by induction:

\[
K[a, b] = \sum_{\Gamma \in \mathcal{R}[a, b]} \prod_{s \in \Gamma} U_{st} = \prod_{s=a}^{b-1} \prod_{t=s+4}^{b} (1 + U_{st}). \tag{2.2.5}
\]

Therefore, the function \( K[a, b] \) is the indicator that a NBW is self-avoiding on \([a, b]\). The function of \( J[a, b] \) and \( C[a, b] \) characterize that walks has a certain structures of self-intersection. We explain this after defining the NoBLE coefficients. The following lemma is the basic ingredient for the NoBLE for SAW:

**Lemma 2.2.6 (Recursive relation on graph level).** For \( a, b \in \mathbb{N} \) with \( a < b \):

\[
\sum_{\Gamma \in \mathcal{R}[a, b]} \prod_{s \in \Gamma} U_{st}(\omega) = \sum_{j=a+4}^{b-1} C[a, j+1] K[j, b] + J[a, b] \tag{2.2.6}
\]

**Proof.** A graph \( \Gamma \in \mathcal{R}[a, b] \setminus \mathcal{R}[a+1, b] \) always has at least one edge \( at \), so \( \Gamma \) restricted to \([a, t]\) is in \( \mathcal{L}^{-1}(\mathcal{L}[a, t]) \). Let \( j(\Gamma) \) be the maximal value of \( j \) such that \( \Gamma[a, j] \in \mathcal{L}^{-1}(\mathcal{L}[a, j]) \). If \( j(\Gamma) = b \), then \( \Gamma \) contributes to \( J[a, b] \), which produces the last term in (2.2.6).

Thus, we restrict to the case \( j(\Gamma) < b \) from now on. We conclude from the definition of \( j(\Gamma) \) that \( \Gamma[a, j(\Gamma) + 1] \in \mathcal{R}[a, j(\Gamma) + 1] \) and that \( \Gamma \) has no edges \( st \) with \( s < j(\Gamma) \) and \( t > j(\Gamma) + 1 \), which implies that \( \Gamma \setminus \Gamma[a, j(\Gamma) + 1] \in \mathcal{R}[j(\Gamma), b] \). Consequently, we can split any \( \Gamma \) with \( j(\Gamma) < b \) into a graph in \( \mathcal{R}[a, j(\gamma) + 1] \) and a graph \( \mathcal{R}[j(\Gamma), b] \).

As this split describes a bijective relation we know that

\[
\sum_{\Gamma \in \mathcal{R}[a, b] \setminus \mathcal{R}[a+1, b]} \prod_{s \in \Gamma} U_{st}(\omega)
= \sum_{j=4}^{b-1} \sum_{\Gamma \in \mathcal{R}[a, j+1]} \prod_{s \in \Gamma} U_{st}(\omega) \sum_{\Gamma' \in \mathcal{R}[j, b]} \prod_{s' \in \Gamma'} U_{s't'}(\omega) + J[a, b].
\]

Inserting the definitions of \( C[0, j + 1] \) and \( K[j, b] \) completes the proof. \( \square \)
2.2.2 Completion of the derivation of the expansion

Now we use Lemma 2.2.6 to derive the expansion. We fix a \( i \) and condition the walk on visiting/not-visiting \( e_i \):

\[
c_n(x) = \sum_{\omega \in \mathcal{W}_{n}^{\text{NBW}}(x)} K[0, n] \prod_{t=1}^{n} (1 - \delta_{\omega_t, e_i}) + K[0, n] \sum_{t=0}^{2} \delta_{\omega_t, e_i} + K[0, n] \sum_{t=3}^{n} \delta_{\omega_t, e_i}.
\]

The first part equals \( c_i^n(x) \) as it describes the contribution of SAWs that never visit \( e_i \), see (2.1.10). Further, we investigate (2.1.11) and see that the contribution for \( t = 1 \) equals \( c_{i-1}^n(x - e_i) \). Moreover, a walk can only reach \( e_i \) in an odd number of steps, so that \( t = 0 \) and \( t = 2 \) do not contribute. We rewrite the last term in (2.2.7) by adding an artificial step at the beginning of the walk. By the translation invariance of NBW we can replace a walk \( \omega \in \mathcal{W}_{n}^{\text{NBW}}(x) \) by a walk \( \omega \in \mathcal{W}_{n+1}^{\text{NBW}}(x - e_i) \), that is conditioned to visit \( -e_i \) in its first step. The artificial step and the shift of the walk by \( -e_i \) allows us to rewrite the constrain \( \delta_{\omega_t, e_i} \) in (2.2.7) to \(-U_0t\):

\[
\sum_{\omega \in \mathcal{W}_{n}^{\text{NBW}}(x)} K[0, n] \sum_{t=3}^{n} \delta_{\omega_t, e_i} = \sum_{\omega \in \mathcal{W}_{n+1}^{\text{NBW}}(x - e_i)} \delta_{\omega_1, -e_i} K[1, n + 1] \sum_{t=4}^{n+1} \delta_{\omega_t, 0} \quad (2.2.8)
\]

Hence, \( K[1, n + 1] \) is the indicator that the walk is self-avoiding on \([1, n + 1]\), and we know that only walks contribute that visit \( e_i \) exactly once. Thereby, the following is only a rewrite of (2.2.8):

\[
\sum_{\omega \in \mathcal{W}_{n}^{\text{NBW}}(x)} K[0, n] \sum_{t=3}^{n} \delta_{\omega_t, e_i} = - \sum_{\omega \in \mathcal{W}_{n+1}^{\text{NBW}}(x - e_i)} \delta_{\omega_1, -e_i} K[1, n + 1] \sum_{m=1}^{n} \sum_{t_1, t_2, t_3} \prod_{i=1}^{m} U_{\omega_{t_i}, 0} \sum_{\Gamma \in \mathcal{B}[0, n+1] \setminus \mathcal{B}[1, n+1]} \prod_{s,t \in \Gamma} U_{s,t}(\omega). \quad (2.2.9)
\]
Then, we apply Lemma 2.2.6 to (2.2.9) and split the \( \omega \) into two pieces, \( \omega^1 \) and \( \omega^2 \), to obtain:

\[
\sum_{\omega \in \mathcal{L}_{n+1}^{NBW}(x-e_i)} K[0, n] \sum_{t=3}^{n} \delta_{\omega_t, e_t} \\
= - \sum_{\omega \in \mathcal{L}_{n+1}^{NBW}(x-e_i)} \delta_{\omega_1, -e_i} \sum_{m=4}^{n} C[0, m+1] K[m, n] + J[0, n+1] \\
= - \sum_{y \in \mathbb{Z}^d} \sum_{k} \sum_{m=4}^{n} \mathcal{L}^1_1(y) \sum_{m+1}^{n+1} \delta_{\omega_t, -e_i} C[0, m+1] \omega^{1}_t \delta_{\omega_m, e_x} C[0, m+1] (\omega^1) \\
\times \sum_{\omega \in \mathcal{L}_{n}^{NBW}(x-y)} \delta_{\omega_t, -e_i} J[0, n+1] \\
= \sum_{y \in \mathbb{Z}^d} \sum_{k} \sum_{m=3}^{n} \pi_{n}^{1, K}(y) \epsilon_{n-m}(x) + \xi_{n}^{1}(x), \tag{2.2.10}
\]

with

\[
\pi_{n}^{1, K}(x) = - \sum_{\omega \in \mathcal{L}_{n+1}^{NBW}(x-e_k-e_i)} C[0, n] \delta_{\omega_1, -e_i} \delta_{\omega_n, x-e_i}, \tag{2.2.11}
\]

\[
\xi_{n}^{1}(x) = - \sum_{\omega \in \mathcal{L}_{n+1}^{NBW}(x-e_i)} J[0, n+1] \delta_{\omega_1, -e_i}. \tag{2.2.12}
\]

### 2.2.3 Interpretation of the lace-expansion coefficients

To interpret the structure of walks counted in \( \pi_{n}^{1, K} \) and \( \xi_{n}^{1} \), we define the following sets of graphs:

**Definition 2.2.7** (Laces with a fixed number of edges). Fix \( n \geq 3 \) and \( N \geq 1 \).

Let \( \mathcal{L}^{[N]}[a, b] \subset \mathcal{L}[a, b] \) be the set of all laces \( L \subset \mathcal{L}[a, b] \) that consist of exactly \( N \) edges and

\[
\mathcal{R}^{[N]}[a, b] = \left\{ \Gamma \in \mathcal{B}[a, b] \mid L(\Gamma) \in \mathcal{L}^{[N]}[a, b-1] \right\}
\]

For \( N \geq 1 \), let

\[
C^{[N]}[a, b](\omega) = \sum_{\Gamma \in \mathcal{R}^{[N]}[a, b]} \prod_{s \in \Gamma} \mathcal{U}_{st}(\omega), \tag{2.2.13}
\]

\[
J^{[N]}[a, b](\omega) = \sum_{\Gamma \in \mathcal{L}^{-1}[\mathcal{L}^{[N]}[a, b]]} \prod_{s \in \Gamma} \mathcal{U}_{st}(\omega), \tag{2.2.14}
\]

\[
\pi_{n}^{(N-1), K}(x) = (-1)^{N} \sum_{\omega \in \mathcal{L}_{n}^{NBW}(x-e_k-e_i)} C^{[N]}[0, n+1](\omega) \delta_{\omega_1, -e_i} \delta_{\omega_n, x}, \tag{2.2.15}
\]

\[
\xi_{n}^{(N-1), t}(x) = (-1)^{N} \sum_{\omega \in \mathcal{L}_{n+1}^{NBW}(x-e_i)} J^{[N]}[0, n+1](\omega) \delta_{\omega_1, -e_i}. \tag{2.2.16}
\]
The reason that we define the coefficient $\xi_n^{(N-1),i}$ and $\pi_n^{(N-1),i,k}$ using laces with $N$ edges is that the first edge $0t_1$ was created artificially. We know that

$$\xi_n^{(N),i}(x) = \sum_{N=0}^{\infty} (-1)^N \xi_n^{(N),i}(x), \quad \pi_n^{(N),i,k}(x) = \sum_{N=0}^{\infty} (-1)^N \pi_n^{(N),i,k}(x). \quad (2.2.17)$$

To interpret the structure of the walks counted in $\xi_n^{(N),i}$ and $\pi_n^{(N),i,k}$, we define the concept of compatible edges:

**Definition 2.2.8 (Compatible edges).** Fix $n \geq 4$ and $L \in \mathcal{L}[0,n]$. We call an edge $st$, $s, t \in [0,n]$, compatible with a lace $L$ if $st \not\in L$ and $L(L \cup \{st\}) = L$. We denote the set of all edges that are compatible to $L$ by $\mathcal{C}(L)$.

![Figure 2.4: An example of a lace and the edges that are compatible to this lace.](image)

We group the graphs $\Gamma \in \mathbb{L}^{-1}(\mathcal{L}^N[0,n])$ by their image under $L$:

$$J^{[0,n]} = \sum_{L \in \mathcal{L}^N[0,n]} \sum_{\Gamma \in \mathbb{L}^{-1}(L)} \prod_{st \in \Gamma} \mathcal{U}_{st} = \sum_{L \in \mathcal{L}^N[0,n]} \prod_{st \in L} \mathcal{U}_{st} \sum_{\Gamma \in \mathbb{L}^{-1}(L)} \prod_{s't' \in \mathcal{C}(L)} \mathcal{U}_{s't'}. \quad (2.2.18)$$

There is an one-to-one relation between a lace $L \in \mathcal{L}^N[0,n]$, a set compatible edges to this lace $\Gamma^* \subset \mathcal{C}(L)$, and a graph $\Gamma \in \mathbb{L}^{-1}(L)$. The reason is that each set of compatible edges identifies a unique graph $\Gamma = \Gamma^* \cup L$ and for each $\Gamma \in \mathbb{L}^{-1}(\mathcal{L}^N[0,n])$, we know that $\Gamma \setminus L(\Gamma) = \Gamma^*$ is a unique set of compatible edges. Therefore,

$$\sum_{\Gamma \in \mathbb{L}^{-1}(L)} \prod_{st \in \Gamma \setminus L} \mathcal{U}_{st} = \sum_{\Gamma \in \mathcal{C}(L)} \prod_{st \in \Gamma \setminus L} \mathcal{U}_{st} = \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}).$$
so that
\[ \xi^{(N)}_n(x) = - \sum_{\omega \in \mathcal{N}^{NW}_{n+1}(x-e_1)} \delta_{\omega_1,e_{i_1}} \prod_{L \in \mathcal{L}^{(N)}[0,n]} \prod_{s \in L} \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s'}) \tag{2.2.19} \]

In a similar way we group the graphs \( \Gamma \in \mathcal{R}^{(N)}[0,n+1] \) by their corresponding lace \( \mathcal{L}^{(N)}[0,n] \) to obtain:
\[ \pi^{(N),L,K}_n(x) = - \sum_{\omega \in \mathcal{N}^{NW}_{n+1}(x-e_{i_1})} \delta_{\omega_1,e_{i_1}} \prod_{L \in \mathcal{L}^{(N)}[0,n]} \prod_{s \in L} \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s'}) \times \delta_{\omega_n,x} \prod_{a=t_{N-1}(L)+1}^n (1 + \mathcal{U}_{a(n+1)}). \tag{2.2.20} \]

This characterization allows us to understand the shape of a walk \( \omega \) contributing to \( \xi^{(N)}_n(x) \) and \( \pi^{(N),L,K}_n(x) \). In (2.2.19) and (2.2.20), it is easy to see that the only difference between them is that \( \pi^{(N),L,K}_n(x) \) has the additional step from \( x \) to \( x - e_K \), which needs to avoid the path between the times \( t_{N-1}(L) \) to \( n \).

For a lace \( L = \{s_1,t_1,\ldots,s_N,t_N\} \in \mathcal{L}^{(N)}[0,n] \), the product \( \prod_{s \in L} \mathcal{U}_{s} \) with \( \mathcal{U}_{s} \delta_{\omega_s,\omega_t} \) guarantees that \( \omega_{s_i} = \omega_{t_i} = x_i \) for \( i \in \{2,\ldots,N\} \).

**Figure 2.5:** Structure of a walk contributing to \( \pi^{(0),L,K}_n(x), \pi^{(1),L,K}_n(x), \pi^{(2),L,K}_n(x) \) and \( \pi^{(4),L,K}_n(x) \)

All edges \( s't' \) with \( t_{i-1} < s' < t' < t_i \) are compatible to the lace \( L \), so that the walk between \( t_{i-1} \) and \( t_i \) is self-avoiding, e.g., the walk on the way from \( x_2 = \omega_{t_2} \) to \( \omega_{t_{i-1}} \) over \( x_4 = \omega_{s_4}, x_3 = \omega_{s_3}, \) and \( x_5 = \omega_{s_5} \) is self-avoiding.

We know that \( t_i - s_i \geq 4 \) so that all loops have length at least four. By definition of a lace we know that \( s_i < t_{i-1} \) for all \( i = 2,\ldots,N \), so that all direct connections between a \( x_{i+1} \) and \( x_i \), being all diagonal connections in Figure 2.5 have at least length one. Further, we know that \( t_i - t_{i-1} \geq 2 \), thus the connection between \( x_i \) to \( x_{i+1} \) has at least length two. This also implies that the last line of the diagram always consists of at least two steps. The dashed lines in Figure 2.5 could have length zero.
2.3 Lattice trees and animals

In this section we derive the NoBLE for lattice trees and lattice animals using graphs. We begin by looking at a lattice tree that contains $0, x$. As a lattice tree does not contain any loops there exists a unique path connecting $0$ and $x$. We call this path the backbone. A lattice animal containing $0, x$ can contain loops. Therefore, a connection between $0$ and $x$ is not necessarily unique. To create an analogue to the backbone for lattice trees we define the following:

**Definition 2.3.1** (Double connections and pivotal elements). Let $A$ be a lattice animal that contains $x, y \in \mathbb{Z}^d$. We say that $x$ and $y$ are doubly connected in $A$ if there exist two edge-disjoint paths $p_1, p_2 \subset A$ connecting $x$ and $y$. We say that $x \in A$ is doubly connected to itself. We call a bond $b \in A$ pivotal for $x$ and $y$ if the removal of $b$ would disconnect $A$ into two disjoint animals, one containing $x$ and the other $y$.

![Lattice animal diagram](image)

Further, $A_3 = A_4 = A_6 = A_7 = A_9 = \emptyset$.

Figure 2.6: A lattice animal containing $x$ and $y$. All bonds of the backbone $(b_i)_{i=1,...,10}$ and all sausages $(A_i)_{i=0,...,10}$ are labeled in the picture.

Let us consider a lattice animal containing $x$ and $y$ as given in Figure 2.6. Then $x$ and $y$ are either doubly connected or there exists at least one pivotal bond $b$. If there are multiple pivotal bonds, then they have a natural order for the connection from $x$ to $y$, because every self-avoiding path from $x$ to $y$ has to pass the pivotal bonds $(b_i)_{i}$ in the same order.

The removal of the bonds $(b_i)_{i}$ disconnects the animal into mutually non-intersecting pieces, which we denote in Figure 2.6 by $(A_i)_{i}$. These pieces form a double connections between the end of one pivotal bond $\bar{b}_i$ to the beginning of the following pivotal bond $\bar{b}_{i+1}$. We call these doubly connected pieces $(A_i)_{i}$ sausages and the sequence of pivotal bonds $(b_i)_{i}$ the backbone of the animal.

We define the rib walk and sausage walk to characterize the combination of a
backbone and a set of ribs/sausages. Then, we create the NoBLE for LT and LA by expanding a graph-based description of the avoidance constraint of the sausages and of the backbone. We use the same family of graphs as use in by Hara and Slade in [39] for the classical lace expansion (LE). The formal difference to the LE is that we encode the non-backtracking condition into the definition of the rib/sausage walk.

**Definition 2.3.2** (Ribs and Rib walks for lattice trees).

(i.) We call a lattice tree $R$ that contains $x \in \mathbb{Z}^d$ a rib for $x$ and define that the empty set is also a rib for all $x \in \mathbb{Z}^d$.

(ii.) For $x, y \in \mathbb{Z}^d$ and $n \geq 1$ we call a collection of $n$ oriented nearest-neighbor bonds $(b_i)_{i=1,\ldots,n}$ and $n+1$ ribs $(R_i)_{i=0,\ldots,n}$ an $n$-step rib walk from $x$ to $y$, if $R_0$ is a rib for $b_1 = x$, $R_n$ is a rib for $b_n = y$, and for $i = 1, \ldots, n - 1$ the lattice tree $R_i$ is a rib for $b_i = b_{i-1}$. We call $(b_i)_{i=1,\ldots,n}$ the backbone of the rib walk.

(iii.) For a rib walk $\omega = ((b_i)_{i=1,\ldots,n}, (R_i)_{i=0,\ldots,n})$ we define $|\omega|$ to be the number of bonds in the backbone. We denote by $b_i^\omega$ the $i$th bond of the backbone and by $R_i^\omega$ the backbone of the rib walk.

(iv.) We say that any lattice tree is a zero step rib walk to the origin.

(v.) We call a rib walk $\omega$ non-backtracking, if $b_i^\omega \notin R_i^\omega, b_i^\omega \notin R_0^\omega$ and $b_i^\omega \neq b_{i+1}^\omega$ for all $i$.

(vi.) We define $\mathcal{W}^T(x)$ as the set of all rib walks from the origin 0 to $x$ and $\mathcal{W}^{T,i}(x)$ to be the set of all rib walks $\omega$ from 0 to $x$ such that $e_i \notin R_0^\omega$ and $b_1^\omega \neq e_i$, if $|\omega| > 0$.

**Definition 2.3.3** (Sausages and Sausage walks for lattice animals).

(i.) We call a lattice animal $S$ a sausage for $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, if $x$ and $y$ are doubly connected in $S$. Further, we define that the empty set is a sausage for every $(x, x)$ with $x \in \mathbb{Z}^d$.

(ii.) For $x, y \in \mathbb{Z}^d$ and $n \geq 1$ we call a collection of $n$ oriented nearest-neighbor bonds $(b_i)_{i=1,\ldots,n}$ and $n+1$ sausages $(A_i)_{i=0,\ldots,n}$ an $n$-step sausage walk from $x$ to $y$, if $A_0$ is a sausage for $(x, b_1)$, $S_n$ is a sausage for $(b_n, y)$ and $S_i$ is a sausage for $(b_i, b_{i+1})$ for $i = 1, \ldots, n - 1$.

(iii.) For a sausage walk $\omega = ((b_i)_{i=1,\ldots,n}, (A_i)_{i=0,\ldots,n})$ we define $|\omega|$ to be the number of steps of $\omega$. We denote by $b_i^\omega$ the $i$th bond of the backbone and by $A_i^\omega$ the $i$th sausage of $\omega$. We call $(b_i)_{i=1,\ldots,n}$ the backbone of the sausage walk.

(iv.) For $x \in \mathbb{Z}^d$ we define any sausage for $(0, x)$ to be a zero step sausage walk from the origin to $x$. 
We call a sausage walk non-backtracking, if \( \bar{b}_{i+1} \not\in S_i \), \( b_i \not\in S_i \) and \( \bar{b}_i \neq b_{i-1} \) for all \( i \).

We define \( \mathcal{W}^A(x) \) as the set of all sausage walks from the origin 0 to \( x \) and \( \mathcal{W}^{A,T}(x) \) to be the set of all sausage walks \( \omega \) from 0 to \( x \) such that \( e_i \not\in A_0^\omega \) and \( \bar{b}_1 \neq e_i \), if \( |\omega| > 0 \).

For the following common analysis we will also use the notation \( A_i^\omega \) for the \( i \)th rib of a rib walk \( \omega \). The rib and sausage walks are only another way to rewrite lattice trees/animals. To make this clear, we define for a rib/sausage walk \( \omega \),

\[
K[a, b](\omega) = \prod_{s=a}^{b-1} \prod_{s=t+1}^b \left( 1 + \mathcal{U}_{s,t}(\omega) \right),
\]

(2.3.1)

where \( -\mathcal{U}_{s,t}(\omega) \) is the indicator that the rib/sausage \( A_s(\omega) \) and \( A_t(\omega) \) intersect at some point in \( \mathbb{Z}^d \). Then \( K[0, |\omega|] \) is the indicator that all ribs/sausages of the walk are self-avoiding. We enforce that the union of the oriented bonds and all ribs/sausages of \( \omega \) is a disjoint union and that the resulting object is a lattice tree/animal. We define

\[
Z[a, b](\omega) := \prod_{i=a}^b \left( z^{A_i^\omega} \right),
\]

(2.3.2)

and remark that \( Z[a, b](\omega) = Z[a, c](\omega)Z[c + 1, b](\omega) \) for \( c \in [a, b) \). If possible, we drop the argument \( \omega \) for \( \mathcal{U}_{st}, K[a, b] \) and \( Z[a, b] \). Further, we drop the superscript \( A \) and \( T \) for \( \mathcal{W} \) when we consider both models simultaneously. We can then write the two-point function as

\[
\tilde{G}_z(x) = \sum_{n=0}^{\infty} t_n(x) z^n = \sum_{\omega \in \mathcal{W}(x)} z^{|\omega|} Z[0, |\omega|] K[0, |\omega|].
\]

(2.3.3)

We use the following adaption of the two-point function in the expansion

\[
\tilde{G}_z^K(x) = \sum_{\omega \in \mathcal{W}^{A,T}(x)} z^{|\omega|} Z[0, |\omega|] K[0, |\omega|]
\]

(2.3.4)

We expand this product using the same set of graphs and laces as introduced by Hara and Slade in [39]:

**Definition 2.3.4 (Graphs and connected graphs).** Let \( a, b \in \mathbb{N} \) with \( a < b \). For \( s, t \in [a, b] \cap \mathbb{N} \) with \( s < t \) the edge between \( s \) and \( t \) is the tuple \((s, t)\). We abbreviate \((s, t)\) instead of \((s, t)\). We call a set of edges a graph. We call a graph connected, if for all \( c \in [a, b] \) there exists an edge \( s \in \Gamma \) such that \( s \leq c \leq t \). Let \( \mathcal{B}[a, b] \) be the set of all graphs on \([a, b]\) and \( \mathcal{G}[a, b] \) the set of all connected graphs on \([a, b]\).
**Definition 2.3.5** (Laces and compatible edges). We call a graph minimally connected or a lace if the removal of any edge would disconnect the graph and define \( \mathcal{L}[0,n] \) as the set of all minimally connected graphs on \([a,b]\). We define the function \( L : \mathcal{G}[a,b] \to \mathcal{L}[a,b] \) in a constructive manner as follows: For a \( \Gamma \in \mathcal{G}[a,b] \) let:

\[
\begin{align*}
    s_1 &= a, \\
    t_i &= \max \{ t : \exists s \leq t_{i-1} \text{ such that } st \in \Gamma \}, \\
    s_i &= \min \{ s : st_i \in \Gamma \}.
\end{align*}
\]

This procedure ends after a finite number of steps \( N \). We denote by \( L(\Gamma) \) the resulting graph \( L = \{ s_1, t_1, s_2, t_2, \ldots, s_N, t_N \} \). We call an edge \( st \neq L \) compatible to a lace \( L \) if \( L(L \cup \{ st \}) = L \). We denote by \( C(L) \) the set of all edges that are compatible with \( L \).

We define for \( a > b \)

\[
J[a,b] = \sum_{\Gamma \in \mathcal{G}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st} = \sum_{L \in \mathcal{L}[a,b]} \prod_{st \in L} \mathcal{U}_{st} \prod_{s't' \in C(L)} (1 + \mathcal{U}_{s't'}). \tag{2.3.5}
\]

Further, we see that

\[
K[a,b] = \prod_{i=a}^{b-1} \prod_{s=t+1}^{b} (1 + \mathcal{U}_{s,t}(\omega)) = \sum_{\Gamma \in \mathcal{G}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st} \tag{2.3.6}
\]

and \( K[a,a] = J[a,a] = 1 \). Then it can be shown that for \( a < b \)

\[
K[a,b] = \sum_{i=a}^{b-1} J[a,i]K[i+1,b] + J[a,b], \tag{2.3.7}
\]

see e.g. [39, Lemma 3.4]. We apply (2.3.7) to (2.3.3) with \( a = 0 \) and \( b = |\omega| > 0 \) to obtain

\[
\tilde{G}_z(x) = \sum_{\omega \in \mathcal{W}(x)} z^{|\omega|} Z[0,|\omega|] \sum_{i=0}^{|\omega|-1} J[0,i]K[i+1,|\omega|] \tag{2.3.8}
\]

\[
+ \sum_{\omega \in \mathcal{W}(x)} z^{|\omega|} Z[0,|\omega|] J[0,|\omega|]. \tag{2.3.9}
\]

We define the contribution of (2.3.9) to be \( \tilde{\Xi}_z(x) \):

\[
\tilde{\Xi}_z(x) = \sum_{\omega \in \mathcal{W}(x)} z^{|\omega|} Z[0,|\omega|] J[0,|\omega|]. \tag{2.3.10}
\]

To characterize (2.3.8) we cut the non-backtracking rib/sausage walk at the \( i \)th
We apply (2.3.7) to obtain
\[
(2.3.8) = \sum_{\omega \in \mathcal{W}(\omega)} z^{[\omega]} Z[0, |\omega|] \sum_{i=0}^{[\omega]-1} J[0, i] K[i+1, |\omega|]
\]
\[
= \sum_{y, k} \omega_{1} \in \mathcal{W}(\omega) \sum_{y, k} z^{[\omega]+1} Z[0, |\omega^{1}|] J[0, |\omega^{1}|] \mathbb{I}_{\{y + e_{k} \not\in A_{\omega_{1}}^{\omega_{1}}\}}
\times \sum_{\omega_{2} \in \mathcal{W}^{\omega}(x - y - e_{k})} z^{[\omega_{2}]} Z[0, |\omega_{2}|] K[0, |\omega_{2}|]
\]
\[
= z \sum_{y, k} \bar{\Psi}^{k}(y) \bar{G}_{x}^{k}(x - y + e_{k}),
\]
with
\[
\bar{G}_{x}^{k}(x) = \sum_{\omega \in \mathcal{W}^{\omega}(x)} z^{[\omega]} Z[0, |\omega|] K[0, |\omega|],
\]
\[
\bar{\Psi}^{k}(x) = \sum_{\omega \in \mathcal{W}(\omega)} z^{[\omega]} Z[0, |\omega|] J[0, |\omega|] \mathbb{I}_{\{x - e_{k} \not\in A_{\omega}^{\omega}\}}.
\]

In this way we have obtained the following characterization for the two-point function
\[
\tilde{G}_{x}(x) = \tilde{\Psi}_{x}(x) + z \sum_{y, k} \bar{\Psi}^{k}(y) \bar{G}_{x}^{k}(x - y + e_{k}),
\]
which is the analogous to the first NBW-equation in (1.5.13). To obtain a relation similar to the second equation of (1.5.13) we look at
\[
\tilde{G}_{x}(x) - \tilde{G}_{x}^{l}(x) = \sum_{\omega \in \mathcal{W}(\omega)} z^{[\omega]} Z[0, |\omega|] K[0, |\omega|].
\]
As \(\omega \in \mathcal{W}(\omega) \setminus \mathcal{W}^{l}(\omega)\) we know that \(e_{i} \in A_{0}^{\omega}\) or \(\tilde{b}_{1} = e_{i}\). It is convenient to define an abbreviation for the indicator of this condition:
\[
\mathbb{I}_{l}(\omega) = (\mathbb{I}_{|\omega|>0} \mathbb{I}_{\{\tilde{b}_{1}=e_{i}\}} + \mathbb{I}_{\{e_{i} \in A_{0}^{\omega}\}}).
\]
We remark that the non-backtracking condition of the rib/sausage walk exclude the case that \(e_{i} \in A_{0}^{\omega}\) and \(\tilde{b}_{1} = e_{i}\) for the same walk. We see that
\[
\tilde{G}_{x}(x) - \tilde{G}_{x}^{l}(x) = \sum_{\omega \in \mathcal{W}(\omega)} z^{[\omega]} Z[0, |\omega|] K[0, |\omega|] \mathbb{I}_{l}(\omega).
\]
We apply (2.3.7) to obtain
\[
\tilde{G}_{x}(x) - \tilde{G}_{x}^{l}(x) = \sum_{\omega \in \mathcal{W}(\omega)} z^{[\omega]} Z[0, |\omega|] \mathbb{I}_{l}(\omega)
\times \left( \sum_{i=0}^{[\omega]-1} J[0, i] K[i+1, |\omega|] + J[0, |\omega|] \right).
\]

bond of the backbone into a walk \(\omega^{1} \in \mathcal{W}\) and a second walk \(\omega^{2} \in \bigcup_{x} \mathcal{W}^{x}\).
We define
\[ \tilde{\Xi}^i_\omega(x) = \sum_{\omega \in W(x)} \mathbb{I}_i(\omega) z^{[\omega]} Z[0,|\omega|] J[0,|\omega|]. \] (2.3.19)
so that the contribution of \(2.3.18\) due to \(J[0,|\omega|]\) equals \(\tilde{\Xi}^i_\omega(x)\). To capture the contribution of \(J[0,i] K[i+1,|\omega|]\) we recall the definition \(\tilde{G}^i_\omega\) in (2.3.4) and see that
\[ \sum_{\omega^2 \in W^t(x-y-e_x)} z^{[\omega^2]} Z[0,|\omega^2|](\omega^2) K[0,|\omega^2|](\omega^2) = \tilde{G}^i_\omega(x-y-e_x) \] (2.3.20)
The dominant contribution to \(2.3.18\) is given by the case \(|\omega| \geq 1, i = 0\) and \(b^0_0 = (0, e_i)\), for which we see that
\[ \sum_{\omega \in W(x)} \mathbb{I}_{b^0_0=(0,e_i)} z^{[\omega]} Z[0,|\omega|] J[0,0] K[1,|\omega|] = z g^i_0 \tilde{G}^i_\omega(x-e_i), \]
with \(g^i_0 = \tilde{G}^i_\omega(0)\). To rewrite the other contribution to \(2.3.18\) we split the walk after the \(i\)th pivotal bound, which we identify to be \((y, y-e_x)\). This leads to
\[ \sum_{\omega \in W(x)} z^{[\omega]} Z[0,|\omega|] \sum_{i=0}^{[\omega]-1} J[0,i] K[i+1,|\omega|] \mathbb{I}_i(\omega)(1 - \delta_{i,0}) \mathbb{I}_{i}(y,y-e_x)=(0,e_i)) \] (2.3.21)
\[ = \sum_{y \in Z^d} \sum_{\omega^1 \in W(y)} \sum_{\omega^2 \in W^t(x-y+e_x)} z^{[\omega^1]+[\omega^2]+1} Z[0,|\omega^1|](\omega^1) Z[0,|\omega^2|](\omega^2) \times J[0,|\omega^1|](\omega^1) K[0,|\omega^2|](\omega^2) (\mathbb{I}_i(\omega^1) + \mathbb{I}_{i}(|\omega^1|=0) \mathbb{I}_{i}(y \neq y-e_x) \mathbb{I}_{i}(y-e_x=e_i) \mathbb{I}_{i}(y-e_x \notin \omega^1{1})). \]
We define
\[ \Pi^i_\omega(\omega) = (\mathbb{I}_i(\omega) + \mathbb{I}_{i}(|\omega|=0) \mathbb{I}_{i}(x \neq x-e_x=e_i)), \]
\[ \Pi^i_{z,K}(x) = \sum_{\omega \in W(x)} \Pi^i_\omega(\omega) \mathbb{I}_{i}(x-e_x \notin \omega^1) z^{[\omega]+1} Z[0,|\omega|] J[0,|\omega|], \] (2.3.22)
and use \(2.3.20\), together with the fact that the sums over \(\omega^1\) and \(\omega^2\) factorize, to see that
\[ \tilde{G}^i_\omega(x) - \tilde{G}^i_\omega(x) = z g^i_0 \tilde{G}^i_\omega(x-e_i) + \Xi^i_\omega(x) + \sum_{y,K} \Pi^i_{z,K}(y) \tilde{G}^i_\omega(x-y+e_k). \] (2.3.23)
This completes the derivation of the expansion for LT and LA.

**Interpretation of the lace-expansion coefficients.** The lace-expansion coefficients can be represented by alternating series of positive functions:

**Definition 2.3.6** (Laces for with fixed number of edges). For \(n \geq 1\) and \(N \geq 1\), let \(\mathcal{L}^{(N)}[0,n] \subset \mathcal{L}[0,n]\) be the set of all laces \(L \in \mathcal{L}[0,n]\) that consists of exactly \(N\) edges.
For $N \geq 0$ and $x \in \mathbb{Z}^d$ let

\[
J^{(N)}[a, b](\omega) = \sum_{L, \omega \in \mathcal{Z}^{(N)}[a, b]} \prod_{s \in L} U_{st} \prod_{s' \in \mathcal{E}(L)} (1 + U_{s't'}),
\]

(2.3.24)

\[
\tilde{\Xi}^{(N)}_z(x) = (-1)^N \sum_{\omega \in \mathcal{W}(x)} z^{\omega} \bar{Z}[0, |\omega|] J^{(N)}[0, |\omega|],
\]

(2.3.25)

\[
\bar{\Psi}^{(N)}_z(x) = (-1)^N \sum_{\omega \in \mathcal{W}(x)} z^{\omega} \bar{Z}[0, |\omega|] J^{(N)}[0, |\omega|] \mathbb{I}_i(\omega),
\]

(2.3.26)

\[
\tilde{\Psi}^{(N)}_z(x) = (-1)^N \sum_{\omega \in \mathcal{W}(x)} z^{\omega} \bar{Z}[0, |\omega|] J^{(N)}[0, |\omega|] \mathbb{I}_{|x-e_s \notin A^w_0|},
\]

(2.3.27)

\[
\Pi^{(N),l,k}_z(x) = (-1)^N \sum_{\omega \in \mathcal{W}(x)} z^{\omega+1} \bar{Z}[0, |\omega|] J^{(N)}[0, |\omega|] \mathbb{I}_{|x-e_s \notin A^w_0|} \mathbb{I}_l \mathbb{I}_k(\omega).
\]

(2.3.28)

In $J^{(N)}$ the sum over laces forces $N$ ribs/sausages to intersect. As we sum over non-backtracking rib/sausage walks all loops formed by the intersections contain at least four bonds. We will discuss this further in Sections 4.3 and 4.4.

For $N = 0$ we investigate the difference between the lattice trees and animals. For $N = 0$, only $\omega$ with $|\omega| = 0$ contribute and thereby the coefficients for $N = 0$ describe trees/animals consisting of only one rib/sausage. For lattice trees, we see that

\[
\tilde{\Xi}^{(0)}_z(x) = \delta_{0,x} \tilde{G}_z(0) = \delta_{0,x} g_z,
\]

(2.3.29)

\[
\tilde{\Psi}^{(0),l}_z(x) = \delta_{0,x} \tilde{G}_z^l(0) = \delta_{0,x} g_z^l,
\]

(2.3.30)

For a lattice animal $A$ and $x, y \in \mathbb{Z}^d$ we denote by $x \leftrightarrow y$ in $A$ the event that $x$ and $y$ are doubly connected via bonds in $A$. For the NoBLE coefficients of LA we know that

\[
\tilde{\Xi}^{(0)}_z(x) = \sum_{A \text{ animal}} \mathbb{I}_{[0 \leftrightarrow x \text{ in } A]} z^{A},
\]

(2.3.31)

\[
\tilde{\Xi}^{(0),l}_z(x) = \sum_{A \text{ animal}} \mathbb{I}_{[e_s \in A]} \mathbb{I}_{[0 \leftrightarrow x \text{ in } A]} z^{A},
\]

(2.3.32)

\[
\tilde{\Psi}^{(0),l}_z(x) = \sum_{A \text{ animal}} \mathbb{I}_{[x-e_s \notin A]} \mathbb{I}_{[0 \leftrightarrow x \text{ in } A]} z^{A},
\]

(2.3.33)

\[
\tilde{\Xi}^{(0),l,k}_z(x) = \sum_{A \text{ animal}} \mathbb{I}_{[x-e_s \notin A]} \mathbb{I}_{[0 \leftrightarrow x \text{ in } A]} (\mathbb{I}_{[e_s \in A]} + \mathbb{I}_{[x-e_s = e_t]} \mathbb{I}_{[x \neq 0]} z^{A} + 1).
\]

(2.3.34)

**Normalization for lattice trees and animals.** In Section 1.4 we already remarked that we will normalize the two-point function $\hat{G}_z$ of LT and LA. Let $g_z = \hat{G}_z(0)$ then we define

\[
G_z(x) = \frac{1}{g_z} \hat{G}_z(x),
\]

(2.3.35)

\[
G^l_z(x) = \frac{1}{g_z} \hat{G}^l_z(x),
\]

(2.3.35)

\[
\Xi_z(x) = \frac{1}{g_z} \tilde{\Xi}_z(x),
\]

(2.3.36)

\[
\Xi^l_z(x) = \frac{1}{g_z} \tilde{\Xi}^l_z(x),
\]

(2.3.36)
Then, the derived NoBLE relations (2.3.23) and (2.3.23) can be rewritten to

\[
G_z(x) = \Xi_z(x) + zg_z \sum_{y \in \mathbb{Z}^d} \sum_{\kappa \in \{\pm 1, \ldots, \pm d\}} \bar{\Psi}_z^\kappa(y) G_z^\kappa(x - y - e_\kappa),
\]

(2.3.37)

\[
G_z(x) = G_z^i(x) + zg_z G_z^{-i}(x - e_i)
+ \sum_{y \in \mathbb{Z}^d} \sum_{\kappa \in \{\pm 1, \ldots, \pm d\}} \Pi_z^{i,\kappa}(y) G_z^\kappa(x - y - e_\kappa) + \Xi^i(x).
\]

(2.3.38)

## 2.4 Percolation

We derive the NoBLE for percolation using the inclusion-exclusion approach already explained in Section 2.1. Since, in contrast to the other models, percolation is not a combinatorial model, the inclusion-exclusion is performed in terms of events. In Section 2.4.1 we introduce the necessary notation and a restricted two-point function that we use for the expansion. In Section 2.4.2 we prove an expansion for this restricted two-point function. In Section 2.4.3 we use this expansion of the restricted two-function to obtain two relations similar to the NBW-relations (1.5.13). Parts of this section are taken almost verbatim from [50, Section 2].

### 2.4.1 Notation

Fix \( p \in [0, 1] \). We write \( \tau(x) = \tau_p(x) \) for brevity, and generally drop subscripts indicating dependence on \( p \).

**Definition 2.4.1.**

(i) Given a bond configuration \( \omega \) and two points \( x, y \in \mathbb{Z}^d \), we say that \( x \) and \( y \) is connected, and write \( x \leftrightarrow y \), if there exists a path of occupied edges connecting \( x \) and \( y \). Further, we say that \( x \) and \( y \) are doubly connected, and write \( x \iff y \), if there exists two bond-disjoint path of occupied bonds connecting \( x \) and \( y \). We adopt the convenient convention that \( x \) is double connected to itself.

(ii) Given a (deterministic or random) set of undirected bonds \( B \) and a bond configuration \( \omega \), we define \( \omega_B \), the restriction of \( \omega \) to \( B \), to be

\[
\omega_B([x, y]) = \begin{cases} 
\omega([x, y]) & \text{if } [x, y] \in B, \\
0 & \text{otherwise},
\end{cases}
\]

(2.4.1)

for every nearest-neighbor pair \( x, y \). In other words, \( \omega_B \) is obtained from \( \omega \) by making every bond that is not in \( B \) vacant.

(iii) Given a (deterministic or random) set of vertices \( A \) we define \( B(A) \) to be the set of all bonds that have at least one endpoint in \( A \).
(iv) Given a (deterministic or random) set of bonds $B$ and an event $E$, we say that $E$ occurs in $B$, and write $\{ E \text{ in } B \}$, if $\omega_B \in E$. In other words, $\{ E \text{ in } B \}$ means that $B$ occurs on the (possibly modified) configuration in which every bond that is not in $B$ is made vacant. We further say that $E$ occurs off $B$ when $E$ occurs in $B^c$.

(v) Given a (deterministic or random) set of vertices $A$ and an event $E$, we say that $E$ occurs in $A$, and write $\{ E \text{ in } A \}$, if $E$ occurs in $B(A)$. We adopt the convenient convention that $x \leftrightarrow x$ in $A$ occurs if and only if $x \in A$. We further say that $E$ occurs off $A$ when $E$ occurs in $A^c$.

(vi) Given a bond configuration and $x \in \mathbb{Z}^d$, we define $C(x)$ to be the set of vertices to which $x$ is connected, i.e., $C(x) = \{ y \in \mathbb{Z}^d : x \leftrightarrow y \}$. Given a bond configuration and a bond $b$, we define $\mathcal{C}^b(x)$ to be the set of vertices $y \in \mathcal{C}(x)$ to which $x$ is connected in the (possibly modified) configuration in which $b$ is made vacant.

(vii) Given a deterministic set of bonds $B$ we define the probability measure $P_B$ by

$$P_B(E) = P(E \text{ occurs off } B). \quad (2.4.2)$$

For $i \in \{ \pm 1, \ldots, \pm d \}$ we define the modified two-point function as:

$$\tau^i_p(x,y) = P_p(x \leftrightarrow y \text{ off } x + e_i) = P_p^{B(x+e_i)}(x \leftrightarrow y), \quad (2.4.3)$$

$$\tau^i_p(x) = \tau^i_p(0,x). \quad (2.4.4)$$

Further, we denote by $b_i$ the bond $b_i = \{0, e_i\}$ and write $P_p^{b_i}$ for $P_p^{\{b_i\}}$. We note that $P_p = P_p^{\emptyset}$ and that for all events $E$ and sets of bonds $A, B$,

$$\{ E \text{ off } A \} \text{ off } B \} = \{ E \text{ in } A \cup B \}. \quad (2.4.5)$$

and thereby

$$P_B(E \text{ off } A) = P(E \text{ occurs off } A \cup B). \quad (2.4.6)$$

We next define the notion of connected through:

**Definition 2.4.2 (Connections through).** Given a bond configuration and a set of bonds $B \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$, we say that $x$ is connected to $y$ through $B$, and write $x \leftrightarrow^B_y$, if every occupied path connecting $x$ to $y$ contains at least one bond in $B$. Given a bond configuration and a set $A \subseteq \mathbb{Z}^d$, we say that $x$ is connected to $y$ through $A$, and write $x \leftrightarrow^A_y$, if $x$ is connected to $y$ through $B(A)$. By convention, $x \leftrightarrow^A_y$ holds if and only if $x \in A$.

In terms of events, it is clear that

$$\{ x \leftrightarrow^A_y \} = \{ x \leftrightarrow^B_y \} \setminus \{ x \leftrightarrow^B_y \text{ off } A \}. \quad (2.4.7)$$
Using the notation (2.4.2) we see that
\[ \tau(x) = \mathbb{P}(0 \leftarrow x) = \mathbb{P}(0 \leftrightarrow x), \]  
(2.4.8)
\[ \tau(x) - \tau^t(x) = \mathbb{P}([0 \leftrightarrow x] \setminus [0 \leftrightarrow x \; \text{off} \; \{e_i\}]) = \mathbb{P}(0 \leftarrow e \; x). \]  
(2.4.9)

When deriving an expansion for \( \mathbb{P}(x \leftarrow A \leftrightarrow y) \) we immediately obtain an expansion for both \( \tau(x) \) and \( \tau(x) - \tau^t(x) \). For technical reasons we prove the more general expansion: Let \( M \in \mathbb{N} \), \( A \) be any deterministic set of vertices and \( B \) be any deterministic set of bonds, then
\[ \mathbb{P}^B(x \leftarrow A \leftrightarrow y) = \Xi_M^B(x, y; A) + \sum_{u, \kappa} p \Psi_M^{B, \kappa}(x, w - e_{\kappa}; A) \tau^\kappa(x - w) + R_M^B(x, y; A). \]  
(2.4.10)

The functions \( \Xi_M^B \) and \( \Psi_M^{B, \kappa} \) are the key quantity in the expansion, and \( R_M^B \) is a remainder term. The dependence of \( \Xi_M^B \) and \( \Psi_M^{B, \kappa} \) on \( M \) is given by
\[ \Xi_M^B(x, y; A) = \sum_{N=0}^{M} (-1)^N \Xi_M^{B, (N)}(x, y; A), \]  
(2.4.11)
\[ \Psi_M^{B, \kappa}(x, w; A) = \sum_{N=0}^{M} (-1)^N \Psi_M^{B, (N), \kappa}(x, w; A), \]  
(2.4.12)

with \( \Xi_M^{B, (N)}(x, y; A) \) and \( \Psi_M^{B, (N), \kappa}(x, w; A) \) independent of \( M \). The alternating signs in (2.4.11) arise via repeated inclusion-exclusion. The next section is devoted to the proof of (2.4.10).

### 2.4.2 Expansions for restricted two-point functions

We need the following definitions:

**Definition 2.4.3 (Pivotal bonds).** Given a bond configuration, a bond \( \{u, v\} \) (occupied or not) is called pivotal for the connection from \( x \) to \( y \), if (i) either \( x \leftarrow u \) and \( y \leftarrow v \), or \( x \leftarrow v \) and \( y \leftarrow u \), and (ii) \( y \not\in \tilde{C}^{(u, v)}(x) \). Bonds are not usually regarded as directed. However, it will be convenient at times to regard a bond \( \{u, v\} \) as directed from \( u \) to \( v \), and we will emphasize this point of view with the notation \((u, v)\). A directed bond \((u, v)\) is pivotal for the connection from \( x \) to \( y \), if \( x \leftarrow u \), \( v \leftarrow y \) and \( y \not\in \tilde{C}^{(u, v)}(x) \). We denote with \( b \) the starting point and \( \bar{b} \) the ending point of the bond \( b \).

In terms of Definition 2.4.1 we have a characterization of a pivotal bond for \( v \leftarrow y \) as
\[ \{b \; \text{pivotal for} \; v \leftarrow y\} = \{v \leftarrow b \} \]
\[ = \{v \leftarrow \overline{b}, \; \overline{b} \not\in \tilde{C}^b(v) \} \cap \{\overline{b} \leftarrow y \; \text{in} \; \tilde{C}^b(v)\}. \]  
(2.4.13)
For a set of vertices $A$, we define the events

$$E'(v, y; A) = \{v \leftarrow^A y\} \cap \{\exists b' \text{ occupied and pivotal for } v \leftarrow y \text{ such that } v \leftarrow^A b'\}$$

(2.4.14)

and

$$E(x, b, y; A) = E'(x, b; A) \cap \{b \text{ is occupied and pivotal for } x \leftarrow y\}.$$  

(2.4.15)

Given a configuration in which $x \leftarrow^A y$, the cutting bond $b$ is defined to be the first bond which is pivotal for $x \leftarrow y$ such that $x \leftarrow^A b$. It is possible that no such bond exists. By partitioning $\{x \leftarrow^A y\}$ according to the location of the cutting bond (or the lack of a cutting bond), we obtain the partition

$$\{x \leftarrow^A y\} = E'(x, y; A) \bigcup \bigcup_b E(x, b, y; A),$$

(2.4.16)

which implies that

$$\mathbb{P}^\beta(x \leftarrow^A y) = \mathbb{P}^\beta(E'(x, y; A)) + \sum_b \mathbb{P}^\beta(E(x, b, y; A)).$$

(2.4.17)

The following lemma is the major tool we use to derive the expansion:

**Lemma 2.4.4** (The cutting lemma). Let $p < p_c(d)$, $x, y \in \mathbb{Z}^d$, and let $A \subseteq \mathbb{Z}^d$. Then, for all bonds $b$,

$$\mathbb{P}\left(E(x, b, y; A)\right) = p\mathbb{E}_0\left(\mathbb{I}_{E(x, b; A)}\mathbb{I}_{b \not\in \widehat{C}^b(x)}\right)\mathbb{P}_1\left(\tilde{b} \leftarrow y \text{ off } \widehat{C}^b(x)\right).$$

(2.4.18)

**Proof.** The lemma is proved e.g. in [92, Lemma 10.1], with the exception that the indicator $b \not\in \widehat{C}^b(x)$ is absent on the right-hand side there. When $\tilde{b} \in \widehat{C}^b(x)$, then $\mathbb{P}_1(\tilde{b} \leftarrow y \text{ off } \widehat{C}^b(x)) \equiv 0$, so the statement is also true with the indicator.

We remark here that [92, Lemma 10.1] proves Lemma 2.4.4 for percolation on all graphs. As a result, Lemma 2.4.4 also applies to the measure $\mathbb{P}^\beta$ for all deterministic
bond sets $B$ and we obtain that for every $p < p_c(d)$, $x, y, z \in \mathbb{Z}^d$, set of bonds $B$, set of vertices $A$ and bonds $b$,

$$\mathbb{P}^b(E(x, b, y; A)) = p\mathbb{E}_0^b\left(\text{1}_{E(x,b:A)}\text{1}_{\{b\notin \hat{\mathcal{E}}^b(x)\}}\mathbb{P}^a_1(\hat{b} \rightarrow y \text{ off } \hat{\mathcal{E}}^b(x))\right). \quad (2.4.19)$$

From (2.4.6) we conclude that

$$\mathbb{P}^b_1(\hat{b} \rightarrow y \text{ off } \hat{\mathcal{E}}^b(x)) = \mathbb{P}^b(\hat{b} \rightarrow y \text{ off } \mathcal{E}^b(x) \cup B), \quad (2.4.20)$$

where we write $\mathcal{E}^b(x) \cup B$ as abbreviation for $B(\hat{\mathcal{E}}^b(x)) \cup B$. Combining this with (2.4.7) we obtain that

$$\mathbb{P}^b(x \xrightarrow{A} y) = \mathbb{P}^b(E'(x, y; A))$$

$$+ \sum_b p\mathbb{E}_0^b\left(\text{1}_{E'(x,b:A)}\text{1}_{\{b\notin \hat{\mathcal{E}}^b(x)\}}\mathbb{P}^a_1(\hat{b} \rightarrow y \text{ off } \hat{\mathcal{E}}^b(x) \cup B)\right). \quad (2.4.21)$$

On the right side, $\mathbb{P}^b_1(\hat{b} \rightarrow y \text{ off } \hat{\mathcal{E}}^b(x) \cup B)$ is the restricted two-point function given the bond boundary $\mathcal{E}^b(x) \cup B$ of the outer expectation, so that in the expectation defining $\mathbb{P}^b_1(\hat{b} \rightarrow y \text{ off } \hat{\mathcal{E}}^b(x) \cup B)$, $\mathcal{E}^b(x)$ should be regarded as a fixed set. We stress this delicate point here, as it is also crucial for the further expansion. The inner expectation on the right side effectively introduces a second percolation model on a second graph, which depends on the original percolation model via the set $\mathcal{E}^b(x)$. For any bond $b'$ and set of bonds $B'$ such that $B(b) \subset B'$ we can generalize (2.4.7) to

$$\{x \xrightarrow{A} y \text{ off } B'\} = \{x \xrightarrow{A} y \text{ off } B(b')\} \setminus \{x \xrightarrow{A} y \text{ off } B(b')\}, \quad (2.4.22)$$

which implies that

$$\mathbb{P}(\hat{b}' \rightarrow y \text{ off } B') = \tau_{\hat{b}'}(\hat{b}', y) - \mathbb{P}_{\hat{b}'}(\hat{b}' \xrightarrow{B'} y). \quad (2.4.23)$$

As the indicator $\text{1}_{E'(x,b:A)}$ is present we know that only configurations with $b \in \mathcal{E}^b(x)$ contribute and we can apply (2.4.23) with $B' = \mathcal{E}^b(x) \cup B$ and $b' = b$ for (2.4.21) to obtain

$$\mathbb{P}^b(x \xrightarrow{A} y) = \mathbb{P}^b(E'(x, y; A))$$

$$+ \sum_b p\mathbb{E}_0^b\left(\text{1}_{E'(x,b:A)}\text{1}_{\{b\notin \hat{\mathcal{E}}^b(x)\}}\tau_{\hat{b}}(\hat{b}, y)\right)$$

$$- \sum_b p\mathbb{E}_0^b\left(\text{1}_{E'(x,y;A)}\text{1}_{\{b\notin \hat{\mathcal{E}}^b(x)\}}\mathbb{P}_1^b(\hat{b} \rightarrow \hat{\mathcal{E}}^b(x) \cup B)\right) \quad \tau_{\hat{b}}(\hat{b}, y) \quad (2.4.24)$$

$$= \Xi^{B,(0)}(x, y; A) + \sum_{w,\kappa} p\mathbb{P}_{B,(0),\kappa}(x, w - e_k; A)\tau_{\kappa}(y - w) - R^b_0(x, y; A),$$
We introduce subscripts for where we define

\[ \Xi^{b,(0)}(x, y; A) = \mathbb{P}^b(E'(x, y; A)), \]  
\[ \Psi^{b,(0),\kappa}(x, w; A) = p \mathbb{P}^b(E'(x, w; A) \cap \{ w + e_\kappa \not\in \bar{\mathcal{C}}(w, w + e_\kappa)(x) \}), \]

\[ R^b_0(x, y; A) = \sum_b p \mathbb{E}^b \left( \mathbb{I}_{E'(x, y; A)} \mathbb{I}_{\bar{b} \not\in \mathcal{C}(x)} \mathbb{P}^b(\bar{B} \rightarrow \bar{\mathcal{C}}(x) \cup B, y) \right). \]  

This proves (2.4.10) for \( M = 0 \). To continue the expansion, we use (2.4.24) to rewrite \( \mathbb{P}^b(\bar{B} \rightarrow \bar{\mathcal{C}}(x) \cup B, y) \) in \( R^b_0(x, y; A) \) as

\[ \mathbb{P}^b(\bar{B} \rightarrow \bar{\mathcal{C}}(x) \cup B, y) = \mathbb{P}^b(E'(\bar{B}, y; \bar{\mathcal{C}}(x) \cup B)) + \sum_{b_1} p \mathbb{E}^{b_1} \left( \mathbb{I}_{E(\bar{B}, \bar{B}'; \bar{\mathcal{C}}(x) \cup B)} \mathbb{I}_{\bar{b}_1 \not\in \bar{\mathcal{C}}(B)} \mathbb{P}^{b_1}(\bar{B} \rightarrow \bar{\mathcal{C}}(x) \cup B, y) \right). \]

We introduce subscripts for \( \bar{\mathcal{C}} \), the expectations and the bonds to indicate to which expectation they belong. For brevity we write \( \mathcal{C}_0 = \bar{\mathcal{C}}_{0,0}(x) \) and \( \mathcal{C}_i = \bar{\mathcal{C}}_{i,0}(\bar{B}_{i-1}) \) for \( i \geq 1 \). We insert (2.4.28) into \( R^b_0(x, y; A) \) and obtain (2.4.10) for \( M = 1 \) with

\[ \Xi^{b,(1)}(x, y; A) = \sum_b p \mathbb{E}^b \left( \mathbb{I}_{E'(x, y; A)} \mathbb{I}_{\bar{b} \not\in \mathcal{C}(x)} \mathbb{P}^b(\bar{B} \rightarrow \bar{\mathcal{C}}_0 \cup B) \right), \]  
\[ \Psi^{b,(1),\kappa}(x, w; A) = \sum_b p \mathbb{E}^b \left( \mathbb{I}_{E'(x, y; A)} \mathbb{I}_{\bar{b} \not\in \mathcal{C}(x)} \mathbb{P}^b(\bar{B} \rightarrow \bar{\mathcal{C}}_0 \cup B) \times \mathbb{P}^{b_1}(E(\bar{B}, w; \mathcal{C}_0 \cup B) \cap \{ w + e_i \not\in \bar{\mathcal{C}}_1(w, w + e_\kappa)(\bar{B}) \}) \right) \]

and

\[ R^b_1(x, y; A) = \sum_{b_0, b_1} p^2 \mathbb{E}^b \left( \mathbb{I}_{E'(x, y; A)} \mathbb{I}_{\bar{b}_0 \not\in \mathcal{C}(x)} \mathbb{P}^{b_0} E(\bar{B}, \bar{B}' \cup B) \mathbb{I}_{\bar{b}_1 \not\in \mathcal{C}(x)} \mathbb{P}^{b_1}(\bar{B}_1 \rightarrow \bar{\mathcal{C}}_1(\bar{B}_0) \cup B, y) \right). \]
This proves (2.4.10) for \( M = 1 \). We now repeat using (2.4.28) recursively, for
\[
\mathbb{P} \begin{array}{c}
\mathcal{B}_M \\
\mathcal{B}_{M+1}
\end{array}
\xrightarrow{\hat{\mathcal{E}}_M (\mathcal{B}_{M-1} \cup \{ b_{M-1} \})} y)
\]
(2.4.31)
in the created remained term \( R^B_M(x, y; A) \). This leads to (2.4.10) for all \( M \) with \( \Xi^{B,(N)} \) and \( \Psi^{B,(N),K} \) given by
\[
\Xi^{B,(N)}(x, y; A) = p^N \sum_{b_0, \ldots, b_{N-1}} \mathbb{E}^B_{E(x, b_0; A)} \mathbb{P} \begin{array}{c}
\mathcal{B}_M \\
\mathcal{B}_N
\end{array}
\xrightarrow{\hat{\mathcal{E}}_M (\mathcal{B}_{N-1} \cup \{ b_{N-1} \})} y)
\] (2.4.32)
\[
\Psi^{B,(N),K}(x, y; A) = p^N \sum_{b_0, \ldots, b_{N-1}} \mathbb{E}^B_{E(x, b_0; A)} \mathbb{P} \begin{array}{c}
\mathcal{B}_M \\
\mathcal{B}_N
\end{array}
\xrightarrow{\hat{\mathcal{E}}_M (\mathcal{B}_{N-1} \cup \{ b_{N-1} \})} y)
\] (2.4.33)
\[
R^B_M(x, y; A) = (-1)^{M+1} p^{M+1} \sum_{b_0, \ldots, b_M} \mathbb{E}^B_{E(x, b_0; A)} \mathbb{P} \begin{array}{c}
\mathcal{B}_M \\
\mathcal{B}_{M+1}
\end{array}
\xrightarrow{\hat{\mathcal{E}}_M (\mathcal{B}_{M+1} \cup \{ b_M \})} y)
\] (2.4.34)
for \( N \geq 2 \). Since
\[
\mathbb{P} \begin{array}{c}
\mathcal{B}_M \\
\mathcal{B}_{M+1}
\end{array}
\xrightarrow{\hat{\mathcal{E}}_M (\mathcal{B}_{M+1} \cup \{ b_{M+1} \})} y) \leq r^B_M (\mathcal{B}_M, x),
\] (2.4.35)
it follows from (2.4.33) – (2.4.34) that
\[
|R^B_M(x, y; A)| \leq \sum_{i \in \mathbb{K}} \Psi^{(M+1),i}(w - \epsilon_x) p r^\mathcal{K}(y - w - x).
\] (2.4.36)

When we take \( M \to \infty \), and assume that \( \lim_{M \to \infty} |R^B_M(x, y; A)| = 0 \), we arrive at
\[
\mathbb{P}^B(x \xrightarrow{A} y) = \Xi^B(x, y; A) + \sum_{y, i} p \Psi^{B,K}(x, y - \epsilon_x; A) r^\mathcal{K}(x - y),
\] (2.4.37)
where
\[
\Xi^B(x, y; A) = \sum_{N=0}^\infty (-1)^N \Xi^{B,(N)}(x, y; A), \quad \Psi^{B,K}(x, w; A) = \sum_{N=0}^\infty (-1)^N \Psi^{B,(N),K}(x, w; A).
\] (2.4.38)

Naturally, the convergence of the expansion needs to be obtained to reach the above conclusion. This convergence follows from (2.4.36) and the bounds on \( \Psi^{(M+1),i} \) that we prove in Section 4.5 by showing that the remainder term \( R^B_M \) converges to zero.
The expansion developed here is different from the traditional lace expansion as it expands in terms of $\tau'(x)$ rather than $\tau(x)$. This difference causes that the formulas (2.4.32), (2.4.33) and (2.4.34) involve $E_j^{b_j-1}$ rather than just $E_j$ as in [38]. Further, we explicitly keep the factors $\Pi_{[\tilde{b}_j \neq \tilde{\mathcal{C}}_j]}$. Finally, we have defined $\tilde{\mathcal{C}}_j = \tilde{\mathcal{C}}_j^{b_j} (\tilde{b}_j - 1) \cup \{b_j-1\}$, while in the classical lace expansion $\tilde{\mathcal{C}}_j = \tilde{\mathcal{C}}_j^{b_j} (\tilde{b}_j)$, see [38]. These differences ensure, as will argue below, that each loop in the lace-expansion coefficients now involve paths of at least four steps, whereas in [38], they can have length equal to two.

Let us now show that each loop in a coefficient consists of at least four steps. By the parity of the hypercubic lattice a loop consists of an even number of steps. On the lattice there exists only one possibility for a two-step loop, when $b_j - 1 = b_j$ and $b_j = b_j - 1$. We will now argue by contradiction that $b_j \neq b_j - 1$ does not contribute. Let us assume instead that $b_j = b_j - 1$. Then, we know by $E'(x, x; A) = \{x \xrightarrow{A} x\} = \{x \in A\}$ that
\[
E'(\tilde{b}_{j-1}, b_j; \tilde{\mathcal{C}}_{j-1}) = \{\tilde{b}_{j-1} \in \tilde{\mathcal{C}}_{j-1}\},
\] (2.4.39)
which can not contribute to the lace-expansion coefficients, due to the presence of the indicator $\Pi_{[\tilde{b}_j \neq \tilde{\mathcal{C}}_j]}$.

2.4.3 Completion of the expansion

In this section, we complete the expansion. We first characterize (2.4.8) using (2.4.10). We choose $B = \emptyset$ and $A = \{0\}$ and obtain
\[
\tau(x) = \mathbb{P}^\emptyset(x \leftrightarrow y) = \Xi_M(x) + p \sum_{y, \kappa} \Psi^K_M(y - e_\kappa) \tau^K(x - y - e_\kappa) + R^\emptyset_M(x, y; A)
\] (2.4.40)
with
\[
\Xi^{(N)}(x) = \Xi^{\emptyset, (N)}(0, x; \{0\}), \quad \Xi_M(x) = \sum_{N=0}^{M} (-1)^M \Xi^{(N)}(x),
\] (2.4.41)
\[
\Psi^{(N), K}(x) = \Psi^{\emptyset, (N), K}(0, x; \{0\}), \quad \Psi^K_M(x) = \sum_{N=0}^{M} (-1)^M \Psi^{(N), K}(x),
\] (2.4.42)
\[
R_M(x) = R^\emptyset_M(0, x; \{0\}).
\] (2.4.43)
Further, we note that
\[
\Xi^{(0)}(x) = E'(0, x; \{0\}) = \{0 \leftrightarrow x\},
\] (2.4.44)
which equals 1 when $x = 0$. This is the main contribution to $\Xi_M(x)$. Similarly, the main contribution to $\Psi^K_M(x)$ arises from $\Psi^{(0), K}(x)$, which equals
\[
\Psi^{(0), K}(x) = \mathbb{P} \left( 0 \leftrightarrow x, x + e_\kappa \notin \tilde{\mathcal{C}}(x, x + e_\kappa) \{0\} \right)
\] (2.4.45)
being also dominated by the contribution at \( x = 0 \):
\[
\Psi^{(0),\kappa}(0) = \mathbb{P}\left( e_\kappa \not\in \tilde{\mathcal{E}}^{(0,e_\kappa)}(0) \right)
\]  \hspace{1cm} (2.4.46)

To obtain the second relation of the NoBLE we will not apply (2.4.10) directly on \( \mathbb{P}(0 \overset{e_i}{\rightarrow} x) \) in (2.4.9). Instead, we first rewrite (2.4.9). We recall \( b_i = \{0, e_i\} \) and we see that
\[
\tau(x) - \tau^i(x) = \mathbb{P}(0 \overset{e_i}{\rightarrow} x) = \mathbb{P}(0 \overset{b_i}{\rightarrow} x) + \mathbb{P}^{b_i}(0 \overset{e_i}{\rightarrow} x).
\]  \hspace{1cm} (2.4.47)

Then, we apply (2.4.10) only on the second part \( \mathbb{P}^{b_i}(0 \overset{e_i}{\rightarrow} x) \). From \( 0 \overset{b_i}{\rightarrow} x \) we want to extract the NBW-like contribution \( p \tau^{-i}(x - e_i) \). We see that \( \{0 \overset{b_i}{\rightarrow} x\} = E(0, b_i, x; \{0\}) \) as it is equivalent to \( b_i \) being pivotal and occupied. Thus, we apply Lemma 2.4.4 (Cutting Lemma) with \( x = 0, b = b_i, y = x \) and \( A = \{0\} \) to obtain
\[
\mathbb{P}(0 \overset{b_i}{\rightarrow} x) = p \mathbb{E}^{b_i}_0 \left[ \mathbb{1}_{\{e_i \not\in \tilde{\mathcal{E}}^{b_i}(0)\}} \mathbb{P}^{b_i}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } \tilde{\mathcal{E}}^{b_i}(0)) \right].
\]  \hspace{1cm} (2.4.48)

Next, we analyze \( \mathbb{P}^{b_i}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } \tilde{\mathcal{E}}^{b_i}(0)) \) within the outside expectation \( \mathbb{E}^{b_i}_0 \). For this we can consider \( \tilde{\mathcal{E}}^{b_i}(0) \) to be fixed deterministic set. First, we see that the origin has to be in \( \tilde{\mathcal{E}}^{b_i}(0) \) and recall (2.4.2) to obtain
\[
\mathbb{P}^{b_i}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } \tilde{\mathcal{E}}^{b_i}(0)) = \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } B(\tilde{\mathcal{E}}^{b_i}(0) \cup \{0\})) = \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } \tilde{\mathcal{E}}^{b_i}(0)).
\]  \hspace{1cm} (2.4.49)

Then, we use an inclusion-exclusion argument to obtain
\[
\mathbb{P}^{b_i}_1(e_i \overset{\cdot}{\rightarrow} x \text{ off } \tilde{\mathcal{E}}^{b_i}(0)) = \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} x) - \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} \tilde{\mathcal{E}}^{b_i}(0)) = \tau^{-i}(x - e_i) - \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} x),
\]  \hspace{1cm} (2.4.50)

and compute
\[
\mathbb{P}(0 \overset{b_i}{\rightarrow} x) = p \tau^{-i}(x - e_i) \mathbb{P}^{b_i}_1(e_i \not\in \tilde{\mathcal{E}}^{b_i}(0)) - p \mathbb{E}^{b_i}_0 \left[ \mathbb{1}_{\{e_i \not\in \tilde{\mathcal{E}}^{b_i}(0)\}} \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} x) \right].
\]  \hspace{1cm} (2.4.51)

Now, we apply (2.4.10) on \( \mathbb{P}^{0}_1(e_i \overset{\cdot}{\rightarrow} \tilde{\mathcal{E}}^{b_i}(0)) \). As (2.4.10) only holds for deterministic sets \( A \) we need to remark here that \( \tilde{\mathcal{E}}^{b_i}(0) \) is within the outer expectation \( \mathbb{E}^{b_i}_0 \) deterministic. In this way we obtain
\[
\tau(x) = \tau^i(x) + p \mathbb{P}^{b_i}_1(e_i \not\in \tilde{\mathcal{E}}^{b_i}(0)) \tau^{-i}(x - e_i) + \sum_{j,b} \Pi^{b_i}_j(y) \tau^j(x - y + e_j) + \mathbb{E}^i(x) + R^i(x).
\]  \hspace{1cm} (2.4.52)
2.5 Discussion

In this chapter we have derived the NoBLE for the different models following the ideas explained in Section 2.1. We have modified the classical lace expansion by remembering the direction $\kappa$ when cutting a bond after a self-intersection, so that the following part does not immediately make the mistake of jumping back. The information of the direction of the bond that is cut is incorporated in the definition of the adapted two-point function $G^{\kappa}_{z}(x)$.

For SAW we modified the approach of [22] by reorganizing only the avoidance constraint necessary for NBW in terms of graphs. Then, we created a lace expansion of this set of graphs and obtain in (2.2.10) the NoBLE for SAW. For LT and LA we define rib and sausage walks to characterize connections between two points by a lattice trees/animals. We integrated the non-backtracking avoidance constraints into the definition of the rib and sausage walk. To describe the self-avoidance constraints of the rib/sausage walks we used the graph-based description of the classical lace expansion, see [39], [92]. When splitting a walk in (2.2.10) and (2.3.18) we have extracted the avoidance constraints between the neighboring ribs/sausages and made them part of the NoBLE coefficients.

For percolation we have derived the NoBLE using an inclusion-exclusion approach. We adapted the proof of the classical lace expansion of [17] by including the direction $\kappa$ of the bond cut. We applied the Cutting Lemma (Lemma [2.4.4], proven in [92] Theorem 10.1), on a lattice where the bond $b_i$ that we cut is removed from the lattice and then include the avoidance constraint of the bond that is cut in the inclusion-exclusion step.

This completes the derivation of the NoBLE for percolation.

with

$$Ξ^{(0),i}(x) = Ξ^{b_i,0}(0, x; \{e_i\}), \quad Π^{(0),i,κ}(x) = Ψ^{b_i,0}(0, x; \{e_i\}),$$

(2.4.53)

$$R_i(x) = R_0^{b_i}(0, x; \{e_i\}).$$

(2.4.54)

and, for $N, M \geq 1$,

$$Ξ^{(N),i}(x) = Ξ^{b_i,0}_M(0, x; \{e_i\}) + p \mathbb{P}^{b_i}_0 \left[ \mathbf{1}_{(e_i, x) \not\in \tilde{C}^{b_i}_0(0)} Ξ^{b_i,0}_{M-1}(e_i, x; \mathbf{c}_0^{b_i}(0)) \right],$$

(2.4.55)

$$Π^{(N),i,κ}(x) = Ψ^{b_i,0}_M(0, x; \{e_i\}) + p \mathbb{P}^{b_i}_0 \left[ \mathbf{1}_{(e_i, x) \not\in \tilde{C}^{b_i}_0(0)} Ψ^{b_i,0}_{M-1}(e_i, x; \mathbf{c}_0^{b_i}(0)) \right],$$

(2.4.56)

$$R_i(x) = R_0^{b_i}(0, x; \{e_i\}) + p \mathbb{P}^{b_i}_0 \left[ \mathbf{1}_{(e_i, x) \not\in \tilde{C}^{b_i}_0(0)} R^{b_i}_{M-1}(e_i, x; \mathbf{c}_0^{b_i}(0)) \right].$$

(2.4.57)
The generalizes NoBLE recursive scheme. For all four models the NoBLE creates relations of the form:

\[
G_z(x) = \Xi_z(x) + \bar{\alpha}_z \sum_{y \in \mathbb{Z}^d \kappa \in \{\pm 1, \ldots, \pm d\}} \Psi^\kappa_z(y) G^\kappa_z(x - y + e_\kappa), \tag{2.5.1}
\]

\[
G_z(x) = G^\iota_z(x) + \alpha_z G^{-\iota}_z(x - e_\iota) + \Xi^\iota_z(x), \tag{2.5.2}
\]

which resemble the NBW-relations (1.5.13). In the next chapter we use this relation to perform a unified analysis.

The dominant contributions. Before ending this chapter let us take a look at the NoBLE for the different models. For NBW (2.5.1)-(2.5.2) holds with \( \bar{\alpha}_z = \alpha_z = z \) and

\[
\Xi_z(x) = \Psi^K_z(x) = \delta_{0,x}, \quad \Xi^\iota_z(x) = \Pi^\iota_z(x) = 0. \tag{2.5.3}
\]

The NoBLE coefficients, \( \Xi_z, \Xi^\iota_z, \Psi^K_z \) and \( \Pi^\iota_z \), are designed to only capture the perturbations of (2.5.3). For all models the coefficients \( \Xi^\iota_z \) and \( \Pi^\iota_z \) include at least four steps. The reason for this is that all contributions to \( \Xi^\iota_z \) and \( \Pi^\iota_z \) include at least four steps. The coefficients \( \Xi_z \) and \( \Psi^K_z \) also include the contributions (2.5.3). For the analysis in the next chapter we extract this contribution and bound the remainder as perturbation. For SAW, the dominant contributions of \( \Xi_z(x) \) and \( \Psi^K_z(x) \) are given by \( \Xi^{(0)}_z(x) = \Psi^{(0),K}_z(x) = \delta_{0,x} \). For percolation \( \Xi_z \) and \( \Psi_z \) we see that

\[
\Xi^{(0)}_z(x) = \mathbb{P}_z(0 \leftrightarrow x), \quad \Psi^{(0)}_z(x) = \mathbb{P}_z\left(\{0 \leftrightarrow x\} \cap \{x - e_\kappa \not\in \tilde{C}_b^i(0)\}\right). \tag{2.5.4}
\]

\[
\Psi^{(0)}_z(x) = \mathbb{P}_z\left(\{0 \leftrightarrow x\} \cap \{x - e_\kappa \not\in \tilde{C}_b^i(0)\}\right). \tag{2.5.5}
\]

As a connection \( 0 \leftrightarrow x \) without any pivotal bonds is not very likely, the dominant contribution of this is given for \( x = 0 \). Similarly, the dominant contribution for LT and LA is \( \Xi^{(0)}_z(0) = 1 \) and \( \Psi^{(0),K}_z(0) = g^\iota_z / g_z \). We summarize this in the following Table 2.1.

<table>
<thead>
<tr>
<th>Model</th>
<th>NBW</th>
<th>SAW</th>
<th>Percolation</th>
<th>LT and LA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Xi^{(0)}_z(0) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Psi^{(0),K}_z(0) )</td>
<td>1</td>
<td>1</td>
<td>( \mathbb{P}<em>z^{b_i}(e</em>\iota \not\in \tilde{C}_b^i(0)) )</td>
<td>( g^\iota_z / g_z )</td>
</tr>
<tr>
<td>( \bar{\alpha}_z )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z^g )</td>
</tr>
<tr>
<td>( \alpha_z )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z ) ( \mathbb{P}<em>z^{b_i}(e</em>\iota \not\in \tilde{C}_b^i(0)) )</td>
<td>( z g_z )</td>
</tr>
</tbody>
</table>

Table 2.1: The dominant contributions of the NoBLE coefficients \( \Xi_z \) and \( \Psi^K_z \) and the factors \( \alpha_z \) and \( \bar{\alpha}_z \).
Chapter 3
Analysis

In this chapter we describe the analysis which we use to prove Theorem 1.7.1. We perform the analysis in a general setting which we will introduce in Section 3.2. This allows us to use the same analysis for self-avoiding walk, percolation, lattice trees and lattice animals.

We begin with an explanation of the analysis on a heuristic level. Then, we introduce the general setting and the required assumptions. These assumptions include bounds on the lace-expansion coefficients which are proven in Chapter 4 and 5. In Section 3.3 we state and prove the infrared bound for the general model. Proving the infrared bound we assume a number of technical bounds which we derive in Section 3.4.

In addition to the analysis shown in Sections 3.3-3.4 we show a second analysis. This analysis is analytically and computationally more involved, but is able to prove mean-field behavior in dimension closer to the upper critical dimension. This analysis is based upon ideas from [40] and is performed in Sections 3.5-3.6.

<table>
<thead>
<tr>
<th>mean-field behavior</th>
<th>SAW</th>
<th>LT</th>
<th>LA</th>
<th>percolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>expected for</td>
<td>$d \geq 5$</td>
<td>$d \geq 9$</td>
<td>$d \geq 9$</td>
<td>$d \geq 7$</td>
</tr>
<tr>
<td>been proven in</td>
<td>$d \geq 5$</td>
<td>sufficiently high</td>
<td>sufficiently high</td>
<td>$d \geq 19$</td>
</tr>
<tr>
<td>using Section 3.3</td>
<td>$d \geq 8$</td>
<td>$d \geq 29$</td>
<td>$d \geq 49$</td>
<td>$d \geq 38$</td>
</tr>
<tr>
<td>using Section 3.5</td>
<td>not computed</td>
<td>$d \geq 20$</td>
<td>$d \geq 21$</td>
<td>$d \geq 15$</td>
</tr>
</tbody>
</table>

Table 3.1: Overview in which dimension we can expect mean-field behavior, in which it has been proven and for which the two analyses can be used to prove it.
In Section 3.7 we discuss the assumptions stated in Section 3.2 that are not related to the bounds on the coefficients. We conclude the chapter with a general discussion and a comparison of the two analytic methods.

3.1 Structure of the analysis

In this chapter we explain the two analyses performed in this chapter. Our aim is to prove the existence of a constant $\gamma_2 > 0$, depending on the dimension and the model, such that

$$f_2(z) = \sup_{k \in (-\pi, \pi)^d} \left| \frac{\hat{G}_z(k)}{\hat{B}_{1/(2d-1)}(k)} \right|$$

is bounded by $\gamma_2$ for all $z \leq z_c$.

We work with a model for which there exists an expansion as derived in the preceding chapter. The NoBLE coefficients $\Xi_z, \Xi^l_z, \Psi_z^\kappa$ and $\Pi_z^{\kappa l}$ can be bounded in terms of simple diagrams (Chapter 4). These simple diagrams are combinations of the two-point function $G_z$, so that a bound on the two-point function also implies bounds on the simple diagrams (Chapter 5). We will use this relation between $G_z$, the simple diagrams, and the coefficients to conclude bounds on $G_z$, see Figures 3.1 and 3.2 below.

The analysis begins at a $z_I < z_c$, where the inequality $G_{z_I}(x) \leq B_{1/(2d-1)}(x)$ holds for all $x$. This inequality is simple to obtain for all models and will be established in Section 3.7. For self-avoiding walk and percolation we choose $z_I = (2d - 1)^{-1}$, while for lattice trees and animal we choose $z_I = (2d - 1)^{-1} e^{-1}$. We numerically compute the value of the non-backtracking two-point function $B_{1/(2d-1)}$ (see Section 5.2). Then, we use the inequality $G_{z_I}(x) \leq B_{1/(2d-1)}(x)$ to obtain bounds on the simple diagrams, which in turn can be used to create bounds on the coefficients. Using the analysis shown in Section 3.3 we conclude that $f_2(z_I) \leq \gamma_2$ holds.

![Figure 3.1: Structure of analysis at the initial $z = z_I$.](image)

The next step is to prove that $f_2(z) \leq \gamma_2$ also for $z \in (z_I, z_c)$. We make the following Ansatz: We assume that there exists a $\Gamma_2 > \gamma_2$ with $f_2(z) \leq \Gamma_2$. From this assumption
we conclude bounds on the simple diagrams which in turn create bounds on the coefficients. We use the bounds on the coefficients to obtain bounds on $\hat{G}_z(k)$ and thus on $f_2(z)$. The created bound on $f_2$ might be better than the assumed bound $f_2(z) \leq \Gamma_2$.

If we can prove for all $z \in (z_l, z_c)$ that the assumption $f_2(z) \leq \Gamma_2$ implies that $f_2(z) \leq \gamma_2$, then we can use the continuity of $f_2$ to conclude that $f_2(z) < \gamma_2$ for all $z \in (z_l, z_c)$. This technique is called bootstrap and is commonly used in lace expansion analyses. We obtain the final result $f_2(z_c) \leq \gamma_2$ using a technical assumption stated in Section 3.2.

To close this cycle from an assumed bound on $f_2$ to an improved bound on $f_2$ (Figure 3.2), we not only have to assume a bound on $f_2$. Additionally, we require bounds on $\bar{\alpha}_z$ and on $G_z(x)[1 - \cos(k \cdot x)]$. We define appropriate functions $f_1$ and $f_3$ and then perform the improvement of bounds for all three functions simultaneously. This analysis is performed in Section 3.3.

In Section 3.3 we use an $f_3$ that creates a bound on the Fourier transform of $G_z(x)[1 - \cos(k \cdot x)]$. This $f_3$ was first used for percolation on the finite tori in [17] and is further developed in [92] and [47]. Unfortunately, this $f_3$ does not produce good numerical bounds. Nevertheless we use this method to explain the analysis as it offers an elegant and numerically relatively simple way to prove the infrared bound.

In Section 3.5 we use another $\tilde{f}_3$ to bound $G_z(x)[1 - \cos(k \cdot x)]$ in $x$-space. This technique is an adaptation of the proof by Hara and Slade [42] used to prove that mean-field behavior holds for SAW for all dimensions above the upper critical dimension $d_c = 4$. The improvement of $\tilde{f}_3(z)$ for $z \in (z_l, z_c)$ will require extensive bounding of error terms (Section 3.6) and bounding of additional simple random walk integrals (Section 5.2.3).

We give a brief comparison of the quality of bounds produced by $f_3$ and $\tilde{f}_3$ at the end of this chapter.
3.2 The general setting

3.2.1 The NoBLE relation

For a model with two-point function \( G_z \) and for \( d \geq 2 \), the NoBLE creates functions \( G^t_z, \Xi_z, \Xi^t_z, \Psi^\kappa_z \) and \( \Pi^{t,\kappa}_z \) for \( t, \kappa \in \{\pm 1, \ldots, \pm d\} \), all mapping from \( \mathbb{Z}^d \) to \( \mathbb{R} \) and functions \( \bar{a}_z, \alpha_z : \mathbb{R}_+ \to \mathbb{R}_+ \), such that

\[
G_z(x) = \Xi_z(x) + \bar{a}_z \sum_{y \in \mathbb{Z}^d} \sum_{\kappa \in \{\pm 1, \ldots, \pm d\}} \Psi^\kappa_z(y) G_z^t(x - y + e_\kappa), \tag{3.2.1}
\]

\[
G_z(x) = G_z^t(x) + \alpha_z G^{-t}_z(x - e_t) + \sum_{y \in \mathbb{Z}^d} \sum_{\kappa \in \{\pm 1, \ldots, \pm d\}} \Pi^{t,\kappa}_z(y) G^\kappa_z(x - y + e_\kappa) + \Xi(t), \tag{3.2.2}
\]

hold for all \( x \in \mathbb{Z}^d \) and \( z \in [0, z_c) \). The functions \( \Xi_z, \Xi^t_z, \Psi^\kappa_z, \Pi^{t,\kappa}_z \) are characterized as alternating sums of non-negative real-valued functions \( \Xi^{(N)}_z, \Xi^{(N)}_z, \Psi^{(N),\kappa}_z, \Pi^{(N),t,\kappa}_z \):

\[
\Xi_z(x) = \sum_{N=0}^{\infty} (-1)^N \Xi^{(N)}_z(x), \quad \Xi^t_z(x) = \sum_{N=0}^{\infty} (-1)^N \Xi^{(N),t}_z(x), \tag{3.2.3}
\]

\[
\Psi^\kappa_z = \sum_{N=0}^{\infty} (-1)^N \Psi^{(N),\kappa}_z(x), \quad \Pi^{t,\kappa}_z(x) = \sum_{N=0}^{\infty} (-1)^N \Pi^{(N),t,\kappa}_z(x). \tag{3.2.4}
\]

Further, we use the common feature of the NoBLE discussed in Section 2.5. We assume that \( \bar{a}_z \geq \alpha_z > 0 \) and \( \Psi^{(0)}(0) = \bar{a}_z \alpha_z = \rho \in (0, 1) \), see Table 2.1. Our goal is to understand the behavior of \( G_z \), considering the functions \( \alpha_z, \bar{a}_z, \Xi^{(N)}, \Xi^{(N),t}, \Psi^{(N),\kappa}, \Pi^{(N),t,\kappa} \), and \( \bar{a}_z^{(N),t,\kappa} \) as given.

General formula of the two-point function. In the following we repeat the steps used to obtain a closed form for the NBW two-point function \((1.5.12)\) to \((1.5.16)\). We apply the Fourier transform to \((3.2.1)\) and \((3.2.2)\):

\[
\hat{G}_z(k) = \hat{\Xi}_z(k) + \bar{a}_z \sum_{t \in \{\pm 1, \ldots, \pm d\}} \hat{\Psi}^t_z(k) e^{-ik_t \hat{\Lambda}^t_z(k)}, \tag{3.2.5}
\]

\[
\hat{G}_z(k) = \hat{G}^t_z(k) + \alpha_z e^{ik_t \hat{G}^{-t}_z(k)} + \sum_{\kappa \in \{\pm 1, \ldots, \pm d\}} \hat{\Pi}^{t,\kappa}_z(k) e^{-ik\kappa \hat{G}^\kappa_z(k)} + \hat{\Xi}(k). \tag{3.2.6}
\]

Then we define the vectors \( \vec{G}_z(k), \vec{\Xi}(k) \) and \( \vec{\Psi}(k) \) and the matrix \( \hat{\Pi}_z(k) \) with entries

\[
\left(\vec{G}_z(k)\right)_t = \hat{G}^t_z(k), \quad \left(\vec{\Psi}(k)\right)_t = \hat{\Psi}^t_z(k), \tag{3.2.7}
\]

\[
\left(\vec{\Xi}(k)\right)_t = \hat{\Xi}_z(k), \quad \left(\hat{\Pi}_z(k)\right)_{t,\kappa} = \hat{\Pi}^{t,\kappa}_z(k), \tag{3.2.8}
\]

and recall the notation of Section 1.5.2. We write \((3.2.5)-(3.2.6)\) in matrix form as

\[
\hat{G}_z(k) = \vec{\Xi}(k) + \bar{a}_z \vec{\Psi}(k) \hat{\Delta}(-k) \vec{G}_z(k), \tag{3.2.9}
\]

\[
\hat{G}_z(k) \vec{1} = \vec{G}_z(k) + \alpha_z \hat{\Delta}(k) \vec{G}_z(k) + \hat{\Pi}_z(k) \hat{\Delta}(-k) \hat{G}^t_z(k) + \vec{\Xi}(k). \tag{3.2.10}
\]
We need to impose a number of assumptions on the general models. First, we assume that there exists a value \( z \in I \) such that \( \tilde{G}_z(k) \) is only a perturbation of \( G_z(k) \). If the perturbation is small enough to perform the improvement of bounds outlined in Section 3.1, then we are able to prove that \( \tilde{B}_{\frac{1}{2}(d-1)}(k) \) and \( \tilde{G}_z(k) \) are comparable for \( z \in [z_I, z_c] \).

### 3.2.2 Assumptions

We need to impose a number of assumptions on the general models. First, we assume that there exists a value \( z_I \), where the two-point function can be bounded by the NBW two-point function:

**Assumption 3.2.1** (Bound for the initial value). There exists a \( z_I \in [0, z_c) \) such that

\[
G_z(x) \leq B_{\frac{1}{2}(d-1)}(x) = \frac{2d-2}{2d-1} C_{\frac{1}{2}d}(x)
\]  

holds for all \( x \) and \( z \in [0, z_I] \).

To control the growth of the system as we approach the critical value \( z_c \) we use the following two assumptions:

**Assumption 3.2.2** (Growth of \( \tilde{a}_z \)). For all \( z \in [0, z_c) \), \( \tilde{a}_z \) and \( \alpha_z \) are non-decreasing and continuous in \( z \).
Assumption 3.2.3 (Growth of the two-point function). The two-point function $G_z(x)$ is non-decreasing and differentiable in $z \in (0, z_c)$. For all $\varepsilon > 0$ there exists a constant $c_\varepsilon \geq 0$ such that for all $z \in (0, z_c - \varepsilon)$ and $x \in \mathbb{Z}^d \setminus \{0\}$

$$\frac{d}{dz} G_z(x) \leq c_\varepsilon (G_z \star D \star G_z)(x)$$

and therefore $$\frac{d}{dz} \hat{G}_z(0) \leq c_\varepsilon \hat{G}_z(0)^2. \quad (3.2.17)$$

For each $z \in (0, z_c)$, there exists a constant $K(z) < \infty$ such that $\sum_{x \in \mathbb{Z}^d} \|x\|^2 G_z(x) < K(z)$.

We only consider models having the following symmetries:

Assumption 3.2.4 (Symmetry of the models). Let $i, \kappa \in \{\pm 1, \pm 2, \ldots, \pm d\}$. The following symmetries hold for all $x \in \mathbb{Z}^d$, $z \leq z_c$, $N \in \mathbb{N}$ and $i, \kappa$:

$$G_z(x) = G_z(-x), \quad G_z^i(x) = G_z^{-i}(-x),$$

$$\Xi_z^{(N)}(x) = \hat{\Xi}_z^{(N)}(-x), \quad \Xi_z^{(N)}(x) = \hat{\Xi}_z^{(N), -i}(-x),$$

$$\Psi_z^{(N), i}(x) = \hat{\Psi}_z^{(N), -i}(-x), \quad \Pi_z^{(N), i, \kappa}(x) = \hat{\Pi}_z^{(N), -i, -\kappa}(-x).$$

The dimensions are exchangeable, i.e., for all $i, \kappa$:

$$\hat{G}_z^i(0) = \hat{G}_z^{\kappa}(0), \quad \hat{\Psi}_z^{(N), i}(0) = \hat{\Psi}_z^{(N), \kappa}(0), \quad \sum_{\kappa'} \hat{\Pi}_z^{(N), i, \kappa'}(0) = \sum_{\kappa'} \hat{\Pi}_z^{(N), i, \kappa}(0).$$

To simplify our analysis we assume the following bound which holds for all models under consideration:

Assumption 3.2.5 (Relation between coefficients). Let $i, \kappa \in \{\pm 1, \pm 2, \ldots, \pm d\}$. For all $x \in \mathbb{Z}^d$, $z \leq z_c$, $N \in \mathbb{N}$ and $i, \kappa$ the following holds:

$$\Psi_z^{(N), i, \kappa}(x) \leq \Xi_z^{(N)}(x) \quad \Pi_z^{(N), i, \kappa}(x) \leq \tilde{\alpha}_z \Xi_z^{(N), i}(x). \quad (3.2.18)$$

Further,

$$\sum_{\kappa} \Psi_z^{(N), i, \kappa}(x) \leq (2d - 2) \Xi_z^{(N)}(x), \quad (3.2.19)$$

$$\sum_{i} \Pi_z^{(N), i, \kappa}(x) = \sum_{\kappa} \Pi_z^{(N), i, \kappa}(x) \leq (2d - 2) \alpha_z \Xi_z^{(N), i}(x), \quad (3.2.20)$$

holds for all $x \in \mathbb{Z}^d$ and $N \geq 0$, with the exception that (3.2.19) does not hold in the case $x = 0$ and $N = 0$. 
Definition of the functions $f_1, f_2$ and $f_3$. As explained in Section 3.1, three functions are the central figures of our analysis:

- a function $f_1$ to bound $\bar{a}_z$, and the value of $z \in (z_l, z_c)$ and the one-point function $g_z$;
- a function $f_2$ to bound the two-point function $G_z(x)$ in Fourier space;
- a function $f_3$ to bound the Fourier transform of $[1 - \cos(k \cdot x)]G_z(x)$ in Fourier space.

To define $f_3$ we introduce, for a function $\hat{A} : (-\pi, \pi)^d \to \mathbb{C}$, the discrete Laplace operator $\Delta_k$ by

$$-\frac{1}{2} \Delta_k \hat{A}(l) = \hat{A}(l) - \frac{1}{2} (\hat{A}(l + k) + \hat{A}(l - k)),$$

which is closely related to the second derivative of $\hat{A}$ with respect to $l$. Let $\hat{C}(k) = \hat{C}_{1/(2d)}(k) = [1 - \hat{D}(k)]^{-1}$ and $c_1, c_2, c_3, c_4 > 0$, then we use the functions

$$\hat{W}(k, l) := [1 - \hat{D}(k)] \left( (c_1 + c_2 \hat{C}(l)) [\hat{C}(l - k) + \hat{C}(l + k)] + c_3 \hat{C}(l - k) \hat{C}(l + k) \right),$$

$$\hat{U}(k, l) := [1 - \hat{D}(k)] \left( (c_1 + c_2 \hat{C}(l)) [\hat{C}(l - k) + \hat{C}(l + k)] + c_4 \hat{C}(l - k) \hat{C}(l + k) \right),$$

as bounds for $\Delta_k \hat{G}_z(l)$. The constants $c_i$ are introduced to tune $f_3$ for the different models. Their usage will be explained at the end of Section 3.3.5. We define the bootstrap function to be

$$f_1(z) := (2d - 1) \bar{a}_z, \quad (3.2.21)$$

$$f_2(z) := \sup_{k \in (-\pi, \pi)^d} \frac{\hat{G}_z(k)}{B_{\mu_z}(k)} = \frac{2d - 1}{2d - 2} \sup_{k \in (-\pi, \pi)^d} [1 - \hat{D}(k)] \hat{G}_z(k), \quad (3.2.22)$$

$$f_3(z) := \max \left\{ \sup_{l, k \in (-\pi, \pi)^d} \frac{1}{2} \Delta_k \hat{G}_z(l), \sup_{l, k \in (-\pi, \pi)^d} \frac{-1}{2} \Delta_k \hat{G}_z(l) \right\}. \quad (3.2.23)$$

In the following we assume that if $f_1(z), f_2(z), f_3(z)$ are bounded for a given $z \in [0, z_c)$ then the lace-expansion coefficients $\Xi_z^{(N)}$ and $\Xi_z^{(N),l}$ obey certain diagrammatic bounds. The form of these bounds is delicate and depends sensitively on the precise model under consideration. Further, we assume that the same bounds hold for $z_l$ regardless of the values $f_1(z_l), f_2(z_l), f_3(z_l)$.

Assumption 3.2.6 (Diagrammatic bounds). Let $\Gamma_1, \Gamma_2, \Gamma_3 \geq 0$. Assume that $z \in (z_l, z_c)$ such that $f_1(z) \leq \Gamma_i$ holds. Then $\hat{G}_z(k) \geq 0, \hat{\Phi}_z(k) \geq 0$ for all $k \in (-\pi, \pi)^d$, and the following bounds hold with $\beta_\ast$ depending only on $\Gamma_1, \Gamma_2, \Gamma_3, d$ and the model: There exist $\beta_\ast^{(N)}, \beta_\ast^{(N),l} \geq 0$, such that

$$\sum_{x \neq 0} \Xi_z^{(0)}(x) \leq \beta_\ast^{(0)}, \quad \hat{\Xi}_z^{(0),l}(0) \leq \beta_\ast^{(0)}, \quad (3.2.24)$$

$$\hat{\Xi}_z^{(N)}(0) \leq \beta_\ast^{(N)}, \quad \hat{\Xi}_z^{(N),l}(0) \leq \beta_\ast^{(N)}, \quad (3.2.25)$$
for $N \geq 1$. There exist $\beta_{\rho}, \bar{\beta}_{\rho} > 0$, such that

$$\beta_{\rho} \leq \Psi^{(0)}(0) = \rho \leq \bar{\beta}_{\rho}.$$  \hfill (3.2.26)

Further, there exist $\beta^{(N)}_{\Delta \Xi}, \beta^{(N)}_{\Delta \Xi, 0}, \beta^{(N)}_{\Delta \Xi, t}$ such that:

$$\hat{\Xi}^{(N)}(0) - \hat{\Xi}^{(N)}(k) \leq \beta^{(N)}_{\Delta \Xi} [1 - \hat{D}(k)],$$  \hfill (3.2.27)

$$\sum_{l, x} \Xi^{(N), l}(x) [1 - \cos(k \cdot x)] \leq \beta^{(N)}_{\Delta \Xi, 0} [1 - \hat{D}(k)],$$  \hfill (3.2.28)

$$\sum_{l, x} \Xi^{(N), l}(x) [1 - \cos(k \cdot (x - e_i))] \leq \beta^{(N)}_{\Delta \Xi, t} [1 - \hat{D}(k)],$$  \hfill (3.2.29)

for all $k \in (-\pi, \pi)^d$ and $N \geq 0$. Moreover, we assume that $\sum_{N=0}^{\infty} \beta^{(N)}_{\bullet} < \infty$ for $\bullet \in \{\Xi, \Xi', \Delta \Xi, \{\Delta \Xi', 0\}, \{\Delta \Xi', 1\}\}$. If Assumption 3.2.1 holds, then the bounds stated above holds for $z = z_1$, where in this case the constants $\beta_{\bullet}$ only depend on the dimension $d$ and the model.

We denote by $\beta^{\text{abs}}_{\bullet}, \beta^{\text{odd}}_{\bullet}$ and $\beta^{\text{even}}_{\bullet}$ the sum over all (resp. odd/even) $N$ of $\beta^{(N)}_{\bullet}$:

$$\beta^{\text{abs}}_{\bullet} = \sum_{N=0}^{\infty} \beta^{(N)}_{\bullet}, \quad \beta^{\text{odd}}_{\bullet} = \sum_{N=0}^{\infty} \beta^{(2N+1)}_{\bullet}, \quad \beta^{\text{even}}_{\bullet} = \sum_{N=0}^{\infty} \beta^{(2N)}_{\bullet},$$  \hfill (3.2.30)

for $\bullet \in \{\Xi, \Xi', \Delta \Xi, \{\Delta \Xi', 0\}, \{\Delta \Xi', 1\}\}$. By (3.2.3) the values $\beta^{\text{even}}_{\Xi}$ and $(-\beta^{\text{odd}}_{\Xi})$ are explicit upper and lower bounds on $\hat{\Xi}(0)$ and imply by Assumption 3.2.5 also bounds on $\Pi^{i, \kappa}$. The coefficients $\Xi_{z}$ and $\Psi^{\kappa}_{z}$ are bounded by

$$1 - \beta^{\text{odd}}_{\Xi} \leq \hat{\Xi}_{z}(0) \leq 1 + \beta^{\text{even}}_{\Xi},$$  \hfill (3.2.31)

$$\frac{1 - (2d - 2)}{2d} \beta^{\text{odd}}_{\Xi} \leq \hat{\Psi}^{\kappa}_{z}(0) \leq \frac{1}{2d} \sum_{\kappa} \hat{\Psi}^{\kappa}_{z}(0) \leq \frac{(2d - 2)}{2d} \beta^{\text{even}}_{\Xi}. $$  \hfill (3.2.32)

as $\beta^{\text{abs}}_{z}, \beta^{\text{even}}_{z}, \beta^{\text{odd}}_{z}$ only bound the remainder term.

Our final assumption allows us to bound the critical point:

**Assumption 3.2.7** (The grows at the critical point). Let $\Gamma_1, \Gamma_2, \Gamma_3 \geq 0$ such that $f_i(z) \leq \Gamma_i$ and that Assumption 3.2.6 holds. Then $G_z(x), G^z_\Xi(x), \Xi^{(N)}_{z}(x), \Xi^{(N), l}_{z}(x), \Psi^{(N), \kappa}_{z}(x), \Pi^{(N), l, \kappa}_{z}(x)$ are left-continuous in $z_c$ with a finite limit $z \nearrow z_c$ for all $x \in \mathbb{Z}^d$.

### 3.3 Analysis

**3.3.1 The main result**

Under the assumptions stated in the last section we prove the following lemma which is the technical cornerstone of the NoBLE analysis:
Proposition 3.3.1 (Bound on key quantities). Let \( z = z_I \) or \( z \in (z_I, z_c) \) such that \( f_i(z) \leq \Gamma_i \). Let \( \Phi(z) \) and \( \bar{F}(z) \) be the Fourier inverse of the functions \( \Phi(z) \) and \( \bar{F}(z) \) defined in \( 3.2.14 \) and \( 3.2.15 \). If Assumptions \( 3.2.2, 3.2.6 \) hold, then there exist constants \( K_\Phi > 0 \) that are all independent of \( z \), such that

\[
\sum_x |F_x(x)| \leq \frac{K}{|f|}, \tag{3.3.1}
\]

\[
K_\Phi \leq \Phi(z(0)) \leq \frac{K}{\Phi}, \quad \sum_{x \neq 0} |\Phi_x(x)| \leq \frac{K}{|\Phi|,} \tag{3.3.2}
\]

\[
K_{|\Phi|} \leq \sum_x |\Phi_x(x)| \leq K_{|\Phi|}, \tag{3.3.3}
\]

and, for all \( k \in (-\pi, \pi)^d \),

\[
(\hat{F}(0) - \hat{F}(k))^{-1} = \left( \sum_x F_x(x)[1 - \cos(k \cdot x)] \right)^{-1} \leq K[1 - \hat{D}(k)]^{-1}, \tag{3.3.4}
\]

\[
|\hat{F}(l) - \hat{F}(l + k)| \leq \sum_x |F_x(x)|[1 - \cos(k \cdot x)] \leq K_{\Delta,}[1 - \hat{D}(k)], \tag{3.3.5}
\]

\[
|\Phi_x(l) - \Phi_x(l + k)| \leq \sum_x |\Phi_x(x)|[1 - \cos(k \cdot x)] \leq K_{\Delta,}[1 - \hat{D}(k)]. \tag{3.3.6}
\]

The proof of this proposition is deferred to Section 3.4. There we also give explicit forms of the constants \( K_\Phi \), arising in Proposition 3.3.1, in terms of \( \beta_* \), given in Assumption 3.2.6. Now we use Proposition 3.3.1 to formulate the following condition:

Definition 3.3.2 (Sufficient condition to improved bounds). Let \( K_* \) be the constants given in Proposition 3.3.1. For a \( \gamma, \Gamma \in \mathbb{R}^3 \) and \( z \in (z_I, z_c) \), we say that \( P(\gamma, \Gamma, z) \) holds, if \( f_i(z) \leq \Gamma_i \) for \( i = 1, 2, 3 \) and the following conditions hold:

\[
0 \leq \gamma_i < \Gamma_i \quad \text{for} \quad i = 1, 2, 3, \tag{3.3.7}
\]

\[
0 < \frac{\gamma_i}{K_\Phi} \tag{3.3.8}
\]

\[
\gamma_1 \geq \max \left\{ f_1(z_I), \frac{1 + \frac{2d-2}{2d-1}\Gamma_1}{\beta}, \frac{2K^2}{c_1} \right\}, \tag{3.3.9}
\]

\[
\gamma_2 \geq \frac{2d-2}{2d-1} \frac{K_{\Delta,}}{K_*}, \tag{3.3.10}
\]

\[
\gamma_3 \geq \max \left\{ \frac{1}{2c_1} K_{\Delta,}, \frac{1}{2c_2} \frac{K_{\Delta,}}{K_*}, \frac{2K^2}{c_3} \sqrt{K_{\Delta,} K_{\Delta,} K_{\Delta,}}, \right\}, \tag{3.3.11}
\]

These conditions are chosen such that we can successfully apply the bootstrap argument in Section 3.3.2. The form of \( 3.3.9 - 3.3.11 \) will be explained in Sections 3.3.3 - 3.3.5. Our main result is the following:
Theorem 3.3.3 (Infrared bound). Let $k \in [-\pi, \pi]^d$, $z_I \in [0, z_c)$ and $\gamma, \Gamma \in \mathbb{R}^3$. If Assumptions 3.2.1-3.2.7 and $P(\gamma, \Gamma, z)$ hold for all $z \in [z_I, z_c]$, then

\[
\hat{G}_z(k) [1 - \hat{D}(k)] \leq \frac{2d - 2}{2d - 1} \gamma_2, \quad (3.3.12)
\]

\[
\hat{G}_z(k) \leq \frac{\bar{K}_{\phi} \max\{K, K^{-1}\}}{(\hat{G}_z(0))^{-1} + [1 - \hat{D}(k)]} \quad (3.3.13)
\]

for all $z \in [z_I, z_c]$.

3.3.2 Overview of the proof

To prove Theorem 3.3.3, we apply the bootstrap argument which is a common tool in the lace expansion, see e.g. [22], [38], [39]. We use the following modification of the classical bootstrap argument:

Lemma 3.3.4 (The bootstrap argument). For $i = 1, 2, 3$ let $z \mapsto f_i$ be a continuous function on the interval $[z_I, z_c)$ and $\gamma_i, \Gamma_i \in \mathbb{R}$ with $1 \leq \gamma_i < \Gamma_i$ and $f_i(z_I) \leq \gamma_i$. If for $z \in (z_I, z_c)$ the condition $f_i(z) \leq \Gamma_i$ for all $i \in \{1, 2, 3\}$ implies that $f_i(z) \leq \gamma_i$ for all $i = 1, 2, 3$, then in fact $f_i(z) \leq \gamma_i$ for all $z \in [z_I, z_c)$ and $i \in \{1, 2, 3\}$.

Proof. We consider the continuous function $z \mapsto \max_{i=1,2,3} \frac{f_i(z) - \gamma_i}{\Gamma_i - \gamma_i}$ and see that the lemma follows directly from the intermediate value theorem for continuous functions.

We prove that we can perform the bootstrap on the functions $f_i$ defined in (3.2.21)-(3.2.23), provided that Assumptions 3.2.2-3.2.7 and $P(\gamma, \Gamma, z)$ hold for all $z \in (z_I, z_c)$. In Sections 3.3.3-3.3.5 we show that the functions $f_i$ satisfy the conditions stated in Lemma 3.3.4 one function at a time. Before doing this, we show how we obtain Theorem 3.3.3

Lemma 3.3.5 (Conclusion for the critical point). If Assumptions 3.2.1-3.2.7 and $P(\gamma, \Gamma, z)$ hold for all $z \in (z_I, z_c)$ then the bounds stated in Assumption 3.2.6 and Proposition 3.3.1 remain to hold for $z = z_c$. Further, $\hat{G}_z(k)$ is left-continuous at $z = z_c$ for any $k \neq 0$.

Proof. We prove this lemma using the arguments of [35, Appendix A]. As the two-point functions $G_z(x), G_z'(x)$ and the coefficients are continuous for every $x \in \mathbb{Z}^d$ we know that (2.1.2) and (2.1.3) also hold at $z = z_c$. The bounds on the coefficients $\beta_\bullet$ are independent of the value of $z \in (z, z_c)$ and the coefficients are left-continuous in $z = z_c$. The dominated convergence theorem implies that the Fourier transform of the coefficients are also left-continuous at $z = z_c$ and that the bounds stated in Assumption 3.2.6 also hold for $z = z_c$. This in
We have just shown in Lemma 3.3.5 that $G_z(x) \leq \lim_{z' \to z} G_z(x) = \lim_{z' \to z_c} \int_{(-\pi,\pi)} \frac{\hat{\Phi}_z(k)}{1 - \hat{F}_z(k)} e^{ikx} \frac{d^d k}{(2\pi)^d} = \int_{(-\pi,\pi)} \frac{\hat{\Phi}_z(k)}{1 - \hat{F}_z(k)} e^{ikx} \frac{d^d k}{(2\pi)^d}. \quad (3.3.14)

Thus, we still have the Fourier representation (3.2.13), with the understanding that $\hat{G}_{z_c}(k)$ is defined by

$$\hat{G}_{z_c}(k) = \frac{\hat{\Phi}_{z_c}(k)}{1 - \hat{F}_{z_c}(k)} \quad (3.3.15)$$

for $k \neq 0$. As $\hat{F}_{z_c}(0) = 1$ this characterization cannot be used for $k = 0$. \qed

Proof of Theorem 3.3.3 subject to Proposition 3.3.1 and Lemma 3.3.4. By Lemma 3.3.4 we know that

$$2d - 2 \hat{G}_z(k)[1 - \hat{D}(k)] = f_z(z) \leq \gamma_2 \quad \text{for all } z \in (z_l, z_c). \quad (3.3.16)$$

We have just shown in Lemma 3.3.5 that $\hat{G}_z$, can be defined to be left-continuous at $z = z_c$ for $k \neq 0$. From this we conclude that (3.3.16) also holds for $z = z_c$, which proves (3.3.12).

To prove (3.3.13), we begin by rearranging $\hat{G}_z(k)$. By (3.3.8) we know that $\hat{\Phi}_z(0) \geq K_\phi > 0$ and we can rewrite (3.2.13) as

$$\hat{G}_z(k) = \frac{\hat{\Phi}_z(k) / \hat{\Phi}_z(0)}{1 - \hat{F}_z(0) + 1 / \hat{\Phi}_z(0) [\hat{F}_z(0) - \hat{F}_z(k)].} \quad (3.3.17)$$

In Lemma 3.3.4 we proved that $f_i(z) \leq \gamma_i < \Gamma_i$ for $z \in (z_l, z_c)$. Thus, we can apply Lemma 3.3.1 for all $z \in (z_l, z_c)$ as well as the equality $\hat{G}_z(0) = \hat{\Phi}_z(0) / (1 - \hat{F}_z(0)) > 0$, to conclude

$$\hat{G}_z(k) \leq \frac{\hat{\Phi}_z(k) / \hat{\Phi}_z(0)}{\hat{G}_z^{-1}(0) + 1 / \hat{\Phi}_z(0) K [1 - \hat{D}(k)].} \quad (3.3.17)$$

In the following we consider the cases $\hat{\Phi}_z(0) K \leq 1$ and $\hat{\Phi}_z(0) K > 1$. We know that $[1 - \hat{D}(k)] \geq 0$ and $\hat{G}_z^{-1}(0) > 0$. Thus, if $\hat{\Phi}_z(0) K \leq 1$ then we can bound

$$\hat{G}_z(k) \leq \frac{K_{\phi}/K_\phi}{\hat{G}_z^{-1}(0) + [1 - \hat{D}(k)]}. \quad (3.3.18)$$

For the case $K_{\phi}/K_\phi > 1$ we rewrite (3.3.17) to:

$$\hat{G}_z(k) \leq \frac{\hat{\Phi}_z(k) K}{\hat{\Phi}_z(0) K \hat{G}_z^{-1}(0) + [1 - \hat{D}(k)]} \leq \frac{K_{\phi}/K_\phi}{\hat{G}_z^{-1}(0) + [1 - \hat{D}(k)]}. \quad (3.3.18)$$

Together with (3.3.18) this completes the proof. \qed
3.3.3 The bootstrap for \( f_1 \)

We know that \( f_1(z) = (2d - 1)\tilde{\alpha}_z \) is continuous by Assumption [3.2.2] since \( \tilde{\alpha}_z \) is continuous. From (3.3.9) we conclude that \( f(z_I) \leq \gamma_1 \). To show that \( f_1(z) \leq \Gamma_1 \) implies \( f_1(z) \leq \gamma_1 \) for \( z \in (z, z_c) \) we prove a relation between \( \hat{B}_\mu(0) \) and \( \hat{G}_z(0) \). We use the abbreviations \( \psi_z = \hat{\Psi}'(0) \) and \( \pi'_z = \sum_k \hat{\Psi}'_{1,k}(0) \), where the choice of \( i \in \{ \pm 1, \ldots, \pm d \} \) is by Assumption [3.2.4] not relevant. Further, we recall that \( \alpha_z / \tilde{\alpha}_z = \rho \).

**Lemma 3.3.6** (Link between NBW and general susceptibility). Let Assumption [3.2.4] hold and define

\[
\mu_z = \frac{\psi_z \tilde{\alpha}_z}{1 + \pi'_z - \tilde{\alpha}_z(\psi_z - \rho)} \quad \text{and} \quad \tilde{\alpha}_z = \frac{(1 + \pi'_z)}{(\psi_z / \mu_z) + \psi_z - \rho}.
\]

Then \( \hat{B}_{\mu_z}(0) = \hat{\Phi}_z(0)\hat{G}_z(0) \) for all \( z < z_c \).

**Proof.** Since \( \hat{D}(0) = I \) and \( \hat{\Psi}(0) = \psi_z I \) the two-point function \( \hat{G}_z \) in the form of (3.2.13) simplifies for \( k = 0 \) to

\[
\hat{G}_z(0) = \frac{\hat{\Phi}_z(0)}{1 - \tilde{\alpha}_z \hat{\Psi}(0) \left[ \hat{D}(0) + \alpha_z J + \hat{\Pi}_z(0) \right]^{-1}} = \frac{\hat{\Phi}_z(0)}{1 - \tilde{\alpha}_z \psi_z \left[ I + \alpha_z J + \hat{\Pi}_z(0) \right]^{-1}}.
\]

By the simple form of \( I \) and \( J \) and the symmetry of \( \Pi_z^{1,k} \) stated in Assumption [3.2.4], it is clear that the sum of each column and the sum of each rows of \( I + \alpha_z J + \hat{\Pi}_z(0) \) equals \( 1 + \alpha_z + \pi_i \). For this follows that the one vector \( \tilde{I} \) is an eigenvector of \( I + \alpha_z J + \hat{\Pi}_z(0) \) to the eigenvalue \( 1 + \alpha_z + \pi_i \). We compute

\[
\hat{G}_z(0) = \frac{\hat{\Phi}_z(0)}{1 - \tilde{\alpha}_z \psi_z \frac{2d}{1 + \alpha_z + \pi_i}} \quad \text{and} \quad \hat{B}_\mu(0) = \frac{1}{1 - \frac{2d\mu}{1 + \mu}}.
\]

Solving \( \hat{B}_{\mu_z}(0)\hat{\Phi}_z(0) = \hat{G}_z(0) \) for \( \mu_z \) and \( \tilde{\alpha}_z \) respectively gives the desired result. □

**Lemma 3.3.7** (Improvement of \( f_1 \)). Let \( z \in (z_1, z_c) \) and \( \gamma, \Gamma \in \mathbb{R}^3 \). If Assumptions [3.2.4][3.2.6] and condition \( P(\gamma, \Gamma, z) \) hold, then \( f_1(z) \leq \gamma_1 \).

**Proof.** Using Assumptions [3.2.4][3.2.6] and recalling (3.2.32) yields

\[
\pi'_z = \sum_k \hat{\Psi}'_{1,k}(0) \leq (2d - 2)\tilde{\alpha}_z(\psi_z - \rho) = \frac{2d - 2}{2d - 1} \Gamma_1 \beta_{\text{even}} \]

\[
\psi_z \geq \hat{\Psi}_{z}^{(0),k}(0) - \sum_{N \geq 1, \text{odd}} \hat{\Psi}_{z}^{(N),k}(0) \geq \frac{\rho}{2} - \frac{(2d - 2)}{2d} \beta_{\text{odd}}.
\]

Then we select \( \mu_z \) as in Lemma [3.3.6] and note that \( \mu_z < (2d - 1)^{-1} \) if \( z < z_c \) as \( \hat{G}_z(0) = \hat{B}_{\mu_z}(0)\hat{\Phi}_z(0) < \infty \). Hence, we can compute that

\[
\tilde{\alpha}_z(2d - 1)^{-1} \text{ (3.3.19)} \quad \frac{(1 + \pi'_z)(2d - 1)}{\psi_z - \rho + \psi_z \mu_z} = \frac{(1 + \pi'_z)(2d - 1)\mu_z}{\rho + (\psi_z - \rho)(1 + \mu_z)}.
\]

\[
\psi_z = \frac{\mu_z}{\rho + (\psi_z - \rho)(1 + \mu_z)}.
\]
We use \((2d - 1)\mu_z \leq 1\) and Assumption 3.2.6 to conclude
\[
\tilde{a}_z(2d - 1) \leq \frac{1 + \frac{2d - 2}{2d - 1} \Gamma \beta_{z}^{\text{even}}}{\beta_{z} - \frac{2d - 2}{2d} \beta_{z}^{\text{odd}}} \leq \gamma_1. \tag{3.3.23}
\]

### 3.3.4 The bootstrap for \(f_2\)

**Lemma 3.3.8 (Continuity of \(f_2\)).** The function \(f_2\) defined in (3.2.22) is continuous in \(z\) on \([0, z_c)\).

**Proof.** We prove the statement using the proof of [47, Lemma 5.3]. To show that \(f_2\) is continuous on \([0, z_c)\) we prove that it is continuous on the closed interval \([0, z_c - \varepsilon]\) for any \(\varepsilon > 0\). Using Assumption 3.2.3 we know that for any \(k\) and \(z \in [0, z_c - \varepsilon]\) the following bound holds
\[
\left| \frac{d}{dz} \hat{G}_z(k) \right| = \left| \sum_x e^{ikx} \frac{d}{dz} G_z(x) \right| \leq \sum_x \frac{d}{dz} G_z(x) = \frac{d}{dz} \hat{G}_z(0) \leq c_\varepsilon (\hat{G}_z(0))^2 \leq c_\varepsilon (\hat{G}_{z_\varepsilon}(0))^2, \tag{3.3.24}
\]
where we can interchange differentiation and summation as the sum is bounded in absolute value, as just shown. From this we conclude that the derivative of \(f_2(z)\) is uniformly bounded on \([0, z_c - \varepsilon]\), which implies the continuity of \(f_2\) on \([0, z_c - \varepsilon]\). \(\square\)

**Lemma 3.3.9 (Improvement of \(f_2\)).** Let \(z \in [z_I, z_c)\) such that Assumptions 3.2.4-3.2.6 and \(P(\gamma, \Gamma, z)\) hold. Then \(f_2(z) \leq \gamma_2\).

**Proof.** Let \(\hat{F}(k)\) be as defined in (3.2.15). Then
\[
\hat{G}_z(k)[1 - \hat{D}(k)] = \frac{\hat{F}_z(k)[1 - \hat{D}(k)]}{1 - \hat{F}_z(k)} = \frac{\hat{F}_z(k)[1 - \hat{D}(k)]}{1 - \hat{F}_z(0) + \hat{F}_z(0) - \hat{F}_z(k)}.
\]
By Lemma 3.3.6 we know that \(1 - \hat{F}_z(0) = (\hat{F}_z(0)\hat{G}_z^{-1}(0)) = \hat{B}_{\mu_z}^{-1}(0) \geq 0\). Using the bounds of Lemma 3.3.1 in the second step we obtain
\[
\hat{G}_z(k)[1 - \hat{D}(k)] \leq \frac{\hat{F}_z(k)[1 - \hat{D}(k)]}{\hat{F}_z(0) - \hat{F}_z(k)} \leq \frac{1}{\hat{F}_z(0)} K \leq 2d - 2 \gamma_2. \tag{3.3.25}
\]
\(\square\)
3.3.5 The bootstrap for $f_3$

**Lemma 3.3.10** (Continuity). *The function $f_3$ as defined in (3.2.23) is continuous in $z \in [1/(2d-1), z_c]$.\'*

**Proof.** This proof is similar to the proof of Lemma 3.3.8. We fix an $\varepsilon > 0$ and a $z \in [0, z_c - \varepsilon]$ and note that

\[
\left| \frac{1}{2} \Delta_k \hat{G}_z(l) \right| = \sum_{x \in \mathbb{Z}^d} \left| 1 - \cos(k \cdot x) \right| e^{il \cdot x} G_z(x) \leq \sum_{x \in \mathbb{Z}^d} \left| 1 - \cos(k \cdot x) \right| G_z(x)
\]

\[
= \hat{G}_z(0) - \hat{G}_z(k) \leq 2 \hat{G}_z(0) \leq 2 \hat{G}_{z-\varepsilon}(0) < \infty.
\]

(3.3.26)

Since $\hat{C}(l) \geq 0.5$ for all $l$,

\[
\hat{U}(k, l) \geq \left[ 1 - \hat{D}(k) \right] \left( \frac{1}{2} c_1 + \frac{2}{4} c_2 + \frac{1}{4} c_3 \right),
\]

(3.3.27)

and thereby

\[
f_3(z) \leq \sup_{k \in (-\pi, \pi)^d} \frac{\hat{G}_z(0) - \hat{G}_z(k)}{\left[ 1 - \hat{D}(k) \right]^{1/2} (c_1 + c_2)}.
\]

(3.3.29)

In the following we prove that the derivative of $f_3(z)$ is uniformly bounded on $[0, z_c - \varepsilon]$ for all $\varepsilon > 0$, which implies the continuity of $f_3$ on $[0, z_c]$. Using Assumption 3.2.3 we know that

\[
\sum_{x \in \mathbb{Z}^d} \frac{d}{dz} \left[ 1 - \cos(k \cdot x) \right] G_z(x) \leq c_{\varepsilon} \sum_{x \in \mathbb{Z}^d} \left[ 1 - \cos(k \cdot x) \right] (G_z \ast D \ast G_z)(x).
\]

(3.3.30)

It is easy to obtain a uniform bound for “non-small” $k$, e.g. for $\delta > 0$ and $k \in (-\pi, \pi)^d \setminus (-\delta, \delta)^d$ we know that

\[
\frac{d}{dz} \frac{\hat{G}_z(0) - \hat{G}_z(k)}{1 - \hat{D}(k)} \leq c_{\varepsilon} \frac{\hat{G}^2_{z-\varepsilon}(0)}{1 - \hat{D}(\varepsilon)} < \infty
\]

(3.3.31)

for all $z \leq z_c - \varepsilon$. For a bound for small $k$ we note that for all $u, w, x \in \mathbb{Z}^d$:

\[
1 - \cos(u + w + x) \leq 3(1 - \cos(u)) + 1 - \cos(w) + 1 - \cos(x)),
\]

(3.3.32)

see e.g. Lemma 3.4.1. We apply this to (3.3.30) and see that

\[
\frac{d}{dz} \frac{\hat{G}_z(0) - \hat{G}_z(k)}{1 - \hat{D}(k)} \leq 6c_{\varepsilon} \frac{\hat{G}_z(0)}{1 - \hat{D}(k)} \sum_{x \in \mathbb{Z}^d} \left[ 1 - \cos(k \cdot x) \right] G_z(x) \leq 6c_{\varepsilon} \frac{\hat{G}_z(0)}{1 - \hat{D}(k)} + 3c_{\varepsilon} \frac{\hat{G}^2_z(0)}{1 - \hat{D}(k)}.
\]

(3.3.33)
By the symmetry of $G_z$ stated in Assumption 3.2.4 we know that
\[
\frac{d}{dk_i} \hat{G}_z(k) = 0 \quad \text{for } i = 1, \ldots, d, \quad (3.3.34)
\]
\[
\sum_{i,j=1}^{d} \frac{d}{dk_i} \frac{d}{dk_j} \hat{G}_z(k) = \sum_{i} \frac{d^2}{dk_i^2} \hat{G}_z(k) = -\sum_{x \in \mathbb{Z}^d} \|x\|^2 \cos(k \cdot x)G_z(x). \quad (3.3.35)
\]
We use this to compute the second order Taylor series of $\hat{G}_z(0) - \hat{G}_z(k)$ as
\[
\hat{G}_z(0) - \hat{G}_z(k) = \sum_{x \in \mathbb{Z}^d} \|x\|^2 \cos(k^* \cdot x)G_z(x)\|k\|^2 \quad (3.3.36)
\]
for some $k^* \in (-\pi, \pi)^d$. Therefore, we can bound (3.3.33) by
\[
\frac{d}{dz} \left( \hat{G}_z(0) - \hat{G}_z(k) \right) \leq 6c_c \hat{G}_z(0) \sum_{x \in \mathbb{Z}^d} \|x\|^2 G_z(x) \frac{\|k\|^2}{1 - \hat{D}(k)} + 3c_c \hat{G}_z^2(0).
\]
It is easy to show that for all dimension $d$ there exists a constant $c_d$ such that $\|k\|^2(1 - \hat{D}(k))^{-1} < c_d$, so that we can use Assumption 3.2.3 to conclude
\[
\sum_{x \in \mathbb{Z}^d} \frac{d}{dz} \left( 1 - \cos(k \cdot x) \right) G_z(x) \frac{1}{1 - \hat{D}(k)} \leq 6c_c c_d \hat{G}_z(0)K(z_c - \epsilon) + 3c_c \hat{G}_z^2(0) \quad (3.3.37)
\]
for all $z \leq z_c - \epsilon$. Therefore, we know that
\[
\frac{d}{dz} \left( \hat{G}_z(0) - \hat{G}_z(k) \right) = \sum_{x \in \mathbb{Z}^d} \frac{d}{dz} \left( 1 - \cos(k \cdot x) \right) G_z(x) \frac{1}{1 - \hat{D}(k)} \leq 6c_c c_d \hat{G}_z(0)K(z_c - \epsilon) + 3c_c \hat{G}_z^2(0). \quad (3.3.38)
\]
where we can interchange summation and differentiation as both terms are bounded in absolute value, see (3.3.26) and (3.3.37).

To improve the bound of $f_3$ we use the following lemma:

**Lemma 3.3.11** (Bound on discrete Laplace). Suppose $a(-x) = a(x)$ for all $x \in \mathbb{Z}^d$, and let
\[
\hat{A}(k) = \frac{1}{1 - \hat{a}(k)}.
\]
Then for all $k \in [-\pi, \pi]^d$,
\[
\frac{1}{2} |\Delta_k \hat{A}(l)| \leq \frac{1}{2} \left[ \hat{A}(l-k) + \hat{A}(l+k) \right] \hat{A}(l)[\hat{a}^\text{av}(0) - \hat{a}^\text{av}(k)]
\]
\[
+ 4 \hat{A}(l-k) \hat{A}(l+k) [\hat{a}^\text{av}(0) - \hat{a}^\text{av}(k)] [\hat{a}^\text{av}(0) - \hat{a}^\text{av}(l)]
\]
where $a^\text{av}$ is defined by $a^\text{av}(x) := |a(x)|$. Further,
\[
- \frac{1}{2} \Delta_k \hat{A}(l) \leq \frac{1}{2} \left[ \hat{A}(l-k) + \hat{A}(l+k) \right] \hat{A}(l)[\hat{a}^\text{av}(0) - \hat{a}^\text{av}(k)].
\]
We use Lemma 3.3.11, (3.3.39)

To bound the second term of (3.3.40) we note that

Then we rewrite

part of the form

(Improvement of Assumptions 3.2.4-3.2.6 and \( P(\gamma, \Gamma, z) \) hold. Then \( f_3(z) \leq \gamma_3 \).

Proof. We begin by noting the following bound that will be used multiple times in this proof:

Then we rewrite \( \Delta_k \hat{G}_z(l) \) as

We use Lemma 3.3.11, (3.3.39) and \( \hat{\Phi}_z(l) \geq K_{[\phi]} > 0 \) to obtain a bound on the first part of the form

To bound the second term of (3.3.40) we note that

and use this to compute

Proof. The inequality (3.3.39) is proven in [92, Lemma 5.7]. Moreover, (3.3.39) is derived within the proof of [3.3.39] at line (5.21) of [92].

Lemma 3.3.12 (Improvement of \( f_3 \)). Let \( z = z_I \) or \( z \in (z_I, z_c) \) be such that Assumptions 3.2.4-3.2.6 and \( P(\gamma, \Gamma, z) \) hold. Then \( f_3(z) \leq \gamma_3 \).
We compute

\[
\sum_{\sigma \in \{-1, 1\}} \frac{\sigma}{1 - \hat{F}_z(l + \sigma k)} = \frac{\hat{F}_z(l + k) - \hat{F}_z(l - k)}{(1 - \hat{F}_z(l + k))(1 - \hat{F}_z(l - k))}
\]

\[
= \sum_x F(x) [\cos((l + k) \cdot x) - \cos((l - k) \cdot x)]
\]

\[
= -2 \sum_x F(x) \sin(l \cdot x) \sin(k \cdot x)
\]

and use Cauchy-Schwarz and \(\sin^2(x) \leq 2[1 - \cos(x)]\) for all \(x \in \mathbb{R}\) to compute

\[
\sum_x \Phi(x) \sin(l \cdot x) \sin(k \cdot x) \sum_x F(x) \sin(l \cdot x) \sin(k \cdot x)
\]

\[
\leq \sqrt{4 \sum_{x \neq 0} |\Phi(x)| \sum_x |\Phi(x)||1 - \cos(k \cdot x)| \sum_x |F(x)| \sum_x |F(x)||1 - \cos(k \cdot x)|}.
\]

Using the bounds stated in Lemma 3.3.1, we bound the second term of (3.3.40) by

\[
\sum_{\sigma \in \{-1, 1\}} \frac{\hat{\Phi}_z(l) - \hat{\Phi}_z(l + \sigma k)}{1 - \hat{F}_z(l + \sigma k)} \leq K_{\Delta_k} K(C(l + k) + C(l - k))[1 - \hat{D}(k)]
\]

\[
+ 4 \sqrt{K_{\|\Phi\|_1} K_{\|\Phi\|_F} K_{\Delta_k} K^2 C(l - k) C(l + k)[1 - \hat{D}(k)]}.
\]

Combining (3.3.41), (3.3.42) and (3.3.43) and recalling the definition of \(\hat{U}(l, k)\) and \(\hat{W}(l, k)\) gives that

\[
f_3(z) \leq \max \left\{ \frac{1}{2c_1} K K_{\Delta_k}, \frac{1}{2c_2} K_{\|\Phi\|_1} K_{\Delta_k} K^2, \frac{2K^2}{c_3} \sqrt{K_{\|\Phi\|_1} K_{\|\Phi\|_F} K_{\Delta_k} K_{\Delta_k}}, \frac{2K^2}{c_4} (2K_{\|\Phi\|_1} K_{\|\Phi\|_F} + \sqrt{K_{\|\Phi\|_1} K_{\|\Phi\|_F} K_{\Delta_k} K_{\Delta_k}}) \right\}.
\]

From condition (3.3.11) follows that \(f_3(z) < \gamma_3\). The constants \(c_i\) should be chosen such that all four terms in the maximum have a comparable value. This tuning of the \(c_i\)'s allows us to use smaller \(\gamma_3, \Gamma_3\) and gives a better bound on \(\Delta_k \hat{G}_z(l)\). For example, for SAW in \(d = 8\) we choose the values \(c_1 = 0.143307, c_2 = 0.5, c_3 = 0.0371683, c_4 = 4.6\).

\[\square\]

### 3.4 Proof of key inequalities in Proposition 3.3.1

In this section we prove Proposition 3.3.1. We first compute the Fourier inverses of \(\hat{F}_z\) and \(\hat{\Phi}_z\) and bound the result. We will omit \(z\) from our notation and write e.g. \(\hat{F}_z(k) = \hat{F}(k)\).
3.4.1 Computation of the Fourier inverse of $\hat{F}$ and $\hat{\Phi}$

To compute the Fourier inverse we rewrite $\hat{F}$ and $\hat{\Phi}$ into a form without matrices. We use that $(\hat{D}(k) + \alpha J)^{-1} = 1/(1 - \alpha^2)(\hat{D}(-k) - \alpha J)$ and use the Neumann-series to rearrange $\hat{F}(k)$ to

$$\hat{F}(k) = \hat{\alpha} \tilde{\Psi}(k) \left[ \hat{D}(k) + \alpha J + \hat{\Pi}_z(k) \right]^{-1} \tilde{1}$$

$$= \frac{\hat{\alpha}}{1 - \alpha^2} \tilde{\Psi}(k) \left[ I + \frac{1}{1 - \alpha^2} (\hat{D}(-k) - \alpha J) \hat{\Pi}_z(k) \right]^{-1} (\hat{D}(-k) - \alpha J) \tilde{1}$$

$$= \frac{\hat{\alpha}}{1 - \alpha^2} \tilde{\Psi}(k) \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{1 - \alpha^2} (\hat{D}(-k) - \alpha J) \hat{\Pi}_z(k) \right)^n (\hat{D}(-k) - \alpha J) \tilde{1}$$

$$= \frac{\hat{\alpha}}{1 - \alpha^2} \sum_{n=0}^{\infty} \sum_{\ell_0, \ldots, \ell_n} \Psi^{t_0}(k) \frac{(-1)^n}{(1 - \alpha^2)^n}$$

$$\times \left( \prod_{s=1}^{n} (e^{-ik_{s-1} \hat{\Pi}_{t_{s-1}, t_s}(k)} - \alpha \hat{\Pi}^{-t_{s-1}, t_s}(k)) \right) (e^{-ik_n} - \alpha) \tilde{1}. \quad (3.4.1)$$

We define $\hat{F}_1, \hat{F}_2, \hat{F}_3$ as the contributions of $n = 0, n = 1$ and $n \geq 2$ in the sum in [3.4.1] and analyze them separately. Then we compute the corresponding $x$-space functions to be

$$F(x) = F_1(x) + F_2(x) + F_3(x), \quad (3.4.2)$$

$$F_1(x) = \frac{\hat{\alpha}}{1 - \alpha^2} \sum_i (\Psi^i(x + e_i) - \alpha \Psi^i(x)), \quad (3.4.3)$$

$$F_2(x) = -\frac{\hat{\alpha}}{(1 - \alpha^2)^2} \sum_{l,k} \sum_{x_0, x_1 : x_0 + x_1} \Psi^l(x_0)$$

$$\times \left( \hat{\Pi}^{l,k}(x_1 + e_l + e_k) - \alpha \Pi^{l,k}(x_1 + e_l) - \alpha \Pi^{l,k}(x_1 + e_k) + \alpha^2 \Pi^{-l,k}(x_1) \right), \quad (3.4.4)$$

$$F_3(x) = \tilde{\alpha} \sum_{n=2}^{\infty} \sum_{l_0, \ldots, l_n} \sum_{x_i : x_i = x} (-1)^n \Psi^{l_0}(x_0)$$

$$\times \left( \prod_{s=1}^{n-1} (\Pi^{l_{s-1}, l_s}(x_s + e_{l_{s-1}}) - \alpha \Pi^{-l_{s-1}, l_s}(x_s)) \right)$$

$$\times \left( \Pi^{l_{n-1}, l_n}(x_n + e_{l_{n-1}} + e_{l_n}) - \alpha \Pi^{l_{n-1}, l_n}(x_n + e_{l_{n-1}}) - \alpha \Pi^{l_{n-1}, l_n}(x_n + e_{l_n}) + \alpha^2 \Pi^{-l_{n-1}, l_n}(x_n) \right). \quad (3.4.5)$$

To avoid confusion we emphasize that $F_1(x) = F_1(x) \neq F_2(x)|_{z=1}$. In the same way we compute the Fourier inverse of $\hat{\Phi}_z$ to be

$$\Phi(x) = \Phi_1(x) + \Phi_2(x) + \Phi_3(x),$$
with
\[
\Phi_1(x) = \Xi(x) - \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{i,y} \Psi^i(y)(\Xi^i(x - y + e_i) - \alpha \Xi^{-i}(x - y)),
\]
(3.4.6)
\[
\Phi_2(x) = \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{i_0,i_1} \sum_{x_1 \in \mathcal{N}_1} \Psi^{i_0}(x_0) \left( (\Pi^{i_0,i_1}(x_1 + e_{i_0}) - \alpha \Pi^{-i_0,i_1}(x_1)) \right)
\times (\Xi^i(x_2 + e_i) - \alpha \Xi^{-i}(x_2)),
\]
(3.4.7)
\[
\Phi_3(x) = -\tilde{\alpha} \sum_{n=2}^\infty \sum_{x \in \mathcal{N}_1} \sum_{x_i = x} \Psi^{i_0}(x_0) \frac{(-1)^n}{(1 - \alpha^2)^{n+1}}
\times \prod_{s=1}^n (\Pi^{i_s-1,i_s}(x_s + e_{\iota_{s-1}}) - \alpha \Pi^{-i_s-1,i_s}(x_s)) (\Xi^{i_n}(x_{n+1} + e_{\iota_n}) - \alpha \Xi^{-i_n}(x_{n+1})).
\]
(3.4.8)

### 3.4.2 Bound on the absolute value of \( \hat{F} \) and \( \hat{\Phi} \)

To obtain a bound of \(|\hat{F}(k)|\) and \(|\hat{\Phi}(k)|\) we bound the functions in \(x\)-space. Therefore, we make use of Assumptions [3.2.5 and 3.2.6] to obtain
\[
|\hat{F}_1(k)| \leq \sum_x |F_1(x)| \leq \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{x,i} |\Psi^{i_0}(x + e_i) - \alpha \Psi^i(x)| \leq \frac{\tilde{\alpha}}{1 - \alpha} \sum_{x,i} |\Psi^i(x)|
\]
\[
\leq \frac{\tilde{\alpha}}{1 - \alpha} \left( \sum_i |\Psi^{i_0,i}(0)| + (2d - 2) \sum_{x \in \mathcal{N}} (1 - \delta_{0,x} \delta_{0,N}) \Xi^{(N)}(x) \right)
\]
\[
\leq \frac{2d \tilde{\alpha}}{1 - \alpha} \left( \beta + \frac{2d - 2}{2d} \beta_{\Xi}^{\text{abs}} \right),
\]
(3.4.9)

For abbreviation we define
\[
K_{|\Psi|} := \beta + \frac{(2d - 2)}{2d} \beta_{\Xi}^{\text{abs}}.
\]
(3.4.10)

In a similar way we obtain that
\[
|\hat{F}_2(k) + \hat{F}_3(k)| \leq \sum_x |F_2(x)| + |F_3(x)|
\]
\[
\leq 2d \tilde{\alpha} \sum_{n=1}^\infty \sum_{x,i} |\Psi^i(x)| \frac{1}{(1 - \alpha)^{n+1}} \left( \sum_{x,i} |\Pi^{i_K}(0)| \right)^n
\]
\[
\leq 2d \tilde{\alpha} K_{|\Psi|} \sum_{n=1}^\infty \left( \frac{(2d - 2) \tilde{\alpha} \beta_{\Xi}^{\text{abs}}}{1 - \alpha} \right)^n.
\]
(3.4.11)

We use the geometric sum to compute
\[
\Rightarrow |\hat{F}(k)| \leq \sum_x |F(x)| \leq \frac{2d \tilde{\alpha} K_{|\Psi|}}{1 - \alpha} \sum_{n=0}^\infty \left( \frac{(2d - 2) \tilde{\alpha} \beta_{\Xi}^{\text{abs}}}{1 - \alpha} \right)^n
\]
\[
= \frac{2d \tilde{\alpha}}{1 - \alpha - (2d - 2) \tilde{\alpha} \beta_{\Xi}^{\text{abs}}} K_{|\Psi|} := \overline{K}_{|\Psi|}.
\]
(3.4.12)
In the same way we obtain a bound on $|\hat{\Phi}_z(k)|$:

$$|\hat{\Phi}_z(k)| \leq \sum_x |\Phi(x)| \leq 1 + \beta_{z}^{\text{abs}} + \overline{K}_{\vert \phi \vert} \beta_{z}^{\text{abs}} := \overline{K}_{\vert \phi \vert},$$  \hspace{1cm} (3.4.13)

$$\sum_{x \neq 0} |\Phi(x)| \leq \beta_{z}^{\text{abs}} + \overline{K}_{\vert \phi \vert} \beta_{z}^{\text{abs}} \iota := \overline{K}_{\vert \phi \vert},$$  \hspace{1cm} (3.4.14)

### 3.4.3 Bounds on differences

Now we want to compute the bounds (3.3.4)-(3.3.6).

**Split of the cosines.** We use the following lemma:

**Lemma 3.4.1** (Split of cosines). Let $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$ for $i = 1, \ldots, J$ such that $t = \sum_{i=1}^{J} t_i$. Then

$$1 - \cos(t) \leq \sum_{i=1}^{J} [1 - \cos(t_i)] + \sum_{i=2}^{J} \sin(t_i) \sin \left( \sum_{j=1}^{i-1} t_j \right),$$  \hspace{1cm} (3.4.15)

$$1 - \cos(t) \leq J \sum_{i=1}^{J} [1 - \cos(t_i)].$$  \hspace{1cm} (3.4.16)

The inequality (3.4.16) with a factor $(2J + 1)$ is commonly used in the lace expansion literature. While reviewing the proof the authors found that a minor adaptation improves the leading factor to be $J$. In this chapter we only use (3.4.15). The inequality (3.4.16) is used in Chapter 4.

**Proof.** We obtain (3.4.15) by taking the real part of the telescoping sum:

$$1 - e^{it} = \sum_{i=1}^{J} [1 - e^{it_i}] \prod_{j=1}^{i-1} e^{it_j}. \hspace{1cm} (3.4.17)$$

In the following we use $|\sin(x + y)| \leq |\sin(x)| + |\sin(y)|$, $\vert ab \vert \leq (a^2 + b^2)/2$ and $1 - \cos^2(a) \leq 2[1 - \cos(a)]$ to conclude from (3.4.15) that

\[
1 - \cos(t) \leq \sum_{i=1}^{J} [1 - \cos(t_i)] + \sum_{i=2}^{J} \sum_{j=1}^{i-1} |\sin(t_i)||\sin(t_j)| \\
\leq \sum_{i=1}^{J} [1 - \cos(t_i)] + \frac{1}{2} \sum_{i=2}^{J} \sum_{j=1}^{i-1} [\sin^2(t_i) + \sin^2(t_j)] \\
= \sum_{i=1}^{J} [1 - \cos(t_i)] + \frac{J-1}{2} \sum_{i=1}^{J} \sin^2(t_i) \leq J \sum_{i=1}^{J} [1 - \cos(t_i)]. \hspace{1cm} (3.4.18)
\]
Conclusion of bounds on the other coefficients. In Assumption 3.2.6 we assumed a bound on $\hat{\xi}^{(N)}(0) - \hat{\xi}^{(N)}(k)$ and $\hat{\xi}^{(N),t}(0) - \hat{\xi}^{(N),t}(k)$. Now we discuss how to conclude similar bounds for $\Psi^{(N),K}$ and $\Gamma^{(N),t,K}$ from these bounds and Assumption 3.2.5. The following three bounds follow directly from (3.2.19) and (3.2.27):

\[
\sum_k \Psi^{(N),K}(x) [1 - \cos(k \cdot x)] \leq (2d - 2) \beta^{(N)}_{\Delta^2} [1 - \hat{D}(k)],
\]

(3.4.19)

\[
\sum_{i,k} \Psi^{(N),K}(x) [1 - \cos(k \cdot x)] \leq 2d(2d - 2) \alpha_z \beta^{(N)}_{\Delta^2,0} [1 - \hat{D}(k)],
\]

(3.4.20)

\[
\sum_{i,k} \Pi^{(N),t,K}(x) [1 - \cos(k \cdot x)] \leq 2d(2d - 2) \alpha_z \beta^{(N)}_{\Delta^2,t} [1 - \hat{D}(k)].
\]

(3.4.21)

We use (3.4.15) with $J = 2$, $t_1 = x$, $t_2 = -e_k$ to obtain the following bounds for $N \geq 1$:

\[
\sum_k \sum_{x \in \mathbb{Z}^d} \Psi^{(N),K}(x) [1 - \cos(k \cdot (x - e_k))]
\]

\[
\leq \sum_k \sum_{x \in \mathbb{Z}^d} \Xi^{(N)}(x) [(1 - \cos(k \cdot x)] + [1 - \cos(k \cdot e_k)] - \sin(k \cdot x) \sin(k \cdot e_k)]
\]

\[
\leq 2d[1 - \hat{D}(k)](\beta^{(N)}_{\Delta^2} + \beta^{(N)}).
\]

(3.4.22)

where we use that $\sum_k \sin(k \cdot e_k) = 0$ in the second step. For $N = 0$ we recall that $\beta^{(0)}_\Delta$ does not bound $\Xi^{(0)}(0)$ and proceed as above to obtain:

\[
\sum_k \sum_{x \in \mathbb{Z}^d} \Psi^{(0),K}(x) [1 - \cos(k \cdot (x - e_k))]
\]

\[
= 2d\Psi^{(0),K}(0) [1 - \hat{D}(k)] + \sum_k \sum_{x \in \mathbb{Z}^d \backslash \{0\}} \Psi^{(0),K}(x) [1 - \cos(k \cdot (x - e_k))]
\]

\[
\leq 2d[1 - \hat{D}(k)](\beta^{(0)} + \beta^{(0)}_\Delta).
\]

(3.4.23)

We use (3.2.18) and (3.2.28) - (3.2.29) to compute for $N \geq 0$ that

\[
\sum_{i,k} \sum_{x \in \mathbb{Z}^d} \Pi^{(N),t,K}(x) [1 - \cos(k \cdot (x - e_k))]
\]

(3.4.24)

\[
\leq \alpha_z \sum_{i,k} \sum_{x \in \mathbb{Z}^d} \Xi^{(N),t}(x) [1 - \cos(k \cdot (x - e_k))] \leq 2d \alpha_z [1 - \hat{D}(k)](\beta^{(N)}_{\Delta^2,0} + 2d \beta^{(N)}_\Delta)
\]

(3.4.25)

Bounds on differences of $F_1$ and $\Phi_1$. To obtain a lower bound for $F_1$ we recall that $\Psi^{(0),K}(0) \geq \beta_{\phi_2} > 0$ and that, for odd $N$, $\Psi^{(N)}$ creates a negative contribution and for even $N$ a positive contribution. Using this and the inequalities (3.4.19), (3.4.22),

\[
\leq \alpha_z \sum_{i,k} \sum_{x \in \mathbb{Z}^d} \Xi^{(N),t}(x) [1 - \cos(k \cdot (x - e_t - e_k))] \leq 2d \alpha [1 - \hat{D}(k)](\beta^{(N)}_{\Delta^2,t} + 2d \beta^{(N)}_\Delta).
\]
We define

\[ I = \text{Bounds on differences of } F \]

For a bound on the absolute value we compute:

\[
\sum_{x} F_1(x)[1 - \cos(k \cdot x)] = \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{x} \Psi^x(x) [(1 - \cos(k \cdot (x - e_k))] - \alpha [1 - \cos(k \cdot x)]
\]

\[
\geq [1 - \hat{D}(k)] \frac{2d\tilde{\alpha}}{1 - \alpha^2} \left( \beta_\rho^l - \beta_\alpha^\text{odd} - \beta_\alpha^\text{odd} - \alpha \beta_\alpha^\text{even} \right). \tag{3.4.26}
\]

For a bound on the absolute value we compute:

\[
\sum_{x} |F_1(x)||1 - \cos(k \cdot x)| \leq [1 - \hat{D}(k)] \frac{2d\tilde{\alpha}}{1 - \alpha^2} \left( \beta_\rho^l + \beta_\alpha^\text{abs} + \beta_\alpha^\text{abs} + \alpha \beta_\alpha^\text{abs} \right). \tag{3.4.27}
\]

For \( \Phi_1 \) we first use Assumption 3.2.5 to bound \( \Psi^x_{\Xi} \) by \( \Xi \) and \( \Pi_{\Xi}^{i,k} \) by \( \Xi^i_{\Xi} \):

\[
\sum_{x} |\Phi_1(x)||1 - \cos(k \cdot x)| \leq \sum_{x} |\Xi(x)||1 - \cos(k \cdot x)|
\]

\[
+ \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{i,x,y} |\Xi(x)||\Xi^i(y)||1 - \cos(k \cdot (x + y - e_i))|
\]

\[
+ \frac{\tilde{\alpha} \alpha}{1 - \alpha^2} \sum_{i,x,y} |\Xi(x)||\Xi^{-i}(y)||1 - \cos(k \cdot (x + y))|. \tag{3.4.28}
\]

Then, we split the cosines using 3.4.15 and see that the sines terms cancel

\[
\sum_{x} |\Phi_1(x)||1 - \cos(k \cdot x)|
\]

\[
\leq \beta_\alpha^\text{abs} [1 - \hat{D}(k)] + \frac{\tilde{\alpha}}{1 - \alpha^2} \sum_{i,x,y} |\Xi(x)||\Xi^i(y)|
\]

\[
\times ([1 - \cos(k \cdot x)] + [1 - \cos(k \cdot (y - e_i))]) \sin(k \cdot x) \sin(k \cdot (y - e_i))
\]

\[
+ \frac{\tilde{\alpha} \alpha}{1 - \alpha^2} \sum_{i,x,y} |\Xi(x)||\Xi^i(y)||([1 - \cos(k \cdot x)] + [1 - \cos(k \cdot y)]) \sin(k \cdot x) \sin(k \cdot y)
\]

\[
\leq [1 - \hat{D}(k)] \beta_\alpha^\text{abs}
\]

\[
+ \frac{\tilde{\alpha}}{1 - \alpha^2} [1 - \hat{D}(k)] \left( 2d\beta_\alpha^\text{abs} \beta_\alpha^\text{abs} + (1 + \beta_\alpha^\text{abs}) \beta_\alpha^\text{abs} + 2d \alpha \beta_\alpha^\text{abs} \beta_\alpha^\text{abs} + \alpha (1 + \beta_\alpha^\text{abs}) \beta_\alpha^\text{abs} \right). \tag{3.4.29}
\]

**Bounds on differences of \( F_2 \) and \( \Phi_2 \).** For a lower bound on \( \hat{F}_2(0) - \hat{F}_2(k) \) we see that \( \Psi^{(N),i}(x)\Pi^{(M),j,k} (y) \) gives a non-negative contribution whenever \( N + M \) is even. We define \( \mathbb{I}(M,N) = \mathbb{I}_{[N+M \text{ even}]} \) and use Assumption 3.2.5 to obtain

\[
- \sum_{x} [1 - \cos(k \cdot x)] F_2(x)
\]

\[
\leq \frac{\tilde{\alpha}}{(1 - \alpha)^2} \sum_{i,x,N} \Xi^{(N)}(x) \sum_{\kappa,y,M} \Xi^{(M),i}(y)
\]

\[
\times \left( [1 - \cos(k \cdot (x + y - e_i - e_k))] \mathbb{I}(M,N) + \alpha^2 [1 - \cos(k \cdot (x + y))] \mathbb{I}(M,N)
\]

\[
+ \alpha (1 - \mathbb{I}(M,N)) ([1 - \cos(k \cdot (x + y - e_i))] + [1 - \cos(k \cdot (x + y - e_k))]) \right). \tag{3.4.30}
\]
Then, we use the inequality (3.4.15) and proceed as in (3.4.22):

\[-\sum_x [1 - \cos(k \cdot x)] F_2(x) \leq \frac{\tilde{a}}{(1 - \alpha)^2} \sum_{l,x,N} |\Xi^{(N)}(x)||[1 - \cos(k \cdot x)] \sum_{\kappa,y,M} |\Xi^{(M)}_\kappa(y)||M,N) + \frac{\tilde{a}}{(1 - \alpha)^2} \sum_{\kappa,y,M} |\Xi^{(N)}(x)| \sum_{M,N} |\Xi^{(M)}_\kappa(y)| \times \left( [1 - \cos(k \cdot (y - e_i - e_k))] + \alpha^2 [1 - \cos(k \cdot y)] \right). \]

We use diagrammatic bounds (3.4.22) - (3.4.25) to obtain:

\[-\sum_x [1 - \cos(k \cdot x)] F_2(x) \leq \frac{(2d\tilde{a})^2}{(1 - \alpha^2)} \left( \beta_{\Lambda^\xi}^\text{odd} \beta_{\Lambda^\xi}^\text{odd} + \beta_{\Lambda^\xi}^\text{even} \beta_{\Lambda^\xi}^\text{even} \right) [1 - \hat{D}(k)] + \frac{2d\tilde{a}^2}{(1 - \alpha^2)} \left( \beta_{\rho} + \frac{(2d - 2)}{2d} \beta_{\Lambda^\xi}^\text{even} \right) [1 - \hat{D}(k)] \times \left( \beta_{\Lambda^\xi,t}^\text{even} + 2d \beta_{\Lambda^\xi}^\text{even} + \alpha^2 \beta_{\Lambda^\xi,0}^\text{even} + \alpha(\beta_{\Lambda^\xi,t}^\text{odd} + \beta_{\Lambda^\xi,0}^\text{odd} + 2d \beta_{\Lambda^\xi}^\text{odd}) \right) + \frac{2d\tilde{a}^2}{(1 - \alpha^2)} \beta_{\Lambda^\xi}^\text{odd} \beta_{\Lambda^\xi}^\text{odd} [1 - \hat{D}(k)] \times \left( \beta_{\Lambda^\xi,t}^\text{odd} + 2d \beta_{\Lambda^\xi,t}^\text{even} + \alpha^2 \beta_{\Lambda^\xi,0}^\text{odd} + \alpha(\beta_{\Lambda^\xi,t}^\text{even} + \beta_{\Lambda^\xi,0}^\text{even} + 2d \beta_{\Lambda^\xi}^\text{even}) \right). \]

In the same way we compute:

\[
\sum_x [1 - \cos(k \cdot x)] |F_2(x)| \leq \frac{2d\tilde{a}^2}{(1 - \alpha^2)} \left[ 2d \beta_{\Lambda^\xi}^\text{abs} \beta_{\Lambda^\xi}^\text{abs} + K_{|y|} \beta_{\Lambda^\xi,t}^\text{abs} + \alpha \beta_{\Lambda^\xi,0}^\text{abs} + 2d \beta_{\Lambda^\xi}^\text{abs} \right]. \tag{3.4.32}
\]

To bound $\Phi_2$ we take the absolute value of $\Phi_z$ and bound $\Psi_z^{K,\tilde{t}}$ by $\Xi_z$ and $\Xi_z^{\tilde{t}}$. After this we use (3.4.15) to split the cosines into three parts.

\[
\sum_x [1 - \cos(k \cdot x)] |\Phi_2(x)| = \frac{\tilde{a}^2}{(1 - \alpha^2)^2} \sum_{l,k,x_1,x_2,x_3} |\Xi(x_1)| \left( |\Xi^{\tilde{t}}(x_2 + e_i)| + |\Xi^{\tilde{t}}(x_1)| \right) \left( |\Xi^{K}(x_3 + e_k)| + |\Xi^{-K}(x_3)| \right) \times \left( \sum_{i=1}^3 [1 - \cos(k \cdot x_i)] + \sin(k \cdot x_1) \sin(k \cdot (x_2 + x_3)) + \sin(k \cdot x_2) \sin(k \cdot x_3) \right). \tag{3.4.33}
\]
Then, we use that

$$\sum_{x_i} \sin(k \cdot x_i) \Xi(x_1) = \sum_{x_2, i} \sin(k \cdot x_2) \Xi' (x_2) = \sum_{x_2, i} \sin(k \cdot (x_2 + e_i)) \Xi' (x_2) = 0, \quad (3.4.34)$$

to obtain

$$\sum_x [1 - \cos(k \cdot x)]|\Phi_2 (x)| = \frac{2d \bar{a}^2}{(1 - \alpha)^2} \left( 2d \beta_{\Delta^2}^{\text{abs}} \rho_{\Delta^2}^{\text{abs}} \beta_{\Delta^2}^{\text{abs}} + 2(1 + \beta_{\Delta^2}^{\text{abs}}) \beta_{\Delta^2}^{\text{abs}} \frac{\beta_{\Delta^2}^{\text{abs}}}{1 + \alpha} \right). \quad (3.4.35)$$

**Bounds on differences of $F_3$ and $\Phi_3$.** As a lower bound on $F_3$ we use

$$\sum_x [1 - \cos(k \cdot x)]\Phi_3 (x) \geq - \sum_x [1 - \cos(k \cdot x)]|\Phi_3 (x)|.$$

We use the idea already displayed in the bound on $\Phi_3$. We take the absolute value of all terms, then bound $\Psi^\kappa, \Pi^\kappa, k$ by $\Xi, \Xi'$ and use Lemma 3.4.1

$$\sum_x [1 - \cos(k \cdot x)]|\Phi_3 (x)|$$

$$\leq \sum_{n=0}^{\infty} \sum_{x_0, x_1 \ldots x_n} |\Xi(x_0)| \frac{\bar{a}^{n+1}}{(1 - \alpha^2)^{n+1}} \prod_{s=1}^{n-1} |\Xi'(x_s)| + |\alpha \Xi'(x_s)|$$

$$\times \left( |\Xi'(x_n)| + |\alpha \Xi'(x_n)| + |\alpha |\Xi'(x_n)| + |\alpha^2 |\Xi'(x_n)| \right)$$

$$\times \left( \sum_{s=0}^{l} [1 - \cos(k \cdot x_s)] + \sum_{s=0}^{l} \sin(k \cdot x_s) \sin \left( \sum_{j=1}^{s} k \cdot x_j \right) \right)$$

We see that the sines cancels out when summing over $\iota, \kappa, x$ and rearrange the rest:

$$\leq \sum_{n=2}^{\infty} \sum_x |\Xi(x)| [1 - \cos(k \cdot x)] \frac{\bar{a}^{n+1}(1 + \alpha)}{(1 - \alpha^2)^{n+1}} \prod_{s=1}^{n} (1 + \alpha)|\Xi'(y)| \sum_{t_n} 1$$

$$+ \sum_{n=0}^{\infty} \sum_{x_0} |\Xi(x_0)| \frac{\bar{a}^{n+1}(1 + \alpha)^2}{(1 - \alpha^2)^{n+1}} \prod_{x_n \neq x_{n-1}} |\Xi'(x_n)| \sum_{t_n} 1$$

$$\times \left( \sum_{i=1}^{n-1} \prod_{s=1}^{n-1} (1 + \alpha)|\Xi^\kappa(y)\| \sum_{x, \iota} |\Xi'(x)\| (1 - \cos(k \cdot x_e + e_i)) + \alpha (1 - \cos(k \cdot x)) \right)$$

$$+ \sum_{n=0}^{\infty} \sum_{x_0} |\Xi(x_0)| \frac{\bar{a}^{n+1}(1 + \alpha)^2}{(1 - \alpha^2)^{n+1}} \left( \sum_{i=1}^{n-1} \prod_{s=1}^{n-1} (1 + \alpha)|\Xi^\kappa(y)\| \sum_{x, \iota} |\Xi'(x)\| \left( 1 - \cos(k \cdot (x + e_i)) \right) + \alpha (1 - \cos(k \cdot x)) \right) \right) \sum_{t_n} 1$$

$$\times \left( \sum_{x, \iota} |\Xi'(x)| \left( 1 - \cos(k \cdot (x + e_i + e_k)) \right) + \alpha (1 - \cos(k \cdot (x + e_k))) \right) \right) \sum_{t_n} 1.$$ For all $c \in (0, 1)$ we recall the following geometric sums

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1 - c}, \quad \sum_{n=0}^{\infty} (n+1) c^n = \frac{1}{(1 - c)^2}. \quad (3.4.36)$$
Using these sums and the pre-computed bounds \((3.4.19)-(3.4.25)\) we obtain
\[
\sum_x [1 - \cos(k \cdot x)] |F_3(x)| \\
\leq 2d \beta^\text{abs}_{\alpha/2} \frac{\alpha}{(1-\alpha)^3} \frac{(2d\bar{\alpha})^2}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} + (1 + \beta^\text{abs}_\alpha) \frac{\alpha(2d\bar{\alpha})^2 \beta^\text{abs}_{\alpha/2}}{(1-\alpha)^2(1-\alpha^2)} \frac{\beta^\text{abs}_{\alpha/2},t}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} + (1 + \beta^\text{abs}_\alpha) \frac{\alpha(2d\bar{\alpha})^2 \beta^\text{abs}_{\alpha/2}}{(1-\alpha)^2(1-\alpha^2)} \frac{\beta^\text{abs}_{\alpha/2},0}{1 - \frac{2d\bar{\alpha}}{1-\alpha}}.
\]
\[(3.4.37)\]

In the same way we obtain the bound on \(\Phi_3\):
\[
\sum_x [1 - \cos(k \cdot x)] |\Phi_3(x)| \\
\leq 2d \beta^\text{abs}_{\alpha/2} \frac{(2d\bar{\alpha})^2 \bar{\alpha}}{(1-\alpha)^3} \frac{(\beta^\text{abs}_{\alpha/2})^3}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} + (1 + \beta^\text{abs}_\alpha) \frac{\alpha(2d\bar{\alpha})^2 \beta^\text{abs}_{\alpha/2}}{(1-\alpha)^2(1-\alpha^2)} \frac{\beta^\text{abs}_{\alpha/2},t}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} + (1 + \beta^\text{abs}_\alpha) \frac{\alpha(2d\bar{\alpha})^2 \beta^\text{abs}_{\alpha/2}}{(1-\alpha)^2(1-\alpha^2)} \frac{\beta^\text{abs}_{\alpha/2},0}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} + \frac{1}{1 - \frac{2d\bar{\alpha}}{1-\alpha}} \left(1 - \frac{2d\bar{\alpha}}{1-\alpha}\right).
\]
\[(3.4.38)\]

### 3.5 Analysis with an alternative bound on the displacement

We require a bound on \(G_z(x;k) = G_z(x)[1 - \cos(k \cdot x)]\) to perform the bootstrap as explained in Section 3.1. Such a bound is necessary to create the bounds \((3.2.27)-(3.2.29)\). The bootstrap function \(f_3\), defined in \((3.2.23)\), states that the Fourier transform of \(G_z(x;k)\) can be bounded by simple random walk (SRW) quantities. These SRW-quantities can be computed numerically. This relatively simple method has two major drawbacks: We require a bound that is uniform in \(k\) and we also bound the negative contribution of \(\Delta_k \hat{G}_z(l)\), see \((3.3.42)\), which we expect to play only a minor role for the value of \(G_z(x;k)\).

In this section we develop an alternative version of \(f_3\), which gives better numerical results.

#### 3.5.1 Introduction of the alternative bootstrap function

We create a second analysis for the NoBLE by replacing \(f_3\) in the bootstrap by a different function \(\tilde{f}_3\). This function is an adaptation of the function used by Hara and Slade to prove the mean-field behavior for self-avoiding walk in \(d \geq 5\), see \([42]\). The central idea needed for the adaptation was created in discussions with Takashi.
We require a bound on this for \( n = 0 \) for self-avoiding walk, \( n = 0, 1 \) for percolation and \( n = 1, 2 \) for lattice trees and animals. Further, we compute the bound for different values of \( l \in \{0, 1, 2, 3\} \). We define

\[
\tilde{f}_{3,l,n}(z) = \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \sum_y \|y\|_2^2 G_z(y)(G_{z^*}^{*n} * D^*)^l(x - y).
\]  

(3.5.2)

and will prove in Lemma 3.5.2 that

\[
\sum_y \{1 - \cos(k \cdot y)\} G_z(y)(G_{z^*}^{*n} * D^*)^l(x - y) \leq [1 - \hat{D}(k)] \tilde{f}_{3,l,n}(z).
\]  

(3.5.3)

Thus, we replace the function \( f_3 \), defined in (3.2.23), not by just with one function but with a number of functions \( \tilde{f}_{3,l,n} \) and then preform the bootstrap using more than three bootstrap functions. Another way to think of this is that we choose as bootstrap function

\[
\tilde{f}_3(z) = \sup_{(l,n) \text{ used in bounds}} \frac{\tilde{f}_{3,l,n}(z)}{\Gamma_{3,l,n}}
\]  

(3.5.4)

with an appropriate choice of \( \Gamma_{3,l,n} > 0 \). In Section 5.4 we give the values of \( n \) and \( l \) for which we use this bootstrap function. For the following arguments it is not relevant whether we use one or more for these functions and the specific choice of \( n \) and \( l \) does not influence the analysis, we assume from now on that we work with one fixed pair \( n, l \) and drop the subscripts \( n, l \).

We bound \( \tilde{f}_3(z) \) using the continuous Laplace operator. For a differentiable function \( f \) and \( \mu \in \{1, 2, \ldots, d\} \), let \( \delta_{\mu} f(k) = \frac{\partial}{\partial k_{\mu}} f(k) \) and \( \Delta f(k) = \sum_{\mu=1}^d \delta_{\mu}^2 f(k) \). Then

\[
\delta_{\mu}^2 \hat{G}_z(k) = \sum_{x \in \mathbb{Z}^d} G_z(x) \delta_{\mu}^2 e^{i k \cdot x} = - \sum_{x \in \mathbb{Z}^d} x_{\mu}^2 \hat{G}_z(k) e^{i k \cdot x},
\]  

(3.5.5)

\[
\Delta \hat{G}_z(k)|_{k=0} = - \sum_{x \in \mathbb{Z}^d} \|x\|_2^2 G_z(x) e^{i k \cdot x}.
\]  

(3.5.6)

This means that we can compute a bound on \( (3.5.2) \) using

\[
\tilde{f}_3(z) = \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \int_{\{0, \ldots, \pi\}^d} (- \Delta \hat{G}_z(k)) \hat{D}(k) \hat{G}_z^{n}(k) e^{i k \cdot x} \frac{d^d k}{(2\pi)^d}.
\]  

(3.5.7)

If we would replace \( G_z \) with \( C_{1/2d} = C \) in (3.5.7), then we could compute the value directly. This is also how we prove \( \tilde{f}_3(z_1) \leq \Gamma_3 \) (see Section 3.6.2). In Sections 3.6.3 and 3.6.4 we obtain a bound for \( z \in (z_1, z_c) \) by extracting a dominant SRW-like contribution from \( G_z \), that we bound as for \( z = z_1 \), and bounding the remainder terms separately. The bound on the error term require bounds on several SRW-integrals, that are compute in Section 5.2.3.

Before explaining the changes to the analysis, we prove (3.5.3) and start this with a definition:
Definition 3.5.1. We denote by $\mathcal{P}_d$ the set of all permutations of $\{1, 2, \ldots, d\}$. For a $\nu \in \mathcal{P}_d$, $\delta \in \{-1, 1\}^d$ and $x \in \mathbb{Z}^d$ we define $p(x; \nu, \delta) \in \mathbb{Z}^d$ to be the vector with entries $(p(x; \nu, \delta))_j = \delta_j x_{\nu_j}$.

Lemma 3.5.2. For a summable, non-negative function $f$ such that $f(x) = f(p(x; \nu, \delta))$ for all $\nu \in \mathcal{P}_d$, $\delta \in \{-1, 1\}^d$, the following holds:

$$\sum_x f(x) [1 - \cos(k \cdot x)] \leq [1 - \hat{D}(k)] \sum_x f(x) \|x\|_2^2.$$  \hfill (3.5.8)

Proof. Using a telescoping sum, we write

$$1 - e^{i k \cdot x} = 1 - e^{i k_1 x_1} + e^{i k_1 x_1} (1 - e^{i k_2 x_2}) + \cdots + e^{i \sum_{j=1}^{d-1} k_j x_j} (1 - e^{i k_d x_d}).$$  \hfill (3.5.9)

We reorder the sum over $x$ using the symmetry of $f$ to obtain

$$\sum_x f(x) [1 - \cos(k \cdot x)] = \sum_x f(x) \sum_{\mu=1}^d \cos \left( \sum_{j=1}^{\mu-1} k_j x_j \right) [1 - \cos(k_\mu \cdot x_\mu)]$$

$$\leq \sum_x f(x) \sum_{\mu=1}^d [1 - \cos(k_\mu \cdot x_\mu)].$$  \hfill (3.5.10)

We use $1 - \cos(nt) \leq n^2 [1 - \cos(t)]$ to see that

$$\sum_x f(x) [1 - \cos(k \cdot x)] \leq \sum_x f(x) \sum_{\mu=1}^d x_\mu^2 [1 - \cos(k_\mu)]$$

$$= [1 - \hat{D}(k)] \sum_x f(x) \|x\|_2^2.$$  \hfill (3.5.11)

Further, we note that for a function $f$, that has the symmetry assumed in Lemma 3.5.2 the following holds:

$$\delta_\mu \hat{f}(k) = i \sum_x x_\mu f(x) e^{i k \cdot x} = - \sum_x f(x) x_\mu \sin(k_\mu x_\mu) \prod_{v \neq \mu} \cos(k_v x_v)$$

$$\leq \sum_x |f(x)| x_\mu^2 \sin(k_\mu x_\mu).$$  \hfill (3.5.12)

Since $|\sin(nt)| \leq n |\sin(t)|$ we obtain for integers $n$ that

$$\delta_\mu \hat{f}(k) \leq |\sin(k_\mu)| \sum_x |f(x)| x_\mu^2.$$  \hfill (3.5.13)
3.5.2 The modified analysis

As we only replace $f_3$ by $\tilde{f}_3$ most of the analysis performed in Section 3.3 remains unchanged. Here we explain the modifications to the analysis of Section 3.3 and show in Section 3.6 that $\tilde{f}_3$ has the properties required to perform the bootstrap. To simplify the computations for the improvement of $\tilde{f}_3$ in Section 3.6 we assume that the two-point function is of the form:

$$\hat{G}_z(k) = \frac{\Phi_z(k) + c \alpha \hat{D}(k) + \hat{R}_\Phi(k)}{c F - \alpha \hat{D}(k) - \hat{R}_F(k)},$$  \hspace{1cm} (3.5.14)

with $c \Phi, c_F \simeq 1, \alpha_F > 1$. The choice of $c_\Phi, \alpha_\Phi$ and $R_\Phi$ are discussed in Section 3.5.3.

We decompose $\hat{G}_z$ in the following way

$$\hat{G}_z(k) = \Phi_z(k) \hat{C}^*(k) - \Delta \hat{R}_F(k) \hat{C}^*(k) \hat{G}_z(k),$$  \hspace{1cm} (3.5.15)

with

$$\hat{C}^*(k) = \frac{1}{1 - \hat{F}(0) + \alpha_F(1 - \hat{D}(k))},$$  \hspace{1cm} (3.5.16)

$$\Delta \hat{R}_F(k) = \hat{R}_F(0) - \hat{R}_F(k).$$  \hspace{1cm} (3.5.17)

We replace Assumptions 3.2.6 and 3.2.7 with the following similar assumptions:

**Assumption 3.5.3** (Diagrammatic bounds, version 2). Let $\Gamma_1, \Gamma_2, \Gamma_3 \geq 0$. Assume that $z \in (z_I, z_c)$ is such that $f_1(z) \leq \Gamma_1, f_2(z) \leq \Gamma_2$ and $f_3(z) < \Gamma_3$ holds. Then $\hat{G}_z(k) \geq 0, \Phi_z(k) \geq 0$ for all $k \in (-\pi, \pi)^d$, and the following bounds hold with $\beta_\Phi$ depending only on $\Gamma_1, \Gamma_2, \Gamma_3, d$ and the model:

There exist $\beta_{\Xi}^{(N)}, \hat{\beta}_{\Xi}^{(N)} \geq 0$, such that

$$\sum_{x \neq 0} \Xi_z^{(0)}(x) \leq \beta_{\Xi}^{(0)}, \hspace{1cm} \hat{\beta}_{\Xi}^{(0)} = 0, \hspace{1cm} (3.5.18)$$

$$\hat{\Xi}_z^{(N)}(0) \leq \beta_{\Xi}^{(N)}, \hspace{1cm} \hat{\beta}_{\Xi}^{(N)} = 0, \hspace{1cm} (3.5.19)$$

for $N \geq 1$. There exist $\beta_{\rho}, \hat{\beta}_{\rho} > 0$, such that

$$\beta_{\rho} \leq \Psi_z^{(0)}(0) = \rho \leq \hat{\beta}_{\rho}.$$  \hspace{1cm} (3.5.20)

There exist $\beta_{A_F}, \beta_{A_\Phi}, \beta_{C_\Phi} \geq 0$, such that

$$1 \leq \alpha_F \leq \beta_{A_F}, \hspace{1cm} |\alpha_\Phi| \leq \beta_{A_\Phi}, \hspace{1cm} 0 \leq c_\Phi \leq \beta_{C_\Phi}.$$  \hspace{1cm} (3.5.21)

Further, there exist $\beta_{\Delta z}^{(N)}, \beta_{\Delta z_0}^{(N)}, \beta_{\Delta z_1}^{(N)}$, such that:

$$\sum_x \|x\|^2 \Xi_z^{(N)}(x) \leq \beta_{\Delta z}^{(N)}[1 - \hat{D}(k)],$$  \hspace{1cm} (3.5.22)

$$\sum_{i,x} \|x\|^2 \Xi_{z_0}^{(N)}(x) \leq \beta_{\Delta z_0}^{(N)}[1 - \hat{D}(k)],$$  \hspace{1cm} (3.5.23)

$$\sum_{i,x} \|x - e_i\|^2 \Xi_{z_1}^{(N)}(x) \leq \beta_{\Delta z_1}^{(N)}[1 - \hat{D}(k)],$$  \hspace{1cm} (3.5.24)
3.5 Analysis with an alternative bound on the displacement 97

for all \( k \in (-\pi, \pi)^d \). Moreover, we assume that \( \sum_{N=0}^{\infty} \beta^{(N)}_* < \infty \) for \( \bullet \in \{ \Xi, \Xi', \Delta \Xi, \{ \Delta \Xi^t, 0 \}, \{ \Delta \Xi^t, 1 \} \} \).

There exist \( \beta_{R,F}, \beta_{R,\Phi}, \beta_{\Delta R,F}, \beta_{\Delta R,\Phi} \) such that

\[
|\hat{R}_F(k)| \leq \beta_{R,F}, \quad |\hat{R}_\Phi(k)| \leq \beta_{R,\Phi},
\]

\[
|\delta_\mu \hat{R}_F(k)| \leq \frac{1}{d} \beta_{\Delta R,F} |\sin(k_\mu)|, \quad |\Delta \hat{R}_F(k)| \leq \beta_{\Delta R,F} [1 - \hat{D}(k)],
\]

\[
|\delta_\mu \hat{R}_\Phi(k)| \leq \frac{1}{d} \beta_{\Delta R,\Phi} |\sin(k_\mu)|, \quad |\Delta \hat{R}_\Phi(k)| \leq \beta_{\Delta R,\Phi} [1 - \hat{D}(k)].
\]

If Assumption 3.2.1 holds, then the bounds stated above hold for \( z = z_I \), where in this case the constants \( \beta_* \) only depend on the dimension \( d \) and the model.

Assumption 3.5.4. We assume that Assumption 3.2.7 holds when the condition \( f_3(z) \leq \Gamma_3 \) is replaced by the condition that \( \bar{f}_3(z_I) < \Gamma_3 \).

From this assumption we conclude that Lemma 3.3.5 which states that \( \hat{G}_z(k) \) is left-continuous for \( k \neq 0 \), also holds in this setting. We use the following modification of Proposition 3.3.1:

Proposition 3.5.5 (Bound on key quantities, version 2). Let \( z = z_I \) or \( z \in (z_I, z_c) \) such that \( f_1(z) \leq \Gamma_1, f_2(z) \leq \Gamma_2 \) and \( \bar{f}_3(z_I) < \Gamma_3(z) \). If Assumptions 3.2.2-3.2.5 and 3.5.3 hold, then there exist constants \( K_* > 0 \) such that (3.3.1) – (3.3.6) hold.

Proof. We see that all assumptions required to perform the computations of Section 3.4 are satisfied. Therefore, we can simply use these computations to obtain the bounds stated in this proposition. \( \square \)

Definition 3.5.6 (Sufficient condition to improved bounds, version 2). Let \( K_* \) be the constants given in Proposition 3.5.5. For \( \gamma, \Gamma \in \mathbb{R}^3 \) and \( z \in [z_I, z_c] \), we say that \( Q(\gamma, \Gamma, z) \) holds, if

- \( f_1(z) \leq \Gamma_1, f_2(z) \leq \Gamma_2, \) and \( \bar{f}_3(z) \leq \Gamma_3 \),

- the inequalities (3.3.7) – (3.3.10) hold,

- \( \gamma_3 \) is smaller than the right hand side of (3.6.20) and (3.6.84) below.

The last condition states that the initial condition \( \bar{f}_3(z_I) \leq \gamma_3 \) is fulfilled and that the improvement of bounds succeeds for \( \bar{f}_3 \). We do not give a more formal statement at this point, as these statements require notation that has not yet been introduced. If \( Q(\gamma, \Gamma, z) \) holds, then we are able to prove the main result:
Theorem 3.5.7 (Infrared bound). Let $k \in [-\pi, \pi]^d$, $z_I \in [0, z_c)$ and $\gamma, \Gamma \in \mathbb{R}^3$. If Assumptions 3.2.2-3.2.5, 3.5.3, 3.5.4 and $Q(\gamma, \Gamma, z)$ hold for all $z \in [z_I, z_c)$, then

\[
\hat{G}_z(k)[1 - \hat{D}(k)] \leq \frac{2d - 2}{2d - 1} \gamma^2,
\]

(3.5.28)

\[
\hat{G}_z(k) \leq \frac{\overline{K}_{|\phi|} \max\{|K, K^{-1}_{\phi}|\}}{(\hat{G}_z(0))^{-1} + [1 - \hat{D}(k)]},
\]

(3.5.29)

for all $z \in [z_I, z_c]$.

We prove Theorem 3.5.7 by adapting the analysis shown in Section 3.3.2-3.3.4. The only additional step needed is to prove that $\bar{f}_3$ can be used for the bootstrap (Lemma 3.3.4). We prove this in Section 3.6.

3.5.3 Rewrite of the two-point function

In this section we define the functions that we used in (3.5.14) to characterize the two-point function. We extract a dominant SRW-like contribution from $\hat{\Phi}_z(k)$ and $\hat{F}_z(k)$, as we can bound only those contributions efficiently. It is possible to extract even more SRW-like contributions from the remainder terms $R_*$ than we show here. However, we consider the decomposition as given below to be sufficient for our purposes. As in Section 3.4 we omit the subscript $z$ and leave the dependence on $z$ implicit. We begin with $\Phi$ and extract the dominant contribution of

\[
\hat{\Phi}_1(k) = \sum_x \Xi(x) e^{i k \cdot x} - \frac{\bar{\alpha}}{1 - \alpha^2} \sum_{i, y, x} \Psi^i(y) \left[ \Xi^i(x - y + e_i) - \alpha \Xi^{-i}(x - y) \right] e^{i k \cdot (x + y)},
\]

(3.5.30)

see (3.4.6). From the first summand we extract $\Xi_z^{(0)}(0) = 1$, $\Xi_z^{(0)}(e_i)$ and $\Xi_z^{(1)}(e_i)$. From the second summand we extract the contributions of $\Psi_z^{(0), i}(0)$ and of $\Xi_z^{(0), i}(e_i)$. We choose

\[
c_{\Phi} = \Xi_z^{(0)}(0) = 1,
\]

(3.5.31)

\[
\alpha_{\Phi} = \sum_i \left( \Xi_z^{(0)}(e_i) - \Xi_z^{(1)}(e_i) - \frac{\bar{\alpha}}{1 - \alpha^2} \Psi_z^{(0), i}(0) \Xi_z^{(0), i}(e_i) \right),
\]

(3.5.32)

\[
\hat{R}_\Phi(k) = \hat{\Phi}(k) - c_{\Phi} - \alpha_{\Phi} \hat{D}(k).
\]

(3.5.33)

The Fourier transform of the functions $F_1$ defined in (3.4.3) is given by

\[
\hat{F}_1(k) = \frac{\bar{\alpha}}{1 - \alpha^2} \sum_i (\hat{\Psi}_i^i(k) e^{-i k \cdot e_i} - \alpha \hat{\Psi}_i(k)).
\]

(3.5.34)

The dominant contribution to $F_1$ is

\[
\frac{\bar{\alpha}}{1 - \alpha^2} \sum_i \Psi_z^{(0), i}(0) (e^{-i k \cdot e_i} - \alpha) = 2d \frac{\bar{\alpha}}{1 - \alpha^2} \rho [\hat{D}(k) - \alpha],
\]

(3.5.35)
with $\Psi^{(0),t}(0) = \rho$. As a minor contribution we extract from $\hat{F}_1(k)$ for $N = 0, 1$:

$$
\frac{\alpha}{1 - \alpha^2} \sum_{x, y} \sum_{l} \|y\| \sqrt{G_{x}^* x (G_x^* x^l)(x - y)}
\Psi^{(0),t}(e_i + e_l) e^{-i k \cdot e_l} - \alpha \Psi^{(0),t}(e_k) e^{i k \cdot e_k}
= 2d \alpha \sum_{N=0}^{1} (-1)^{N} (\Psi^{(1),1}(e_1 + e_2) - (2d - 2) \Psi^{(1),1}(e_2) - \alpha \Psi^{(1),1}(e_1)) \hat{D}(k).
$$

(3.5.36)

These contributions are of order $1/d^2$. Further, we extract the following factor from $\hat{F}_1(k)$:

$$
- \frac{\alpha}{1 - \alpha^2} \sum_{t} \Psi^{t}(0) = - \frac{2d \alpha}{1 - \alpha^2} \rho.
$$

(3.5.37)

We define

$$
c_F = 1 - \frac{2d \alpha}{1 - \alpha^2} \rho,
$$

(3.5.38)

$$
\alpha_F = 2d \frac{\alpha}{1 - \alpha^2} \rho + \frac{2d \alpha}{1 - \alpha^2} \sum_{N=0}^{1} (-1)^{N} (\Psi^{(1),1}(e_1 + e_2) - (2d - 2) \Psi^{(1),1}(e_2) - \alpha \Psi^{(1),1}(e_1)),
$$

(3.5.39)

$$
\hat{R}_F(k) = \hat{F}(k) - 1 - (c_F - 1) - \alpha_F \hat{D}(k).
$$

(3.5.40)

### 3.6 The bootstrap for $\tilde{f}_3$

In this section we show that the function $\tilde{f}_3$, defined in (3.5.2), satisfies the conditions of the Bootstrap Lemma (Lemma 3.3.4). Namely, we prove that $\tilde{f}_3$ is continuous, that $\tilde{f}_3(z_I) \leq \gamma_3$, and that $\tilde{f}_3(z) \leq \Gamma_3$ implies that $\tilde{f}_3(z) \leq \gamma_3$ for all $z \in (z_I, z_c)$.

#### 3.6.1 Continuity

**Lemma 3.6.1** (Continuity). The function $\tilde{f}_3$ as defined in (3.5.2) is continuous in $z \in [1/(2d - 1), z_c]$.

**Proof.** We fix an $\varepsilon > 0$ and define

$$
h_x(z) = \sum_{y} \|y\| \sqrt{G_{z}^* x (G_x^* x^l)(x - y)}.
$$

(3.6.1)

We will prove that $(h_x)_{x \in \mathbb{Z}^d}$ is an equicontinuous family of functions and uniformly bounded for all $x$ on $z \in [0, z_c - \varepsilon)$. This allows us to obtain the continuity of $\tilde{f}_3(z) = \sup_x h_x(z)$ directly from the Arzela-Ascoli Theorem. By Assumption 3.2.3 we know that there exists a constant $K(z_c - \varepsilon) < \infty$ such that

$$
\sum_{x \in \mathbb{Z}^d} \|x\| \sqrt{G_{z_c}^* x (G_x^* x^l)(x - y)} < K(z_c - \varepsilon).
$$

(3.6.2)
Hence \( \hat{G}_{z_c-\epsilon}(0) = \chi(z_c - \epsilon) < \infty \) we know that \( h_x(z) \) is uniformly bounded:

\[
\sum_y \| y \|^2 \bar{G}_z(y)(G_z^* \ast D^*)(x - y) \leq \sup_y (G_z^* \ast D^*)(y) K(z_c - \epsilon) \\
\leq \hat{G}_{z_c-\epsilon}(0)^n K(z_c - \epsilon). \tag{3.6.3}
\]

By Assumption 3.2.3 we know that

\[
\frac{d}{dz} h_x(z) \leq c \varepsilon \sum_y \| y \|^2 (G_z \ast D \ast G_z)(y)(G_z^* \ast D^*)(x - y) \\
+ n c \varepsilon \sum_y \| y \|^2 G_z(y)(G_z^{(n+1)} \ast D^{(l+1)})(x - y). \tag{3.6.4}
\]

We use that \( \| w + x + y \|^2 \leq 3 \| w \|^2 + \| x \|^2 + \| y \|^2 \) for all \( w, x, y \in \mathbb{Z}^d \) to obtain

\[
\sum_{y \in \mathbb{Z}^d} \| y \|^2 (G_z \ast D \ast G_z)(y) \\
\leq \sum_w \| w \|^2 G_z(w)(D \ast G_z)(y - w) + (G_z \ast G_z)(y - w) D(w) \| w \|^2 \\
+ \sum_w (G_z \ast D)(w) G_z(y - w) \| y - w \|^2 \\
\leq 6 K(z_c - \epsilon) \hat{G}_{z_c-\epsilon}(0) + 3 \hat{G}_{z_c-\epsilon}(0)^2. \tag{3.6.5}
\]

We conclude that

\[
\frac{d}{dz} h_x(z) \leq c \varepsilon (6 K(z_c - \epsilon) + 3 \hat{G}_{z_c-\epsilon}(0) + n K(z_c - \epsilon)) \hat{G}_{z_c-\epsilon}(0)^n + 1 < \infty. \tag{3.6.6}
\]

As this bound is uniform in \( x \) we conclude that \( (h_x)_{x \in \mathbb{Z}^d} \) is equicontinuous. \( \Box \)

### 3.6.2 Bound for the initial point

In this section, we prove that \( \bar{f}_3(z_l) \leq \gamma_3 \). By Assumption 3.2.1 we can bound \( G_z \) by the critical SRW two-point function. Thus, we can bound \( \bar{f}_3(z_l) \) using only SRW-quantities. For the Fourier transform of the critical SRW two-point function \( \hat{C}_{1/2d}(k) = \hat{C}(k) = [1 - \hat{D}(k)]^{-1} \) we compute

\[
\sum_{\mu=1}^d \delta_{\mu} \hat{C}(k) = \sum_{\mu=1}^d \frac{\delta_{\mu} \hat{D}(k)}{[1 - \hat{D}(k)]^2} = - \frac{1}{d} \sum_{\mu=1}^d \sin(k_{\mu}) \tag{3.6.7}
\]

\[
\Delta \hat{C}(k) = \sum_{\mu=1}^d \frac{2 (\delta_{\mu} \hat{D}(k))^2}{[1 - \hat{D}(k)]^3} + \frac{\delta_{\mu}^2 \hat{D}(k)}{[1 - \hat{D}(k)]^2}, \tag{3.6.8}
\]
and
\[ \sum_{\mu=1}^{d} \delta_{\mu} \hat{D}(k) = -\frac{1}{d} \sum_{\mu=1}^{d} \sin(k_{\mu}), \quad (3.6.9) \]
\[ \Delta \hat{D}(k) = \sum_{\mu=1}^{d} \delta_{\mu}^2 \hat{D}(k) = -\frac{1}{d} \sum_{\mu=1}^{d} \cos(k_{\mu}) = -\hat{D}(k), \quad (3.6.10) \]
\[ \sum_{\mu=1}^{d} (\delta_{\mu} \hat{D}(k))^2 = \frac{1}{d^2} \sum_{\mu=1}^{d} \sin^2(k_{\mu}) := \hat{D}^{\sin}(k). \quad (3.6.11) \]

We use
\[ \sin^2(k_{\mu}) = -\frac{1}{4} \left( e^{ik_{\mu}} - e^{-ik_{\mu}} \right)^2 = \frac{1}{2} - \frac{1}{4} e^{ik_{\mu}} + \frac{1}{4} e^{-ik_{\mu}} \quad (3.6.12) \]
to compute that
\[ \hat{D}^{\sin}(k) = \frac{d}{2d^2} - \frac{1}{4d^2} \sum_{i} e^{-2ik_{i}} = \frac{1}{2d} [1 - \hat{D}(2k)] \quad (3.6.13) \]

Then we define \( \hat{M}(k) = \hat{D}(k) - 2\hat{D}^{\sin}(k)\hat{C}(k) \) and see that \( (3.6.8) \) and \( (3.6.13) \) imply that
\[ \Delta \hat{C}(k) = -\hat{C}(k)^2 \hat{M}(k) = -\hat{D}(k)\hat{C}(k)^2 + \frac{1}{d} \hat{C}(k)^3 - \frac{1}{2d^2} \sum_{i} e^{-2ik_{i}} \hat{C}(k)^3. \quad (3.6.14) \]

Thus, we obtained that
\[ \sum_{y} \|y\|_2^2 \hat{C}(y)(D^{*l} \ast C^{*n})(x - y) \]
\[ = \int_{(\pi, -\pi)^d} \hat{D}^{l}(k)\hat{C}^{n+2}(k) \hat{M}(k) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d} \quad (3.6.15) \]
\[ = \int_{(\pi, -\pi)^d} \hat{D}^{l}(k)\hat{C}^{n+2}(k) \left( \hat{D}(k) - \frac{1}{d} \hat{C}(k) + \frac{1}{2d^2} \sum_{i} e^{2ik_{i}} \hat{C}(k) \right) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d}. \quad (3.6.16) \]

As we explain in Section 5.2, we can numerically compute the following SRW-integral:
\[ I_{n,l}(x) := \frac{1}{(2d)^l} (D^{*l} \ast C^{*n}_{1/(2d)}) (x) = \int_{[\pi, \pi]^d} \hat{D}^{l}(k) \frac{d^d k}{[1 - \hat{D}(2k)]^n e^{ik \cdot x} (2\pi)^d}. \quad (3.6.17) \]

The function \( I_{n,m} \) is non-negative and monotone in the sense that \( I_{n,l}(x) \geq I_{n,l}(x + e_{i}) \) for all \( x \in \mathbb{Z}^d \) and \( i \in \{1, 2, \ldots, d\} \) with \( x_i \geq 0 \). This is proven in [42, Lemma B.3]. For \( (3.6.16) \) we conclude
\[ (3.6.16) = I_{n+2,l+1}(x) - \frac{1}{d} I_{n+3,l}(x) + \frac{1}{2d^2} \sum_{i} I_{n+3,l}(x + 2e_{i}) := I_{n,l}(x), \quad (3.6.18) \]
where we introduce the abbreviate this term by $\mathcal{M}^M_{n,l}(x)$. Using the monotonicity of $I_{n,m}$ and symmetry we obtain:

$$\sup_{x \neq 0} \{\mathcal{M}^M_{n,l}(x)\} = \max\{\mathcal{M}^M_{n,l}(e_1), \mathcal{M}^M_{n,l}(2e_1), \mathcal{M}^M_{n,l}(e_1 + e_2)\} \tag{3.6.19}$$

This yields that

$$\tilde{f}_3(z_I) \leq \left(\frac{2d - 2}{2d - 1}\right)^{n+1} \max\{\mathcal{M}^M_{n,l}(e_1), \mathcal{M}^M_{n,l}(2e_1), \mathcal{M}^M_{n,l}(e_1 + e_2)\}. \tag{3.6.20}$$

We have assumed that $Q(\gamma, \Gamma, z)$ holds. Thereby, we know that the right hand side is smaller than $\gamma_3$ and we conclude $\tilde{f}_3(z_I) \leq \gamma_3$. We can use this argument to bound $\tilde{f}_3(z_I)$ only in dimension $d \geq 2(n + 3) + 1$ as we use $I_{n+3,l+1}(0)$, which is only finite in these dimensions. This restricts the analysis shown here to dimensions $d \geq d_c + 3$, e.g. for percolation we can only use this bound for $d \geq 9$ as we require a bound on the $\tilde{f}_{3,1,0}$ for the bootstrap. This problem can be avoided using a different bound for the integral. For example using the bound

$$D^{\sin}(k) = \frac{1}{d^2} \sum_{\mu=1}^{d} \left[1 - \cos^2(k multiplicity of $

in (3.6.15)$ we obtain that, for even $l$,

$$\sum_y \|y\|_2^2 C(y)(D^{*l} \star C^{*n})(x - y) \leq I_{n+2,l+1}(x) + \frac{4}{d} I_{n+2,l}(0), \tag{3.6.22}$$

which implies

$$\tilde{f}_3(z_I) \leq I_{n+2,l+1}(0) + \frac{4}{d} I_{n+2,l}(0), \tag{3.6.23}$$

for even $l$. This and other bounds known to us that work in $d = d_c + 1, d_c + 2$, perform numerically worse than the bound in $(3.6.20)$. As we are not able to prove mean-field behavior in dimension $d_c + 1, d_c + 2$ for LT, LA and percolation we use the numerically better bound $(3.6.20)$.

### 3.6.3 The idea of the improvement of bounds

We want to prove that $\tilde{f}_3(z) \leq \Gamma_3$ implies that $\tilde{f}_3(z) \leq \gamma_3$ for all $z \in (z_I, z_c)$. We prove this statement by creating a bound on

$$\tilde{f}_3(z) = \sup_{x \in Z^{d \setminus \{0\}}} \int_{(\pi, -\pi)^d} (-\Delta \hat{G}_z(k)) \hat{D}^l(k) \hat{G}^n_z(k) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d}. \tag{3.5.7}$$

We extract SRW-like contributions from $\Delta \hat{G}_z(k)$ and $\hat{G}^n_z(k)$ and bound them as in the preceding section. Further, we bound the remainder terms carefully to obtain a good numerical bound. We decompose $\Delta \hat{G}_z(k)$ into five terms $H_1, \ldots, H_5$ and then bound the contribution of each term. We bound the contributions using the bounds stated in Assumption $\ref{3.5.3}$ and Proposition $\ref{3.5.5}$. 


SRW-integrals.

Using the terminology introduced in Definition 3.5.1 we define

$$\hat{D}^{(x)}(k) = \frac{1}{2^d d!} \sum_{\mu \in \mathcal{P}_d} \sum_{\delta \in [-1, 1]^d} e^{ik \cdot p(x; v, \delta)}.$$  \hspace{1cm} (3.6.24)

Performing the sum over $\delta$ gives cosines, so $\hat{D}^{(x)}(k)$ is real. We define

$$\hat{C}^{(x)}(k) = \frac{1}{\varepsilon + 1 - \hat{D}(k)},$$  \hspace{1cm} (3.6.25)

$$\hat{M}^{(x)}(k) = \hat{D}(k) - 2\hat{D}^{(x)}(k)\hat{C}^{(x)}(k),$$  \hspace{1cm} (3.6.26)

and the following SRW-integrals which are adaptations of the integrals used in \[41\] Section 1.6: for $x \in \mathbb{Z}^d$ and $n, l \in \mathbb{N}$ let

$$I_{n,l}^{(e)}(x) = \int_{(\pi, -\pi)^d} \hat{D}(k)^l \hat{C}^{(x)}(k)^n \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (3.6.27)

$$K_{n,l}^{(e)}(x) = \int_{(\pi, -\pi)^d} |\hat{D}(k)|^l |\hat{C}^{(x)}(k)|^n |\hat{D}^{(x)}(k)| |\hat{M}^{(x)}(k)| \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (3.6.28)

$$T_{n,l}^{(e)}(x) = \int_{(\pi, -\pi)^d} |\hat{D}^{(x)}(k)| |\hat{C}^{(x)}(k)|^n |\hat{D}^{(x)}(k)| |\hat{M}^{(x)}(k)| \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (3.6.29)

$$U_{n,l}^{(e)}(x) = \int_{(\pi, -\pi)^d} |\hat{D}^{(x)}(k)| |\hat{C}^{(x)}(k)|^n |\hat{D}^{(x)}(k)| |\hat{D}^{(x)}(k)| |\hat{M}^{(x)}(k)| \frac{d^d k}{(2\pi)^d}.$$  \hspace{1cm} (3.6.30)

When $\varepsilon = 0$, we drop the superscript $(\varepsilon)$. For $\varepsilon = 1 - \hat{F}_z(0)$ we use the superscript $\ast$.

For any function $f$ such that $f(x) = f(p(x; v, \delta))$ for all $v, \delta$ (see Definition 3.5.1) we see that

$$\int_{(\pi, -\pi)^d} \hat{f}(k) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d} = \int_{(\pi, -\pi)^d} \hat{f}(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d}.$$  \hspace{1cm} (3.6.31)

The two-point function $G_z$ and $D$ have these symmetries, so that we can replace $e^{ik \cdot x}$ in (3.5.7) by $\hat{D}^{(x)}(k)$. In Section 5.4 we show how to compute the value of $I_{n,l}^{(e)}(x)$ and in Section 5.2.3 we bound the other integrals in terms of $I_{n,l}^{(e)}(x)$.

Decomposition of the two-point function.

In (3.5.15) we have already shown how we decompose $G_z$ into a SRW part and a remainder term. In this section we decompose $\hat{G}_z(k)$ and $\Delta \hat{G}_z(k)$ into several pieces which we then bound in the next section. We start with some preparations for this decomposition. We define

$$\hat{E}(k) = \frac{\Delta \hat{R}_F(k) \hat{C}^\ast(k)}{1 - \hat{F}_z(k)},$$  \hspace{1cm} (3.6.32)
We assume that 
\[
\frac{1}{1 - \hat{F}_z(k)} = \frac{1}{1 - \hat{F}(0) + \alpha_F[1 - \hat{D}(k)] + \Delta \hat{R}_F(k)} = \hat{C}^*(k) - \hat{E}(k). \tag{3.6.33}
\]

By Proposition 3.5.5 we know that 
\[
\frac{1}{1 - \hat{F}_z(k)} \leq K\hat{C}(k). \tag{3.6.34}
\]

We assume that \(f_2(z) \leq \Gamma_2\), i.e. that 
\[
|\hat{G}_z(k)| \leq \frac{2d - 2}{2d - 1} \Gamma_2 \hat{C}(k). \tag{3.6.35}
\]

and abbreviate from now on \(\Gamma'_2 = \frac{2d - 2}{2d - 1} \Gamma_2\). We use Assumption 3.5.3 and Proposition 3.5.2 to bound 
\[
|\hat{E}(k)| \leq \left| \frac{\Delta \hat{R}_F(k) \hat{C}^*(k)}{1 - \hat{F}_z(k)} \right| \leq \beta_{\Delta R_F} K\hat{C}^*(k) \leq \beta_{\Delta R_F} K\hat{C}(k), \tag{3.6.36}
\]

\[
|\hat{G}_z(k)\hat{C}^*(k)| \leq (\beta_{\Delta R_F} + \beta_{\Delta R,F} \Gamma'_2) \hat{C}(k). \tag{3.6.37}
\]

Next we decompose \(\Delta \hat{G}_z(k)\). Therefore, we first recall (3.5.14) with 
\[
\hat{\Phi}(k) = c_\Phi + \alpha_\Phi \hat{D}(k) + \hat{R}_\Phi(k), \tag{3.6.38}
\]

\[
1 - \hat{F}_z(k) = c_F - \alpha_F \hat{D}(k) - \hat{R}_F(k), \tag{3.6.39}
\]

and see that 
\[
\Delta \hat{\Phi}_z(k) = -\alpha_\Phi \hat{D}(k) + \Delta \hat{R}_\Phi(k), \tag{3.6.40}
\]

\[
\Delta \hat{F}_z(k) = -\alpha_F \hat{D}(k) + \Delta \hat{R}_F(k). \tag{3.6.41}
\]

Then we compute \(\Delta \hat{G}_z(k)\) to be 
\[
\Delta \hat{G}_z(k) = \frac{\Delta \hat{\Phi}_z(k)}{1 - \hat{F}_z(k)} + \hat{\Phi}_z(k) \left( \Delta \frac{1}{1 - \hat{F}_z(k)} \right) + 2 \sum_{\mu=1}^{d} \partial_\mu (\hat{\Phi}_z(k)) \partial_\mu \left( \frac{1}{1 - \hat{F}_z(k)} \right). \tag{3.6.42}
\]

We bound the three terms of (3.6.42) individually. For the first term we compute that 
\[
\frac{\Delta \hat{\Phi}_z(k)}{1 - \hat{F}_z(k)} = -\alpha_\Phi \hat{D}(k) \hat{C}^*(k) + \alpha_\Phi \hat{D}(k) \hat{E}(k) + \frac{\Delta \hat{R}_\Phi(k)}{1 - \hat{F}_z(k)}. \tag{3.6.43}
\]

For the second term we note that 
\[
\left( \Delta \frac{1}{1 - \hat{F}_z(k)} \right) = \frac{\Delta \hat{F}(k)}{(1 - \hat{F}_z(k))^2} + 2 \sum_{\mu=1}^{d} \left( \partial_\mu \hat{F}(k) \right)^2 \frac{1}{(1 - \hat{F}_z(k))^3}. \tag{3.6.44}
\]
and then compute

\[ \hat{\Phi}_z(k) \frac{\Delta \hat{F}(k)}{(1 - \hat{F}_z(k))^2} = -\alpha_F \hat{D}(k)(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k)^2 \]
\[ + \alpha_F \hat{D}(k)(c_\Phi + \alpha_\Phi \hat{D}(k)) \left( \hat{E}(k) \hat{C}^*(k) + \frac{\hat{E}(k)}{1 - \hat{F}_z(k)} \right) \]
\[ - \hat{R}_\Phi(k) \alpha_F \hat{D}(k) \frac{(1 - \hat{F}_z(k))^2}{(1 - \hat{F}_z(k))} + \frac{\Delta \hat{R}_F(k)}{(1 - \hat{F}_z(k))} \hat{G}_z(k), \]

(3.6.45)

and

\[ 2 \hat{\Phi}_z(k) \sum_{\mu=1}^d (\partial_\mu \hat{F}(k))^2 = 2\alpha_F^2 \hat{D}\sin(k)(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k)^3 \]
\[ - 2\alpha_F^2 \hat{D}\sin(k)(c_\Phi + \alpha_\Phi \hat{D}(k)) \]
\[ \times \left( \frac{\hat{E}(k) \hat{C}^*(k)^2 + \hat{E}(k) \hat{C}^*(k) + \hat{E}(k)}{1 - \hat{F}_z(k)} \right) \]
\[ + 2\alpha_F^2 \hat{R}_\Phi(k) \frac{\hat{D}\sin(k)}{(1 - \hat{F}_z(k))^3} \]
\[ + 2 \sum_{\mu=1}^d (\partial_\mu \hat{R}_F(k))^2 + 2\alpha_F \partial_\mu \hat{D}(k) \partial_\mu \hat{R}_F(k) \]
\[ \frac{1}{(1 - \hat{F}_z(k))^2} \hat{G}_z(k). \]

(3.6.46)

The last term of (3.6.42) is given by

\[ 2 \sum_{\mu=1}^d \partial_\mu (\hat{\Phi}_z(k)) \partial_\mu \left( \frac{1}{1 - \hat{F}_z(k)} \right) \]
\[ = 2\alpha_F \alpha_\Phi \hat{D}\sin(k) \left( \frac{C^*(k)^2 - \hat{E}(k) \hat{C}^*(k) - \hat{E}(k)}{1 - \hat{F}_z(k)} \right) \]
\[ + 2 \frac{1}{(1 - \hat{F}_z(k))^2} \sum_{\mu=1}^d (\partial_\mu \hat{R}_\Phi(k) \alpha_F \partial_\mu \hat{D}(k) + \partial_\mu \hat{\Phi}_z(k) \partial_\mu \hat{R}_F(k)) . \]

(3.6.47)

We group the terms in (3.6.43) and (3.6.45) - (3.6.47) into five groups \( H_i \) which we explain after defining them. The dominant contribution

\[ \hat{H}_1(k) = (\alpha_F(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \alpha_\Phi) \hat{C}^*(k) \hat{M}^*(k), \]

(3.6.48)
and the remainder terms are

\[
\begin{align*}
\hat{H}_2(k) &= -\alpha_F \left((c_\Phi + \alpha_\Phi \hat{D}(k)) \left(\hat{C}^*(k) + \frac{1}{1 - \hat{F}_z(k)}\right) + \alpha_\Phi\right) \hat{E}(k) \hat{M}^*(k) \\
&\quad + \alpha_F \frac{\hat{R}_\Phi(k)}{(1 - \hat{F}_z(k))^2} \hat{M}^*(k), \\
\hat{H}_3(k) &= 2 \frac{\hat{D}^{\sin}(k)}{1 - \hat{F}_z(k)} \left(\hat{E}(k) - \frac{\alpha_F - 1}{1 - \hat{F}_z(k)} \right) \left(\alpha_\Phi + \alpha_F - \frac{\alpha_\Phi \hat{C}(k)}{1 - \hat{F}_z(k)}\right) \\
&\quad + 2\alpha_F \hat{R}_\Phi(k) \hat{D}^{\sin}(k) \left(\frac{\hat{E}(k)}{(1 - \hat{F}_z(k))^2} - \frac{(\alpha_F - 1)}{(1 - \hat{F}_z(k))^3}\right), \\
\hat{H}_4(k) &= -\frac{\Delta \hat{R}_\Phi(k)}{1 - \hat{F}_z(k)} - \frac{\Delta \hat{R}_F(k)}{1 - \hat{F}_z(k)} \hat{G}_z(k), \\
\hat{H}_5(k) &= -2 \frac{\sum_{\mu=1}^d (\partial_\mu \hat{R}_\Phi(k))^2 - 2\alpha_F \partial_\mu \hat{D}(k) \partial_\mu \hat{R}_F(k)}{(1 - \hat{F}_z(k))^2} \hat{G}_z(k) \\
&\quad - \frac{2}{(1 - \hat{F}_z(k))^2} \sum_{\mu=1}^d (\partial_\mu \hat{R}_\Phi(k) \partial_\mu \hat{D}(k) + \partial_\mu \hat{\Phi}_z(k) \partial_\mu \hat{R}_F(k)).
\end{align*}
\]  

(3.6.49)

We leave it to the reader to check that

\[
-\Delta \hat{G}_z(k) = \sum_{i=1}^5 \hat{H}_i(k). 
\]  

(3.6.50)

The first term \(H_1\) is a SRW-like that contribution that can be bounded in the same way as shown in Section 3.6.2. The second term \(H_2\) corresponds to everything that has \(\hat{M}^*(k)\) and an error term. In \(H_3\) we collect remaining \(\hat{D}^{\sin}\) contributions. We put into \(H_4\) the contributions of \(\Delta \hat{R}_\Phi(k)\) and \(\Delta \hat{R}_F(k)\), and \(H_5\) represents all products of single derivatives.

### 3.6.4 Improvement of bounds

In this section we bound

\[
\sum_{i=1}^5 \int_{(\pi, -\pi)^d} \hat{H}_i(k) \hat{D}^l(k) \hat{G}_z^n(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d}. 
\]  

(3.6.51)

by bounding the contributions of the \(H_i\) one at a time.

**Step 1: Bound on contribution due to \(\hat{H}_1\).**

We want to bound

\[
\int_{(\pi, -\pi)^d} \hat{H}_1(k) \hat{D}^l(k) \hat{G}_z^n(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d}. 
\]  

(3.6.52)
where
\[
\hat{H}_1(k) = (\alpha_F(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \alpha_\Phi) \hat{M}^*(k) \hat{C}^*(k)
\]  
(3.6.53)
for \(n = 0, 1, 2\). We begin by computing a bound on
\[
\int_{(\pi, -\pi)^d} \hat{D}(k)^l \hat{C}^*(k)^m \hat{M}^*(k) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d}.
\]  
(3.6.54)
As in (3.6.14) we rewrite \(\hat{D}^\text{sin}(k)\) and proceed as in (3.6.16)
\[
= \int_{(\pi, -\pi)^d} \hat{D}(k)^l \hat{C}^*(k)^m \left( \hat{D}(k) + \frac{1}{d} \hat{C}^*(k) - \frac{1}{2d^2} \sum I \ e^{2ik \cdot x} \hat{C}^*(k) \right) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d}
\]  
(3.6.55)

**Bound on a weighted line** \((n = 0)\). To bound (3.6.52) for \(n = 0\) we simply use (3.6.55) to compute
\[
\| \alpha_F c_\Phi \mathcal{S}_0,1^m(x) + \alpha_F |\alpha_\Phi| \mathcal{S}_0,l+1^m(x) + |\alpha_\Phi| \mathcal{S}_1,l^m(x) \|.
\]  
(3.6.56)

**Bound on a weighted bubble** \((n = 1)\). To bound (3.6.52) for \(n = 1\) we expand \(\hat{G}_z(k)\):
\[
\hat{G}_z(k) = (c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + (\hat{R}_0(k) - \Delta \hat{R}_F(k) \hat{G}_z(k)) \hat{C}^*(k)
\]  
(3.6.57)
so that
\[
\| \alpha_F c_\Phi \mathcal{S}_{0,1}^m(x) + \alpha_F |\alpha_\Phi| \mathcal{S}_{0,l+1}^m(x) + 2\alpha_F |\alpha_\Phi| c_\Phi \mathcal{S}_{1,l+1}^m(x)
\]  
(3.6.58)
\[
+ \alpha_\Phi^2 \mathcal{S}_{0,l+1}^m(x) + \alpha_F \alpha_\Phi^2 \mathcal{S}_{1,l+2}^m(x)
\]  
(3.6.59)
\[
+ (\beta_{r,\Phi} + \beta_{\Delta R_F} \Gamma_2) (c_\Phi \alpha_F T_{3,1}(x) + |\alpha_\Phi| \alpha_F T_{3,l+1}(x) + |\alpha_\Phi| T_{2,l}(x)),
\]
with \(T_{n,l}\) as defined in (3.6.29).
Bound on a weighted triangle \((n = 2)\). Now we want to bound \((3.6.52)\) for \(n = 2\). We decompose \(\hat{G}_z^2(k)\) into two terms

\[
\hat{G}_z^2(k) = \hat{C}^*(k)^2(c_\Phi + \alpha_\Phi \hat{D}(k))^2 + \left[ \hat{R}_\Phi(k) - \Delta \hat{R}_F(k) \hat{G}_z(k) \right] \hat{C}^*(k) \left[ (c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \hat{G}_z(k) \right]
\]

We compute the contribution of line \((3.6.60)\) to be

\[
\hat{H}_1(k)(z) \hat{C}^*(k)^2(c_\Phi + \alpha_\Phi \hat{D}(k))^2 = (c_\Phi + \alpha_\Phi \hat{D}(k))^2 \left[ \alpha_F(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \alpha_\Phi \right] \hat{M}^*(k) \hat{C}^*(k)^3
\]

We expand the brackets and then use \((3.6.19)\) to bound the created terms and obtain:

\[
\int_{(\pi, -\pi)^d} \hat{H}_1(k)(z) \hat{C}^*(k)^2(c_\Phi + \alpha_\Phi \hat{D}(k))^2 \hat{D}^{(x)}(k) \hat{D}^{(y)}(k) \frac{d^d k}{(2\pi)^d} \leq \alpha_F^3 |c_\Phi|^2 \mathcal{G}^M_{2,1}(x) + |\alpha_\Phi|^2 |c_\Phi| \mathcal{G}^M_{1,1}(x) + 3 \alpha_F |c_\Phi|^2 \mathcal{G}^M_{2,1}(x)
\]

\[
+ 2 |\alpha_\Phi|^2 |c_\Phi| \mathcal{G}^M_{1,1}(x) + 3 \alpha_F^2 |c_\Phi| \mathcal{G}^M_{2,1}(x) + |\alpha_\Phi|^2 |\mathcal{G}^M_{1,1}(x)
\]

\[
+ \alpha_F |c_\Phi|^3 \left( \mathcal{G}^M_{2,1}(x) \right). \quad (3.6.63)
\]

We bound the absolute value of the minor contributions, given in \((3.6.61)\), using \((3.6.37)\) as follows

\[
(\beta_{r,\Phi} + \beta_{\Delta r,\Phi} \Gamma'_2) \left( (c_\Phi + |\alpha_\Phi||\hat{D}(k)| + \Gamma'_2) \hat{C}(k)^2 \right). \quad (3.6.64)
\]

The contributions due to \((3.6.61)\) are bounded by

\[
\int_{(\pi, -\pi)^d} \alpha_F(c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \alpha_\Phi \right] \hat{R}_\Phi(k) - \Delta \hat{R}_F(k) \hat{G}_z(k) \right]
\]

\[
\times \left[ (c_\Phi + \alpha_\Phi \hat{D}(k)) \hat{C}^*(k) + \hat{G}_z(k) \right] \hat{D}^{(x)}(k) \hat{D}^{(y)}(k) \hat{M}^*(k) \hat{C}^*(k)^2 \frac{d^d k}{(2\pi)^d} \leq \alpha_F c_\Phi T_{4,1}(x) + |\alpha_\Phi| T_{3,1}(x) \left( \beta_{r,\Phi} + \beta_{\Delta r,\Phi} \Gamma'_2 \right) (c_\Phi + \Gamma'_2)
\]

\[
+ \left[ \alpha_F |c_\Phi| (2c_\Phi + \Gamma'_2) T_{4,1}(x) + \alpha_\Phi^2 T_{3,1}(x) + \alpha_F^2 T_{4,1}(x) + \alpha_F^2 T_{4,1}(x) \right] \left( \beta_{r,\Phi} + \beta_{\Delta r,\Phi} \Gamma'_2 \right)
\]

Conclusion of step 1. We have just bounded the contribution due to \(H_1\) and obtained the following bound:

\[
\int_{(\pi, -\pi)^d} \hat{H}_1(k) \hat{G}_z^n(k) \hat{D}^{(x)}(k) \hat{D}^{(y)}(k) \frac{d^d k}{(2\pi)^d} \leq \left\{ \begin{array}{ll} (3.6.56) & n = 0, \\ (3.6.59) & n = 1, \\ (3.6.63) + (3.6.65) & n = 2. \end{array} \right. \quad (3.6.66)
\]

By the sum of two line numbers we denote the sum of the terms given in the corresponding lines. As for \(z = z_I\) this bound uses \(I_{n+2,1}(x)\) and can therefore not be used in \(d = d_c + 1, d_c + 2\). We have chosen to use these bounds, even if other bounds would be available, as they give numerically better bounds.
Step 2: Bound on contribution due to $\hat{H}_2$.

We next bound

$$\int_{(\pi, -\pi)^d} \hat{H}_2(k) \hat{D}^l(k) \hat{G}_z^\alpha(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (3.6.67)

where

$$\hat{H}_2(k) = -\alpha_F \left( (c_\Phi + \alpha_\Phi \hat{D}(k)) \left( \hat{C}^* (k) + \frac{1}{(1 - \hat{F}_z(k))} \right) + \alpha_\Phi \right) \hat{E}(k) \hat{M}^* (k)$$

$$+ \alpha_F \frac{\hat{R}_\Phi (k)}{(1 - \hat{F}_z(k))^2} \hat{M}^* (k)$$  \hspace{1cm} (3.6.68)

for $n = 0, 1, 2$. We first bound that absolute value of $\hat{H}_2(k)$:

$$|\hat{H}_2(k)| \leq \alpha_F \left( (c_\Phi + |\alpha_\Phi| |\hat{D}(k)|) |\hat{C}(k)| (1 + K) + |\alpha_\Phi| \right) \beta_{\Delta, F} K \hat{C}(k) |\hat{M}^* (k)|$$

$$+ \alpha_F \beta_{\Delta, F} K^2 \hat{C}(k)^2 |\hat{M}^* (k)|.$$  \hspace{1cm} (3.6.69)

We use that $|\hat{G}_z(k)| \leq \Gamma'_2 \hat{C}(k)$ and $T_{n,l}$ defined [3.6.29] to bound [3.6.67] by

$$\alpha_F \beta_{\Delta, F} (\Gamma'_2)^n K \left( (c_\Phi T_{n+2,l}(x) + |\alpha_\Phi| T_{n+2,l+1}(x)) (1 + K) + |\alpha_\Phi| T_{n+1,l}(x) \right)$$

$$+ |\alpha_F| \beta_{\Delta, F} K^2 T_{n+2,l}(x).$$  \hspace{1cm} (3.6.70)

Step 3: Bound on contribution due to $\hat{H}_3$.

We now bound

$$\int_{(\pi, -\pi)^d} \hat{H}_3(k) \hat{D}^l(k) \hat{G}_z^\alpha(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (3.6.71)

where

$$\hat{H}_3(k) = 2 \frac{\hat{D}\sin(k)}{1 - \hat{F}_z(k)} \left( \hat{E}(k) - \frac{\alpha_F - 1}{1 - \hat{F}_z(k)} \left( \alpha_\Phi + \alpha_F c_\Phi + \alpha_\Phi \hat{D}(k) \right) \right)$$

$$+ 2 \alpha_F \hat{R}_\Phi (k) D\sin(k) \left( \frac{\hat{E}(k)}{(1 - \hat{F}_z(k))^2} - \frac{(\alpha_F - 1)}{(1 - \hat{F}_z(k))^3} \right).$$  \hspace{1cm} (3.6.72)

We first bound $\hat{H}_3(k)$ as

$$|\hat{H}_3(k)| \leq 2 K^2 \hat{C}(k)^2 \hat{D}\sin(k) \left( \beta_{\Delta, F} + |\alpha_F - 1| \right) \left( |\alpha_\Phi| + \alpha_F K \hat{C}(k) (c_\Phi + |\alpha_\Phi| |\hat{D}(k)|) \right)$$

$$+ 2 \alpha_F \beta_{\Delta, F} K \hat{D}\sin(k) K^3 \hat{C}(k)^3 \left( \beta_{\Delta, F} + |\alpha_F - 1| \right).$$  \hspace{1cm} (3.6.73)

Then we use $U_{n,l}$ defined in [3.6.30] to bound [3.6.71] by

$$\leq 2 (\Gamma'_2)^n K^2 \left( \beta_{\Delta, F} + |\alpha_F - 1| \right) \times \left( |\alpha_\Phi| U_{n+2,l}(x) + \alpha_F K \left( (c_\Phi + \beta_{\Delta, F}) U_{n+3,l}(x) + |\alpha_\Phi| U_{n+3,l+1}(x) \right) \right).$$  \hspace{1cm} (3.6.74)
**Step 4: Bound on contribution due to \( \hat{H}_4 \).**

We proceed by bounding

\[
\int_{(\pi,-\pi)^d} \hat{H}_4(k) \hat{D}^l(k) \hat{G}_z^n(k) e^{ik \cdot x} \frac{d^d k}{(2\pi)^d},
\]  

(3.6.75)

where

\[
\hat{H}_4(k) = \frac{-\Delta \hat{R}_\Phi(k) + \Delta \hat{R}_F(k) \hat{G}_z(k)}{1 - \hat{F}_z(k)}.
\]  

(3.6.76)

We first bound \( \hat{H}_4(k) \) in Fourier space

\[
|\hat{H}_4(k)| \leq K\left(\beta_{\Delta R, \Phi} + \beta_{\Delta R, \Phi} \Gamma_2 \hat{C}(k)\right).
\]

Then we use \( K_{n,1} \) defined (3.6.28) to bound (3.6.75) is bounded by

\[
(3.6.75) \leq K\left(\beta_{\Delta R, \Phi} K_{n,1}(x) + \beta_{\Delta R, \Phi} \Gamma_2 K_{n+1,1}(x)\right).
\]  

(3.6.77)

**Step 5: Bound on contribution due to \( \hat{H}_5 \).**

We finally bound

\[
\int_{(\pi,-\pi)^d} \hat{H}_5(k) \hat{D}^l(k) \hat{G}_z^n(k) \hat{D}^{(x)}(k) \frac{d^d k}{(2\pi)^d},
\]

(3.6.78)

where

\[
\hat{H}_5(k) = -2 \sum_{\mu=1}^{d} \partial_\mu \hat{R}_F(k) (2\alpha_F \partial_\mu \hat{D}(k) + \partial_\mu \hat{R}_F(k))
\]

\[
\frac{(1 - \hat{F}_z(k))^2}{(1 - \hat{F}_z(k))^2} \sum_{\mu=1}^{d} (\partial_\mu \hat{R}_\Phi(k) \alpha_F \partial_\mu \hat{D}(k) + \partial_\mu \hat{R}_\Phi(k) \partial_\mu \hat{R}_F(k)).
\]  

(3.6.79)

By direct computation and Assumption [3.5.3] respectively, we know that

\[
|\partial_\mu \hat{D}(k)| = \frac{1}{d} |\sin(k_\mu)|, \quad |\delta_\mu \hat{R}_\Phi(k)| \leq \frac{1}{d} \beta_{\Delta R, \Phi} |\sin(k_\mu)|,
\]  

(3.6.80)

\[
|\delta_\mu \hat{R}_F(k)| \leq \frac{1}{d} \beta_{\Delta R, \Phi} |\sin(k_\mu)|,
\]  

(3.6.81)

so that all terms of (3.6.79) create a \( \hat{D}^\sin(k) \) in a bound:

\[
|\hat{H}_5(k) \hat{G}_z^n(k)| \leq 2K^2 \Gamma_2^{n+1} \hat{C}(k)^{n+3} \hat{D}^\sin(k) (2\alpha_F \beta_{\Delta R, \Phi} + \beta_{\Delta R, \Phi}^2)
\]

\[
+ 2K^2 \Gamma_2^n \hat{C}(k)^{n+2} \hat{D}^\sin(k) (\alpha_F |\beta_{\Delta R, \Phi}| + |\alpha_F| \beta_{\Delta R, \Phi} + \beta_{\Delta R, \Phi} \beta_{\Delta R, \Phi}).
\]  

(3.6.82)

We use this to bound (3.6.78) as

\[
(3.6.78) \leq 2K^2 \Gamma_2^{n+1} (2\alpha_F \beta_{\Delta R, \Phi} + \beta_{\Delta R, \Phi}^2) U_{n+3,1}(x)
\]

\[
+ 2K^2 \Gamma_2^n (\alpha_F |\beta_{\Delta R, \Phi}| + |\alpha_F| \beta_{\Delta R, \Phi} + \beta_{\Delta R, \Phi} \beta_{\Delta R, \Phi}) U_{n+2,1}(x).
\]  

(3.6.83)
3.7 Proof of model assumptions

In this section we have bounded $\tilde{f}_3$ as follows:

$$\tilde{f}_3(z) \leq \sup_{x \in \mathbb{Z}^d \setminus \{0\}} (3.6.70) + (3.6.74) + (3.6.77) + (3.6.83) \quad \text{for } n = 0,$$

$$+ \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \begin{cases} 
(3.6.56) & n = 1, \\
(3.6.63) + (3.6.65) & n = 2.
\end{cases} \quad (3.6.84)$$

By the sum of several line numbers we denote the sum of the terms given in the corresponding lines. We assume that $Q(\gamma, \Gamma, z)$ holds, so that this term is smaller than $\gamma_3$, so that the improvement of bounds is successful and we have successfully performed the bootstrap. In Section 5.2 we show how to bound the integrals $I_{n,l}, K_{n,l}, T_{n,l}, U_{n,l}$ and explain for which $x$ the supremum is obtained.

3.7 Proof of model assumptions

The analysis explained in this chapter can be applied to all models for which we have recursive relations like those given in (3.2.1) and (3.2.2) and for which Assumptions 3.2.1-3.2.7 and Assumptions 3.5.3-3.5.4 hold. In this section we show that self-avoiding walk (SAW), percolation, lattice tree (LT) and lattice animals (LA) satisfy these requirements.

We have derived the NoBLE for these four models in Chapter 2 and have in Section 2.5 given a generalized form of the recursive relations that the NoBLE produces. This general form corresponds to the forms given in (3.2.1) and (3.2.2).

From the definition of the coefficients in Chapter 2 we conclude the symmetries stated in Assumption 3.2.4 and the relations between the coefficients given in Assumption 3.2.5 all hold.

The bounds on the coefficients given in Assumption 3.2.6 and Assumption 3.5.3 will be discussed at the end of the corresponding section in Chapter 4.

In the following, we show for each model that Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.7, and 3.5.4 are satisfied.

3.7.1 Self-avoiding walk

The aim of this section is to prove that SAW satisfies the assumptions stated in Section 3.2.

Initial point $z_I$. For SAW we choose $z_I = 1/(2d - 1)$ to be the critical point for the NBW and note that $c_n(x) \leq b_n(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, as each self-avoiding walk
is also a non-backtracking walk. From this follows that

\[ G_{z_i}(x) = \sum_{n=0}^{\infty} c_n(x) z^n_i \leq \sum_{n=0}^{\infty} b_n(x) z^n_i = B_{z_i}(x) \]

so that the Assumption 3.2.1 holds for SAW.

**Continuity of \( \tilde{\alpha}_z \).** As we choose \( \tilde{\alpha}_z = \alpha_z = z \) the continuity of \( z \mapsto \tilde{\alpha}_z \) in \( z \) is obvious.

**Growth of the two-point function.** We have defined \( G_z(x) \) to be the Green’s function of the sequence \( (c_n(x))_n \):

\[ G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n. \]

We know that this sum is convergent for all \( z < z_c \). Since \( c_n(x) \) is non-negative we know that \( G_z(x) \) is non-decreasing in \( z \). Next, we prove that \( G_z \) is differentiable on \([0, z_c)\) and prove the bound on the derivative (3.2.17) stated in Assumption 3.2.3. This is an adaptation of the proof of [92, Theorem 2.3], where upper and lower bounds on the derivative of \( \chi_{\text{SAW}}(z) \) are proven. We compute

\[ \sum_{n=0}^{\infty} \frac{d}{dz} c_n(x) z^n = \sum_{n=1}^{\infty} n c_n(x) z^{n-1} \leq \sum_{n=1}^{\infty} \sum_{m=0}^{n} c_n z^{n-1}. \]

Then, we split each \( n \)-step SAW after the \( m \)-step into two independent SAWs

\[ c_n(x) \leq 2d \sum_{u, v \in \mathbb{Z}^d} c_m(u) D(v - u) c_{n-m-1}(v) = 2d (c_m \star D \star c_{n-m-1})(x) \]

and use this to bound

\[ \sum_{n=0}^{\infty} \frac{d}{dz} c_n(x) z^n \leq 2d \sum_{n=0}^{\infty} \sum_{m=0}^{n} (c_m \star D \star c_{n-m-1})(x) z^{n-1} \]

\[ = 2d \sum_{n=0}^{\infty} \sum_{m=0}^{n} (c_m \star D \star c_n)(x) z^{n+m} = 2d (G_z \star D \star G_z)(x). \]

Therefore,

\[ \frac{d}{dz} G_z(x) = \sum_{n=0}^{\infty} \frac{d}{dz} c_n(x) z^n \leq 2d (G_z \star D \star G_z)(x), \]

where the interchange of summation and differentiation is justified by the fact that both sums are uniformly absolute convergent series of functions.
In the next step we prove that for all \( z < z_c \) there exists \( K(z) < \infty \) such that
\[
\sum_{x \in \mathbb{Z}^d} \|x\|_2^2 G_z(x) < K(z).
\]
The two-point function is exponentially decaying in \( x \), i.e.,
\[
G_z(x) = B^{\text{SAW}}(z)^{1/2} e^{-m(z)\|x\|_\infty},
\]
where \((B^{\text{SAW}}(z))^{1/2}\) is the SAW-bubble (analog to the SRW-bubble defined \([1.5.35]\)),
\[
\|x\|_\infty = \sup_j |x_j|,
\]
and \( m(z) > 0 \) is the so-called mass. This inequality is proven in \([74]\) Theorem 4.1.5]. Further, it is known that \( m(z) \in (0, \infty) \), see \([74]\) Section 1.3]. As a crude bound on the bubble we use \( B^{\text{SAW}}(z)^{1/2} \leq \hat{\gamma}_z(0) \) and obtain the bound
\[
\sum_{x \in \mathbb{Z}^d} \|x\|_2^2 G_z(x) \leq \hat{\gamma}_z(0) \sum_{x \in \mathbb{Z}^d} \|x\|_2^2 e^{-m(z)\|x\|_\infty} \leq d^2 \sum_{x \in \mathbb{Z}^d} \|x\|_\infty^2 e^{-m(z)\|x\|_\infty}
\]
\[
\leq \sum_{n=1}^\infty n^2 e^{-m(z)n} \sum_{x : \|x\|_\infty = n} 1 \leq d \sum_{n=1}^\infty n^{d+1} e^{-m(z)n} := K(z) < \infty.
\]
At the critical point. The two-point functions \( G_z, G'_z \) and the coefficients \( \Xi, \Xi' \), and \( \Psi_{z,k}^{\kappa}, \Pi_{z,k} \) are defined by infinite series. As \( f_2 \leq \Gamma_2 \) we know that, for \( d > 2 \),
\[
G_z(x) = \int_{(-\pi,\pi)^d} \hat{\gamma}_z(k) e^{-k \cdot x} \frac{d^d k}{(2\pi)^d}
\]
\[
\leq \Gamma_2 \int_{(-\pi,\pi)^d} \hat{B}_{1/(2d-1)}(k) \frac{d^d k}{(2\pi)^d} = \Gamma_2 B_{1/(2d-1)}(0) < \infty.
\]
Thereby, \( G_z(x) \) is uniformly bounded for \( x \in \mathbb{Z}^d \) and \( z \in (z_l, z_c) \). As \( G_z \) is defined by a power series we know that \( G_z \) is continuous for \( z < z_c \) and left-continuous in \( z = z_c \). As \( G'_z \) is \( x \)-wise bounded by \( G_z \) we conclude the same for this function. In Chapter \([4]\) we prove that the coefficients are bounded by simple diagrams, which in turn can be bounded by the two-point function. We conclude that also the coefficients are continuous for \( z < z_c \) and left-continuous in \( z = z_c \). We omit further details.

### 3.7.2 Lattice trees and animals

Now we prove that LT and LA satisfy the assumptions stated in Section \([3.2]\).

Initial point \( z_l \). For lattice trees (LT) and lattice animals (LA) we choose \( z_l = ((2d-1)e)^{-1} \). To prove Assumption \([3.2.1]\) we adapt an argument used in \([39]\) Proof of Lemma 3.1]. We know that each lattice tree/animal containing 0 and \( x \) contains a path of bonds that connects 0 and \( x \). We can bound the weight of each rib/sausage by the one-point function \( g_z = G_z(0) \). For \( x \neq 0 \) we can bound the ribs by \( g'_z = G'_z(0) \), as each rib/sausage has to avoid the backbone. Therefore,
\[
\hat{G}_z(x) \leq g_z^l \sum_{\omega \in \mathcal{W}^{\text{NWV}}(x)} (z g_z^l)^{|\omega|} = g_z^l B z g_z^l(x),
\]
which holds for all \( z \) with \( z g_z^l \leq (2d-1)^{-1} \) and \( x \neq 0 \). Next we bound \( g_z \) and \( g'_z \).
Lemma 3.7.1 (Bound on the number of $n$-bond lattice animals). The number of $n$-bond lattice animals that contain the origin is bounded by

$$2d(2d - 1)^{n-1} \frac{(n+1)^n}{(n+1)!}$$

(3.7.11)

A similar argument is also used in [39] Proof of Lemma 3.1 and could be adapted to prove Lemma 3.7.1. Unfortunately, one step in the [39] Proof of Lemma 3.1 is not correct. Namely, it is used that there are $(n+1)^{n-1}/(n+1)!$ abstract unlabeled trees with $n$ edges. However, while Cayley’s theorem states that there are $(n+1)^{n-1}/(n+1)!$ labeled trees the removal of the labels does not create a factor $1/(n+1)!$ as stated. Indeed, there are less than $(n+1)!$ ways to label an abstract tree, e.g., for a tree with two edges there are only 3 different ways to label the tree as the only difference is the label of the vertex that is part of two edges. We will adapt arguments of [16] to prove this result. Let us first produce the bound on $g_z$ and $g'_z$ and prove Lemma 3.7.1 thereafter. We see that (3.7.11) implies that

$$g_z \leq 1 + \sum_{n=2}^{\infty} z^n (2d)(2d - 1)^{n-1} \frac{(n+1)^n}{(n+1)!}.$$  

(3.7.12)

We note that

$$\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n} = 1,$$

(3.7.13)

which is known to hold as the distribution of the total progeny of a branching process $\mathcal{F}$ with Poisson offspring distribution with parameter one has distribution

$$\mathbb{P}(\mathcal{F} \text{ consists of } n \text{ individuals}) = \frac{n^{n-1}}{n!} e^{-n},$$

(3.7.14)

and $\mathbb{P}(|\mathcal{F}| < \infty) = 1$. We obtain for $z \leq ((2d - 1)e)^{-1}$:

$$g_z \leq 1 + \sum_{n=2}^{\infty} z^n (2d)(2d - 1)^{n-1} \frac{n^{n-1}}{n!} \leq 1 + \sum_{n=2}^{\infty} \frac{(2d)(2d - 1)^{n-1}}{(2d - 1)^{n-1}} \frac{n^{n-1}}{n!}\frac{2d}{2d - 1} e^{-n} = 1 + \frac{2de}{(2d - 1)} \sum_{n=2}^{\infty} \frac{n^{n-1}}{n!} e^{-n} = 1 + \frac{2de}{(2d - 1)} \left(1 - \frac{1}{e}\right) = e + \frac{e - 1}{2d - 1}.$$  

(3.7.15)

Since $g'_z \leq g_z$ this is also a bound for $g'_z$. We can get a better bound on $g'_z$ by using the condition that $e_i$ is not part of the tree/animal. A non-trivial lattice tree/animal contains at least one edge connecting the origin to a neighboring point. Therefore, at least at 1 out of $2d$ lattice trees/animals counted in $g_z$ does not contribute to $g'_z$. We conclude that

$$g'_z \leq 1 + \frac{2d - 1}{2d} (g_z - 1) = 1 + \frac{(2d - 1)(e - 1)}{2d} \left(1 + \frac{1}{2d - 1}\right) = e,$$

(3.7.16)
and thus, for all \( z \leq z_I \)

\[
(2d - 1)g_z^I z \leq 1, \quad \quad (2d - 1)\tilde{a}_z = (2d - 1)g_z z = 1 + \frac{1 - e^{-1}}{2d - 1}. \tag{3.7.17}
\]

Therefore, we know that \( zg_z^I z \leq (2d - 1)^{-1} \), so that we can use (3.7.10) for all \( z \leq z_I \).

**Proof of Lemma 3.7.1.** We first use the techniques of [16, Sections 2 and 5] to show the lemma only for LTs. Then we adapt the arguments for LAs.

We begin by defining a non-backtracking branching random walk with Poisson offspring distribution, similar to the NBBRW introduced in Section 1.5.3. Abstract trees are the family trees of a critical branching process with Poisson offspring distribution. In more detail, we begin with a single individual having \( \xi \) offspring, where \( \xi \) is a Poisson random variable of mean 1, i.e., \( \mathbb{P}(\xi = m) = (em!)^{-1} \). Each of the offspring then independently has offspring of its own with the same critical Poisson distribution. We denote by \( |T| \) the number of bonds of the tree \( T \). For an abstract tree \( T \), with the \( i \)th individual having \( \xi_i \) offspring, this associates to \( T \) the weight

\[
\mathbb{P}(T) = e^{-(|T|+1)} \prod_{i \in T} \frac{1}{\xi_i!}. \tag{3.7.18}
\]

We define an embedding of \( \varphi \) of \( T \) into \( \mathbb{Z}^d \) to be a mapping of the vertices of \( T \) into \( \mathbb{Z}^d \), such that the root is mapped to the origin and adjacent vertices in the tree are mapped to nearest-neighbors in \( \mathbb{Z}^d \). Further, we restrict to embeddings in which the children are not mapped to the location of the grandparents. We define the pair \((T, \varphi)\) to be a non-backtracking branching random walk with Poisson offspring distribution (Poisson-NBBRW). Given an abstract tree \( T \) with \( n \) bonds there are \( 2d(2d - 1)^{n-1} \) possible embeddings \( \varphi \) of \( T \). The Poisson-NBBRW measure is given by

\[
\mathbb{P}(T, \varphi) = \frac{1}{2d(2d - 1)^{|T|-1}} \mathbb{P}(T) = \frac{1}{2d(2d - 1)^{|T|-1}} e^{-(|T|+1)} \prod_{i \in T} \frac{1}{\xi_i!}. \tag{3.7.19}
\]

for all \( T \) and \( \varphi \). We define \( \mathbb{I}^{(t)}(T, \varphi) \) to be the indicator that the embedding \( \varphi \) of \( T \) is a lattice tree, i.e., \( \varphi : T \to \mathbb{Z}^d \) is injective. For a lattice tree \( T \) we write \( T = \varphi(T) \) if the embedding \( \varphi \) of \( T \) equals \( T \). For abstract trees with \( n \) bonds we define the measure

\[
\mathbb{Q}_n^{(t)}(T, \varphi) = \frac{1}{Z_n^{(t)}} \mathbb{P}(T, \varphi) \mathbb{I}^{(t)}(T, \varphi), \tag{3.7.20}
\]

with normalization

\[
Z_n^{(t)} = \sum_{(T, \varphi) : |T| = n} \mathbb{P}(T, \varphi) \mathbb{I}^{(t)}(T, \varphi). \tag{3.7.21}
\]
We prove that the number of $n$-bond lattice trees that contain the origin is given by

$$t_n^{(t)}(0) = Z_n^{(t)} e^{(n+1)} 2d(2d-1)^{n-1}. \quad (3.7.22)$$

To do that we show that for each $n$-bond lattice tree $T$ the following holds:

$$\sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} Q_n^{(t)}(\mathcal{F}, \varphi) = \frac{e^{-(n+1)}}{2d(2d-1)^{n-1}}, \quad (3.7.23)$$

which implies that the measure $Q_n^{(t)}$ corresponds to the uniform measure on all $n$-bond lattice trees and thus implies \[3.7.22\]. From \[3.7.19\] we conclude that, for all $T$ with $|T| = n$,

$$\sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} \mathbb{P}(\mathcal{F}, \varphi) = \frac{e^{-(n+1)}}{2d(2d-1)^{n-1}} \sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} \prod_{i \in \mathcal{F}} \frac{1}{\xi_i!}. \quad (3.7.24)$$

Then, we prove that, for any lattice tree $T$,

$$\sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} \prod_{i \in \mathcal{F}} \frac{1}{\xi_i!} = 1. \quad (3.7.25)$$

Let $b_0$ be the degree of 0 in $T$, and given non-zero $x \in T$, let $b_x$ be the degree of $x$ in $T$ minus 1. Then the set $\{b_z : x \in T\}$ must be equal to the set of $\xi_i$ for any $\mathcal{F}$ that can be mapped to $T$. Defining $\nu(T) = \#\{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T\}$, \[3.7.25\] is therefore equivalent to

$$\nu(T) = \prod_{x \in T} b_x!. \quad (3.7.26)$$

We prove \[3.7.26\] by induction on the number $N$ of generations of $T$. By this, we mean the length of the longest self-avoiding path in $T$, starting from the origin. The identity \[3.7.26\] clearly holds for $N = 0$. Our induction hypothesis is that \[3.7.26\] holds if there are $N-1$ or fewer generations. Suppose $T$ has $N$ generations, let $T_1, ..., T_{b_0}$ denote the lattice trees resulting from deleting from $T$ all bonds incident on the origin. We regard each $T_a$ as rooted at the neighbor of the origin in the corresponding deleted bond. It suffices to show that $\nu(T) = b_0! \prod_{a=1}^{b_0} \nu(T_a)$, since each $T_a$ has fewer than $N$ generations.

To prove this, we note that each $(\mathcal{F}, \varphi)$ with $\varphi(\mathcal{F}) = T$ induces a Poisson-NBBRW $(\mathcal{F}_a, \varphi_a)$ such that $\varphi_a(\mathcal{F}_a) = T_a$. This correspondence is $b_0!$ to 1, since $(\mathcal{F}, \varphi)$ is determined by the set of $(\mathcal{F}_a, \varphi_a)$, up to permutations of the branches of $\mathcal{F}$ at its root. This proves $\nu(T) = b_0! \prod_{a=1}^{b_0} \nu(T_a)$.

Now we complete the proof of the lemma for LITs. We rearrange \[3.7.22\] and use \[3.7.19\] to obtain

$$t_n^{(t)}(0) = 2d(2d-1)^{n-1} e^{(n+1)} \sum_{T : |T| = n} \sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} \mathbb{P}(\mathcal{F}, \varphi)$$

$$= e^{(n+1)} \sum_{T : |T| = n} \sum_{(\mathcal{F}, \varphi) : \varphi(\mathcal{F}) = T} \mathbb{P}(\mathcal{F}) \leq e^{(n+1)} \sum_{(\mathcal{F}, \varphi) : |\mathcal{F}| = n} \mathbb{P}(\mathcal{F}). \quad (3.7.27)$$
As there are $2d(2d - 1)^{n-1}$ embeddings $\varphi$ we then know that

$$t_n^{(i)}(0) \leq 2d(2d - 1)^{n-1} e^{(n+1)} \sum_{|\mathcal{T}| = n} P(\mathcal{T})$$

$$= 2d(2d - 1)^{n-1} e^{(n+1)} P(|\mathcal{T}| = n). \quad (3.7.28)$$

The probability distribution of the total progeny of a Poisson branching process is given in (3.7.14). We insert this into (3.7.28) and obtain the claimed bound for LTs.

The claim for LA is obtained using the same ideas but requires some additional ideas as animals can contain loops. The main difficulty is to obtain a relation similar to (3.7.26).

We define $I^{(a)}(\mathcal{T}, \varphi)$ to be the indicator for the event that

1. no two bonds of the abstract tree $\mathcal{T}$ are mapped to the same bond in $\mathbb{Z}^d$ by $\varphi$,
2. for all $i \in \mathcal{T}$ either $\xi_i = 0$ or the following two conditions hold
   a. there exist no $j \in \mathcal{T} \setminus \{i\}$, such that $\varphi(i) = \varphi(j), \xi_j > 0$ and $\text{height}(j) = \text{height}(i)$,
   b. there exist no $j \in \mathcal{T} \setminus \{i\}$, such that $\varphi(i) = \varphi(j)$ and $\text{height}(j) < \text{height}(i)$,

where the height of a individual point $a \in \mathcal{T}$ is intrinsic distance of $a$ to the root in $\mathcal{T}$.

For a Poisson-NBBRW $(\mathcal{T}_a, \varphi_a)$ (1.) guarantees that each bond is only used once by the process and (2.) is the condition that whenever a point is visited by a multiple individual only the first of them can have offspring. For a lattice animal $A$ we write $A = \varphi(\mathcal{T})$ if the embedding $\varphi$ of $\mathcal{T}$ equals $A$. Now we proceed as for the LT by defining

$$Z_n^{(a)} = \sum_{(\mathcal{T}, \varphi):|\mathcal{T}|=n} P(\mathcal{T}, \varphi) I^{(a)}(\mathcal{T}, \varphi), \quad (3.7.29)$$

$$Q_n^{(a)}(\mathcal{T}, \varphi) = \frac{1}{Z_n^{(a)}} P(\mathcal{T}, \varphi) I^{(a)}(\mathcal{T}, \varphi). \quad (3.7.30)$$

We next prove that the number of $n$-bond lattice animals containing the origin is given by

$$t_n^{(a)}(0) = Z_n^{(a)} e^{(n+1)} 2d(2d - 1)^{n-1}. \quad (3.7.31)$$

This is analog for LT, which we exception of the proof of

$$\sum_{(\mathcal{T}, \varphi):\varphi(\mathcal{T})=A} I^{(a)}(\mathcal{T}, \varphi) \prod_{i \in \mathcal{T}} \frac{1}{\xi_i!} = 1. \quad (3.7.32)$$
To prove this we define, as for LT, $b_x$ such they correspond to the number of offspring of the Poisson branching process. In contrast to the LT, it can happen that $\varphi$ maps multiple individuals of $\mathcal{T}$ to $x$. Therefore, it is not obvious how to choose $b_x$ such that $(\xi_i)_i$ and $(b_x)_x$ are equivalent. For a lattice animal $A$, a vertex $x \in A$ and a bond in $b \in A$ we define $d_A(x, b)$ to be the intrinsic distance between $x$ and $b$, meaning the length of the shortest path in $A$ from $x$ to one endpoint of $b$. We call the distance $d_A(0, b)$ the age of $b$ and for $N \in \mathbb{N}$ denote by the generation $N$ the set of all bonds with age $N$. We define an exploration process, that we use to define $(b_x)_x$, as follows:

1.) We define $b_0$ to be the degree of 0 in $A$.

2.) We define $S$ to be the set of all vertices directly connected to the origin in $A$.

3.) We define $B \subset A$ to be the set of all bonds $A$ that do not contain the origin.

4.) Let $N$ be the age of the youngest vertex in $S$.

5.) For the unique $x$ in the $N$th generation with the lowest lexicographic order:
   i.) we define $b_x$ to be the degree of 0 in $B$;
   ii.) we update $S$ by adding the endpoints of the bonds in $B$ adjacent to $x$;
   iii.) we remove $x$ from $S$ and the bonds that contain $x$ from $B$.

6.) If $S$ is non-empty then we repeat the procedure starting in 4.) with the updated sets $S$ and $B$.

This procedure terminates when $b_x$ has been defined for all vertices of $A$. As for LT we define $\nu(A) = \#\{(\mathcal{T}, \varphi) : \varphi(\mathcal{T}) = A, \|^{(a)}(\mathcal{T}, \varphi) = 1\}$ and now show that

$$\nu(A) = \prod_{x \in T} b_x!,$$

(3.7.33)

which implies (3.7.32).

We again prove (3.7.33) by induction on the number $N$ of generations of $A$. The number of generations corresponds to the age of the oldest particle in $A$. The identity (3.7.26) clearly holds for $N = 0$. Our induction hypothesis is that (3.7.26) holds if there are $N - 1$ or fewer generations.

For a LT the removal of the $b_0$ bonds adjacent to the origin splits the tree into $b_0$ unique, non-intersecting subtrees $T_1, \ldots, T_{b_0}$. Due to possible double connections in a LA the removal of the bonds adjacent to the origin does not automatically split the animal into $b_0$ uniquely defined subanimals $A_a$. We will now create a unique split of the animal $A$, that is coherent with the exploration process we used to define $(b_x)_x$.

Therefore, we first remove some bonds to create a lattice tree $T \subset A$. Then we split the $T$ into $b_0$ subtrees as done above. In the last step we add each of the removed bonds subtrees to create a unique decomposition $A_1, \ldots, A_{b_0}$. This procedure is visualized in Figure 3.3. We fix a lattice animal $A$. 
Figure 3.3: The first picture shows how we orient the bonds of $A$ to create $\tilde{A}$. In the second picture we see the lattice tree create by the removal of the bonds. The coordinates of the vertices that we compare in step 2.) of the algorithm are indicated. The last picture shows the result of the deconstruction of $\tilde{A}$.

1.) We orient all bonds of $A$ away from the origin in the intrinsic sense, see Figure 3.3 and denote by $\tilde{A}$ the created LA of oriented bonds.

2.) For all $x \in \mathbb{Z}^d$ which have multiple ingoing bonds we remove all ingoing bonds from $\tilde{A}$ with the exception of the ingoing bond $(b_i = (b_i, \tilde{b}_i = x))$ that contains the starting point $\tilde{b}_i$ with the lowest lexicographic order.

3.) We denote the resulting animal by $\tilde{T}$ and the set of all removed oriented bonds by $\mathcal{S}$.

4.) We know that $\tilde{A} = \tilde{T} \cup \mathcal{S}$ and that $\tilde{T}$ is a lattice tree as 3.) left only one ingoing bond for every $x$. Further, we see that $\tilde{T}$ contains the same set of vertices as $A$.

5.) We delete from $\tilde{T}$ all bonds incident to the origin and denote the created lattice trees by $\tilde{T}_1, \ldots, \tilde{T}_{b_0}$.

6.) For $i = 1, \ldots, b_0$ we define

$$\tilde{A}_i = \tilde{T}_i \cup \{ b \in \mathcal{S} | b_i \in \tilde{T}_i \} \quad (3.7.34)$$

and $A_i$ to be the LA obtained by removing the orientation of the bonds of $\tilde{A}_i$ again.

We regard $A_i$ as rooted at the vertex that was directly connected to the origin in $A$. We see that for for all $i = 1, \ldots, b_0$ and $x \in A_i$ the number $b_x$ corresponds to the number of outgoing edges of $x$ in $A_i$. We note that for all bonds $b \in \mathcal{S} \cup \tilde{A}_i$ the end-vertex $\tilde{b}$ has not outgoing bonds. This is related to the condition (2.) of the definition of $\mathbb{I}^{(a)}(\mathcal{F}, \varphi)$. 
It suffices to show that \( \nu(A) = b_0! \prod_{a=1}^{b_0} \nu(A_a) \), since each \( A_i \) has fewer at most \( N - 1 \) generations. We note that each \( (\mathcal{T}, \varphi) \) with \( \varphi(\mathcal{T}) = A \) induces a Poisson-NBBRW \( (\mathcal{T}_a, \varphi_a) \) such that \( \varphi_a(\mathcal{T}_a) = A_a \). This correspondence is \( b_0! \) to 1, since \( (\mathcal{T}, \varphi) \) is determined by the set of \( (\mathcal{T}_a, \varphi_a) \), up to permutations of the branches of \( \mathcal{T} \) at its root. This proves \( \nu(A) = b_0! \prod_{a=1}^{b_0} \nu(A_a) \).

This completes the proof of (3.7.31). Knowing that (3.7.31) holds we repeat the steps between (3.7.28) and (3.7.27) for LA and obtain the claimed bound on \( t^{(a)}_n(0) \), which completes the proof of Lemma 3.7.1.

**Continuity of \( \tilde{\alpha}_z \).** We choose \( \tilde{\alpha}_z = zg_z \). By Abel’s theorem the one-point function is continuous on \((z_l, z_c)\). Thus, \( \tilde{\alpha}_z \) clearly is also continuous in \( z \) on \((z_l, z_c)\).

**Growth of the two-point function.** As a generating function of a non-negative sequence (see (1.4.2)) the two-point function is clearly non-decreasing in the parameter \( z \). First, we prove the bound on the derivative (3.2.17) only for lattice trees. We know that a lattice tree \( T \) with \(|T|\) edges contains \(|T| + 1\) vertices. We use this property to compute for \( x \neq 0 \)

\[
\sum_{T \ni x} \frac{d}{dz} z^{|T|} = \sum_{T \ni x} |T|z^{|T|-1} = \sum_{y \neq 0} \sum_{T \ni x, y} z^{|T|-1}.
\] (3.7.35)

As a lattice tree \( T \ni x \), \( y \) contains no loops the path from \( x \) to \( y \) is unique. We denote this path by \( b^T(x, y) \). By \( u \) we denote the last point that the paths from 0 to \( x \) and from 0 to \( y \) have in common. For \( u \neq x \) we split the walk at \( u \) into three individual trees and bound the individual tree by two-point functions. Doing this, we have to take into account that in (3.7.35) the tree \( T \) is only weighted by \( z^{|T|-1} \), so that we have to choose one bond of the tree, that does not carry the weight \( z \). We choose the first step of the path from \( u \) to \( x \) to be this bond. For \( u = x \) there exists no first step. In this case we choose the last step of the path from 0 to \( x \) to be the bond without weight \( z \). Using that \( \tilde{G}_z(0) \geq 1 \) we obtain the bound

\[
\sum_{T \ni x} \frac{d}{dz} z^{|T|} \leq \sum_{y \neq 0} \sum_{u \neq x, y} \tilde{G}_z(t)(u)2dD(v-u)\tilde{G}_z(t)(x-v)\tilde{G}_z(t)(u-y) + \sum_{y \neq 0} \sum_{v \neq v} \tilde{G}_z(t)(v)2dD(v-x)\tilde{G}_z(t)(x-y)
\leq 4d(\tilde{G}_z(t) \star D \star \tilde{G}_z(t))(x)\sum_{y} \tilde{G}_z(t)(y) = 2d\tilde{G}_z(t)(0)2d(\tilde{G}_z(t) \star D \star \tilde{G}_z(t))(x).
\] (3.7.36)

From this we conclude that

\[
\frac{d}{dz} G_z(t)(x) = \frac{1}{G_z(t)} \sum_{T \ni x} \frac{d}{dz} z^{|T|} \leq 2d(G_z(t)^2 \tilde{G}_z(t)(0)(G_z(t) \star D \star G_z(t))(x)
\] (3.7.37)
To conclude such an inequality for LAs we note that an animal with $|A|$ bonds contains at least $|A|/d$ vertices and as for the LT compute that

$$
\frac{d}{dz} G_z^{(a)}(x) = \frac{1}{g_z^{(a)}} \sum_{A \ni x} |A|z^{[A-1]} \leq \frac{d}{g_z^{(a)}} \sum_{A \ni x,v} z^{[A-1]}
\leq 2d^2 (g_z^{(a)})^2 G_z^{(a)}(0)(G_z^{(a)} \ast D \ast G_z^{(a)})(x).
$$

(3.7.38)

In the next step we prove that for all $z < z_c$ there exists $K(z) < \infty$ such that $\sum_{x \in \mathbb{Z}^d} \|x\|_2^2 G_z(x) < K(z)$. In [39] it is proven that the connectivity constants $\lambda = 1/z_c$ can be used in the following bound

$$
\frac{t_n(0)}{n+1} \leq \lambda^n \Rightarrow t_n(0) \leq \lambda^n(n+1).
$$

(3.7.39)

We use that to compute

$$
\hat{G}_z(x) = \sum_{n=\|x\|_\infty} \infty t_n(x)z^n \leq \sum_{n=\|x\|_\infty} \infty t_n(0)z^n \leq \sum_{n=\|x\|_\infty} \infty (z/z_c)^n(n+1) = \sum_{n=\|x\|_\infty} \infty (\lambda/z_c)^n\frac{-\log(n+1)}{\log(1/z_c)}.
$$

(3.7.40)

From this we conclude that $\hat{G}_z(x)$ decays exponentially, i.e. there exists $c, m(z) > 0$ such that

$$
\hat{G}_z(x) \leq \sum_{n=\|x\|_\infty} \infty (z/z_c)^n\frac{-\log(n+1)}{\log(1/z_c)} \leq ce^{m(z)\|x\|_\infty}.
$$

(3.7.41)

We use this exponential decay of $G_z$ as in (3.7.8) to conclude the existence of $K(z) < \infty$ such that $\sum_{x \in \mathbb{Z}^d} \|x\|_2^2 G_z(x) < K(z)$.

At the critical point. The two-point functions $G_z, G_z^i$ and the coefficients $\Xi_z, \Xi_z^i$, and $\Psi_k^z, \Pi_k^z$ are defined by infinite power series in $z$. As for the SAW we conclude from the assumed uniform bounds that the two-point functions and the coefficients are continuous for $z < z_c$ and left-continuous in $z = z_c$.

3.7.3 Percolation

Initial point $z_I$. For percolation we choose $z_I = 1/(2d - 1)$ to be the critical point for the NBW. Whenever $0 \rightarrow x$ then there exists a path of open bonds connecting 0 and $x$. Hence this path is a SAW we know that

$$
\tau_{z_I}(x) = G_{z_I}(x) \leq \sum_{n=0} ^\infty c_n(x)z_I^n \leq \frac{2d-2}{2d-1} C_{1/2d}(x)
$$

(3.7.42)

as for (3.7.1).
Continuity of $\tilde{a}_z$. As we choose $\tilde{a}_z = z$, the continuity of $z \mapsto \tilde{a}_z$ is obvious.

Growth of the two-point function. The two-point function $\tau_z(x)$ is defined as the probability that the origin is connected to $x$ by a path of open bonds in a percolation configuration in which we open a bond with probability $z$. As opening a bond can only increase the probability that a path of open bonds from 0 to $x$ exists, $\tau_z(x)$ is non-decreasing in $z$. The differentiability of $\tau_z(x)$ is well known for $z \in (0, z_c)$ and the bound on the derivative is obtained using Russo’s Formula ([86], Lemma 3) or [29] Theorem 2.25) and the BK-inequality. As these are standard arguments in percolation theory we will not comment on them further.

From [29], Theorem 6.1] follows that

$$\tau_z(x) = G_z(x) \leq e^{-\sigma(z)\|x\|_\infty}, \quad (3.7.43)$$

with $\sigma(z) > 0$ for every $z < z_c$. From this we conclude the existence of $K(z) < \infty$ such that $\sum_{x \in \mathbb{Z}^d} \|x\|_\infty^2 G_z(x) < K(z)$, see (3.7.8).

At the critical point. The two-point function $G_z$, $G_\iota$ and the coefficients $\Xi^{(N)}_z$, $\Xi^{(N),I}_z$, $\Psi^{(N),K}_z$, $\Pi^{(N),I}_z, \kappa$ characterize the probability of certain events. Restricted on a finite graph the functions are clearly continuous. The continuity can be obtained using a finite-volume approximation which is not trivial. In particular, the left-continuity of the functions at $z = z_c$ requires some justification. We omit the proof of this and refer the reader to [35], Appendix A.2] where such a statement is proven for the coefficient of the classical lace expansion. The extension to our setting is straightforward.

3.8 Discussion

In this chapter we have explained the analysis used to prove the infrared bound. We have performed this analysis in a generalized setting and argued in the preceding section that SAW, LT, LA and percolation satisfy most of the assumed properties. As discussed in Section 3.1 the analysis is founded on two central ideas. The first idea is that for $z_I < z_c$ the two-point function $G_{z_I}(x)$ is bounded by the critical NBW two-point function in $x$-space. Thus, we can bound the coefficients using the NBW relations. The second idea is the application of the Bootstrap Lemma (Lemma 3.3.4) on the functions $f_i$ which creates a priori bounds on the coefficients. These bounds on the coefficients $\beta_\ast$ are given in Assumptions 3.2.6 and 3.5.3. Using these bounds on the coefficient we compute several quantities $K_\ast$, see Propositions 3.3.1 and 3.5.5 and use these quantities to formulate the sufficient conditions under which we can prove the infrared bound. To check whether the sufficient conditions are satisfied we need to numerically compute the bounds $\beta_\ast$ stated in Assumptions 3.2.6 and 3.5.3. In the next two chapters we explain how to compute these bounds on the coefficients.
Result for all $d \geq d_0$. The Theorems 3.3.3 and 3.5.7 state the infrared bound only in a given dimension. We obtain the infrared bound for all $d \geq d_0$ as stated in Theorem 1.7.1 as follows: First, we prove the infrared bound for one dimension $d' \geq d_0$ using the analysis of Section 3.3. Then, we use the monotonicity of bounds argument to conclude that this implies the result for all $d \geq d'$. In the last step, we use the analysis of Section 3.5 to prove the infrared bound for all dimensions $d \in [d_0, d')$ one at a time.

To the monotonicity argument: Once we derived the bounds $\beta_*, K_*$ used for the analysis of Section 3.3 it will be easy to see that these bounds are monotone decreasing in $d$. We discuss this in Section 5.5. The monotonicity of the bounds implies that the right-hand side of (3.3.9)-(3.3.11) is monotone decreasing in $d$, so that when the sufficient condition $P(\gamma, \Gamma, \cdot)$ holds in a dimension $d'$ then it will also holds for all $d \geq d'$ with the same constants $\gamma_i, \Gamma_i$ and $c_i$. As the Assumptions 3.2.1, 3.2.7 did not state any further assumption on the dimension this implies that the infrared bound (3.3.12)-(3.3.13) holds in all $d \geq d'$.

In the implementation we see that also the bounds used for the analysis of Section 3.5 have actually the same monotonicity. Proving this is not trivial. For this reason we use for $d \geq d'$ the analysis of Section 3.3 as the monotonicity of its bounds is next to trivial.

Comparison of $f_3$ and $\hat{f}_3$. We close this chapter with a comparison of the two bootstrap functions $f_3$ and $\hat{f}_3$. We defined $G_z(x; k) = G_z(x) [1 - \cos(k \cdot x)]$ and will now review the bound on sup$_{x \in \mathbb{Z}^d} G_z(x; k)$ which is the central bound for the SAW. We use $f_3(z) \leq \Gamma_3$ to obtain the following bound:

$$
\sup_{x \in \mathbb{Z}^d} G_z(x; k) = \sup_{x \in \mathbb{Z}^d} \int_{(-\pi, \pi)} (-\frac{1}{2} \Delta_k \hat{G}_z(l)) e^{i \cdot x} \frac{d^d k}{(2\pi)^d} \leq \Gamma_3 \int_{(-\pi, \pi)} \hat{U}(k,l) \frac{d^d k}{(2\pi)^d}
$$

$$
= \Gamma_3 [1 - \hat{D}(k)] \int_{(-\pi, \pi)} (c_1 + c_2 \hat{C}(l)) [\hat{C}(l-k) + \hat{C}(l+k)] \frac{d^d k}{(2\pi)^d}
$$

$$
+ c_4 \Gamma_3 [1 - \hat{D}(k)] \int_{(-\pi, \pi)} \hat{C}(l-k) \hat{C}(l+k) \frac{d^d k}{(2\pi)^d}.
$$

(3.8.1)

For this we use Cauchy-Schwarz as follows

$$
\int_{(-\pi, \pi)} \hat{C}(l) \hat{C}(l-k) \frac{d^d k}{(2\pi)^d} \leq \left( \int_{(-\pi, \pi)} \hat{C}^2(l) \frac{d^d k}{(2\pi)^d} \right)^{1/2} \left( \int_{(-\pi, \pi)} \hat{C}^2(l-k) \frac{d^d k}{(2\pi)^d} \right)^{1/2}
$$

$$
= \int_{(-\pi, \pi)} \hat{C}^2(l) \frac{d^d k}{(2\pi)^d} = I_{2,0}(0),
$$

(3.8.2)
We can compute
\[ \sup_{x \in \mathbb{Z}^d} G_z(x; k) \leq [1 - \hat{\Delta}(k)] \Gamma_3 \left( 2c_1 I_{1,0}(0) + 2c_2 I_{2,0}(0) + c_4 I_{2,0}(0) \right). \]  

(3.8.3)

We choose \( c_i \) close to the SRW value: \( c_1 = 0, c_2 = 0.5, c_3 = 0 \) and \( c_4 = 4 \) and we know that \( I_{2,0}(0) = 1 + O(d^{-1}) \). Thus, using \( f_3 \) we expect the following bound:
\[ \sup_{x \in \mathbb{Z}^d} G_z(x; k) \leq [1 - \hat{\Delta}(k)] 5 \Gamma_3 (1 + O(d^{-1})). \]  

(3.8.4)

Using \( \tilde{f}_3 \) we obtain directly a bound on \( \sup_{x \in \mathbb{Z}^d} G_z(k; x) \). As derived in Section 3.6.2 the following bound holds for \( z = z_I \), see (3.8.6):
\[ \sup_{x \in \mathbb{Z}^d} G_z(x; k) \leq [1 - \hat{\Delta}(k)] \frac{2d - 2}{2d - 1} \left( I_{2,2}(0) + \frac{1}{d} I_{3,1}(0) \right). \]  

(3.8.5)

We can compute \( I_{2,2}(0) \) and \( I_{3,1}(0) \) explicitly, see Section 5.2 and they are both of the order \( O(1/d) \). Thus, the bound is of the form
\[ \sup_{x \in \mathbb{Z}^d} G_z(x; k) \leq [1 - \hat{\Delta}(k)] O(1/d). \]  

(3.8.6)

The computation in Sections 3.6.3, 3.6.4 are designed to show that \( \tilde{f}_3 \) obeys for \( z \in (z_I, z_c) \) a similar bound. Thus, the difference between the bounds (3.8.3) and (3.8.6) is clearly visible. While \( f_3 \) produces a bound with a factor 5, \( \tilde{f}_3 \) creates a bound with a factor \( O(1/d) \). For all weighed diagrams we find a similar difference of the bounds create by the analyses of Section 3.3 and Section 3.5.

**Bounds for closed weighted.** We compensated this poor performance of the bound \( f_3 \) for \( \sup_{x \in \mathbb{Z}^d} G_z(x; k) \) and \( \sup_{x \in \mathbb{Z}^d \setminus \{0\}} G_z(\cdot; k) \ast G_z(x) \) by defining \( \tilde{f}_3 \) as containing two bounds. The first bound states that the absolute value of \( -\frac{1}{2} \Delta_k \hat{G}_z(l) \) is bounded by \( \hat{\Delta}(k, l) \) and the bound that \( \hat{W}(k, l) \) is an upper bound for \( -\frac{1}{2} \Delta_k \hat{G}_z(l) \). Indeed, we expect the negative part of \( -\frac{1}{2} \Delta_k \hat{G}_z(l) \) to make only a minor contribution to the integral in (3.8.1). For closed bubbles and triangles like:
\[ (G_z(\cdot; k) \ast G_z \ast G_{2,z})(0), \]  

(3.8.7)

we do not need to take the absolute value in the integral to obtain a bound. Therefore, the second bound in \( f_3 \) allows us to obtain better bounds for closed weighted bubble and triangle, which are the dominant contribution to the bounds we prove in the next chapter.

We define \( \tilde{f}_{3,n,1} \) to bound open weighted diagrams. We can improve the analysis of Section 3.5 by adding an additional bootstrap function
\[ \tilde{f}_{4,n,1}(z) = \int_{(\pi, \pi)^d} (-\Delta \hat{G}_z(k)) \hat{D}^l(k) \hat{G}_z^n(k) \frac{d^d k}{(2\pi)^d}, \]  

(3.8.8)

which only produces bounds on closed diagrams. Since all arguments of Section 3.6 also hold for \( x = 0 \) the function \( \tilde{f}_{4,n,1}(z) \) can clearly be used as a bootstrap function.
Chapter 4
Diagrammatic Bounds

In this chapter we bound the non-backtracking lace-expansion coefficients that we have derived in Chapter 2. We begin with an explanation of the bounding procedure. Then we derive the bounds for the different models. Several steps of the bounding procedure are very similar for all the four models considered in this thesis. Therefore, we will give the complete proof only for SAW. In Section 3.7 we have already shown that we can bound $\Psi^\kappa_z$ by $\Xi_z$ and $\Pi^\kappa_z$ by $\Xi_z^\kappa$. Therefore, we only need to discuss bounds on $\Xi_z$ and $\Xi_z^\kappa$ in this chapter.

4.1 The structure of the bounding procedure

For each model the lace-expansion coefficients represent intertwined combinations of intersections. For self-avoiding walk (SAW) these are self-intersections of the non-backtracking walk (NBW). For lattice trees (LT) and lattice animals (LA) ribs/sausages intersect. In the percolation coefficients the intersections are created by intersections of the nested clusters $\tilde{C}_i$ and $\tilde{C}_{i+1}$.

The paths between points of intersection can be bounded by the two-point functions $G_z$ and visualized in terms of diagrams. Therefore, we refer to the two-point functions also as lines and to combinations of two-point functions as bubbles, triangles and squares.

The diagrams are very similar to the diagrams used in the classical lace expansion. The difference is that in the diagrams of NoBLE all loops in the diagram have a length of at least four. To make use of this information, we introduce repulsive diagrams. These are diagrams consisting of multiple connections that are pairwise avoiding, where each of the connections has at least a certain length.
For all four models the bounding procedure has the following four steps:

1. We define repulsive diagrams and use them to define building blocks.

2. We combine these building blocks to bounding diagrams and prove that these bounding diagrams give pointwise bounds for $\Xi^{(N)}_z(x)$ and $\Xi^{(N),\iota}_z(x)$.

3. We bound $\hat{\Xi}^{(N)}_z(0)$ and $\hat{\Xi}^{(N),\iota}_z(0)$ by bounding the bounding diagrams by products of repulsive diagrams.

4. Then we bound $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$ and similar differences. We call these diagrams weighted as we add the weight $[1 - \cos(k \cdot x)]$ to the diagram of the NoBLE. We use the $x$-wise bounds obtained in the second step and apply Lemma 3.4.1 to split the weight $[1 - \cos(k \cdot x)]$, such that each individual weight corresponds to the displacement of a line of the bounding diagram. Finally, we bound the created diagrams by bounding them by products over repulsive diagrams.

As the bounds are given in terms of products of repulsive diagram we refer to the bounds as diagrammatic bounds.

When creating the bounding structures we introduce constraints on the length of lines that are shared by two loops and make use of this information when bounding the coefficients. These constraints on the lengths of shared lines are the reason that most bounds are stated in terms of matrices.

### 4.2 Self-avoiding walk

#### 4.2.1 Repulsive diagrams

When bounding the diagrams we want to use the information that a loop consists of at least four steps. Therefore, we will introduce constraints on the length of a path. For $m \geq 0$ we define the modified two-point functions:

$$ G_{m,z}(x) = \sum_{n=m}^{\infty} c_n(x) z^n \quad \text{and} \quad G'_{m,z}(x) = \sum_{n=m}^{\infty} c_n'(x) z^n. \quad (4.2.1) $$

We note that for $m = 0$ these functions coincide with $G_z(x)$ and $G'_z(x)$ respectively, and that, for $m \geq 1$,

$$ G'_{m,z}(x) \leq G_{m,z}(x) \leq 2 dz (D \star G_{m-1,z})(x) \leq (2 dz)^m (D^m \star G_z)(x). \quad (4.2.2) $$

In the following we define the repulsive bubble to describe the combinations of two mutually avoiding SAWs. For $x, y \in \mathbb{Z}^d$ we define $\mathcal{W}^{\text{SAW}}(x, y)$ to be the set of all SAWs that start in $\omega_0 = x$ and end in $\omega_{|\omega|} = y$. In the same way, we define $\mathcal{W}^{\text{NBW}}(x, y)$
as the set of all NBWs that start in $\omega_0 = x$ and end in $\omega_{|\omega|} = y$. For $i = 1, 2$ let $m_i \in \mathbb{N}$ and $j_i \in \{m_i, m_i \}$, we define the repulsive bubble by

$$B_{j_1, j_2}(y, x) = \sum_{\omega^1 \in \mathbb{W}^{SAW}(0, y)} \sum_{\omega^2 \in \mathbb{W}^{SAW}(y, x)} \left( \delta_{0, x} \mathbb{I}_{\omega^1 \cap \omega^2 = \{y, 0\}} + (1 - \delta_{0, x}) \mathbb{I}_{\omega^1 \cap \omega^2 = \{y\}} \right)$$

\[\times \prod_{i=1}^{2} \left( \mathbb{I}_{j_i = m_i} \mathbb{I}_{|\omega_i| = m_i} + \mathbb{I}_{j_i = m_i} \mathbb{I}_{|\omega_i| \geq m_i} \right) \] \hspace{1cm} (4.2.3)

and

$$B_{j_1, j_2}(x) = \sum_{y \in \mathbb{Z}^d} B_{j_1, j_2}(y, x). \hspace{1cm} (4.2.4)$$

### 4.2.2 The diagrammatic bounds

As discussed in Section 3.7 we only require bounds on $\Xi_z$ and $\Xi^l_z$ for the NoBLE. The coefficient is trivial for the SAW:

$$\Xi^{(N)}_z = \delta_{0, x} \delta_{0, N}. \hspace{1cm} (4.2.5)$$

By its definitions in (2.1.15) or (2.2.15) we know that $\Xi^{(0), l}_z(x) = \delta_{x, e_i} G_{3, z}(e_i)$ which implies that:

**Lemma 4.2.1** (Diagrammatic estimates for $N = 0$). For $z < z_c$ and $k \in (-\pi, \pi)^d$, the following holds:

\[\sum_{x \in \mathbb{Z}^d} \Xi^{(0), l}_z(x) = G_{3, z}(e_1), \hspace{1cm} (4.2.6)\]

\[\sum_{i} \sum_{x \in \mathbb{Z}^d} \Xi^{(0), l}_z(x) [1 - \cos(k \cdot x)] \leq 2d G_{3, z}(e_1) [1 - \hat{D}(k)], \hspace{1cm} (4.2.7)\]

\[\sum_{i} \sum_{x \in \mathbb{Z}^d} \Xi^{(0), l}_z(x) [1 - \cos(k \cdot (x - e_i))] = 0. \hspace{1cm} (4.2.8)\]

As $\Xi^{(l), l}_z$ gives a substantial contribution to $\Xi^l_z$ we create detailed bounds for this coefficient.

**Lemma 4.2.2** (Diagrammatic estimates for $N = 1$). For $z < z_c$ and all $i$ the following holds:

\[\sum_{x} \Xi^{(1), l}_z(x) \leq 2G_{3, z}(e_1)^2 + (2d - 2) z G_{4, z}^1(e_1 + e_2) (z^2 + G_{2, z}^1(e_1 + e_2)) + z G_{4, z}^1(2e_1)^2 + \left( \sup_{x \neq 0} G_{2, z}(x) \right) \mathcal{R}_{2, 2}(e_1). \hspace{1cm} (4.2.9)\]
Let \( k \in (-\pi, \pi)^d \), then
\[
\sum_{\mathbf{x} \in \mathbb{Z}^d} \Xi_z^{(1), d}(\mathbf{x})[1 - \cos(k \cdot (\mathbf{x} - e_i))]
\leq \frac{1}{z} \sum_{\mathbf{x} \in \mathbb{Z}^d} [1 - \cos(k \cdot \mathbf{x})] G_z(x) G_{2,z}(x)^2 + 2d [1 - \hat{D}(k)] G_{3,z}(e_1)^2. \tag{4.2.10}
\]

Consequently,
\[
\sum_{\mathbf{x} \in \mathbb{Z}^d} \Xi_z^{(1), d}(\mathbf{x})[1 - \cos(k \cdot \mathbf{x})]
\leq 2 \sum_{\mathbf{x} \in \mathbb{Z}^d} \Xi_z^{(1), d}(\mathbf{x})[1 - \cos(k \cdot (\mathbf{x} - e_i))] + 4d \sum_{\mathbf{x}} \Xi_z^{(1), d}(\mathbf{x})[1 - \hat{D}(k)]. \tag{4.2.11}
\]

To create a bound for \( N \geq 2 \) we decompose the coefficients as indicated in Figure 4.1. In order to be able to use the property that all loops have length four we split between the case that a line shared by two loops consists of either exactly one step or of more than one. As in the classical lace expansion these lines can not be trivial. In Figure 4.1 we label the lengths of these lines by \( \theta_i \). We introduce the following notation to state the bounds for \( N \geq 2 \). Let
\[
\mathbf{\tilde{w}}_1 = \left( G_{3,z}(e_i), \mathcal{B}_{0,2}(e_i) \right), \tag{4.2.12}
\]
\[
\mathbf{\tilde{w}}_2 = \left( G_{3,z}(e_i), \sup_{\mathbf{x} \in \mathbb{Z}^d \setminus \{0\}} G_{2,z}(\mathbf{x}) \right), \tag{4.2.13}
\]
\[
\mathbf{\tilde{w}}_3 = \left( 2d z G_{3,z}(e_i), \mathcal{B}_{2,2}(0) \right). \tag{4.2.14}
\]

The vector \( \mathbf{\tilde{w}}_1 \) bounds the contribution of the first bubble that starts in the origin and ends at \( e_i \). The entry \((\mathbf{\tilde{w}}_1)_1\) bounds the case that \( \theta_1 = 1 \), so that the diagonal line consists of exactly one step. As we sum over the last but one point of the connection we can bound this bubble by the two point function \( G_{3,z}(e_i) \). The entry \((\mathbf{\tilde{w}}_1)_2 = \mathcal{B}_{0,2}(e_i) \) is a bound for the case that the diagonal line consists of more than two steps. The vector \( \mathbf{\tilde{w}}_2 \) describes a single line as given at the end of the left diagram in Figure 4.1. The vector \( \mathbf{\tilde{w}}_3 \) is a bound on the final closed loop, as
draw at the end of the right diagram in Figure 4.1. We capture contributions of the intermediate pieces by

\[ B = \left( \begin{array}{cc} G_{3,z}(e_i) & \mathcal{B}_{2,0}(e_i) \\ \sup_{x \in \mathbb{Z}^d \setminus \{0\}} G_{2,z}(x) & \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \mathcal{B}_{2,0}(x) \end{array} \right), \]  

(4.2.15)

\[ \tilde{B} = \left( \begin{array}{cc} G_{3,z}(e_i) & \sup_{x \in \mathbb{Z}^d \setminus \{0\}} G_{1,z}(x) \\ \mathcal{B}_{2,0}(e_i) & \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \mathcal{B}_{2,0}(x) \end{array} \right). \]  

(4.2.16)

The term \((\mathbf{B})_{\theta_1,\theta_2}\) describes the intermediate open bubbles, where the right line has length \(\theta_2\) and the left line, that does not contribute to the bubble, has length \(\theta_1\). For example, \((\mathbf{B})_{1,2} = \mathcal{B}_{2,0}(e_i)\) bounds bubbles where one line has length at least 2 = \(\theta_1\). The left line has length 1 = \(\theta_1\) and does not contribute to the bubble. However, we know that the bubble connects the starting point of the bubble with a direct neighbor of the starting point. For the bubbles in \(\tilde{B}\) the left line is counted and the right line does not contribute to the bubble.

As we have to consider all possible combinations of the lengths \(\theta_i\) our bounds are given as product of matrices:

**Proposition 4.2.3** (Diagrammatic estimates for \(N \geq 2\)). For \(N \geq 2\) the following holds:

\[ \sum_{x \in \mathbb{Z}^d} \Xi^{(N),l}_z(x) \leq \bar{w}_1 \mathbf{B}^{N-1} \bar{w}_2. \]  

(4.2.17)

Let \(\bar{\mathbf{I}} = (1,1), Z(k) = 2d \sup_{x \in \mathbb{Z}^d} G_{0,z}(x)[1 - \cos(k \cdot x)].\) Then,

\[ \sum_{i} \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot (x - e_i))] \Xi^{(N),t}_z(x) \]  

(4.2.18)

and

\[ \sum_{i} \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Xi^{(N),i,K}_z(x) \]  

(4.2.19)

We prove Lemmas 4.2.1, 4.2.2 in Section 4.2.3 and Proposition 4.2.3 in Section 4.2.4.
4.2.3 Proof of the bounds for \( N = 0, 1 \)

Proof of Lemma 4.2.1. The stated bounds follow trivially from the definition of \( \Xi_{\bar{z}}^{(0),t} \) in (2.2.19), see also (2.1.15). \( \square \)

Proof of Lemma 4.2.2. We recall the characterization of \( \xi_{\bar{z}}^{(1),t}(x) \) in (2.2.19) and see that all laces \( L \in \mathcal{L}^{(2)}[0,|\omega|] \) are of the form \( L = \{0,t,|\omega|\} \) with \( 4 \leq t \leq |\omega| - 2 \) and \( s < t \). Thus,

\[
\Xi_{\bar{z}}^{(1),t}(x) = \sum_{\omega \in \mathcal{W}_{\text{NBW}}(e_{t,0})} \sum_{t=4}^{t-2} \delta_{\omega_1,0} \delta_{\omega_t,e_t} \delta_{\omega_s,x} \\
\times \prod_{s',t' \in \mathcal{E}([0,t,|\omega|])} (1 + \mathcal{U}_{s',t'}) z^{[\omega]-1} \quad (4.2.20)
\]

We note that \( \Xi_{\bar{z}}^{(1),t}(e_t) = 0 \) and that \( x = 0 \) contributes to (4.2.20) only for \( s = 1 \). We extract the special cases \( s = 1, 2, t - 1 \) to prove the claimed bound:

- When \( s = 1 \) then \( x = 0 \). First, the walk goes from 0 to \( \omega_t = e_t \) in at least three steps. Then, it goes back to the origin without intersecting the former piece of the walk. As \( t \leq |\omega| - 2 \), the last piece consists of more than one step. We bound contribution for \( s = 1 \) by \( G_{3,\bar{z}}(e_t)^2 \).

- \( s = 2 \) and \( x \neq -e_t \). After the first step from 0 to \( x \) we go to \( e_t \), to then return to \( x \). If the walk goes from \( x \) to \( e_t \) in only two steps, the path back would need at least four steps. If it goes to \( e_t \) in more than two steps, the walk could go directly back to \( x \). Keeping in mind that there are \( (2d-2) \) possible \( x \) with \( \|x\| = 1 \) and \( x \notin \{-e_t, e_t\} \), we obtain the following bound for this case:

\[
(2d - 2)z \left[ z^2 G_{4,\bar{z}}^1(e_t + e_\rho) + G_{4,\bar{z}}^1(e_t + e_\rho) G_{2,\bar{z}}^1(e_t + e_\rho) \right]. \quad (4.2.21)
\]

- \( s = 2 \) and \( x = -e_t \). The walk goes to \( -e_t \) in one step. Then the walk goes to \( e_t \) without using the origin. In the last piece, the walk goes back from \( e_t \) to \( -e_t \), without intersecting the former piece and without using the origin. As the walks from \( -e_t \) to \( e_t \) and back need to avoid the origin, both consist of at least four steps and can be bounded by \( z G_{4,\bar{z}}^1(2e_t)^2 \).

- \( s = t - 1 \). The walk goes to a point \( x \), then goes directly to \( e_t \), and returns to \( x \) in at least 3 steps. Summing over all possible \( x \) we bound the first part of the walk by \( G_{3,\bar{z}}(e_t) \). The piece from \( e_t \) back to \( x \) is bounded in the same way. Thus, we bound the contribution of \( s = t - 1 \) by \( G_{3,\bar{z}}(e_t)^2 \).

- otherwise. We already considered the case that \( s = 0, s = 1 \) and \( s = t - 1 \). We simply ignore the non-avoidance constraints between the first loop that goes
from 0 to $x$ to 0, and the last part that goes from 0 to $x$ and obtain the bound:

$$
\sum_{x \in \mathbb{Z}^d} \sum_{\omega \in \mathcal{W}_{NBW}(e_i, x)} \sum_{t=4}^{[|\omega|-2]} \sum_{s=3}^{t-2} \delta_{\omega_t,0} \delta_{\omega_t, \epsilon_i} \delta_{\omega_s, x} \prod_{s' \in s'(\{0t,s|\omega]\})} (1 + \mathcal{U}_{s', t'}) z^{[|\omega|-1} \leq \sum_{x \in \mathbb{Z}^d} G_{2,z}(x) G_{2,z}(e_i - x) G_{2,z}(x - e_i) \leq \left( \sup_{x \in \mathbb{Z}^d} G_{2,z}(x) \right) \mathcal{B}_{2,z}(e_i) \tag{4.2.22}
$$

This proves [4.2.9]. To prove [4.2.10] we add the factor $[1 - \cos(k \cdot (x - e_i))]$ to [4.2.20]. We bound the contribution of $s = 1$ as follows:

$$
\sum_{i} \sum_{\omega \in \mathcal{W}_{NBW}(e, 0)} \sum_{t=4}^{[|\omega|-2]} \sum_{s=3}^{t-2} \delta_{\omega_t,0} \delta_{\omega_t, \epsilon_i} \prod_{s' \in s'(\{0t,s|\omega]\})} (1 + \mathcal{U}_{s', t'}) z^{[|\omega|-1} [1 - \cos(k \cdot (-e_i))] \leq \sum_{i} G_{3,z}(e_i) G_{3,z}(-e_i) \sum_{i} [1 - \cos(k_i)] = 2d [1 - \hat{D}(k)] G_{3,z}(e_i)^2 \tag{4.2.23}
$$

For $s \geq 2$ we shift the sum from $x$ to $x - e_t$, and obtain the contribution:

$$
\sum_{i,x} \sum_{\omega \in \mathcal{W}_{NBW}(0, x - e_i)} \sum_{t=4}^{[|\omega|-2]} \sum_{s=2}^{t-1} \delta_{\omega_t, -e_i} \delta_{\omega_t, 0} \delta_{\omega_s, x} \prod_{s' \in s'(\{0t,s|\omega]\})} (1 + \mathcal{U}_{s', t'}) z^{[|\omega|-1} [1 - \cos(k \cdot x)] \leq \frac{1}{2} \sum_{x \in \mathbb{Z}^d} G_{2,z}(x)^2 G_{1,z}(x) [1 - \cos(k \cdot x)] \tag{4.2.24}
$$

which proves [4.2.10]. To prove [4.2.11] we use that $ab \leq (a^2 + b^2)/2$ and $\sin^2(a) \leq 2[1 - \cos(a)]$ for all $a, b \in \mathbb{R}$ to obtain

$$
[1 - \cos(k \cdot x)] \leq [1 - \cos(k \cdot (x - e_i))] + [1 - \cos(k_i)] + \sin(k \cdot (x - e_i)) \sin(-k_i) \leq 2[1 - \cos(k \cdot (x - e_i))] + 2[1 - \cos(k_i)].
$$

Applying this relation to the left-hand side of [4.2.11] proves [4.2.11], thus completing the proof.  

\[\square\]

### 4.2.4 Proof of the bounds for $N \geq 2$

To prove Proposition 4.2.3 we adapt the technique of Hara and Slade shown in [92 Chapter 4], where a similar statement for the classical lace expansion is proven. While some parts are copied even verbatim from [92], others have been significantly changed.

**The bounding diagrams**

In Figure 4.2 we show how we split a walk counted in $\Xi^{[N]}(x)$. We label the point that is visited at the $s_i$th and the $t_i$th step by $x_i$. We split depending on whether the
walk takes one or more than one step to go from a $x_{i+1}$ to $x_i$. We describe the first part of the diagram in Figure 4.2 by

$$
\begin{align*}
P^{1,1}_z(x_1, x_2) &:= \delta_{x_1, e} B_{2,1}(x_2, x_1), \\
P^{1,2}_z(x_1, x_2) &:= \delta_{x_1, e} B_{0,2}(x_2, x_1), \\
P^{1}_z(x_1, x_2) &:= P^{1,1}_z(x_1, x_2) + P^{1,2}_z(x_1, x_2),
\end{align*}
$$

(4.2.25)

where the repulsive bubble $B$ was defined in (4.2.3). We define the intermediate pieces by:

$$
\begin{align*}
A^1_z(x_{r-1}, x_r, x_{r+1}) &:= (1 - \delta_{x_{r-1}, x_r}) B_{1,1}(x_{r+1} - x_{r-1}, x_r - x_{r-1}), \\
A^2_z(x_{r-1}, x_r, x_{r+1}) &:= (1 - \delta_{x_{r-1}, x_r}) B_{0,2}(x_{r+1} - x_{r-1}, x_r - x_{r-1}), \\
A_z(x_{r-1}, x_r, x_{r+1}) &:= A^1_z(x_{r-1}, x_r, x_{r+1}) + A^2_z(x_{r-1}, x_r, x_{r+1}).
\end{align*}
$$

(4.2.26)

For $N \geq 2$ and $r = 1, 2$ we combine these pieces by

$$
\begin{align*}
P^{(N), r}_z(x_N, x_{N+1}) &= \sum_{x_{N-1}} P^{(N-1), r}_z(x_{N-1}, x_N) A_z(x_{N-1}, x_N, x_{N+1}) , \\
P^{(N)}_z(x_N, x_{N+1}) &= P^{(N), 1}_z(x_N, x_{N+1}) + P^{(N), 2}_z(x_N, x_{N+1}).
\end{align*}
$$

(4.2.27, 4.2.28)

Proof of an $x$-space bound.

**Lemma 4.2.4.** Let $N \geq 2, 0 \leq z \leq z_c$. Then,

$$
\Xi^{(N), l}_z(x) \leq \sum_{x_{N-1}} P^{(N), l}_z(x_{N-1}, x) G_{2, z}(x - x_{N-1}).
$$

(4.2.29)

**Proof.** Let $N \geq 1$ and $n \geq 0$. For a lace $L = \{s_1 t_1, \ldots, s_N t_N\} \in \mathcal{L}^{(N)}[0, n]$ we define $t_{N-1}(L) = t_{N-1}$, and $t_0 = 0$ for $N = 1$ respectively. Further, we define

$$
J_y^{(N)}[0, n] = \sum_{L \in \mathcal{L}^{(N)}[0, n]} \sum_{s_{N+1} = t_{N-1}(L)}^{n-1} \delta_{\omega s_{N+1}, y} \prod_{s \in L} (-U_{s,t}) \prod_{s' \in \mathcal{E}(L)} (1 + U_{s', t'}).
$$

(4.2.30)
We begin by proving that for all $n$-step NBW $\omega$:

$$0 \leq (-1)^N j^{(N)}_x[0, n] \leq \sum_{m=0}^{n-2} j^{(N-1)}_{\omega_n}[0, m] K[m, n]. \quad (4.2.31)$$

The first inequality is immediate, and the second becomes clear once we write the right hand side as

$$\sum_{m=0}^{n-2} j^{(N-1)}_{\omega_n}[0, m] K[m, n] = \sum_{m=0}^{n-2} \sum_{L \in L^{(N-1)}[0, m]} \sum_{s_N=t_N-2(L)}^{m-1} \delta_{\omega_{s_N}, \omega_n} \prod_{s \in L} (-U_{s, t}) \times \prod_{s' \in \mathcal{E}(L)} (1 + U_{s', t'}) \prod_{s''=m}^{n-4} \prod_{t''=s+4}^{n} (1 + U_{t''}). \quad (4.2.32)$$

We compare a lace $L' \in L^{(N)}[0, n]$ in (2.2.18), with $L \cup \{s_N, n\}$ in (4.2.32). The set of compatible edges $\mathcal{E}(L)$ and the set of edges between $n$ and $m$ are disjoint subsets of $\mathcal{E}(L')$. This means that (4.2.31) has weaker self-avoidance constraints than (2.2.18). Thereby, (4.2.32) holds, as relaxing self-avoidance constraints can only increase the quantity. We define

$$\Xi^{[N], l}_{z}(x, y) = \sum_{\omega \in \mathcal{W}_{NBW}(e_1, x)} \delta_{\omega_{1}, 0} j^{[N]}_{y}[0, |\omega|](\omega) z^{[\omega]-1}. \quad (4.2.33)$$

From (2.2.16) and (4.2.31) it follows that

$$\Xi^{[N], l}_{z}(x) \leq \sum_{y \in \mathbb{Z}^d \setminus \{x\}} \Xi^{[N], l}_{z}(y, x) G_{2, z}(x - y). \quad (4.2.34)$$

We complete the proof the lemma by proving that, for $N \geq 1$,

$$\Xi^{[N], l}_{z}(x, y) \leq P^{[N], l}_{z}(x, y) \quad (4.2.35)$$

by induction on $N$. We start the induction at $N = 1$. We know that $j^{(1)}[0, |\omega|](\omega)$ is the indicator for the event that the NBW $\omega$ is a self-avoiding polygon:

$$\Xi^{(1), l}_{z}(x, y) = \sum_{\omega \in \mathcal{W}_{SBW}(e_1, x)} \delta_{\omega_{1}, 0} z^{[\omega]-1} \sum_{s_2=0}^{[\omega]-1} \delta_{\omega_{s_2}, e_1} \delta_{s_2, y} \prod_{s' \in \mathcal{E}([-1|\omega|])} (1 + U_{s'}) \prod_{\omega \in \mathcal{W}_{SAW}(e_1, e_i)} z^{[\omega]-1} \delta_{\omega_{1}, 0} \sum_{s_2=0}^{[\omega]-1} \delta_{s_2, y} = P^{(1), l}_{z}(x, y).$$

This initiates the induction hypothesis.

To advance the induction we see that for $N \geq 2$:

$$j^{[N]}_{x}[0, n] \leq \sum_{t_{N-1}=3}^{n-2} j^{(N-1)}_{\omega_n}[0, t_{N-1}] \sum_{s_{N+1}=t_{N-1}}^{n-1} \delta_{x, \omega_{s_N+1}} K[t_{N-1}, n]. \quad (4.2.36)$$
This means that
\[
\Xi^{(N),d}_n(x, y) \leq \sum_{\omega \in \mathcal{W}^{NBW}(e_1, x)} z^{\vert \omega_1 \vert - 1} \delta_{\omega_1, 0} \sum_{t_{N-1}=3}^{t_N - 1} J^{(N-1)}_{\omega_{0t}}[0, t_{N-1}]
\times \sum_{s_{N+1}=t_{N-1}}^{t_N - 1} \delta_{s_{N+1}, \omega_{0t}} K[t_{N-1}, \vert \omega \vert].
\]

Then, we split the NBW \(\omega\) at \(t_{N-1}\) into a walk \(\omega^1\) from 0 to \(x_{N-1}\) and a walk \(\omega^2\) from \(x_{N-1}\) to \(x\) to obtain
\[
\Xi^{(N),d}_n(x, y) \leq \sum_{x_{N-1} \in \mathbb{Z}^d \setminus \{x\}} \sum_{\omega^1 \in \mathcal{W}^{NBW}(e_1, x_{N-1})} \delta_{\omega^1, 0} J^{(N-1)}_x[0, \vert \omega^1 \vert] (\omega^1) z^{\vert \omega^1 \vert - 1}
\times \sum_{\omega^2 \in \mathcal{W}^{NBW}(x_{N-1}, x)} \sum_{n=0}^{\vert \omega^2 \vert - 1} \delta_{\omega^2, n} K[0, n - t_{N-1}] (\omega^2) z^{\vert \omega^2 \vert}
\leq \sum_{x_{N-1} \in \mathbb{Z}^d \setminus \{x\}} \Xi^{(N-1),d}_n(x_{N-1}, x) A_z(x_{N-1}, x, y).
\]

Applying the induction hypothesis and using (4.2.27) gives (4.2.35) and completes the proof. \(\square\)

**Bound on the absolute values.**

As the SAW is translation invariant we know that \(A_z(u, x, y) = A_z(u - x, 0, y - x)\) for all \(u, x, y \in \mathbb{Z}^d\). We decompose \(P^{(N),d}_z\) using its recursive definition in (4.2.27) and obtain the following rewrite:
\[
\sum_{x, y : y \neq 0} P^{(N),d}_z(x, x + y) = \sum_{x, y : y \neq 0} \sum_{u \in \mathbb{Z}^d \setminus \{x\}} P^{(N-1),d}_z(u, x) (A^1_z(u, x, x + y) + A^2_z(u, x, x + y))
= \sum_{u, w \in \mathbb{Z}^d \setminus \{0\}} P^{(N-1),d}_z(u, u + w) \sum_{y : y \neq 0} (A^1_z(w, 0, y) + A^2_z(w, 0, y))
= \sum_{u, y_1, y_2 \in \mathbb{Z}^d \setminus \{0\}} (P^{(1),d,1}_z(u, u + y_2) + P^{(1),d,2}_z(u, u + y_2)) \sum_{y_3 : y_3 \neq 0} \cdots \sum_{y_N : y_N \neq 0}
\times \prod_{r=3}^N (A^1_z(y_{r-1}, 0, y_r) + A^2_z(y_{r-1}, 0, y_r)).
\]

We expand this and use a sum over \(\theta \in \{1, 2\}^N\) to represent all possible combinations to obtain the following characterization of \(P^{(N),d}_z\):
\[
\sum_{x, y : y \neq 0} P^{(N),d}_z(x, x + y) = \sum_{\theta \in \{1, 2\}^N} \sum_{u \in \mathbb{Z}^d} \sum_{y_1 \in \mathbb{Z}^d \setminus \{0\}} \cdots \sum_{y_N \in \mathbb{Z}^d \setminus \{0\}} P^{(1),d,\theta}_z(u, u + y_2) \prod_{r=2}^N A^\theta_{z,r-1}(y_{r-1}, 0, y_r).
\]

We bound this \(\theta\)-wise in the following way:
Lemma 4.2.5. For $t$, $N \geq 2$, $\theta \in \{1,2\}^N$, $0 \leq z \leq z_c$. Then
\begin{equation}
\sum_{u,y_2,\ldots,y_N \in \mathbb{Z}^d \setminus \{0\}} P^{(1,4,\theta_1)}_{z,t}(u, u + y_2) \prod_{r=2}^{N} A^{\theta_{r-1}}_{z}(y_{r-1}, 0, y_r) \leq (\bar{w}_1)_{\theta_1} \prod_{n=1}^{N-1} (B)_{\theta_n, \theta_{n+1}},
\end{equation}
\begin{equation}
\sum_{u,y_2,\ldots,y_N \in \mathbb{Z}^d \setminus \{0\}} P^{(1,4,\theta_1)}_{z,t}(u, u + y_2) \prod_{r=2}^{N} A^{\theta_{r-1}}_{z}(y_{r-1}, 0, y_r) G_{z,2}(y_r) \leq (\bar{w}_1)_{\theta_1} \prod_{n=1}^{N-1} (B)_{\theta_n, \theta_{n+1}} (\bar{w}_2)_{\theta_N}.
\end{equation}

We sum over $\theta \in \{1,2\}^N$ and use Lemma 4.2.5 to obtain the bound stated in 4.2.17.

Proof. As the functions $P^{(1,4,1)}_{z,t}$ and $A^1_{z}$ have only one step between the last points, we know that the sum over $y_{r+1} \in \mathbb{Z}^d$ reduces to a sum over nearest neighbors:
\begin{equation}
\sum_{y_2 : y_2 \neq 0} P^{(1,4,1)}_{z,t}(u, u + y_2) = \sum_{\kappa} P^{(1,4,1)}_{z,t}(u, u + e_\kappa),
\end{equation}
\begin{equation}
\sum_{y_r : y_r \neq 0} A^1_{z}(y_{r-1}, 0, y_r) = \sum_{\kappa} A^1_{z}(y_{r-1}, 0, e_\kappa).
\end{equation}

We note that for $f : \mathbb{Z}^d \to \mathbb{R}_+$ and $g : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$:
\begin{equation}
\sum_{x,y \in \mathbb{Z}^d} f(x) g(x, y) \leq \sum_{x} f(x) \left( \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} g(x, y) \right).
\end{equation}

We use this $(N - 1)$ times to obtain
\begin{equation}
\sum_{u \in \mathbb{Z}^d} \sum_{\kappa \in \mathbb{Z}^d} \ldots \sum_{\kappa \in \mathbb{Z}^d} P^{(1,4,\theta_1)}_{z,t}(u, u + y_2) \prod_{r=2}^{N} A^{\theta_{r-1}}_{z}(y_{r-1}, 0, y_r) G_{z,2}(y_r)
\leq \sum_{u \in \mathbb{Z}^d} \sum_{\kappa \in \mathbb{Z}^d} P^{(1,4,\theta_1)}_{z,t}(u, u + y_2)
\times \prod_{r=2}^{N} \left[ \delta_{\theta_{r-2}, 1} \sup_{\kappa \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d \setminus \{0\}} A^{\theta_{r-1}}_{z}(e_\kappa, 0, y_r) + \delta_{\theta_{r-2}, 2} \sup_{y \in \mathbb{Z}^d \setminus \{0\}} A^{\theta_{r-1}}_{z}(y, 0, y_r) \right]
\times \left( \delta_{\theta_{N-1}, 1} \sup_{\kappa \in \mathbb{Z}^d} G_{z,2}(e_\kappa) + \delta_{\theta_{N-2}, 2} \sup_{y \in \mathbb{Z}^d \setminus \{0\}} G_{z,2}(y) \right).
\end{equation}

In the same way we see that the left-hand side of 4.2.37 is bounded by the sum of 4.2.40 and 4.2.41. We proceed by bounding the three lines individually. We begin with line 4.2.42. We see that $\sup_{y \in \mathbb{Z}^d} G_{z,2}(y_N) = (\bar{w}_2)_{2}$. The point $e_\kappa$ can only be reached in an odd number of steps, so that:
\begin{equation}
\sup_{\kappa} G_{z,2}(e_\kappa) = \sup_{\kappa} G_{3,2}(e_\kappa) = G_{3,2}(e_1) = (\bar{w}_2)_{1}.
\end{equation}
Therefore, \((4.2.42)\) equals \(\vec{w}_2\). We bound \((4.2.40)\) in the following way:

\[
\sum_{u \in \mathbb{Z}^d} \sum_{y_2 \in \mathbb{Z}^d \setminus \{0\}} P_{\vec{z}}^{(0),1,l}(u, u + y_2) = \sum_{k} P_{\vec{z}}^{(1),1,l}(e_l, e_l + e_k) = G_{3,z}(e_l) = (\vec{w}_1)_1, \tag{4.2.44}
\]

\[
\sum_{u \in \mathbb{Z}^d} \sum_{y_2 \in \mathbb{Z}^d \setminus \{0\}} P_{\vec{z}}^{(0),1,2}(u, u + y_2) = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} P_{\vec{z}}^{(0),2,l}(e_l, x) = B_{0,2}(e_l) = (\vec{w}_1)_2. \tag{4.2.45}
\]

For the elements in \((4.2.41)\) we see that:

\[
\sup_{\kappa} \sum_{y_r \in \mathbb{Z}^d \setminus \{0\}} A_{\vec{z}}^{1}(e_{\kappa}, 0, y_r) = \sum_{\rho} \sup_{\omega \in \mathcal{W}(\mathbb{Z}^d, e_{\rho}, 0)} \|\omega\| \geq 2 z^{\|\omega\|} \delta_{\omega_{\|\omega\|-1}, e_p} = G_{3,z}(e_l) = (\mathbf{B})_{11} \tag{4.2.46}
\]

\[
\sup_{\kappa} \sum_{y_r \in \mathbb{Z}^d \setminus \{0\}} A_{\vec{z}}^{2}(e_{\kappa}, 0, y_r) = \sum_{\rho} \sup_{\omega \in \mathcal{W}(\mathbb{Z}^d, e_{\rho}, 0)} \|\omega\| \geq 2 z^{\|\omega\|} \delta_{\omega_{\|\omega\|-1}, y} \sum_{s=0}^{\|\omega\|-2} \delta_{\omega_{s}, y} = B_{0,2}(e_l) = (\mathbf{B})_{12} \tag{4.2.47}
\]

and

\[
\sup_{\kappa} \sum_{y \in \mathbb{Z}^d \setminus \{0\}} A_{\vec{z}}^{1}(y, 0, y_r) = \sup_{\kappa} \sum_{\omega \in \mathcal{W}(\mathbb{Z}^d, y, 0)} \|\omega\| \geq 2 z^{\|\omega\|} \delta_{\omega_{\|\omega\|-1}, y_r} = * \sup_{y \in \mathbb{Z}^d} G_{2,z}(y) = (\mathbf{B})_{21} \tag{4.2.48}
\]

\[
\sup_{\kappa} \sum_{y \in \mathbb{Z}^d \setminus \{0\}} A_{\vec{z}}^{2}(y, 0, y_r) = \sup_{\kappa} \sum_{\omega \in \mathcal{W}(\mathbb{Z}^d, y, 0)} \|\omega\| \geq 2 z^{\|\omega\|} \sum_{s=0}^{\|\omega\|-2} \delta_{\omega_{s}, y} = \sup_{y \in \mathbb{Z}^d} B_{2,0}(y) = (\mathbf{B})_{22} \tag{4.2.49}
\]

Therefore, we know that the terms in \((4.2.40)-(4.2.42)\) satisfy the bounds stated in Lemma \(4.2.5\). \(\square\)

**Preparation for a bound on weighted diagrams.**

To prove \((4.2.18)-(4.2.19)\) we need to change the order in which we separate the diagram. We define

\[
R_{\vec{z}}^{(1),x,1}(v, u) := \delta_{x,u} B_1(v - x, 0), \tag{4.2.50}
\]

\[
R_{\vec{z}}^{(1),x,2}(v, u) := \delta_{x,u} (1 - \delta_{x,v}) B_2(x, v), \tag{4.2.51}
\]

\[
R_{\vec{z}}^{(1),x}(v, u) := R_{\vec{z}}^{(1),x,1}(v, u) + R_{\vec{z}}^{(1),x,2}(v, u), \tag{4.2.52}
\]
and, for $\theta = 1, 2,$

$$\ilde{A}_z^{1,\theta} (v, u, w) := \sum_{\omega \in \mathcal{W}_{SAW}(u, w)} \mathbb{1}_{|\omega| \geq 3-\theta} \delta_{\omega, v}, \quad (4.2.53)$$

$$\ilde{A}_z^{2,\theta} (v, u, w) := \sum_{\omega \in \mathcal{W}_{SAW}(u, w)} \mathbb{1}_{|\omega| \geq 4-\theta} \sum_{s=2}^{|\omega|-2+\theta} \delta_{\omega, v}, \quad (4.2.54)$$

$$\ilde{A}_z^{\theta} (v, u, w) := \ilde{A}_z^{1,\theta} (v, u, w) + \ilde{A}_z^{2,\theta} (v, u, w). \quad (4.2.55)$$

We combine these diagrams by recursively defining, for $N \geq 2$ and $\theta = 1, 2,$

$$R_z^{(N), x, \theta} (v, u) := \sum_w \sum_{\theta' = 1}^2 \ilde{A}_z^{\theta, \theta'} (v, u, w) R_z^{(N-1), x, \theta'} (u, w). \quad (4.2.56)$$

Figure 4.3: A bound on $\Xi_z^{(5), x_6}$ as proved in Lemma 4.2.6.

**Lemma 4.2.6.** Let $t, M, N \geq 1,$ and $x, y \in \mathbb{Z}^d.$ Then

$$\Xi_z^{(N+M), x} (y) = \sum_{u, v, w} \Xi_z^{(M), x} (u, v) G_{0, z} (w-u) R_z^{(N), y} (v, w). \quad (4.2.57)$$

**Proof.** Let $a, b, n, N, M \in \mathbb{N}$ with $N > 0, M > 0$ and $0 \leq a < b.$

For a lace $L = \{s_1 t_1, s_2 t_2, \ldots, s_N t_N\}$ with $s_i < s_{i+1}$ we define

$$\rho(L) = \begin{cases} \min \{s_2, t_1 - 2\}, & \text{for } N \geq 2, \\ t_1 - 2, & \text{for } N = 1. \end{cases} \quad (4.2.58)$$

Let

$$\tilde{J}_x^{(N)} [a, b] := \sum_{L \in \mathcal{L}^{(N)} [a, b]} \prod_{st \in L} \mathcal{W}_{st} \sum_{\Gamma \in \mathcal{L}^{(N)} [a+1]} \prod_{st' \in \Gamma} (1 + \mathcal{W}_{st'}) \prod_{i=a+1}^{\rho(L)} \delta_{\omega_i, x}. \quad (4.2.59)$$

This is closely related to $J_y^{(N)} [a, b]$ defined in 4.2.30. With $J_y^{(N)} [a, b]$ we condition that the point $y$ should be visited at a time $s \in (t_{N-1}, b).$ With $\tilde{J}_x^{(N)} [a, b]$ we introduce
the condition that a point $x$ should be visited at a time $t \in (a, t_1 - 2)$.
We recall the definition of $J^{(N+M+1)}[a, b]$ in (4.2.18), remove from a lace $L \in \mathcal{L}^{(N)}[0, n]$ the edges $s_{N+1}t_{N+1}$, and weaken the self-avoidance constraints, expressed by the compatible edges, to obtain

$$(-1)^N J^{(N+M+1)}[0, n] = \sum_{L \in \mathcal{L}^{(N+M+1)}[0, n]} \prod_{s \in L} |\mathcal{U}_{st}| \sum_{L' \in L^{-1}(L)} \prod_{s't' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'})$$

$$\leq \sum_{n} \sum_{t_N=0}^{n} \prod_{s \in L} |\mathcal{U}_{st}| \sum_{L' \in L^{-1}(L)} \prod_{s't' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'})$$

$$\times \sum_{s_{N+1}=t_{N-1}+1} \sum_{n} \sum_{t_{N+1}=\max(s_{N+4}, t_{N+2})} \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'}) \times |\mathcal{U}_{s_{N+1}t_{N+1}}| [K[t_N, s_{N+2}](1)^M f^{(M)}[s_{N+2}, n]].$$

(4.2.60)

We identify the point of intersection $\omega_{s_M} = \omega_{t_M} = v$ and obtain

$$(-1)^{N+M+1} J^{(N+M+1)}[0, n] \leq \sum_{v} \sum_{n} \sum_{t_{N+2}=t_{N}} \prod_{s \in L} |\mathcal{U}_{st}| \sum_{L' \in L^{-1}(L)} \prod_{s't' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'})$$

$$\times f^{(N)}[0, t_N] K[t_N, s_{N+2}] f^{(M)}[s_{N+2}, n].$$

(4.2.61)

We recall the definitions $\Xi^{(N)}_Z(x), \Xi^{(N)}_Z(x, y)$ in (2.2.19), (4.2.33) and see that

$$\Xi^{(M+N)}_Z(x) \leq \sum_{u, v, w} \sum_{\omega \in \mathcal{W}^{(N)}[a_1, x]} |\omega| \sum_{\omega} \delta_{\omega_{1,0}} \delta_{\omega_{1,0}} \delta_{\omega_{N+2,0}}$$

$$\times (-1)^{N+M+1} f^{(N)}[0, t_N] K[t_N, s_{N+2}] f^{(M)}[s_{N+2}, |\omega|] z^{(|\omega| - 1)}$$

$$\leq \sum_{u, v, w} P^{(N)}[w, v] G_0(x, w) \sum_{\omega \in \mathcal{W}^{(N)}[a_1, x]} |f^{(M)}[0, |\omega|] z^{(|\omega| - 1)},$$

(4.2.62)

where we use (4.2.35) to bound $\Xi^{(N)}_Z(w, v)$ by $P^{(N)}[w, v]$. We complete the proof by proving that for $\theta \in [1, 2]$: 

$$\sum_{\omega \in \mathcal{W}^{(N)}[x, \theta]} |f^{(N)}[0, |\omega|] z^{(|\omega|)} \leq R^{(N)}_Z(x, \theta)(w, v),$$

(4.2.63)

with

$$J^{(N)}_x[a, b] = \sum_{L \in \mathcal{L}^{(N)}[a, b]} \prod_{s \in L} |\mathcal{U}_{st}| \sum_{L' \in L^{-1}(L)} \prod_{s't' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'}) \delta_{\omega_{a+1}, x} \delta_{\omega_{a+1}, \rho(L)},$$

(4.2.64)

$$J^{(N)}_x[a, b] = \sum_{L \in \mathcal{L}^{(N)}[a, b]} \prod_{s \in L} |\mathcal{U}_{st}| \sum_{L' \in L^{-1}(L)} \prod_{s't' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'}) \delta_{\omega_{a+1}, x} \delta_{\omega_{a+1}, \rho(L)},$$

(4.2.65)

with $\rho(L)$ as defined in (4.2.58). We prove (4.2.63) by induction over $N$. We begin at $N = 1$. Since only $\omega$ with $\omega_0 = \omega_{|\omega|}$ give non-trivial contributions to $f^{(N)}[0, |\omega|] z^{(|\omega|)}$ we
know that
\[
\sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(1)}[0, 1, 0, |\omega|] z^{\omega} = \delta_{w, x} \sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(1)}[0, 1, 0, |\omega|] z^{\omega} \delta_{\omega, v} \mathbb{1}_{1 \leq |\omega| - 2} = \delta_{w, x} \mathcal{B}_1(0, v, 0) = R_z^{(1), x, 1}(v, w)
\]
(4.2.66)

and
\[
\sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(1), 2}[0, 1, 0, |\omega|] z^{\omega} = \delta_{w, x} \sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(1)}[0, 1, 0, |\omega|] z^{\omega} \sum_{i=2}^{|\omega| - 1} \delta_{\omega, v} = \delta_{w, x} \sum_{\omega \in \mathcal{W}^N(0, 0)} \tilde{f}^{(1)}[0, 1, 0, |\omega|] z^{\omega} \sum_{i=2}^{|\omega| - 1} \delta_{\omega, v} - \mathcal{B}_2(0, v, 0) = R_z^{(1), x, 2}(v, w).
\]
(4.2.67)

For \(N \geq 2\) we relax the self-avoidance constraints and see that
\[
|\tilde{f}^{(N), 1}_v[0, 1, 0, |\omega|]| \leq \delta_{\omega, v} \sum_{s_2=1}^{|\omega| - 1} \mathbb{1}_{|\omega| = 3} [\tilde{f}^{(N), 1}_v[s_2, |\omega|] + \tilde{f}^{(N), 1}_v[s_2, |\omega|]] \quad (4.2.68)
\]
\[
|\tilde{f}^{(N), 2}_v[0, 1, 0, |\omega|]| \leq \sum_{s_2=2}^{|\omega| - 1} \sum_{i=2}^{|\omega| - 1} \delta_{\omega, v} \mathcal{K}[0, s_2] \mathbb{1}_{|\omega| = 3} [\tilde{f}^{(N), 1}_v[s_2, |\omega|] + \tilde{f}^{(N), 1}_v[s_2, |\omega|] + \tilde{f}^{(N), 1}_v[s_2, |\omega|]] \quad (4.2.69)
\]

We sum (4.2.68) over all \(\omega \in \mathcal{W}^N(w, x)\) and drop the non-backtracking constraint of the \(s_2\)th step:
\[
\sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(N), 1}_v[0, 1, 0, |\omega|] z^{\omega} \leq \sum_{u \in \mathcal{W}^N(w, x)} \sum_{\omega \in \mathcal{W}^N(w, x)} \sum_{\omega \in \mathcal{W}^N(w, x)} \delta_{\omega, v} K[0, 1, |\omega|] |(\omega^1)| z^{\omega^1} \chi \{\mathbb{1}_{|\omega^1| = 3} \tilde{f}^{(N), 1}_v[0, 1, 0, |\omega^2|] + \tilde{f}^{(N), 2}_v[0, 1, 0, |\omega^2|] \} z^{\omega^2} \leq \sum_{u \in \mathcal{W}^N(w, x)} \tilde{A}^{(1)}(w, w, u) R^{(N), x, 1}(w, u) \quad (4.2.70)
\]

having applied the induction hypothesis (4.2.63) in the last step. In the same way, we bound
\[
\sum_{\omega \in \mathcal{W}^N(w, x)} \tilde{f}^{(N), 2}_v[0, 1, 0, |\omega|] \leq \sum_{u \in \mathcal{W}^N(w, x)} \sum_{\omega \in \mathcal{W}^N(w, x)} \sum_{i=2}^{|\omega| - 1} \delta_{\omega, v} K[0, 1, |\omega|] [\mathbb{1}_{|\omega| = 3} \tilde{f}^{(N), 1}_v[s_2, |\omega|] + \tilde{f}^{(N), 1}_v[s_2, |\omega|] + \tilde{f}^{(N), 1}_v[s_2, |\omega|]] \leq \sum_{u \in \mathcal{W}^N(w, x)} \tilde{A}^{(2)}(w, w, u) R^{(N), x, 1}(w, u) \quad (4.2.71)
\]

This completes the proof of Lemma 4.2.6. □
Bound on weighted diagrams.

Here we want to bound

$$
\sum_{i} \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Xi^{(N,i)}_z(x),
$$

$$
\sum_{i} \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot (x - e_i))] \Xi^{(N,i)}_z(x).
$$

We distribute the weight $[1 - \cos(k \cdot x)]$ by rewriting $x$, using the labeling of Figure 2.5 and then applying Lemma 3.4.1 to split the cosine. In Figure 4.4 and Figure 4.4 we give a graphic representation of these rewrites of $x$.

#### y

$$
y_1 = x_3 - e_t
$$

$$
y_t = x_{2r+3} - x_{2r+1}
$$

$$
v_1 = \begin{cases} 
  e_t & \text{N is odd} \\
  x_2 & \text{N is even} 
\end{cases}
$$

for $t = 2, \ldots, \lfloor (N + 1)/2 \rfloor$. Then

$$
\sum_{i=1}^{\lfloor N/2 \rfloor} y_i = x - e_t
$$

and

$$
\sum_{i=1}^{\lfloor N/2 \rfloor + 1} v_i = x.
$$

![Figure 4.4: The displacements $y_i$](image)

![Figure 4.5: The displacements $v_i$](image)
Using Lemma 3.4.1 and Lemmas 4.2.4-4.2.6 we see that for even $N$:

\[
\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot (x - e_i))] \Xi^{(N),d}_z(x) \leq \frac{N}{2} \sum_{M=1}^{N/2} \sum_{\theta=1}^{2} \sum_{u,v,w,x} P^{(2M-1),d}_z(u, v) [1 - \cos(k \cdot (w - u))] G_z(w - u) R_z^{(N-2M+1),x,\theta}(v, w)
\]

\[
\leq \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N}{2} \sum_{M=1}^{N/2} \sum_{\theta=1}^{2} \sum_{u,v,w,x} P^{(2M-1),d}_z(u, v) R_z^{(N-2M+1),x,\theta}(v, w),
\]

\[
\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Xi^{(N),d}_z(x) \leq \frac{N+2}{2} [1 - \cos(k \cdot e_i)] \sum_{x} \Xi^{(N),d}_z(x)
\]

\[
+ \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N+2}{2} \sum_{M=1}^{N/2} \sum_{\theta=1}^{2} \sum_{u,v,w,x} P^{(2M-1),d}_z(u, v) R_z^{(N-2M+1),x,\theta}(v, w).
\]

For odd $N$ we obtain the bounds

\[
\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot (x - e_i))] \Xi^{(N),d}_z(x) \leq \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N+1}{2} \sum_{M=1}^{(N-1)/2} \sum_{\theta=1}^{2} \sum_{u,v,w,x} P^{(2M-1),d}_z(u, v) R_z^{(N-2M+1),x,\theta}(v, w)
\]

\[
+ \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N+1}{2} \sum_{w,x,y} P^{(N-1),d}_z(w, y) \beta_{0,2}(x - w, y - x)
\]

and

\[
\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Xi^{(N),d}_z(x) \leq \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N+1}{2} \sum_{\theta=1}^{2} \sum_{x,y} R_z^{(N),x,\theta}(e_i, v)
\]

\[
+ \left( \sup_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] G_z(x) \right) \frac{N+1}{2} \sum_{M=1}^{(N-1)/2} \sum_{\theta=1}^{2} \sum_{u,v,w,x} P^{(2M-1),d}_z(u, v) R_z^{(N-2M),x,\theta}(v, w).
\]

To bound the sums in (4.2.72)-(4.2.77) we use the translation invariance of $R_z^{(N),x,\theta}$.
Diagrammatic Bounds

\[ \sum_{\theta=1}^{2} \sum_{u,v,w,x} P_{z}^{(2M-1),I}(u,v) R_{z}^{(N-2M+1),x,\theta}^{} (v,w) \]

\[ = \sum_{\theta=1}^{2} \sum_{u,v,w,x} B_{z}^{(2M-1),I}(u,v) R_{z}^{(N-2M+1),x,v,\theta}^{} (v-v,w-v) \]

\[ = \sum_{u,v} P_{z}^{(2M-1),I}(u,v) \sum_{w,x} \sum_{\theta=1}^{2} R_{z}^{(N-2M+1),x,\theta}^{} (0,w) \] (4.2.78)

We use \(4.2.37\) of Lemma 4.2.5 to obtain

\[ \sum_{u,v} P_{z}^{(N),I}(u,v) \leq \vec{w}_{3}^{T} \vec{B}^{N-1} \vec{1}. \] (4.2.79)

In the lemma below we will prove a similar bound on the sum over \(R_{z}^{(N),x,\theta}\). Combining this bound with \(4.2.79\) we obtain the bounds stated in \(4.2.18\)-\(4.2.19\).

**Lemma 4.2.7.** Let \(z < z_{c}\) and \(N \geq 1\) then

\[ \sum_{\theta=1}^{2} \sum_{w,x} R_{z}^{(N),x,\theta}^{} (0,w) \leq \vec{1}^{T} (\vec{B})^{N-1} \vec{w}_{3}. \] (4.2.80)

**Proof.** This proof is very similar to the proof of Lemma \(4.2.5\). We will show that

\[ \sum_{w,x} R_{z}^{(N),x,\theta}^{} (0,w) \leq (\vec{B})^{N-1} \vec{w}_{3}\] (4.2.81)

by induction over \(N\). Starting with \(N = 1\) we recall the definition of \(R_{z}^{(1),x,\theta}\) \(4.2.50\)-\(4.2.51\) and see that

\[ \sum_{w,x} R_{z}^{(1),x,1}(0,w) = \sum_{x} \vec{B}_{1,2}(w-x,0) = \vec{B}_{1,2}(0) = (\vec{w}_{3})_{1}, \] (4.2.82)

\[ \sum_{w,x} R_{z}^{(1),x,2}(0,w) = \vec{B}_{2,2}(0) = (\vec{w}_{3})_{2}. \] (4.2.83)

This initializes the induction and proves the lemma for \(N = 1\). For \(N \geq 2\) we use the recursive definition of \(R_{z}^{(N),x,\theta}\) and the translation invariance of \(\vec{A}\) and \(R_{z}^{(N),x,\theta}\) to obtain

\[ \sum_{u,x} R_{z}^{(N),x,\theta}^{} (0,u) = \sum_{u,w,x} \sum_{\theta'=1}^{2} \vec{A}_{z}^{} \theta,\theta' (0,u,w) R_{z}^{(N-1),x,\theta'}^{} (u,w) \] (4.2.84)

\[ = \sum_{u,w,x} \sum_{\theta'=1}^{2} \vec{A}_{z}^{} \theta,\theta' (-u,0,w-u) R_{z}^{(N-1),x-u,\theta'}^{} (0,w-u) \] (4.2.85)

\[ = \sum_{u,w,x} \sum_{\theta'=1}^{2} \vec{A}_{z}^{} \theta,\theta' (u,0,w) R_{z}^{(N-1),x,\theta'}^{} (0,w). \] (4.2.86)
The we use the inequality \(4.2.39\) to split the summands

\[
\sum_{u,x} \sum_{\theta' \in \{0\}} R_{z}^{(N-1), x, \theta'}(u, 0, w) \sum_{w, x} \bar{A}_{z}^{\theta, \theta'}(u, 0, w),
\]

where we can exclude that case that \(w = 0\) as \(R_{z}^{(N-1), x, \theta'}(0, 0) = 0\) for all \(x, \theta'\). For \(\theta' = 1\) the supremum over \(w\) reduces to a maximum over the neighbor of the origin. We compute:

\[
\begin{align*}
\sup_{\kappa} \sum_i \bar{A}_{z}^{1,1}(e_i, 0, e_{\kappa}) &= G_{3,z}(e_1) = (\bar{B})_{1,1}, \\
\sup_{\kappa} \sum_{w \in \mathbb{Z}^d} \bar{A}_{z}^{1,2}(e_i, 0, w) &= \sup_{x \neq 0} G_{1,z}(x) = (\bar{B})_{1,2}, \\
\sup_{\kappa} \sum_{u, x, y} \bar{A}_{z}^{2,1}(u, 0, e_{\kappa}) &= \beta_{2,0}(e_i) = (\bar{B})_{2,1}, \\
\sup_{w} \sum_u \bar{A}_{z}^{2,2}(u, 0, w) &= \sup_{x \neq 0} \beta_{2,0}(x) = (\bar{B})_{2,2}.
\end{align*}
\]

We use these bounds and apply the induction hypothesis on \(4.2.87\) to complete the induction and also the proof.

\[\square\]

### 4.2.5 On the bounds assumed in Chapter 3

We have now derived diagrammatic bounds on the lace-expansion coefficients, which can be bounded numerically using the techniques shown in Chapter 5. We used these bounds in Chapter 3 to prove mean-field behavior for SAW. In this section we explain how to derive the bounds assumed in Assumption 3.2.6 and Assumption 3.5.3.

As \(\Xi^{(N)}(x) = \delta_{N,0} \delta_{0,x}\) is it is clear that, for all \(N\),

\[
\beta^{(N)}_{z} = \beta^{(N)}_{\Lambda z} = 0.
\]

Further, we define \(\Psi^{(N)}_{z}(x) = \delta_{N,0} \delta_{0,x}\), so that

\[
\beta_{\Psi,2} = \Psi^{(0)}_{z}(0) = K_{\Psi} = \beta_{\Psi,1} = 1.
\]

The diagrammatic bound on \(\Xi^{(N)}_{z}\) are given in Section 4.2.2. The largest entry of \(\bar{B}\) and \(\bar{B}\) is the open bubble \(\sup_{x \in \mathbb{Z}^d \setminus \{0\}} \beta_{2,0}(x)\), which we bound numerically by a quantity of the order \(O(d^{-1})\). In particular, the bound on the bubble is for \(d \geq 7\) smaller than one. Considering the form of the bound \(\beta^{(N)}_{z}, \beta^{(N)}_{\Lambda z}, \beta^{(N)}_{z/d'}\), given in Lemma 4.2.3, we see that these bound are of the order \(O(d^{-N})\). We conclude that the sums

\[
\sum_{N=0}^{\infty} \beta^{(N)}_{z}, \sum_{N=0}^{\infty} \beta^{(N)}_{\Lambda z}, \sum_{N=0}^{\infty} \beta^{(N)}_{z/d'}.
\]
are finite. Next we discuss the additional bounds stated in Assumption 3.5.3 as they simplify for the SAW. The quantities we use to characterize the two-point function (defined in Section 3.5.3) are given for the SAW by

\[ \alpha_\Phi = 2d G_{3,z}(e_i) \left( 1 - \frac{z}{1 - z^2} \right), \quad (4.2.95) \]

\[ \hat{R}_\Phi(k) = -z \hat{\Phi}(k) \left[ \hat{D}(k) + \alpha_z J + \hat{\Pi}_z(k) \right]^{-1} \hat{\Sigma}(k) - \alpha_\Phi \hat{D}(k). \quad (4.2.96) \]

and

\[ c_F = 1 - \frac{2dz^2}{1 - z^2}, \quad \alpha_F = 2d \left( \frac{z}{1 - z^2} \right), \quad (4.2.97) \]

\[ \hat{R}_F = z \hat{\Phi}(k) \left[ \hat{D}(k) + \alpha_z J + \hat{\Pi}_z(k) \right]^{-1} \hat{\Sigma}(k) + \frac{2dz^2}{1 - z^2} - \alpha_\Phi \hat{D}(k). \quad (4.2.98) \]

\textbf{Proof of bounds on }\alpha_F\text{ and }\alpha_\Phi.\text{ We know that }z \in [1/(2d - 1), 1/(2d - 1))\text{ and we will compute upper bounds on }G_{3,z}(e_i)\text{ in Chapter 5}. \text{ As a lower bound for }G_{3,z}(e_i)\text{ we use}

\[ G_{3,z}(e_i) \geq c_3(e_i) \left( \frac{1}{2d - 1} \right)^3 + c_5(e_i) \left( \frac{1}{2d - 1} \right)^5 + c_7(e_i) \left( \frac{1}{2d - 1} \right)^7, \quad (4.2.99) \]

where we give the value of }c_3(e_i), c_5(e_i)\text{ and }c_7(e_i)\text{ in Section 5.1.3.}

\textbf{Proof of bounds on }\hat{R}_F(k)\text{ and }\hat{R}_\Phi(k).\text{ Let use review the computations of Section 3.4 and the explanation of the decomposition in Section 3.5.3}. \text{ For SAW we capture with }\alpha_F\text{ the contribution }\hat{F}_1,\text{ so that }\hat{R}_F\text{ only bounds }\hat{F}_2 + \hat{F}_3. \text{ Thereby,}

\[ |\hat{R}_F(k)| \leq \frac{2dz}{1 - z} \sum_{n=1}^{\infty} \left( \frac{(2d - 2)z \beta_{\text{abs}}}{1 - z} \right)^n := \beta_{R,F}, \quad (4.2.100) \]

\text{see (3.4.11)}, \text{ and we define }\beta_{\Lambda,R,F}\text{ to be the sum of (3.4.32) and (3.4.37)}\text{. With }\alpha_\Phi\text{ and }c_\Phi\text{ we remove the dominant contribution of }\Phi_1. \text{ We can bound the absolute value of }\hat{R}_\Phi(k)\text{ by}

\[ |\hat{R}_\Phi(k)| \leq \beta_{R,F} \beta_{\text{abs}} + \sum_{N=1}^{\infty} \beta_{\text{abs}}^N := \beta_{R,\Phi}. \quad (4.2.101) \]

\text{From the difference }\Phi_1(0) - \Phi_1(k)\text{ we remove the contribution of }\Psi^{(0),\ell}(x) \Xi^{(0),\ell}(y),\text{ which is trivial as }\Psi^{(0),\ell}(x) = \delta_{x,0}\text{ and }\Xi^{(0),\ell}(y) = \delta_{y,e_i}\text{. }G_{3,z}(e_i). \text{ Thus, we have to bound from }\Phi_1(0) - \Phi_1(0)\text{ in }\hat{R}_\Phi\text{ only}

\[ \frac{z}{z^2} \left( 1 - \hat{D}(k) \right) \sum_{N=1}^{\infty} \left( \beta_{\Lambda,\Xi}^N + z \beta_{\Lambda,\Xi}^{N+1} \right). \quad (4.2.102) \]
and define $\beta_{\Delta R,\Phi}$ as the sum of (4.2.102), (3.4.35), and (3.5.10). In the following for this choice $\beta_{\Delta R,F}$ and $\beta_{\Delta R,\Phi}$ the following equations holds:

$$|\delta \mu \hat{R}_F(k)| \leq \frac{1}{d} \beta_{\Delta R,F} |\sin(k_\mu)|,$$

$$|\Delta \hat{R}_F(k)| \leq \beta_{\Delta R,F} [1 - \hat{D}(k)], \quad (4.2.103)$$

$$|\delta \mu \hat{R}_\Phi(k)| \leq \frac{1}{d} \beta_{\Delta R,\Phi} |\sin(k_\mu)|,$$

$$|\Delta \hat{R}_\Phi(k)| \leq \beta_{\Delta R,\Phi} [1 - \hat{D}(k)]. \quad (4.2.104)$$

By Lemma 3.5.2 and the discussion thereafter we know that

$$\hat{R}_F(0) - \hat{R}_F(k) = [1 - \hat{D}(k)] \sum_x R_F(x) \|x\|^2_2 = \Delta \hat{R}_F(k),$$

$$\delta \mu \hat{R}_F(k) \leq \sum_x |x_\mu| R_F(x) \leq \frac{1}{d} |\sin(k_\mu)| \sum_x \|x\|^2_2 \mu R_F(x). \quad (4.2.105)$$

For the weight $\|x\|^2_2$ we see that for sequence of vertices $(x_i)_i$ with $\sum x_i = x$ the following holds

$$\|x\|^2_2 \leq J \sum_{i=1}^J \|x_i\|^2_2. \quad (4.2.106)$$

We have bounded $\hat{\Xi}^{(N),J}(0) - \hat{\Xi}^{(N),J}(k)$ using a a similar statement for the weight $[1 - \cos(k \cdot x)]$ (Lemma 3.4.1). Therefore, the argument we use to bound $\hat{\Xi}^{(N),J}(0) - \hat{\Xi}^{(N),J}(k)$ still holds when we replace the weight by $[1 - \cos(k \cdot x)]$ by $\|x\|^2_2$.

### 4.3 Lattice trees

In this section we bound the NoBLE coefficients for lattice trees defined in (2.3.25)-(2.3.26). For this we have to address three problems, that were not present for the SAW: How do we split a tree/rib walk? How do we allocate the weight of the ribs/sausages when we split the walk? In what sense would a repulsive diagram be repulsive? Before defining the repulsive diagrams and the building block we will address these questions.

#### 4.3.1 Rib weight

We have already seen in Section 3.7.2 that $\tilde{G}_z(0) = g_z \approx e \neq 1$ needs special attention. This is especially true for the bounding of the coefficients.

**The problem with the rib weights.** To identify the problem of the ribs weights, let us review how we bound the contribution of lattice trees that contain the vertices
0, x, v ∈ Z^d. We identify the last vertex u that the path 0 ↔ x and the path 0 ↔ v use, and then split the tree into three independent trees and obtain the bound:

\[
\sum_{\text{lattice tree } T} \mathbb{1}_{|0, x, v \in T|} z^{||T||} \leq \sum_{\text{lattice tree } T_1, T_2, T_3} \mathbb{1}_{|0, u \in T_1|} \mathbb{1}_{|x, u, v \in T_2|} \mathbb{1}_{|u, v \in T_3|} z^{||T_1|| + ||T_2|| + ||T_3||} = \sum_u \tilde{G}_z(u) \tilde{G}_z(x - u) \tilde{G}_z(v - u). \tag{4.3.1}
\]

This clearly overcounts as the independent trees T_1, T_2, T_3 might intersect. However, the possible intersections are not the biggest source of overcounting. The biggest error is created by bounding the rib at u three times. To avoid this we define an adapted version of the two-point function that only counts the trivial first rib R_0^ω:

\[
\tilde{G}_z(x) = \sum_{\omega \in W^T(x)} z^{||\omega||} Z[0, ||\omega||] K[0, ||\omega||] \mathbb{1}_{[R_0^\omega = \emptyset]}, \tag{4.3.2}
\]

We bound these walks as follows

\[
\sum_{\text{lattice tree } T} \mathbb{1}_{|0, x, v \in T|} z^{||T||} \leq \sum_u \tilde{G}_z(u) \tilde{G}_z(x - u) \tilde{G}_z(v - u), \tag{4.3.3}
\]

which is approximately \(e^2\) smaller than (4.3.1).

**The split of a rib walk.** The lace-expansion coefficients describe combinations of intersecting lattice trees. As in the preceding example we want to prevent the overcounting of individual ribs in a bound on the coefficients. Thus, we have to choose to which connection of the coefficient we associate a rib. This needs to be done carefully, as the rib walk does not characterize a lattice tree if any ribs intersect.

We first show how we split the rib walks and then explain how we associate a rib in a given diagram. To rigourously state how we split the contributions of a rib walk we give a formal definition of the backbone and introduce the concept of a connecting planted tree and the first point of intersection. These concepts are illustrated in Figure 4.6.

**Definition 4.3.1 (Backbone).** A path from to x to y is a sequence of bonds \((b_i)_{i=1, \ldots, N}\) such that \(b_1 = x, b_i = b_{i+1}, b_N = y\) for \(i = 1, \ldots, N - 1\) and \(b_i = b_j\) for all \(i \neq j\). For a lattice tree \(T\) containing \(x, y \in Z^d\) we define \(b^T(x, y)\) to be the unique path connecting \(x\) and \(y\) via bonds in \(T\). We call \(b^T(x, y)\) the backbone between \(x\) and \(y\). We define \(b^T_1(x, y)\) to be the first bond of the backbone from \(x\) to \(y\). For a rib walk \(\omega\) from \(x\) to \(y\) we denote by \(b^\omega_1(x, y)\) the backbone of \(\omega\), see also Definition 2.3.2.

When removing \(b^T_1(x, y)\) from \(T\) we split the tree into two subtrees. We define the connecting planted tree \(B^T(x, y)\) to be the subtree that contains \(y\) plus the bond \(b^T_1(x, y)\).
We split the tree into three parts: the tree $B^T(x, y)$ from $x$ to $y$ is drawn in red/grey on the right. The set of first intersection points $W_N(\omega) = \{w_1, w_2, w_3\}$ is marked in the left picture. We choose the point $w_N(\omega)$ to be the unique representative of $W_N(\omega)$.

**Definition 4.3.2** (Connecting planted tree). Let $T$ be a lattice tree containing $x, y \in \mathbb{Z}^d$, with $x \neq y$. We define $B^T(x, y)$ to be the set of all bonds in $T$ for which at least one point of the bond is connected to $y$ via bonds in $T \setminus B^T_1(x, y)$. Thus, $b^T_1(z, y)$ is also in $B^T(x, y)$. We call $B^T(x, y)$ the connecting planted tree from $x$ to $y$.

For a rib walk $\omega$ we define $B^\omega_i(x, y) = B^{\omega_1}(x, y)$.

Further, we define a unique vertex $w_N$ to characterize that $R^\omega_{s_N}$ and $R^\omega_{t_N}$ intersect.

**Definition 4.3.3** (First intersection point). For a rib walk $\omega$ we define $W_N(\omega)$ to be the set of vertices $w$ that are contained in $R^\omega_{s_N}$ and $R^\omega_{t_N}$, such that $b^\omega_{s_N}(\bar{b}^\omega_{s_N}, w)$ and $R^\omega_{t_N}$ only intersect at $w$. If $W_N(\omega)$ is non-empty then we define $w_N(\omega)$ to be one fixed representative of the set $W_N(\omega)$, e.g., the first in lexicographic order. We call $w_N(\omega)$ the first intersection point of $R^\omega_{s_N}$ and $R^\omega_{t_N}$. We note that if the end of the $s_N$th pivotal bond $\bar{b}^\omega_{s_N}$ is in $R^\omega_{t_N}$ then $W_N(\omega) = \{\bar{b}^\omega_{s_N}\}$.

We now explain how we split a rib walk contribution to $\Xi^{(1)}_z(x)$. We know that $\mathcal{L}^{(1)}[0, n] = \{(0, n)\}$, so that the indicator $J^{(1)}[a, b](\omega)$, defined in (2.3.24), is the indicator for the event that $R^\omega_0$ and $R^\omega_{[a]}$ intersect, but all other ribs do not intersect. We split the tree into three parts: the tree $B^\omega_0(0, w_1(\omega))$ that connects 0 and $w_1(\omega)$, the tree $B^\omega_{[a]}(x, w_1(\omega))$ that connects $x$ and $w_1(\omega)$, and the remaining rib walk with a trimmed first and last ribs. This trimmed rib walk does not need to characterize a lattice tree as the trimmed first and last ribs might still have intersects. We bound the remaining walk by bounding the contribution of the first trimmed rib $R^\omega_0 \setminus (B^\omega_0(0, w) \cup b^\omega_{[a]})$ by $g^x_z$, so that we can bound the remainder of the trimmed rib walk by $2dz(D \ast G_z)(x)$. In this way we conclude that:

$$\Xi^{(1)}_z(x) \leq g^x_z \sum_w G_z(w)G_z(w-x)2dz(D \ast G_z)(x) \quad (4.3.4)$$
The allocation of rib contributions. We have to decide how to distribute the rib weights over the diagram. To show how we will do this let us briefly discuss how we split a tree contribution to $\Xi_z^{(2)}(x)$. This will be done in more detail in Section 4.3.5. We assume that $s_2 < t_1$. We first remove the intersection rib: $B_0(0, w_1(\omega))$, $B_{s_2}(b_{s_2+1}, w_1(\omega))$, $B_{t_1}(b_{t_1+1}, w_2(\omega))$ and $B_{|\omega|}(b_{|\omega|}, w_2(\omega))$. We can bound each of these connecting planted trees by $\tilde{G}_z$. We call the rib walks obtained after the removal of these parts trimmed rib walk. We cut the trimmed rib walk, that can still contain intersections, into four pieces:

i.) a rib walk from $b_{t_1+1}$ to $b_{|\omega|}$ with trivial first rib,

ii.) a rib walk from $b_{s_2+1}$ to $b_{t_1+1}$ with trivial first rib,

iii.) a rib walk from 0 to $b_{s_2+1}$ with trivial first rib,

iv.) the first trimmed rib, that we bound by $g_z^t$.

We always bound the weight of the trimmed first rib by $g_z^t$ and then we distribute the rib weight in the diagrams as explained for $\Xi_z^{(1)}(x)$. In Figure 4.7 we show how we associate the weights ribs to bound $\Xi_z^{(4)}(x)$. We see that each single line can be bounded by $\tilde{G}_z$ and note that the points $w_i(\omega)$ are the only points where we bound the contribution of two ribs.

![Diagram](image)

Figure 4.7: Picture of a possible $\Xi_z^{(4)}$ diagram. The backbone is marked by a thicker line and the ribs of $x_i$ have different colors. The arrow tip indicates that the rib at this endpoint is bound by the corresponding line. The trimmed first rib is bounded by $g_z^t$. The small numbers are labels for the corresponding connection.

Discussion. If we bound each line simply by $\tilde{G}_z(k)$ then we bound many ribs multiple times. This multiple bounding of the ribs would unnecessarily increase a bound on $\hat{\Xi}_z^{(N)}$, by a factor of about $e^{4N+2}$. In the bound that we derive in this section, we bound each rib only once. In order to do this we need to decide how we
4.3 Lattice trees

bound the contribution of the ribs. For convenience we decided to bound the very first rib by a factor \( g_1 \approx e \), extract the contribution of all connecting ribs and then split the rib walk, as explained for \( \Xi_z^{(2)} \). There are other ways possible to archive this. For example we can bound the first rib \( R_0^\omega \ni 0, w_1 \) by \( \bar{G}_z(w_1) \).

If we would know that the trimmed rib walk characterizes a LT then we could actually choose any line of the diagram, to carry the weight of the ribs at both endpoint of the connection and be bounded by \( \bar{G}_z \). This is desirable as this would improve our bound on \( \hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \). The problem with a bound on \( \hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \) is that \( f_3 \) and \( \bar{f}_3 \) only provides bound on \( \bar{G}_z(x)[1 - \cos(k \cdot x)] \) and not on \( \tilde{G}_z(x)[1 - \cos(k \cdot x)] \), which we need to bound our diagrams. Since \( \bar{G}_z(x) < \tilde{G}_z(x) \) for all \( z \) and \( x \) we can still use \( f_3 \) and \( \bar{f}_3 \) to obtain a bound on \( \bar{G}_z(x)[1 - \cos(k \cdot x)] \). However, this is not a very good bound as we expect that \( e\tilde{G}_z(x) \approx \tilde{G}_z(x) \).

4.3.2 Repulsive diagrams

Before defining repulsive diagrams for LT we have to define what repulsive means in the context of LT. To motivate the definition of repulsiveness of LT we review the avoidance structure of the first square \((1,2,3,4)\) in Figure 4.7. In the diagrams the vertices \( x_i \) denote points on the backbone, the vertices \( w_i \) denote the \( i \)th intersection and \( u_i \) vertices where we have to split a rib. In the split explained above the lines have the following interpretation:

- Line 2 is the connected planted tree \( B_0^\eta(0, w_1) \).
- Line 3 is the connected planted tree \( B_1^\eta(x_1, w_1) \).
- Line 4 is the connected planted tree \( B_1^\eta(w_1, v_1) \).
- Line 1 is the backbone \( b_1^\eta(0, x_1) \), the ribs \((R_i^\eta)_{i=1,\ldots,t_1-1}\) and trimmed version of \( R_{t_1}^\eta \).

These lines obey the following avoidance constraints:

- Relation between 1 and 2: By the properties of \( J[0,|\omega|] \) the rib \( R_{t_1}^\eta \) is the first rib to intersect with the rib \( R_0^\eta \). Thus, 2 intersects 1 only at the common starting point 0.
- Relation between 1 and 3, 4: As in the case of 1 and 2, we know that the rib \( R_{t_1}^\eta \) does not intersect any of the former pieces except \( R_0^\eta \). Therefore 3, 4 avoid 1, except at the point \( x_1 \).
- Relation between 3 and 4: As a rib is also a LT we know that 3 and 4 only intersect at \( u_1 \).
• Relation between 2 and 3, 4: We know that \( R_0^\omega \) and \( R_{t_1}^\omega \) intersect in at least one point. The LT drawn as 2 and 3 might intersect at multiple points. To preserve some of the avoidance structure we choose \( w_1 = w_1(\omega) \) to be the first intersection point, as defined in Definition 4.3.3. Therefore, the backbones \( b_{R_0^\omega}(0, w_1) \) and \( b_{R_{t_1}^\omega}(x_1, w_1) \) only intersect at \( w_1 \).

We define repulsive diagrams to bound such diagrams. As we have seen for the relation between 2 and 3, 4 the constraint that part of a diagram do not intersect at all is too strong. We define a repulsive diagram for LT to be a combination of LTs in which the backbones connecting the corner point of the diagram do not intersect.

To define the repulsive diagrams we also need a notion of the distance on a LT.

**Definition 4.3.4** (Intrinsic distance on a tree). For \( x, y \in \mathbb{Z}^d \) and a lattice tree \( T \) containing both points we define \( d_T(x, y) \) as the intrinsic distance between \( x \) and \( y \) in \( T \), i.e., the length of the backbone \( b^T(x, y) \). For a rib walk \( \omega \) from \( x \) to \( y \) we define the distance between \( x \) and \( y \) to be \( d_{\omega}(x, y) = |\omega| \).

In the following we define the skeleton of a diagram which encodes the avoidance constraint of the backbones, the length constraints on the connections and the information whether the first or the last rib of a connection will be bounded.

**Definition 4.3.5** (Mutually avoiding skeleton for lattice trees). Let \( x_0 = 0 \in \mathbb{Z}^d \) and \( n \in \{2, 3, 4\} \). For each \( i \in \{1, \ldots, n\} \), let \( x_i \in \mathbb{Z}^d, l_i \in \mathbb{N}, \) the index \( j_i \in \{l_i, l_i\}, s_i \in \{+, -\}, \) and the rib walks \( \omega^i \) from \( x_{i-1} \) to \( x_i \). We define \( S_{j_1, \ldots, j_n}^{x_1, \ldots, x_n}(\omega_1, \ldots, \omega_n) \) to be the indicator that the following holds:

1. Each backbone \( b^{\omega_i} \) describes a self-avoiding path that does not intersect any of the other \( n-1 \) backbones with the exception of the common starting- and end-point, i.e., for \( i \in \{1, \ldots, n-1\} \): \( b^{\omega_i}_{|\omega_i|} = x_i = b^{\omega_i+1}_{|\omega_i|} \) holds and it is allowed that \( b^{\omega_n}_{|\omega_n|} = x_0 = 0 = b^{\omega_1}_{|\omega_1|} \).

2. For each \( i \in \{1, \ldots, n\} \), the ribs \( (R^\omega_{j})_{j=0,\ldots,|\omega|} \) do not intersect, i.e. \( K[0, |\omega^i|](\omega^i) = 1 \) so that \( \omega^i \) describes a lattice tree.

3. For each \( i \in \{1, \ldots, n\} \), if \( j_i = l_i \) then \( |\omega^i| \geq l_i \), while \( |\omega^i| = l_i \) when \( j_i = l_i \).

4. For each \( i \in \{1, \ldots, n\} \), if \( s_i = + \) then \( R^\omega_{0} = \emptyset \), while \( R^\omega_{|\omega^i|} = \emptyset \) when \( s_i = - \).
We use this skeleton to define the repulsive bubble, triangle and square diagrams. For $i = 1, 2, 3, 4$ let $x_i \in \mathbb{Z}^d$, $l_i \in \mathbb{N}$, $j_i \in \{l_i, l_i^{-1}\}$, $s_i \in \{+, -, 0\}$, and $x_0 = 0$. We define

$$B_{s_1, s_2}(x_1, x_2) = \sum_{\omega^1 \in \mathcal{W}(x_0, x_1)} \sum_{\omega^2 \in \mathcal{W}(x_1, x_2)} \left( \prod_{i=1}^{2} z^{||\omega^i||} Z[0, ||\omega^i||] \right) S_{j_1, j_2}^{s_1, s_2}(\omega^1, \omega^2) \quad \text{(4.3.5)}$$

$$J_{s_1, s_2, s_3}(x_1, x_2, x_3) = \sum_{\omega^1 \in \mathcal{W}(x_0, x_1)} \sum_{\omega^2 \in \mathcal{W}(x_1, x_2)} \sum_{\omega^3 \in \mathcal{W}(x_2, x_3)} \left( \prod_{i=1}^{3} z^{||\omega^i||} Z[0, ||\omega^i||] \right) S_{j_1, j_2, j_3}^{s_1, s_2, s_3}(\omega^1, \omega^2, \omega^3) \quad \text{(4.3.6)}$$

$$J_{s_1, s_2, s_3, s_4}(x_1, x_2, x_3, x_4) = \sum_{\omega^1 \in \mathcal{W}(x_0, x_1)} \sum_{\omega^2 \in \mathcal{W}(x_1, x_2)} \sum_{\omega^3 \in \mathcal{W}(x_2, x_3)} \sum_{\omega^4 \in \mathcal{W}(x_3, x_4)} \left( \prod_{i=1}^{4} z^{||\omega^i||} Z[0, ||\omega^i||] \right) S_{j_1, j_2, j_3, j_4}^{s_1, s_2, s_3, s_4}(\omega^1, \omega^2, \omega^3, \omega^4) \quad \text{(4.3.7)}$$

The repulsive diagrams are bounded by combinations of the two-point functions $\tilde{G}_{m,z}$, e.g. for $m_1, m_2, m_3 \in \mathbb{N}$:

$$J_{m_1, m_2, m_3}(x_1, x_2, x_3) \leq \tilde{G}_{m_1, z}(x_1) \tilde{G}_{m_2, z}(x_2 - x_1) \tilde{G}_{m_3, z}(x_2 - x_3). \quad \text{(4.3.8)}$$

### 4.3.3 Building blocks

**Encoding of the line length and backbone path.**

In Figure 4.7 we have shown the diagrammatic representation of one possible form of $\Xi(z)^i$. As the backbone does not need to follow the pattern shown in Figure 4.7 there are seven other possible forms. In Figure 4.8 we show all possibilities for the backbone for the $\Xi(z)$-diagrams.

When creating a bound on the coefficients we condition on the lengths of the paths that are shared by two squares. To be able to bound all the patterns of the backbone (see Figure 4.8), we include the information whether the shared path is a piece of the backbone or not. We encode the information on the length and the path of the backbone using the indices $a, b \in \{-2, -1, 0, 1, 2\}$. We explain the meaning of these indices using the example of the diagram of Figure 4.9. As indicated in Figures 4.7, 4.8 and 4.9 we draw the diagrams in such a way that the origin is in the lower left corner and the first piece of the backbone corresponds to the lower line of the first square. A positive index $a_i$ indicates that the backbone starts resp. ends on the upper part of a piece of the diagram. By $|a_i| = 1$ we denote that the vertical line consists of exactly one step and $|a_i| = 2$ denotes that the vertical line consists of at
Informal introduction of the building blocks.

Before giving a rigorous definition of the building blocks, we give an informal introduction. We draw a small sketch and list the properties of the building blocks. The properties of the building blocks originate from the properties of the NoBLE coefficients and are used to obtain good bounds on the coefficients. As in Figure 4.7, we use arrow tips to indicate where the ribs are bounded. For each of the building block listed below we define an adapted version of the diagram in which we swap the orientation of the arrows on the backbone.

(i.) Closed square $P^{(n)}_1$:

This diagram is used to bound the first square of $\Xi_z^{(N)}$ and the last square of $\Xi_z^{(N)}$ and $\Xi_z^{(N),4}$. Its shape is indicated as follows:
with $b \geq 0$. The index $b$ identifies the length of the connection between $x$ and $y$ and whether the backbone moves up or not. If $b = 0$ then $x = y$. If $|b| = 1$ then $x$ and $y$ are direct neighbors and are directly connected and $|b| = 2$ characterizes all the other cases.

We denote by $v \in \mathbb{Z}^d$ the point where the $R_0$ and the last rib intersect. The square consists of at least four edges. If the backbone between 0 and $x$ consists of only one step then $v \not\in \{0, x\}$. For $b \leq 0$ the backbone $b(0, x)$ is non-trivial, i.e., 0 is connected to $x \neq 0$. If $b > 0$ then the backbone pieces $b(0, y), b(x, y)$ are both non-trivial.

(ii.) Closed square with fixed first step $P^{(1), \text{step}, b}$:
This diagram is one of two possible “initial” diagrams of $\Xi^{(N), f}$:

with $b \geq 0$. It has the same properties as the closed square $P^{(1), b}$ with the additional condition that the first step of the backbone is fixed.

(iii.) Closed square where $e_i$ is contained in the first rib $P^{(1), \text{rib}, b}$:
Second diagram of the part of the “initial” diagram of $\Xi^{(N), f}$.
with \( b \geq 0 \). The pentagon consists of at least four edges and if the backbone \( b(0, x) \) consists of only one step then \( v \not\in \{0, x\} \). Also the pentagon \((0, u, v, x, y)\) consists of at least four edges. If \( b \leq 0 \) then \( b(0, x) \) is non-trivial, while if \( b > 0 \) then \( b(0, y) \) and \( b(x, y) \) are non-trivial.

(iv.) Open square \( A^{a,b} \):
With this building block we bound the intermediate pieces of the coefficient diagrams.

\[
A^{a,b}(u, v, x, y) := \sum_w
\]

A positive index \( a, b \) indicates that the backbone starts/ends at the top. This means that if \( ab \geq 0 \) then the connection \( x \) to \( y \) is not part of the backbone. In this case \( b(0, x) \) is non-trivial. If \( ab < 0 \) and \( |a| \neq 0 \) then the backbones \( b(0, y), b(x, y) \) are non-trivial. The closed pentagon, including \( u \leftarrow v \), consists of at least four edges.
We always label the points such that \( u \) is the beginning and \( x \) is the end of the backbone. If the backbone consists of only one step, then we know that \( w \not\in \{u, x\} \).

(v.) Double open triangle \( \tilde{A}^{a,b} \).
We use this building block to bound the next-to-last square of the coefficient diagrams. It will play a special role for bounding the weighted diagrams.

\[
\tilde{A}^{a,b}(u, v, x, y) := \sum_w
\]

This diagram follows the same rules as the open square \( A^{a,b} \). We only draw the example with \( a, b > 0 \). The difference to \( A^{a,b} \) is that the connection \( x \) to \( y \) does not contribute.
**4.3 Lattice trees**

In this section we define the diagrams that we have just introduced informally. When defining the diagrams $P^{(i),b}$, $P^{(i),\text{step},b}$ and $P^{(i),\text{rib},b}$, we want to use the properties listed above. The most instructive way to do this is to define the quantities in a table where also a small sketch can be displayed. Therefore, we define these diagrams via the Tables 4.11-4.10. To give an example we define in Table 4.3 and 4.14 the following building blocks:

$$P^{(1),2}(x, y) = (1 - \delta_{x,y}) \delta_{0,x}(B_{3,1}^{+,-}(y,0) + B_{2,2}^{+,-}(y,0))$$

$$+ (1 - \delta_{x,y})(1 - \delta_{0,x}) \left( T_{1,2,1}^{+,-,-}(x,y,0) + \sum_{v \in Z^d \setminus \{0\}} T_{1,0,2,1}^{+,-,-}(v,x,y,0) \right),$$

$$P^{(1),i,\text{rib},-1}(x, y) = G_{1,z}(e_i) P^{(i),-1}(x, y) + G_{1,z}(y - e_i) T_{2,1,1}^{+,-,-}(y, x, 0)$$

$$+ \sum_{v \in Z^d \setminus \{y\}} T_{1,1,1}^{+,-,-}(v, y, x, 0)$$

$$+ \sum_{v \in Z^d \setminus \{y\}} \sum_{u \in Z^d \setminus \{v\}} 2dzg_z D(x) G_{1,z}(u - e_i) T_{1,0,1}^{+,-,-}(u, v, y, x)$$

$$+ \sum_{u, v \in Z^d} T_{1,0}^{+,-,+}(u, e_i) T_{2,1,0,0}^{+,-,+}(x, y, v, u).$$

Further, we define

$$P^{(1),i,b}(x, y) = P^{(i),\text{step},b}(x, y) + P^{(i),i,\text{rib},b}(x, y).$$

**Definition of the open square.** For $a = 0$ the open square $A^{a,b}$ is actually a closed square, so that we can use $P^{(i),b}$ (Tables 4.1-4.5) to define $A^{a,b}$:

$$A^{0,b}(u, v, x, y) = \delta_{u,v} P^{(1),b}(x - u, y - u)$$

for $b \in \{-2, -1, 0, 1, 2\}$. We define $A^{1,0}$ and $A^{2,0}$ using Table 4.16. We define

$$A^{1,1}(u, v, x, y) = \sum_{w \in Z^d} 2dD(v - u) T_{1,1,0,0}^{+,-,+}(x - u, y - u, w - u, v - u),$$

$$A^{1,2}(u, v, x, y) = \sum_{w \in Z^d} 2dD(v - u) T_{1,2,0,0}^{+,-,+}(x - u, y - u, w - u, v - u),$$

$$A^{2,1}(u, v, x, y) = \sum_{w \in Z^d} T_{1,1,0,0}^{+,-,+}(x - u, y - u, w - u, v - u),$$

$$A^{2,2}(u, v, x, y) = \sum_{w \in Z^d} T_{1,2,0,0}^{+,-,+}(x - u, y - u, w - u, v - u),$$

and for $a, b \in \{1, 2\}$

$$A^{a,-b}(u, v, x, y) = A^{a,b}(u, v, y, x),$$

$$A^{-a,0}(u, v, x, y) = A^{a,0}(u, v, x, y).$$
We define the remaining cases using symmetry. Let \( a \in \{1, 2\} \) and \( b \in \{-2, -1, 1, 2\} \), then

\[
A^{-a,-b}(u, v, x, y) = A^{a,b}(u, v, x, y).
\] (4.3.19)

**Definition of the double open square.** For \( a \in \{-2, \ldots, 2\} \) we define

\[
\tilde{A}^{a,0}(u, v, x, y) = A^{a,0}(u, v, x, y), \tilde{A}^{0,a}(u, v, x, y) = \tilde{A}^{a,0}(x, y, u, v),
\] (4.3.20)

and for \( a, b \in \{1, 2\} \):

\[
\tilde{A}^{a,b}(u, v, x, y) = (1 - \delta_{v,x})G_{1,2}(x - u) \sum_{w \in \mathbb{Z}^d} B_{0,0}(w - v, y - v),
\] (4.3.21)

\[
\tilde{A}^{-a,-b}(u, v, x, y) = \tilde{A}^{a,b}(u, v, x, y),
\] (4.3.22)

\[
\tilde{A}^{-a,-b}(u, v, x, y) = \tilde{A}^{a,b}(u, v, y, x),
\] (4.3.23)

\[
\tilde{A}^{-a,-b}(u, v, x, y) = \tilde{A}^{-a,b}(u, v, x, y).
\] (4.3.24)

We define \( \tilde{A}^{0,b}(u, v, x, y) \) using symmetry.

**Swapped backbone.** In the diagrams we have defined so far, the ribs of the backbone, \( 0 \leftrightarrow x \) and \( 0 \leftrightarrow y \leftrightarrow x \), are bounded at \( x \) and \( x, y \) respectively. To state the bound we also need to define similar objects in which we bound the ribs at \( 0 \) and \( 0, x \) respectively. In the representation as diagrams this means that we want to swap the orientation of the arrow at all lines that are part of the backbone. Instead of defining completely new diagrams to express this we define for the diagrams \( A^{a,b} \) and \( P^{(1),b} \) the diagrams in which the backbone are swapped by \( A^{a,b} \) and \( P^{(1),b} \).

For example the swapped version of \( P^{(1),2}(x, y) \) defined in (4.3.10) is given by

\[
P^{(1),2}(x, y) = (1 - \delta_{x,y})\delta_{0,x}(B_{-1,2}^{+,+}(y, 0) + B_{2,2}^{+,+}(y, 0))
\]

\[
+ (1 - \delta_{x,y})(1 - \delta_{0,x}) \left( B_{-1,2}^{+,+}(x, y, 0) + \sum_{v \in \mathbb{Z}^d \setminus \{0\}} B_{1,0,2,1}^{+,+,+,+}(v, x, y, 0) \right).
\] (4.3.25)

The difference between the original diagram and the swapped diagram is very small and we even use the same numerical bound for them. We use this swapped diagram to define the missing double open squares:

\[
\tilde{A}^{0,b}(u, v, x, y) = \tilde{A}^{b,0}(x, y, u, v),
\] (4.3.26)

for \( b = \{-2, -1, 1, 2\} \).
Table 4.1: Definition of $P^{(1),0}(x,y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$\Rightarrow v = 0, d_{\omega}(0,x) \geq 4$</td>
<td>$0 \geq 4$</td>
<td>$\sum_{i} B_{3,1}^{+,+}(e_i,0)$</td>
</tr>
<tr>
<td>$d_{\omega}(0,x) = 1$</td>
<td>$\Rightarrow v \notin [0,x]$</td>
<td>$v \geq 2$</td>
<td>$B_{1,3}^{+,+}(x,v,0) + B_{1,3}^{+,+}(x,v,0)$</td>
</tr>
<tr>
<td>$d_{\omega}(0,x) \geq 2$</td>
<td>$v \in [0,x]$</td>
<td>$0 \geq 2$</td>
<td>$B_{2,2}^{+,+}(x,v,0) + B_{2,2}^{+,+}(x,v,0)$</td>
</tr>
<tr>
<td>$v \notin [0,x]$</td>
<td>$0 \geq 2$</td>
<td>$x \geq 1$</td>
<td>$T_{2,1,1}^{+,+}(x,v,0)$</td>
</tr>
</tbody>
</table>

Table 4.2: Definition of $P^{(1),1}(x,y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$\Rightarrow v = 0, d_{\omega}(0,x) \geq 4$</td>
<td>$0 \geq 3$</td>
<td>$B_{3,1}^{+,+}(y,0)$</td>
</tr>
<tr>
<td>$x, y \neq 0$</td>
<td>$v \in [0,x]$</td>
<td>$y \geq 2$</td>
<td>$\delta_{v,x} T_{1,1,2}^{+,+}(x,y,0) + \delta_{v,0} T_{1,1,2}^{+,+}(x,y,0)$</td>
</tr>
<tr>
<td>$v \notin [0,x]$</td>
<td>$y \geq 2$</td>
<td>$x \geq 1$</td>
<td>$T_{1,1,1}^{+,+}(v,x,y,0)$</td>
</tr>
</tbody>
</table>

Table 4.3: Definition of $P^{(1),2}(x,y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$\Rightarrow v = 0$</td>
<td>$0 \geq 1$</td>
<td>$B_{3,1}^{+,+}(y,0) + B_{2,2}^{+,+}(y,0)$</td>
</tr>
<tr>
<td>$x \neq 0$</td>
<td>$v = 0$</td>
<td>$x \geq 1$</td>
<td>$T_{1,2,1}^{+,+}(x,y,0)$</td>
</tr>
<tr>
<td>$v \neq 0$</td>
<td>$v \geq 0$</td>
<td>$x \geq 1$</td>
<td>$T_{1,0,2,1}^{+,+}(v,x,y,0)$</td>
</tr>
</tbody>
</table>
### Table 4.4: Definition of $P^{(1),-1}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_\omega(0, x) = 1$ \Rightarrow $v \notin {0, x}$</td>
<td>$v = y$</td>
<td>[\begin{array}{c} \geq 2 \ \geq 1 \ 0 \end{array} \begin{array}{c} y \ = 1 \ = 1 \ 0 \end{array} = 1 ]</td>
<td>$T_{1,1,2}^{+,+,+}(x, y, 0)$</td>
</tr>
<tr>
<td>$v \neq y$</td>
<td>$\geq 1$</td>
<td>[\begin{array}{c} \geq 1 \ \geq 0 \ 0 \ 0 \end{array} \begin{array}{c} y \ y \ = 1 \ = 1 \end{array} = 1 ]</td>
<td>$T_{1,1,1}^{+,+,+}(x, y, 0)$</td>
</tr>
<tr>
<td>$d_\omega(0, x) \geq 2$</td>
<td>$v = 0$</td>
<td>[\begin{array}{c} \geq 1 \ \geq 2 \ 0 \end{array} \begin{array}{c} y \ y \ = 1 \end{array} = 1 ]</td>
<td>$T_{2,1,1}^{+,+,+}(x, y, 0)$</td>
</tr>
<tr>
<td>$v \neq 0$</td>
<td>$\geq 1$</td>
<td>[\begin{array}{c} \geq 1 \ \geq 2 \ 0 \end{array} \begin{array}{c} y \ y \ = 1 \end{array} = 1 ]</td>
<td>$T_{2,1,0,1}^{+,+,+}(x, y, v, 0)$</td>
</tr>
</tbody>
</table>

### Table 4.5: Definition of $P^{(1),-2}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = 0$</td>
<td>$\geq 1$</td>
<td>[\begin{array}{c} \geq 2 \ \geq 1 \ 0 \end{array} \begin{array}{c} y \ y \ = 1 \end{array} = 1 ]</td>
<td>$T_{1,2,1}^{+,+,+}(x, y, 0)$</td>
</tr>
<tr>
<td>$v \neq 0$</td>
<td>$\geq 1$</td>
<td>[\begin{array}{c} \geq 2 \ \geq 1 \ 0 \end{array} \begin{array}{c} y \ y \ = 1 \end{array} = 1 ]</td>
<td>$T_{1,2,0,1}^{+,+,+}(x, y, v, 0)$</td>
</tr>
</tbody>
</table>
### Table 4.6: Definition of $P^{(1),i,\text{step},0}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td></td>
<td><img src="x=0" alt="Diagram" /></td>
<td>$\mathcal{B}^{+,−}_{3,1}(e_i, 0)$</td>
</tr>
<tr>
<td>$x = e_i$</td>
<td></td>
<td><img src="x=e_i" alt="Diagram" /></td>
<td>$\mathcal{B}^{+,−}<em>{1,2,1}(v, e_i, 0)$ + $\mathcal{B}^{+,−}</em>{2,1}(v, e_i, 0)$</td>
</tr>
<tr>
<td>$x \not\in {0, e_i}$</td>
<td>$v \in {0, x}$</td>
<td>![Diagram](x=0, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−}_{1,1,1,1}(v, x, e_i, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \not\in {0, x}$</td>
<td>![Diagram](x=0, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−,−}_{1,1,1,1}(v, x, e_i, 0)$</td>
</tr>
</tbody>
</table>

### Table 4.7: Definition of $P^{(1),i,\text{step},1}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = e_i$</td>
<td>$x = 0$</td>
<td>![Diagram](y=e_i, x=0)</td>
<td>$\mathcal{B}^{+,−}_{3,1}(e_i, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \in {0, x}$</td>
<td>![Diagram](y=e_i, v=x)</td>
<td>$\delta_{0,v} \mathcal{J}^{+,−,−,−}<em>{2,1,1}(x, e_i, 0)$ + $\delta</em>{x,v} \mathcal{J}^{+,−,−,−,−}_{2,1,1}(x, e_i, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \not\in {0, x}$</td>
<td>![Diagram](y=e_i, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−,−}_{1,1,1,1}(v, x, e_i, 0)$</td>
</tr>
<tr>
<td>$y \neq e_i$</td>
<td>$v = x = 0$</td>
<td>![Diagram](y=e_i, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−,−}_{1,2,1}(y, e_i, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v = x \neq 0$</td>
<td>![Diagram](y=e_i, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−,−}_{1,1,1,1}(x, y, e_i, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \neq x$</td>
<td>![Diagram](y=e_i, v=x)</td>
<td>$\mathcal{J}^{+,−,−,−,−,−}_{0,1,1,1,1}(v, y, x, e_i)$</td>
</tr>
</tbody>
</table>

---

4.3 Lattice trees

159
### Table 4.8: Definition of $P^{(1), t, \text{step}_2}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td></td>
<td><img src="image1" alt="Diagram" /></td>
<td>$\mathcal{B}<em>{3,1}^{-,\text{--}}(e_t, 0) + \mathcal{J}</em>{2,1,1}^{-,\text{--}}(y, e_t, 0)$</td>
</tr>
<tr>
<td>$x \neq 0$</td>
<td>$v = x$</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\mathcal{J}_{1,2,0,1}^{+,\text{--},\text{--}}(x, y, e_t, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \neq x$</td>
<td><img src="image3" alt="Diagram" /></td>
<td>$z g^{t}<em>{z} \mathcal{J}</em>{0,1,2,0}^{+,\text{--},\text{--}}(v, x, y, e_t)$</td>
</tr>
</tbody>
</table>

### Table 4.9: Definition of $P^{(1), t, \text{step}_{-1}}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 0$</td>
<td>$\Rightarrow v = 0$</td>
<td><img src="image4" alt="Diagram" /></td>
<td>$\mathcal{J}_{1,2,1}^{-,\text{--},\text{--}}(x, e_t, 0)$</td>
</tr>
<tr>
<td>$x = e_t$</td>
<td>$\Rightarrow v \neq 0$</td>
<td><img src="image5" alt="Diagram" /></td>
<td>$\mathcal{J}<em>{2,1,1}^{+,\text{--},\text{--}}(y, e_t, 0) + \mathcal{J}</em>{1,1,1,1}^{+,\text{--},\text{--}}(v, y, e_t, 0)$</td>
</tr>
<tr>
<td>$x \neq e_t$</td>
<td>$v = 0$</td>
<td><img src="image6" alt="Diagram" /></td>
<td>$\mathcal{J}_{1,1,1,1}^{-,\text{--},\text{--}}(y, x, e_t, 0)$</td>
</tr>
<tr>
<td></td>
<td>$v \neq 0$</td>
<td><img src="image7" alt="Diagram" /></td>
<td>$z g^{t}<em>{z} \mathcal{J}</em>{1,0,1,1}^{+,\text{--},\text{--}}(v, y, x, e_t)$</td>
</tr>
</tbody>
</table>
### Table 4.10: Definition of $P^{(1),\text{step},-2}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = e_t$</td>
<td>$y \neq 0$</td>
<td><img src="image1" alt="" /></td>
<td>$\mathcal{F}^{+,-,-}_{1,0,2,1}(v, y, e_t, 0)$</td>
</tr>
<tr>
<td>$v \neq 0$</td>
<td></td>
<td><img src="image2" alt="" /></td>
<td>$zg^{+}_{2,2,0,2,1}(v, y, x, e_t)$</td>
</tr>
</tbody>
</table>

### Table 4.11: Definition of $P^{(1),\text{rib},0}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td></td>
<td><img src="image3" alt="" /></td>
<td>$\tilde{g}_{1,2}(e_t)P^{(1),0}(x, x)$</td>
</tr>
<tr>
<td>$v = u$</td>
<td>$v = x = e_t$</td>
<td><img src="image4" alt="" /></td>
<td>$\mathcal{B}_{3,1}^{+,+}(e_t)$</td>
</tr>
<tr>
<td></td>
<td>$d_\omega(0, x) \geq 2$</td>
<td></td>
<td>$\tilde{g}<em>{2,2}(x - e_t)\mathcal{B}</em>{1,3}^{+,+}(x)$ + $\tilde{g}<em>{1,2}(x - e_t)\mathcal{B}</em>{2,2}^{+,+}(x)$</td>
</tr>
<tr>
<td>$v \neq x$</td>
<td>$d_\omega(0, x) = 1$ $d_{R_{0}}(0, v) = 1$</td>
<td><img src="image5" alt="" /></td>
<td>$\tilde{g}<em>{2,2}(v - e_t)\mathcal{F}^{+,+,+}</em>{1,2,1}(x, v, 0)$</td>
</tr>
<tr>
<td></td>
<td>$d_\omega(0, x) = 1$ $d_{R_{0}}(0, v) \geq 2$</td>
<td><img src="image6" alt="" /></td>
<td>$(\delta_{u,e_t}\tilde{g}<em>{3,2}(e_t) + \mathcal{B}</em>{2,1}^{+,+}(u, e_t))\times\mathcal{B}_{1,1}^{+,+}(x, u)$</td>
</tr>
<tr>
<td>$v \neq x$</td>
<td>$d_\omega(0, x) \geq 2$</td>
<td><img src="image7" alt="" /></td>
<td>$\mathcal{B}<em>{1,0}^{+,+}(u, e_t)\mathcal{B}</em>{2,1}^{+,+}(x, u)$</td>
</tr>
<tr>
<td>$v \neq u$</td>
<td>$v = x$ $d_\omega(0, x) \geq 2$</td>
<td><img src="image8" alt="" /></td>
<td>$\mathcal{B}<em>{1,0}^{+,+}(u, e_t)\mathcal{B}</em>{2,1}^{+,+}(x, u)$</td>
</tr>
<tr>
<td></td>
<td>$v \neq x$</td>
<td><img src="image9" alt="" /></td>
<td>$\mathcal{B}<em>{1,0}^{+,+}(u, e_t)\mathcal{F}^{+,+,+}</em>{1,1,1}(x, v, u)$</td>
</tr>
</tbody>
</table>
Table 4.12: Definition of $P^{(1),\text{rib},1}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td></td>
<td><img src="image1" alt="Diagram" /></td>
<td>$\tilde{G}_{1,2}(e_i)P^{(1),1}(x, y)$</td>
</tr>
<tr>
<td>$x = e_i$</td>
<td>$\Rightarrow v = e_i$</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\mathcal{T}_{2,1,1}^+(y, e_i, 0)$</td>
</tr>
<tr>
<td>$d_\omega(0, x) = 2$</td>
<td>$x \neq e_i$</td>
<td>$u = e_i$</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$u \neq e_i$</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>$d_\omega(0, x) \geq 3$</td>
<td></td>
<td></td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Table 4.13: Definition of $P^{(1),\text{rib},2}(x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td></td>
<td><img src="image6" alt="Diagram" /></td>
<td>$\tilde{G}_{1,2}(e_i)P^{(1),2}(x, y)$</td>
</tr>
<tr>
<td>$u \neq 0$</td>
<td></td>
<td><img src="image7" alt="Diagram" /></td>
<td>$\mathcal{T}<em>{1,2,0,0}^-(y, x, v, u)$ $\times \mathcal{T}</em>{1,0}^+(u, e_i)$</td>
</tr>
</tbody>
</table>
### Table 4.14: Definition of $P^{(1), \iota, \text{rib}, -1} (x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td></td>
<td><img src="image1" alt="Diagram" /></td>
<td>$\tilde{G}_{1,z}(e_1)P^{(1), -1}(x, y)$</td>
</tr>
<tr>
<td>$d_\omega (0, x) = 1$</td>
<td>$u = v = y$</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\tilde{G}<em>{1,z}(y - e_1)\mathcal{J}^{+, -, -}</em>{2, 1, 1}(y, x, 0)$</td>
</tr>
<tr>
<td></td>
<td>$u = v \neq y$</td>
<td><img src="image3" alt="Diagram" /></td>
<td>$\tilde{G}<em>{1,z}(v - e_1) \times \mathcal{J}^{+, -, -, -}</em>{1, 1, 1, 1}(v, y, x, 0)$</td>
</tr>
<tr>
<td></td>
<td>$u \neq v$</td>
<td><img src="image4" alt="Diagram" /></td>
<td>$2d z g_1 D(x) \tilde{G}<em>{1,z}(u - e_1) \times \mathcal{J}^{+, +, -, -}</em>{1, 1, 0, 1}(u, v, y, x)$</td>
</tr>
<tr>
<td>$d_\omega (0, x) \geq 2$</td>
<td></td>
<td><img src="image5" alt="Diagram" /></td>
<td>$\mathcal{B}^{+, +}<em>{1, 0}(u, e_1) \times \mathcal{J}^{+, +, +, -, -}</em>{2, 1, 0, 0}(x, y, v, u)$</td>
</tr>
</tbody>
</table>

### Table 4.15: Definition of $P^{(1), \iota, \text{rib}, -2} (x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition 2</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td></td>
<td><img src="image6" alt="Diagram" /></td>
<td>$\tilde{G}_{1,z}(e_1)P^{(1), -2}(x, y)$</td>
</tr>
<tr>
<td></td>
<td>$u \neq 0$</td>
<td><img src="image7" alt="Diagram" /></td>
<td>$\mathcal{B}^{+, +}<em>{1, 0}(u, e_1) \times \mathcal{J}^{+, +, +, -, -}</em>{1, 2, 0, 0}(x, y, v, u)$</td>
</tr>
</tbody>
</table>
### Table 4.16: Definition of $A^{a,b}(u, u + v, u + x, u + y) = A^{a,b}(0, v, x, y)$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1, b = 0$</td>
<td>$x = v \Rightarrow w = 0,</td>
</tr>
<tr>
<td>$x \neq v, d_\omega(0, x) = 1$</td>
<td>$u = 0$ $\Rightarrow w = 0,</td>
</tr>
<tr>
<td>$x \neq v, d_\omega(0, x) \geq 2$</td>
<td>$d_\omega(0, x) = 1$ $\Rightarrow w \neq x$</td>
</tr>
<tr>
<td>$a \geq 2, b = 0$</td>
<td>$d_\omega(0, x) = 1$ $\Rightarrow w \neq x$</td>
</tr>
<tr>
<td>$d_\omega(0, x) \geq 2$</td>
<td>$d_\omega(0, x) \geq 2$</td>
</tr>
</tbody>
</table>

Diagram:

```
\[ 2dD(x) \tilde{G}_{3,z}(x) \]
```

```
\[ 2dD(v)\delta_{v,w}\tilde{\mathcal{R}}_{1,2}^{-,+}(x,v) + 2dD(v)\tilde{\mathcal{R}}_{1,1,1}^{-,+}(x,w,v) \]
```

```
\[ 2dD(v)\delta_{v,w}\tilde{\mathcal{R}}_{2,1}^{-,+}(x,v) + 2dD(v)\tilde{\mathcal{R}}_{2,0,1}^{-,+}(x,w,v) \]
```

```
\[ \tilde{\mathcal{R}}_{1,1,1}^{+,+,-}(x,w,v) \]
```

```
\[ \tilde{\mathcal{R}}_{2,0,0}^{+,+,-}(x,w,v) \]
```
Matrix formulation of the diagrammatic bound.

We state the bound on \( \hat{\Xi} \) and \( \hat{\Xi}^l \) using vector and matrix notation. We define the vectors \( \vec{P}, \vec{P}^l \in \mathbb{R}^5 \) by their entries: for \( a \in \{-2, -1, 0, 1, 2\} \) let

\[
(\vec{P})_a = \sum_{x,y} p^{(i,a)}(x,y),
\]

\[
(\vec{P}^l)_a = \sum_{i,x,y} \left( p^{(i,l,\text{step},a)}(x,y) + p^{(i,l,\text{rib},a)}(x,y) \right).
\]

In the same way we define the matrices \( \mathbf{A}, \tilde{\mathbf{A}} \in \mathbb{R}^{5 \times 5} \): for \( a, b \in \{-2, -1, 0, 1, 2\} \) let

\[
(\mathbf{A})_{a,b} = \sup_v \sum_{x,y} A^{a,b}(0, v, x, y),
\]

\[
(\tilde{\mathbf{A}})_{a,b} = \sup_{v,w} \sum_x A^{a,b}(0, v, x, x + w).
\]

Further, we define \( \vec{P} \) and \( \mathbf{A} \) as the analog bounds on the swapped diagrams. These bounds are sufficient to state the bounds on \( \hat{\Xi}^{(N)}(0) \) and \( \hat{\Xi}^{(N),l}(0) \). To formulate bounds on \( \hat{\Xi}^{(N)}(0) - \hat{\Xi}^{(N)}(k) \), we define \( G_z(x;k) = \tilde{G}_z(x)[1 - \cos(k \cdot x)] \) and use the following versions of weighted triangles:

\[
\Delta^1_0(k) = 4d g_z^2 \sum_{x} [1 - \hat{D}(k)] \hat{G}_{3,z}(e_1) + 2(G_z(;k) \star \tilde{G}_{2,z})(0)
\]

\[
+ (G_z(;k) \star \tilde{G}_{1,z}^2)(0),
\]

\[
\Delta^1_1(k) = [1 - \hat{D}(k)] \hat{G}_{3,z}(e_1) + (G_z(;k) \star \hat{G}_{1,z} \star D)(0)
\]

\[
+ (G_z(;k) \star \hat{G}_{1,z} \star \tilde{G}_{0,z} \star D)(0),
\]

\[
\Delta^1_2(k) = \sup \{ G_z(;k) \star \tilde{G}_{0,z}^2(x) \},
\]

\[
\Delta^1_3(k) = 2(D \star \hat{G}_{1,z} \star G_z(;k))(0) + 2(D \star \hat{G}_{1,z} \star \hat{G}_{1,z} \star G_z(;k))(0),
\]

\[
\Delta^1_4(k) = \sup \{ \hat{G}_{1,z} \star G_z(;k)(x) \} + (\tilde{G}_{2,z} \star G_z(;k))(0) + 2(\tilde{G}_{1,z}^2 \star G_z(;k))(0),
\]

\[
\Delta^1_5(k) = (D \star 2 \star \tilde{G}_{0,z}^2 \star G_z(;k))(0),
\]

\[
\Delta^1_6(k) = (D \star 2 \star \tilde{G}_{1,z} \star G_z(;k))(0) + (D \star 2 \star \tilde{G}_{1,z}^2 \star G_z(;k))(0).
\]

These quantities are bounds on weighted triangle. We explain how these arise in the proof of the bound on \( \hat{\Xi}_z(0) - \hat{\Xi}_z(k) \) in Section 4.3.7.

We use these quantities to define the vectors \( \vec{\Delta}^{\text{start}}(k), \vec{\Delta}^{\text{end}}(k) \in \mathbb{R}^5 \) with entries

\[
(\vec{\Delta}^{\text{start}})_a(k) = \Delta^1_{|a|}(k),
\]

for \( a \in \{-2, -1, 0, 1, 2\} \), and

\[
(\vec{\Delta}^{\text{end}}(k))_2 = \Delta^1_4(k), \quad (\vec{\Delta}^{\text{end}}(k))_1 = \Delta^1_3(k),
\]

\[
(\vec{\Delta}^{\text{end}}(k))_0 = \Delta^1_0(k), \quad (\vec{\Delta}^{\text{end}}(k))_{-1} = \Delta^1_1(k),
\]

\[
(\vec{\Delta}^{\text{end}}(k))_{-2} = \Delta^1_2(k).
\]
Further, we define the matrix $\Delta(k) \in \mathbb{R}^{5 \times 5}$ with entries:

\[
\begin{align*}
(\Delta(k))_{0,0} &= \Delta_{0}^{1}(k), \\
(\Delta(k))_{-1,0} &= (\Delta(k))_{0,1} = (\Delta(k))_{0,-1} = \Delta_{1}^{1}(k), \\
(\Delta(k))_{1,0} &= \Delta_{3}^{1}(k), \\
(\Delta(k))_{-1,1} &= \Delta_{2}^{1}(k), \\
(\Delta(k))_{1,1} &= \Delta_{6}^{1}(k),
\end{align*}
\]

and, for $a, b$, where $a$ and $b$ satisfy $\max(|a|, |b|) = 2$ we define

\[
(\Delta(k))_{a,b} = \begin{cases} \\
\Delta_{2}^{1}(k) & a \leq 0, \\
2\Delta_{6}^{1}(k) & a > 0.
\end{cases}
\]

To state the bound on

\[
\sum_{i} \left( \hat{\Xi}_{z}^{(N),i}(0) - \tilde{\Xi}_{z}^{(N),i}(k) \right) \quad \text{and} \quad \sum_{i} \left( \hat{\Xi}_{z}^{(N),i}(0) - \tilde{\Xi}_{z}^{(N),i}(k)e^{ikv} \right),
\]

we need some more notation. We define

\[
\begin{align*}
\Delta_{0}^{II}(k) &= 2dz[1 - \hat{D}(k)]\tilde{G}_{3}(\epsilon_{1}) + (G_{z}(\cdot;k) \ast \tilde{G}_{2}(z))(0) + (G_{z}(\cdot;k) \ast \tilde{G}_{1,z}^{*}(z))(0), \\
\Delta_{1}^{II}(k) &= (G_{z}(\cdot;k) \ast \tilde{G}_{1}(z) \ast \tilde{G}_{0}(z) \ast D)(0), \\
\Delta_{2}^{II}(k) &= \sup_{x \neq 0}(G_{z}(\cdot;k) \ast \tilde{G}_{0}(z) \ast \tilde{G}_{1}(z)(x).
\end{align*}
\]

and the vectors $\tilde{\Delta}_{\text{tota},I}(k), \tilde{\Delta}_{\text{tota},II}(k) \in \mathbb{R}^{5}$ with entries

\[
\begin{align*}
(\tilde{\Delta}_{\text{tota},I}(k))_{a} &= \Delta_{|a|}^{I}(k) + \Delta_{|a|}^{II}(k) + \tilde{G}_{1}(z)(\epsilon_{1})(\Delta_{0,a}) \\
&\quad + \sum_{v} B_{1,2}^{+,+} (v, \epsilon_{1})(\Delta(k))_{-1,a} + \sum_{v} B_{2,1}^{+,+} (v, \epsilon_{1})(\Delta(k))_{-2,a} \\
(\tilde{\Delta}_{\text{tota},II}(k))_{a} &= \Delta_{|a|}^{I}(k) + \Delta_{|a|}^{II}(k) + \tilde{G}_{1}(z)(\epsilon_{1})(\Delta(k))_{0,a} \\
&\quad + 2 \sum_{v} B_{1,2}^{+,+} (v, \epsilon_{1})(\Delta(k))_{-1,a} + 2 \sum_{v} B_{2,1}^{+,+} (v, \epsilon_{1})(\Delta(k))_{-2,a} \\
&\quad + 4dz[1 - \hat{D}(k)]g_{z}^{4} \sum_{l,x,y} P_{l,x,y}^{(1),l,ib}(x, y).
\end{align*}
\]

for $a \in \{-2, -1, 0, 1, 2\}$ to bound diagrams in which the initial square $P_{l,x,y}^{(1),l,ib}(x, y)$ has a weighted line.

### 4.3.4 The diagrammatic bounds

In this section we list all the bounds that we are going to prove. We treat $N = 0, 1, 2$ separately, before stating a final bound for all $N \geq 3$. 

Lemma 4.3.6 (Bounds on the lattice tree coefficients for $N = 0$). Let $z < z_c$. Then,

\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0)}(x) = 1, \quad \sum_{x \in \mathbb{Z}^d} \Psi_z^{(0),K}(x) = \frac{g_z^l}{g_z} = \rho, \quad (4.3.54)
\]

\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0),l}(x) = G_z(e_l), \quad \sum_{x \in \mathbb{Z}^d} \Pi_z^{(0),l,K}(x) \leq (1 - \delta_{l,K}) z g_z G_z^K(e_l), \quad (4.3.55)
\]

and

\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0)}(x)[1 - \cos(k \cdot x)] = \sum_{x \in \mathbb{Z}^d} \Xi_z^{(0),l}(x)[1 - \cos(k \cdot x)] = 0. \quad (4.3.56)
\]

Further,

\[
\sum_{k} \sum_{x \in \mathbb{Z}^d} \Psi_z^{(0),K}(x)[1 - \cos(k \cdot (x - e_K))] = 2d[1 - \hat{D}(k)]K_P, \quad (4.3.57)
\]

\[
\sum_{l} \sum_{x \in \mathbb{Z}^d} \Xi_z^{(0),l}(x)[1 - \cos(k \cdot (x - e_l))] = 2dG_z(e_l)[1 - \hat{D}(k)]. \quad (4.3.58)
\]

Lemma 4.3.7 (Bounds on the lattice tree coefficients for $N = 1$). Let $0 \leq z \leq z_c$. Then,

\[
\hat{\Delta}_z^{(1)}(0) \leq \sum_{x,y} P_{z,0}(x,y) = \rho \langle \bar{P} \rangle_0, \quad (4.3.59)
\]

\[
\hat{\Delta}_z^{(1),l}(0) \leq \frac{1}{2d} \sum_{x,y} P_{z,0}(x,y) = \frac{\rho}{2d} \langle \bar{P}^l \rangle_0, \quad (4.3.60)
\]

\[
\hat{\Delta}_z^{(1)}(0) - \hat{\Delta}_z^{(1)}(k) \leq \rho \Delta_0^1(k), \quad (4.3.61)
\]

\[
\sum_{l} \left( \hat{\Delta}_z^{(1),l}(0) - \hat{\Delta}_z^{(1),l}(k) \right) \leq \rho \Delta_0^{\text{total}}(k)_0, \quad (4.3.62)
\]

\[
\sum_{l} \left( \hat{\Delta}_z^{(1),l}(0) - \hat{\Delta}_z^{(1),l}(k) e^{-ik \cdot e_l} \right) \leq \rho \Delta_0^{\text{total}}(k)_0. \quad (4.3.63)
\]

Lemma 4.3.8 (Bounds on the lattice tree coefficients for $N = 2$). Let $0 \leq z \leq z_c$. Then,

\[
\hat{\Delta}_z^{(2)}(0) \leq \rho \sum_{a=2}^{2} (\bar{P})_a (A)_a(0), \quad (4.3.64)
\]

\[
\hat{\Delta}_z^{(2),l}(0) \leq \frac{\rho}{2d} \sum_{a=2} (\bar{P}^l)_a (A)_a(0), \quad (4.3.65)
\]

\[
\hat{\Delta}_z^{(2)}(0) - \hat{\Delta}_z^{(2)}(k) \leq 2 \rho \Delta_{\text{start}}^2(k) \bar{P} + 2 \rho \bar{P} \Delta_{\text{end}}^2(k), \quad (4.3.66)
\]

\[
\sum_{l} \left( \hat{\Delta}_z^{(2),l}(0) - \hat{\Delta}_z^{(2),l}(k) \right) \leq 2 \rho \Delta_{\text{total}}^1(k) \bar{P} + 2 \rho \bar{P} \Delta_{\text{end}}^2(k), \quad (4.3.67)
\]

\[
\sum_{l} \left( \hat{\Delta}_z^{(2),l}(0) - \hat{\Delta}_z^{(2),l}(k) e^{-ik \cdot e_l} \right) \leq 2 \rho \Delta_{\text{total}}^1(k) \bar{P} + \frac{2}{g_z} \bar{P} \Delta_{\text{end}}^2(k). \quad (4.3.68)
\]
Figure 4.10: Schematic representation of the bound on $\hat{\Xi}_z^{(N)}(0)$ for $N \geq 3$. The backbone is marked by a thicker line.

**Proposition 4.3.9** (Bounds on the lattice tree coefficients for $N \geq 3$). Let $0 \leq z \leq z_c$ and $N \geq 3$, then

$$\hat{\Xi}_z^{(N)}(0) \leq \rho \bar{P}^T A^{N-3} \bar{A} \bar{P},$$  \hspace{1cm} (4.3.69)

$$\hat{\Xi}_z^{(N),d}(0) \leq \frac{\rho}{2d} \bar{P}^T A^{N-3} \bar{A} \bar{P},$$ \hspace{1cm} (4.3.70)

and

$$\hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \leq N \rho \left( \bar{\Delta}^{\text{start}, T}(k) A^{N-2} \bar{P} + \bar{P}^T A^{N-2} \bar{A} \bar{P} \right),$$ \hspace{1cm} (4.3.71)

$$\sum_l \left( \hat{\Xi}_z^{(N),d}(0) - \hat{\Xi}_z^{(N),d}(k) e^{-ik \cdot e_l} \right) \leq N \rho \left( \bar{\Delta}^{\text{iota}, I}(k) A^{N-2} \bar{P} + (\bar{P})^T A^{N-2} \bar{A} \bar{P} \right).$$ \hspace{1cm} (4.3.72)

4.3.5 Proof of $x$-space bounds

We use the building blocks to define the bounding diagrams: For $N \geq 2$, $b \in \{-2, -1, \ldots, 2\}$ we define recursively

$$P^{(N),b}(x, y) = \sum_{u, v \in \mathbb{Z}^d} \sum_{a=2}^{2} P^{(N-1),a}(u, v) A^{a,b}(u, v, x, y),$$ \hspace{1cm} (4.3.75)

$$P^{(N),d,b}(x, y) = \sum_{u, v \in \mathbb{Z}^d} \sum_{a=2}^{2} P^{(N-1),d,a}(u, v) A^{a,b}(u, v, x, y).$$ \hspace{1cm} (4.3.76)
Further, we define
\[ P^{(N),b}_{(x,y)}(u,v) = \sum_{u,v}^{2} P^{(N),a}_{(u,v)} A^{a,b}_{2}(u,v,x,y), \] (4.3.77)
and see that \( P^{(N),b} \) corresponds the swapped version of \( P^{(N),b} \), i.e. the diagram in which the arrows of all backbone pieces is inverted. We next prove that \( P^{(N),d}_{b}, P^{(N),b}_{b} \) and \( P^{(N),d}_{b} \) can be used to create \( x \)-space bounds for the lace-expansion coefficients:

**Lemma 4.3.10** (Pointwise bounds for lattice trees.) **For all** \( i \) **and** \( 0 \leq z \leq z_c, \)
\[
\begin{align*}
\Xi^{(1)}_{z}(x) & \leq \rho P^{(1),0}_{z}(x,x), \quad (4.3.78) \\
\Xi^{(1),d}_{z}(x) & \leq \rho P^{(1),0}_{z}(x,x), \quad (4.3.79) \\
\Xi^{(2)}_{z}(x) & \leq \rho \sum_{u,v,a} P^{(1),a}_{z}(u,v) A^{a,0}_{2}(u,v,x,x), \quad (4.3.80) \\
\Xi^{(2),d}_{z}(x) & \leq \rho \sum_{u,v,a} P^{(1),d}_{z}(u,v) A^{a,0}_{2}(u,v,x,x). \quad (4.3.81)
\end{align*}
\]

**Further, for** \( N, M \geq 1, \)
\[
\begin{align*}
\Xi^{(N+M+1)}_{z}(x) & \leq \rho \sum_{u,v,\omega,\gamma} P^{(N),a}_{z}(u,\omega) A^{a,b}_{2}(u,\omega,\gamma) P^{(M),b}_{z}(x-\omega,x-\gamma), \quad (4.3.82) \\
\Xi^{(N+M+1),d}_{z}(x) & \leq \rho \sum_{u,v,\omega,\gamma} P^{(N),d}_{z}(u,\omega) A^{a,b}_{2}(u,\omega,\gamma) P^{(M),b}_{z}(x-\omega,x-\gamma). \quad (4.3.83)
\end{align*}
\]

**Proof.** In Section 4.3.1 we have already discussed how to bound \( \Xi^{(1)}_{z}(x) \) and obtained the bound \( 4.3.4 \). Reviewing the decomposition we used to obtain this bound we see that three parts are repulsive and we obtain the bound
\[
\Xi^{(1)}_{z}(x) \leq g^{(i)}_{z} \sum_{(x,y)} \mathcal{F}_{1,0,0}^{+,+,-}(x,\omega,0), \quad (4.3.84)
\]
This bound does not make use of the fact that the total loop consists of four steps and that if \( d_{\omega}(0,x) = 1 \) then \( w \not\in (0,x) \), which follows from the non-backtracking condition of the rib walk. Considering the special case for \( x \) and using these properties, as shown Table 4.1 we obtain the stated improved bound.

For \( \Xi^{(1),d} \) we see that the indicator \( I_{i}(\omega) \) defined in 2.3.16 introduces additional constraints on the diagrams. In Table 4.7 and Table 4.12 we show how to integrate these constraints into the diagram and conclude the stated bound. For the bound created in Table 4.12 we choose \( u \) to be the last point that the paths \( b^{R_{0}}_{\omega}(0,e_{i}) \) and \( b^{R_{0}}_{\omega}(0,\omega_{1}(\omega)) \) have in common.

For \( N = 2, \) we label the six relevant points as given in Figure 4.11. For a lace \( L = \{0t_{1},s_{2}|\omega|\} \) we define \( v = w_{1}(\omega), \omega = w_{2}(\omega) \) and \( y = b_{s_{2}}^{\omega} \). If \( s_{2} < t_{1} \) then we define \( y' = s_{2} \omega_{s_{j}} \) and \( a = 1 \) if \( t_{1} - s_{2} = 1 \) and \( a = 2 \) if \( t_{1} - s_{2} \geq 2 \). For \( s_{2} = t_{1} \) we identify the last point \( u \) that the paths \( b^{R_{0}}_{\omega}(b_{s_{2}},v) \) and \( b^{R_{0}}_{\omega}(b_{s_{2}},w) \) have in common and
Diagrammatic Bounds

Figure 4.11: Labeling in the proof of the bound on $\Xi^{(2)}_z$

Define $a = 1$ if $d_{R^o_{s_2}}(\overline{b}_{s_2}, u) = 1$ and $a = -2$ if $d_{R^o_{s_2}}(\overline{b}_{s_2}, u) \geq 2$.

Let us collect all the information we have about the lines of the diagram: We know that each square is repulsive and consists of at least four steps. By the non-backtracking condition we know that if $t_1 = 1$ then $v \not\in \{0, y\}$ and if $|\omega| - s_2 = 1$ then $w \not\in \{y, x\}$.

We let the line that is shared by the two squares contribute to the left square. Then the left square is bounded by $P^{(1)}_{z}(u, y)$ for $a \leq 0$ and $P^{(1)}_{z}(y', y)$ for $a \geq 0$, while the right square is bounded by the double open square $\overline{A}^{a,0}$. Going through all cases for $a$ it is not difficult to see that (4.3.80) holds.

The bound on $\Xi^{(2),i}_z$ in (4.3.81) is obtained in the same way.

We extend the analysis to $N \geq 3$: The proof of this lemma is an adaptation of the proofs of Lemma 4.2.4 and Lemma 4.2.6. We will only give an outline of the proof, as the full proof is not very insightful.

Let $N \geq 1$, $y \in \mathbb{Z}^d$ and fix a rib walk $\omega$ and a lace $L = \{s_1 t_1, \ldots, s_N t_N\} \in \mathcal{L}^{(N)}[0, |\omega|]$. We define the endpoint of the next-to-last edge by $t_{N-1}(L) = t_{N-1}$ with $t_0(L) = 0$. We define $E(\omega, L, N, a, x, y)$ to be the indicator that the following holds: $\overline{b}^o_{t_N} = x$, the ribs $R^o_{s_N}$ and $R^o_{t_N}$ intersect, and

i.) for $a = 2$: there exists a $s < t_N - 1$ such that $\overline{b}^i_s = y$;

ii.) for $a = 1$: $\overline{b}^i_{t_{N-1}} = y$;

iii.) for $a = 0$: $x = y$;

iv.) for $a = -1$: the backbone from $x$ to $w_N(\omega)$ on $B^o_{t_N}(x, w_N(\omega))$ passes through $y$ and $d_{R^o_{t_N}}(x, y) = 1$;

v.) for $a = -2$: the backbone from $x$ to $w_N(\omega)$ on $B^o_{t_N}(x, w_N(\omega))$ passes through $y$ and $d_{R^o_{t_N}}(x, y) \geq 2$.

The indicator $E$ allows us to identify the form of the $N$th intersection rib, and whether the backbone moves up/down ($a = 1, 2$) at the end of the $N$-square of the
We prove (4.3.90)-(4.3.92) by proving the inequality we sum over in $J_R$ where we are sloppy in our notation as

$$J^{(1)}_{b,y}[0,n] = \sum_{L \in \mathcal{L}^{(1)}[0,n]} \prod_{s \in L} (-\mathcal{U}_{s,t}) \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'}) E(\omega, L, 1, b, \bar{b}_{[\omega]}, y). \quad (4.3.85)$$

As explained at the beginning of Section 4.3.3 for a diagram $P^{(N),b}$ the sign of the index $b$ encodes whether the backbone ends at the top or bottom of the right-most square. The indicator $E$ does not contain the information whether the rightmost line is part of the backbone or not. Next we create the indicator $F$ to combine these different notions: For lace $L \in \mathcal{L}^{(M)}[0, |\omega|]$ and a $N \leq M$ we define $F(L, N) = 1$ if the number of $j \in \{2, \ldots, N\}$ such that $t_j > s_{j-1}$ is odd. Otherwise, we define $F(L, N) = 0$. If $F(L, N) = 1$ then the backbone in the diagram ends at the rightmost top point. For $N \geq 2, b \in \{-2, -1, \ldots, 2\}$ and a rib walk $\omega$ we define

$$J^{(N)}_{b,y}[0,n] = \sum_{L \in \mathcal{L}^{(N)}[0,n]} \prod_{s \in L} (-\mathcal{U}_{s,t}) \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s't'})$$

$$\times \left( F(L, N-1) E(\omega, L, N, -b, \bar{b}_{[\omega]}, y) + (1 - F(L, N-1)) E(\omega, L, N, b, \bar{b}_{[\omega]}, y) \right) \quad (4.3.86)$$

We define

$$\hat{\Xi}^{(N),a}_{z}(x, y) = \sum_{\omega \in \mathcal{W}^T(x)} z^{[\omega]} Z_{R}[0, |\omega|] J^{(N-1)}_{a,y}[0, |\omega|], \quad (4.3.87)$$

$$\hat{\Xi}^{(N),i,a}_{z}(x, y) = \sum_{\omega \in \mathcal{W}^T(x)} 1_{\omega}(\omega) z^{[\omega]} Z_{R}[0, |\omega|] J^{(N-1)}_{a,y}[0, |\omega|]. \quad (4.3.88)$$

Further, we define

$$\Xi^{(M),a}_{z}(x, y) = \sum_{\omega \in \mathcal{W}^T(x)} z^{[\omega]} Z_{R}[0, |\omega| - 1] J^{(N-1)}_{a,y}[0, |\omega|] z^{R_{t_N}}_{z}^{(\bar{b}_{t_N}, w_N(\omega))}, \quad (4.3.89)$$

where we are sloppy in our notation as $R_{t_N}^{(\bar{b}_{t_N}, w_N(\omega))}$ depends on the lace that we sum over in $J^{(N-1)}_{a,y}$. We next prove that

$$\hat{\Xi}^{(N),b}_{z}(x, y) \leq \rho \hat{\Xi}^{(N),b}_{z}(x, y), \quad (4.3.90)$$

$$\hat{\Xi}^{(N),i,b}_{z}(x, y) \leq \rho \hat{\Xi}^{(N),i,b}_{z}(x, y), \quad (4.3.91)$$

$$\hat{\Xi}^{(N),i,b}_{z}(x, y) \leq \rho \hat{\Xi}^{(N),i,b}_{z}(x, y). \quad (4.3.92)$$

We prove 4.3.90 - 4.3.92 by proving the inequality

$$\hat{\Xi}^{(N),b}_{z}(x, y) \leq \sum_{u,v \in \mathbb{Z}^d} \sum_{a=-2}^{2} \hat{\Xi}^{(N-1),a}_{z}(u, v) A^{a,b}(u, v, x, y). \quad (4.3.93)$$

using induction. This uses the same ideas in the proof of Lemma 4.2.4. We prove an inequality of $J^{(N)}_{y}[0,n]$ by a combination of $J^{(N)}_{a,y}[0,n]$ and events described by
E. Then we extract the last square of a $\Xi^{(N),b}_z(x, y)$ by identifying the vertices $u, v$ where we can cut the connections. In this way (4.3.90) - (4.3.92) can be obtained using induction. In the derivation of the bound for $\Xi^{(2)}_z$ we have already seen how to initialize this induction for $b = 0$. While the idea of this deconstruction is quite simple, the rigorous proof is very elaborate and will be omitted.

Before proceeding we need to comment on a feature of the proof of (4.3.92) that is not straightforward. When proving that

$$\Xi^{(N),a,b}_z(x, y) \leq \sum_{u, v \in Z^d} \sum_{a=-2}^2 \Xi^{(N-1),a}_z(u, v) A^{a,b}(u, v, x, y). \tag{4.3.94}$$

for an $N \geq 2$ we extract the last square of the diagram and bound it by $A^{a,b}(u, v, x, y)$. In this extraction we have to be careful how we distribute the weights of the ribs. We explain how we do this by an example. Let us assume $a, b > 0$. Then the diagram begins and ends at the top part of the diagram, which implies that $t_{N-1} = s_N$ and $t_{N-2} = s_{N-1}$. In this case $u$ and $v$ are points on the ribs $R^o_{t_{N-1}}$ and $R^o_{t_N}$ respectively. The diagram $\Xi^{(N),a,b}_z(x, y)$ contains the weight of the connecting planted tree $z_{t_N}^{R^o_{t_N}}(b_{t_{N-1}}, w_{N}(\omega))$ (see Definition 4.3.2) and we know that $y$ is on the path from 0 to $u \omega_N(\omega)$. We split the connected planted tree $B^o_{t_N}(x, w_{N}(\omega))$ at $y$ into two parts: $P_1 = B^o_{t_N}(x, w_{N}(\omega)) \setminus P_2 = B^o_{t_N}(x, w_{N}(\omega)) \setminus P_1$. Therefore, we identify $u = \overline{b}_{t_{N-1}}$ and $v$ to be the last vertex that the paths $b^o_{t_{N-1}}(u, w_{N-1}(\omega))$ and $b^o_{t_{N-1}}(u, w_{N}(\omega))$ have in common. Then we split the rib $R^o_{t_{N-1}}$ into three paths: $P_3 = B^o_{t_{N-1}}(u, w_{N}(\omega)), P_3 = B^o_{t_{N-1}}(v, w_{N}(\omega)) \setminus P_3$ and the trimmed rib $R^o_{t_{N-1}} \setminus B^o_{t_{N-1}}(v, w_{N}(\omega))$. We combine the trimmed rib with that rib walk from $u$ to $y$ to create the fourth piece $P_4$. The combination of $P_1, P_2, P_3, P_4$ is a repulsive open square that we bound by $A^{a,b}(u, v, x, y)$. The weight of $B^o_{t_{N-1}}(v, w_{N}(\omega)) \setminus P_3$ is bounded in $\Xi^{(N-1),a}_z(u, v)$. The other cases are bounded in a similar way.

To complete the proof we need to show that:

$$\Xi^{(N+1),a,b}_z(x) \leq \sum_{a,b,u,v,w,y} \Xi^{(N),a}_z(u, v) A^{a,b}(u, v, w, y) \Xi^{(M),b}_z(x - w, x - y), \tag{4.3.95}$$

$$\Xi^{(N+1),a,b}_z(x) \leq \sum_{a,b,u,v,w,y} \Xi^{(N),a}_z(u, v) A^{a,b}(u, v, w, y) \Xi^{(M),b}_z(x - w, x - y). \tag{4.3.96}$$

This is done as shown in the proof of Lemma 4.2.6. For a lace $L = [s_1 t_1, \ldots, s_N t_N]$ we define $u = \overline{b}_{t_N}$ and $w = \overline{b}_{s_{N+2}}$. Further, we identify by $y$ and $\omega$ the other two points for $A^{a,b}$. For $s_{N+1} = t_N$ the point $\omega$ is the last common point of $b^o_{s_{N+1}}(u, w_{N+1}(\omega))$ and $B^o_{s_{N+1}}(u, w_{N+1}(\omega))$ and for $s_{N+1} < t_N$ we define $\omega = \overline{b}_{s_{N+1}}$. Similarly, $y$ is either $\overline{b}_{s_{N+2}}$ or the last common point on $R^o_{t_{N+1}}$.

Then we extract the edge $s_{N+1} t_{N+1}$ from a lace $L \in \mathcal{L}^{(N+1)}[0, |\omega|]$ and split $L$ into two laces, $L_1 \in \mathcal{L}^{(N)}[0, t_N]$ and $L_2 \in \mathcal{L}^{(M)}[s_{N+2}, |\omega|]$. We weaken the avoidance constraint and obtain (4.3.95) and (4.3.96). \qed
4.3.6 Proof of the bounds on the absolute value.

Here we explain how to obtain the bounds on $\hat{\Xi}^{(N)}$ and $\hat{\Xi}^{(N),t}$. We combine the $x$-space bound (4.3.78) and (4.3.79) with the definitions (4.3.27) and (4.3.28) and immediately obtain the bounds for $N = 1$: (4.3.59)-(4.3.60).

For $N = 2$ we use (4.3.80) to compute

$$\sum_{x} \Xi_{x}^{(2)}(x) \leq \sum_{u,v,a,x} P_{z}^{(1),a}(u, v) \bar{A}^{a,0}(u, v, x, x)$$

$$= \sum_{u,v,a,x} P_{z}^{(1),a}(u, u + v) \bar{A}^{a,0}(0, v, x, x)$$

$$\leq \sum_{a} \left( \sum_{u,v} P_{z}^{(1),a}(u, u + v) \right) \left( \sup_{v \in \mathbb{Z}^{d}} \sum_{x} \bar{A}^{a,0}(0, v, x, x) \right), \quad (4.3.97)$$

which corresponds to the bound given in Lemma 4.5.3. The bound on $\Xi_{z}^{(2),t}$ is obtained in the same way. For a bound for $N \geq 3$ we use the same argument already used for SAW, see (4.3.95). We use the translation invariance of the blocks to compute

$$\sum_{x \in \mathbb{Z}^{d}} \Xi_{x}^{(N+1)}(x) \leq \sum_{a,b,u,v,w,y,x} \Xi_{x}^{(N),a}(u, u + v) \bar{A}^{a,b}(0, v, w - u, w - u + y)$$

$$\times \Xi_{x}^{(M),b}(w - x, w + y - x)$$

$$= \sum_{a,b,u,v,w,y,x} \Xi_{x}^{(N),a}(u, u + v) \bar{A}^{a,b}(0, v, w, w + y) \Xi_{x}^{(M),b}(x, x + y)$$

$$\leq \sum_{a,b} \left( \sum_{u,v} P_{z}^{(N),a}(u, v) \right) \left( \sup_{y \in \mathbb{Z}^{d}} \sum_{w} \bar{A}^{a,b}(0, v, w, w + y) \right) \left( \sum_{x,y} P_{z}^{(M),b}(x, y) \right). \quad (4.3.98)$$

For the first term we know by the construction in (4.3.75) that for $N \geq 1$:

$$\sum_{x,y \in \mathbb{Z}^{d}} P_{z}^{(N),b}(x, y) = \sum_{a=-2}^{2} \sum_{u,v,x,y \in \mathbb{Z}^{d}} P_{z}^{(N+1),a}(u, u + v) A^{a,b}(u, u + v, x, y)$$

$$\leq \sum_{a=-2}^{2} \left( \sum_{u,v \in \mathbb{Z}^{d}} P_{z}^{(N+1),a}(u, v) \right) \left( \sup_{x,y \in \mathbb{Z}^{d}} A^{a,b}(0, v, x, y) \right) \quad (4.3.100)$$

We use this recursively and see that $\bar{P}_{z}^{T}$ is a bound on the contribution of $P_{z}^{(1),b}$. The sums over $P_{z}^{(M),b}$ and $P_{z}^{(M),t,b}$ are bounded in the same way. Before bounding the middle term of (4.3.99) by $(\bar{A})_{a,b}$ and the last term of (4.3.100) by $(A)_{a,b}$, we show how to obtain an improved bound for the case $|a| = 1$ or $|b| = 1$ using symmetry. To show this idea assume in (4.3.98) that $a = 1$. Then we know that only those $v$...
contribute that are neighbors of the origin. We use the spatial symmetry of $\Xi$ to see that

$$\sum_u \Xi^{(N),1}_z(u, u + v) = \sum_u \Xi^{(N),1}_z(u, u + e_1) = \frac{1}{2d} \sum_{u,v} \Xi^{(N),1}_z(u, u + v) \quad (4.3.101)$$

is independent of $v$. Thus, we know for a summand in $\Xi^{(N),a}$ with $a = 1$ that

$$\sum_{a,b} \sum_{u,v,w,y,x} \Xi^{(N),a}_z(u, u + v) \tilde{A}^{a,b}(0, v, w, w + y) \Xi^{(M),b}(x, x + y)$$

$$\leq \frac{1}{2d} \left( \sum_{u,v} \Xi^{(N),1}_z(u, v) \right) \left( \sup_{y \in \mathbb{Z}^d} \sum_{v,w} \tilde{A}^{1,b}(0, v, w, w + y) \right) \left( \sum_{x,y} \Xi^{(M),b}(x, y) \right). \quad (4.3.102)$$

Using this argument for all shared lines with length one we see that the middle term of $\Xi^{(N),a}$ is bounded by $(\tilde{A})_{a,b}$ and the last term of $(4.3.100)$ is bounded by $(A)_{a,b}$.

The bound $\Xi^{(N)}$ is obtained in the same way, where $P^{(N),a}_z$ is replaced with $P^{(N),a,1}_z$. Therefore, we note that

$$\sum_{u,t} \Xi^{(N),1}_z(u, u + e_k) = \sum_{u,t} \Xi^{(N),1}_z(u, u + e_1), \quad (4.3.103)$$

so that we can also spatial symmetry for the case $|a| = 1$, as done in $(4.3.101)$ - $(4.3.102)$.

### 4.3.7 Bound on weighted diagram for $\hat{\Xi}^{(N)}_z$

In this section we prove the bound on $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$. For $N = 0$ the bound is trivial, so we begin by proving the bound for $N \geq 1$.

**Bound for $N = 1$.** To obtain a bound on $\hat{\Xi}^{(1)}_z(0) - \hat{\Xi}^{(1)}_z(k)$ we recall the bound on $\Xi_z(x)$ $(4.3.84)$ and multiply this bound with $[1 - \cos(k \cdot x)]$ to obtain:

$$\hat{\Xi}^{(1)}_z(0) - \hat{\Xi}^{(1)}_z(k) \leq g_t^z \sum_{x,w \in \mathbb{Z}^d} G_z(x; k) \tilde{G}_0,z(w) \tilde{G}_0,z(w - x)$$

$$= 2 g_t^z \sum_{x \in \mathbb{Z}^d} G_z(x; k) \tilde{G}_0,z(x) \quad (4.3.104)$$

$$+ g_t^z \sum_{w,x \in \mathbb{Z}^d} G_z(x; k) \tilde{G}_1,z(w) \tilde{G}_1,z(w - x), \quad (4.3.105)$$

where we recall that $G_z(x; k) = \tilde{G}_z(x)[1 - \cos(k \cdot x)]$. Then we investigate $(4.3.104)$, which corresponds to the case that either $0 = w$ or $x = w$. For the discussion we assume that $w = 0$. The term $x = 0$ does not contribute and if $x$ is a neighbor of the origin then we know that the backbone consists of more than one step. Thereby,

$$g_t^z \sum_{x \in \mathbb{Z}^d} G_z(x; k) \tilde{G}_0,z(x) \leq 2d z (g_t^z)^2 \tilde{G}_3,z(e_1)[1 - \hat{D}(k)] + g_t^z \sum_{x \in \mathbb{Z}^d} G_z(x; k) \tilde{G}_2,z(x). \quad (4.3.106)$$
Adding the bounds \(4.3.105\)–\(4.3.106\) we obtain the bound stated in \(4.3.61\).

**Distribution of the displacement.** We want to distribute the weight \([1 - \cos(k \cdot x)]\) along the lines of the diagram using Lemma 3.4.1. Therefore, we define \((x_i)\) as given in Figure 4.12. As you see in Figure 4.12 we always distribute the displacement along the bottom-line of the diagram. We first split \(x = \sum_i x_i\), where each \(x_i\) characterizes the displacement of a square. For each \(x_i\) we use the bounds \(4.3.82\) and \(4.3.83\) of Lemma 4.3.10, where we choose \(M\) and \(N\) so that the weight \([1 - \cos(k \cdot x_i)]\) is on the bottom lines of the intermediate piece \(\bar{A}\).

![Figure 4.12: The choice of the element for the rewrite of \(x_i\) at the example of the diagrammatic representation of \(\Xi^{(i)}(x)\).](image)

**Bounds on the weighted double-open square.** After creating the bounding diagrams in which only bottom lines are weighted, we bound these diagrams as already seen in Section 4.3.6. To bound these weighted squares we define the building block \(H^{a,b}\), which is closely related to the double-open square. A sketch of this new building block is given in Figure 4.13. As we see in Figure 4.13 the block \(H^{a,b}(k)\) is very alike the double open square \(\bar{A}^{a,b}\) for \(a \leq 0\). Thus, we define for \(a \leq 0\) and \(b \leq 0\)

\[
H^{a,b}(k; u, v, x, y) = \bar{A}^{a,b}(u, v, x, y)[1 - \cos(k \cdot (x - u))], \quad \tag{4.3.107}
\]

\[
H^{a,-b}(k; u, v, x, y) = \bar{A}^{a,-b}(u, v, x, y)[1 - \cos(k \cdot (y - u))]. \quad \tag{4.3.108}
\]
For $a = 1, 2$ we define

\[
H^{1,0}(k; u, v, x, y) = 2dD(u - v)\delta_{y,x}(\tilde{G}_{1,z}(x - u) + \tilde{G}_{2,z}(x - u))G_z(k; x - v) + 4dD(u - v)\tilde{G}_{1,z}(x - u)(\tilde{G}_{1,z} \ast G_z(:; k))(x - v),
\]

\[(4.3.109)\]

\[
H^{2,0}(k; u, v, x, y) = \delta_{y,x}(\tilde{G}_{1,z}(x - u) + \tilde{G}_{2,z}(x - u))G_z(k; x - v) + 2\tilde{G}_{1,z}(x - u)(\tilde{G}_{1,z} \ast G_z(:; k))(x - v),
\]

\[(4.3.110)\]

\[
H^{1,1}(k; u, v, x, y) = 2(2d)^2 D(u - v)D(x - y)\tilde{G}_z(x - u)(G_z(:; k) \ast \tilde{G}_z)(v - y),
\]

\[(4.3.111)\]

\[
H^{1,2}(k; u, v, x, y) = 4dD(u - v)\tilde{G}_z(x - u)(G_z(:; k) \ast \tilde{G}_z)(v - y),
\]

\[(4.3.112)\]

\[
H^{2,2}(k; u, v, x, y) = 2\tilde{G}_z(x - u)(G_z(:; k) \ast \tilde{G}_z)(v - y).
\]

\[(4.3.113)\]

We use symmetry to define the remaining cases as

\[
H^{1,-1}(k; u, v, x, y) = H^{1,1}(k; u, v, y, x), \quad H^{2,-2}(k; u, v, x, y) = H^{2,2}(k; u, v, y, x),
\]

\[
H^{1,-2}(k; u, v, x, y) = H^{1,2}(k; u, v, y, x), \quad H^{-1,2}(k; u, v, x, y) = H^{1,2}(k; v, u, x, y), \quad H^{-1,-2}(k; u, v, x, y) = H^{1,2}(k; v, u, y, x).
\]

\[(4.3.114)\]

Next, we show how the building block $H^{a,b}$ can be bounded by the entries of $\Delta$. We begin with $H^{0,0}$. By definition

\[
\sup_{v,y} \sum_x H^{0,0}(k; 0, v, x + y, x) = \sum_x A^{0,0}(0, 0, x, x)[1 - \cos(k \cdot x)]
\]

\[
= \sum_x P^{(1),0}(x, x)[1 - \cos(k \cdot x)] = \Delta^1_0(k).
\]

\[(4.3.115)\]

We review $P^{(1),0}$ as defined in Table 4.1 and see that the weighted version is bounded by $\Delta^1_0(k)$. Then we look at $H^{0,-1}$ and see that this can be bounded in a similar way.
we bound this simply by twice the bound we obtained for the weight along two lines

\[ \frac{1}{2d} \sum_{x,y} R_{0}^{-1}(0, v, x + y, x) \]

\[ = \sum_{x,k} A_{0,0}(0, 0, x + e_x, x) [1 - \cos(k \cdot x)] \leq \frac{1}{2d} \sum_{x,k} (\tilde{G}_{0,z} \ast \tilde{G}_{0,z})(x + e_k) G_z(x; k) \]

\[ = [1 - \hat{D}(k)] \tilde{G}_z(e_1) + (\tilde{G}_{1,z} \ast D \ast G_z(; k))(0) + (\tilde{G}_{0,z} \ast \tilde{G}_{1,z} \ast D \ast G_z(; k))(0) = \Delta_1^1(k). \] (4.3.116)

By symmetry we obtain the same bound for \( R_{0,1} \) and \( R_{1,0} \). For \( R_{1,0} \) we distribute the weight along two lines \( v \leftrightarrow w \) and \( w \leftrightarrow x \). We bound first bound case that one of the non-weighted bottom line is trivial and obtain the bound:

\[ \frac{1}{2d} \sum_{x,k} R_{1,0}(k; 0, e_k, x, x) = 2(D \ast \tilde{G}_{1,z} \ast G_z(; k))(0) \]

\[ + 2(D \ast \tilde{G}_{1,z} \ast \tilde{G}_{1,z} \ast G_z(; k))(0) = \Delta_3^1(k). \] (4.3.117)

Next we compute a bound for \( R_{a,b} \) with \( a, b \in \{-1, 1\} \). We begin with \( R_{-1,-1} \). We used \( A^{-1,-1} \), see [4.3.13], to define \( R_{-1,-1} \). Therefore, it is not difficult to see that

\[ \frac{1}{(2d)^2} \sum_{i,k} R_{-1,-1}(k; 0, e_t, x, x + e_k) \leq (D \ast 2 \ast \tilde{G}_{0,z} \ast G_z(; k))(0) = \Delta_3^1(k). \] (4.3.118)

By symmetry we conclude the same bound for \( R_{-1,1} \). For \( R_{1,-1} \) we also have two possible weights. As for \( R_{1,0} \) we consider the special case that one of the non-weighted bottom lines is trivial and obtain the bound:

\[ \frac{1}{(2d)^2} \sum_{i,k} R_{1,-1}(k; 0, e_t, x, x + e_k) \leq 2(D \ast 2 \ast \tilde{G}_{1,z} \ast G_z(; k))(0) \]

\[ + (D \ast 2 \ast \tilde{G}_{1,z} \ast G_z(; k))(0) = \Delta_5^1(k). \] (4.3.119)

For all \( a \leq 0 \) and \( b \leq 0 \) we see that

\[ \sup_{y,v} \sum_x R_{a,b}(k; 0, v, x, x + y) \leq \sup_{y,v} \sum_x G_z(x; k) \tilde{G}_{0,z}^2(x + y - v) \]

\[ = \sup_{v} \sum_x G_z(x; k) \tilde{G}_{0,z}^2(x - v) = \Delta_2^1(k). \] (4.3.120)

For \( b > 0 \) we see that the same bound holds where in this case the role of \( x \) and \( x + y \) are interchanged. For \( a < 0 \) we have to split the weight into two pieces. Therefore, we bound this simply by twice the bound we obtained for \( a \leq 0 \), i.e. \( \Delta_2^1(k) \).
Bound for $N = 2$. We will discuss the case $N = 2$ in all detail and show for $N \geq 3$ how to extend the ideas shown here. In the following we use the notation of the proof of Lemma 4.3.10, see also Figure 4.11. By Lemma 4.3.10 we know that

$$\hat{\Xi}_z^{(2)}(0) - \hat{\Xi}_z^{(2)}(k) = \rho \sum_{x \in \mathbb{Z}^d} \Xi_z^{(2)}(x) [1 - \cos(k \cdot x)]$$

$$\leq \sum_{x,u,v,a} P_z^{(1),a}(u,v) \tilde{A}^{a,0}(u,v,x,x) [1 - \cos(k \cdot x)]. \quad (4.3.121)$$

By symmetry we conclude that also

$$\hat{\Xi}_z^{(2)}(0) - \hat{\Xi}_z^{(2)}(k) \leq \rho \sum_{x,u,v,a} \tilde{A}^{0,0}(0,0,0) H_z^{(1),a}(u - x, v - x) [1 - \cos(k \cdot x)]. \quad (4.3.122)$$

Before using these bound we split the weight using

$$[1 - \cos(k \cdot x)] \leq 2 [1 - \cos(k \cdot u)] + 2 [1 - \cos(k \cdot (x - u))] \quad \text{for } a \leq 0 \quad (4.3.123)$$

$$[1 - \cos(k \cdot x)] \leq 2 [1 - \cos(k \cdot v)] + 2 [1 - \cos(k \cdot (x - v))] \quad \text{for } a > 0 \quad (4.3.124)$$

and then use (4.3.121) and (4.3.122) to obtain

$$\hat{\Xi}_z^{(2)}(0) - \hat{\Xi}_z^{(2)}(k) \leq 2 \rho \sum_{x,u,v,a} \frac{2}{2} H^{0,a}(k;0,0,u,v) P_z^{(1),a}(u - x, v - x)$$

$$+ 2 \rho \sum_{x,u,v,a} \frac{2}{2} P_z^{(1),a}(u,v) H^{a,0}(k;u,v,x,x). \quad (4.3.125)$$

For $a \in \{-2, 0, 2\}$ we split this, as already seen in (4.3.97), e.g.,

$$\sum_{x,u,v} H^0,2(k;0,0,u,v) P_z^{(1),2}(u - x, v - x)$$

$$\leq \left( \sup_{u \in \mathbb{Z}^d} \sum_{v} H^0,2(k;0,0,u,v + u) \right) \left( \sum_{x,u} P_z^{(1),2}(x, x + u) \right) \quad (4.3.126)$$

For $a \in \{-1, 1\}$ we use symmetry to obtain a bound of the form:

$$\sum_{x,u,v} H^0,1(k;0,0,u,v) P_z^{(1),1}(u - x, v - x)$$

$$= \sum_{v,k} H^0,1(k;0,0,u,v + e_k) \sum_{x} P_z^{(1),1}(x + e_k, x)$$

$$= \frac{1}{2d} \sum_{v,k} H^0,1(k;0,0,u,v + e_k) \sum_{x,k} P_z^{(1),1}(x, x + e_k). \quad (4.3.127)$$

We defined $\bar{P}_a$ and $\bar{P}_a$ to be the sum over $x, \kappa$, as in the former inequality. In the preceding paragraph we have already discussed that the sum over $H^{a,b}$ is bounded by the corresponding element of $\tilde{A}^{\text{start}}(k)$ and $\tilde{A}^{\text{end}}(k)$, which completes the proof of the bound on $\hat{\Xi}_z^{(2)}(0) - \hat{\Xi}_z^{(2)}(k)$. 

---

*Diagrammatic Bounds*
Bound for $N \geq 3$. For a bound on $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$ for $N \geq 3$ we first split the weight $[1 - \cos(k \cdot x)]$ into $N$ pieces, create $N$ bounding diagrams. Then we obtain that $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$ is bounded by

$$N \rho \sum_{b, u, v, x} H^{0, b}(k; 0, 0, u, v, u, v) \Xi^{(N-1), b}(u - x, v - x)$$

$$+ N \rho \sum_{M=1}^{N-2} \sum_{a, b, u, v, w, y} \Xi^{(M), a}(u, v) H^{a, b}(k; u, v, w, y) \Xi^{(M), b}(w - x, y - x)$$

$$+ N \rho \sum_{a, u, v, x} \Xi^{(N-1), a}(u, v) H^{a, 0}(k; u, v, x, x).$$

(4.3.128)

We bound the three pieces as in (4.3.99) and obtain the bound stated in (4.3.72).

4.3.8 Bound on weighted diagram for $\hat{\Xi}^{(N), t}_z$

In this section we show how to modify the arguments of the preceding section to obtain bounds on

$$\sum_i \left( \hat{\Xi}^{(N), t}_z(0) - \hat{\Xi}^{(N), t}_z(k) \right), \quad \sum_i \left( \hat{\Xi}^{(N), t}_z(0) - \hat{\Xi}^{(N), t}_z(k) e^{-i k \cdot e_t} \right).$$

(4.3.129)

Since the only difference between $\Xi^{(N)}_z$ and $\Xi^{(N), t}_z$ is the form of the initial diagram, the bound will also only differ in how we bound the very first square. For $\Xi^{(N), t}_z$ we know that either the first step of the backbone is to $e_t$ or $e_t$ is a vertex of the first rib. For the $x$-space bound we bounded these two contributions by $P^{(1), t, \text{step}, a}$ and $P^{(1), t, \text{rib}, a}$, respectively.

In the following we bound the weighted versions of these diagrams.

![Figure 4.14: The weighted initial block. Picture is given only for $a \leq 0$.](image)

Fixed initial step to $e_t$. We begin with $P^{(1), t, \text{step}, a}$. If we sum over $t$ then the constraint of the first fixed step can be omitted, see the first line of Figure 4.15. Thereby, we can bound the diagram with weight $[1 - \cos(k \cdot x)]$ by the building block $H^{0, a}$. For the total weight $[1 - \cos(k \cdot (x - e_t))]$ we change the way in which we bound the first square. We bound this weighted open square into three parts:
Figure 4.15: Graphical representation of two bounds on the weighted initial block. The pictures are given for $a \leq 0$.

1. the backbone from $e_i$ to $\vec{b}_{s_2}^\omega$,

2. the first step of the backbone and the first rib $R_0^\omega$,

3. the part of the rib leading to an intersection point with $R_0^\omega$: $B_{t_1}(\vec{b}_{s_2}^\omega, \nu_1(\omega))$.

This idea corresponds to the second line of Figure 4.15. We sum over $\iota$ and shift the diagram by $e_i$. The resulting diagram corresponds to the diagram with weight $[1 - \cos(k \cdot x)]$, with the exception that we know that the connection $0 \rightarrow v$ has at least one step. We define $H_{\text{step}, a}$ as the bounding diagram for this case:

$$H_{\text{step},0}(k; x, y) = \delta_{x,y}G_z(x; k)(\tilde{G}_{1,z} \ast \tilde{G}_{0,z})(x),$$
$$H_{\text{step},-1}(k; x, y) = 2dD(x - y)G_z(x; k)(\tilde{G}_{1,z} \ast \tilde{G}_{0,z})(y),$$
$$H_{\text{step},-2}(k; x, y) = G_z(x; k)(\tilde{G}_{1,z} \ast \tilde{G}_{0,z})(y),$$

and for $a = 1, 2$:

$$H_{\text{step},a}(k; u, v) = H_{\text{step},-a}(k; v, u).$$

The first rib contains $e_i$. Here we bound the weighted version of $P^{(1),i,\text{rib},a}$. We will first discuss the case that $e_i$ is a part of the connection from 0 to $v$. In Figure 4.14 this corresponds to the case $e_i = u$. We split between 4 cases:
(a) $e_i = u$ and the weight $[1 - \cos(k \cdot x)]$: After summing over $i$, we omit the constraint of visits $e_i$, as the connection $b_{R_0}^{0, a}(0, w_i(\omega))$ has to pass at least one neighbor of the origin anyway. We only keep the information that the connection $b_{R_0}^{0, a}(0, w_i(\omega))$ is non-trivial and bound the diagram by $H^{\text{step}, a}(u, v)$, see also the second line of Figure 4.15.

(b) $e_i = u$ and the weight $[1 - \cos(k \cdot (x - e_i))]$: As indicated in the first line of Figure 4.15, we sum over $i$, then shift the resulting diagram and see that the resulting diagram is bounded by $H^{0, a}$.

(c) $e_i \neq u$ and the weight $[1 - \cos(k \cdot x)]$: For this case we split the connection into two parts: One bubble from 0 to $u$ to $e_i$ and the remainder which correspond to $H^{b, a}(0, u, x, y)$. In this way we obtain the bounding diagram

$$
\sum_u \delta_{0,u} G_{1,z}(e_i) H^{0, a}(k; 0, 0, x, y) + 2D(u) \rho \mathcal{B}^{+, +}_{1, 2}(u, e_i) H^{-1, a}(k; 0, u, x, y) \\
+ \rho \sum_u \mathcal{B}^{+, +}_{1, 2}(u, e_i) H^{-2, a}(k; 0, u, x, y). \quad (4.3.134)
$$

(d) $e_i \neq u$ and the weight $[1 - \cos(k \cdot (x - e_i))]$: We bound this diagram by splitting the cosine into

$$
[1 - \cos(k \cdot (x - e_i))] \leq 2[1 - \cos(k \cdot x)] + 2[1 - \cos(k \cdot e_i)]. \quad (4.3.135)
$$

The first term is bounded as just discussed, where we note that this contribution carries the factor 2 from the split of the cosine. For the second term we note that $\sum_i \Xi^{(N), i}(x)$ is symmetric in $x$, so that

$$
\sum_{x, i} \Xi^{(N), i}(x)[1 - \cos(k \cdot e_i)] = \frac{1}{2d} \sum_i [1 - \cos(k \cdot e_i)] \sum_{x, k} \Xi^{(N), K}(x) \\
= 2d[1 - \hat{D}(k)] \sum_x \Xi^{(N), K}(x) \quad (4.3.136)
$$
Combining these steps we obtain a bound for (4.3.129) of the form

\[ \sum_{\ell} \tilde{\hat{\xi}}^{(N),\ell}_z(0) - \tilde{\hat{\xi}}^{(N),\ell}_z(k) \leq N \rho \sum_{b,u,v,x} \big( H^{0,b}(k;0,0,u,v) + H^{\text{step},b}(k;0,0,u,v) + \delta_{0,0} G_1(z_1) H^{0,a}(k;0,0,x,y) \big) \]

\[ + \sum_{y} 2 D(y) \tilde{\beta}^{+,+,1}_z(y,e_i) H^{-1,a}(k;0,y,u,v) + \tilde{\beta}^{+,+,1}_z(y,e_i) H^{-2,a}(k;0,y,u,v) \big) \times \hat{\Xi}^{(N-1),b}(u-x, v-x) \]

\[ + N \rho \sum_{M=1}^{N-2} \sum_{a,b,u,v,w,x,y} \Xi^{(M),l,a}(u,v) H^{a,b}(k;u,v,w,y) + \Xi^{(M),b}(w-x, y-x) \]

\[ + N \rho \sum_{a,u,v,x} \Xi^{(N-1),l,a}(u,v) H^{a,0}(k;u,v,x,x), \quad (4.3.137) \]

and a similar bound for (4.3.129). Splitting the element as we have done for the bound on the absolute value (see (4.3.99)) we obtain the claimed bounds.

### 4.3.9 On the bounds assumed in Chapter 3

In this section we explain where to find the bounds assumed in Assumption 3.2.6 and Assumption 3.5.3. The diagrammatic bounds on $\Xi^{(N)}_z$, $\Xi^{(N),\ell}_z$ are given in Section 4.3.4. Computing the value of these diagrams, as described in Chapter 5, creates the bounds $\beta^{(N)}_*$. The biggest entry of $\Lambda^M$ is a bound on the open square with at least 2 steps and is of order $1/d$. The bound on this square is in the dimension that we consider smaller than one. We conclude from this that $\beta^{(N)}_*$ is of order $1/d^N$ and that

\[ \sum_{N=0}^{\infty} \beta^{(N)}_z, \sum_{N=0}^{\infty} \beta^{(N)}_{\Lambda^M,0}, \sum_{N=0}^{\infty} \beta^{(N)}_{\Lambda^M,\ell}, \quad (4.3.138) \]

are finite. Further, we discuss the bounds stated in Assumption 3.5.3 as some of these simplify for LT. The quantities we use to characterize the two-point function (defined in Section 3.5.3) are given for LT by

\[ \alpha_\Phi = 2 d G_3(z_1) \left( 1 - \rho \frac{z g_z}{1 - (z g_z^t)^2} \right) = 2 d G_3(z_1) \left( 1 - \frac{z g_z^t}{1 - (z g_z^t)^2} \right), \quad (4.3.139) \]

\[ \hat{R}_\Phi(k) = \sum_{N=1}^{\infty} \hat{\xi}^{(N)}_z(k) - z g_z \hat{\Psi}(k) \left[ \hat{D}(k) + \alpha_z \hat{J} + \hat{\Pi}_z(k) \right]^{-1} \hat{Z}(k) - \alpha_\Phi \hat{D}(k), \quad (4.3.140) \]
and
\[
c_F = 1 - \frac{2d(zg_z^i)^2}{1 - (zg_z^i)^2},
\]
(4.3.141)
\[
\alpha_F = 2d \frac{zg_z^i}{1 - (zg_z^i)^2} - 2d(2d - 2) \frac{zg_z}{1 - (zg_z^i)^2} \Psi^{(i),1}(e_1 + e_2)
\]
\[
+ 2d \frac{z^2 g_z g_z^i}{1 - (zg_z^i)^2} \left( \Psi^{(i),1}(e_2) + zg_z \Psi^{(i),1}(e_1) \right),
\]
(4.3.142)
\[
\hat{R}_F = z \tilde{T} \left[ \hat{D}(k) + \alpha_z \mathbf{J} + \hat{N}_z(k) \right]^{-1} \tilde{T} + \frac{2d(zg_z^i)^2}{1 - (zg_z^i)^2} - \alpha_F \hat{D}(k).
\]
(4.3.143)

For a bound on \( \alpha_F \) and \( \alpha_\Phi \) we review \( \Psi_z^{(i),1} \) and see that
\[
\Psi^{(i),1}(e_1 + e_2) \leq 2z \mathcal{B}_{2,2}(e_1 + e_2, 0) + z \sum_v \mathcal{T}_{2,1,1}(e_1 + e_2, v, 0)
\]
\[
+ z \sum_v \mathcal{B}_{4,0,0}(e_1 + e_2, v, 0),
\]
(4.3.144)
\[
\max \{ \Psi^{(i),1}(e_2), \Psi^{(i),1}(e_1) \} \leq 2z (\mathcal{B}_{1,3}(e_2, 0) + \mathcal{B}_{3,1}(e_2, 0)) + z \sum_v \mathcal{T}_{1,1,1}(e_2, v, 0).
\]
(4.3.145)

We will bound these repulsive diagrams and \( G_{3,z}(e_i) \) in Chapter 5. We bound \( G_{3,z}(e_i) \), \( \Psi^{(i),1}(e_1 + e_2), \Psi^{(i),1}(e_1) \) and \( \Psi^{(i),1}(e_2) \) from below by zero. The constants \( \alpha_F, \alpha_\Phi, c_F \) and \( c_\Phi \) that we use in Section 3.5.3 to rewrite \( \hat{G}_z(k) \) capture the dominant contribution of \( F_1 \) and \( \Phi_1 \), see Section 3.4. We compute
\[
|\hat{R}_F(k)| \leq (2d - 2) zg_z \left( \sum_{N=1}^{\infty} \beta_N^{(N)} \right) \left( \sum_{N=1}^{\infty} \beta_N^{(N)} \right) + (3.4.11) := \beta_{R,F},
\]
(4.3.146)

where we use the line number (3.4.11) to denote the term given in this line. We compute that
\[
|\hat{R}_\Phi(k)| \leq \sum_{N=1}^{\infty} \left( \beta_N^{(N)} + \frac{zg_z}{1 - (zg_z^i)^2} \beta_N^{(N)} \right) + \left( \frac{zg_z}{1 - (zg_z^i)^2} \sum_{N=1}^{\infty} \beta_N^{(N)} + (3.4.11) \right) \beta_{z^{\text{abs}}} := \beta_{R,\Phi}.
\]
(4.3.147)

As already discussed in Section 4.2.5 the bounds on \( \hat{e}_1^{(N)}(0) - \hat{e}_2^{(N)}(x) \) also imply bounds on \( \sum_x \| x \|^2 \hat{e}_2^{(N)}(x) \) required for Assumption 3.5.3. We bound the part of \( \Phi_1(0) - \Phi_1(k) \) that is not extracted by \( c_\Phi \) and \( \alpha_\Phi \) by
\[
\sum_{N=2}^{\infty} \beta_N^{(N)} + \frac{zg_z}{1 - (zg_z^i)^2} \left( 2d \sum_{N=1}^{\infty} \beta_N^{(N)} \beta_{z^{\text{abs}}} + \beta_{z^{\text{abs}}} \beta_{\Delta z^{(i)}} \right)
\]
\[
+ \frac{zg_z}{1 - (zg_z^i)^2} \left( 2d \alpha \beta_{\Delta z^{(i)}} \beta_{z^{\text{abs}}} + \alpha (1 + \beta_{z^{\text{abs}}} \beta_{\Delta z^{(i)}}) \right).
\]
(4.3.148)
We define $\beta_{\Delta R}$ as the sum of $(4.3.148)$, $(3.4.35)$, and $(3.5.10)$. The part of $\hat{F}_1(0) - \hat{F}_1(k)$ that is not extracted by $c_F$ and $\alpha_F$ is bounded by

$$
\frac{2d z g_z}{1 - (z g_z)^2} \sum_{N=2}^{\infty} \left( \beta_{\Delta R}^N + \beta_{\Xi}^N + z g_z^2 \beta_{\Delta R}^N \right) + \frac{2d z g_z}{1 - (z g_z)^2} \sum_{i=1}^{\infty} \sum_{x: \|x\|_2 > 1} \Psi_i^{(i),c}(x + e_i) \left( \|x\|_2^2 + z g_z^2 \|x + e_i\|_2^2 \right) .
$$

(4.3.149)

We define $\beta_{\Delta R,F}$ as the sum of $(4.3.149)$, $(3.4.32)$, and $(3.4.37)$. This completes the bounding of the NoBLE coefficients for lattice trees.

### 4.4 Lattice animals

In this section we explain how to bound the coefficients of the non-backtracking lace expansion $\Xi_{z}^{(N)}$ and $\Xi_{z}^{(N),c}$ for lattice animals (LA) using the techniques of the preceding section. We will see that the bounds for the LA coefficients are extremely similar to the LT bounds. As for the LT we first explain how we split a LA containing three points. Then we discuss how to separate a sausage walk contributing to a coefficient, into repulsive contributions.

In Section 4.4.2 we define the repulsive diagram and use these diagrams to define the building blocks that are very similar to the building blocks of the LT and then state the bounds on the coefficients. After stating the bounds we explain briefly how to prove the bounds, but omit the proof as it is extremely similar to the proof for LT. We close this section with a brief review of where to find the bounds assumed in Chapter 3.

#### 4.4.1 The split of a lattice animal

Here we introduce the basic idea of the bounds of the LA diagrams. We will explain how we split a LA and how to split the contribution of a sausage walk. This is very similar to the proofs in Section 4.3.1.

**Connecting planted animal.** We used the connected planted tree to describe how we split a LT and a rib walk, see Definition 4.3.2. In contrast to the LT, such a split is not unique for LA as a connection between two points is not necessarily unique. Moreover, cutting the first step of the connection does not split the LA into two unique LA’s. To make clear how we split the animal we create an analog of a connecting planted tree for LA, the planted animal which is illustrated in Figure 4.16.

**Definition 4.4.1** (Path, backbone and connecting planted animal). Let $A$ be a lattice animal containing $x, y \in \mathbb{Z}^d$, with $x \neq y$. A path from to $x$ to $y$ in $A$ is a sequence of
bonds \((b_i)_{i=1,\ldots,N}\) such that \(b_i \in A, \bar{b}_i = \bar{b}_j\) for all \(i \neq j\), and \(b_1 = x, \bar{b}_i = \bar{b}_{i+1}, \bar{b}_N = y\) for \(i = 1,\ldots,N-1\). We define \(b^{A}(x, y)\) to be a unique path from to \(x\) to \(y\) in \(A\), e.g. the one with the lowest lexicographic order. See below for definition of the lexicographic order of a path. For \(x = y \in A\) we define \(b^{A}(x, y) = \emptyset\).

Let \(S^{A}(x, y)\) be the subset of \(A \setminus b^{A}(x, y)\) of for all bonds for which at least one point of the bond is connected to \(x\) via bonds in \(A \setminus b^{A}(x, y)\). We call \(B^{A}(x, y) = A \setminus S^{A}(x, y)\) the connecting planted animal from \(x\) to \(y\).

We note that \(B^{A}(x, y)\) contains at most one bond containing \(x\) and that for a lattice tree the connecting planted animal is the same as the connecting planted tree.

We have chosen \(b^{A}(x, y)\) to be the path with lowest lexicographic order. As it might not be clear what lexicographic order means for path we explain this now. We say that \(x \in \mathbb{Z}^d\) has a lower lexicographic order than \(y \in \mathbb{Z}^d\), if there exits an \(i \in \{1, 2, \ldots, d\}\) with \(x_i < y_i\) and \(x_j = y_j\) for \(j = 1, 2, \ldots, i\). A bond \(b\) is defined as a tuple of two vertices \(b = (\bar{b}, \bar{b})\), so that we can view \(b\) also to be a vector in \(\mathbb{Z}^2\) and use the same order relations as for vertices. A path \((b_1, b_2, \ldots, b_n)\) has a lower lexicographic order than the path \((t_1, t_2, \ldots, t_m)\), if either there exists a \(i \in \{1, 2, \ldots, \min(n, m)\}\), such that \(b_i\) has a lower order than \(t_i\) and \(b_j = t_j\) for \(j = 1, 2, \ldots, i-1\) or if \(n < m\) and \(b_i = t_i\) for \(j = 1, 2, \ldots, n\).

As for LT we define the concept of first intersection point to characterize a point at which two sausages intersect:

**Definition 4.4.2** (First intersection point). For a sausage walk \(\omega\) we define \(W_N(\omega)\) to be the set of vertices \(w\) that are contained in \(A^{\omega}_{s_N}\) and \(A^{\omega}_{t_N}\), such that \(b^{A(\omega)}_{s_N}(\bar{b}^{\omega}_{s_N}, w)\) and \(A^{\omega}_{t_N}\) only intersect at \(w\). If \(W_N(\omega)\) is non-empty then we define \(w_N(\omega)\) to be one fixed representative of the set \(W_N(\omega)\), e.g. the first in lexicographic order. We call \(w_N(\omega)\) the first intersection point of \(A^{\omega}_{s_N}\) and \(A^{\omega}_{t_N}\).
We note that if \( \tilde{b}^枉_{sn} \in A^枉_{ln} \) then \( W_N(\omega) = \{ \tilde{b}^枉_{sn} \} \), as for any other point \( w \) the path would \( B^枉_{sn}(\tilde{b}^枉_{sn}, w) \) intersect \( A^枉_{ln} \).

**Split for an animal containing three points.** Now we bound lattice animals \( A \) containing \( 0, x, v \in \mathbb{Z}^d \), using the concepts of connecting planted animals. We define \( u(A, v, x) \) to be the last point that \( b^A(0, v) \) and \( b^A(0, x) \) have in common. Then, we can split \( A \) into three bond-disjoint animals \( B^A(u(A, v, x), x) \) and \( A \setminus (B^A(u(A, v, x), v) \cup B^A(u(A, v, x), x)) \) and obtain the following bound

\[
\sum_{A:0,x,v\in A} \lambda |A| = \sum_{A:0,x,v\in A} \lambda |B^A(u(A,v,x),x)| \lambda |B^A(u(A,v,x),v)| \lambda |A \setminus (B^A(u(A,v,x),v) \cup B^A(u(A,v,x),x))| \leq \sum_u \tilde{G}_z(u) \tilde{G}_z(x - u) \tilde{G}_z(v - u) \leq \lambda^d \sum_u \tilde{G}_z(u) \tilde{G}_z(x - u) \tilde{G}_z(v - u) \tag{4.4.1}
\]

with

\[
\tilde{G}_z(x) = \sum_{\omega \in \mathbb{W}^A(x)} \lambda |\omega| \mathbb{Z}[0,|\omega|] K[0,|\omega|] \mathbb{I}_{A^枉_{sn} = \emptyset}, \tag{4.4.2}
\]

which has the same form as the corresponding bounds for the LTs in (4.3.3).

**Repulsiveness for LAs.** For LT a diagram is repulsive when the backbone connecting the corner points of the diagrams are disjoint. For LA we define repulsive diagrams such that the backbones might intersect, but do not use the same bonds. Let us formalize that:

**Definition 4.4.3** (Repulsiveness for lattice animals). For \( i = 1,2 \) let \( A_i \) be a lattice animal and \( p^i \) be a path in \( A_i \). We say that two path \( p^1 \) and \( p^2 \) are repulsive if \( p^1 \) and \( p^2 \) are bond-disjoint, i.e. \( p^1_i \neq p^2_j \) for all \( i,j \).

Let \( x_0, \ldots, x_n \in \mathbb{Z}^d \) and animals \( A_1, \ldots, A_n \) such that \( x_i, x_{i-1} \in A_i \) for \( i = 1, \ldots, n \). Then we say that \( (A_j)_{j=1,\ldots,n} \) is a repulsive structure with corner points \( (x_i)_{i=0,\ldots,n} \) if there exists paths \( p^i \) from \( x_{i-1} \) to \( x_i \) in \( A_i \) for \( i = 1, \ldots, n \) such that the paths \( p^i \) are pairwise repulsive.

To avoid more technicalities at this point we postpone the definition of the repulsive diagram to Section 4.4.2. Let us immediately use the concept of repulsiveness by discussing how to bound an animal \( A \) that contains 0, \( x, v \) such that 0 to \( x \) are double connected in \( A \). We define the point \( u \) to be a point that every path from 0 to \( v \) and \( x \) to \( v \) share. Then 0, \( u, x \) are pairwise doubly connected and there exist three bond-disjoint paths: \( p_1(0, x), p_2(0, u) \) and \( p_3(x, u) \) connecting the indicated points in \( A \). We decompose this lattice animal into four parts:

i.) a connecting planted animal \( B^A(u, w) \), with is trivial for \( u = v \),

ii.) a connecting planted animal \( A_2 = B^A(0, u) \) that uses the path \( p_2(0, v) \) to connect 0, \( v \),
iii.) a connecting planted animal $A_3 = B^{A\setminus B^4}(0,u)(x,u)$ that uses the path $p_3(u,x)$ to connect $z,u,$

iv.) and the remaining animal $A_1$ that contains $0,x$ and the path $p_1(0,x)$.

We know that $(A_i)_{i=1,2,3}$ with corner points $0,x,u$ is repulsive. We can bound the contribution of these animals as follows

$$\sum_{A:0,x,v\in A} 1_{[0 \text{ and } x \text{ are doubly connected}]} |\mathcal{A}| \leq \sum_{u} \bar{G}_c(u-w)\bar{G}_c(u)\bar{G}_c(u-x)\bar{G}_c(x).$$

(4.4.3)

**Bound on the sausage walk in $\Xi_z^{(1)}(x)$** We now explain how we split a sausage walk $\omega$ contributing to $\Xi_z^{(1)}(x)$. We know that only the first $A_0^\omega$ and last sausage $A_{|\omega|}^\omega$ intersect. We first discuss the case $0 = b_0$, to show the similarity of the bound to the rib walk. In this case we can split the animal into four parts: the connecting planted animal $B_0^\omega(0, w_1(\omega))$, the connecting planted animal $B_n^\omega(x, w_1(\omega))$, the trimmed first sausage $A_0^\omega \setminus B_0^\omega(0, w_1(\omega))$ and the remaining sausage walk. The contributions of the connecting planted animals are bounded by $\bar{G}_c$, the trimmed first sausage is bounded by $g_z^l$, and the contributions of the remaining sausage walk can be bounded by $2dz(D \ast \bar{G}_c)(x)$. This corresponds to the bound for LT’s.

If $b_0 \neq 0$ then we can not always split the first sausage such that a trimmed sausage walk is a LA. Therefore we split a first sausage, that contains $0, b_0$ and $w_1(\omega)$, as explained in the preceding paragraph and obtain the following bound

$$\Xi_z^{(1)}(x) \leq \sum_w g_z^l \bar{G}_c(w)G_z(w-x)\bar{G}_c(x)$$

$$+ g_z^l \sum_{u,w|b_0} \bar{G}_c(u-w)\bar{G}_c(u)\bar{G}_z(u-b_0)\bar{G}_z(b_0)G_z(w-x)\bar{G}_z(x-b_0).$$

(4.4.4)

Thus, we can deconstruct the sausage walks such that we can always find bond-disjoint connection between $u,w,x,b_0$, so that we can bound $\Xi_z^{(1)}(x)$ by a combination of repulsive diagrams.

**Bound on the sausage walk in $\Xi_z^{(2)}(x)$**. Now we discuss the bound on a sausage walk contributing to $\Xi_z^{(2)}(x)$ and also explain how we decompose the coefficient $\Xi_z^{(N)}(x)$ for $N \geq 2$. For a $\omega$ that contributes to $\Xi_z^{(2)}(x)$ we know that $A_0^\omega, A_n^\omega, A_{s_2}^\omega, A_{|\omega|}^\omega$ intersect. For $b_0$, we split the diagram in the same way as described in the proof of Lemma 4.3.10 for LT. We use now discuss an example for the case that $b_0 \neq 0$.

For our example we assume that $s_2 < t_1$ and $b_1 \neq 0$. We label the points and lines in the diagram as shown in Figure 4.17. We decompose the first sausage as discussed for $\Xi_z^{(1)}$: I$=B_0^\omega(u_0, v_0)$, II$=B_0^\omega(0, u_0)$\ I, III$=B_0^\omega(b_1, u)$\ II, IV$=A_0^\omega \setminus (I\cup II\cup III)$. 


Next, we remove the intersecting part of the sausages:
\[ V = B_{s_2}^\omega (\bar{b}_{s_2 + 1}, w_2(\omega)), \quad VI = B_{t_1}^\omega (\bar{b}_{t_1 + 1}, w_1(\omega)), \quad VII = B_{|\omega|} (x, w_2(\omega)). \]
Further, we split the remaining part of the sausage walk into three pieces VIII, IX and X. Therefore, we remark each trimmed sausage, except I, is still connected to the backbone, e.g. \( A_{s_2}^\omega \setminus B_{s_2}^\omega (\bar{b}_{s_2 + 1}, w_2(\omega)) \) is connected and contains \( \bar{b}_{s_2} \) and \( \bar{b}_{s_2 + 1} \). By this construction, the ten pieces are bond-disjoint and can be bounded by two-point functions. In contrast to the bounds derived in [39], where the classical lace expansion for LT and LA was first derived, our bounding diagrams include at most one trivial diagram.
Further, we see that except the very first triangle the diagrams have the same form as for the LT. We will define a new building block to bound this first piece and bound the other pieces using blocks that are defined in the same way as the LT-diagrams.

### 4.4.2 Repulsive diagrams and building blocks

We have already defined what it means for a set of animals to be repulsive. Now we modify the *mutually avoiding skeleton* that we define for LT in Definition 4.3.5.

**Definition 4.4.4** (Mutually avoiding skeleton for lattice animals). Let \( x_0 = 0 \in \mathbb{Z}^d \) and \( n \in \{2, 3, 4\} \). For each \( i \in \{1, \ldots, n\} \), let \( x_i \in \mathbb{Z}^d, l_i \in \mathbb{N}, \) the index \( j_i \in \{l_i, l_i^{-}\}, s_i \in \{+,-\}, \) and \( \omega^i \) be a sausage walk from \( x_{i-1} \) to \( x_i \). We define \( S_{f_1, \ldots, f_n}^{s_1, \ldots, s_n}(\omega^1, \ldots, \omega^n) \) to be the indicator that the following holds:

1. There exists a sequence of paths \( (p^i)_{i=1,\ldots,n} \) such that \( p^i \) is a path from \( x_{i-1} \) to \( x_i \) and \( p^i \) uses only edges in \( \omega^i \), each path \( p^i \) describes a self-avoiding walk and all paths are pairwise bond-disjoint.

2. For each \( i \in \{1, \ldots, n\} \) we can choose \( p^i \) in point (1.) such that, if \( j_i = l_i \) then \( |p^i| \geq l_i \), while \( |p^i| = l_i \) when \( j_i = l_i^{-} \).
(3.) For each \( i \in \{1, \ldots, n\} \) the sausages of a walk \((A^0_j)_{j=0,\ldots,|\omega^i|}\) do not intersect, i.e. \(K[0,|\omega^i|](\omega^i) = 1\), so that \( \omega^i \) describes a lattice animal.

(4.) For each \( i \in \{1, \ldots, n\} \), if \( s_i = + \) then \( A^0_0 = \emptyset \), while \( A^0_{|\omega^i|} = \emptyset \) when \( s_i = - \).

As in (4.3.5) - (4.3.7) we define the repulsive bubble, triangle and square diagrams using these skeletons. We define the building block \( P^{(1),b} \), \( P^{(1),t,b} \), \( A^{a,b} \), \( \bar{A}^{a,b} \) in exactly the same way as for LT, with the exception that we use LA repulsive diagrams.

Using these diagrams we define all building blocks we have defined in Section 4.3.3, including the vectors and matrices that we used to state the bounds. We add the superscript \((a)\) to highlight that the quantities are defined in terms of lattice animal diagrams, e.g., we write
\[
P^{(1),b,(a)}, \bar{P}^{(a),b}, P^{(a),t,b}, A^{(a),b}, \bar{A}^{(a),b} \quad \text{and} \quad \bar{A}^{(a),b,(a)}.
\]

**Building block for the unweighted diagram.** We define an additional diagram \( Q^{(1),b} \) to bound the initial non-trivial triangle by
\[
Q^{(0),0}(x, y) = \delta_{x,y} \left( B_{1,3}^{+,+}(x, 0) + B_{2,2}^{+,+}(x, 0) \right),
\]
\[
Q^{(0),1}(x, y) = \delta_{0,y} B_{1,3}^{+,-}(x, 0) + T_{1,1}^{+,-}(y, x, 0),
\]
\[
Q^{(0),2}(x, y) = \delta_{0,y} B_{2,2}^{+,+}(x, 0) + T_{1,1}^{+,-}(y, x, 0),
\]
and then define the first block as
\[
Q^{(1),b}(x, y) = P^{(1),b,(a)}(x, y) + \sum_{c=0}^{2} \sum_{u,v \in \mathbb{Z}^d} Q^{(0),0}(u, v) A^{-(c,b),(a)}(u, v, x, y),
\]
for \( b = -2, -1, \ldots, 2 \). To bound the contribution of these first blocks we define the vector \( \bar{Q} \in \mathbb{R}^5 \) with entries:
\[
(\bar{Q})_b = \sum_{x,y} Q^{(1),b}(x, y).
\]

Next we define the diagram that we use to bound \( \Xi^{(N),t}_z \). For sausage walk \( \omega \) that contributes to \( \Xi^{(N),t}_z \) we know that either \( \bar{G}_1 = e_i \) or \( e_i \in A^0_i \). We define \( Q^{(0),t,\text{sausage},b}(x, y) \) to bound the contribution of the initial triangle in the case that \( e_i \in A^0_i \):
\[
Q^{(0),t,\text{sausage},0}(x, y) = \delta_{x,y} \left( G_{1,2}(e_i) Q^{(0),0}(x, y) + \delta_{e_i,x} B_{1,3}^{+,+}(x, 0) + T_{1,1}^{+,-}(e_i, x, 0) \right)
\]
\[
+ \delta_{x,y} \sum_u G_{2,2}(e_i - u) \left( \delta_{u,x} B_{1,3}^{+,+}(x, 0) + T_{1,1}^{+,-}(u, x, 0) \right)
\]
\[
+ \delta_{x,y} \sum_u G_{1,2}(e_i - u) \left( \delta_{u,x} B_{2,2}^{+,-}(x, 0) + T_{2,1}^{+,-}(u, x, 0) \right)
\]
\[
(4.4.11)
\]
\[Q^{(0),\text{sausage},1}(x,y) = \tilde{G}_{1,z}(e_i)Q^{(0),1}(x,y) + \mathcal{J}^{+,+,+,-}_{1,0,1,1}(e_i, y, x, 0) + \mathcal{J}^{+,+,+,-}_{1,0,0,0}(e_i, x, y, 0) + \sum_u G_{2,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{1,0,1,1}(u, y, x, 0) \]
\[+ \sum_u G_{2,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{1,0,2,1}(u, y, x, 0) + \mathcal{J}^{+,+,+,-}_{1,0,0,2}(y, x, u, 0) \]
\[+ \sum_u G_{1,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{2,0,1,1}(u, y, x, 0) + \mathcal{J}^{+,+,+,-}_{0,1,0,2}(y, x, u, 0) \]
\[= (4.4.12) \]

\[Q^{(0),\text{sausage},2}(x,y) = \tilde{G}_{1,z}(e_i)Q^{(0),2}(x,y) + \mathcal{J}^{+,+,+,-}_{1,0,2,1}(e_i, y, x, 0) + \mathcal{J}^{+,+,+,-}_{1,0,2,0}(e_i, x, y, 0) + \sum_u G_{2,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{1,0,2,1}(u, y, x, 0) \]
\[+ \sum_u G_{2,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{1,0,2,0}(u, y, x, 0) + \mathcal{J}^{+,+,+,-}_{2,0,2,0}(y, x, u, 0) \]
\[+ \sum_u G_{1,z}(e_i - u)\mathcal{J}^{+,+,+,-}_{2,0,2,0}(u, y, x, 0) + \mathcal{J}^{+,+,+,-}_{0,2,0,2}(y, x, u, 0) \]
\[= (4.4.13) \]

and combine these with an open square to create \(Q^{(1),\text{sausage},b}, \text{for } b = \{-2, -1, \ldots, 2\},\)

\[Q^{(1),\text{sausage},b}(x,y) = \sum_{c=0}^{2} \sum_{u,v \in \mathbb{Z}^d} Q^{(0),\text{sausage},0}(u,v)A^{-c,b,(a)}(u,v,x,y), \]  \[= (4.4.14) \]

To bound the contribution due to the case that \(b_1 \neq 0\) and \(\bar{b}_1 = e_i\) we define

\[Q^{(0),\text{step},0}(x,y) = \delta_{x,y}2dD(x - e_i)\mathcal{J}^{+,+}_{2,0,1,1}(x,0), \]
\[Q^{(0),\text{step},1}(x,y) = 2dD(x - e_i)\left(\mathcal{J}^{+,+,+,-}_{1,1,1,1}(y, x, 0) + \mathcal{J}^{+,+,+,-}_{2,1,3}(y, x, 0)\right), \]
\[Q^{(0),\text{step},2}(x,y) = 2dD(x - e_i)\mathcal{J}^{+,+,+,-}_{2,2,0,0}(x, 0). \]
\[= (4.4.15, 4.4.16, 4.4.17) \]

and

\[Q^{(1),\text{step},-2}(x,y) = zg^\ell_x \left( \sum_{c=0}^{2} Q^{(0),\text{step},c}(u,v) \sum_w \mathcal{J}^{+,+,+,-}_{0,2,0,0}(x - e_i, y - e_i, w - e_i, v - e_i) \right), \]
\[= (4.4.18) \]

\[Q^{(1),\text{step},-1}(x,y) = zg^\ell_x \left( \sum_{c=0}^{2} Q^{(0),\text{step},c}(u,v) \sum_w \mathcal{J}^{+,+,+,-}_{0,1,0,0}(x - e_i, y - e_i, w - e_i, v - e_i) \right), \]
\[= (4.4.19) \]

\[Q^{(1),\text{step},0}(x,y) = \delta_{x,y}zg^\ell_x \left( \sum_{c=0}^{2} Q^{(0),\text{step},c}(u,v) \sum_w \mathcal{J}^{+,+,+,-}_{0,0,0,0}(x - e_i, w - e_i, v - e_i) \right), \]
\[= (4.4.20) \]

For \(b = 1, 2\) we define

\[Q^{(1),\text{step},-b}(x,y) = Q^{(1),\text{step},b}(y,x). \]
\[= (4.4.21) \]
4.4 Lattice animals

We can then bound the first part of the diagram of $\Xi^{(N),i}_{z}$ by

$$Q^{(1),i,b}(x,y) = P^{(1),i,b,(a)}(x,y) + Q^{(1),i,\text{step},b}(x,y) + Q^{(1),i,\text{sausage},b}(x,y)$$

(4.4.22)

for $b \in \{-2,-1,\ldots,2\}$. To bound the contribution of the first block we define the vector $\vec{Q} \in \mathbb{R}^5$ with entries:

$$(\vec{Q})_b = \sum_{i,x,y} Q^{(1),i,b}(x,y).$$

(4.4.23)

**Building block for the weighted diagram.** We bound the weighted version of this block using the vector $\vec{\Delta}_Q^{\text{start}}(k)$ with entries

$$((\vec{\Delta}_Q^{\text{start}}(a)(k)))_b + 2 \sum_{c=0}^2 \left( (\vec{Q})_c (\Delta^{(a)}(k))_{-c,b} + h_Q^0(k)(\Delta^{(a)}(k))_{-c,b} \right),$$

(4.4.24)

where

$$h_Q^0(k) = d z g_z^i \bar{G}_{3,z}(e_1)[1 - \hat{D}(k)] + (G_z(\cdot;k) \star \bar{G}_{2,z}(0)),$$

(4.4.25)

$$h_Q^1(k) = \bar{G}_{3,z}(e_1)[1 - \hat{D}(k)] + (G_z(\cdot;k) \star \bar{G}_{1,z} \star D)(0),$$

(4.4.26)

$$h_Q^2(k) = \sup_{x \in \mathbb{Z}^d} (G_z(\cdot;k) \star \bar{G}_{0,z}(x)).$$

(4.4.27)

To bound the first weighted block of $\hat{\Xi}^{(N),i}_{z}$ we define the vectors $\vec{\Delta}_Q^{\text{iota},I}, \vec{\Delta}_Q^{\text{iota},II} \in \mathbb{R}^5$. The entries are defined to be bounds on the diagrams shown in Figure 4.18. This bound is created by splitting the weight along the line and then bound the terms individual by the already defined diagrams. We omit stating the precise form of $\vec{\Delta}_Q^{\text{iota},I}, \vec{\Delta}_Q^{\text{iota},II}$ as this is very cumbersome and not very interesting.

### 4.4.3 The diagrammatic bounds

Here we state the bound for coefficient for the lattice animal. Before doing to we recall that we normalized the coefficient by the one-point function $g_z$ and that we defined $\rho = g_z^i / g_z$.

**Lemma 4.4.5** (Diagrammatic estimates for $N = 0$). For $z < z_c$ the following holds, for all $i$

$$\sum_{x \in \mathbb{Z}^d} \Xi_{z}^{(0)}(x) = 1 + \rho(\mathcal{B}^{+,+}_{1,3}(0) + \mathcal{B}^{+,+}_{2,2}(0))$$

(4.4.28)

$$\sum_{x \in \mathbb{Z}^d} \Xi_{z}^{(0),i}(x) = \rho \mathcal{B}^{+,+}_{1,3}(0) + \rho \bar{G}_{1,i}(e_i)(1 + \mathcal{B}^{+,+}_{1,3}(0) + \mathcal{B}^{+,+}_{2,2}(0)),$$

(4.4.29)
Figure 4.18: The form of the diagrams we bound with $\Delta_Q^{\text{iota, I}}, \Delta_Q^{\text{iota, II}}$. For the bound we sum over $u, v, w, x, \iota$ and take the supremum over $x - y$. For $\Delta_Q^{\text{iota, I}}$ we add the weight $[1 - \cos(k \cdot x)]$ and for $\Delta_Q^{\text{iota, II}}$ the weight $[1 - \cos(k \cdot (x - e_i))]$.

and

$$\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0)}(x)[1 - \cos(k \cdot x)] \leq \rho 2d z[1 - \hat{D}(k)] \tilde{G}_{3,z}(e_i) + \rho(G_z(; k) \star \tilde{G}_{2,z})(0)$$

(4.4.30)

$$\sum_{\iota} \sum_{x \in \mathbb{Z}^d} \Xi_z^{(0, \iota)}(x)[1 - \cos(k \cdot x)] \leq (1 + 2d \rho \tilde{G}_{1,z}(e_i))(G_z(; k) \star \tilde{G}_{2,z})(0).$$

(4.4.31)

Further,

$$\sum_{\iota} \sum_{x \in \mathbb{Z}^d} \Xi_z^{(0, \iota)}(x)[1 - \cos(k \cdot (x - e_i))]$$

$$\leq (G_z(; k) \star \tilde{G}_{2,z})(0) + 2d[1 - \hat{D}(k)] \tilde{G}_{1,z}(e_i)$$

$$+ 2\tilde{G}_{1,z}(e_i) \left\{ 2d[1 - \hat{D}(k)](\mathcal{B}^{+,+}_{1,3}(0) + \mathcal{B}^{+,+}_{2,2}(0)) + (G_z(; k) \star \tilde{G}_{2,z})(0) \right\}.$$  

(4.4.32)

We obtain the equality (4.4.32) using that, for $a, b \in \mathbb{R}$,

$$[1 - \cos(a + b)] \leq [1 - \cos(a)] + [1 - \cos(b)] + \sin(a) \sin(b).$$  

(4.4.33)

The sines terms cancel when we sum over $x$ due to the symmetry of $\sum_i \Xi_z^{(0, \iota)}(x)$. 
Lemma 4.4.6 (Bounds on the lattice animal coefficients for $N = 1$). Let $0 \leq z \leq z_c$. Then

\[ \hat{\Sigma}_z^{(1)}(0) \leq \rho(\bar{Q}^{(a)})_0, \quad (4.4.34) \]
\[ \hat{\Sigma}_z^{(1),t}(0) \leq \frac{\rho}{2d}(\bar{Q}^{(l,a)})_0, \quad (4.4.35) \]
\[ \hat{\Sigma}_z^{(1)}(0) - \hat{\Sigma}_z^{(1)}(k) \leq \rho(\bar{\Delta}_Q(k))_0, \quad (4.4.36) \]
\[ \sum_i (\hat{\Sigma}_z^{(1),t}(0) - \hat{\Sigma}_z^{(1),t}(k)) \leq \rho(\bar{\Delta}_{Q}^{\text{total},I}(k))_0, \quad (4.4.37) \]
\[ \sum_i \left( \hat{\Sigma}_z^{(1),t}(0) - \hat{\Sigma}_z^{(1),t}(k)e^{-ik\cdot e_i} \right) \leq \rho(\bar{\Delta}_{Q}^{\text{total},II}(k))_0. \quad (4.4.38) \]

Lemma 4.4.7 (Bounds on the lattice animal coefficients for $N = 2$). Let $0 \leq z \leq z_c$. Then

\[ \hat{\Sigma}_z^{(2)}(0) \leq \rho \sum_{b=-2}^{2} (\bar{Q}^{(a)})_b (\bar{A}^{(a)})_{b,0}, \quad (4.4.39) \]
\[ \hat{\Sigma}_z^{(2),t}(0) \leq \frac{\rho}{2d} \sum_{b=-2}^{2} (\bar{Q}^{(l,a)})_b (\bar{A}^{(a)})_{b,0}, \quad (4.4.40) \]
\[ \hat{\Sigma}_z^{(2)}(0) - \hat{\Sigma}_z^{(2)}(k) \leq 2\rho \bar{\Delta}_Q^{\text{start}}(k) \bar{P}^{(a)} + 2\rho (\bar{Q}^{(a)})^T \bar{\Delta}_Q^{\text{end},(a)}(k), \quad (4.4.41) \]

and

\[ \sum_i (\hat{\Sigma}_z^{(2),t}(0) - \hat{\Sigma}_z^{(2),t}(k)) \leq 2\rho \bar{\Delta}_Q^{\text{total},I}(a)(k) \bar{P}^{(a)} + 2\rho (\bar{Q}^{(l,a)})^T \bar{\Delta}_Q^{\text{end},(a)}(k), \quad (4.4.42) \]
\[ \sum_i (\hat{\Sigma}_z^{(2),t}(0) - \hat{\Sigma}_z^{(2),t}(k)e^{-ik\cdot e_i}) \leq 2\rho \bar{\Delta}_Q^{\text{total},II}(k) \bar{P}^{(a)} + \frac{2}{g_z} (\bar{Q}^{(l,a)})^T \bar{\Delta}_Q^{\text{end},(a)}(k). \quad (4.4.43) \]

Proposition 4.4.8 (Bounds on the lattice animal coefficients for $N \geq 3$). Let $0 \leq z \leq z_c$ and $N \geq 3$, then

\[ \hat{\Sigma}_z^{(N)}(0) \leq \rho(\bar{Q}^{(a)})^T (\bar{A}^{(a)})_{N-3} \bar{A}^{(a)} \bar{P}^{(a)}, \quad (4.4.44) \]
\[ \hat{\Sigma}_z^{(N),t}(0) \leq \frac{\rho}{2d} (\bar{Q}^{(l,a)})^T (\bar{A}^{(a)})_{N-3} \bar{A}^{(a)} \bar{P}^{(a)}, \quad (4.4.45) \]
and

\[
\hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \leq N \rho \Delta_{\text{start}}(k)^T (\bar{A}^{(a)})^{N-2} \bar{P}^{(a)} + N \rho (\tilde{Q})^T (\bar{A}^{(a)})^{N-2} \bar{\Delta}_{\text{end}}(a)(k) + N \rho \sum_{M=0}^{N-2} (\tilde{Q}^{(a)})^T ((\bar{A}^{(a)})^M \Delta^{(a)}(k) \bar{A}^{(a)})^{N-2-M} \bar{P}^{(a)}.
\]

\[
\sum_i (\hat{\Xi}_z^{(N),i}(0) - \hat{\Xi}_z^{(N),i}(k)) \leq N \rho \Delta_{\text{iota,I}}(k)^T (\bar{A}^{(a)})^{N-2} \bar{P}^{(a)} + N \rho (\tilde{Q}^{(a)})^T (\bar{A}^{(a)})^{N-2} \bar{\Delta}_{\text{end}}(a)(k) + N \rho \sum_{M=0}^{N-2} (\tilde{Q}^{(a)})^T ((\bar{A}^{(a)})^M \Delta^{(a)}(k) \bar{A}^{(a)})^{N-2-M} \bar{P}^{(a)}.
\]

\[
\sum_i (\hat{\Xi}_z^{(N),i}(0) - \hat{\Xi}_z^{(N),i}(k) e^{-k \cdot e_1}) \leq N \rho \Delta_{\text{iota,II}}(k)^T (\bar{A}^{(a)})^{N-2} \bar{P}^{(a)} + N \rho (\tilde{Q}^{(a)})^T (\bar{A}^{(a)})^{N-2} \bar{\Delta}_{\text{end}}(a)(k) + N \rho \sum_{M=0}^{N-2} (\tilde{Q}^{(a)})^T ((\bar{A}^{(a)})^M \Delta^{(a)}(k) \bar{A}^{(a)})^{N-2-M} \bar{P}^{(a)}.
\]

### 4.4.4 On the proof of the bounds

In Section 4.4.1, we have already shown how to decompose a sausage walk into piecewise repulsive diagrams. These constructs can be bounded by the defined building blocks. The remainder of the proof is very similar to the proof of the bounds for LT and will be omitted.

The only real difference between the LT and LA diagram is the case \( b_1 \neq 0 \) in which the first piece of the diagram has different form, see Figure 4.18. We bound these diagrams using the ideas of Section 4.4.1 and then bound each diagram one-by-one. This is elaborate, but straightforward.

Further, we want to highlight that the dominant contribution comes from diagrams with \( b_1 = 0 \) that we bound in the same way as the LT coefficients. Reason is that \( b_1 \neq 0 \) requires an additional double connection and is therefore by an order \( 1/d^2 \) smaller than the contributions from \( b_1 = 0 \).

### 4.4.5 On the bounds assumed in Chapter 3

The diagrammatic bound on \( \Xi_z^{(N)}, \Xi_z^{(N),i} \), assumed in Assumption 3.2.6 and Assumption 3.5.3 are given in Section 4.4.3. Computing the value of these diagrams, as described in Chapter 5, creates the bonds \( \beta^{(N)}_i \). These bounds \( \beta^{(N)}_i \) are of order \( 1/d^N \) since the biggest entry of \( A, \bar{A} \) is of the order \( 1/d \). In the dimension that we consider...
this square is smaller than one. Thereby,
\[
\sum_{N=0}^{\infty} \beta_{\Xi(N)}^{(0)}, \quad \sum_{N=0}^{\infty} \beta_{\Delta \Xi,0}^{(N)}, \quad \sum_{N=0}^{\infty} \beta_{\Delta \Xi,1}^{(N)},
\]
are all finite in the dimension that we consider. As already discuss in Section 4.2.5 the bounds on \( \hat{\Xi}^{(N)}(0) - \hat{\Xi}^{(N)}(k) \) also imply bounds on \( \sum_{x} \|x\|_2^2 \Xi_z^{(N)}(x) \) required for Assumption 3.5.3.

### 4.5 Percolation

For percolation the lace-expansion coefficients are not combinatorial objects. Instead, they are obtained form the probability of intertwined events. Consequently, the bounds on the coefficients is more elaborate than for the other models.

We begin by defining repulsive diagrams in Section 4.5.1 and use these to define the building blocks in Section 4.5.2. Then, we state the bounds on the coefficients in Section 4.5.3.

These bounds are proven in three steps. First we bound the coefficients on the level of events Section 4.5.4. Then, we bound the probability of these events by bounding diagrams, which are combinations of building blocks, in Section 4.5.5. In the third step we decompose the bounding diagrams and obtain the stated bound. As the last step is similar to the correspond part of SAW and LT, we omit it here.

In Section 4.5.6 we discuss bounds on some complicated building blocks. these are bounded in a slightly different way than for the other models. In essence our approach is an adaption of [38 Section 2.2] and [17 Section 4] respectively.

#### 4.5.1 Repulsive diagrams

As for the other models we bound the lace-expansion coefficients using diagrams. We prove that \( \Xi_z^{(N)}(x) \) can be bounded by diagrams as shown in Figure 4.19. When bounding the coefficients we use the information that \( z_{i+1} \notin b_i \) and that all non-trivial triangles/squares consist of at least four occupied bonds. We denote the length of a line that two squares share by \( l \) and decompose the diagrams as shown in Figure 4.20.

We create such a diagrams using repulsive diagrams, that we are going to define now. For \( m \geq 0 \) we denote by \( \{0 \overset{m}{\rightarrow} x\} \) the event that 0 and \( x \) are connected and that there exists a path of open, disjoint bonds between 0 and \( x \) that consists of at least \( m \) bonds. Further, we define \( \{0 \overset{m}{\leftarrow} x\} \) as the event that 0 and \( x \) are connected by a path of exactly \( m \) occupied bonds. We define for \( m \geq 0 \)

\[
\tau_{m,z}^{(N)}(x) = \mathbb{P}_z(0 \overset{m}{\rightarrow} x), \quad \tau_{m,z}^{(N)}(x) = \mathbb{P}_z(0 \overset{m}{\leftarrow} x), \quad \tau_{m,z}^{(N)}(x) = \mathbb{P}_z(0 \overset{m}{\rightarrow} x), \quad \tau_{m,z}^{(N)}(x) = \mathbb{P}_z(0 \overset{m}{\leftarrow} x).
\]
we omit a formal definition at this point. A definition can be found in, e.g., \[\text{Figure 4.19: Diagrammatic representations of the bound on } \Xi_z^{(4)}(x). \text{ Lines indicate disjoint connections. Filled triangles might be trivial. The } F_i \text{ denote events, that we define later.}\]

\[
\sum_x \Xi_z^{(N)}(x) = \sum_{\ell} 0 \quad l_0 = \begin{array}{c}
\vdots \\
| l_{r-1} \\
| l_r \\
\vdots \\
| l_{N-1} \\
\end{array} + \begin{array}{c}
\vdots \\
| l_{r-1} \\
| l_r \\
\vdots \\
| l_{N-1} \\
\end{array} \quad N - 1
\]

\[
\text{Figure 4.20: Diagrammatic decomposition of } \Xi_z^{(N)}(x). \text{ The numbers } l_i \text{ denote the length of a connection.}
\]

For \( m \geq 1 \) and \( x \neq 0 \) we know that

\[
\begin{align*}
\tau^{i}_{m,z}(x) & \leq \tau_{m,z}(x) \leq 2dz(D \ast \tau_{m-1,z})(x) \leq (2dz)^m(D^m \ast \tau_z)(x), \\
\tau^{i}_{m,z}(x) & \leq \tau_{m,z}(x) \leq 2dz(D \ast \tau_{m-1,z})(x) \leq (2dz)^m(D^m)(x).
\end{align*}
\tag{4.5.3, 4.5.4}
\]

For \( i = 1, \ldots, 5 \) let \( x_i \in \mathbb{Z}^d \), and indices \( j_i \in \{0, 1, \ldots\} \cup \{0, 1, \ldots\} \) we define the non-repulsive diagrams

\[
\begin{align*}
\mathcal{B}^\ast_{j_1,j_2}(x_1,x_2) &= \tau_{j_1,z}(x_1)\tau_{j_2,z}(x_2-x_1), \\
\mathcal{F}^\ast_{j_1,j_2,j_3}(x_1,x_2,x_3) &= \tau_{j_1,z}(x_1)\tau_{j_2,z}(x_2-x_1)\tau_{j_3,z}(x_3-x_2), \\
&= \mathcal{B}^\ast_{j_1,j_2}(x_1,x_2)\tau_{j_3,z}(x_3-x_2), \\
\mathcal{F}^\ast_{j_1,j_2,j_3,j_4}(x_1,x_2,x_3,x_4) &= \mathcal{F}^\ast_{j_1,j_2,j_3}(x_1,x_2,x_3)\tau_{j_4,z}(x_4-x_3), \\
\mathcal{P}^\ast_{j_1,j_2,j_3,j_4,j_5}(x_1,x_2,x_3,x_4,x_5) &= \mathcal{F}^\ast_{j_1,j_2,j_3,j_4}(x_1,x_2,x_3,x_4)\tau_{j_5,z}(x_5-x_4).
\end{align*}
\tag{4.5.5-4.5.8}
\]

To define the repulsive diagrams we use the notion of two-events occurring disjointly which we denote by the symbol \( \circ \). Loosely speaking for two connections to exist disjointly \( \{(u \leftrightarrow v) \circ (x \leftrightarrow y)\} \) denotes that there exist two non-intersecting paths that connect the points. As this is a standard concept in percolation theory we omit a formal definition at this point. A definition can be found in, e.g., [29].
Section 2.3]. We define the repulsive diagrams:

\[
\mathcal{B}_{j_1, j_2}(x_1, x_2) = \mathbb{P}_z(0 \xrightarrow{j_1} x_1 \circ (x_1 \xrightarrow{j_2} x_2)),
\]

(4.5.9)

\[
\mathcal{T}_{j_1, j_2, j_3}(x_1, x_2, x_3) = \mathbb{P}_z(0 \xrightarrow{j_1} x_1 \circ (x_1 \xrightarrow{j_2} x_2) \circ (x_2 \xrightarrow{j_3} x_3)),
\]

(4.5.10)

\[
\mathcal{S}_{j_1, j_2, j_3, j_4}(x_1, x_2, x_3, x_4) = \mathbb{P}_z(0 \xrightarrow{j_1} x_1 \circ (x_1 \xrightarrow{j_2} x_2) \circ (x_2 \xrightarrow{j_3} x_3) \circ (x_3 \xrightarrow{j_4} x_4)),
\]

(4.5.11)

\[
\mathcal{P}_{j_1, j_2, j_3, j_4, j_5}(x_1, x_2, x_3, x_4, x_5) = \mathbb{P}_z(0 \xrightarrow{j_1} x_1 \circ (x_1 \xrightarrow{j_2} x_2) \circ (x_2 \xrightarrow{j_3} x_3) \circ (x_3 \xrightarrow{j_4} x_4) \circ (x_4 \xrightarrow{j_5} x_5)).
\]

(4.5.12)

### 4.5.2 Building blocks

Now we define the building blocks that we combine to obtain a bound on \(\Xi^{(N)}_z\) and \(\Xi^{(N), f}_z\). We begin with six blocks that we use to bound the absolute value of \(\hat{\Xi}^{(N)}_z\). We first introduce these diagrams informally in Table 4.17 and then define these diagrams using Tables 4.18-4.24.
Table 4.17: Description of building blocks for percolation: $P^a$, $A^{i,a,b}$, $A^{a,b}$, $\bar{A}^{a,b}$, $B^{2,i,a,b}$, $\bar{B}^{2,i,a,b}$

<table>
<thead>
<tr>
<th>Building Block</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>repulsive triangle $P^{0,a}(x,y)$</td>
<td>The distance between $x$ and $y$ is described by $a$. We know that if $x = 0$ then the triangle shrinks to a point (only $y = 0, a = 0$ contributes). If $x \neq 0$ then the triangle will consists of at least four bonds.</td>
</tr>
<tr>
<td>open triangle $A^{a,b}(0,v,x,y)$</td>
<td>The distance between $0, v$ and $x, y$ are denoted by $a$ and $b$ respectively. We know that the complete square has at least four steps and that $x, y \neq 0$ and $x \neq v$.</td>
</tr>
<tr>
<td>open repulsive triangle with one pivotal edge $A^{i,a,b}(0,v,x,y)$</td>
<td>This describe an open repulsive triangle. We denote $a$ and $b$ the distance between $0, v$ and $x, y$. If we complete the square by adding the connection $0 \leftrightarrow v$ then the created square consists of at least four bonds. Further, we know that $x, y \neq 0, x \neq v$ and $y, v \neq e^i$.</td>
</tr>
<tr>
<td>double-open triangle $A^{i,a,b}(0,v,x,y)$</td>
<td>The only difference to $A^{i,a,b}(0,v,x,y)$ is that the connection $x \leftrightarrow y$ is not present.</td>
</tr>
<tr>
<td>double non-trivial triangle $B^{2,i,a,b}(0,v,x,y)$, right</td>
<td>A combination of a closed triangle and an open square. All points $(u, w, y)$ of the small triangle are distinct and $u, w \not\in {0, x}$.</td>
</tr>
<tr>
<td>double non-trivial triangle $B^{2,i,a,b}(0,v,x,y)$, left</td>
<td>A combination of a closed triangle and an open square. All points $(u, w, y)$ of the small triangle are distinct and $x \neq u$ and $0 \not\in {w, w + e_i}$.</td>
</tr>
</tbody>
</table>
Table 4.18: Diagrams and definition of $P^{0,a}(x, y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0$</td>
<td>$x = 0$</td>
<td>$x = y = 0$</td>
<td>$\delta_{0,x}$</td>
</tr>
<tr>
<td>$\Rightarrow x = y$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d(0, x) = 1$</td>
<td>$\geq 3$</td>
<td>$x = y$</td>
<td>$B_{1,3}(x, 0)$</td>
</tr>
<tr>
<td>$d(0, x) \geq 2$</td>
<td>$\geq 2$</td>
<td>$x = y$</td>
<td>$B_{2,2}(x, 0)$</td>
</tr>
<tr>
<td>$a = 1$</td>
<td>$y = 0$</td>
<td>$0 = y$</td>
<td>$\delta_{0,y}B_{3,1}(x, 0)$</td>
</tr>
<tr>
<td>$\Rightarrow x \neq 0$</td>
<td></td>
<td>$\geq 3$</td>
<td></td>
</tr>
<tr>
<td>$d(0, x) = 1$, $y \neq 0$</td>
<td>$\geq 2$</td>
<td>$y \leq 1$</td>
<td>$T_{1,1,2}(x, y, 0)$</td>
</tr>
<tr>
<td>$d(0, x) \geq 2$, $y \neq 0$</td>
<td>$\geq 1$</td>
<td>$y \leq 1$</td>
<td>$T_{2,1,1}(x, y, 0)$</td>
</tr>
<tr>
<td>$a = 2$</td>
<td>$y = 0$, $d(0, x) = 1$</td>
<td>$0 = y$</td>
<td>$\delta_{0,y}B_{1,3}(x, 0)$</td>
</tr>
<tr>
<td>$\Rightarrow x \neq 0$</td>
<td></td>
<td>$\geq 3$</td>
<td></td>
</tr>
<tr>
<td>$y = 0$, $d(0, x) \geq 2$</td>
<td>$\geq 2$</td>
<td>$y \leq 1$</td>
<td>$\delta_{0,y}B_{2,2}(x, 0)$</td>
</tr>
<tr>
<td>$y \neq 0$</td>
<td>$\geq 1$</td>
<td>$y \geq 2$</td>
<td>$T_{1,2,1}(x, y, 0)$</td>
</tr>
</tbody>
</table>
Table 4.19: Diagrams and definition of $A^{a,b}(0, \nu, x, y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
</table>
| $a = b = 0$  
$\Rightarrow x = y, \nu = 0$  
$\Rightarrow \nu \neq y, x \neq e$ | ![Diagram](image) | $B_{1,3}(x, 0)$  
$+ B_{2,2}(x, 0)$ |
| $a = 0, b = 1$  
$\Rightarrow \nu = 0, y \neq \nu$ | ![Diagram](image) | $T_{1,1,2}(x, y, 0)$  
$+ T_{2,1,1}(x, y, 0)$ |
| $a = 0, b \geq 2$  
$\Rightarrow \nu = 0, y \neq 0, x \neq 0$ | ![Diagram](image) | $T_{1,2,1}(x, y, 0)$ |
| $a = 1, b = 0$ | ![Diagram](image) | $2dD(\nu)B_{1,2}(x, \nu)$  
$+ 2dD(\nu)B_{2,1}(x, \nu)$ |
| $a = b = 1$ | ![Diagram](image) | $2dD(\nu)T_{1,1,1}(x, y, \nu)$  
$+ 2dD(\nu)T_{2,1,0}(x, y, \nu)$ |
| $a = 1, b \geq 2$ | ![Diagram](image) | $2dD(\nu)T_{1,2,0}(x, y, \nu)$ |
| $a \geq 2, b = 0$ | ![Diagram](image) | $B_{1,0}(x, \nu)$ |
| $a \geq 2, b = 1$ | ![Diagram](image) | $T_{1,1,0}(x, y, \nu)$ |
| $a \geq 2, b \geq 2$ | ![Diagram](image) | $T_{1,2,0}(x, y, \nu)$ |
### Table 4.20: Diagrams and definition of $A^{a,b}(0, v, x, y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
</table>
| $a = b = 0$  
$\Rightarrow x = y, v = 0$  
$\Rightarrow v \neq y, x \neq e$ | ![Diagram for $a = b = 0$] | $\mathcal{F}_{1,1,2}(e, x, 0)$  
$+ \mathcal{F}_{1,2,1}(e, x, 0)$ |
| $a = 0, b = 1$  
$\Rightarrow v = 0, y \neq v$ | ![Diagram for $a = 0, b = 1$] | $\delta_{x,e}\mathcal{F}_{1,1,2}(e, y, 0)$  
$+ \mathcal{F}_{1,1,1}(e, x, y, 0)$ |
| $a = 0, b \geq 2$  
$\Rightarrow v = 0, y \neq 0, x \neq 0$ | ![Diagram for $a = 0, b \geq 2$] | $\mathcal{F}_{1,0,2,1}(e, x, y, 0)$ |
| $a = 1, b = 0$ | ![Diagram for $a = 1, b = 0$] | $2dD(v)\mathcal{F}_{1,2,1}(e, x)$  
$+ 2dD(v)\mathcal{F}_{1,1,1}(e, x, v)$ |
| $a = b = 1$ | ![Diagram for $a = b = 1$] | $2dD(v)\mathcal{F}^*_{1,0,1,0}(e, x, y, v)$ |
| $a = 1, b \geq 2$ | ![Diagram for $a = 1, b \geq 2$] | $2dD(v)\mathcal{F}^*_{1,0,2,0}(e, x, y, v)$ |
| $a \geq 2, b = 0$ | ![Diagram for $a \geq 2, b = 0$] | $\mathcal{F}^*_{1,1,0}(e, x, v)$ |
| $a \geq 2, b = 1$ | ![Diagram for $a \geq 2, b = 1$] | $\mathcal{F}^*_{1,0,1,0}(e, x, y, v)$ |
| $a \geq 2, b \geq 2$ | ![Diagram for $a \geq 2, b \geq 2$] | $\mathcal{F}^*_{1,0,2,0}(e, x, y, v)$ |
Table 4.21: Diagrams and definition of $B^{2,i,a,b}(0, v, x, y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0, b \geq 2$</td>
<td>$d(w, u) = 1$</td>
<td>$w \geq 1$</td>
<td>$2dD(w-u)$ $\times \mathcal{I}<em>{1,1,0,1}(w, u, x, e_i)$ $\times \left( \mathcal{I}</em>{1,2,1}(u, v, w, y, 0) + \mathcal{I}_{2,1,1}(u, v, w, y, 0) \right)$</td>
</tr>
<tr>
<td>$0 = v \neq w$</td>
<td>$\geq 1$</td>
<td>$0 = v$</td>
<td>$d(w, u)$ $\geq 2$</td>
</tr>
<tr>
<td>$a \geq 2, b \geq 2$</td>
<td>$d(u, w) = 1$</td>
<td>$v \geq 0$</td>
<td>$\frac{1}{p} \mathcal{I}<em>{1,2,1}(y, u, w, v)$ $\times \mathcal{P}</em>{1,0,1,2,0}(e_i, x, u, w, v)$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$u \geq 1$</td>
<td>$0 = v$</td>
<td>$\frac{1}{p} \mathcal{I}<em>{1,2,1}(y, u, w, v)$ $\times \mathcal{P}</em>{1,0,1,2,0}(e_i, x, u, w, v)$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$x \geq 0$</td>
<td>$0 = v$</td>
<td>$\frac{1}{p} \mathcal{I}<em>{1,2,1}(y, u, w, v)$ $\times \mathcal{P}</em>{1,0,1,2,0}(e_i, x, u, w, v)$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$y \geq 0$</td>
<td>$0 = v$</td>
<td>$\frac{1}{p} \mathcal{I}<em>{1,2,1}(y, u, w, v)$ $\times \mathcal{P}</em>{1,0,1,2,0}(e_i, x, u, w, v)$</td>
</tr>
</tbody>
</table>
Table 4.22: Diagram of the different cases of $B^{2,l,a,b}(0,v,x,y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1, b = 0$</td>
<td>$d(w, y) = 1$</td>
<td>$d(w, y) = 1$</td>
<td>$\Rightarrow y = v$</td>
</tr>
<tr>
<td>$a = 1, b = 1$</td>
<td>$d(w, y) = 1$</td>
<td>$d(w, y) = 1$</td>
<td>$\tau_{3, p}(w - y) \times P_{1,1,1,0}(x, y, w, w + e, 0)$</td>
</tr>
<tr>
<td>$a = 1, b = 2$</td>
<td>$d(w, y) = 1$</td>
<td>$d(w, y) = 1$</td>
<td>$\tau_{3, p}(w - y) \times P_{0,1,1,0}(x, y, w, w + e, 0)$</td>
</tr>
<tr>
<td>$\Rightarrow y = v$</td>
<td>$d(w, y) \geq 2$</td>
<td>$d(w, y) \geq 2$</td>
<td>$\tau_{2, p}(w - y) \times P_{0,1,2,1}(x, y, w, w + e, 0)$</td>
</tr>
<tr>
<td>$\Rightarrow y = v$</td>
<td>$d(w, y) \geq 2$</td>
<td>$d(w, y) \geq 2$</td>
<td>$\tau_{2, p}(w - y) \times P_{0,1,2,1}(x, y, w, w + e, 0)$</td>
</tr>
</tbody>
</table>
### Table 4.23: Diagram of the different cases of $B_{2,l,a,b}^{2,l,a,b}(0, v, x, y)$ (continued)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \geq 2, b = 0$</td>
<td>$u = y$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$R_{1,2}(y - w, 0) \times T_{1,2}(e, -w, x - w, y - w)$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) = 1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{p}(T_{1,2}(y - w, u - w, 0) + T_{2,1}(y - w, u - w, 0)) \times R_{0,1,1,1}(x, u, w + e, 0)$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) \geq 2$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$R_{1,2}(y - w, u - w, 0) \times T_{1,1,1}(e, -w, x - w, u - w)$</td>
</tr>
<tr>
<td>$a \geq 2, b = 1$</td>
<td>$u = y$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$2dD(v)R_{1,2}(y - w, 0) \times \frac{1}{p} T_{0,1,0,2}(e, -w, v - w, x - w, y - w)/p$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) = 1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$2dD(v)T_{1,2}(y - w, u - w, 0) + T_{2,1}(y - w, u - w, 0)) \times R_{0,1,1,1}(x, u, w + e, v)$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) \geq 2$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$2dD(v)T_{1,1,2}(y - w, u - w, 0) \times R_{0,1,2,0}(x, u, w, w + e, v)$</td>
</tr>
<tr>
<td>$a \geq 2, b \geq 2$</td>
<td>$u = y$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$2dD(w - y)T_{3,p}(w - y) \times R_{2,1,1,0}(x, y, w, w + e, v) + T_{2,p}(w - y) \times R_{0,2,2,0}(x, y, w, w + e, v)$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) = 1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$(T_{1,2,1}(y - w, u - w, 0)/p + T_{2,1,1}(y - w, u - w, 0)/p) \times R_{0,1,2,0}(x, u, w + e, v)$</td>
</tr>
<tr>
<td>$d(u, y) \geq 1$</td>
<td>$d(w, u) \geq 2$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$pT_{0,p}(w + e_1)R_{0,1}(x, u) R_{1,1,2}(y - w, u - w, 0)$</td>
</tr>
</tbody>
</table>
Table 4.24: Diagrams and definition of $P_{i,(0),a}(x, y)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
<th>Diagram</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0$</td>
<td>$x = y$</td>
<td>$\tau_{3,p}(e_i) p^{(0),0} (x - e_i, x - e_i)$</td>
<td></td>
</tr>
<tr>
<td>$a = 1$</td>
<td>$y$ on sausage</td>
<td>$\tau_{3,p}(e_i) p^{(0),1} (x - e_i, y - e_i)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x = e_i$</td>
<td>$p \delta_{y,0} + \mathcal{B}_{2,1}(y, x)$</td>
<td></td>
</tr>
<tr>
<td>$a = 2$</td>
<td>$y$ on sausage</td>
<td>$\tau_{3,p}(e_i) p^{(0),2} (x - e_i, y - e_i)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x = e_i$</td>
<td>$\delta_{0, y} \tau_{3,p}(e_i) + \mathcal{B}_{1,2}(y, x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x \neq e_i$</td>
<td>$\mathcal{B}<em>{1,2}(y, e_i) + \mathcal{B}</em>{2,1}(y, e_i)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y$ not on sausage</td>
<td>$\mathcal{B}_{3,1}(x - e_i, 0)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathcal{B}_{2,2}(x - e_i, 0)$</td>
<td></td>
</tr>
</tbody>
</table>
We give an example of how we use a table to define the diagram: in Table 4.18 we define $B^{(i,a)}(x,y)$ to be

\[
P^{(0,0)}(x,y) = \delta_{x,y}(\delta_{0,x} + B_{1,3}(x,0) + B_{2,2}(x,0)),
\]
\[
P^{(0,1)}(x,y) = \delta_{0,y}(B_{1,3}(x,0) + T_{1,1,2}(x,y,0) + T_{2,1,1}(x,y,0)),
\]
\[
P^{(0,2)}(x,y) = \delta_{0,y}(B_{1,3}(x,0) + B_{2,2}(x,0)) + T_{2,1,1}(x,y,0).
\]

We have defined $A^{i,a,b}(0,v,x,y)$ using the non-repulsive diagram for $b > 0$. The reason for this is that we can not guarantee that the connection $x \to y$ and $e_i \to x$ are mutually avoiding. We will explain this in Section 4.5.4. We define the double-open triangle $\tilde{A}^{i,a,b}(0,v,x,y)$ to be

\[
\tilde{A}^{i,a,0}(0,v,x,y) = A^{i,a,0}(0,v,x,y) \quad \text{for } a = 0, 1, 2,
\]
\[
\tilde{A}^{i,a,1}(0,v,x,y) = \frac{1}{p} A^{i,a,1}(0,v,x,y) \quad \text{for } a = 0, 1, 2,
\]
\[
\tilde{A}^{i,0,2}(0,v,x,y) = \delta_{0,y} T_{1,1,0}(-y, e_i - y, x - y),
\]
\[
\tilde{A}^{i,1,2}(0,v,x,y) = \frac{1}{p} T_{0,1,1,0}(v - y, -y, e_i - y, x - y),
\]
\[
\tilde{A}^{i,2,2}(0,v,x,y) = p \tau_{0,p}(x - e_i) \tau_{0,p}(v - y).
\]

Further, we define the diagrams $B^{2,i,a,b}, B^{2,i,a,b},$ that are not yet defined in the tables 4.21 4.23 to be zero, i.e., for $b = 0, 1$ we let $B^{2,i,a,b}(0,v,x,y) = 0$ and for $a = 0$ we let $B^{2,i,a,b}(0,v,x,y) = 0$. We bound the intermediate pieces in Figure 4.20 by

\[
B^{i,a,b}(0,v,x,y) = \sum_{u,w} \sum_{c=0}^{2} A^{i,a,c}(0,v,u,w) A^{c,b}(u,w,x,y)
\]
\[
+ \sum_{u} A^{i,a,b}(0,v,u,x) P^{(0,0)}(x-u, x-u) + B^{2,i}(0,v,x,y).
\]

Further, we define $\tilde{B}^{i}$ by

\[
\tilde{B}^{i,a,b}(0,v,x,y) = \sum_{u,w} \sum_{c=0}^{2} A^{i,a,c}(0,v,u,w) A^{c,b}(u,w,x,y)
\]
\[
+ A^{i,a,b}(0,v,x,y) + \tilde{B}^{2,i,a,b}(0,v,x,y).
\]

Building blocks with weight.

To bound $\hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k)$ we define weighted diagrams, that are diagrams in which one line obtain the weight $[1 - \cos(k \cdot x)]$. In Table 4.25 we give as small overview of these diagrams.
4.5 Percolation

<table>
<thead>
<tr>
<th>weighted diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v ullet y$</td>
</tr>
<tr>
<td>$a$ $b$</td>
</tr>
<tr>
<td>$0 ullet x$</td>
</tr>
<tr>
<td>$[1 - \cos(k \cdot x)]$</td>
</tr>
</tbody>
</table>

weighted double open triangle $H^{1,a,b}(0, v, x, y)$. We know that the closed diagram consists of at least four bonds.

<table>
<thead>
<tr>
<th>weighted diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v ullet y$</td>
</tr>
<tr>
<td>$a$ $b$</td>
</tr>
<tr>
<td>$0 \uparrow e_i$</td>
</tr>
<tr>
<td>$x$ $[1 - \cos(k \cdot x)]$</td>
</tr>
</tbody>
</table>

weighted double open triangle $H^{2,a,b}(0, v, x, y)$, where the weighted line has a fixed step.

<table>
<thead>
<tr>
<th>weighted diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v \uparrow e_i$</td>
</tr>
<tr>
<td>$a$ $b$</td>
</tr>
<tr>
<td>$0$ $x$</td>
</tr>
<tr>
<td>$[1 - \cos(k \cdot x)]$</td>
</tr>
</tbody>
</table>

weighted double open triangle $H^{3,a,b}(0, v, x, y)$, where the unweighted line has a fixed step.

<table>
<thead>
<tr>
<th>Intermediate piece with triangle on top</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{t,w,z,u} v \quad t \quad u \quad y$</td>
</tr>
<tr>
<td>$a \quad b \quad + \quad a \quad t \quad u \quad y$</td>
</tr>
<tr>
<td>$0 \quad 1 \quad 1 \quad 0 \quad x \quad 1 \quad 1 \quad x$</td>
</tr>
<tr>
<td>$[1 - \cos(k \cdot w)]$</td>
</tr>
</tbody>
</table>

$C^{1,\kappa,a,b}(0, v, x, y)$

<table>
<thead>
<tr>
<th>Intermediate piece with triangle on button</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{t,w,z,u} v \quad t \quad w \quad y$</td>
</tr>
<tr>
<td>$a \quad b \quad + \quad a \quad t \quad w \quad y$</td>
</tr>
<tr>
<td>$0 \quad 1 \quad 1 \quad 0 \quad x \quad 1 \quad 1 \quad x$</td>
</tr>
<tr>
<td>$[1 - \cos(k \cdot u)]$</td>
</tr>
</tbody>
</table>

$C^{2,\kappa,a,b}(0, v, x, y)$

<table>
<thead>
<tr>
<th>The initial piece weighted piece of $\Xi , h^{i,a,b}(0, v, x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y \quad b + 0 \quad e_i \quad x$</td>
</tr>
<tr>
<td>$[1 - \cos(k \cdot (x-e_i))]$</td>
</tr>
</tbody>
</table>

Table 4.25: Description of $H^1, H^{i,2}, H^{i,3}, C^1, C^2$ and $h^i$

We use the building blocks defined in the previous section to define the weighted diagrams. For $a, b \in \{0, 1, 2\}$, let

\[
H^{1,a,b}(u, v, x, y; k) = [1 - \cos(k \cdot (u - x))]| \tilde{A}^{a,b}(u, v, x, y), 
\]

\[
H^{2,a,b}(u, v, x, y; k) = [1 - \cos(k \cdot (u - x))]| \tilde{A}^{i,a,b}(u, v, x, y), 
\]

\[
H^{3,a,b}(u, v, x, y; k) = [1 - \cos(k \cdot (v - y))]| \tilde{A}^{i,a,b}(u, v, x, y), 
\]

(4.5.23) (4.5.24) (4.5.25)
These will be sufficient to state the bounds on $\hat{\Sigma}_z^{(N)}(0)$ and $\hat{\Sigma}_z^{(N),i}(0)$ To bound $\hat{\Sigma}_z^{(N)}(0) - \hat{\Sigma}_z^{(N)}(k)$ we further define the matrices $H^1, H^2, H^3, C^1$ and $C^2$ with entries

$$
(H^1(k))_{a,b} = \sup_{v,y \in \mathbb{Z}^d} \sum_x H^{1,a,b}(0,v,x,y; k),
$$

$$
(H^2(k))_{a,b} = \sup_{v,y \in \mathbb{Z}^d} \sum_x H^{2,a,b}(0,v,x,y; k),
$$

$$
(H^3(k))_{a,b} = \sup_{v,y \in \mathbb{Z}^d} \sum_x H^{3,a,b}(0,v,x,y; k),
$$

$$
(C^1(k))_{a,b} = \sup_{v,y \in \mathbb{Z}^d} \sum_x C^{1,a,b}(0,v,x,y; k),
$$

$$
(C^2(k))_{a,b} = \sup_{v,y \in \mathbb{Z}^d} \sum_x C^{2,a,b}(0,v,x,y; k).
$$
Further, we define the vectors \( \vec{h}(k) \), \( \vec{h}^i(k) \) and \( \vec{h}^{i,II}(k) \) with entries
\[
(\vec{h}(k))_b = (H^1(k))_{0,b}, \quad (\vec{h}^i(k))_b = \sum_{l, x, y} h^{i,k,b}(x, y; k),
\]
\[
(\vec{h}^{i,II}(k))_b = \sum_{l, x, y} h^{i,k,II,b}(x, y; k).
\] (4.5.38)

The entries of \( H^1(k), H^2(k), H^3(k) \) and \( \vec{h}(k) \) are bounded in the same way as for LTs. The bounding of \( \vec{h}^i(k), C^1(k) \) and \( C^2(k) \) is difficult and will therefore be discussed in Section 4.5.6.

### 4.5.3 The diagrammatic bounds

In this section we list the bounds on the coefficients that we are going to prove:

**Lemma 4.5.1** (Bounds on the percolation coefficients for \( N = 0 \)). Let \( z < z_c \). Then,
\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0)}(x) \leq \sum_{x \in \mathbb{Z}^d} P_z^{(0),0}(x, x) \] (4.5.39)
\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0),i}(x) \leq \tau_{3,z}(e_1) \sum_{x \in \mathbb{Z}^d} P_z^{(0),i,0}(x, x), \] (4.5.40)
\[
\sum_{x \in \mathbb{Z}^d} \Psi_z^{(0),k}(x) \leq (1 - \tau_{1,p}(e_1)) + \sum_{x \in \mathbb{Z}^d \setminus \{0\}} P_z^{(0),0}(x, x), \] (4.5.41)

and
\[
\sum_{x \in \mathbb{Z}^d} \Xi_z^{(0)}(x)[1 - \cos(k \cdot x)] \leq 2d p \tau_{3,p}(e_1)[1 - \hat{D}(k)] + \sum_{x \in \mathbb{Z}^d} \tau_{2,p}(x)^2[1 - \cos(k \cdot x)]. \] (4.5.42)

Further,
\[
\sum_{l, x \in \mathbb{Z}^d} \Xi_z^{(0),i}(x)[1 - \cos(k \cdot (x - e_i)))] \] (4.5.43)
\[
= 2d(2d - 1) p \tau_{3,p}(e_1)^2[1 - \hat{D}(k)] + (2d - 1) \tau_{3,p}(e_1) \sum_{x \in \mathbb{Z}^d} \tau_{2,p}(x)^2[1 - \cos(k \cdot x)], \] (4.5.44)
\[
\sum_{l, x \in \mathbb{Z}^d} \Xi_z^{(0),i}(x)[1 - \cos(k \cdot x)] \] (4.5.45)
\[
= 2 \sum_{l, x \in \mathbb{Z}^d} \Xi_z^{(0),i}(x)[1 - \cos(k \cdot (x - e_i)))] + 4d[1 - \hat{D}(k)] \sum_{x \in \mathbb{Z}^d} \Xi_z^{(0),i}(x). \] (4.5.46)

**Lemma 4.5.2** (Bounds on the percolation coefficients for \( N = 1 \)). Let \( 0 \leq z \leq z_c \). Then,
\[
\hat{\Xi}_z^{(1)}(0) \leq \vec{P}^T \vec{A} \vec{P}, \] (4.5.45)
\[
\hat{\Xi}_z^{(1),i}(0) \leq \frac{1}{2d} (\vec{P}^i)^T \vec{A} \vec{P}, \] (4.5.46)
and
\[
\hat{\Xi}_z^{(1)}(0) - \hat{\Xi}_z^{(1)}(k) \leq (H^2(k))_{0,0} + 8 \sum_{a=0}^2 (\tilde{h}(k))_a (A')_{a,0} + 6 \sum_{a,b=0}^2 (\tilde{h}(k))_a (A')_{a,b} [(\bar{p})_b - \delta_{0,b}]
+ 3 \sum_{a,b=0}^2 [(\tilde{p})_a - \delta_{0,a}] (H^2)_{a,b} [(\bar{p})_b - \delta_{0,b}],
\]
(4.5.47)
\[
\sum_i (\hat{\Xi}_z^{(i,1)}(0) - \hat{\Xi}_z^{(i,1)}(k) e^{-ik} e_i)
\leq (\tilde{h}'(k))_0 + 2\tilde{h}'(k)^T (\bar{p} - (1,0,0)) + 2(\bar{p})^T A' h(k).
\]
(4.5.48)
Consequently,
\[
\sum_i (\hat{\Xi}_z^{(i,1)}(0) - \hat{\Xi}_z^{(i,1)}(k) ) \leq 2\sum_i (\hat{\Xi}_z^{(i,1)}(0) - \hat{\Xi}_z^{(i,1)}(k) e^{-ik} e_i) + 4d(1 - \hat{D}(k)]\hat{\Xi}_z^{(i,1)}(0).
\]
(4.5.49)

**Proposition 4.5.3** (Bounds on the percolation coefficients for \(N \geq 2\)). Let \(0 \leq z \leq z_c\). Then,
\[
\hat{\Xi}_z^{(N)}(0) \leq \bar{p}^T (B')^{N-1} A' \bar{p},
\]
(4.5.50)
\[
\hat{\Xi}_z^{(N,1)}(0) \leq \frac{1}{2d} (\bar{p})^T (B')^{N-1} A' \bar{p}.
\]
(4.5.51)
For \(N \geq 2\) even,
\[
\hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \leq (N + 2) \left[ (\tilde{h}(k))^T A' (\tilde{B}')^{N-1} \bar{p} + \bar{p}^T (B')^{N-1} (H^3(k) \bar{p} + A' \tilde{h}(k)) \right]
+ (N + 2) \sum_{M=0}^{N/2-1} \bar{p}^T (B')^{2M} C_1(k)(\tilde{B}')^{N-2-2M} \bar{p}
+ (N + 2) \sum_{M=0}^{N/2-2} \bar{p}^T (B')^{2M+1} C_2(k)(\tilde{B}')^{N-3-2M} \bar{p},
\]
(4.5.52)
and for \(N \geq 2\) odd,
\[
\hat{\Xi}_z^{(N)}(0) - \hat{\Xi}_z^{(N)}(k) \leq (N + 2) \left[ (\tilde{h}(k))^T A' (\tilde{B}')^{N-1} \bar{p} + \bar{p}^T (B')^{N-1} (H^2(k) \bar{p} + A' \tilde{h}(k)) \right]
+ (N + 2) \sum_{M=0}^{(N-3)/2} \bar{p}^T (B')^{2M} C_1(k)(\tilde{B}') + B' C_2(k)(\tilde{B}')^{N-3-2M} \bar{p}.
\]
(4.5.53)
For \(N \geq 2\) even,
\[
\sum_i (\hat{\Xi}_z^{(N,1)}(0) - \hat{\Xi}_z^{(N,1)}(k) e^{ik})
\leq (N + 1) \left[ (\tilde{h}'(k))^T (B')^{N-1} \bar{p} + (\bar{p})^T (B')^{N-1} (H^3(k) \bar{p} + A' \tilde{h}(k)) \right]
+ (N + 1) \sum_{M=0}^{(N-4)/2} (\bar{p})^T (B')^{2M+1} (C_2(k)(\tilde{B}') + B' C_1(k)(\tilde{B}')^{N-4-2M} \bar{p},
\]
(4.5.54)
and
\[
\sum_i (\hat{\Xi}_z^{(N),i}(0) - \hat{\Xi}_z^{(N),i}(k)) \leq (N + 2) \left[ \left( \tilde{\eta}^{(k)}(k) \right)^T (\mathbf{B}^i)^{N-1} \tilde{\mathbf{p}} + (\mathbf{p}^i)^T (\mathbf{B}^i)^{N-1} (\mathbf{H}^3(k) \tilde{\mathbf{p}} + \mathbf{A}^i \tilde{\mathbf{h}}(k)) \right] \\
+ (N + 2) \sum_{M=0}^{(N-4)/2} (\mathbf{p}^i)^T (\mathbf{B}^i)^{2M+1} (\mathbf{C}^2(k) \tilde{\mathbf{B}}^i + \mathbf{B}^i \mathbf{C}^1(k)) (\mathbf{B}^i)^{N-4-2M} \tilde{\mathbf{p}} \\
+ (N + 2)(\mathbf{p}^i)^T (\mathbf{B}^i)^{N-1} \mathbf{A}^i \tilde{\mathbf{p}}. \tag{4.5.55}
\]

For \(N \geq 2\) odd,
\[
\sum_i (\hat{\Xi}_z^{(N),i}(0) - \hat{\Xi}_z^{(N),i}(k)) e^{ik} \leq (N + 1) \left[ \left( \tilde{\eta}^{(k)}(k) \right)^T (\mathbf{B}^i)^{N-1} \tilde{\mathbf{p}} + (\mathbf{p}^i)^T (\mathbf{B}^i)^{N-1} (\mathbf{H}^2(k) \tilde{\mathbf{p}} + \mathbf{A}^i \tilde{\mathbf{h}}(k)) \right] \\
+ (N + 1) \sum_{M=0}^{(N-3)/2} (\mathbf{p}^i)^T (\mathbf{B}^i)^{2M+1} \mathbf{C}^2(k) (\mathbf{B}^i)^{N-3-2M} \tilde{\mathbf{p}} \\
+ (N + 1) \sum_{M=0}^{(N-5)/2} (\mathbf{p}^i)^T (\mathbf{B}^i)^{2M+2} \mathbf{C}^1(k) (\mathbf{B}^i)^{N-4-2M} \tilde{\mathbf{p}}. \tag{4.5.56}
\]

\[
\sum_i (\hat{\Xi}_z^{(N),i}(0) - \hat{\Xi}_z^{(N),i}(k)) = (N + 2) \left[ \left( \tilde{\eta}^{(k)}(k) \right)^T (\mathbf{B}^i)^{N-1} \tilde{\mathbf{p}} + (\mathbf{p}^i)^T (\mathbf{B}^i)^{N-1} (\mathbf{H}^2(k) \tilde{\mathbf{p}} + \mathbf{A}^i \tilde{\mathbf{h}}(k)) \right] \\
+ (N + 2) \sum_{M=0}^{N/3-1} (\mathbf{p}^i)^T (\mathbf{B}^i)^{2M} \mathbf{C}^1(k) (\mathbf{B}^i)^{N-2-2M} \tilde{\mathbf{p}} \\
+ (N + 2)(\mathbf{p}^i)^T (\mathbf{B}^i)^{N-1} \mathbf{A}^i \tilde{\mathbf{p}}. \tag{4.5.57}
\]

### 4.5.4 Bounding events

The NoBLE coefficients for percolation are defined in terms of the probability of events. First we create bounding events and then the bounding diagrams that bound the probability of these events. We adapt arguments that can be found in either [38 Lemma 2.5] or [74 Lemma 5.5.8].

Let \(\mathbb{P}^{(N)}\) denote the product measure on \(N + 1\) copies of percolation on \(\mathbb{Z}^d\), where in the \(i\)th copy, all bonds emanating from \(\mathbf{b}_i \) are made vacant, i.e.,
\[
\mathbb{P}^{(N)} = \mathbb{P}_0 \times \mathbb{P}_1^{\mathbf{b}_0} \times \cdots \times \mathbb{P}_N^{\mathbf{b}_{N-1}}. \tag{4.5.58}
\]

Using Fubini’s Theorem and \(2.4.32\) we conclude that
\[
\Xi^{(N),i}(x, y; A) = \sum_{b_0, \ldots, b_{N-1}} p^{N} \mathbb{P}^{(N)} \left( E^i(x, b_0; A) \cap \mathbf{b}_0 \notin \mathbf{\tilde{C}}_0 \cup B \cap \mathbf{E}^i(\mathbf{b}_0, b_1; \mathbf{\tilde{C}}_0 \cup B) \cap \mathbf{E}^i(\mathbf{b}_1, \mathbf{\tilde{C}}_1; \mathbf{b}_0 \notin \mathbf{\tilde{C}}_1) \cap \mathbf{E}^i(\mathbf{\tilde{b}}_{i-1}, \mathbf{\tilde{b}}_i; \mathbf{\tilde{C}}_{i-1}) \cap \mathbf{E}^i(\mathbf{b}_N, y; \mathbf{\tilde{C}}_{N-1}) \right), \tag{4.5.59}
\]
where, for an event $F$, we write $F_i$ to denote that $F$ occurs on graph $i$. Then, we define events to bound $E'(\overline{b}_{i-1}, b_i; \overline{e}_{i-1})_i$. Therefore, we recall the that $E \circ F$ denotes the event that $E$ and $F$ occur disjointly. We first focus on the bounding event for $\Xi^{(N)}_z(x)$. We define the events

\[
F_0(x, b_0, w_0, z_1) = \{x \leftrightarrow b_0\} \circ \{x \leftrightarrow w_0\} \circ \{w_0 \leftrightarrow b_0\} \circ \{w_0 \leftrightarrow z_1\} \cap \{z_1 \notin b\},
\]
\[
F_N(\overline{b}_{N-1}, t_N, z_N, y) = \{\overline{b}_{N-1} \leftrightarrow t_N\} \circ \{t_N \leftrightarrow z_N\} \circ \{t_N \leftrightarrow x\} \circ \{z_N \leftrightarrow y\} \cap \{\overline{b}_{N-1} \notin \{t_N, z_N, y\}\},
\]
for \( N \geq 1 \) and

\[
\begin{align*}
F'(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) &= \{ \overline{b}_{i-1} \leftarrow w_i \} \circ \{ w_i \leftarrow b_i \} \circ \{ w_i \leftarrow z_i \} \\
&\quad \cap \{ z_i \neq b_i \} \cap \{ z_i = b_i \} \cap \{ z_i = t_i \} \\
&\quad \cap \{ z_{i+1} \neq b_i \} \cap \{ b_{i-1} \neq \{ t_i, w_i, z_i, b_i \} \},
\end{align*}
\]

(4.5.62)

\[
\begin{align*}
F''(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) &= \{ \overline{b}_{i-1} \leftarrow w_j \} \circ \{ w_i \leftarrow t_i \} \circ \{ t_i \leftarrow z_i \} \circ \{ t_i \leftarrow b_i \} \\
&\quad \circ \{ z_i \leftarrow b_j \} \circ \{ w_i \leftarrow z_i \} \cap \{ z_i \neq b_i \} \\
&\quad \cap \{ z_{i+1} \neq b_i \} \cap \{ b_{i-1} \neq \{ t_i, w_i, z_i, b_i \} \} \cap \{ w_i \neq t_i \},
\end{align*}
\]

(4.5.63)

\[
\begin{align*}
F'''(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) &= \{ \overline{b}_{i-1} \leftarrow t_i \} \circ \{ t_i \leftarrow z_i \} \circ \{ t_i \leftarrow w_i \} \circ \{ z_i \leftarrow b_i \} \\
&\quad \circ \{ w_i \leftarrow b_j \} \circ \{ w_i \leftarrow z_i \} \cap \{ z_i \neq b_i \} \cap \{ z_{i+1} \neq b_i \} \\
&\quad \cap \{ b_{i-1} \neq \{ t_i, w_i, z_i, b_i \} \},
\end{align*}
\]

(4.5.64)

\[
\begin{align*}
F(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) &= F'(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) \\
&\quad \cup F''(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) \\
&\quad \cup F'''(b_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}),
\end{align*}
\]

(4.5.65)

for \( i \in [1, N - 1] \). The events \( F_0, F, F_N \) are depicted in Figure 4.23. These diagrams are similar to the diagrams used for the classical lace expansion, see [38, Section 2.2]
that (4.5.66) holds. Next we argue for right hand side of there exists a path of open edges connecting them as shown in Figure 4.23. In the event that all bonds that contain \( x \) or \( \{ t_i, z_i, b_i \} \) contribute. Since \( t_N, w_N, z_N, y \in \mathcal{C}_N \) we know that there exists a path of open edges connecting them as shown in Figure 4.23 in the right hand side of (4.5.66) we simply sum over all possible \( t_N \) and \( z_N \) and conclude that (4.5.66) holds. Next we argue for \( N \geq 1 \) and \( i \in [1, N-1] \) that:

\[
E'(\tilde{b}_{i-1}, b_i; \mathcal{C}_{i-1}) \cap \{ z_i+1 \in \mathcal{C}_i \} \cap \{ \tilde{b}_i \notin \mathcal{C}_i \} \cup E_{\text{vac}}(b_{i-1})
\]

(4.5.66)

\[
\subseteq \bigcup_{z_N \in \tilde{\mathcal{C}}_{N-1}} \bigcup_{t_N \in \mathbb{Z}^d} F_N(b_{N-1}, t_N, z_N, w) \cap E_{\text{vac}}(b_{N-1}),
\]

where we defined \( E' \) in (2.4.14). For the event \( E' \) to occur there must exist at least one vertex in \( z_N \in \tilde{\mathcal{C}}_{N-1} \) that lies on the last sausage. We denote by \( t_N \) the first point of the last sausage. As we restrict to the configurations in which \( E_{\text{vac}}(b_{N-1}) \) holds we know that only \( b_{N-1} \notin \{ t_N, z_N, y \} \) contribute. Since \( t_N, w_N, z_N, y \in \mathcal{C}_N \) we know that (4.5.66) holds. Next we argue for (4.5.66) we simply sum over all possible \( t_N \) and \( z_N \) and conclude that (4.5.66) holds. Next we argue for \( N \geq 1 \) and \( i \in [1, N-1] \) that:

\[
E'(\tilde{b}_{i-1}, b_i; \mathcal{C}_{i-1}) \cap \{ z_i+1 \in \mathcal{C}_i \} \cap \{ \tilde{b}_i \notin \mathcal{C}_i \} \cup E_{\text{vac}}(b_{i-1})
\]

(4.5.67)
If \( E'(\vec{b}_{i-1}, b_i; \tilde{\mathcal{C}}_{i-1}) \) occurs then a string of sausages connects \( \vec{b}_{i-1} \) and \( b_i \) and the last sausage of the string is cut through by \( \tilde{\mathcal{C}}_{i-1} \). We denote the “first” point of the last sausage by \( t_i \). We identify the first point that every path from \( \vec{b}_{i-1} \) to \( b_i \) and from \( \vec{b}_{i-1} \) to \( z_{i+1} \) share by \( w_i \). By \( z_i \in \tilde{\mathcal{C}}_{i-1} \) we identify one point in the last sausage where \( \vec{b}_{i-1} \rightarrow b_i \) is cut through. The event \( F' \) characterizes the cases that \( z_i = b_i \), \( F'' \) that \( w_i \) is in the last sausage and \( F''' \) that \( w_i \) is on the last sausage. If \( w_i \) is in the last sausage then we choose \( z_i \) such that it is on the opposite side of the sausage, i.e., we choose \( z_i \) such that there exist two bond-disjoint paths from \( t_i \) to \( \vec{b}_i \) such that \( z_i \) lies on the path and \( w_i \) on the other path. This is always possible as a sausage characterizes a double connection and as this sausage is cut through by \( \tilde{\mathcal{C}}_{i-1} \) all connections contain an element of \( \tilde{\mathcal{C}}_{i-1} \). The restriction on configurations with \( E_{\text{vac}}(\vec{b}_{i-1}) \) guarantees that \( \{b_{i-1} \notin \{t_i, w_i, z_i, b_i\}\} \).

The graph 0 is bounded in the same way using \( F_0 \). As this is completely analogous, we omit this here. We conclude from (4.5.66) and (4.5.67) that

\[
\Xi^{(N)}_z(x) \leq \sum_{i, w, z, \vec{b}} p^N p_z(F(0, t_0, a, b_0, w_0, z_1) \cap \{b_0 \notin \tilde{\mathcal{C}}_0\}) \prod_{i=1}^{N-1} p^b_z b_{i-1}(F(\vec{b}_{i-1}, t_i, z_i, b_i, w_i, z_{i+1}) \cap \{b_i \notin \tilde{\mathcal{C}}_i\}) b^b_z(N-1)(F_0(\vec{b}_{N-1}, t_N, z_N, \vec{x})),
\]

where the summation is over \( \vec{t} = (t_0, \ldots, t_N) \), \( \vec{w} = (w_0, \ldots, w_{N-1}) \), \( \vec{z} = (z_1, \ldots, z_N) \) and \( \vec{b} = (b_0, \ldots b_{N-1}) \). The probabilities in (4.5.66) factor as the events \( F_0, \ldots, F_N \) occur on different percolation configurations. If we would at this point follow the classical lace expansion then we would apply the BK-inequality on (4.5.66) and obtain a bound on \( \Xi^{(N)}_z \) in terms of a combinations of two-point functions \( \tau_z \). In doing so, we would lose the information that all loops have length at least four and that the intersection at \( z_{i+1} \) can not occur at the bond \( b_i \). In Section 4.5.5 we show how to create a bound on (4.5.66) that uses this extra information.

For the bound we use another property of the diagram. We want to choose \( z_i \) such that there exists a path from \( w_{i-1} \) to \( z_i \) that intersects \( \tilde{\mathcal{C}}_i \) only at its endpoint \( z_i \). In this case we can bound the triangle/square using a repulsive diagram. If \( w_i \) is not on the last sausage then we simply define \( z_i \) to be that a point in \( \tilde{\mathcal{C}}_{i-1} \cap \tilde{\mathcal{C}}_i \) with the smallest intrinsic distance between \( b_{i-1} \) and \( z_i \) on \( \tilde{\mathcal{C}}_{i-1} \). If however \( w_i \) is in the last sausage then it can occur that, for every choice of \( z_i \), every path from \( z_i \) to \( b_{i-1} \) has to pass the point \( w_{i-1} \) and thereby also \( \tilde{\mathcal{C}}_{i-1} \), see Figure 4.24 for a trivial example. This is the reason why we define the diagram \( A^{i,a,b} \) by non-repulsive diagrams. We will now derive a relation similar to (4.5.66) for \( \Xi^{(N)}_z \). We review the definition of \( \Xi^{(N)}_z \) and \( \Xi^{(N)}_{\mathcal{C}} \) in (2.4.41) and (2.4.55). In (2.4.55) we define \( \Xi^{(N)}_{\mathcal{C}}(x) \) for \( N \geq 1 \) as the sum of two terms. The only formal difference of the first term \( \Xi_{\mathcal{C}}^{b,v}(0, x; |e_i|) \) to \( \Xi^{(N)}_z(x) = \Xi^{b,N}_{\mathcal{C}}(0, x; |0|) \) is at the level of graph 0 and 1. Thus, we can use (4.5.66) and (4.5.67) to estimate the event defining \( \Xi_{\mathcal{C}}^{b,v}(0, x; |e_i|) \) on the
Figure 4.24: A configuration in which we define \( z_{sisi} \) and \( w_i \) such that the path \( b_{i-1} \leftarrow z_i \) and the path \( z_{i+1} \leftarrow w_i \) are bond-disjoint.

Graph level \( i = 2, \ldots, N \). The condition \( E'(0, b_0; \{e_i\})_0 \) on the graph 0 states that \( b_i \in \mathcal{C}_1 \) and thereby \( E'(\overline{b_0, b_1}; \overline{\mathcal{C}_0} \cup \{b_i\})_1 = E'(\overline{b_0, b_1}; \overline{\mathcal{C}_0})_1 \) and we can use (4.5.67) also to bound the event on level \( i = 1 \). To estimate the event on graph 0 we define

\[
F_{0}^{I}(b_0, w_0, z_1) = \{0 \leftarrow e_i \circ \{e_i \leftarrow w_0 \circ \{w_0 \leftarrow b_0 \circ \{w_0 \leftarrow z_1 \} \right. \} \}
\cap \{z_1 \notin b_0 \} \cap \{b_i \text{ is vacant}\},
\]

(4.5.69)

\[
F_{0}^{II}(b_0, w_0, z_1) = \{0 \leftarrow w_0 \circ \{w_0 \leftarrow e_i \circ \{e_i \leftarrow b_0 \circ \{w_0 \leftarrow z_1 \} \right. \} \}
\cap \{w_0 \neq e_i \} \cap \{z_1 \notin b_0 \} \cap \{b_i \text{ is vacant}\},
\]

(4.5.70)

\[
F_{0}^{III}(b_0, w_0, z_1) = \{0 \leftarrow z_1 \circ \{b_i \text{ is occupied} \} \cap \{z_1 \notin b_0 \}
\cap \{w_0 = 0 \} \cap \{b_0 = (0, e_i)\}.
\]

(4.5.71)

See Figure 4.25 for diagrammatic representations of these events.

Now, we will argue that

\[
F_{0}^{I} = (b_0, w_0, z_1) = \quad 0 \quad e_i \quad w_0 \quad b_0 \quad z_1 \notin b_0
\]

\[
F_{0}^{II} = (b_0, w_0, z_1) = \quad 0 \quad e_i \quad b_0 \quad z_1 \notin b_0, w_0 \neq e_i
\]

\[
F_{0}^{III} = (b_0, w_0, z_1) = \quad 0 = w_0 \quad e_i = \overline{b_0} \quad z_1 \notin b_0
\]

Figure 4.25: Diagrammatic representations of the events \( F_{0}^{I}(b_0, w_0, z_1) \). Lines indicate disjoint connections. Filled cycles might be trivial.
Thus, (4.5.68) hold for $\Xi$. The second part of (4.5.66) with

$$E' \left(0, b_0; \{e_i\}_0 \cap \{z_1 \in \hat{C}_0 \} \cap \{b_i \text{ is vacant} \} \subseteq \bigcup_{w_1 \in \mathbb{Z}^d} \left( F_{0}^{l,l}(b_0, w_0, z_1) \cup F_{0}^{l,\Pi}(b_0, w_0, z_1) \right).$$

(4.5.72)

If $E'(0, b_0; \{e_i\}_0$ occurs then there exists a path of occupied edges to $e_i$. As $e_i$ cuts the connection $0 \rightarrow b_0$ either $b_0 = e_i$ or $e_i$ and $b_0$ are connected by a sausage. We denote by $w_0$ the last point that a connection $0 \rightarrow b_0$ and $0 \rightarrow z_1$ share. If $w_1$ is on the last sausage then the event is part of $F'$ and otherwise part of $F''$. The second part of $\Xi^{(N),l.\Pi}(x)$ is given by

$$p_{0}^{(0),l.\Pi}(\{z \in \hat{C}_0 \} \cap \{b_i \text{ is vacant} \} \subseteq F_{0}^{l,\Pi}(0, 0, z_1).$$

(4.5.73)

We use (4.5.59) with $A = \hat{C}_0(0)$ and $B = B(0)$ the set bond connected to the origin. Then we use (4.5.66) to bound the event at level $i = 2, \ldots, N$. For $i = 1$ we see that

$$E'(e_i, b_1; e \hat{C}_1(0) \cap \{z_1 \in \hat{C}_1 \} \cap \{b_1 \notin \hat{C}_1 \} \cap F_{0}^{\text{vac}}(0)
\subseteq \bigcup_{z_2 \in \hat{C}_1} \bigcup_{i_1, i_2 \in \mathbb{Z}^d} F(e_i, i_1, z_1, b_1, w_1, z_2) \cap F_{0}^{\text{vac}}(0).$$

(4.5.74)

For the initial graph in (4.5.73) we see that

$$\{z_1 \in \hat{C}_0 \} \cap \{e_i \notin \hat{C}_0 \} \cap \{b_i \text{ is vacant} \} \subseteq F_{0}^{l,\Pi}(0, 0, z_1).$$

(4.5.75)

Thus, (4.5.68) hold for $\Xi^{(N),l.\Pi}$ when replacing $F$ with $F^l = F_{0}^{l,\Pi} \cup F_{0}^{l,\Pi} \cup F_{0}^{l,\Pi}$.  

### 4.5.5 Proof of $x$-space bounds

Here, we combine the building blocks defined in Section 4.5.2 to construct the bounding diagrams. For $b = 0, 1, 2$ and $x, y \in \mathbb{Z}^d$ let

$$R_{z}^{(0),b}(x, y) = P_{z}^{(0),b}(x, y),$$

(4.5.76)

and for $N \geq 1$

$$P_{z}^{(N),b}(u_N, w_N) = \sum_{u_{N-1}, w_{N-1} \in \mathbb{Z}^d} \sum_{a=0}^{2} \sum_{a=0}^{2} P_{z}^{(N-1),b}(u_{N-1}, w_{N-1}) \times B_{z}^{a,b}(u_{N-1}, w_{N-1}, w_N, u_N),$$

(4.5.77)

$$P_{z}^{(N),l.\Pi}(u_N, w_N) = \sum_{u_{N-1}, w_{N-1} \in \mathbb{Z}^d} \sum_{a=0}^{2} \sum_{a=0}^{2} P_{z}^{(N-1),l.\Pi}(u_{N-1}, w_{N-1}) \times B_{z}^{a,b}(u_{N-1}, w_{N-1}, w_N, u_N),$$

(4.5.78)

$$R_{z}^{(N),a}(x, y) = \sum_{u, v \in \mathbb{Z}^d} \sum_{b=0}^{2} \sum_{b=0}^{2} \tilde{B}_{z}^{a,b}(x, y, u, v) R_{z}^{(N-1),b}(u, v).$$

(4.5.79)
Further, we define a modification of \( P^{(0),t,a}(x, y) \) in which the connection \( x \leftrightarrow y \) does not contribute:

\[
Q^t,0(x, y) = P^{(0),t,0}(x, y), \quad Q^t,1(x, y) = \frac{1}{p} P^{(0),t,1}(x, y),
\]

\[
Q^t,2(x, y) = \tau_{3,p}(e_i) \sum_{\kappa} A^{\kappa,0,2}(e_i, e_i, x, y) + \delta_{0,y}(\delta_{x,e_i} + (1 - \delta_{0,x}) \tau_{2,p}(x - e_i)) + \tau_{1,p}(w)(\mathcal{B}_{3,1}(e_i, x) + \mathcal{B}_{2,2}(e_i, x)).
\]

Then, we briefly describe how to prove an analog of Lemma 4.2.57 for percolation.

**Lemma 4.5.4 (x-space bounds).** For every \( x \in \mathbb{Z}^d, N \geq 1 \) and \( 0 \leq M \leq N - 1 \) the following bounds hold

\[
\Xi_z^{(N)}(x) \leq \sum_{u_M, w_M, w_{M+1}, z_{M+1} \in \mathbb{Z}^d} \sum_{a, b = 1}^2 \sum_{k} P^{(M),a}(u_M, w_M) A^{\kappa,0,a}(u_M, w_M, w_{M+1}, z_{M+1}) R^{(N-M-1),b}(z_{M+1} - x, w_{M+1} - x)
\]

\[
\Xi_z^{(N)}(x) \leq \sum_{u_{N-1}, w_{N-1}, z_{N-1}} \sum_{a, b = 0}^2 \sum_{k} P^{(N-1),a}(u_{N-1}, w_{N-1}) A^{\kappa,0,a}(u_{N-1}, w_{N-1}, w_N, u_N)(\delta_{b,0}\delta_{w_N,x} + A^{b,0}(u_N, w_N, x, x))
\]

\[
\Xi_z^{(N)}(x) \leq \sum_{u_0, w_0, z_1} \sum_{a, b = 0}^2 \sum_{k} (\delta_{a,0}\delta_{w_0,0} + A^{a,0}(u_0, w_0, 0, 0)) A^{\kappa,a,b}(z_1, w_1, u_0, w_0) R^{(N-1),b}(u_1 - x, w_1 - x)
\]

For \( \Xi_z^{(N),t}(x) \) the bound (4.5.82) and (4.5.83) hold where \( P^{(M),a} \) is replaced with \( P^{(M),t,a} \).

Finally,

\[
\Xi_z^{(N),t}(x) \leq \sum_{u_0, w_0, z_1} \sum_{a, b = 0}^2 \sum_{k} Q^{t,a}(u_0, w_0) A^{\kappa,a,b}(z_1, w_1, u_0, w_0) R^{(N-1),b}(u_1 - x, w_1 - x)
\]

**Proof.** We prove this lemma in the same way as Lemma 4.2.6 and Lemma 4.3.10. We use induction. We start at (4.5.68) and deconstruct the events from right to left. When removing a block we label the points such that we can choose paths between the points that do not intersect. Then we see that the existence of these connections is contained in the event that we use to define the building block. As this is very similar to Lemma 4.3.10 we will omit the details of the proof.

To give the idea of the proof, we now briefly discuss how to the bound diagrams as shown in Figure 4.26. In Section 4.5.4 we have proven that a diagram as shown in Figure 4.26 is bounded by

\[
p^2 P^{(N)}[ F_0(0, b_0, w_0, z_1) \cap E_{\text{vac}}(\overline{b}_0) \cap F(\overline{b}_i, t_1, z_1, b_1, w_1, z_2) \cap E_{\text{vac}}(\overline{b}_1) \cap F(\overline{b}_1, t_2, z_2, y_2)]
\]
Figure 4.26: The combination of events that we used to bound $\Xi_z^{(2)}(x)$ and the corresponding bounding diagram. Lines indicate disjoint connections. A filled triangles might be trivial.

In this example we can choose $z_1$ and $z_2$ such that the path from $w_i$ to $z_i$ intersects $\tilde{C}_i$ only at $z_i$. Thereby, we know that each triangle and pentagon of the diagram is repulsive and we can deconstruct the diagram to obtain the desired bound. We define

- $a_0$ to be the length of a path in $\tilde{C}_0$ from $b_0$ to $w_0$ that does not pass the origin,
- $a_1$ to be the length of a path in $\tilde{C}_1$ from $b_1$ to $w_0$ that does not path $z_1$,
- $a_2$ to be the length of a path in $\tilde{C}_2$ from $z_2$ to $t_2$ that does not path $x$,

and then go through the distinguish between the case $a_i = 1$ and $a_i \geq 2$. We can bound the last and the final triangle by $P^{(0),a_0}(b_0, w_0)$ and $P^{(0),a_2}(t_2 - x, z_2 - x)$, respectively. Then, we bound the left open square including the triangle by $B^{2,i,a_0,a_1}(b_0, w_1, b_1, w_0)$ and the double open triangle by $\tilde{A}^{i,a_1,a_2} (b_1, t_2, z_2, w_1)$. The building blocks were defined such that this last step is possible for all possible contributions to the coefficients.

**4.5.6 Final step of the bounding procedure**

We next comment on the proofs of Lemmas 4.5.1-4.5.2 and Proposition 4.5.3 and discuss some features that are not present for the other models.
We recall that $\Xi^{(0)}(x)$ corresponds to the probability that $0$ and $x$ are double connected and $\Xi^{(0),i}(x)$ to the probability that the connection $0$ to $x$ is cut through by $e_i$, such that $e_i$ is doubly connected to $x$. Thus, the bounds on $\Xi^{(0)}_z$ and $\Xi^{(0),i}_z$ for $N=0$ in Lemma 4.5.1 follow from definition. For the bound on $\Psi^{(0),K}_z$ we recall that

$$\Psi^{(0),K}_z(0) = \rho = \mathbb{P}(e_k \not\in \mathcal{E}_0(0)) = 1 - \mathbb{P}(e_k \in \mathcal{E}_0(0)) = 1 - \tau_1, p(e_k) \quad (4.5.87)$$

For $x \neq 0$ we use the bound $\Psi^{(0),K}_z(x) \leq \Xi^{(0),K}_z(x)$ and obtain the stated bound. The bounds on the absolute value of $\Xi^{(N)}_z$ and $\Xi^{(N),i}_z$ for $N \geq 1$ follow from the $x$-space bound of Lemma 4.5.4 and a decomposition of the bounding diagrams that we omit as a similar decompositions is already shown in detail for SAW and LT. The bounds on $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$ and similar differences for $\Xi^{(i),j}_z$ follows the same scheme as for LT. We decompose the weight $[1 - \cos(k \cdot x)]$ as shown in Figure 4.21 and 4.22 and then use the $x$-space bounds to prove the statement.

As the bound for the weighted pieces is a like the bound on $H^{a,b}$ for LT we also omit this here and focus on two aspects that are not present for LT. Since the bounds on $\hat{\Xi}^{(1)}_z(0) - \hat{\Xi}^{(1)}_z(k)$ and $\Sigma_1(\hat{\Xi}^{(1),i}_z(0) - \hat{\Xi}^{(1),i}_z(k)e^{-ik_i})$ are the major contributions to the perturbation of the expansion we consider several special cases for these diagrams. For $\hat{\Xi}^{(1)}_z(0) - \hat{\Xi}^{(1)}_z(k)$, we consider three cases:

- The left and right triangle are trivial. Then, we bound the intermediate weighted triangle by $(H^2(k))_{0,0}$.
- One of the triangles is trivial. In this case we apply Lemma 3.4.1 with $J = 2$ and use symmetry to obtain the bound $8\Sigma_2^2(\tilde{H}(k))_a(A^i)_a,0$.
- All triangles are nontrivial, so that we can bound them by $[\{\tilde{P}\}_b - \delta_{0,b}]$. We apply Lemma 3.4.1 with $J = 3$ and obtain the stated bound.

For $\Sigma_1(\hat{\Xi}^{(i),j}_z(0) - \hat{\Xi}^{(i),j}_z(k)e^{-ik_i})$ we distinguish whether the final triangle is trivial or not.

While it is straightforward to bound $H^{1,a,b}, H^{2,i,a,b}$ and $H^{3,i,a,b}$, it is a bit more demanding to bound the blocks $C^{1,i,k,a,b}, C^{2,i,k,a,b}$ and the entries of $\tilde{H}_i(k), \tilde{H}_i,\tilde{H}_i(k)$. For the blocks $C^{1,i,k,a,b}, C^{2,i,k,a,b}$ we need to compute a bound on

$$(C^1(k))_{a,b} = \sup_{v,y} \sum_{e=0,i,k} B^{1,i,a,c}_v(0,v,w,u) H^{2,k,c,b}_v(u,w,x+y,x)[1 - \cos(k \cdot w)], \quad (4.5.88)$$

$$(C^2(k))_{a,b} = \sup_{v,y} \sum_{e=0,i,k} B^{1,i,a,c}_v(0,u,w) A^{k,c,b}_v(u,x+y)[1 - \cos(k \cdot u)]. \quad (4.5.89)$$
The diagram $B^i$ is defined in (4.5.21) as sum of three parts. For $C^1(k)$ we bound each term independently and obtain the bound

$$(C^1(k))_{a,b} \leq \left( \left[ 2H^2(k)A + 2A'H^1(k) + H^2(k) \right] A^i \right)_{a,b} + \sum_{c=0}^{2} H^2(k)_{a,c} \sup_v \sum_{i,x,y} B^{2,i,c,b}(0,v,x,y). \quad (4.5.90)$$

Computing a bound on the entries of $C^2(k)$ is complicated for the case that the triangle $(z, t, u)$ is non-trivial. Therefore, we define $C^{3,i,a,b}(0, v, x, y; k)$ to be the diagram $C^{2,i,a,b}(0, v, x, y; k)$ in which we replace the weight $[1 - \cos(k \cdot u)]$ by $[1 - \cos(k \cdot (z - u))]$, see Figure 4.28. Thus, $C^{3,i,a,b}$ corresponds to $C^{2,i,a,b}$ with the exception that only the small triangle $(z, t, u)$ has a weight. We define

$$(C^3(k))_{a,b} = \sup_{v,y} \sum_{i,x} C^{3,i,k,a,b}(0,v,x,x+y;k) \quad (4.5.91)$$

We can bound $C^2$ by

$$(C^2(k))_{a,b} \leq \left( \left[ 2H^3(k)A + 2A'H^1(k) + H^3(k) \right] A^i + 2C^3(k) \right)_{a,b} + \sum_{c=0}^{2} H^3(k)_{a,c} \sup_v \sum_{i,x,y} B^{2,i,c,b}(0,v,x,y) \quad (4.5.92)$$

To bound $C^3(k)$ we bound $C^{3,i,k}$ using simple diagrams:

$$\sum_{i,k} C^{3,i,k}(0, v, x, y; k) \leq \sum_{t,u,w,z \in \mathbb{Z}^d} \mathcal{S}_{0,1,0,1}(z, t, w, v) \tau_z(z - u)[1 - \cos(k \cdot (z - u))] \times \tau_{1,p}(u - x) \tau_{1,p}(t - x) \tau_{0,p}(w - y) \quad (4.5.93)$$
We decomposing this diagram to create a bound is quite delicate, so we show each step:

\[
\sup_{v, y, t, w, x} C^3_{v, y, t, k, a, b}(0, v, x, y; k) \leq \sup_{r \in \mathbb{Z}^d} \tau_z(r)[1 - \cos(k \cdot r)]
\times \sup_{v, y, t, w, z, x \in \mathbb{Z}^d} \left[ \mathcal{J}_{0,1,0,1}(z, t, w, v) \tau_{1, p}(t - u) \tau_{1, p}(u - x) \tau_{0, p}(w - x - y) \right]
\]

We relabel \( w' = w - t \) to denote only the displacement of the line \( w \rightarrow t \):

\[
\sup_{v, y, t, w, x} C^3_{v, y, t, k, a, b}(0, v, x, y; k) \leq \sup_{r \in \mathbb{Z}^d} \tau_z(r)[1 - \cos(k \cdot r)] \sup_{v, y, t, w', z \in \mathbb{Z}^d} \left[ \mathcal{J}_{0,1,0,1}(z, t, t + w', v) \times \sum_{u, x \in \mathbb{Z}^d} \tau_{1, p}(t - u) \tau_{1, p}(u - x) \tau_{0, p}(t + w - x - y) \right]
\]

and then relabel \( x' = x - t, u' = u - t \) to shift the diagram in the second sum by \( t \) and obtain

\[
\leq \sup_{r \in \mathbb{Z}^d} \tau_z(r)[1 - \cos(k \cdot r)] \sup_{v, y, t, w', z \in \mathbb{Z}^d} \left[ \mathcal{J}_{0,1,0,1}(z, t, t + w', v) \times \sum_{u', x' \in \mathbb{Z}^d} \tau_{1, p}(u') \tau_{1, p}(u' + x') \tau_{0, p}(x' + w' - y) \right].
\]

For the term in the second line we take the supremum over \( u' \) and obtain the bound

\[
\leq \sup_{r \in \mathbb{Z}^d} \tau_z(r)[1 - \cos(k \cdot r)] \sup_{v, t, w, z \in \mathbb{Z}^d} \mathcal{J}_{0,1,0,1}(z, t, w, v) \times \sup_{y, z \in \mathbb{Z}^d} \sum_{x'} (\tau_{1, p}^{*2}(u') \tau_{0, p}(y - x' - w))
\]

with

\[
\sup_{y, z \in \mathbb{Z}^d} \sum_{x'} (\tau_{1, p}^{*2}(u') \tau_{0, p}(y - x' - w)) = \sup_{y \in \mathbb{Z}^d} (\tau_{1, p}^{*2} \ast \tau_{0, p})(y)
\]

This bound holds for all \( a, b \). For \( a, b = 0, 1 \) we can improve the bound as we know that the complete square \( \mathcal{J}_{0,1,0,1}(z, t, w, v) \) contains at least four steps.

At last we explain how we bound \( \tilde{h}^I(k) \) and \( \tilde{h}^{I, II}(k) \). We recall the definition of \( \tilde{h}^{I, II}(k) \) in (4.5.29) to see that:

\[
(\tilde{h}^{I, II}(k))_b \leq (\tilde{h}^I(k))_b + 2(\tilde{h}^I(k)A^I)_b + 2((\tilde{B}^I)^T A^I H^2(k))_b.
\]

(4.5.99)
Figure 4.29: The possible forms of a diagram in $\tilde{h}'(k)$. We take the supremum over $(x - y)$ and sum over $x$ and $\iota$.

The diagrams $\tilde{h}'(k)$ can be considered to be the sum of three diagrams, see Figure 4.29. We will show how to bound the three terms individually. We bound the first term we know that only $y \neq 0$ can contribute. This means that the connection $e_\iota \rightarrow 0 \rightarrow y$ consists of at least two steps. We bound this contribution by $2dp \sum_{\kappa} H^{3,k,1,b}(e_{\iota}, 0, y; x; k)$, where the factor $2dp$ counts the connection $e_\iota \rightarrow 0$ that is not contribute to $H^{3,k,1,b}$. The second and third diagrams are decomposed as shown in Figures 4.30 and 4.31.

We bound the diagrams shown in Figure 4.30 by

$$2d\tau_3,p(e_{\iota})(H^2(k))_{0,b} + 4d\tau_3,p(e_{\iota})((\vec{P} - (1,0,0))^T H^2(k))_b$$

$$+ 4d\tau_3,p(e_{\iota})(H^1(k)A^1(k))_{0,b} \quad (4.5.100)$$
The diagrams of Figure 4.31 are bounded by

$$\sum_w B_{2,1}(w, e_i)(H^2(k))_{1,b} + \sum_w B_{0,2}(w, e_i)(H^2(k))_{2,b}$$

$$+ 2 \sum_w (B_{2,1}(w, e_i) + B_{0,2}(w, e_i))(A^t(k))_{0,0} \max \{(H^2(k))_{1,b}, (H^2(k))_{2,b}\}$$

$$+ 2 \sum_w (B_{2,1}(w, e_i) + B_{0,2}(w, e_i))(H^1(k))_{0,0} \max \{(A^t(k))_{1,b}, (A^t(k))_{2,b}\}$$ (4.5.101)

Combining these bounds we obtain the following bound

$$\langle h^t(k) \rangle_b \leq 2d z(H^3(k))_{1,b} + (4.5.100) + (4.5.101)$$ (4.5.102)

where the two line numbers denote the terms given in the corresponding lines.

### 4.5.7 To the bounds assumed in Chapter 3

In this section we comment on the bounds assumed in Assumption 3.2.6 and Assumption 3.5.3. The diagrammatic bound on $\Xi^{(N)}_z \cdot \xi^{(N)}_z$ are given in Section 4.5.3. Computing the value of these diagrams, as described in Chapter 5, creates the bonds $\beta^{(N)}_\star$. The largest entry of $B$ and $\bar{B}$ is a bound on the open triangle with at least 2 steps, and is thereby of the order $1/d$. The bound on this open triangle in the dimension that we consider smaller than one. We conclude from this that $\beta^{(N)}_\star$ is of order $1/d^N$ and that

$$\sum_{N=0}^\infty \beta^{(N)}_\star \Xi^{(N)}_z \cdot \xi^{(N)}_z \sum_{N=0}^\infty \beta^{(N)}_\star \Delta \Xi^{(N)}_z \cdot \xi^{(N)}_z \sum_{N=0}^\infty \beta^{(N)}_\star \Delta \Xi^{(N)}_z \cdot \xi^{(N)}_z$$ (4.5.103)

are all finite in the dimension that we consider. As for LA we will not discuss the bounds of Assumption 3.5.3, as they do not simplify for percolation. With the recharacterization in Section 3.5.3 we remove from the leading order terms of $\hat{\Phi}_1$ and $\hat{F}_1$. Let us review a short example. We bounded $\Xi^{(0)}_z$ in Lemma 4.5.1 by

$$\sum_{x \in \mathbb{Z}^d} \Xi^{(0)}_z(x) \leq \sum_{x \in \mathbb{Z}^d} \delta_{0,x} + B_{1,3}(x, 0) + B_{2,2}(x, 0)$$ (4.5.104)

$$\sum_{x \in \mathbb{Z}^d} \Xi^{(0)}_z(x) \leq 2d \tau_{3,p}(e_i) [1 - \hat{D}(k)] + \sum_{x \in \mathbb{Z}^d} \tau_{2,p}(x)^2 [1 - \cos(k \cdot x)].$$ (4.5.105)

For the $\Phi_1(k)$ we exact the leading behavior into $c_\Phi$ and $\alpha_\Phi$ and bound from the terms in (4.5.104) and (4.5.104) the terms $B_{2,2}(x, 0)$ and $\sum_{x \in \mathbb{Z}^d} \tau_{2,p}(x)^2 [1 - \cos(k \cdot x)]$ in $R_\Phi$. As already discussed in 4.2.5 the bounds on $\hat{\Xi}^{(N)}_z(0) - \hat{\Xi}^{(N)}_z(k)$ also imply the bounds on $\sum_x \|x\|_2^2 \Xi^{(N)}_z(x)$ required for Assumption 3.5.3.
4.6 Discussion

In this chapter we have derived the diagrammatic bounds on the lace-expansion coefficients, which can be bounded numerically using the techniques of Chapter 5. We use these bounds in Chapter 3 to prove mean-field behavior, more precisely the infrared bound. We have given a complete proof for the SAW and have explained in detail how to adapt the proof for SAW to obtain the diagrammatic bound for LT. Although we not even give the full proof for the LT bounds in Section 4.3, this section is still 50% longer than the corresponding section for SAW (Section 4.2). The reason for this is that the lace-expansion coefficients for LT have a more complex structure. The coefficients for LA and percolation have an even richer structure, so that we do not give the full proof of the bounds. We only explain the general structure of the proof and features of LA and percolation that were not present for SAW and LT. This should enable the devoted reader to complete the proofs for LA and percolation.

The form of the bounds. To state the bounds on $\Xi^{(N)}_z$ and $\Xi^{(N)}_z,\iota$ we use the recursive structure of the diagrams to bound it by a combination of building blocks. In the classical lace expansion (LE) these building blocks are very simple diagrams. For example the building block of the LE for percolation would be a simple nontrivial triangle, i.e. $\sup_x (\tau^*_{x3} \star D)(x)$ in our notation. The coefficients of the NoBLE have some structural properties which we use for our bounds, most prominently the fact that all loops consist of at least four bonds. To use the structural properties we define complex building blocks that we combine to create bounding diagrams. When combining the blocks we add a condition on the length of the shared line to make full use of the structural properties. This creates matrix-valued bounds.

Possible improvements. The created bounds use the dominant avoidance properties of the coefficients to create good bounds. However, they can be further improved. We will now discuss ways to obtain better bounds on the NoBLE coefficients.

The most beneficial way to improve the numerical bounds is to avoid the use of open weighted diagrams, as most bounds using of open weighted diagrams creates bad bounds. We can avoid using them by splitting the diagrams in a different way. For this, we need to rely on bounds on closed weighted triangles, squares and pentagons. The problem thereby is that the bound on the weighted pentagon is only finite in dimensions $d > 12$, so that we can the bound value of $d_0$ (see Table 1.3), but we would have no chance to prove the result down to the upper critical dimension.
Another way to significantly improve the bounds is to extract more explicit contributions from the biggest coefficients $\Xi^{(0)}_z, \Xi^{(1)}_z, \Xi^{(0)}_{\iota z}$. In the accompanying Mathematica notebook we give a table of the derived bounds on the coefficients. We can improve these bounds either by a computer-based enumeration of the dominant contributions or by an analytic extraction of special cases for $x$. The values of the coefficients at $x \in \{0, e_i, e_\iota, e_i + e_\iota\}$ are of special interest. These values create the largest contributions to the coefficients and allow us to obtain better bounds on the coefficients in the following way:

$$
\sum_i (\hat{\Xi}^{(0)}_{\iota}(k) - \hat{\Xi}^{(0)}_{\iota}(k)) = 2d[1 - \hat{D}(k)](\Xi^{(0),1}_z(e_1) + (2d - 2)\Xi^{(0),1}_z(e_2))
+ 4d(2d - 2)[1 - \hat{D}(k)]\Xi^{(0),1}_z(e_1 + e_2))
+ \Xi^{(0),d}_z(2e_1) \sum_{\mu=1}^d [1 - \cos(2k\mu)]
+ \sum_i \sum_{x ||x||_1 > 2} \Xi^{(0),d}_z(x)[1 - \cos(k \cdot x)].
$$

(4.6.1)

Additionally, we can improve the bounds by considering more cases for the length of shared lines $a_i$. For instance, we can distinguish between the lengths $a_i = 0, \ldots, 5$ and $a_i \geq 6$ and consider special cases for the displacement of the shared line. We recall that the coefficients are defined as an alternating sequence

$$(\Xi_z(x) = \sum_{N=0}^{\infty} (-1)^N \Xi^{(N)}_z(x))$$

of non-negative functions. We bound $\Xi_z$ by using either the positive part or the negative part of the sum. In particular, we bound $\hat{\Xi}_z(k)$ by the sum of the positive and negative part. If we could use the alternating nature of the coefficient we would obtain much better bounds.

**Bounds with different weights.** In the lemmas and propositions in this chapter we only stated bounds of the type $\Xi^{(0)}_z(0) - \Xi^{(0)}_z(k)$. These are the bounds required for Assumption 3.2.6. For the analysis of Section 3.5 we require bounds on quantities like $\sum_x \|x\|_2^2 \Xi^{(N)}_z(x)$, see Assumption 3.5.3. To bound these sums we note that for sequences of vertices $(x_i)_i$ with $\sum_{i=1}^J x_i = x$ the following two relations hold:

$$
[1 - \cos(k \cdot x)] \leq \sum_{i=1}^J [1 - \cos(k \cdot x_i)],
$$

(4.6.2)

$$
\|x\|_2^2 \leq \sum_{i=1}^J \|x_i\|_2^2.
$$

(4.6.3)

We bounded differences of the coefficients by extracting either nearest-neighbor contributions or by using (4.6.2) to split the weight $[1 - \cos(k \cdot x)]$. We can do this also for the weight $\|x\|_2^2$, so that the proof for both weights are completely identical.
The different performances of the bounds for LT and LA. We should also explain why the analysis of Section 3.5 give similar result of LA and LT for $d_0$, while the results using the analysis of Section 3.3 are so different. The bound derived on the coefficients for LA in Section 4.4 are supposed to be only a factor $(d+1)/d$ bigger than the bounds on the corresponding LT coefficients, since the only difference of the bounds arise due to a possible triangle at the beginning of the diagram for the respective coefficient. Therefore, we can expect that the value of $d_0$, see Table 1.3, for LA should only be slightly worse than for LT. This is also what we see for the results obtained using the analysis of Section 3.5.

The reason why this is so different for the analysis of Section 3.3, is due to the bootstrap function $f_3$. Namely $f_3$ produces only a very poor bound on weighted open diagrams, see Section 3.8. While the bound on dominant contributions $\Xi^{(0)}_z, \Xi^{(1)}_z, \Xi^{(0,\d)}_z$ and $\Xi^{(1,\d)}_z$ do not involve weighted open diagrams for LT, the bounds for LA do. Therefore, the bounds on the perturbation for LA are quite bad. For the same reason the analysis of Section 3.3 is not as effective for percolation.
Numerical features

In the preceding chapter, we have bounded the coefficients of the non-backtracking lace expansion (NoBLE) by combinations of repulsive diagrams. In this chapter we bound these diagrams numerically.

We first explain the basic idea of the bounds. Then we derive the bounds on the two-point function and the repulsive diagrams that we use in our computations. These bounds will be stated in terms of SRW-integrals in Section 5.1.2. In Section 5.2 we explain how to compute the value of these SRW-integrals numerically. Moreover, we compute the SRW-integrals that we use in the improvement of \( \tilde{f}_3 \) in Section 3.6. Thereafter, we discuss some model-dependent aspects of the bounds and close this chapter with a general discussion.

5.1 Numerical bounds on the diagrams

5.1.1 Idea of the numerical bounds

We bound the two-point function \( G_z \) and the repulsive diagrams using the critical SRW two-point function \( C_{1/2d} \).

For \( z = z_I \) we bound \( G_{z_I}(x) \) pointwise by \( \frac{2d-2}{2d-1} C_{1/2d}(x) \), see Assumption 3.2.1.

For \( z \in (z_I, z_c) \) we assume in the bootstrap that \( f_2 \leq \Gamma_2 \), which states that \( \hat{G}_z(k) \) is bounded by \( \frac{2d-2}{2d-1} \Gamma_2 \hat{C}_{1/2d}(k) \).

Bound on unweighted diagrams. We first discuss how we bound the repulsive diagram for \( z \in z_I, z_c \) and then argue that we can use similar bounds for \( z = z_I \). In Section 3.5 we have already introduced the function \( I_{n,m}(x) \) given by

\[
I_{n,m}(x) = \frac{1}{(2d)^m} (D^* \star C_{1/(2d)}^n)(x) = \int_{[\pi, \pi]^d} \hat{D}^m(k) \frac{e^{ik \cdot x}}{(2\pi)^d} \frac{d^d k}{[1 - \hat{D}(k)]^n}, \tag{5.1.1}
\]
We show in Section 5.1.2 how to obtain a better bound. This holds only for $f$. For LT and LA similar relations holds, namely,

$$G_{n,z}(x) \leq (2d z(D \star G_{n-1,z})(x) \leq (2d z)^n (D^{*n} \star G_z)(x). \quad (5.1.2)$$

For LT and LA similar relations holds, namely,

$$G_{n,z}(x) \leq (2d \hat g_z(D \star G_{n-1,z})(x) \leq \alpha_z^n (D^{*n} \star G_{n-1,z})(x), \quad (5.1.3)$$

$$\hat G_{n,z}(x) \leq (2d \hat z(D \star \hat G_{n-1,z})(x) = 2d z g_z(D \star G_z)(x) \leq (2d \hat a_z)^n (D^{*n} \star G_z)(x). \quad (5.1.4)$$

For SAW and percolation we have chosen $\hat a_z = z$, so that (5.1.4) holds for all models. We can bound the repulsive diagram by combinations of two-point functions, e.g.

$$B_{n,m}(x) \leq (2d \hat a_z)^m (D^{*m} \star G_{n,z})(x) \leq (2d \hat a)^{m+n} (D^{*(m+n)} \star G_z)(x), \quad (5.1.5)$$

$$T_{n_1,n_2,n_3}(x) \leq (2d \hat a_z)^{n_1+n_2+n_3} (D^{*(n_1+n_2+n_3)} \star G_z^3)(x), \quad (5.1.6)$$

$$S_{n_1,n_2,m,n_3}(x) \leq (2d \hat a_z)^{n_1+n_2+n_3+m} (D^{*(n_1+n_2+n_3+m)} \star G_z^3)(x). \quad (5.1.7)$$

For the bootstrap we assume that $f_1(z) \leq \Gamma_1$ and $f_2(z) \leq \Gamma_2$, so that

$$\hat a_z \leq \frac{\Gamma_1}{2d-1}, \quad (5.1.8)$$

$$|\hat G_z(k)| \leq \frac{2d-2}{2d-1} \Gamma_2 \hat C_{1/2d}(k). \quad (5.1.9)$$

If these assumptions hold, we can bound $G_{n,z}(x)$ and the square in (5.1.7) as follows:

Let $m, n, n_1, n_2, n_3 \in \mathbb{N}$ be such that $n$ and $n_1 + n_2 + n_3 + m$ are even. Then

$$G_{n,z}(x) \leq (2d \hat a_z)^n (D^{*n} \star G_z)(x) = \int_{(-\pi,\pi)} \hat D^n(k) \hat G_z(k) e^{ik.x} \frac{d^d k}{(2\pi)^d}$$

$$= \left( \frac{2d}{2d-1} \Gamma_1 \right)^n \left( \frac{2d-2}{2d-1} \Gamma_2 \right) I_{1,n}(0), \quad (5.1.10)$$

$$S_{n_1,n_2,m,n_3}(x) \leq (2d \hat a_z)^{n_1+n_2+n_3+m} \int_{(-\pi,\pi)} \hat D^{n_1+n_2+n_3+m}(k) \hat G_z^3(k) e^{ik.x} \frac{d^d k}{(2\pi)^d}$$

$$\leq \left( \frac{2d}{2d-1} \Gamma_1 \right)^{n_1+n_2+n_3+m} \left( \frac{2d-2}{2d-1} \Gamma_2 \right)^3 I_{3,n_1+n_2+n_3+m}(0). \quad (5.1.11)$$

This holds only for $n$ and $n_1 + n_2 + n_3 + m$ even as we take the absolute value within the integral in order to use $f_2(z) \leq \Gamma_2$. This is not a good bound, especially for $x \neq 0$. We show in Section 5.1.2 how to obtain a better bound.
For \( z = z_I \) we bound the repulsive diagrams using the \( x \)-space bound stated in Assumption 3.2.1. For example, we bound the square in (5.1.6) as follows
\[
\mathcal{J}_{n_1,n_2,m,n_3}(x) \leq (2d \bar{d}_{z_I})^{n_1+n_2+n_3+m} (D^* \Gamma_{n_1+n_2+n_3+m} \ast \mathcal{B}_{1/2d-1}^3)(x)
\leq \left( \frac{2d}{2d-1} f_1(z_I) \right)^{n_1+n_2+n_3+m} \left( \frac{2d-2}{2d-1} \right)^3 I_{3,n_1+n_2+n_3+m}(x)
\leq \left( \frac{2d}{2d-1} f_1(z_I) \right)^{n_1+n_2+n_3+m} \left( \frac{2d-2}{2d-1} \right)^3 I_{3,n_1+n_2+n_3+m}(0),
\]
where we use that \( \sup_x I_{n,m}(x) = I_{n,m}(0) \) for all \( m \) even in the last step. The relation \( \sup_x I_{n,m}(x) = I_{n,m}(0) \) follows from the monotonicity of \( I_{n,m} \) explained after line 3.6.17 and proven in [42, Lemma B.3]. As we will see, the bound for \( z = z_I \) and \( z \in (z_I, z_c) \) has a very similar form. We define the abbreviations
\[
\Gamma_{1,o} = \frac{1}{2d-1} \Gamma_1, \quad \Gamma_{2,o} = \frac{2d-2}{2d-1} \Gamma_2, \quad \Gamma_{1,l} = \frac{1}{2d-1} f_1(z_I), \quad \Gamma_{2,l} = \frac{2d-2}{2d-1},
\]
and see that the only difference in the bounds for \( z = z_I \) and \( z \in (z_I, z_c) \) is whether we use \( \Gamma_{1,l}, \Gamma_{2,l} \) or \( \Gamma_{1,o}, \Gamma_{2,o} \).

**Bound on weighted diagrams.** We have bounded the two-point functions and the repulsive diagrams using only \( f_1 \) and \( f_2 \). The bootstrap functions \( f_3 \) and \( \bar{f}_3 \) are used to bound weighted diagrams. The function \( \bar{f}_3 \), used in the analysis of Section 3.5 directly gives us bounds on the weighted diagrams. Therefore, we do not need to discuss these bounds here. We define \( G_z(x; k) = G_z(x)[1 - \cos(k \cdot x)] \) and see that the Fourier transform of \( G_z(x; k) \) is given by:
\[
\hat{G}_z(l; k) = \sum_x G_z(x; k) e^{il \cdot x}
= \sum_x G_z(x) e^{il_{x}x} - \frac{1}{2} G_z(x) e^{i(l+k) \cdot x} - \frac{1}{2} G_z(x) e^{i(l-k) \cdot x}
= \hat{G}_z(k) - \frac{1}{2} \hat{G}_z(k+l) - \frac{1}{2} \hat{G}_z(l-k) = - \frac{1}{2} \Delta_k \hat{G}_z(l).
\]

The function \( f_3 \) defined in [3.2.23] gives us a bound on \( \Delta_k \hat{G}_z(l) \) in term of combinations of SRW two-point functions. The assumption \( f_3(z) \leq \Gamma_3 \) allows us to bound the weighted line as follows:
\[
\sup_{x \in \mathbb{Z}^d} G_{k,z}(x) = \sup_{x \in \mathbb{Z}^d} \int_{(-\pi,\pi)} \left( -\frac{1}{2} \Delta_k \hat{G}_z(l) \right) e^{i k \cdot x} \frac{d^d k}{(2\pi)^d} \leq \Gamma_3 \int_{(-\pi,\pi)} \hat{U}(k, l) \frac{d^d k}{(2\pi)^d}
= \Gamma_3 [1 - \hat{D}(k)] \int_{(-\pi,\pi)} (c_1 + c_2 \hat{C}(l)) [\hat{C}(l-k) + \hat{C}(l+k)] \frac{d^d k}{(2\pi)^d}
+ c_4 \Gamma_3 [1 - \hat{D}(k)] \int_{(-\pi,\pi)} \hat{C}(l-k) \hat{C}(l+k) \frac{d^d k}{(2\pi)^d}.
\]
Then, we use Cauchy-Schwarz as follows
\[
\int_{(-\pi,\pi)} \hat{C}(l) \hat{C}(l-k) \frac{d^dk}{(2\pi)^d} \leq \left( \int_{(-\pi,\pi)} \hat{C}^2(l) \frac{d^dk}{(2\pi)^d} \right)^{1/2} \left( \int_{(-\pi,\pi)} \hat{C}^2(l-k) \frac{d^dk}{(2\pi)^d} \right)^{1/2} = \int_{(-\pi,\pi)} \hat{C}^2(l) \frac{d^dk}{(2\pi)^d} = I_{2,0}(0), 
\]
and obtain
\[
\sup_{x \in \mathbb{Z}^d} G_z(x;k) \leq [1 - \tilde{D}(k)] \bar{G}_3 \{ 2c_1 I_{1,0}(0) + 2c_2 I_{2,0}(0) + c_4 I_{2,0}(0) \}. \tag{5.1.18}
\]

We bound the weighted bubbles and triangles in the same way, e.g.
\[
\sup_{x \in \mathbb{Z}^d} (G_z(\cdot;k) \ast G_{1,z}^2)(x) 
\leq \sup_{x \in \mathbb{Z}^d} (2d \tilde{\alpha}_z)^2 \int_{(-\pi,\pi)} (-\frac{1}{2} \Delta_k \hat{G}_z(l)) \hat{D}^2(l) \hat{G}_z^2(l) e^{il \cdot x} \frac{d^dl}{(2\pi)^d} 
\leq (2d \Gamma_{1,0})^2 \Gamma_{2,0}^2 \Gamma_3 \int_{(-\pi,\pi)} \hat{U}(k,l) \hat{D}^2(l) \hat{C}^2(l) \frac{d^dl}{(2\pi)^d} \tag{5.1.19} 
\leq [1 - \tilde{D}(k)] (2d \Gamma_{1,0})^2 \Gamma_{2,0}^2 \Gamma_3 \{ 2c_1 I_{3,2}(0) + 2c_2 I_{4,2}(0) + c_4 I_{4,2}(0) \}.
\]

For \( z = z_f \) we can bound the weighted diagrams in two ways. Either we use the bound on the weighted diagrams \( \text{(3.5.7)} \) as explained in Section \( \text{3.5} \) and computed in Section \( \text{3.6.2} \). Or we first use the pointwise bound on \( G_z \) and bound the \( -\frac{1}{2} \Delta_k \hat{C}(l) \) using Lemma \( \text{3.3.11} \) by
\[
\frac{1}{2} |\Delta_k \hat{C}(l)| \leq [1 - \tilde{D}(k)] \frac{1}{2} (\hat{C}(l-k) + \hat{C}(l-k)) \hat{C}(l) 
+ [1 - \tilde{D}(k)] 4 \hat{C}(l-k) \hat{C}(l-k) \tag{5.1.20} 
\]
and then proceed as shown above.

### 5.1.2 Bounds used

In this section we derive the bounds on the repulsive diagrams we use for our computations. These bounds are obtained by extracting the contributions of short connections. For this section we fix \( n, m_1, m_2, m_3, m_4 \in \mathbb{N} \) with \( n > 0 \) and \( m_1 + m_2 + m_3 + m_4 \leq n \) and \( x \in \mathbb{Z}^d \setminus \{0\} \).

**Bound on the value of the two-point function.** Every contribution to \( G_{n,z}(x) \) is due to a path from 0 to \( x \), so that
\[
G_{m_1,z}(x) \leq \sum_{s=m_1}^{n-1} c_s(x) \tilde{\alpha}_z^s + G_{n,z}(x) \leq \sum_{s=m_1}^{n-1} c_s(x) \tilde{\alpha}_z^s + (2d \tilde{\alpha}_z)^n (D^*n \ast G_z)(x), \tag{5.1.21}
\]
where \( c_s(x) \) is the number of \( s \)-step SAWs starting in \( 0 \) and ending at \( x \). This bound is very efficient, but requires that \( c_s(x) \) is known for all \( s \leq n \). In Section 5.1.3 we compute \( c_s(e_1) \) and \( c_s(e_1 + e_2) \) for \( s \leq 7 \). As we will see these values are sufficient.

**Bound on \( G_{n,z}(e_1) \).** For all models, \( G_{3,z}(e_1) \) is used to bound the leading behavior of \( \Xi^{0,\ell} \). Therefore, a good bound on this quantity is crucial. We use (5.1.21) with \( n = 9 \) and use symmetry to obtain:

\[
G_{9,z}(e_1) = (D * G_{9,z})(0) \leq (2d\bar{\alpha}_z)^9 (D^{*10} \ast G_z)(0)
\]

\[
\leq (2d\Gamma_{1,0})^9 \Gamma_{2,0} I_{1,10}(0),
\]

(5.1.22)

for \( z \in (z_I, z_c) \). For \( z = z_I \) we obtain the same bound where \( \Gamma_{1,0} \) and \( \Gamma_{2,0} \) are replaced by \( \Gamma_{1,1} \) and \( \Gamma_{2,1} \). For our implementation we use the bound iteratively:

\[
G_{7,z}(e_1) \leq c_7(e_1)\bar{\alpha}_z^7 + G_{9,z}(e_1), \quad G_{5,z}(e_1) \leq c_5(e_1)\bar{\alpha}_z^5 + G_{7,z}(e_1), \quad G_{3,z}(e_1) \leq c_3(e_1)\bar{\alpha}_z^3 + G_{5,z}(e_1), \quad G_{1,z}(e_1) \leq c_1(e_1)\bar{\alpha}_z + G_{3,z}(e_1).
\]

(5.1.23)

(5.1.24)

**Bound on \( G_{n,z}(e_1 + e_2) \).** The value of \( G_{n,z}(e_1 + e_2) \) occurs in several bounds and can be bounded using symmetry:

\[
G_{n,z}(e_1 + e_2) = \frac{1}{2d(2d - 2)} \sum_{i} \sum_{x:|x| \neq |i|} G_{n,z}(e_i + e_k) \leq \frac{d}{d - 1} (D^{*2} \ast G_{n,z})(0)
\]

\[
\leq \frac{d}{d - 1} (2d\bar{\alpha}_z)^n \Gamma_{2,0} I_{1,n+2}(0).
\]

(5.1.25)

We will use the bound

\[
G_{2,z}(e_1 + e_2) \leq \sum_{i=1}^3 c_{2i}(e_1 + e_2)\bar{\alpha}_z^{2i} + \frac{d}{d - 1} (2d\bar{\alpha}_z)^8 \Gamma_{2,0} I_{1,8+2}(0).
\]

(5.1.26)

**Bound on \( \sup_{x \in \mathbb{Z}^d} G_{n,z}(x) \).** We use the simple bound (5.1.2) on \( G_{n,z}(x) \) only for \( n = 8 \):

\[
\sup_{x \in \mathbb{Z}^d} G_{8,z}(x) \leq (2d\bar{\alpha}_z)^8 \Gamma_{2,0} I_{1,8}(0).
\]

(5.1.27)

We bound the supremum of \( G_{n,z}(x) \) for \( n \leq 8 \) using (5.1.21) as follows:

\[
\sup_{x \in \mathbb{Z}^d} G_{6,z}(x) \leq \max_{x} \{ \max_{x} c_6(x)\bar{\alpha}_z^6, \max_{x} c_7(x)\bar{\alpha}_z^7 \} + \sup_{x \in \mathbb{Z}^d} G_{8,z}(x),
\]

(5.1.28)

\[
\sup_{x \in \mathbb{Z}^d} G_{4,z}(x) \leq \max_{x} \{ \max_{x} c_4(x)\bar{\alpha}_z^4, \max_{x} c_5(x)\bar{\alpha}_z^5 \} + \sup_{x \in \mathbb{Z}^d} G_{6,z}(x),
\]

(5.1.29)

\[
\sup_{x \in \mathbb{Z}^d} G_{2,z}(x) \leq \max_{x} \{ \max_{x} c_2(x)\bar{\alpha}_z^2, \max_{x} c_3(x)\bar{\alpha}_z^3 \} + \sup_{x \in \mathbb{Z}^d} G_{4,z}(x),
\]

(5.1.30)

\[
\sup_{x \in \mathbb{Z}^d} G_{1,z}(x) \leq \max_{x} \{ \sup_{x \in \mathbb{Z}^d} G_{2,z}(x), G_{1,z}(e_1) \}.
\]

(5.1.31)
While we cannot compute $\sup_x c_n(x)$ for all $n$ it is next-to-trivial to check that

$$
\begin{align*}
\max_x c_2(x) &= c_2(e_1 + e_2), & \max_x c_3(x) &= c_3(e_1), \\
\max_x c_4(x) &= c_4(e_1 + e_2), & \max_x c_5(x) &= c_5(e_1), \\
\max_x c_6(x) &= c_6(e_1 + e_2), & \max_x c_7(x) &= c_7(e_1).
\end{align*}
$$

(5.1.32)

**Bound on repulsive diagrams.** We use the idea of extracting short walks explicitly to bound repulsive diagrams. For SAW and LT the connection paths of the repulsive diagrams are constrained to not intersect, so that the combined walk is also a SAW. Therefore, we know that, for $x \neq 0$,

$$
\sum_{y \in \mathbb{Z}^d} B_{m_1,m_2}(x) \leq \sum_{i=m_1+m_2}^{n-1} c_i(x) \bar{\alpha}_z^i + (2d \bar{\alpha}_z)^n (D^{*n} \star G_z)(x),
$$

(5.1.33)

where we recall that the subscript $m_1$ denotes that the first connection consists of exactly $m_1$ steps and the subscript $m_2$ denoted that the connection consist of at least $m_2$ steps. This bound does not hold for LA and percolation, as the connecting paths are only bond-disjoint and not vertex-disjoint/self-avoiding. However, if we replace $c_i(x)$ in (5.1.33) with the number of walks that do not use the same bonds twice, then such a relation also holds for LA and percolation. In the following we only state the bound for SAW and LT, knowing that we can obtain bounds for LA and percolation in the same way.

The bound on $x = 0$ requires an additional argument as $c_n(0) = 0$ for all $n \geq 1$. We explain this after discussing the bounds on the bubble, triangle and square. We continue the idea of extracting short walks and obtain for the bubble:

$$
B_{m_1,m_2}(x) \leq \sum_{i=m_1}^{n-m_2-1} B_{i,m_2}(x) + B_{n-m_2,m_2}(x)
$$

(5.1.34)

$$
\begin{align*}
&\leq \sum_{s_1=m_1}^{n-m_2-1} \left( \sum_{s_2=m_2}^{n-1-s_1} c_{s_1+s_2}(x) \tilde{\alpha}_z^{s_1+s_2} \right) + (2d \bar{\alpha}_z)^n (D^{*n} \star G_z)(x) \\
&\quad + (2d \bar{\alpha}_z)^n (D^{*n} \star G_z^{*2})(x) \\
&= \sum_{i=m_1+m_2}^{n-1} (i - m_1 - m_2) c_i(x) \bar{\alpha}_z^i \\
&\quad + (n - m_1 - m_2)(2d \bar{\alpha}_z)^n (D^{*n} \star G_z)(x) + (2d \bar{\alpha}_z)^n (D^{*n} \star G_z^{*2})(x).
\end{align*}
$$

We iterate this idea to obtain bounds for triangles and squares. For short walks we count all possible combinations of path-lengths $s_1, s_2, s_3, s_4$ such that $\sum s_i \leq n - 1$. We define for $a, b \in \{1, 2, 3, 4\}$ with $a < b$ the abbreviation $m_{a,b} = \sum_{s=a}^{b} m_s$ and
compute for the triangle:

\[
\mathcal{T}_{m_1, m_2, m_3}(x) \leq \sum_{s=m_1}^{n-m_2-m_3-1} \mathcal{T}_{s, m_2, m_3}(x) + \mathcal{T}_{n-m_2-3, m_2, m_3}(x) \tag{5.1.35}
\]

\[
\leq \sum_{i=m_1,3}^{n-1} c_i(x) \bar{\alpha}_i^i \sum_{s_1=m_1}^{i-m_2,3-s_1} \sum_{s_2=m_2}^{i-m_3-s_1} \sum_{s_3=m_3}^{1} (n-m_2,3-s)(2d\bar{\alpha}_z)^n(D^*n \star G_z)(x)
\]

\[
+ (n-m_1,3)(2d\bar{\alpha}_z)^n(D^*n \star G_z^2)(x) + (2d\bar{\alpha}_z)^n(D^*n \star G_z^3)(x)
\]

\[
= \sum_{i=m_1,3}^{n-1} \frac{(i+1-m_1,3)(i+2-m_1,3)}{2} c_i(x) \bar{\alpha}_i^i
\]

\[
+ \frac{(n-m_1,3)(n-1-m_1,3)}{2} (2d\bar{\alpha}_z)^n(D^*n \star G_z)(x)
\]

\[
+ (n-m_1,3)(2d\bar{\alpha}_z)^n(D^*n \star G_z^2)(x) + (2d\bar{\alpha}_z)^n(D^*n \star G_z^3)(x).
\]

In the same way we obtain the bound on the square:

\[
\mathcal{J}_{m_1, m_2, m_3, m_4}(x) \leq \sum_{i=m_{1,4}}^{n-1} c_i(x) \bar{\alpha}_i^i \sum_{s_1=m_1}^{i-m_{2,4}} \sum_{s_2=m_2}^{i-m_{3,4}} \sum_{s_3=m_3}^{i-m_4} \sum_{s_4=m_4}^{1} (n-m_{2,4}, n-m_{3,4}-s_1, n-m_4-s_1-s_2, n-m_3-s_1-s_2)
\]

\[
+ (2d\bar{\alpha}_z)^n(D^*n \star G_z)(x) \sum_{s_1=m_1}^{n-m_{2,4}} \sum_{s_2=m_2}^{n-m_{3,4}-s_1} \sum_{s_3=m_3}^{n-m_4-s_1-s_2} \sum_{s_4=m_4}^{1} (n-m_{2,4})(n-1-m_{1,4}) (2d\bar{\alpha}_z)^n(D^*n \star G_z^2)(x) \tag{5.1.36}
\]

\[
+ (n-m_{1,4})(2d\bar{\alpha}_z)^n(D^*n \star G_z^3)(x) + (2d\bar{\alpha}_z)^n(D^*n \star G_z^4)(x).
\]

Reviewing the form of these bounds we see that we can bound a diagram in which a line has a fixed length, like a diagram with one corner point less. For example, we bound \(\mathcal{J}_{m_1, m_2, m_3, m_4}(x)\) in the same way as \(\mathcal{J}_{m_1, m_2 + m_3, m_4}(x)\) and we bound \(\mathcal{J}_{m_1, m_2, m_3, m_4}(x)\) in the same way as \(\mathcal{B}_{m_1, m_2, m_3, m_4}(x)\).

**Bound on closed diagrams.** The bounds in \([5.1.21], [5.1.34]-[5.1.36]\) are only valid for \(x \neq 0\). However, \([5.1.34]-[5.1.36]\) remain to hold for \(x = 0\), when \(c_s(x)\) is replaced with the number of \(s\)-step self-avoiding loops. The number of \(s\)-step self-avoiding loops equals \(2dc_{s-1}(e_1)\), as every \(s\)-loop includes a \((s-1)\)-step SAW.
The parameters $c_i$ to avoid confusion we remark here on an notation collision between the two meaning are used in the same context. As it should be clear which meaning

$$D_f$$ from the assumed bound for $m$ even. We bound $G_{m,z}(x) \leq (2d\tilde{a}_z)^m (D^*m \star G_z)(x)$ and rewrite the result in Fourier space as

$$\sum_x G_{m,z}(x)G_z(x; k) \leq (2d\tilde{a}_z)^m \int_{[-\pi,\pi]^d} \hat{D}(l)^m \hat{G}_z(l)(-\frac{1}{2} \Delta_k \hat{G}_z(l)) \frac{d^d l}{(2\pi)^d}. \quad (5.1.40)$$

From the assumed bound $f_3 \leq \Gamma_3$ follows that $-\frac{1}{2} \Delta_k \hat{G}_z(l_1 - l_2) \leq \hat{W}(k, l)$. As $\hat{G}_z$ and $\hat{D}^m$ for even $m$ are non-negative we know that

$$\sum_x G_{m,z}(x)G_{k,z}(x) \leq [1 - \hat{D}(k)](2d\tilde{a}_z)^m \Gamma_2 \int_{[-\pi,\pi]^d} \hat{D}(l)^m \hat{C}(l) \times [(c_1 + c_2 \hat{C}(l)) \hat{C}(l - k) + \hat{C}(k + l)] + c_3 \hat{C}(l - k) \hat{C}(l + k) \frac{d^d l}{(2\pi)^d}. \quad (5.1.41)$$

To avoid confusion we remark here on an notation collision between $c_i$ and $c_i(x)$. The parameters $c_1, \ldots, c_4 \geq 0$ are used in Chapter 3 to define $f_3$, while $c_n(x)$ to number of $n$-step SAW ending at $x$. This section is the only part of the thesis where the two meaning are used in the same context. As it should be clear which meaning
is used in a term and we consider $c_1$ to an otherwise appropriate notation for both meaning we did not resolve this notational collision. Then, we use Cauchy-Schwarz as in (5.1.17) and obtain
\[
\sum_x G_{m,z}(x) G_{k,z}(x) \leq [1 - \hat{D}(k)] (2d\tilde{a}_z)^m \Gamma_2 \left[ 2c_1 I_{2,m}(0) + 2c_2 I_{3,m}(0) + c_3 I_{3,m}(0) \right].
\]
(5.1.42)

For the weighted triangle we obtain in the same way
\[
\sum_x (G_{m_1,z} \ast G_{m_2,z})(x) G_z(x; k) 
\leq (2d\tilde{a}_z)^{m_1 + m_2} \Gamma_2^2 \left[ 2c_1 I_{2,m_1 + m_2}(0) + 2c_2 I_{3,m_1 + m_2}(0) + c_3 I_{4,m_1 + m_2}(0) \right],
\]
(5.1.43)

for $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2$ even. For $m = m_1 = m_2 = 0$ this bound is not very good. The bound (5.1.42) for $m = 0$ is close to $(2c_1 + 2c_2 + c_3) \approx 1$. We improve the bound by extracting the contributions for small $m$ and use (5.1.42) only for $m \geq 4$. We first show how we extract the first two contributions of the closed weighted bubble
\[
\sum_x G_{0,z}(x) G_z(x; k) = G_z(0; k) + (c_1 \ast G_z(;k))(0) + \sum_x G_{2,z}(x) G_z(x; k),
\]
(5.1.44)

We know that $G_z(0; k) = 0$ and see that by symmetry
\[
(c_1 \ast G_z(;k))(0) = 2d z[1 - \hat{D}(k)] G_{3,z}(e_1),
\]
(5.1.45)

which is of the order $O(1/d^2)$. For $m = 2$ the term in (5.1.42) is of the order $O(1/d)$, so that (5.1.44) improves our bound by a factor $1/d$. To improve the bound further we extract two more contributions:
\[
\sum_x G_{2,z}(x) G_z(x; k) = (c_2 \ast G_z(;k))(0) + (c_3 \ast G_z(;k))(0) + \sum_x G_{4,z}(x) G_z(x; k).
\]
(5.1.46)

We bound the last term using (5.1.42). For the first term we compute
\[
(c_2 \ast G_z(;k))(0) = \sum_{i,k:|i|\neq k} c_2(e_i + e_k) G_{2,z}(e_i + e_k) [1 - \cos(k_i + k_k)]
+ \sum_i c_2(2e_i) G_{4,z}(2e_i) [1 - \cos(2k_i)].
\]
(5.1.47)

Then we split the cosines using Lemma 3.4.1. The sinus terms in the first sum cancel and we obtain a factor $2$ in the second sum.

We note that $c_2(e_i + e_k) = 2$ and $c_2(2e_i) = 1$ and conclude:
\[
(c_2 \ast G_z(;k))(0) \leq [1 - \hat{D}(k)] (8d(2d - 2)G_{2,z}(e_i + e_k) + 8dG_{4,z}(2e_i))
\]
(5.1.48)

For $c_3$ we use the same idea to compute
\[
(c_3 \ast G_z(;k))(0) \leq 2d [1 - \hat{D}(k)] \left[ G_{3,z}(e_1) + 3(2d - 2)(2d - 4)G_{3,z}(e_1 + e_2 + e_3) + 2d [1 - \hat{D}(k)] (5(2d - 2)G_{3,z}(2e_1 + e_2) + 3G_{3,z}(3e_1)) \right).
\]
(5.1.49)
5.1.3 Explicit computation of self-avoiding loops

In the bounds stated above we use the values of $c_n(e_1)$ for $n = 1, 3, 5, 7$ and of $c_n(e_1 + e_2)$ for $n = 2, 4, 6$. We compute these values by simple combinatorial means. We will first state the values and then explain at one example how we find these values:

$$c_1(e_1) = 1,$$  \hspace{1cm} (5.1.50)
$$c_2(e_1 + e_2) = 2,$$  \hspace{1cm} (5.1.51)
$$c_3(e_1) = 2d - 2,$$  \hspace{1cm} (5.1.52)
$$c_4(e_1 + e_2) = 4 + 6(2d - 4),$$  \hspace{1cm} (5.1.53)
$$c_5(e_1) = (2d - 2)(3 + 4(2d - 4)),$$  \hspace{1cm} (5.1.54)
$$c_6(e_1 + e_2) = 16 + 84(2d - 4) + 36(2d - 4)(2d - 6),$$  \hspace{1cm} (5.1.55)
$$c_7(e_1) = (2d - 2)(14 + 62(2d - 4) + 27(2d - 4)(2d - 6)).$$  \hspace{1cm} (5.1.56)

The values of $c_1(e_1)$ and $c_2(e_1 + e_2)$ are obvious. For $c_3(e_1)$ we note that a 3-step SAW $\omega$ with $\omega_3 = e_i$ must be of the form $(0, e_\kappa, e_\kappa + e_i, e_i)$ for some $\kappa$. As $\omega$ is self-avoiding we know that $|i| \neq |\kappa|$, so that there are $2d - 2$ choices for $\kappa$. We conclude that $c_3(e_i) = 2d - 2$. The other values are obtained by considering all possible patterns of the walk. In Figure 5.1 we give the example of $c_5(e_1)$ and omit the discussion of $c_4(e_1 + e_2), c_6(e_1 + e_2)$ and $c_7(e_1)$ as these follow the same idea.

![Figure 5.1: Patterns of SAW contributing to $c_5(e_1)$](image)
5.2 Simple random walk integrals

In this section we bound the SRW-integrals $I_{n,l}, K_{n,l}, T_{n,l}, U_{n,l}$ defined in (3.6.27)-(3.6.30). We first compute $I_{n,m}(x)$ and then show that the other integral can be bounded by. We compute $I_{n,m}(x)$ using

$$I_{n,m}(x) = I_{n,m-1}(x) - I_{n-1,m-1}(x),$$  \hspace{1cm} (5.2.1)$$

which is obtained by writing the factor $\hat{D}(k)$ as $1 - [1 - \hat{D}(k)]$ in (5.1.1). Using this relation the problem of computing $I_{n,m}$ for general $n, m \in \mathbb{N}$ simplifies to the computation of $I_{n,0}$ and $I_{0,m}$.

5.2.1 Computation of $I_{n,0}$

We compute $I_{n,0}$ as already done by Hara and Slade in [41, Appendix B]. Let $b(n, s)$ be the modified Bessel function of the first kind and $F(t, d, n)$ the modified Bessel function:

$$b(n, s) = \sum_{k=0}^{\infty} (-1)^k \left[ \frac{s}{2} \right]^{2k+n} \frac{1}{k!(n+k+1)!},$$  \hspace{1cm} (5.2.2)$$

$$F(t, d, n) = e^{-t/d} b(n, t/d),$$  \hspace{1cm} (5.2.3)$$

see e.g. [27] (8.401) and (8.406) or [1, Section 9.6]. Using

$$\frac{1}{[1 - \hat{D}(k)]^n} = \frac{1}{(n-1)!} \int_0^{\infty} t^{n-1} e^{-t[1 - \hat{D}(k)]} \frac{d^d k}{(2\pi)^d},$$  \hspace{1cm} (5.2.4)$$

we compute

$$I_{n,0}(x) = \frac{1}{(n-1)!} \int_0^{\infty} t^{n-1} \prod_{\mu=1}^{d} F(t, d, |x_{\mu}|) dt,$$  \hspace{1cm} (5.2.5)$$

see [41, Appendix B]. Most mathematical software packages, such as Mathematica, Mathlab, and R, come with a method to compute the modified Bessel Integral. We used Mathematica which allows us to compute $I_{n,0}(x)$ for dimensions bigger than $2n+2$, up to a precision of $o(10^{-40})$. This precision is completely satisfying for our purposes and we will not try to compute the value of $I_{n,0}$ in $d = 2n+1$, as we are not able to prove mean-field behavior for just two dimensions above the upper critical dimension anyway. In Mathematica the code, is as follows

\begin{verbatim}
F[ t_, d_, n_ ] := E^(- (t /d )) BesselI [ n, t/d ]
Cz[n_,d_,T_] := 1 /((n - 1)!) NIntegrate [ t^(n - 1) (F[ t, d, 0])^d , { t, 0, T}, WorkingPrecision -> 30]
\end{verbatim}

Thus, the command $Cz[n,d,\infty]$ computes the values of $C_{1/2d}^n(0)$ in dimension $d$ with a precision of at least 30 digits.
5.2.2 Computation of $I_{0,m}$

The computation of $I_{0,m}(x)$ is a purely combinatorial problem as $(2d)^m I_{0,m}(0) = p_m(x)$, where $p_m(x)$ is the number of $m$-step SRWs with $\omega_0 = 0, \omega_m = x$. For the analysis of Section 3.3 we only need to compute the value of $p_n(0)$ for $n \leq 8$. It is trivial that

\[ p_0(0) = 1, \quad p_2(0) = 2d, \]

and that $p_m(0) = 0$ for odd $m$, as a nearest-neighbor walk can reach the origin only in an even number of steps. We use simple combinatorics to compute the number of $m$-step SRW loops in $d \geq 4$ for $m = 4, 6, 8$. For example, we explain the computation of $p_6(0)$. The walker uses at most three different dimensions as it needs to undo all his steps. We distinguish between the number of dimensions used by the walker:

- If the walk has only used one dimension it has stepped three times to the positive direction (right) and three times to the negative direction (left). As any combination of left and right steps is allowed there are $6!/(3!3!)$ different possibilities for that. As there are $d$ choices for the dimension there are $d \frac{6!}{3!3!}$ such walks.

- If the walk uses two dimensions and has 6 steps, it makes 4 steps in one dimensions and 2 in the other. As any combination of moves is allowed there are $6!/(2!2!1!1!)$ different possibilities for that. Further, there are $d$ choices for the dimension in which to take 4 steps and likewise $d - 1$ choices for the dimension where 2 steps are made. Thus, there are $d(d - 1) \frac{6!}{2!2!}$ SRW 6-step loops using steps in exactly two dimensions.

- If the walk uses three dimensions then there are 2 steps in each dimension. There are $6!$ different orders for these 6 steps. Further, we have to choose 3 out of the $d$ dimensions (without repetition). This gives a factor $\frac{d(d-1)(d-2)}{3!} 6!$.

We define the multinomial

\[ \left( \begin{array}{c} m \\ k_1, k_2, \ldots, k_r \end{array} \right) = \frac{m!}{k_1! k_2! \ldots, k_r!}, \]  

\[ (5.2.7) \]
and then compute, as just demonstrated, that

\[
p_4(0) = d \binom{4}{2,2} + \frac{d(d-1)}{2} \binom{4}{1,1,1,1} \tag{5.2.8}
\]

\[
p_6(0) = d \binom{6}{3,3} + d(d-1) \left( \binom{6}{2,2,1,1} + \frac{d(d-1)(d-2)}{3!} \binom{6}{1,1,1,1,1,1} \right) \tag{5.2.9}
\]

\[
p_8(0) = d \binom{8}{4,4} + d(d-1) \left[ \binom{8}{3,3,1,1} + \frac{1}{2} \binom{8}{2,2,2,2} \right] + \frac{d(d-1)(d-2)}{2!} \binom{8}{2,2,1,1,1,1} + \frac{d(d-1)(d-2)(d-3)}{4!} \binom{8}{1,1,1,1,1,1,1,1} \tag{5.2.10}
\]

For our implementation of the analysis of Section 3.5 we compute the value of \( p_n(x) \) for \( n \in \{0,12\} \) and

\[
x \in \{0,e_1,2e_1,3e_1,4e_1,e_1 + e_2,2e_1 + 2e_2 e_1 + e_2,3e_1 + e_2,2e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4\}. \tag{5.2.11}
\]

This is elaborate, but used the same idea for \( p_n(0) \). For example we now describe how we compute

\[
p_6(2e_1) = \frac{6!}{4!2!} + (d-1) \left( \frac{6!}{(2!)^2} + \frac{6!}{3!} + \frac{(d-1)(d-2)}{2!} \right) \tag{5.2.12}
\]

As for \( p_6(0) \) we distinguish between the number of dimensions used by the walker.

- One dimension: then the walk has 4 steps to the right and 2 to the left. Thus, the number of walks is given by the number of permutations 4 right and 2 left steps \( (6!/(4!2!)) \).

- Two dimensions: there are \( (d-1) \) choices for the second dimension. Let say that step in the second dimension are up and down movements. Then we have two possibility combinations to end up at \( 2e_1 \): 2 up, 2 down, 2 right movements \( (6!/(2!2!2!)) \) or 1 up, 1 down, 3 right, 1 left movements \( (6!/(3!1!1!1!)) \).

- Two dimensions: there are \( (d-1)(d-2) \) different choices the second and the third dimension. With six steps in total we can only reach \( 2e_1 \) if we take 2 right steps and 2 steps (forth-back) in the second and the third dimension. Thus there are \( 6!/(2!1!1!1!1!) \) different combinations of these steps. Moreover, we correct this by factor \( 1/2 \) as the choice of the second and the third dimension is interchangeable.
5.2.3 Bounds on involved SRW-Integrals

In the following we show how to bound the integrals defined in (3.6.28)-(3.6.30). The integrals are only required for the analysis of Sections 3.5-3.6. In this section we adapt techniques of [41, Appendix B.1]. For the bounds we use the following integrals:

\[
L_n(x) = \int_{(\pi,-\pi)^d} \hat{C}(k)^n \hat{D}^{(x)}(k)^2 \frac{d^d k}{(2\pi)^d}, \tag{5.2.13}
\]

\[
V_{n,l} = \int_{(\pi,-\pi)^d} \frac{\hat{D}^l(k) [\hat{D}^{\sin}(k)]^2}{[1 - \hat{D}(k)]^n} \frac{d^d k}{(2\pi)^d}. \tag{5.2.14}
\]

**Bound in terms of \( I_{n,l}, L_n, V_n \)**

We use the Cauchy-Schwarz inequality to bound the integrals defined in (3.6.28) and (3.6.28) by

\[
K_{n,l}(x) \leq \left| I_{n,2l}(0) L_n(x) \right|^{1/2}, \tag{5.2.15}
\]

\[
U_{n,l}(x) \leq \left| V_{n,l} L_n(x) \right|^{1/2}. \tag{5.2.16}
\]

For \( x = 0 \) we know that \( L_n(x) = 1 \) and we use the better bound

\[
K_{n,l}(0) \leq \begin{cases} 
I_{n,l}(0) & \text{if } l \text{ is even} \\
I_{n,l-1}(0)^{1/2} I_{n,l+1}(0)^{1/2} & \text{if } l \text{ is odd}
\end{cases}, \tag{5.2.17}
\]

\[
U_{n,l}(0) \leq \frac{1}{d} K_{n,l}(0), \tag{5.2.18}
\]

where the second inequality follows from \( \hat{D}^{\sin}(k) \leq 1/d \). For even \( l \) can further use \( \hat{D}^{\sin}(k) \geq 0 \) and (3.6.13) to obtain the bound

\[
U_{n,l}(0) = \frac{1}{2d} \left( I_{n,l}(0) - I_{n,l}(2e_1) \right) \tag{5.2.19}
\]

Moreover, we used a different bound on \( K_{n,l}(x) \) for the special case of \( l = 0 \) and \( n \geq 1 \). Therefore we note for \( n \geq 1 \) that

\[
\frac{1}{[1 - \hat{D}(k)]^n} = \frac{1}{[1 - \hat{D}(k)]^{n-1}} + \frac{\hat{D}(k)}{[1 - \hat{D}(k)]^{n-1}} + \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^n}, \tag{5.2.20}
\]

which implies that \( K_{n,0}(x) \leq K_{n-1,0}(x) + K_{n-1,1}(x) + K_{n,2}(x) \), and

\[
K_{n,0}(x) \leq K_{n-1,0}(x) + \left| I_{n-1,4}(0) L_{n-1}(x) \right|^{1/2} + \left| I_{n,4}(0) L_n(x) \right|^{1/2}. \tag{5.2.21}
\]

To bound \( T_{n,l} \) we use (3.6.21) and \( \hat{D}^{\sin}(k) \leq 1/d \), respectively, to obtain

\[
|\hat{M}(k)| \leq |\hat{D}(k)| + 2|\hat{D}^{\sin}(k)| \hat{C}(k) \leq |\hat{D}(k)| + \min \left\{ \frac{4}{d}, \frac{1}{d} \right\} \hat{C}(k). \tag{5.2.22}
\]
Thereby, we can bound $T_{n,l}(x)$ as follows:

$$T_{n,l}(x) \leq \int_{[\pi,-\pi]^d} |\hat{D}(k)\hat{\mathcal{C}}(k)\hat{\mathcal{M}}(k)||\hat{D}(x)(k)| \frac{d^d k}{(2\pi)^d}$$

$$\leq K_{n,l+1}(x) + \min \left\{ \frac{4}{d} K_{n,l}(x), \frac{1}{d} K_{n+1,l}(x) \right\}.$$  \hspace{1cm} (5.2.23)

**Computation of $L_n$.** Reviewing the definition of $\hat{D}(x)(k)$ in (3.6.24) it is not difficult to see that

$$L_n(x) = \int_{[\pi,-\pi]^d} \frac{(\hat{D}(x)(k))^2}{|1-\hat{D}(k)|^n} \frac{d^d k}{(2\pi)^d}$$

$$= \frac{1}{2^d d!} \sum_{\mu \in \mathcal{P}_d} \sum_{\delta \in \{-1,1\}^d} I_{n,0}(x + p(x; \nu, \delta)).$$  \hspace{1cm} (5.2.24)

The set $\mathcal{P}_d$ and the operator $p(x; \nu, \delta)$ are defined in Definition 3.5.1 As we can compute $I_{n,0}(x)$, we can also compute the sum in (5.2.24) directly.

As the value of $I_{n,0}(x)$ only depends on the number of entries that have a given absolute value, we can reduce the domain over which we sum. In the end of this section we will show that it is sufficient to compute $L_n(x)$ for $x \in \{e_1,2e_1,e_1+e_2\}$.

We first state the value and then explain how we computed them:

$$L_n(e_1) = \frac{1}{2d} I_{n,0}(0) + \frac{1}{2d} I_{n,0}(2e_1) + \frac{d-1}{d} I_{n,0}(e_1 + e_2),$$  \hspace{1cm} (5.2.25)

$$L_n(2e_1) = \frac{1}{2d} I_{n,0}(0) + \frac{1}{2d} I_{n,0}(4e_1) + \frac{d-1}{d} I_{n,0}(2e_1 + 2e_2),$$  \hspace{1cm} (5.2.26)

$$L_n(e_1 + e_2) = \frac{(d-2)(d-3)}{d(d-1)} I_{n,0}(e_1 + e_2 + e_3 + e_4)$$

$$\quad \quad + \frac{(d-2)}{2d(d-1)} \left( I_{n,0}(e_1 + e_2) + I_{n,0}(2e_1 + e_2 + e_3) \right)$$

$$\quad \quad + \frac{1}{4d(d-1)} \left( I_{n,0}(0) + I_{n,0}(2e_1 + 2e_2) + 2I_{n,0}(2e_1) \right).$$  \hspace{1cm} (5.2.27)

Computation of $L_n(e_1)$: For all $\nu \in \mathcal{P}_d$ with $\nu_1 \neq 1$ we know by symmetry that $I_{n,0}(e_1 + p(e_1; \nu, \delta)) = I_{n,0}(e_1 + e_2)$ for all $\delta \in \{-1,1\}^d$. This explains the third summand of (5.2.25), where we note that there are $(d-1)!(d-1)$ permutations $\nu$ with $\nu_1 \neq 1$. That leaves $(d-1)!$ permutations $\nu$ with $\nu_1 = 1$. As all entries of $p(e_1; \nu, \delta)$ except the first one are trivial, the values $\delta_2, \ldots, \delta_d$ do not affect the summand. If $\delta_1 = 1$ then $e_1 + p(e_1; \nu, \delta) = 2e_1$ and if $\delta_1 = -1$, then $e_1 + p(e_1; \nu, \delta) = 0$. The two cases correspond to the first and second term in (5.2.25). The equality (5.2.26) is obtained in the same way.

Computation of $L_n(e_1 + e_2)$: There are $2(d-2)!$ permutations $\nu$ with $\{\nu_1, \nu_2\} = \{1,2\}$. Further, there are $2(d-2)!(2d-2)$ that map 1 to $\{3, \ldots, d\}$ and 2 to $\{1,2\}$. That leaves $d! - 2(d-2)! - 4(d-2)!(2d-2) = (d-2)!(d-2)(d-3)$
permutations $\nu$ that do not map 1 and 2 to the first coordinates, i.e. $\{\nu_1, \nu_2\} \cap \{1, 2\} = \emptyset$. For these $\nu$ we know that
\[
I_{n,0}(e_1 + e_2 + p(e_1; \nu, \delta)) = I_{n,0}(e_1 + e_2 + e_3 + e_4),
\]
which creates the first summand of \[5.2.27\]. The second corresponds to the case that either $e_1$ or $e_2$ is mapped to one of the first two coordinates. For example let us assume $\nu_1 = 1$ and $\nu_2 = 3$, then
\[
e_1 + e_2 + p(e_1; \nu, \delta) \in \{2e_1 \pm e_2 \pm e_3 \pm e_4\},
\]
depending on the sign of $\delta_1$. This creates the second summand of \[5.2.27\]. If we map $e_1 + e_2$ to both of the first two coordinates, then
\[
e_1 + e_2 + p(e_1; \nu, \delta) \in \{0, 2e_1, 2e_2, 2e_1 + 2e_2\},
\]
which only depends on $\delta_1$ and $\delta_2$. This creates the third summand of \[5.2.27\].

**Computation of $V_{n,l}$.** The equality \[3.6.13\] implies that
\[
\hat{D}^{\sin}(k)^2 = \frac{1}{(2d)^2} [1 - \hat{D}(2k)]^2
\]
\[
e = \frac{1}{(2d)^2} \left[ 1 - \frac{2}{2d} \sum_{i} e^{i2k_i} + \frac{1}{(2d)^2} \sum_{i,k} e^{i2(k_i+k_x)} \right].
\]
Form this we conclude that that
\[
V_{n,l} = \int_{(\pi, -\pi)^d} \frac{\hat{D}(k)^l \hat{D}^{\sin}(k)^2}{[1 - \hat{D}(k)]^n} \frac{d^d k}{(2\pi)^d}
\]
\[
= \frac{1}{(2d)^2} \left( I_{n,l}(0) - 2I_{n,l}(2e_1) + \frac{d-1}{d} I_{n,l}(2e_1 + 2e_2) + \frac{1}{2d} I_{n,l}(0) + \frac{1}{2d} I_{n,l}(4e_1) \right)
\]

**Bond on the supremum.** In Section \[3.6.4\] we derive a bound on $\hat{f}_3$, see \[3.6.84\]. This bounds uses the values of SRW-Integrals
\[
\sup_{x \in \mathbb{Z}^d \setminus \{0\}} I_{n,l}(x), \sup_{x \in \mathbb{Z}^d \setminus \{0\}} K_{n,l}(x), \sup_{x \in \mathbb{Z}^d \setminus \{0\}} T_{n,l}(x), \sup_{x \in \mathbb{Z}^d \setminus \{0\}} U_{n,l}(x).
\]
In the previous section we bounded the $K_{n,l}(x), T_{n,l}(x)$, $U_{n,l}(x)$ for a given $x$ using $I_{n,l}(x), L_n(x), V_n$. Now we explain how we bound the supremums in \[5.2.34\]. The value of $V_n$ is independent of $x$ and the $I_{n,l}, L_n$ are monotone in $x$ in the following sense:

**Lemma 5.2.1.** Let $n$ be a positive integer and consider $x, y \in \mathbb{Z}^d$ with $x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$ and $y_1 \geq y_2 \geq \cdots \geq y_d \geq 0$. Then
\[
I_{n,l}(x) \leq I_{n,l}(x + z),
\]
\[
L_n(x) \leq L_n(x + z).
\]
This lemma is a combination of Lemma B.3 and Lemma B.4 of [41]. We conclude from Lemma 5.2.1 that

\[
\sup_{x \in \mathbb{Z}^d \setminus \{0\}} I_{n,l}(x) = I_{n,l}(e_1) = I_{n,l+1}(0),
\]

(5.2.37)

\[
\sup_{x \in \mathbb{Z}^d \setminus \{0\}} L_n(x) = L_n(e_1),
\]

(5.2.38)

where \( L_n(e_1) \) is given in (5.2.25). We can use these equalities and the bounds we derived in the preceding paragraphs to compute bounds on the terms in (5.2.34). This allows us to compute the bound on \( \bar{f}_3 \).

We use a simple trick to improve the bound on \( \bar{f}_3 \). We produce a bound for \( x = e_1 \) and another bound for all \( x \in \mathbb{Z}^d \) with \( \|x\|_2 > 1 \), and then bound \( \bar{f}_3 \) by the maximum of these two bounds.

We use for \( x \) with \( \|x\|_2 > 1 \) we use the uniform bound

\[
\sup_{x \in \mathbb{Z}^d : \|x\|_2 > 1} I_{n,l}(x) = \max\{I_{n,l}(2e_1), I_{n,l}(e_1 + e_2)\},
\]

(5.2.39)

\[
\sup_{x \in \mathbb{Z}^d : \|x\|_2 > 1} L_n(x) = \max\{L_n(2e_1), L_n(e_1 + e_2)\}.
\]

(5.2.40)

For \( x = e_1 \) we use symmetry to obtain the bounds

\[
K_{n,l}(e_1) \leq K_{n,l+1}(0) \leq \begin{cases} I_{n,l+1}(0) & \text{if } (l + 1) \text{ is even} \\ I_{n,l}(0)^{1/2} I_{n,l+2}(0)^{1/2} & \text{if } (l + 1) \text{ is odd} \end{cases},
\]

(5.2.41)

\[
U_{n,l}(e_1) = U_{n,l+1}(0) = \frac{1}{d} K_{n,l+1}(0),
\]

(5.2.42)

\[
T_{n,l}(e_1) \leq K_{n,l+2}(0) + \frac{4}{d} K_{n,l+1}(0).
\]

(5.2.43)

### 5.3 Series involving matrices

In Chapter 3 we have assumed that the coefficients are bounded by values \( \beta_{\text{abs}}(N) \), \( \beta_{\text{odd}}(N) \), and \( \beta_{\text{even}}(N) \). In Chapter 4 we proved bounds on the coefficients \( (\beta_*)_{N \geq 0} \) in terms of a product of matrices. In this section we explain how we compute the sum over those bounds \( (\beta_{\text{abs}}, \beta_{\text{odd}}, \beta_{\text{even}}) \) when the individual bounds involves matrices.

We explain two bounds used for SAW, see Lemma 4.2.3. Let \( \mathbf{B}, \bar{\mathbf{B}} \in \mathbb{R}^{2 \times 2}_+ \) be matrices and \( w_1, w_2, w_3 \in \mathbb{R}_+^2 \) vectors such that they bound the lace-expansion coefficient \( \Xi^{(\mathbf{t})} \) in the following way:

\[
\sum_x \Xi^{(\mathbf{t})}_z(x) \leq w_1^T \mathbf{B}^{N-1} w_2.
\]

(5.3.1)

\[
\sum_t \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot (x - e_1))] \Xi^{(\mathbf{t})}_z(x) \leq \frac{KN^{(N-2)/2}}{2} \sum_{M=0}^{N-2} w_1^T \mathbf{B}^{2M} \bar{\mathbf{I}}^T (\bar{\mathbf{B}})^{N-2M-2} w_3,
\]

(5.3.2)
with \( \mathbf{I} = (1,1) \).

We sum the bounds in \([5.3.1]\) and \([5.3.2]\) over \( N \) by computing the eigenvectors \( \eta_1 \) and \( \eta_2 \) of \( \mathbf{B} \) and the right eigenvectors \( \mathbf{\bar{\eta}}_1 \) and \( \mathbf{\bar{\eta}}_2 \) of \( \mathbf{\bar{B}} \). These vectors will be independent, so that there exist \( a_1, \ldots, a_4 \in \mathbb{R} \) such that

\[
\begin{aligned}
    w_1 &= a_1 \eta_1 + a_2 \eta_2 = (1 \ 1) = a_3 \mathbf{\bar{\eta}}_1 + a_4 \mathbf{\bar{\eta}}_2. \quad (5.3.3)
\end{aligned}
\]

We compute \( a_1, \ldots, a_4 \in \mathbb{R} \) using relations of the form

\[
\begin{aligned}
    w_1 &= a_1 \eta_1 + a_2 \eta_2 \quad \Rightarrow \quad w_1 = \begin{pmatrix} (\eta_1)_1 & (\eta_2)_1 \\ (\eta_1)_2 & (\eta_2)_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
    &\quad\Rightarrow \begin{pmatrix} (\eta_1)_1 & (\eta_2)_1 \\ (\eta_1)_2 & (\eta_2)_2 \end{pmatrix}^{-1} w_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (5.3.4)
\end{aligned}
\]

We define the eigenvectors

\[
\begin{aligned}
    v_1 &= a_1 \eta_1, \\
    v_2 &= a_2 \eta_2, \\
    v_3 &= a_3 \mathbf{\bar{\eta}}_1, \\
    v_4 &= a_4 \mathbf{\bar{\eta}}_2. \quad (5.3.5-6)
\end{aligned}
\]

and denote the eigenvalue corresponding to the eigenvector \( v_i \) by \( \lambda_i \). Using these notations we compute

\[
\begin{aligned}
    \sum_x \Xi^{(N)}(x) \leq &\quad w_1^T \mathbf{B}^{N-1} \mathbf{\bar{w}}_2 \leq (\mathbf{\bar{v}}_1 + \mathbf{\bar{v}}_2)^T \mathbf{B}^{N-2} \mathbf{\bar{w}}_2 \\
    &= \mathbf{\bar{v}}_1^T \lambda_1^{N-1} \mathbf{\bar{w}}_2 + \mathbf{\bar{v}}_2^T \lambda_2^{N-1} \mathbf{\bar{w}}_2, \quad (5.3.7)
\end{aligned}
\]

\[
\begin{aligned}
    \sum_{N \geq 2 \text{ even}} \sum_x \Xi^{(N)}(x) \leq &\quad \sum_{N=0}^{\infty} \mathbf{\bar{v}}_1^T \mathbf{\bar{w}}_2 \lambda_1^{2N+1} + \mathbf{\bar{v}}_2^T \mathbf{\bar{w}}_2 \lambda_2^{2N+1} = \frac{\mathbf{\bar{v}}_1^T \mathbf{\bar{w}}_2 \lambda_1}{1 - \lambda_1^2} + \frac{\mathbf{\bar{v}}_2^T \mathbf{\bar{w}}_2 \lambda_2}{1 - \lambda_2^2}, \quad (5.3.8)
\end{aligned}
\]

\[
\begin{aligned}
    \sum_{N \geq 2 \text{ odd}} \sum_x \Xi^{(N)}(x) \leq &\quad \sum_{N=0}^{\infty} \mathbf{\bar{v}}_1^T \mathbf{\bar{w}}_2 \lambda_1^{2N+2} + \mathbf{\bar{v}}_2^T \mathbf{\bar{w}}_2 \lambda_2^{2N+2} = \frac{\mathbf{\bar{v}}_1^T \mathbf{\bar{w}}_2 \lambda_1^2}{1 - \lambda_1^2} + \frac{\mathbf{\bar{v}}_2^T \mathbf{\bar{w}}_2 \lambda_2^2}{1 - \lambda_2^2}. \quad (5.3.9)
\end{aligned}
\]
We use the geometric sums already introduced in (3.4.36) to compute the sum of (5.3.2) over odd \(N\):

\[
\sum_{x, N \geq 2 \text{ even}} [1 - \cos (k (x - e_j))] \Xi^{(N),i}(x)
\]

\[
\leq K \sum_{N,M=0}^{\infty} (N + M + 1)(\bar{v}_1 + \bar{v}_2)^T \mathbf{B}^{2N} \mathbf{I} (\bar{v}_3 + \bar{v}_4)^T (\bar{\mathbf{B}})^2M \bar{w}_3
\]

\[
= K \sum_{N=0}^{\infty} (N + 1)(\bar{v}_1 + \bar{v}_2) \mathbf{B}^{2N} \mathbf{I} \sum_{M=0}^{\infty} (\bar{v}_3 + \bar{v}_4)^T (\bar{\mathbf{B}})^2M \bar{w}_3
\]

\[
+ K \sum_{N=0}^{\infty} (N + 1)(\bar{v}_1 + \bar{v}_2)^T \mathbf{B}^{2N} \mathbf{I} \sum_{M=0}^{\infty} M(\bar{v}_3 + \bar{v}_4)^T (\bar{\mathbf{B}})^2M \bar{w}_3
\]

\[
= K \left( \frac{\bar{v}_1^T \bar{1}}{1 - \lambda_1^2} + \frac{\bar{v}_2^T \bar{1}}{1 - \lambda_2^2} \right) \left( \frac{\bar{v}_3^T \bar{w}_3}{1 - \lambda_3^2} + \frac{\bar{v}_4^T \bar{w}_3}{1 - \lambda_4^2} \right)
\]

\[
+ K \left( \frac{\bar{v}_1^T \bar{1}}{1 - \lambda_1^2} + \frac{\bar{v}_2^T \bar{1}}{1 - \lambda_2^2} \right) \left( \frac{\bar{v}_3^T \bar{w}_3}{1 - \lambda_3^2} + \frac{\bar{v}_4^T \bar{w}_3}{1 - \lambda_4^2} \right)
\]

\[
+ K \left( \frac{\bar{v}_1^T \bar{1}}{1 - \lambda_1^2} + \frac{\bar{v}_2^T \bar{1}}{1 - \lambda_2^2} \right) \left( \frac{\bar{v}_3^T \bar{w}_3}{1 - \lambda_3^2} + \frac{\bar{v}_4^T \bar{w}_3}{1 - \lambda_4^2} \right)
\]

\[
(5.3.10)
\]

For the other models we compute \((\beta_{abs}, \beta_{odd}, \beta_{even})\) in the same way. The matrices for percolation are \(3 \times 3\) dimensional and for LT and LA \(5 \times 5\) dimensional. Thus, the decomposition of the vectors, as shown in (5.3.3)-(5.3.6), requires three and five, respectively, eigenvalues and eigenvector. As the entries of the matrices are all real and strictly positive, the existence of three/five independent, possibly complex eigenvectors, is guaranteed.

### 5.4 Model specific features

#### 5.4.1 Comments for self-avoiding walk

In this section we discuss some bounds that are only relevant for the SAW. We begin by explaining a bound on \(G_{4,z}(2e_i)\). We extract the dominant contribution \(c_{4}(2e_i)\) and bound the remainder as given in (5.1.28). Further, we improve the obtained bound by removing some contributions of the 6-step walks to \(2e_i\) using \(z \geq z_I\). We use the following bound:

\[
G_{4,z}(2e_i) \leq (2d - 2)z^4 - (2d - 2)(8(2d - 4) + 3(2d - 3))z_I^6 + \sup_{x \in \mathbb{Z}^d} G_{6,z}(x). \tag{5.4.1}
\]

**Bounds for a double loop with one weighted line.** For the SAW we also bound a closed repulsive diagram not discussed in Section 5.1.2

\[
\sum_x G_{2,z}(x)^3 [1 - \cos(k \cdot (x))]. \tag{5.4.2}
\]
We know that \( \hat{G}_z \) and \( \hat{D}^2 \) are non-negative. The assumption \( f_3(z) \leq \Gamma_3 \) states a bound the absolute value of \( \frac{1}{2} \Delta_k \hat{G}_z(l_1 - l_2) \) by \( \hat{\Delta}(l_1, l_2) \). It also states that \( \hat{\Delta}(k, l) \) is an upper bound on \( -\frac{1}{2} \Delta_k \hat{G}_z(l_1 - l_2) \). We use this upper bound and \( f_i(z) \leq \Gamma_i \) to obtain

\[
\sum_x G_{2,z}(x)^3 [1 - \cos(k \cdot x)] 
= [1 - \hat{\Delta}(k)](2d z)^4 \Gamma_2 \Gamma_3 [1 - \hat{\Delta}(k)] \sum_x (D^{*2} \star C)(x)^2 \cos(k \cdot x) 
= 2c_1 (2d z)^4 \Gamma_2 \Gamma_3 [1 - \hat{\Delta}(k)] \sum_x (D^{*2} \star C)(x)^2 \sum_y \cos(k \cdot y)C(y)C(x - y) 
+ 2c_2 (2d z)^4 \Gamma_2 \Gamma_3 [1 - \hat{\Delta}(k)] \sum_x (D^{*2} \star C)(x)^2 
\times \sum_y \cos(k \cdot y)C(y)\cos(k \cdot (x - y))C(x - y). 
\]

We bound all cosines by one and obtain

\[
\sum_x G_{2,z}(x)^3 [1 - \cos(k \cdot x)] 
\leq (2d z)^4 \Gamma_2 \Gamma_3 [1 - \hat{\Delta}(k)] \sum_x (D^{*2} \star C)(x)^2 [2c_1 C(x) + (2c_2 + c_3)(C \star C)(x)]. 
\]

Then, we compute

\[
\sum_x (D^{*2} \star C)(x)^2 [2c_1 C(x) + (2c_2 + c_3)(C \star C)(x)] 
= I_{1,2}^2(0)(c_1 I_{1,0}(0) + (2c_2 + c_3)I_{2,0}(0)) 
+ \sum_{x \neq 0} (D^{*2} \star C)(x)^2 [2c_1 C(x) + (2c_2 + c_3)(C \star C)(x)] 
\leq I_{1,2}^2(0)(c_1 I_{1,0}(0) + (2c_2 + c_3)I_{2,0}(0)) 
+ \sup_{x \neq 0} [2c_1 C(x) + (2c_2 + c_3)(C \star C)(x)] \sum_{x \neq 0} (D^{*2} \star C)(x)^2. 
\]

We bound one \( G_{2,z}(x) \) by \( G_z(x) \) and the other two by \( G_{2,z}(x) \leq (2d z)^2 (D^{*2} \star G_z)(x) \) and obtain

\[
\sum_x G_{2,z}(x)^3 [1 - \cos(k \cdot x)] 
\leq (2d z)^4 \sum_x (D \star D \star G_z)(x)G_z(x) [1 - \cos(k \cdot x)] 
= (2d z)^4 \int_{[-\pi,\pi]^d} \int_{[-\pi,\pi]^d} \hat{D}(l_1)^2 \hat{G}_z(l_1) \left( -\frac{1}{2} \Delta_k \hat{G}_z(l_1 - l_2) \right) \hat{D}(l_2)^2 \hat{G}_z(l_2) \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d}.
\]
For the SRW the supremum is attained at a neighbor of the origin, see also Lemma 5.2.1. Thus, we know that

\[
\sup_{x \neq 0} C(x) = C(e_1) = \frac{1}{2d} \sum_i C(e_i) = (D^{*2} \ast C)(0) = I_{1,2}(0), \quad (5.4.4)
\]

\[
\sup_{x \neq 0} (C \ast C)(x) = (C \ast C)(e_1) = C(e_1) + (C \ast C \ast D)(e_1) = I_{1,2}(0) + I_{2,2}(0), \quad (5.4.5)
\]

\[
\sum_{x \neq 0} (D^{*2} \ast C)(x)^2 = \sum_x (D^{*2} \ast C)(x)^2 - (D^{*2} \ast C)(0)^2 = I_{2,4}(0) - I_{1,2}^2(0). \quad (5.4.6)
\]

Combining, this, we finally obtain the bound

\[
\sum_x G_{2,z}(x)^3 [1 - \cos(k \cdot (x))] \leq \left[1 - \hat{D}(k) \right] (2dz)^4 G^2 \Gamma^2 \left[2c_1 I_{1,2}(0) + (2c_2 + c_3)(I_{1,2}(0) + I_{2,2}(0)) \right] (I_{2,4}(0) - I_{1,2}^2(0))
\]

\[
+ \left[1 - \hat{D}(k) \right] (2dz)^4 G^2 \Gamma^2 \left(c_1 I_{1,0}(0) + (2c_2 + c_3)I_{2,0}(0) \right) I_{1,2}(0)^2. \quad (5.4.7)
\]

**Bound on sum over** \( N \geq 2 \). As already discussed in Section 5.3, we sum the matrix-valued bounds. We have already computed the bound for \( \hat{z}_{N,N}^{(N)}(0) \), see (5.3.1) - (5.3.2) and (5.3.8) - (5.3.10). Here we give a list of the bounds we compute for SAW. Let \( \vec{w}_1, \vec{w}_2, \vec{w}_3 \in \mathbb{R}^2 \) and \( \vec{B}, \vec{B} \in \mathbb{R}^{2 \times 2} \) and eigenvectors \( \vec{v}_i \) such that

\[
\vec{w}_1 = \vec{v}_1 + \vec{v}_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_3 + \vec{v}_4. \quad (5.4.8)
\]

with corresponding eigenvalues \( \lambda_i \). Then, we compute

\[
\sum_x \Xi^{(N,N)}(x) \leq w_1^T \hat{B}^{N-1} \vec{w}_2 \leq (\vec{v}_1 + \vec{v}_2)^T \hat{B}^{N-2} \vec{w}_2 = \vec{v}_1^T \lambda_1^{N-1} \vec{w}_2 + \vec{v}_2^T \lambda_2^{N-1} \vec{w}_2 \quad (5.4.9)
\]

\[
\sum_{N \geq 2 \text{ even}} \sum_x \Xi^{(N,N)}(x) \leq \sum_{N=0}^{\infty} \vec{v}_1^T \vec{w}_2 \lambda_1^{2N+1} + \vec{v}_2^T \vec{w}_2 \lambda_2^{2N+1} = \frac{\vec{v}_1^T \vec{w}_2 \lambda_1}{1 - \lambda_1^2} + \frac{\vec{v}_2^T \vec{w}_2 \lambda_2}{1 - \lambda_2^2}, \quad (5.4.10)
\]

\[
\sum_{N \geq 2 \text{ odd}} \sum_x \Xi^{(N,N)}(x) \leq \sum_{N=0}^{\infty} \vec{v}_1^T \vec{w}_2 \lambda_1^{2N+2} + \vec{v}_2^T \vec{w}_2 \lambda_2^{2N+2} = \frac{\vec{v}_1^T \vec{w}_2 \lambda_1^2}{1 - \lambda_1^2} + \frac{\vec{v}_2^T \vec{w}_2 \lambda_2^2}{1 - \lambda_2^2}. \quad (5.4.11)
\]
For $N$ even, we then have

\[
\sum_{x,N \geq 2 \text{ even}} [1 - \cos (k \cdot (x - e_i))]\Xi^{(N),d}(x) \leq K \sum_{N \geq 2 \text{ even}} \sum_{M=0}^{(N-2)/2} \bar{w}_1 B^{2M} \bar{1} \bar{1}^T \bar{\left( B \right)}^{N-2M-2} \bar{w}_3
\]

\[
= K \left( \frac{\bar{v}_1}{1-\lambda_1^2} + \frac{\bar{v}_2}{1-\lambda_2^2} \right) \left( \frac{\bar{v}_3 \bar{w}_3}{1-\lambda_3^2} + \frac{\bar{v}_4 \bar{w}_3}{1-\lambda_4^2} \right)
\]

\[
= K \left( \frac{\bar{v}_1}{1-\lambda_1^2} + \frac{\bar{v}_2}{1-\lambda_2^2} \right) \left( \frac{\bar{v}_3 \bar{w}_3}{1-\lambda_3^2} - \frac{\bar{v}_4 \bar{w}_3}{1-\lambda_4^2} \right)
\]

\[
+ K \left( \frac{\bar{v}_1^T \bar{1}}{1-\lambda_1^2} + \frac{\bar{v}_2^T \bar{1}}{1-\lambda_2^2} \right) \left( \frac{\bar{v}_3^T \bar{w}_3}{1-\lambda_3^2} - \frac{\bar{v}_4^T \bar{w}_3}{1-\lambda_4^2} \right)
\]

and

\[
\sum_{x,N \geq 2 \text{ even}} [1 - \cos (k \cdot x)]\Xi^{(N),d}(x) \leq 2d \sum_{N \geq 2 \text{ even}} \frac{N+2}{2} \sum_{M=0}^{(N-2)/2} \bar{w}_1 B^{N-1} \bar{w}_2
\]

\[
+ K \left( \frac{\bar{v}_1}{1-\lambda_1^2} + \frac{\bar{v}_2}{1-\lambda_2^2} \right) \left( \frac{\bar{v}_3 \bar{w}_3}{1-\lambda_3^2} + \frac{\bar{v}_4 \bar{w}_3}{1-\lambda_4^2} \right)
\]

\[
+ K \left( \frac{\bar{v}_1^T \bar{1}}{1-\lambda_1^2} + \frac{\bar{v}_2^T \bar{1}}{1-\lambda_2^2} \right) \left( \frac{\bar{v}_3^T \bar{w}_3}{1-\lambda_3^2} + \frac{\bar{v}_4^T \bar{w}_3}{1-\lambda_4^2} \right)
\]

For $N$ odd we compute

\[
\sum_{x,N \geq 2 \text{ odd}} [1 - \cos (k \cdot (x - e_i))]\Xi^{(N),d}(x) \leq \sum_{N \geq 2 \text{ odd}} \frac{K(N+1)}{2} \sum_{M=0}^{(N-1)/2} \bar{w}_1 B^{2M} \bar{1} \bar{1}^T \bar{\left( B \right)}^{N-2M-2} \bar{w}_3
\]

\[
+ \sum_{N \geq 2 \text{ odd}} \frac{K(N+1)}{2} \bar{w}_1 B^{N-1} \bar{\left( \sup_{x \in Z^d} (G_{2,z} \ast G_z)(x) \right)}
\]

\[
= K \left( \frac{\bar{v}_1}{1-\lambda_1^2} \left( \sum_{j=3}^{4} \bar{v}_j^T \lambda_j \bar{w}_3 \right) \right) + K \left( \frac{\bar{v}_2}{1-\lambda_2^2} \left( \sum_{j=3}^{4} \bar{v}_j^T \lambda_j \bar{w}_3 \right) \right)
\]

\[
+ K \left( \frac{\bar{v}_1^T \lambda_1 \bar{1}}{1-\lambda_1^2} + \bar{v}_1^T \lambda_1 \bar{1} \right) \left( \sum_{x \in Z^d} (G_{2,z} \ast G_z)(x) \right)
\]
\[
\sum_{x,N \geq 2 \text{ odd}} [1 - \cos (k \cdot x)] \Xi^{(N)/l}(x) \\
\leq \sum_{N \geq 2 \text{ odd}} \frac{K(N + 1)}{2} \left( \bar{T}^T \tilde{B}^{N-1} \bar{w}_3 + \sum_{M=0}^{(N-3)/2} \bar{w}_1^T \tilde{B}^{2M+1} \bar{I}^T (\tilde{B})^{N-2M-3} \bar{w}_3 \right)
\]
\[
= K \sum_{N,M=0}^{\infty} (N + M + 2)(\bar{v}_1 + \bar{v}_2)^T \tilde{B}^{2N+1} \bar{I}(\bar{v}_3 + \bar{v}_4)^T (\tilde{B})^{2M} \bar{w}_3
\]
\[
+ \sum_{N=0}^{\infty} (N + 2)(\bar{v}_1 + \bar{v}_2) \tilde{B}^{2N+1} \bar{I} \left( \sup_{x \in \mathbb{Z}^d} (G_{2,z} \ast G_z)(x) \right)
\]
\[
= K \sum_{i=1}^{2} \lambda_i \bar{v}_i^T \bar{I} \left( \sum_{j=3}^{4} (1 - \lambda_i^2)^2 \right) + K \left( \sum_{i=1}^{2} \lambda_i \bar{v}_i^T \bar{I} \left( \sum_{j=3}^{4} (1 - \lambda_i^2)^2 \right) \right)
\]
\[
+ K \sum_{i=1}^{2} \lambda_i \bar{v}_i^T \bar{I} \left( \sum_{j=3}^{4} (1 - \lambda_i^2) \right) \sup_{x \in \mathbb{Z}^d} (G_{2,z} \ast G_z)(x).
\]

5.4.2 Comments for lattice trees and animals

For LA and LT we comment on the bounding of the one-point functions \(g_z\) and \(g_z^l\) for \(z \in (z_1, z_c)\). Then we explain which function \(\tilde{f}_{3,n,l}\) we use for the bootstrap of the analysis of Section 3.5.

**Bound on the one-point function \(g_z\).** The bound \(f_1(z) \leq \Gamma_1\) gives us an upper bound on \(z g_z\). To also obtain a bound on \(g_z\) we use that \(z \geq 1/(2d - 1)e\) and thereby

\[
\frac{g_z}{e} \leq (2d - 1) z g_z \leq \Gamma_1.
\]

We bound \(g_z^l\) use the relation (3.7.16)

\[
g_z^l = 1 + \frac{2d - 1}{2d} (g_z - 1) \leq \frac{2d - 1}{2d} \Gamma_1 e + \frac{1}{2d}.
\]

**Bootstrap function used for Section 3.5.** As already explained in Sections 3.5 and 3.8 the analysis of Section 3.5 might use more the functions

\[
\tilde{f}_3(z) = \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \int_{(\pi, -\pi)^d} (-\hat{G}_z(k)) \hat{D}^l(k) \hat{G}_z^n(k) e^{ik \cdot x} \frac{d^dk}{(2\pi)^d}
\]

for several values of \(n, l\) in the bootstrap. As discussed in Section 3.8 we also use the function:

\[
\tilde{f}_{4,n,l}(z) = \int_{(\pi, -\pi)^d} (-\hat{G}_z(k)) \hat{D}^l(k) \hat{G}_z^n(k) \frac{d^dk}{(2\pi)^d}.
\]

In our implementation we use besides the bootstrap functions \(f_1, f_2\) also the functions \(\tilde{f}_{3,1,0}, \tilde{f}_{3,1,1}, \tilde{f}_{3,1,2}, \tilde{f}_{3,1,3}, \tilde{f}_{3,2,0}, \tilde{f}_{3,2,1}, \tilde{f}_{3,2,2}, \tilde{f}_{3,2,3}\) and \(\tilde{f}_{4,1,4}, \tilde{f}_{4,2,4}\).
5.4.3 Comments for percolation

Regarding the computational aspect for percolation we only want to comment on one feature, namely the bound on $G_{1,z}(e_1) = \tau_{1,z}(e_1)$. We have already seen that $G_{n,z}(e_1)$ occurs in many bounds. For percolation this is especially true as

$$\rho = P_z(e_1 \notin C(0)) = 1 - \tau_{1,z}(e_1). \quad (5.4.18)$$

For a good bound on $\rho$ we need a lower bound on $\tau_{1,p}(e_1)$. We produce such a bound by considering all possible paths from 0 to $e_1$ in at most five steps and using that $z \geq z_I = (2d - 1)^{-1}$. We use the following bound

$$\tau_{1,z}(e_1) \geq c_1(e_i) z_I + c_3(e_i) z_I^3 (1 - z_I) + c_5(e_i) z_I^5 (1 - z_I)^2, \quad (5.4.19)$$

where the factors $(1 - z_I)$ are needed to prevent overcounting: In the bound we can considering an open path with three steps only when the bond $(0,e_1)$, that makes the direct connection, is closed.

Bootstrap function used for Section 3.5

To perform the analysis of Section 3.5 we compute $\bar{f}_{3,n,l}$ for $(n,l) : (0,0), (1,0), (1,2), (1,2), (1,3)$. To improve our bounds further we also use the function

$$\bar{f}_{4,1,4}(z) = \int_{(\pi,-\pi)^d} (- \Delta \hat{G}_z(k)) \hat{D}^l(k) \hat{G}_z^n(k) \frac{d^d k}{(2\pi)^d}. \quad (5.4.20)$$

for the bootstrap. In total we thus use eight function for the bootstrap.

5.5 Discussion

In this chapter we have derived bounds on the two-point function and on the repulsive diagrams. We use the bootstrap assumption ($f_i(z) \leq \Gamma_i$) to bound the diagrammatic bounds, proven in Chapter 4, numerically. These bounds are then used as $\beta_*$, see Assumptions 3.2.8 and 3.3.3, to perform the analysis in Chapter 3. In Section 5.2 we compute the value of the SRW-integral following the ideas of [41] and in Section 5.2.3 we derive bounds on SRW-integrals required for the analysis of Section 3.5. In contrast to the technique that Hara and Slade use to prove mean-field behavior for SAW, we need to compute just a small number of integrals, because we have formulated our bounds such that they have a certain monotonicity in $x$.

Then, we explained how to compute the sum of the matrix-valued bounds by computing the eigensystem of the matrices involved and discussed some model-dependent numerical features.
5.5 Discussion

The monotonicity of the bound in \( d \). To prove the infrared bound for all dimensions \( d \geq d_0 \) we use the monotonicity of the bounds used for the analysis of Section 3.3. We will now explain why the bounds on the coefficients \( \beta \) and the constants of Proposition 3.3.1 are monotone decreasing in \( d \).

We first note that

\[
I_{n,l}(x) = (C^n \star D^l)(x), \quad \frac{2d}{2d-1} \Gamma_1, \quad c_n(x) \left( \frac{\Gamma_1}{2d-1} \right)^n \quad (5.5.1)
\]

are decreasing in \( d \) for all \( x \in \mathbb{Z}^d \) and \( n,l \in \mathbb{N} \).

Let us assume that there exist \( d' \) and \( \Gamma_1 \) such that \((2d - 1)z_c(d) \leq \Gamma_1\) for all \( d \geq d' \). Reviewing the bounds on the repulsive diagram we find that this implies that these bounds are all monotone decreasing in \( d \). Using the analysis of Section 3.3 we bound the weighted diagram, as given in \((5.1.18),(5.1.19)\) and \((5.1.39)-(5.1.49)\). Reviewing these bound, we see that they are also decreasing in \( d \). We compute the bounds \( \beta \) by sums and products of diagrams, so that the bounds \( \beta \) are also decreasing in \( d \). When we review the definition the constants \( K \) in Section 3.4 we see that the monotonicity of \( \beta \) causes the constants \( K \) to be decreasing in the \( d \). Thereby, the right-hand side of \((3.3.9)\) is also decreasing in \( d \), which confirms our assumption that we can use the same \( \gamma_i, \Gamma \) for all \( d \geq d' \).

The choice of the constants \( \gamma_i, \Gamma_i \) and \( c_i \). To perform the bootstrap we need to find appropriate values for the constants \( \gamma_i, \Gamma_i \) and \( c_i \). Therefore, we first note that for the implementation we actually do not need to choose a value for \( \gamma_i \). It is enough to prove that \( f_i(z) \leq \Gamma_i \) implies that \( f_i(z) < \Gamma_i \), as we can define \( \gamma_i \) to be the value of the produced bound on \( f_i(z) \).

We find good values for \( \Gamma_i \) and \( c_i \) by starting our Mathematica program in an very high dimensions, say \( d = 100 \), with values that are a bit bigger than the corresponding NBW values, e.g.

\[
\Gamma_1 = 1.1, \quad \Gamma_2 = 1.1, \quad \Gamma_3 = 1.1, \quad c_1 = 0.1, \quad c_2 = 0.55, \quad c_3 = 0.1, \quad c_4 = 5 \quad (5.5.2)
\]

and for the analysis of Section 3.5 we could choose

\[
\bar{\Gamma}_{3,n,l} = 2^{-(l+1)}, \quad \bar{\Gamma}_{4,n,l} = 2^{-(l+1)} \quad (5.5.4)
\]

Using these values we compute bounds on the coefficients and perform the improvement of bounds, which will succeed since we are in a dimension \( d \gg d_0 \). Then, we use the computed bounds on the coefficients to define new \( \Gamma' \) and \( c'_i \), that
are smaller than the original constants, e.g., we define

\[ \Gamma'_1 = \frac{1 + \frac{2d-2}{2d-1} \Gamma_1 \beta_0^{\text{even}}}{\beta_0^{\beta} - \frac{2d-2}{2d} \beta_0^{\text{odd}}} + \epsilon, \quad (5.5.5) \]

\[ \Gamma'_2 \geq \frac{2d - 2}{2d - 1} K |\Phi| K + \epsilon. \quad (5.5.6) \]

We repeat this procedure a number of times using always the new values for \( \Gamma'_i \) and \( c'_i \) and in this way obtain good values for the constants and very good estimates on the coefficient.

Then, we use these constants for a lower dimension, e.g. \( d = 80 \). In the beginning the bootstrap will not succeed as the constants \( \Gamma_i, c_i \) we be to small for this dimension. We simply redefine the constants to values, as in (5.5.5)-(5.5.6), to values for which the bootstrap might succeed and compute bounds on the coefficient using theses constants. We repeat this procedure a number of times and see that the values of the constants converge to values for which the bootstrap succeed.

After having found values for which the bootstraps succeed, we reduce the dimension again, and repeat the algorithm of redefining the constants to value that might work. We repeat the process of lowering of dimension and redefining the constants until we fail to obtain values for the appropriate constants. In this way we can verify in which dimension the NoBLE can prove the infrared bound and also obtain the values of the constants.

This procedure of finding the right value in a given dimension can be automatized. On the website of the author you can download the implementations of analysis that include a semi-automate version of an algorithm to compute the constants.

**Possible improvements.** The numerical bounds stated in this chapter can be further improved. In Section 5.1.2 we have improved the bounds by extracting explicit terms of the repulsive diagrams. These improved bounds arise from simple combinatorial arguments. Besides this, the author is aware of two other ways to improve the bounds. We can also improve the bounds by creating explicit lower bounds on the two-point functions as done in [41]. To do this for models other than SAW is analytically quite demanding.

Another way to improve the bounds for SAW and percolation is to use the differential inequality (3.2.17) for \( z \in (z_l, z_c - \epsilon) \) as follows:

\[ G_z(x) = G_{z_l}(x) + (z_c - z_l) \sup_{z \in (z_l, z_c - \epsilon)} \frac{d}{dz} G_z(x) \]

\[ = G_{z_l}(x) + 2d(z_c - z_l)(G_{z_c - \epsilon} \star D \star G_{z_c - \epsilon})(x). \quad (5.5.7) \]
**Numerical error.** Using (and trusting) Mathematica we omitted a detailed discussion of the numerical error. Altogether, our implementation include only two computations that could lead to noticeable error, namely the computation of $I_{n,l}(x)$ and the computations of the series explained in Section 5.3. All other steps of the computations require only simple addition and multiplication of positive terms, so that any further error should be negligible.

To evaluate the sum of the bounds $\beta_{\text{abs}}$ we compute the eigensystem of the bounding matrices $\mathbf{B}, \tilde{\mathbf{B}}$ and use the geometric series. When computing the geometric series a small numerical error of the computed eigensystem could create a bigger error. However, the entries of the bounding matrices, and thereby also the biggest eigenvalues, are much smaller than 1. Further, these sums are then multiplied by terms of the order $O(1/d)$, so that any created error should be insignificant. For the computation of $I_{n,l}(x)$ we use build-in function of Mathematica, for which we can specify with which precision we want to perform the computations. We choose a precision of 30 digits, which should be more than sufficient for our computations.

Looking at the result of our computation we also see that the implementations are quite robust. Minor changes of the constants $\Gamma_i, c_i$ and of the form of the bounds on the coefficients, could always be compensated by choosing slightly different values for the constants $\Gamma_i, c_i$. Therefore, we are convinced that the numerical error did not influence our results.
Numerical features
Chapter 6
Alternative expansion for SAW

6.1 The idea

When reviewing the classical lace expansion (LE) for self-avoiding walk (SAW), as introduced by Brydges and Spencer [22], Remco van der Hofstad and the author had the idea to recombine the lace-expansion coefficients to extract all the two-step loop contributions. The expansion obtained in this way turns out to be more efficient than the classical lace expansion and the NoBLE, as derived in Section 2.2. By more efficient, we mean that we can use it to obtain better numerical bounds on the coefficients and on the critical value \( z_c \).

Figure 6.1: Structure of a walk contributing to \( \Pi_z \)

To understand the idea of the recombination of the expansion let us review the coefficients of the classical lace expansion \( \Pi_z(x) \). A walk contributing to \( \Pi_z(x) \) is a simple random walk (SRW) with multiple overlapping intersections. These intersections create a sequence of loops, as displayed in Figure 6.1. While for the NoBLE all the loops consist of at least four steps, the loops of the LE can also consist of only two steps. A two-step loop can only be created by a backtracking of the walk, i.e. a simple back and forth traversal. Removing a two-step loop divides the diagram into two pieces. The only dependence between these two pieces is the direction and the starting point of the two-step loop.
We recombine the coefficients by splitting them at each two-step loop. We denote the coefficient that remains after this extraction of the two-step loop by $\Phi_{i,\kappa}^k$. The loops contributing to $\Phi_{i,\kappa}^k$ consist of at least four steps, so that the coefficient $\Phi_{i,\kappa}^k$ is, like in the NoBLE coefficients, of the order $1/d^2$.

We first introduce the LE in Section 6.2 and then derive the new expansion in Section 6.3. After this, we discuss how to bound the coefficients and explain the analysis needed to obtain the infrared bound.

We should be able to create such an expansion also for lattice trees and animals. It is a priori unclear whether such an expansion for these models is also more efficient than the NoBLE. For percolation, not being a combinatorial model, it is not clear whether such an expansion can be created at all.

6.2 The classical lace expansion

The following definitions are a condensed version of [92, Chapter 3].

**Definition 6.2.1** (Set of simple random walks). Let $\mathcal{W}^{\text{SRW}}$ be the set of all SRWs. We define the subsets of SRWs with a certain length $n$ and/or with end point $x$ to be $\mathcal{W}^{\text{SRW}}_n$, $\mathcal{W}^{\text{SRW}}_n(x)$, and $\mathcal{W}^{\text{SRW}}_n(x)$ respectively.

**Definition 6.2.2** (Edges, graphs, connected graphs, and laces). Let $a, b \in \mathbb{N}$ with $a < b$. For $s, t \in [a, b] \cap \mathbb{N}$ with $s < t$ the edge between $s$ and $t$ is the tuple $(s, t)$. We abbreviate $st$ instead of $(s, t)$. The length of an edge $st$ is given by $t - s$. We call a set of edges a graph. We denote by $\mathcal{B}[a, b]$ the set of all graphs on $[a, b]$. We say that a graph $\Gamma \in \mathcal{B}[a, b]$ is connected, if, for all $i \in (a, b)$, there exists an edge $st \in \Gamma$ such that $s < i < t$. We denote by $\mathcal{G}[a, b]$ the set of all connected graphs on $[a, b]$.

We call a connected graph minimally connected or lace, if the removal of any edge would disconnect the graph. We denote by $\mathcal{L}[a, b]$ the set of all minimally connected graphs on $[a, b]$.

**Definition 6.2.3** (Laces, corresponding graphs, and compatible edges). We define the function $L: \mathcal{G}[a, b] \rightarrow \mathcal{L}[a, b]$ in a constructive manner: Let $\Gamma \in \mathcal{G}[a, b]$. Define $s_1 = a$, $t_1 = \max\{t : at \in \Gamma\}$ and for

$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$$ 

This procedure terminates when $t_{i+1} = b$. We denote by $L(\Gamma)$ the resulting graph $\{s_1 t_1, \ldots, s_N t_N\}$. Given a lace $L$, the set of all edges $st \not\in L$ such that $L(L \cup \{st\}) = L$ is denoted by $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be compatible with $L$. 
For a simple random walk \( \omega \) and \( s, t \geq 0 \) we define \( \mathcal{U}_{s,t}(\omega) = -\delta_{\omega_s, \omega_t} \) and

\[
J[0, n] = \sum_{L \in \mathcal{L}[0, n]} \prod_{s \in L} \mathcal{U}_{s,t} \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s', t'}). \tag{6.2.1}
\]

In [92, Section 3.3] it is shown that the lace-expansion coefficient \( \pi_n \) is given by

\[
\pi_n(x) = \sum_{\omega \in \mathcal{W}_{SR}(x)} J[0, n](\omega). \tag{6.2.2}
\]

The generating function \( \Pi_z \) of \( \pi_n \) is given by

\[
\Pi_z(x) = \sum_{\omega \in \mathcal{W}_{SR}(x)} J[0, |\omega|](\omega) z^{|\omega|} = \sum_{\omega \in \mathcal{W}_{SR}(x)} \prod_{s \in L} \mathcal{U}_{s,t} \prod_{s' \in \mathcal{E}(L)} (1 + \mathcal{U}_{s', t'}) z^{|\omega|}. \tag{6.2.3}
\]

This coefficient can be used to characterize the two-point function as

\[
\hat{\mathcal{G}}_z(k) = \frac{1}{1 - 2dz \hat{D}(k) - \hat{\Pi}_z(k)}. \tag{2.1.9}
\]

In the next section we derive a new expansion by creating a new characterization for \( \hat{\Pi}_z(k) \).

### 6.3 The derivation of the new expansion

We extract the contributions of all two-step loops by splitting each lace \( L \) in (6.2.3) into connected subgraphs that have either only edges of length two or only edges longer than two.

**Definition 6.3.1.** Let \( a, b \in \mathbb{N} \) with \( a < b \). Let \( R^{(N)}[a, b] \subset \mathcal{L}[a, b] \) be the set of laces \( L \) having exactly \( N \) edges, where each edge \( s \in L \) is longer than two, i.e., \( t - s > 2 \). We define \( r^{(N)}[a, b] \) to be the set of all laces in \( \mathcal{L}[a, b] \) that have exactly \( N \) edges and all of these edges have length two.

By definition \( r^{(N)}[0, N + 1] = \{[02, 13, 24, \ldots, (N - 2)N, (N - 1)(N + 1)] \} \) and \( \mathcal{L}^{(0)}[a, a] = \emptyset \).

Any lace in \( \mathcal{L}[a, b] \) can be split into a combination of laces from \( R^{(N)} \) and \( r^{(M)} \). For example, we split the lace \([0, 4), (2, 10), (9, 11), (10, 12), (11, 13), (12, 40), (39, 40)\) into \( L_1 = \{(0, 4), (2, 10)\}, L_2 = \{(12, 40)\} \) and \( L_1 = \{(9, 11), (10, 12), (11, 13)\}, L_2 = \{(39, 40)\}. \) As we can see in this example, the initial lace can be retrieved by recombining the results of the split. Thus, there exists a bijection between \( \mathcal{L}[a, b] \) and

\[
\bigcup_{K=0}^{\infty} \bigcup_{(\vec{a}, \vec{b}, \vec{\beta}) \in \text{AB}(K)} \bigcup_{(\vec{M}, \vec{N}) \in \text{NM}(K)} r^{(M_0)}[a_0, b_0] \cup \left( \bigcup_{i=1}^{K} R^{(N_i)}[\alpha_i, \beta_i] \cup r^{(M_i)}[a_i, b_i] \right), \tag{6.3.1}
\]
We describe the non-trivial two-step loops in the middle of the diagram by weights \( z \) with the coefficient \( \Psi \). For the two-step loops we define for indices \( \iota, \kappa \) to capture the contribution of long loops using the following coefficient: For \( R \)-laces in need to be non-trivial, thus \( M \) for \( K \)-pieces:

\[
\begin{align*}
\Phi_z^{(N),\iota,\kappa}(x) &= \sum_{\omega \in \mathcal{H}^{SRW}(x)} \mathbb{I}_{[0,\omega_1=\iota, \omega_{|\omega|-1}=\kappa, x-e_{\kappa}]} \\
&\times \prod_{L \in R^{(N)}[0,|\omega|]} \prod_{s,t \in L} \mathcal{U}_{st}(\omega) (1 + \mathcal{U}_{s't'}(\omega)) z^{[\omega]}.
\end{align*}
\]

For the two-step loops we define for \( \iota, \kappa \in \{\pm 1, \cdots, \pm d\} \) and \( N \geq 1 \) and \( M \geq 0 \):

\[
\begin{align*}
\Psi_z^{(N),\iota,\kappa}(x) &= \sum_{\omega \in \mathcal{H}^{SRW}(x)} \mathbb{I}_{[0,\omega_1=\iota, \omega_{|\omega|-1}=\kappa, x-e_{\kappa}]} \\
&\times \prod_{L \in R^{(N)}[0,|\omega|]} \prod_{s,t \in L} \mathcal{U}_{st}(\omega) z^{[\omega]-2} \\
\Psi_z^{(N),\iota,\kappa}(x) &= \delta_{0,M} \delta_{0,\omega} \delta_{\iota,\kappa} \psi_{\omega,\iota,\kappa} + (1 - \delta_{0,M}) z \psi_{\iota,\kappa}.
\end{align*}
\]

We use this bijection between \( \mathcal{L}[a, b] \) and the set in \ref{6.3.1} to expand \( \Pi_{z} \). We capture the contribution of long loops using the following coefficient: For \( \iota, \kappa \in \{\pm 1, \cdots, \pm d\} \) and \( N \geq 1 \) we define

\[
\begin{align*}
\Phi_z^{(N),\iota,\kappa}(x) &= \sum_{\omega \in \mathcal{H}^{SRW}(x)} \mathbb{I}_{[0,\omega_1=\iota, \omega_{|\omega|-1}=\kappa, x-e_{\kappa}]} \\
&\times \prod_{L \in R^{(N)}[0,|\omega|]} \prod_{s,t \in L} \mathcal{U}_{st}(\omega) (1 + \mathcal{U}_{s't'}(\omega)) z^{[\omega]}.
\end{align*}
\]

For the two-step loops we define for \( \iota, \kappa \in \{\pm 1, \cdots, \pm d\} \) and \( N \geq 1 \) and \( M \geq 0 \):

\[
\begin{align*}
\Psi_z^{(N),\iota,\kappa}(x) &= \sum_{\omega \in \mathcal{H}^{SRW}(x)} \mathbb{I}_{[0,\omega_1=\iota, \omega_{|\omega|-1}=\kappa, x-e_{\kappa}]} \\
&\times \prod_{L \in R^{(N)}[0,|\omega|]} \prod_{s,t \in L} \mathcal{U}_{st}(\omega) z^{[\omega]-2} \\
\Psi_z^{(N),\iota,\kappa}(x) &= \delta_{0,M} \delta_{0,\omega} \delta_{\iota,\kappa} \psi_{\omega,\iota,\kappa} + (1 - \delta_{0,M}) z \psi_{\iota,\kappa}.
\end{align*}
\]

We describe the non-trivial two-step loops in the middle of the diagram by \( \psi_z^{(N),\iota,\kappa} \). The coefficient \( \psi_z^{(N),\iota,\kappa} \) describes the possible trivial first and last piece. The indices \( \iota, \kappa \) denote the orientation of the last and first step of the two-step loop. For \( \psi_z^{(N),\iota,\kappa}(x) \), one of the steps and in \( \psi_z^{(N),\iota,\kappa} \) two steps are not weighted by \( z \). These weights \( z \) contribute to the neighboring \( \Phi_z^{(N),\iota,\kappa}(x) \). We define

\[
\begin{align*}
\Phi_z^{(N),\iota,\kappa}(x) &= \sum_{N=1}^{\infty} (-1)^N \Phi_z^{(N),\iota,\kappa}(x), \\
\psi_z^{(N),\iota,\kappa}(x) &= \sum_{N=1}^{\infty} (-1)^N \psi_z^{(N),\iota,\kappa}(x) = -\frac{\delta_{\iota,\kappa} \delta_{x,0}}{1 - z} + \frac{\delta_{\iota,\kappa} \delta_{x,e_{\kappa}}}{1 - z^2}, \\
\psi_z^{(N),\iota,\kappa}(x) &= \sum_{N=0}^{\infty} (-1)^N \psi_z^{(N),\iota,\kappa}(x) = \frac{\delta_{\iota,\kappa} \delta_{x,0}}{1 - z} - \frac{\delta_{\iota,\kappa} \delta_{x,e_{\kappa}}}{1 - z^2} = -\psi_z^{(N),\iota,\kappa}(-x).
\end{align*}
\]
We conclude from (6.3.6)-(6.3.7) that

\[ X(K,x) := \{(y_0, y_1, \ldots, y_K, x_1, x_2, \ldots, x_K) \in \mathbb{Z}^{2K+1} | y_0 + \sum_{i=1}^{K} (x_i + y_i) = x \}. \] (6.3.8)

As index set for the orientations of the two-step loops we define

\[ IK(K) := \{(t_0, \ldots, t_K, k_1, k_2, \ldots, k_K) \in \{ \pm 1, \ldots, \pm d \}^{2K+1} \}. \] (6.3.9)

We split the walk at each point where a lace in \( R \) begins/ends and obtain in a straightforward manner that

\[ \Pi_z(x) = \sum_{l_z} z^{2} Y_{z}^{l_z}(x) + \sum_{K=1}^{\infty} \sum_{IK(K)} \sum_{X(K,x)} \Psi_{z}^{l_0,K_1}(y_0) \left( \prod_{s=1}^{K-1} \Phi_{z}^{k_x,s}(x_s) Y_{z}^{l_z,k_{s+1}}(y_s) \right) \]
\[ \times \Phi_{z}^{k_x,k}(x_K)(\Psi_{z}^{l_z,k_x-1}(y_K) + \Psi_{z}^{l_z,k_x}(y_K)), \] (6.3.10)

where we abbreviate, e.g.,

\[ \sum_{IK(K)} \quad \text{to denote} \quad \sum_{(l_0, \ldots, l_K, k_1, k_2, \ldots, k_K) \in IK(K)}. \] (6.3.11)

We apply the Fourier transformation and write the resulting equation in matrix form as

\[ \hat{\Pi}_z(k) = z^{2} \hat{I}^{T}\hat{Y}_z(k)\hat{I} + \sum_{j=0}^{\infty} \hat{I}^{T}\hat{\Psi}_z(k) \left( \hat{\Phi}_z(k) \hat{Y}_z(k) \right)^{j} \hat{\Phi}_z(k) \hat{\Psi}_z(-k) \hat{I}, \] (6.3.12)

with

\[ \left( \hat{\Phi}_z(k) \right)_{l_z} = \hat{\Phi}_{z}^{l_z}(k), \quad \left( \hat{Y}_z(k) \right)_{l_z} = \hat{Y}_{z}^{l_z}(k), \quad \left( \hat{\Psi}_z(k) \right)_{l_z} = \hat{\Psi}_{z}^{l_z}(k). \]

We conclude from (6.3.6)-(6.3.7) that

\[ \hat{Y}_z(k) = \frac{1}{1 - z^{2}} (z \hat{D}(-k) - \hat{I}) = -\hat{\Psi}_z(-k) \hat{J} = -\hat{J} \hat{\Psi}_z(k), \]

and use that \( \hat{J} \hat{I} = \hat{I} \) to obtain

\[ \hat{\Pi}_z(k) = z^{2} \hat{I}^{T}\hat{Y}_z(k)\hat{I} + \sum_{j=1}^{\infty} \hat{I}^{T}\hat{Y}_z(k) \left( \hat{\Phi}_z(k) \hat{Y}_z(k) \right)^{j-1} \hat{I} \] (6.3.13)
\[ = z^{2} \hat{I}^{T}\hat{Y}_z(k)\hat{I} + \hat{I}^{T}\hat{Y}_z(k) \left[ \mathbf{I} - \hat{\Phi}_z(k) \hat{Y}_z(k) \right]^{-1} \hat{\Phi}_z(k) \hat{Y}_z(k) \hat{I}, \] (6.3.14)
where the dominant contribution comes from walks that only backtrack, i.e. \( \omega = (0, e_\iota, 0, e_\iota, \ldots) \) for some \( \iota \). As a result,

\[
z^2 \hat{\Upsilon}_z(k) \hat{\Gamma} = -\frac{2dz^2}{1-z^2} + \frac{2dz^3 \hat{D}(k)}{1-z^2}.
\]

(6.3.15)

Therefore, we know that

\[
\hat{G}_z(k) = \frac{1}{1 - 2d z \hat{D}(k) + \frac{2dz^2}{1-z^2} - \frac{2dz^3 \hat{D}(k)}{1-z^2} - \hat{Q}_z(k)}
\]

\[
= \frac{1}{(1 + (2d - 1)z^2)(1 - 2d z \hat{D}(k)) - (1 - z^2) \hat{Q}_z(k)}.
\]

(6.3.16)

with

\[
\hat{Q}_z(k) = \hat{\Gamma}^T \hat{\Upsilon}_z(k)[I - \hat{\Phi}_z(k) \hat{\Upsilon}_z(k)]^{-1} \hat{\Phi}_z(k) \hat{\Upsilon}_z(k) \hat{\Gamma}.
\]

(6.3.17)

If \( \hat{Q}_z \) is small, then (6.3.16) is only a perturbation of the non-backtracking random walk (NBW) equivalent

\[
\hat{B}_\mu(k) = \frac{1 - \mu^2}{1 + (2d - 1)\mu^2 - 2d\mu \hat{D}(k)}.
\]

(1.5.18)

To make the comparison to the NBW two-point function for the NoBLE, we use the matrix characterization (1.5.16). In this way the expansion introduced here is much more explicit.

### 6.4 Bounds on \( \Phi_z \)

In this section we state the bound \( \Phi_z \) and explain how to prove them. To state the bounds we use the repulsive diagrams defined in Section 4.2.1. For this section we assume that \( |\iota| \neq |\kappa| \).

#### Bounds on \( \Phi_z^{(1)} \)

We know that each walk \( \omega \) contributing to \( \hat{\Phi}_z^{(1),\iota,K}(0) \) is self-avoiding, except at one intersection in the starting and final point. Further, we know that \( n = |\omega| \geq 4 \) and \( \omega_1 = e_\iota, \omega_{n-1} = -e_\kappa \). Thereby, the following bounds hold:

\[
\hat{\Phi}_z^{(1),\iota,\iota}(0) = z^2 G_{4,z}^l(2e_\iota),
\]

\[
\hat{\Phi}_z^{(1),\iota,K}(0) = z^2 G_{2,z}^l(e_\iota + e_\kappa),
\]

\[
\sum_{\iota,\kappa} \Phi_z^{(1),\iota,K}(x)[1 - \cos(k \cdot x)] = 0.
\]

\[
\sum_{\iota,\kappa} \hat{\Phi}_z^{(1),\iota,K}(0) = zG_{3,z}(e_\iota),
\]

\[
\hat{\Phi}_z^{(1),\iota,K}(0) = 0,
\]

\[
\sum_{\kappa} \hat{\Phi}_z^{(1),\iota,K}(0) = zG_{3,z}(e_\iota),
\]
6.4 Bounds on $\Phi^2_z$

The bounds on $\Phi^2_z$ are proven like shown in the proof of Lemma 4.2.2. We first extract the special case that $x \in \mathbb{Z}^d$ is a neighbor of the origin and then treat the general case. To obtain the following bounds we consider the three cases $x = e_i$, $x = -e_i$, $x = e_\rho$ with $|\rho|:

$$
\sum_{x : \|x\|_1 = 1} \Phi^{(2),i,l}(x) \leq z^2 G_{3,z}(e_i) + z^2 G_{4,z}^l(2e_i)^2 G_{1,z}(e_i) \\
+ (2d - 2) z^2 G_{2,z}^l(e_i + e_\kappa)^2 G_{1,z}(e_i) \equiv B^{(2)}_i
$$

(6.4.1)

$$
\sum_{x : \|x\|_1 = 1} \Phi^{(2),l,-l}(x) \leq 2z^2 G_{4,z}^l(2e_i) G_{3,z}(e_i) + (2d - 2) z^2 G_{1,z}(e_i) G_{2,z}^l(e_i + e_\kappa)^2 \equiv B^{(2)}_{-l}.
$$

(6.4.2)

For $\kappa$ with $|\kappa| \neq |l|$ we consider five cases $x = \pm e_i, x = \pm e_\kappa, x = \pm e_\rho, |\rho| \not\in \{|l|, |\kappa|\}$ and obtain the bound

$$
\sum_{x : \|x\|_1 = 1} \Phi^{(2),i,l,K}(x) \leq 2z^2 G_{3,z}(e_i + e_\kappa) G_{1,z}(e_i)(G_{3,z}(e_i) + G_{4,z}^l(2e_i)) \\
+ (2d - 4) z^2 G_{2,z}^l(e_i + e_\kappa)^2 G_{1,z}(e_i) \equiv B^{(2)}_K.
$$

(6.4.3)

Summing over $\kappa$, the constraint $\omega_{|\omega|} = x - e_\kappa$ can be omitted and we obtain the bound

$$
\sum_{\kappa} \sum_{x : \|x\|_1 = 1} \Phi^{(2),i,l,K}(x) \leq z G_{3,z}(e_i) G_{1,z}(e_1) + z G_{4,z}^l(2e_1) G_{1,z}(e_1)^2 \\
+ (2d - 2) z G_{2,z}^l(e_1 + e_2)(z + G_{1,z}(e_1)) G_{3,z}(e_1) \equiv B^{(2)}.
$$

(6.4.4)

Considering these special cases for the nearest-neighbors we obtain the following bound when we consider the sum of $\Phi^{(2),i,l,\rho}(x)$ over all $x \in \mathbb{Z}^d$ we use the bound

$$
- \sum_{x \in \mathbb{Z}^d} \Phi^{(2),i,l,\rho}(x) \leq B^{(2)}_\rho + z^2 \sup_{x \in \mathbb{Z}^d} G_{1,z}(x)(G_{2,z} \ast G_{1,z})(e_\rho),
$$

(6.4.5)

for $\rho \in \{i, -i, \kappa\}$. When summing over $i, \kappa$, the condition on the first and final step become obsolete. Extracting the contributions from the direct neighbors of the origin we obtain the bound

$$
- \sum_{i,\kappa} \sum_{x} [1 - \cos(k \cdot x)] \Phi^{(2),i,l,K}(x) \\
\leq 2dz^2 [1 - \hat{D}(k)] B^{(2)} + \sum_{x \in \mathbb{Z}^d} G_{2,z}(x)^3 [1 - \cos(k \cdot x)].
$$

(6.4.6)
Bounds on $\Phi_{z}^{N_i,k}$ for $N \geq 3$

For $N \geq 3$ we bound the coefficient using the same technique as used for the NoBLE (see Section 4.2). We define the vectors $\vec{v}', \vec{w}, \vec{w}_k, \vec{u}$ and the matrix $A$ by

$$\vec{v}' = \begin{pmatrix} \frac{z}{2d} G_{3,z}(e_i)^2 + z \sup_x G_{1,z}(x) \mathcal{B}_{1,2}(e_i) \\ z(2G_{3,z}(e_i) + \mathcal{B}_{2,1}(e_i)) \sup_x G_z(x) \end{pmatrix}$$  \quad (6.4.7)

$$A = \begin{pmatrix} G_{3,z}(e_1) & \max[zD(e_1), \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \mathcal{B}_{1,1}(x)] \\ G_{3,z}(e_1) + \mathcal{B}_{1,2}(e_1) & \sup_{x \in \mathbb{Z}^d} \mathcal{B}_{0,2}(x) \end{pmatrix}$$  \quad (6.4.8)

$$\vec{w} = \begin{pmatrix} 2dzG_{3,z}(e_1) \\ \mathcal{B}_{2,1}(0) \end{pmatrix}$$  \quad (6.4.9)

$$\vec{u} = \begin{pmatrix} \frac{2dz[1-B(k)]G_{3,z}(e_1)}{\sup_x[1-\cos(k \cdot x)]G_z(x)} + \mathcal{B}_{1,2}(e_1) + \mathcal{B}_{1,2}(e_1) \\ \sup_{x \in \mathbb{Z}^d} \mathcal{B}_{0,1}(x) \end{pmatrix}$$  \quad (6.4.10)

and $\vec{w}_k = \vec{w}/(2d)$. The vector $\vec{v}'$ represents the first four lines of the diagram (the path $0 \to e_i \to x_1 \to 0 \to x_2$). The matrix $A$ represents the intermediate pieces of the diagram. The entry $(A)_{i,j}$ represents a bubble. If $i = 1$, then the length of the first line is 1 and at least two for $i = 2$. If $j = 1$, the bubble ends at a neighbor of the origin. The vector $\vec{w}^k$ represents the final loop of the diagram of $\Phi_{z}^{N_i,k}$. We define $\vec{w}_k$ by $\vec{u}/(2d)$, which is a closed bubble without any constraints on the last step, using a symmetry argument.

**Lemma 6.4.1.** Let $N \geq 0$ and $\vec{v}, A, \vec{w}, \vec{u}$ be as given above. Then

$$|\Phi_{z}^{(N+3),k}(0)| \leq \vec{v}' A^N \vec{w}^k$$  \quad (6.4.11)

and

$$\sum_{x \in \mathbb{Z}^d} \frac{1 - \cos(k \cdot x)}{|\Phi_{z}^{(N+3),k}(x)|} \leq (N + 1) \sum_{i=0}^{N} \sum_{j=0}^{N} \vec{w}^T (A^T)^{2i} \mathbb{1} \mathbb{1}^T A^{2(N-i)} \vec{w},$$  \quad (6.4.12)

$$\sum_{x \in \mathbb{Z}^d} \frac{1 - \cos(k \cdot x)}{|\Phi_{z}^{(N+4),k}(x)|} \leq (N + 2) \sum_{i=0}^{N} \vec{w}^T (A^T)^{2i} \mathbb{1} \mathbb{1}^T A^{2(N-i)+1} \vec{w} + (N + 2) \vec{w}^T (A^T)^{2N+1} \vec{u}.$$  \quad (6.4.13)

We leave it to the reader to adapt the steps in Section 4.2 to prove these bounds.

### 6.5 Bounds on $\hat{\Pi}_z$

#### 6.5.1 Bounds on $\hat{\Pi}_z(k)$

In this section we derive bounds on $\hat{\Pi}_z$ using the bounds for $\Phi_{z}$ stated in the last section. To perform the bootstrap for $f_1$ we create a bound on the special case
6.5 Bounds on $\hat{\Pi}_z$

$k = 0$. The sum of each row and column of the matrix $\hat{\Phi}_z(0)$ is given by $\sum_k \hat{\Phi}^{i,k}_z = \Phi^i$. Thereby, we know that

$$
\hat{Y}_z(0)\mathbf{1} = \frac{-1}{1+z}\mathbf{1}, \quad \hat{\Phi}_z(0)\mathbf{1} = \Phi^i\mathbf{1}, \quad (6.5.1)
$$

$$
[\mathbf{I} - \hat{\Phi}_z(0)\hat{Y}_z(0)]^{-1}\mathbf{1} = \sum_{j=0}^{\infty} (\hat{\Phi}_z(0)\hat{Y}_z(0))^j\mathbf{1} = \sum_{j=0}^{\infty} \left(\frac{-\Phi^i}{1+z}\right)^j\mathbf{1} = \frac{1}{\Phi^i+1}\mathbf{1}. \quad (6.5.2)
$$

We use this and the characterization of $\hat{\Pi}_z$ in \[6.3.14\] to compute

$$
\hat{\Pi}_z(0) = z^2\mathbf{1}^T\hat{Y}_z(0)\mathbf{1} + \mathbf{1}^T\hat{Y}_z(0)\left[\mathbf{I} - \hat{\Phi}_z(0)\hat{Y}_z(0)\right]^{-1}\hat{\Phi}_z(0)\hat{Y}_z(0)\mathbf{1}
$$

$$
= -\frac{2dz^2}{1+z} + \frac{2d}{1+z} + \frac{\Phi^i}{1+z+\Phi^i}. \quad (6.5.3)
$$

To obtain a bound on $\hat{\Pi}_z(k)$ for all $k$ we define a matrix $C$ with entries

$$
C_{i,k} := \sum_{N=1}^{\infty} \text{Bound on } \hat{\Phi}^{(N),i,k}_z(0), \quad (6.5.4)
$$

where we use the bounds on $\hat{\Phi}^{(N),i,k}_z(x)$ given in Section\[6.4\]. By construction of $C$ we know that $|\hat{\Phi}^{i,k}_z(k)| \leq C_{i,k}$ and that the row and column sums of $C$ are constant. In the following, we denote by $C_B$ the row sum of $C$. We take the absolute value in \[6.3.13\] and obtain the following bound:

$$
|\hat{\Pi}_z(k)| \leq z^2\mathbf{1}^T\frac{\mathbf{J} + z\mathbf{I}}{1-z^2}\mathbf{1} + \sum_{j=0}^{\infty} \mathbf{1}^T\frac{\mathbf{I} + z\mathbf{J}}{1-z^2} \left(\frac{\mathbf{J} + z\mathbf{I}}{1-z^2}\right)^j C \frac{\mathbf{I} + z\mathbf{J}}{1-z^2}\mathbf{1}
$$

$$
= \frac{2dz^2}{1-z} + \frac{2dC_B}{(1-z) (1-z-C_B)}. \quad (6.5.5)
$$

6.5.2 Bounds on $\hat{\Pi}_z(0) - \hat{\Pi}_z(k)$

The idea of the bound.

In this section we derive a bound on $\hat{\Pi}_z(0) - \hat{\Pi}_z(k)$ using the characterization \[6.3.14\] and \[6.3.17\]:

$$
\hat{\Pi}_z(0) - \hat{\Pi}_z(k) = z^2\mathbf{1}^T\hat{Y}_z(0)\mathbf{1} + \mathbf{Q}_z(0) - z^2\mathbf{1}^T\hat{Y}_z(k)\mathbf{1} - \mathbf{Q}_z(k) \quad (6.5.6)
$$

The dominant NBW contribution can be easily computed as

$$
z^2\mathbf{1}^T\hat{Y}_z(0)\mathbf{1} - z^2\mathbf{1}^T\hat{Y}_z(k)\mathbf{1} = \frac{2dz^3}{1-z^2}[1 - \hat{D}(k)]. \quad (6.5.7)
$$

To compute a bound for the minor contributions is demanding. We define

$$
\bar{v}_k = \hat{Y}_z(k)\mathbf{1}, \quad A_k = \hat{\Phi}_z(k)\hat{Y}_z(k) \quad B_k = [\mathbf{I} - \hat{\Phi}_z(k)\hat{Y}_z(k)]^{-1}. \quad (6.5.8)
$$
and write the difference of the remainder terms, defined in (6.3.17), as

\[ \hat{Q}_z(0) - \hat{Q}_z(k) = \vec{v}_0^T B_0 \Phi_z(0) \vec{v}_0 - \vec{v}_k^T B_k \Phi_z(k) \vec{v}_k. \]  (6.5.9)

We expand this difference as follows:

\[
\begin{aligned}
\vec{v}_0^T B_0 \Phi_z(0) \vec{v}_0 - \vec{v}_k^T B_k \Phi_z(k) \vec{v}_k &
\leq (\vec{v}_0^T - \vec{v}_k^T) B_0 \Phi_z(0) \vec{v}_0 + \vec{v}_0^T B_0 \Phi_z(0) (\vec{v}_0 - \vec{v}_k) \\
&\quad + \vec{v}_0^T (\Phi_z(0) - \Phi_z(k)) \vec{v}_0 + \vec{v}_0^T (B_0 - B_k) \Phi_z(0) \vec{v}_0 \\
&\quad - (\vec{v}_0^T - \vec{v}_k^T) (B_0 - B_k) \Phi_z(0) \vec{v}_0 - \vec{v}_0^T (B_0 - B_k) (\Phi_z(0) - \Phi_z(k)) \vec{v}_0 \\
&\quad - \vec{v}_0^T (B_0 - B_k) \Phi_z(k) (\vec{v}_0 - \vec{v}_k) - (\vec{v}_0^T - \vec{v}_k^T) B_k \Phi_z(k) (\vec{v}_0 - \vec{v}_k).
\end{aligned}
\]  (6.5.10)

(6.5.11)

\[ \vec{v}_0^T B_0 \Phi_z(0) (\vec{v}_0 - \vec{v}_k) = - \frac{2dz^3}{1 - z^2 1 + z + \Phi} [1 - \hat{D}(k)], \]  (6.5.15)

\[ \vec{v}_0^T B_0 \Phi_z(0) (\vec{v}_0 - \vec{v}_k) = (\vec{v}_0^T - \vec{v}_k^T) B_0 \Phi_z(0) \vec{v}_0, \]  (6.5.16)

\[ \vec{v}_0^T B_0 \Phi_z(0) (\Phi_z(k)) \vec{v}_0 = \frac{z^2}{1 + z^2} \frac{1}{1 + \Phi} \sum_{i,k} [1 - \cos(k \cdot x)] \Phi^{i,k}(x). \]  (6.5.17)

Since we cannot compute \( B_k \) directly, we will bound the other seven contributions in (6.5.11)-(6.5.14) using matrix norms. In the following paragraphs, we define these norms and bound the norms of the terms occurring in (6.5.11)-(6.5.14). Combining these norms to produce the final bound is straightforward, so that we omit this last step.

Matrix norms.

Let \( \langle \cdot, \cdot \rangle \) be the complex inner product, i.e

\[ \langle \vec{v}, \vec{v} \rangle = \vec{v}^H \vec{v} = \sum_{i=1}^d \left[ \text{Re}(\vec{v}_i)^2 + \text{Im}(\vec{v}_i)^2 \right], \]  (6.5.18)

for \( \vec{v} \in \mathbb{C}^d \), where \( \vec{v}^H \) is the Hermitian of the vector \( \vec{v} \). We use the Euclidean norm, the Operator norm that is induced by the Euclidean norm and the Hilbert-Schmidt norm (in this setting also called Frobenius norm). For \( \vec{v} \in \mathbb{C}^d \) and \( A \in \mathbb{C}^{d \times d} \) these norms are defined by

\[ ||\vec{v}||_2 = \sqrt{\langle \vec{v}, \vec{v} \rangle}, \]  (6.5.19)

\[ ||A||_{op} = \max_{\vec{w} \in \mathbb{C}^d : \|\vec{w}\|_2 = 1} \|A \vec{w}\|_2, \]  (6.5.20)

\[ ||A||_{HS} = \sqrt{\sum_{i,k} \left[ \text{Re}(A_{i,k})^2 + \text{Im}(A_{i,k})^2 \right].} \]  (6.5.21)
By Cauchy-Schwarz we know that \( \|A\|_{\text{op}} \leq \|A\|_{\text{HS}} \) for all matrices \( A \). For two vectors \( \vec{v}, \vec{w} \) and matrices \( A \) and \( B \) we use
\[
|\langle \vec{v}, AB \vec{w} \rangle| \leq \|\vec{v}\|_2 \|A\|_{\text{op}} \|B\|_{\text{op}} \|\vec{w}\|_2. \tag{6.5.22}
\]

To compute the Operator norm we make use of the following property:

**Lemma 6.5.1.** For \( d \geq 1 \) let \( A \in \mathbb{C}^{d \times d} \) be a matrix with a system of \( d \) independent, orthogonal eigenvectors \( \vec{v}_1, \ldots, \vec{v}_d \) with eigenvalues \( \lambda_1, \ldots, \lambda_d \). Then \( \|A\|_{\text{op}} = \max_{i=1,\ldots,d} |\lambda_i| \).

Hereby, we note a similar relation that we are not going to use
\[
\|A\|_{\text{HS}} = \sqrt{\|\lambda_1\|_2^2 + \|\lambda_2\|_2^2 + \cdots + \|\lambda_d\|_2^2}. \tag{6.5.23}
\]

**Proof.** As we assume that there exist \( d \) independent eigenvectors we know that each vector \( w \in \mathbb{C}^d \) can be characterized by a convex combination of the eigenvectors: i.e. there exist \( \alpha_1, \ldots, \alpha_n \) with \( w = \sum_i \alpha_i \vec{v}_i \). We use the triangle inequality to conclude
\[
\|A\vec{w}\|_2 \leq \sum_i |\alpha_i A\vec{v}_i|_2 \leq \sum_i |\alpha_i| |\lambda_i|.
\]

We see that the maximum for \( \vec{w} \) with \( \|\vec{w}\|_2 = 1 \) is clearly obtained, if \( \vec{w} \) corresponds to a normalized eigenvector with eigenvalue \( \lambda = \max_{i=1,\ldots,d} |\lambda_i| \). \( \square \)

**Computation of norms.** Now we compute the norm of the terms occurring in \( (6.5.11)-(6.5.14) \). We note that
\[
[1 - \cos(k)]^2 + \sin^2(k) = 2[1 - \cos(k)], \tag{6.5.24}
\]
and compute
\[
\|\vec{v}_0 - \vec{v}_k\|_2^2 = \frac{z^2}{(1-z^2)^2} \|T^T [I - \hat{D}(k)]\|_2^2 = \frac{z^2}{(1-z^2)^2} \sum_{\iota} \left[ [1 - \cos(k_{\iota})]^2 + \sin^2(k_{\iota}) \right]
\]
\[
= 4d \frac{z^2}{(1-z^2)^2} \left[ 1 - \hat{D}(k) \right]. \tag{6.5.25}
\]

In the following computation we first use Cauchy-Schwarz and then \( (6.5.24) \) to bound
\[
\|\left( \Phi_z(0) - \Phi_z(k) \right)\|_{\text{HS}}^2 = \sum_{i,k} \left( \sum_x \left[ 1 - \cos(k \cdot x) \right] \Phi_z^{i,k}(k) \right)^2 + \left( \sum_x \sin(k \cdot x) \Phi_z^{i,k}(k) \right)^2
\]
\[
\leq 2 \max_{i,k} \sum_x |\Phi_z^{i,k}(x)| \sum_{i,k} \left( \sum_x \left[ 1 - \cos(k \cdot x) \right] |\Phi_z^{i,k}(x)| \right). \tag{6.5.26}
\]
In order bound \( \|B_0 - B_k\|_{\text{HS}} \) we expand \( B_0 - B_k \) and find that
\[
\|B_0 - B_k\|_{\text{OP}} = \left\| (I - A_0)^{-1} - (I - A_k)^{-1} \right\|_{\text{OP}}
\]
\[
= \left\| (I - A_0)^{-1} [A_0 - A_k] (I - A_k)^{-1} \right\|_{\text{OP}} \leq \|B_0\|_{\text{OP}} \|A_0 - A_k\|_{\text{OP}} \|B_k\|_{\text{OP}}. \quad (6.5.27)
\]
Then we bound the three norms one at a time. We recall (6.3.13) and compute
\[
\|A_0 - A_k\|_{\text{OP}} = \left\| \frac{1}{1 - z^2} (z\hat{D}(0) - J) \hat{Y}_z(0) - \frac{1}{1 - z^2} (z\hat{D}(-k) - J) \hat{Y}_z(k) \right\|_{\text{OP}}
\]
\[
\leq \frac{1}{1 - z^2} \left\| \hat{\Phi}_z(0) - \hat{\Phi}_z(k) \right\|_{\text{HS}} + \frac{z}{1 - z^2} \left\| \hat{\Phi}_z(0) - \hat{\Phi}_z(-k) \hat{\Phi}_z(k) \right\|_{\text{OP}}. \quad (6.5.28)
\]
We have already bounded the first norm in (6.5.26). For the second norm we see that
\[
\|\hat{\Phi}_z(0) - \hat{\Phi}_z(k)\|_{\text{HS}} \leq \max_{l,k} \sum_x |\Phi_{i,k}^{l,k}(x)| \sqrt{4d (1 - \hat{D}(k))} + \|\hat{\Phi}_z(0) - \hat{\Phi}_z(k)\|_{\text{HS}}.
\]
Next we compute a bound on \( \|B_k\|_{\text{OP}} \). Using the Neumann-Series we compute
\[
\|B_k\|_{\text{OP}} \leq \sum_{j=0}^n \|\hat{\Phi}_z(k)\hat{Y}_z(k)\|_{\text{OP}}. \quad (6.5.29)
\]
We know for all \( i, k \) and \( k \)
\[
|\hat{\Phi}_z(k)_{i,k}| \leq \frac{1}{1 - z^2} (J + zI)_{i,k}, \quad |\hat{\Phi}_z(k)_{i,k}| \leq C_{i,k}, \quad (6.5.30)
\]
with \( C \) defined in (6.5.4). For \( \tilde{v} \in \mathbb{C}^{2d} \) and \( n \in \mathbb{N} \) we know that
\[
\|\hat{\Phi}_z(k)\hat{Y}_z(k)\|_{\text{OP}} \leq \left\| \frac{J + zI}{1 - z^2} \right\|_{\text{OP}} \|\tilde{v}\|_2. \quad (6.5.31)
\]
Thus,
\[
\|B_k\|_{\text{OP}} \leq \sum_{n=0}^\infty \|\mathbf{J} + z\mathbf{I}\mathbf{C}\|_{\text{OP}}^n. \quad (6.5.32)
\]
The matrix \( \mathbf{J} + z\mathbf{I}\mathbf{C} \) has a very special form, created by the symmetry of the coefficient \( \hat{\Phi}^{i,k} \). There exist constants \( a, b, c \geq 0 \) such that
\[
\mathbf{J} + z\mathbf{I}\mathbf{C} = a\mathbf{I} + b\mathbf{J} + c(\mathbf{I}\mathbf{I}^T - \mathbf{J} - \mathbf{I}). \quad (6.5.33)
\]
It is easy to check that such a matrix has only three eigenvalues: \( a + b + (2d - 2)c \), \( a - b \), and \( a + b - 2c \). As \( a, b, c \geq 0 \) the biggest eigenvalue is given by the row sum \( a + b + (2d - 2)c \). We use Lemma 6.5.1 to conclude that
\[
\|B_k\|_{\text{OP}} \leq \sum_{n=0}^\infty \|\mathbf{J} + z\mathbf{I}\mathbf{C}\|_{\text{OP}}^n \leq \sum_{n=0}^\infty \left( \frac{1}{1 - z} \mathbf{C}_B \right)^n = \frac{1}{1 - \frac{z_B}{1 - z}}, \quad (6.5.34)
\]
with $C_B$ as introduced in (6.5.4). To bound $\|B_0\|_{op} = \|I_0 - A_0\|_{op}^{-1}$ we see that also $A_0 = \hat{\Phi}_z(0)\hat{\Upsilon}_z(0)$ is of the form $aI + bJ + c(\hat{\Pi}^T - I)$, with $a, b, c \in \mathbb{R}$, so that also $A_0$ has only three eigenvalues, as given below (6.5.33). We define $\lambda_A^{\text{max}}$ to be absolute value of the maximal eigenvalue $A_0$. Then, we argue in the same way as we did for $\|B_k\|_{op}$ and obtain

$$\|B_0\|_{op} = \sum_{n=0}^{\infty} \|A_0\|_{op}^n = \frac{1}{1 - \lambda_A^{\text{max}}}.$$

(6.5.35)

Using the same argument as for $\|B_k\|_{op}$ and $\|B_0\|_{op}$ we find that

$$\|\hat{\Phi}_z(0)\|_{op} = \max\{|\hat{\Phi}_z(0)|, |\hat{\Phi}_z^{-1}(0)|, |\hat{\Phi}_z^t(0) + \hat{\Phi}_z^{-t}(0) - 2\hat{\Phi}_z(0)|\},$$

(6.5.36)

$$\|\hat{\Phi}_z(k)\|_{op} \leq C_B.$$

(6.5.37)

### 6.5.3 Bounds on $\hat{\Pi}_{z}^{\text{AV}}(0) - \hat{\Pi}_{z}^{\text{AV}}(k)$

To obtain a bound of the bootstrap function $f_3$ we also require a bound on

$$\sum_{i, \kappa} \sum_{x \in \mathbb{Z}^d} |1 - \cos(k \cdot x)||\Pi_{z}^{i, \kappa}(x)|.$$

(6.5.38)

We obtain such a bound by defining

$$\Phi_{z}^{i, \kappa, \text{AV}}(x) = \sum_{N=1}^{\infty} |\Phi_{z}^{(N), i, \kappa}(x)|, \quad \text{and} \quad \Upsilon_{z}^{i, \kappa, \text{AV}}(x) = \sum_{N=1}^{\infty} |\Upsilon_{z}^{(N), i, \kappa}(x)|.$$

(6.5.39)

We take the absolute values in (6.3.10) and see that $|\Pi_{z}^{i, \kappa}(x)|$ is bounded by a corresponding equation, where $\Phi_{z}^{i, \kappa}$ and $\Upsilon_{z}^{i, \kappa}$ are replaced with $\Phi_{z}^{i, \kappa, \text{AV}}$ and $\Upsilon_{z}^{i, \kappa, \text{AV}}$. Then we can compute (6.5.38) using the same technique as shown in Section 6.5.2. The only difference is that we use the coefficients defined in (6.5.39) for the computations.

### 6.6 Analysis

The expansion we have just derived creates an alternative characterization of the classical lace-expansion coefficient. Thus, we can also use the techniques of the classical lace expansion to prove the mean-field behavior.

We modified the analysis shown in Section 3.3 to reestablish the result that the infrared bound holds for SAW in dimensions above seven. Now we explain the necessary modifications:

**The setting.** We define $\Phi_z(x) = 0$ and $F_z(x) = 2dzD(x) + \Pi_z(x)$ and see that the SAW two-point function is given by

$$\hat{G}_z(k) = \frac{1}{1 - 2dz \hat{D}(k) - \hat{\Pi}_z(k)} = \frac{\hat{\Phi}_z(k)}{1 - \hat{f}_z(k)}.$$

(6.6.1)
with equals the form of $G_z$ in Section 3.3 (see (3.2.13)).

We assume that Assumption 3.2.3 holds and use the same functions $f_i$ for the bootstrap. Assumption 3.2.3 is a general assumption we used to control the growth of $G_z$ as $z \uparrow z_c$ and is discussed in Section 3.7. We assume that we can bound $\hat{\Phi}_\iota \kappa z$, and $\hat{Q}_z(0) - \hat{Q}_z(k)$ numerically if $z = z_I = 1/(2d - 1)$ or if $z \in (z_I, z_c)$ and $f_i(z) \leq \Gamma_i$, which is similar to Assumption 3.2.6. Especially, we assume the existence of bound $\beta \Phi_\iota \geq 0$ such that

$$\Phi_\iota = \sum_k \hat{\Phi}^{i_k}_z (0) \leq \beta \Phi_\iota,$$  \hspace{1cm} (6.6.2)

and check numerically that $1 + z + \Phi_\iota > 0$.

In the preceding sections we have created diagrammatic bounds on $\hat{\Phi}_\iota \kappa z$ and $\hat{Q}_z(0) - \hat{Q}_z(k)$. We use the techniques explained in Chapter 5 to compute bounds on the repulsive diagrams and obtain the assumed numerical bounds.

**Key quantities.** We define the constants $K_\Phi$ that are given for the NoBLE in Proposition 3.3.1 as follows: For $\Phi$ we define

$$K_\Phi = K_{\Phi | \Phi} = K_{\Phi | \Phi} = 1,$$  \hspace{1cm} (6.6.3)

$$K_{\Phi | \Phi} = 0,$$  \hspace{1cm} (6.6.4)

Let $z_I = (2d - 1)^{-1}$ and $z_o = (2d - 1)\Gamma_1$. Then we use the bound assumed on $\hat{\Phi}^{i_k}_z$ and $\hat{Q}_z(0) - \hat{Q}_z(k)$ to define

$$\bar{K}_|\iota| = 2dz + \frac{2dz_0^2}{1-z_o} + \frac{2dC_B}{(1-z_o)1-z_o-C_B},$$  \hspace{1cm} (6.6.5)

$$K = 2dz_I + \frac{2dz_I^2}{1-z_I^2} - \text{Bound on } [\hat{Q}_z(0) - \hat{Q}_z(k)],$$  \hspace{1cm} (6.6.6)

$$K_{\Delta_F} = 2dz_o + \frac{2dz_o^3}{1-z_o^2} - \text{Bound on } [\hat{Q}_z^w(0) - \hat{Q}_z^w(k)],$$  \hspace{1cm} (6.6.7)

where $Q^w$ is the error term created when taking the absolute value of the coefficients, see Section 6.5.3. These constants satisfy the inequalities (3.3.1)-(3.3.6) stated in Proposition 3.3.1.

**Changes to the sufficient condition and the result.** We replace the condition (3.3.9) in Definition 3.3.2 with the condition that

$$\frac{2d}{2d-1} \gamma_1 \leq 1 + \frac{2dz_0^2}{1+z_o} - \frac{2d}{1+z_o} \frac{1}{1+z_o + \beta \Phi_\iota},$$  \hspace{1cm} (6.6.8)

where $z_o = (2d - 1)\Gamma_1$. As we explain below, this modified version of $P(\gamma, \Gamma, z)$ is a sufficient condition for the bootstrap.
The proof of the infrared bound in Section 3.3.2 depends only on three ingredients: the existence of constants $K_i$ satisfying (3.3.1) - (3.3.6), that we can apply the boot-strap (Lemma 3.3.4) for the functions $f_1, f_2, f_3$, and that Assumption 3.2.3 holds. As all three ingredients remain to hold no further modifications are required, assuming that the bootstrap succeeds.

**The bootstrap.** In the NoBLE we check the required property of the bootstrap functions $f_1, f_2, f_3$ in Sections 3.3.3-3.3.5. The continuity of $f_2$ and $f_3$ is proven using only Assumption 3.2.3. Further, the improvement of bounds for $f_2$ and $f_3$ use only the abstract representation of $G_z$ by $\Phi_z$ and $F_z$. Therefore, we can use the computations of Section 3.3.4-3.3.5 to prove the required properties for $f_2$ and $f_3$. The only thing we still have to prove that $f_1$ as the properties required for the bootstrap.

The continuity of $f_1(z) = (2d - 1)z$ is obvious and we know that $f_1(z_c) = 1$. To show that for $z \in (z_c, z_c)$ the assumption $f_1(z) \leq \gamma_i$ implies $f_1(z) \leq \gamma_i$ we first see that $1 \leq \hat{G}_z(0) < \infty$ for $z \leq z_c$. From this we conclude that the denominator $1 - 2d z \hat{D}(k) - \hat{\Pi}_z(k)$ of $\hat{G}_z(k)$ (see (6.6.1)) is strictly positive for all $k \in (-\pi, \pi)^d$ and $z \leq z_c$ and know that

$$1 > 2d z \hat{D}(0) + \hat{\Pi}_z(0) \quad \Rightarrow \quad 2d z < 1 - \hat{\Pi}_z(0). \quad (6.6.9)$$

We computed $\hat{\Pi}_z(0)$ in (6.5.3) and conclude

$$2d z < 1 + \frac{2d z_o^2}{1 + z_o} - \frac{2d}{1 + z_o} \frac{1}{1 + z_o + \beta \Phi_i}. \quad (6.6.10)$$

If this is smaller than $\gamma_1 < \Gamma_1$, the bootstrap has succeeded.

**Using the analysis of Section 3.5** Naturally, we can also use the analysis of Sections 3.5-3.6. The required changes are equally simple, but we omit them here as the analysis of Section 3.3 is efficient enough for SAW.

### 6.7 Discussion

In this chapter we have extracted all two-step loops from the coefficients of the classical lace expansion. In this way, we have created a new expansion that combines the simpler form of the classical expansion with the reduced size of the perturbation term of the NoBLE. Using this expansion, we were able to prove the infrared bound for SAW in dimensions bigger than six.

We were not able to reestablish the seminal result of Hara and Slade [42], [41], who have proven mean-field behavior for SAW in all dimensions above four. However,
we were able to prove the infrared bound using a simple analysis and computing only the SRW-integrals $I_{1,0}(0), I_{2,0}(0)$, see Section 5.2. Thus, the required computations are essentially simpler than the computations used by Hara and Slade [41]. We did not try to improve the technique to prove the infrared bound also for dimensions five and six, as we do not expect that we can create a technique that is simpler than that of Hara and Slade [41]. We have no doubt, though, that it is possible.

It remains to be seen, if this simple idea of further expanding the expansion can also be applied to other models and whether such an expansion would be more efficient than the classical lace expansion or the NoBLE. Additionally, it would be an interesting idea to create a memory-$m$ expansion by extracting all loops of length at most $m$ from the classical lace expansion or from the NoBLE.
Chapter 7
Non-backtracking random walk

In this chapter we analyze the non-backtracking walk (NBW) that we introduced in Section 1.5.2. The content of this chapter was published in the Journal of Statistical Physics [25].

The non-backtracking walk (NBW) is a simple random walk that is conditioned not to jump back along the edge it has just traversed. NBW can be viewed as a Markov chain on the set of directed edges where the edge has the interpretation of being the last edge traversed by the NBW, but we will not rely on this interpretation. We study NBWs on \( \mathbb{Z}^d \) in the nearest-neighbor setting and NBWs on tori in various settings, derive the transition matrix for the NBW and analyze its eigensystem in Fourier space. We use this to study asymptotic properties of the NBW, such as its Green’s function and a functional central limit theorem (CLT) on \( \mathbb{Z}^d \), and its convergence towards the stationary distribution on the torus. In particular, our analysis allows us to give an explicit formula for the Fourier transform of the number of \( n \)-step NBWs traversing fixed positions at given times. We use this to prove that the finite-dimensional distributions of the NBW displacements after diffusive rescaling converge to those of Brownian motion. By an appropriate tightness argument, this proves a functional CLT. We further evaluate the Fourier transform of the NBW \( n \)-step transition probabilities on the torus to identify when the NBW transition probabilities are close to the stationary distribution. Our paper is inspired by the study of various high-dimensional statistical mechanical models. For example, our derivation allows us to give detailed estimates of the probability that NBW on a torus is at a given vertex, a fact used in the analysis of hypercube percolation in [51].

NBWs have been investigated on finite graphs in [7], in particular expanders, where it was shown that NBWs mix faster than ordinary random walks. See also [8], where a Poisson limit was proved for the number of visits of an \( n \)-step NBW to a vertex in an \( r \)-regular graph of size \( n \). In the nearest-neighbor setting on \( \mathbb{Z}^d \), NBWs were investigated in [74 Section 5.3], where an explicit expression of its
Green’s function and many related properties are derived (see also Exercise 3.8). Noonan [80] investigates the generating function of NBWs, and his results also apply to walks that avoid their last 7 previous positions (i.e., with memory up to 8), and were used to improve the known upper bounds on the SAW connective constant. See [83] for an extension up to memory 22 for \( d = 2 \), further improving the upper bound on the SAW connective constant. The methods in [80,83] allow to compute the generating function of the number of memory-\( k \) SAWs for appropriate values of \( k \), but do not investigate the number of memory-\( k \) SAWs ending in a particular position in \( \mathbb{Z}^d \). Finally, [81] studies relations between the exponential growth decay of the transition probabilities of NBW and non-amenability of the underlying graph. For interesting connections to zeta functions on graphs, which allow to study the number of NBWs of arbitrary length ending in the starting point, we refer the reader to [67].

This chapter is organized as follows. In Section 7.1 we investigate NBW on \( \mathbb{Z}^d \) and in Section 7.2 we study NBW on tori.

7.1 Non-backtracking random walk on \( \mathbb{Z}^d \)

7.1.1 Setting

In Section 1.5.2 we have define \( b_n(x) \) to be the number of \( n \)-step NBW with \( \omega_0 = 0 \) and \( \omega_n = x \) and \( b'_n(x) \) as the number of \( n \)-step NBW with \( \omega_0 = 0, \omega_1 \neq e_i \) and \( \omega_n = x \). For these function we have proven that For \( n \geq 1 \), the following relations between these objects hold:

\[
\begin{align*}
    b_n(x) &= \sum_{i \in \{\pm 1, \ldots, \pm d\}} b_{n-1}^{-i}(x - e_i), \\
    b_n(x) &= b'_n(x) + \sum_{i \in \{\pm 1, \ldots, \pm d\}} b_{n-1}^{-i}(x - e_i) \quad \forall \, i.
\end{align*}
\]

We have used these equation to compute the two-point function of the NBW in (1.5.18). Equating the right-hand sides of (7.1.2) and (7.1.1) we obtain:

\[
    b'_n(x) = \sum_{\kappa \in \{\pm 1, \ldots, \pm d\} \setminus \{i\}} b_{n-1}^{-\kappa}(x - e_\kappa).
\]

Applying the Fourier transformation to (7.1.1)–(7.1.3) yields

\[
\begin{align*}
    \hat{b}_n(k) &= \sum_{i \in \{\pm 1, \ldots, \pm d\}} \hat{b}_{n-1}^{-i}(k) e^{ik_i}, \\
    \hat{b}_n(k) &= \hat{b}'_n(k) + \sum_{i \in \{\pm 1, \ldots, \pm d\}} \hat{b}_{n-1}^{-i}(k) e^{ik_i} \quad \forall \, i, \\
    \hat{b}'_n(k) &= \sum_{\kappa \in \{\pm 1, \ldots, \pm d\} \setminus \{i\}} \hat{b}_{n-1}^{-\kappa}(k) e^{ik_\kappa}.
\end{align*}
\]
Then, we can rewrite (7.1.4)-(7.1.6) using matrix notation to

\[
\hat{b}_n(k) = \hat{\mathbf{D}}(-k)\vec{b}_{n-1}(k), \\
\hat{b}_n(k)\hat{\mathbf{1}} = \hat{\mathbf{b}}_n(k) + \hat{\mathbf{D}}(k)\vec{b}_{n-1}(k), \\
\vec{b}_n(k) = (\mathbf{C} - \mathbf{J})\hat{\mathbf{D}}(-k)\vec{b}_{n-1}(k) = (\mathbf{C} - \mathbf{J})\hat{\mathbf{D}}(-k)^n\hat{\mathbf{1}}.
\]

(7.1.7) (7.1.8) (7.1.9)

We define the transition matrix

\[
\mathbf{A}(k) = (\mathbf{C} - \mathbf{J})\hat{\mathbf{D}}(-k),
\]

so that

\[
\mathbf{A}(k) = \begin{pmatrix}
e^{-ik_1} & 0 & e^{ik_2} & \cdots & e^{-ik_d} & e^{ik_d} \\
0 & e^{ik_1} & e^{ik_2} & \cdots & e^{-ik_d} & e^{ik_d} \\
e^{-ik_1} & e^{-ik_1} & e^{ik_2} & \cdots & e^{-ik_d} & e^{ik_d} \\
e^{-ik_1} & e^{ik_1} & 0 & \cdots & e^{-ik_d} & e^{ik_d} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e^{-ik_1} & e^{ik_1} & e^{-ik_2} & \cdots & e^{ik_d} & 0 \\
e^{-ik_1} & e^{ik_1} & e^{-ik_2} & \cdots & e^{ik_d} & 0 \\
\end{pmatrix},
\]

(7.1.11)

Thus, \((\mathbf{A}(k))_{i,k} = e^{-ik}(1 - \delta_{i,-k})\).

### 7.1.2 Transition matrix

**The eigensystem.** We start by evaluating the transition matrix (7.1.11) by characterizing its eigenvalues and eigenvectors:

**Lemma 7.1.1 (Dominant eigenvalues).** For \(d \geq 2\) and \(k \in (-\pi, \pi)^d\), let \(\mathbf{A}(k)\) be the matrix given in (7.1.11). Then

\[
\hat{\lambda}_{\pm} = \hat{\lambda}_{\pm}(k) = d\hat{\mathbf{D}}(k) \pm \sqrt{(d\hat{\mathbf{D}}(k))^2 - (2d - 1)}
\]

(7.1.12)

are eigenvalues of \(\mathbf{A}(k)\). For \(k \neq (0,0,\ldots,0)^T\), the right eigenvectors \(\vec{v}^{(\pm)}\) to the eigenvalue \(\hat{\lambda}_{\pm}\) are given by

\[
\vec{v}^{(\pm)} = \hat{\lambda}_{\pm} \hat{\mathbf{1}} \pm \hat{\mathbf{D}}(k) \hat{\mathbf{1}}.
\]

(7.1.13)

For \(k = (0,0,\ldots,0)^T\), the eigenvectors are given by \(\vec{v}^{(1)}(0) = (2d - 2)\hat{\mathbf{1}}\) and \(\vec{v}^{(-1)} := (1, -1, 0, 0, 0, \ldots, 0) \in \mathbb{Z}^{2d}\).

As we will see below, Lemma 7.1.1 yields the two most important eigenvalues. When \(\hat{\lambda}_+ = \hat{\lambda}_-\), which occurs when \((d\hat{\mathbf{D}}(k))^2 - (2d - 1) = 0\), it turns out that \(\vec{v}^{(1)}\) has geometric multiplicity 1, and that \(\hat{\mathbf{1}}\) is a generalized eigenvector satisfying \(\mathbf{A}(k)\hat{\mathbf{1}} = \vec{v}^{(1)} + \hat{\lambda}_+ \hat{\mathbf{1}}\). We continue by computing the remaining eigenvalues and eigenvectors:
Lemma 7.1.2 (Simple eigenvalues). For $d \geq 2$, $k \in (-\pi, \pi)^d$, let $A(k)$ be the matrix given in (7.1.11) and $\vec{e}_i \in \mathbb{C}^{2d}$ the $i$th unit vector, i.e., $(\vec{e}_i)_\kappa = \delta_{i, \kappa}$ for $\kappa \in \{0, \ldots, d\}$. For $i \in \{2, 3, \ldots, d\}$, let
\begin{align*}
\vec{v}^{(0)} &= (1 + e^{ik})(e^{ik}\vec{e}_i + \vec{e}_{-1}) - (1 + e^{ik})(e^{ik}\vec{e}_i + \vec{e}_{-1}) \quad \forall k \in [-\pi, \pi]^d \\
\vec{v}^{(-)} &= (1 - e^{ik})(e^{ik}\vec{e}_i - \vec{e}_{-1}) - (1 - e^{ik})(e^{ik}\vec{e}_i - \vec{e}_{-1}) \quad \forall k \in [-\pi, \pi]^d \text{ if } e^{ik} \neq 1,
\end{align*}
Then, $\vec{v}^{(\pm)}$ is an eigenvector of $A(k)$ to the eigenvalue $\mp 1$. Both eigenvalues have a geometric multiplicity of $d - 1$ for all $k$.

Lemmas 7.1.1–7.1.2 identify a collection of $2d$ independent eigenvectors, and thus the complete eigensystem, of $A(k)$. Now we prove these two lemmas:

Proof of Lemma 7.1.1 Let $\hat{\lambda} \in \{\hat{\lambda}_1, \hat{\lambda}_-\}$ and $\vec{v} = \hat{\lambda} 1 - \hat{\mathbf{D}}(k) 1$. The values $\hat{\lambda}_1$ and $\hat{\lambda}_-$ are the solutions of the quadratic equation
$$\hat{\lambda}^2 = 2d \hat{\lambda} \hat{\mathbf{D}}(k) - (2d - 1).$$
Using $C \hat{\mathbf{D}}(-k) \mathbf{I} = 2d \hat{\mathbf{D}}(k) \mathbf{I}$ and $J \hat{\mathbf{D}}(-k) = \hat{\mathbf{D}}(k) J$, we compute
$$A(k) \vec{v} = (C - J) \hat{\mathbf{D}}(-k)(\hat{\lambda} 1 - \hat{\mathbf{D}}(k)) \mathbf{I} = (2d \hat{\lambda} \hat{\mathbf{D}}(k) 1 - \hat{\mathbf{D}}(k) - (2d - 1) 1) \mathbf{I} \quad \frac{(7.1.14)}{7.1.14} = (\hat{\lambda}^2 1 - \hat{\lambda} \hat{\mathbf{D}}(k) 1) = \hat{\lambda} \vec{v}.$$
This proves that $\vec{v}$ is a eigenvector of $A(k)$ corresponding to the eigenvalue $\hat{\lambda}$ for all $k \neq 0$ and also for the case of $i = 1$ for $k = 0$. For $k = (0, \ldots, 0)$ we note that $\hat{\lambda}_-(0) = 1$ and see that
$$A(0) \vec{v}^{(i)}(0) = (C - J) \vec{v}^{(i)}(0) = -J \vec{v}^{(i)}(0) = \vec{v}^{(i)}(0).$$

Proof of Lemma 7.1.2 For $i \in \{1, 2, \ldots, d\}$, the vectors
$$\vec{u}^{(i)} = e^{ik_i} \vec{e}_i + \vec{e}_{-1} \quad \text{and} \quad \vec{u}^{(-)} = e^{ik_i} \vec{e}_i - \vec{e}_{-1},$$
are eigenvectors of $J \hat{\mathbf{D}}(-k)$, where $\vec{u}^{(\pm)}$ is associated to the eigenvalue $\pm 1$. For $i \in \{2, 3, \ldots, d\}$, we define
$$\vec{u}^{(0)} = \sum_k \vec{u}^{(0)} - \vec{u}^{(i)} \sum_k \vec{u}^{(1)} \quad \text{and} \quad \vec{u}^{(-)} = \sum_k \vec{u}^{(-i)} - \vec{u}^{(-i)} \sum_k \vec{u}^{(-1)}.$$ (7.1.15)
By construction, $\vec{u}^{(0)}$ and $\vec{u}^{(-)}$ are also eigenvalues of $J \hat{\mathbf{D}}(-k)$. For $C \hat{\mathbf{D}}(-k)$, we compute that
$$C \hat{\mathbf{D}}(-k) \vec{u}^{(0)} = C(\vec{e}_i + e^{ik_i} \vec{e}_{-1}) = \vec{u}^{(0)} - \vec{u}^{(i)} \vec{u}^{(-)} = C(\vec{e}_i - e^{ik_i} \vec{e}_{-1}) = -\vec{u}^{(-)}.$$
so that

\[ \mathbf{C} \hat{\mathbf{D}}(-k) \bar{v}^0 = \mathbf{C} \hat{\mathbf{D}}(-k) \bar{v}^{(-)} = 0. \]

Knowing this, it follows that

\[ \mathbf{A}(k) \bar{v}^0 = (\mathbf{C} - \mathbf{J}) \hat{\mathbf{D}}(-k) \bar{v}^0 = -\bar{v}^0, \quad (7.1.16) \]

\[ \mathbf{A}(k) \bar{v}^{(-)} = (\mathbf{C} - \mathbf{J}) \hat{\mathbf{D}}(-k) \bar{v}^{(-)} = \bar{v}^{(-)}. \quad (7.1.17) \]

By (7.1.16), \( \bar{v}^0 \) is an eigenvector for all \( k \). Since the set of vectors \((\bar{v}^0)_{t=2,3,...,d}\) is linearly independent we know that the eigenvalue \(-1\) has geometric multiplicity \( d - 1 \).

From (7.1.17), we conclude the existence of \( d - 1 \) linear independent eigenvalue for 1 only when \( e^{ik_\kappa} \neq 1 \) for all \( t \in \{1,2,...,d\} \). To prove that the eigenvalue 1 has geometric multiplicity \( d - 1 \) for all \( k \), we show how to choose \( d - 1 \) linear independent eigenvectors when \( e^{ik_\kappa} = 1 \) for a \( \kappa \in \{1,2,...,d\} \). For this, let \( S_1 \) be the set of all \( k \in \{1,2,...,d\} \) with \( k_\kappa = 0 \) and \( S_2 \) the set of all \( k \in \{1,2,...,d\} \) with \( k_\kappa \neq 0 \). Let \( s_1 \) and \( s_2 \) be the number of elements in \( S_1 \) and \( S_2 \). Then \( s_1 + s_2 = d \). For all \( t \in S_1 \), we define \( \bar{v}^{(-)} = e_t - e_{-1} \).

If \( s_1 = d \), then \( k = 0 \). In Lemma 7.1.1 we define for this case \( \hat{\lambda}_2 = 1 \) and \( \bar{v}^{(-)} = e_1 - e_{-1} \). Then \( \{\bar{v}^{(-)},\bar{v}^{(-)}_2,...,\bar{v}^{(-)}_d\} \) is a set of independent eigenvectors of \( \mathbf{A}(k) \) to the eigenvalue 1.

If \( s_1 < d \), then let \( \rho \) be the smallest number in \( S_2 \) and define

\[ \bar{v}^{(-)} = \bar{u}^{(-)} \sum_\kappa \bar{u}^{(-)}_\kappa - \bar{u}^{(-)}_\kappa \sum_\kappa \bar{u}^{(-)}_\kappa. \]

for all \( t \in S_2 \setminus \{\kappa\} \). Then it is easy to verify that the vectors \((\bar{v}^{(-)}_t)_{t=2,3,...,d}\) are linearly independent and are eigenvectors of \( \mathbf{A}(k) \) with eigenvalue 1.

We use the eigensystem of the matrix \( \mathbf{A}(k) \) to identify \( \hat{b}_n(k) \) and \( \tilde{b}_n(k) \):

**Lemma 7.1.3** (NBW characterization). Let \( d \geq 2 \), \( n \geq 1 \) and \( k \in (-\pi,\pi^d) \) such that \( \hat{\lambda}_1(k) \neq \hat{\lambda}_-(k) \). Then,

\[ \hat{b}_n(k) = 2d \left( \frac{\hat{\lambda}_1^n(k) - \hat{\lambda}_1^{-1}(k)}{\hat{\lambda}_+(k) - \hat{\lambda}_-(k)} + \frac{\hat{\lambda}_1^{-1}(k) - \hat{\lambda}_1^{-1}(k)}{\hat{\lambda}_+(k) - \hat{\lambda}_-(k)} \right), \quad (7.1.18) \]

\[ \tilde{b}_n(k) = \frac{\hat{\lambda}_1^n(k) - \hat{\lambda}_1^{-1}(k)}{\hat{\lambda}_+(k) - \hat{\lambda}_-(k)} \hat{\lambda}_+(k) \hat{\mathbf{I}} - \frac{\hat{\lambda}_1^{-1}(k) - \hat{\lambda}_1^{-1}(k)}{\hat{\lambda}_+(k) - \hat{\lambda}_-(k)} \hat{\lambda}_-(k) \hat{\mathbf{I}}. \quad (7.1.19) \]

When \( \hat{\lambda}_+(k) = \hat{\lambda}_-(k) \),

\[ \hat{b}_n(k) = 2d[(n - 1)\hat{\lambda}_+(k)^{n-2} + \hat{\mathbf{D}}(k) n\hat{\lambda}_+(k)^{n-1}], \quad (7.1.20) \]

\[ \tilde{b}_n(k) = [(n + 1)\hat{\lambda}_+(k)^n + n\hat{\lambda}_+(k)^{n-1} \hat{\mathbf{D}}(k)] \hat{\mathbf{I}}. \quad (7.1.21) \]
Clearly, one can reprove \(1.5.18\) using Lemma 7.1.3.

**Proof.** For \(\hat{\lambda}_1(k) \neq \hat{\lambda}_-(k)\), we define

\[
\alpha(k) = \frac{1}{\hat{\lambda}_1(k) - \hat{\lambda}_-(k)} = \frac{1}{2\sqrt{(d\hat{D}(k))^2 - (2d - 1)}}.
\]

We can write

\[
\bar{\mathbf{1}} = \alpha(k) \left( \hat{\lambda}_1(k) \mathbf{1} - \hat{\lambda}_-(k) \mathbf{1} + \hat{\mathbf{D}}(k) - \hat{\mathbf{D}}(k) \right) \bar{\mathbf{1}} = \alpha(k) \bar{\mathbf{v}}^{(1)}(k) - \alpha(k) \bar{\mathbf{v}}^{(-1)}(k).
\]

Using \(7.1.9\) and the fact that \(\bar{\mathbf{v}}^{(\pm)}(k)\) are eigenvectors of \(\mathbf{A}(k)\) with eigenvalue \(\hat{\lambda}_\pm(k)\), we obtain

\[
\tilde{b}_n(k) = \mathbf{A}(k)^n \bar{\mathbf{1}} = \alpha(k) \left( \hat{\lambda}^n_1(k) \bar{\mathbf{v}}^{(1)}(k) - \hat{\lambda}^n_-(k) \bar{\mathbf{v}}^{(-1)}(k) \right), \tag{7.1.22}
\]

which proves \(7.1.19\). Combining \(7.1.7\) and \(7.1.22\) gives

\[
\tilde{b}_n(k) = \alpha(k) \bar{\mathbf{1}}^T \hat{\mathbf{D}}(-k) \left( \hat{\lambda}^n_1(k) \bar{\mathbf{v}}^{(1)}(k) - \hat{\lambda}^n_-(k) \bar{\mathbf{v}}^{(-1)}(k) \right). \tag{7.1.23}
\]

Inserting the definition of \(\bar{\mathbf{v}}^{(\pm)}(k)\) gives \(7.1.18\). The proof of \(7.1.20\) is similar, now using that \(\mathbf{A}(k) \bar{\mathbf{1}} = \bar{\mathbf{v}}^{(1)} + \hat{\lambda}_+ \bar{\mathbf{1}}\), which implies that \(\mathbf{A}(k)^n \bar{\mathbf{1}} = n\hat{\lambda}_+(k)^{n-1} \bar{\mathbf{v}}^{(1)} + \hat{\lambda}_+(k)^n \bar{\mathbf{1}}\).

### 7.1.3 Central limit theorem

This section is devoted to the proof of a functional central limit theorem for the NBW. The uniform measures on \(n\)-step NBWs form a consistent family of measures, so that there is a unique law that describes them as a process. We let \(\omega = (\omega_t)_{t \geq 0}\) be distributed according to this law. For a NBW \(\omega\) and \(t \geq 0\), we define

\[
X_n(t) = \frac{\omega_{[nt]}}{\sqrt{n}}, \tag{7.1.24}
\]

where \([x]\) denotes the integer part of \(x \in \mathbb{R}\).

**Theorem 7.1.4** (Functional central limit theorem). The processes \((X_n(t))_{t \geq 0}\) converge weakly to \((B(t))_{t \geq 0}\), where \((B(t))_{t \geq 0}\) is a Brownian motion with covariance matrix \(\mathbf{I}/(d - 1)\).

Our proof of Theorem 7.1.4 is organized as follows. We use Lemma 7.1.3 to prove a CLT for the endpoint in Lemma 7.1.5. We then prove the convergence of the finite-dimensional distributions to the Gaussian distribution (see Lemma 7.1.6), followed by a proof of tightness (see Lemma 7.1.8). This implies Theorem 7.1.4, see e.g. [13] Theorem 15.6. We now fill in the details.
Lemma 7.1.5 (Central limit theorem). Let \( d \geq 2 \) and \( k \in [-\pi, \pi]^d \). Then,
\[
\lim_{n \to \infty} \mathbb{E}[e^{ik \cdot \omega_n / \sqrt{n}}] = e^{-\|k\|_2^2/(2d-2)},
\]
(7.1.25)
where \( \|k\|_2 = \sum_{i=1}^{d} k_i^2 \) is the Euclidean norm of \( k \).

Lemma 7.1.5 implies that the distribution of the endpoint of an \( n \)-step NBW converges in distribution to a normal distribution with mean zero and covariance matrix \((d-1)^{-1}I\).

Proof. We can rewrite the expectation as
\[
\mathbb{E}[e^{ik \cdot \omega_n / \sqrt{n}}] = \sum_{x \in \mathbb{Z}^d} \hat{b}_n(x) e^{ik \cdot x / \sqrt{n}} = \hat{b}_n(k / \sqrt{n})/2d(2d-1)^n.
\]
(7.1.26)
As \( n \to \infty \), we can assume without loss of generality that \( k \) is small and therefore that \( \hat{\lambda}_1(k) > \hat{\lambda}_{-}(k) \). Then we can use (7.1.18) to compute the limit of (7.1.26). For \( \iota = 1, -1 \), we compute
\[
\lim_{n \to \infty} \hat{\lambda}_{\iota}^{-1}(k / \sqrt{n}) = \frac{\mathbb{I}^T \hat{D}(-k / \sqrt{n}) \hat{v}^{(\iota)}(k / \sqrt{n})}{2d}.
\]
(7.1.27)
We first consider the case \( \iota = 1 \). The coefficient \( \alpha(k) \) as well as \( \hat{v}^{(1)}(k) \) are continuous in a neighborhood of \( 0 \), so we can directly compute
\[
\lim_{n \to \infty} \alpha(k) \frac{\mathbb{I}^T \hat{D}(-k / \sqrt{n}) \hat{v}^{(1)}(k / \sqrt{n})}{2d} = \alpha(0) \frac{\mathbb{I}^T \hat{D}(0) \hat{v}^{(1)}(0)}{2d} = 1.
\]
Further, \( \hat{\lambda}_1(k) \) is differentiable in \( k \), so we can Taylor expand \( \hat{\lambda}_1(k) \) at 0 to obtain
\[
\hat{\lambda}_1(k) = 2d - 1 - \frac{2d-1}{2d-2} \|k\|_2^2 + O(\|k\|_4^4).
\]
Therefore,
\[
\lim_{n \to \infty} \hat{\lambda}_1^{-1}(k / \sqrt{n}) = \lim_{n \to \infty} \left(1 - \frac{1}{2d-2} \frac{\|k\|_2^2}{n} + O\left(\frac{\|k\|_4^4}{n^2}\right)\right)^{-1} = e^{-\|k\|_2^2/(2d-2)},
\]
(7.1.28)
so that the limit (7.1.27) for \( \iota = 1 \) is given by \( e^{-\|k\|_2^2/(2d-2)} \).

We next consider the case \( \iota = -1 \), for which we use that \( k \mapsto \hat{\lambda}_-(k) \) is continuous and \( \hat{\lambda}_-(0) = 1 \). Therefore, for \( d \geq 2 \),
\[
\lim_{n \to \infty} \hat{\lambda}_{-1}^{-1}(k / \sqrt{n}) = 0.
\]
(7.1.30)
The second factor in (7.1.27) can easily be bounded uniformly for small \( k \), so that for \( \iota = -1 \) the limit of (7.1.27) is zero. \( \Box \)
Lemma 7.1.6 (Convergence of finite-dimensional distributions). For \( d \geq 2 \) and \( N > 0 \), let \( 0 = t_0 < t_1 < t_2 < \cdots < t_N = 1 \) and \( k^{(r)} \in (-\pi, \pi]^d \) for \( r = 1, \ldots, N \). Then,

\[
\lim_{n \to \infty} \mathbb{E}[e^{i \sum_{r=1}^N k^{(r)}(\omega_{t[rN]} - \omega_{t[r-1]})/\sqrt{n}}] = e^{-\sum_{r=1}^N \|k^{(r)}\|_2^2 (t_r - t_{r-1})/(2d-2)}. \tag{7.1.31}
\]

**Proof.** As we take the limit \( n \to \infty \), without loss of generality we can assume that \( \eta_r(n) := [t_r n] - [t_{r-1} n] \geq 1 \), \( t_r > t_{r-1} \) and each \( k^{(r)} := k^{(r)}/\sqrt{n} \) is so small that \( \hat{\lambda}_1(k) > \hat{\lambda}_-(k) \) for \( r = 1, \ldots, N \) and \( n \in \mathbb{N} \). Recall that \( \mathcal{W}_n^{NBW} \) denotes the set of all \( n \)-step NBW. For any function \( f : \mathbb{Z}^{d \times N} \to \mathbb{C} \), we know that

\[
\mathbb{E}[f(\omega_{t_1 n}, \ldots, \omega_{t_N n})] = \frac{1}{\hat{b}_n(0)} \sum_{\omega \in \mathcal{W}_n^{NBW}} f(\omega_{t_1 n}, \ldots, \omega_{t_N n}) \tag{7.1.32}
\]

\[
= \frac{1}{\hat{b}_n(0)} \sum_{\omega \in \mathcal{W}_n^{NBW}} \sum_{x_1, \ldots, x_N \in \mathbb{Z}} f(x_1, \ldots, x_N) \prod_{i=1,\ldots,N} \delta_{x_i, \omega_{t_i n}}. \]

Let \( b^{i,k}_n(x) \) be the number of \( n \)-step NBW \( \omega \) with \( \omega_1 \neq e_i, \omega_{n-1} = x + e_k \) and \( \omega_n = x \). We define the matrix \( \hat{B}_n(k) \) with entries \((\hat{B}_n(k))_{i,k} = b^{i,k}_n(k)\). By a relation similar to (7.1.6), we conclude that \( \hat{B}_n(k) = \hat{A}(k)^n \). We fix \( N \) points \( x_1, \ldots, x_N \in \mathbb{Z} \), then the number of NBWs \( \omega \) with \( \omega_{t_i n} = x_i \) for all \( i = 1, \ldots, N \) is given by

\[
\sum_{i_1,\ldots,i_N} b^{i_0,i_1}_{\eta_1(n)-1}(x_1 + e_{i_0}) \prod_{r=2,\ldots,N} b^{i_{r-1},i_r}_{\eta_r(n)}(x_r - x_{r-1}). \tag{7.1.33}
\]

We insert \( f(x_1, \ldots, x_N) = e^{i \sum_r k^{(r)}(x_r - x_{r-1})} \) and obtain

\[
\mathbb{E}[e^{i \sum_r k^{(r)}(\omega_{t_r n} - \omega_{t_{r-1} n})}] = \frac{1}{\hat{b}_n(0)} \sum_{i_1,\ldots,i_N} \hat{B}_{\eta_1(n)-1}(k^{(1)}) e^{-ik^{(1)}_1/\sqrt{n}} \prod_{r=2}^N \hat{B}_{\eta_r(n)}(k^{(r)}) \tag{7.1.34}
\]

\[
= \frac{1}{\hat{b}_n(0)} \hat{\mathcal{T}} \hat{D}(-k^{(1)}) \hat{A}(k^{(1)})^{\eta_1(n)-1} \prod_{r=2}^N \hat{B}_{\eta_r(n)}(k^{(r)}) \hat{I} 
\]

To proceed, we define, for \( \tau \in \{1, 2, \ldots, N\} \),

\[
\tilde{h}_n(\tau) = \frac{1}{\hat{b}_{[t_\tau,n]}(0)} \hat{\mathcal{T}} \hat{D}(-k^{(1)}) \hat{A}(k^{(1)})^{\eta_1(n)-1} \cdots \hat{A}(k^{(r)})^{\eta_r(n)}, \quad \tilde{h}_n(0) = \frac{1}{2d} \hat{\mathcal{T}} \hat{D}(-k^{(1)}).
\]

As \( \sum_{r=1}^T \eta_r(n) \leq t_T n \), by construction of \( \tilde{h}_n(\tau) \), for all \( \tilde{v} \in \mathbb{C}^{2d} \)

\[
|\tilde{h}_n(\tau) \tilde{v}| \leq \mathbb{E} \left[ e^{i \left( \sum_{r=1}^T k^{(r)}(\omega_{t_\tau n} - \omega_{t_{r-1} n})/\sqrt{n} \right)} \sum_i |\Pi(\omega_{t_r n} - \omega_{t_{r-1} n} = e_i) | \| \tilde{v} \|_1 \right] 
\]

\[
\leq \mathbb{E} \left[ \sum_i |\Pi(\omega_{t_r n} - \omega_{t_{r-1} n} = e_i) | \| \tilde{v} \|_1 \right] \leq \| \tilde{v} \|_\infty, \tag{7.1.35}
\]

Non-backtracking random walk
where we write $\|v\|_\infty := \max_i |v_i|$. Therefore, we can rewrite

$$7.1.34 \quad = \quad \tilde{h}_n(N) \tilde{I} = \tilde{h}_n(N-1) \left( \frac{A(k_n^{(i)})}{2d-1} \right) \eta_n^{(i)} \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} + \tilde{h}_n(N) \left( \tilde{I} - \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} \right)$$

$$= \quad \tilde{h}_n(N-1) \left( \frac{\tilde{\lambda}_1(k_n^{(i)})}{2d-1} \right) \eta_n^{(i)} \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} + \tilde{h}_n(N) \left( \tilde{I} - \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} \right)$$

$$= \quad \tilde{h}_n(0) \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} \left( \frac{2d-1}{\tilde{\lambda}_1(k_n^{(i)})} \right) \prod_{r=1}^{N} \left( \frac{\tilde{\lambda}_1(k_n^{(i)})}{2d-1} \right) \eta_r^{(i)}$$

$$+ \sum_{r=1}^{N-1} \frac{\tilde{h}_n(r)}{2d-2} \left( \tilde{v}_1^{(i)}(k_n^{(r+1)}) - \tilde{v}_1^{(i)}(k_n^{(r)}) \right) \prod_{i=r+1}^{N} \left( \frac{\tilde{\lambda}_1(k_n^{(i)})}{2d-1} \right) \eta_r^{(i)}$$

$$+ \tilde{h}_n(N) \left( \tilde{I} - \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} \right).$$

(7.1.36)

To compute the limit of (7.1.34), we show that (7.1.37) and (7.1.38) converge to zero and identify the limit of (7.1.36). From (7.1.35), it follows that we can bound (7.1.38) by $\tilde{I} - \tilde{v}_1^{(i)}(k_n^{(i)})/(2d-2)$ and we know that $\tilde{v}_1^{(i)}(k_n^{(i)}) \stackrel{n \to \infty}{\to} (2d-2) \tilde{I}$. Thereby we conclude that (7.1.38) converges to 0.

To compute the limit of (7.1.37) we note that $\tilde{\lambda}_1(k) \leq 2d-1$ and that $\tilde{v}_1^{(r+1)}(k_n^{(r+1)}) - \tilde{v}_1^{(r)}(k_n^{(r)}) \stackrel{n \to \infty}{\to} 0$, so that

$$\lim_{n \to \infty} \|7.1.37\| \leq \lim_{n \to \infty} \left| \sum_{r=1}^{N-1} \frac{\tilde{h}_n(r)}{2d-2} \left( \tilde{v}_1^{(i)}(k_n^{(i)}) - \tilde{v}_1^{(i)}(k_n^{(i)}) \right) \right|$$

$$\leq \lim_{n \to \infty} \frac{1}{2d-2} \sum_{r=1}^{N-1} \| \tilde{v}_1^{(i)}(k_n^{(i)}) - \tilde{v}_1^{(i)}(k_n^{(i)}) \|_\infty = 0.$$

To compute the limit of (7.1.36), we use that

$$\lim_{n \to \infty} \tilde{h}_n(0) \frac{\tilde{v}_1^{(i)}(k_n^{(i)})}{2d-2} \left( \frac{2d-1}{\tilde{\lambda}_1(k_n^{(i)})} \right) = \lim_{n \to \infty} \frac{\tilde{I} \tilde{D}(-k)(\tilde{\lambda}_1(k_n^{(i)}) \tilde{I} + \tilde{D}(k_n^{(i)})) \tilde{I}}{2d(2d-2)} \frac{2d-1}{\tilde{\lambda}_1(k_n^{(i)})} = 1,$$

and, for each $r \in \{1, \ldots, N\}$,

$$\lim_{n \to \infty} \left( \frac{\tilde{\lambda}_1(k_n^{(i)})}{2d-1} \right) \eta_r^{(i)} = e^{-\|k^{(i)}\|^2 / (t_i - t_{i-1}) / (2d-2)}.$$
Lemma 7.1.7 (The second moment). For $d \geq 2$ and $n \in \mathbb{N}$, we have

$$\mathbb{E}[\|\omega_n\|^2_2] = \frac{d}{d-1} n + \frac{4d-1}{2(d-1)^2} + \frac{d}{2(d-1)^2(2d-1)^{n-2}}.$$ 

Proof. We define the differential operator $\nabla^2 := \sum_{i=1}^d \left( \frac{\partial}{\partial x_i} \right)^2$ and see that

$$\mathbb{E}[\|\omega_n\|^2_2] = \frac{1}{b_n(0)} \sum_{x \in \mathbb{Z}} b_n(x) \|x\|^2_2 = -\frac{1}{b_n(0)} \nabla^2 \hat{b}_n(0).$$

To compute the second derivative in a neighborhood of the origin, we recall (7.1.18).

By Lemma 7.1.3 for $k$ such that $\hat{\lambda}_+(k) \neq \hat{\lambda}_-(k)$,

$$\hat{b}_n(k) = \sum_{\sigma \in \{-1,1\}} dI \frac{\hat{\lambda}_n^\sigma(k) \hat{D}(k) - \hat{\lambda}_n^{\sigma-1}(k)}{\sqrt{(d \hat{D}(k))^2 - (2d - 1)}} \equiv \sum_{\sigma \in \{-1,1\}} \hat{g}_\sigma(k). \quad (7.1.39)$$

Since $\hat{\lambda}_+(k) \neq \hat{\lambda}_-(k)$ for $k$ small, (7.1.39) holds in particular for $k$ small. A straightforward computation gives

$$\nabla^2 g_\sigma(k) \bigg|_{k=0} = d\sigma \left( \nabla^2 \hat{\lambda}_\sigma(0) \right) \frac{n\hat{\lambda}_\sigma^{n-1}(0) \hat{D}(0) - (n-1)\hat{\lambda}_\sigma^{n-2}(0)}{\sqrt{(d \hat{D}(0))^2 - (2d - 1)}} + d\sigma \left( \nabla^2 \hat{D}(0) \right) \left( \frac{\hat{\lambda}_\sigma^n(0)}{\sqrt{(d \hat{D}(0))^2 - (2d - 1)}} \right) - d^2 \hat{D}(0) \left( \frac{\hat{\lambda}_\sigma^n(0) \hat{D}(0) - \hat{\lambda}_\sigma^{n-1}(0)}{\sqrt{(d \hat{D}(0))^2 - (2d - 1)}} \right)^3 \equiv -d\sigma \frac{d((1 + \sigma) d - 1)}{d - 1} \frac{n\hat{\lambda}_\sigma^{n-1}(0) - (n-1)\hat{\lambda}_\sigma^{n-2}(0)}{d - 1} + \frac{d\sigma}{d - 1} \hat{\lambda}_\sigma^{n-1}(0) \left( \hat{\lambda}_\sigma(0) - d^2 \hat{\lambda}_\sigma(0) - 1 \right) \left( \frac{\hat{\lambda}_\sigma(n-2)}{(d-1)^2} \right),$$

so that

$$\nabla^2 g_1(k) \bigg|_{k=0} = -d \frac{(2d - 1)^{n-1}}{d - 1} \frac{[2d(d-1)n + (4d - 1)]}{d - 1}$$

$$\nabla^2 g_{-1}(k) \bigg|_{k=0} = d \frac{d((-1) d - 1)}{d - 1} = -d \frac{2d - 1}{(d - 1)^2}.$$ 

We arrive at

$$-\frac{1}{\hat{b}_n(0)} \nabla^2 \hat{b}_n(0) = \frac{1}{2d(2d - 1)^{n-1}} \sum_{\sigma \in \{-1,1\}} -\nabla^2 g_\sigma(k) \bigg|_{k=0} = \frac{d}{d - 1} n + \frac{4d - 1}{2(d-1)^2} + \frac{d}{2(d-1)^2(2d-1)^{n-2}}.$$
Lemma 7.1.8 (Tightness). For \( d \geq 2 \) and \( 0 \leq t_1 < t_2 < t_3 \leq 1 \), there exists a \( K > 0 \), such that, for all \( n \geq 1 \),

\[
\mathbb{E}[\|X_n(t_2) - X_n(t_1)\|_2^2 \|X_n(t_3) - X_n(t_2)\|_2^2] \leq K(t_2 - t_1)(t_3 - t_2).
\]

Proof. We use the same notation as in the proof of Lemma 7.1.6. In (7.1.33), we have seen how to describe the number of NBWs that visit a number of fixed points. This time we forget about the non-backtracking constraint between two subsequent NBWs to upper bound

\[
\mathbb{E}[\|\omega_{[t_2,n]} - \omega_{[t_1,n]}\|_2^2 \|\omega_{[t_3,n]} - \omega_{[t_2,n]}\|_2^2]
\leq \frac{1}{b_n(0)} \sum_{x_1,x_2,x_3,x_4 \in \mathbb{Z}^d} \|x_2 - x_1\|_2^2 \|x_3 - x_2\|_2^2 \eta_n(x_1) \prod_{r=2}^4 b_{\eta_r(n)}(x_r - x_{r-1})
\]

\[
= \frac{1}{b_n(0)} \sum_{x_1,x_2,x_3,x_4 \in \mathbb{Z}^d} \|x_2\|_2^2 \|x_3\|_2^2 \prod_{r=1}^4 b_{\eta_r(n)}(x_r)
\]

\[
= \frac{\hat{b}_{\eta_1(n)} \cdot \hat{b}_{\eta_2(n)}}{b_n} \sum_{x_2 \in \mathbb{Z}} b_{\eta_2(n)}(x_2) \|x_2\|_2^2 \sum_{x_3 \in \mathbb{Z}} b_{\eta_3(n)}(x_3) \|x_3\|_2^2
\]

\[
= \left( \frac{2d - 1}{2d} \right)^3 \mathbb{E}[\|\omega_{\eta_2(n)}\|_2^2] \mathbb{E}[\|\omega_{\eta_3(n)}\|_2^2].
\]

Applying Lemma 7.1.7 completes the proof. \( \square \)

### 7.1.4 Extension to non-nearest-neighbor settings

In this section, we extend the analysis of NBW on \( \mathbb{Z}^d \) to other bond sets. We start by introducing the bond sets that we consider. We let \( \mathcal{B} \subset \mathbb{Z}^d \times \mathbb{Z}^d \) be a translation invariant collection of bonds. Let \( \mathcal{V}_0 = \{x : \{0, x\} \in \mathcal{B}\} \) denote the set of endpoints of bonds containing the origin and write \( m = |\mathcal{V}_0| \). We assume that \( 0 \not\in \mathcal{V}_0 \), and that \( \mathcal{V}_0 \) is symmetric, i.e., \(-x \in \mathcal{V}_0 \) for every \( x \in \mathcal{V}_0 \). Thus, \( m \) is even. We define the simple random walk step distribution by

\[
D(x) = \frac{1}{m} \mathbb{I}_{\{x \in \mathcal{V}_0\}}. \quad (7.1.40)
\]

Define the matrices \( C, J \in \mathbb{C}^{m \times m} \) by \((C)_{x,y} = 1\) and \((J)_{x,y} = \delta_{x,-y}\), and let the diagonal matrix \( \hat{D}(k) \) have entries \((\hat{D}(k))_{x,x} = e^{ik \cdot x}\), where \( x, y \in \mathcal{B}_0 \). Then, we define the matrix \( A(k) \) of size \( m \times m \) by

\[
A(k) = (C - J) \hat{D}(-k). \quad (7.1.41)
\]

With this definition at hand, we see that \((7.1.7) - (7.1.9)\) remain to hold. As a result, also Lemmas \(7.1.1 - 7.1.2\) whose proof only depends on \((7.1.7) - (7.1.9)\), continue to hold when we replace each occurrence of \( 2d \) by \( m \). Since the proof of Lemma \(7.1.3\)
in turn, only depends on Lemmas 7.1.1–7.1.2, also it extends to this setting, so that, for example

\[ \hat{\lambda}_\pm(k) = F_\pm(\hat{D}(k); m), \quad \text{where} \quad F_\pm(x; m) = \frac{1}{2}(mx \pm \sqrt{(mx)^2 - 4(m-1)}), \]  

(7.1.42)

and, when \( \hat{\lambda}_+(k) \neq \hat{\lambda}_-(k) \),

\[ \hat{b}_n(k) = \frac{m}{2} \frac{\hat{D}(k)(\hat{\lambda}_+^n(k) - \hat{\lambda}_-^{n-1}(k)) + (\hat{\lambda}_-^{n-1}(k) - \hat{\lambda}_+^{n-1}(k))}{\hat{\lambda}_+(k) - \hat{\lambda}_-(k)}. \]  

(7.1.43)

Naturally, Theorem 7.1.4 needs to be adapted, and now reads that the processes \((X_n(t))_{t \geq 0}\) converge weakly to a Brownian motion with covariance matrix \(M\) of size \(d \times d\), where, for \(\iota, \kappa \in \{\pm 1, \ldots, \pm d\}\), we define

\[ M_{\iota,\kappa} = \frac{\partial^2 \hat{\lambda}_+(k)}{\partial k_\iota \partial k_\kappa} \bigg|_{k=0} (\hat{\lambda}_1(0))^{-1}. \]  

(7.1.44)

We next compute \(M\) explicitly in terms of the covariance matrix of the transition kernel \(D\). We compute that \(F_+(1; m) = m/2 + (m-2)/2 = m-1\), and

\[ F'_+(x; m) = m/2 + \frac{m^2 x}{2 \sqrt{(mx)^2 - 4(m-1)}}, \quad \text{so that} \quad F'_+(1; m) = m(m-1)/(m-2). \]  

(7.1.45)

By symmetry, the odd derivatives of \(\hat{D}(k)\) are zero, so that a Taylor expansion yields

\[ \hat{D}(k) = 1 - \frac{1}{2} k^T H k + O(\|k\|^3_2), \]  

(7.1.46)

where, for \(\iota, \kappa \in \{1, \ldots, d\}\),

\[ H_{\iota,\kappa} = \sum_x x_\iota x_\kappa D(x) \]  

(7.1.47)

denotes the covariance matrix of SRW. As a result,

\[ M = H \frac{F'_+(1; m)}{F_+(1; m)} = H mL/(m-2). \]  

(7.1.48)

In the nearest-neighbor case, \(m = 2d\) and \(H = I/d\), so that we retrieve the result in Theorem 7.1.4.

### 7.2 Non-backtracking random walk on tori

In this section, we extend the results in Section 7.1 to NBWs on tori. In Section 7.2.1 and 7.2.2, we investigate NBW on a torus of width \(r \geq 2\), and in Section 7.2.3, we investigate NBW on the hypercube, for which \(r = 2\). The study of random walks on various finite transitive graphs has attracted considerable attention. See e.g., [71] for a recent book on the subject, and [6] for a book in preparation. Here we restrict ourselves to NBWs on tori.
7.2 Non-backtracking random walk on tori

285

7.2.1 Setting

For \( d \geq 2 \) and \( r \geq 3 \), we denote by \( \mathbb{T} = \mathbb{T}_{r,d} = (\mathbb{Z}/r\mathbb{Z})^d \) the discrete \( d \)-dimensional torus with side length \( r \). The torus has periodic boundaries, i.e., we identify two points \( x, y \in \mathbb{T}_{r,d} \) if \( x \mod r = y \mod r \) for all \( i = 1, \ldots, d \) where mod denotes the modulus. We define the Fourier dual torus of \( \mathbb{T} \) as

\[
\mathbb{T}_{r,d}^* := \frac{2\pi}{r} \{ -\left\lfloor \frac{r-1}{2} \right\rfloor, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor \}^d,
\]

so that each component of \( k \in \mathbb{T}_{r,d}^* \) is between \(-\pi\) and \( \pi \). The Fourier transform of \( f: \mathbb{T}_{r,d} \to \mathbb{C} \) is defined by

\[
\hat{f}(k) = \sum_{x \in \mathbb{T}_{r,d}} f(x)e^{ik \cdot x}, \quad k \in \mathbb{T}_{r,d}^*.
\]

As in Section 7.1, we define an \textit{n-step random walk} on \( \mathbb{T}_{r,d} \) to be an ordered tuple \( \omega = (\omega_0, \ldots, \omega_n) \), with \( \omega_i \in \mathbb{T}_{r,d} \) and \( \omega_i - \omega_{i+1} \in \mathbb{V}_0 \), where we recall that \( \mathbb{V}_0 = \{ x : \{0, x\} \in \mathbb{B} \} \), and \( \mathbb{B} \) is the translationally invariant bond set on which our random walks moves. We always assume that \( 0 \not\in \mathbb{V}_0 \), and that \( \mathbb{V}_0 \) is symmetric, i.e., if \( x \in \mathbb{V}_0 \), then also \( -x \in \mathbb{V}_0 \). Further, we always assume that \( \omega_0 = (0, \ldots, 0) \). The simple random walk step distribution is given by

\[
D(x) = \frac{1}{m} \mathbb{1}_{\{x \in \mathbb{V}_0\}} \quad \text{and} \quad \hat{D}(k) = \frac{1}{m} \sum_{x \in \mathbb{T}_{r,d}} e^{ik \cdot x}.
\]

If an \( n \)-step random walk on \( \mathbb{T}_{r,d} \) additionally satisfies \( \omega_i \neq \omega_{i-2} \), then we call the walk a non-backtracking walk (NBW) on \( \mathbb{T}_{r,d} \). Let \( b_n(x) \) be the number of \( n \)-step NBWs with \( \omega_n = x \). Further, let \( b'_{n}(x) \) be the number of \( n \)-step NBWs \( \omega \) with \( \omega_n = x \) and \( \omega_1 \neq e_i \).

In this setting, we can express \( b_n(x) \) for NBW on \( \mathbb{T}_{r,d} \) in terms of NBW on \( \mathbb{Z}^d \). Indeed, identify \( \mathbb{T}_{r,d} \) with \( \{0, \ldots, r-1\}^d \subset \mathbb{Z}^d \), and also identify \( \mathbb{V}_0 = \{ x : \{0, x\} \in \mathbb{B} \} \) as a subset of \( \{0, \ldots, r-1\}^d \subset \mathbb{Z}^d \). Define \( \mathbb{V}_0^z = \mathbb{V}_0 \cup (-\mathbb{V}_0) \) (which, by construction, are disjoint subsets of \( \mathbb{Z}^d \)), and define the random walk step distribution \( D_z(x) \) by the uniform distribution on \( \mathbb{V}_0^z \). Then, for \( x \in \mathbb{T}_{r,d} \),

\[
b_n(x) = \sum_{y : x \prec y} b_n^z(y),
\]

where, for \( y \in \mathbb{Z}^d \) and \( x \in \mathbb{T}_{r,d} \), we say that \( x \prec y \) when \( x = y \mod r \), and \( b_n^z(y) \) denotes the number of \( n \)-step NBWs on \( \mathbb{Z}^d \) with step distribution \( D_z(x) \). As a result, \( \hat{b}_n(k) \) for NBW on \( \mathbb{T}_{r,d} \) is equal to \( \hat{b}_n^z(k) \) for every \( k \in \mathbb{T}_{r,d}^* \) as defined in (7.2.1).

Therefore, we can use most results for NBW on \( \mathbb{Z}^d \) to study NBW on \( \mathbb{T}_{r,d} \). We define the probability mass function of the endpoint of an \( n \)-step NBW by

\[
p_n(x) = \frac{b_n(x)}{\sum_y b_n(y)} = \frac{b_n(x)}{m(m-1)^{n-1}}.
\]
In order to study the asymptotic behavior of NBW, we investigate $\lambda_{\pm}(k)$ for $k \neq 0$. Our main result in this section is the following theorem:

**Theorem 7.2.1** (Pointwise bound on $\hat{b}_n(k)$). Let $\mathbb{V}_0$ be symmetric and satisfy $0 \notin \mathbb{V}_0$. Then, for $n \in \mathbb{N}$, NBW with steps in $\mathbb{V}_0$ satisfies

$$|\hat{p}_n(k)| \leq \left(1/\sqrt{m-1} \lor |\hat{D}(k)|\right)^{n-1}. \quad (7.2.5)$$

To prove Theorem 7.2.1, we start by investigating $\lambda_{\pm}(k)$ for $k \neq 0$. For this, we use Lemma 7.1.1 to note that

$$\hat{\lambda}_{\pm}(k) = F_{\pm}(\hat{D}(k); m), \quad \text{where} \quad F_{\pm}(x; m) = \frac{1}{2}\left(mx \pm \sqrt{(mx)^2 - 4(m-1)}\right), \quad (7.2.6)$$

and where $m$ denotes the degree of our graph. We bound $\hat{\lambda}_{\pm}(k)$ in the following lemma:

**Lemma 7.2.2** (Bounds on $\hat{\lambda}_{\pm}(k)$). For any $k \in \mathbb{T}_{r,k}$,

$$|\hat{\lambda}_{\pm}(k)| = \begin{cases} \sqrt{m-1} & \text{when } (m\hat{D}(k))^2 - 4(m-1) \leq 0; \\ (m-1) \left[1 - (1 - \hat{D}(k)) \frac{m}{m-2}\right] & \text{when } \hat{D}(k) > 0, (m\hat{D}(k))^2 - 4(m-1) > 0; \\ 1 & \text{when } \hat{D}(k) \leq 0, (m\hat{D}(k))^2 - 4(m-1) > 0. \end{cases} \quad (7.2.7)$$

**Proof.** The function $x \mapsto F_{\pm}(x; m)$ is real when $(mx)^2 - 4(m-1) \geq 0$, and complex when $(mx)^2 - 4(m-1) < 0$. When $(mx)^2 - 4(m-1) < 0$,

$$|F_{\pm}(x; m)|^2 = m - 1, \quad (7.2.8)$$

so that $|F_{\pm}(x; m)| = \sqrt{m-1}$.

When $(mx)^2 - 4(m-1) \geq 0$, by the symmetry $F_{+}(x; m) = F_{-}(-x; m)$, we only need to investigate $x \in [0, 1]$. We start with $F_{-}(x; m)$, which clearly satisfies $F_{-}(x; m) \geq 0$. Further, we can compute that $F_{-}(1; m) = 1$, and

$$F'_{-}(x; m) = \frac{m^2 x}{2\sqrt{(mx)^2 - 4(m-1)}} = \frac{m}{2} \left[1 - \frac{mx}{\sqrt{(mx)^2 - 4(m-1)}}\right] < 0, \quad (7.2.9)$$

so that $F_{-}(x; m) \leq 1$ for all $x \in [0, 1]$ for which $(mx)^2 - 4(m-1) > 0$.

To bound $F_{+}(x; m)$, we use (7.1.45) as well as

$$F''_{+}(x; m) = \frac{m^2}{2\sqrt{(mx)^2 - 4(m-1)}} - \frac{m^4 x^2}{2((mx)^2 - 4(m-1))^{3/2}} \quad (7.2.10)$$

$$= \frac{m^2}{2((mx)^2 - 4(m-1))^{3/2}} \left\{((mx)^2 - 4(m-1)) - (mx)^2\right\}$$

$$= - \frac{2m^2(m-1)}{((mx)^2 - 4(m-1))^{3/2}} < 0.$$
As a result, a Taylor expansion yields
\[ F_+(x; m) \leq F_+(1; m) + (x - 1)F'_+(1; m) = (m - 1) + (x - 1)m(m - 1)/(m - 2) \] (7.2.11)
\[ = (m - 1)[1 - (1 - x)m/(m - 2)]. \]

**Proof of Theorem 7.2.1** By (7.1.7) and (7.1.9),
\[ |\hat{b}_n(k)| \leq \|\hat{D}(-k)\hat{I}\|_2\|\hat{b}_{n-1}(k)\|_2 \leq \|\hat{I}\|_2\|A(k)\|_{op}^{n-1}\|\hat{I}\|_2, \] (7.2.12)
where we write \(\|M\|_{op} = \sup\|Mx\|_2/\|x\|_2\) for the operator norm of the matrix \(M\). We next use that \(A(k)\) has eigenvalues \(\hat{\lambda}^+_m(k), \hat{\lambda}^-_m(k)\) and \(\pm 1\) by Lemmas 7.1.1-7.1.2, so that
\[ \|A(k)\|_{op} = |\hat{\lambda}^+_m(k)| \vee |\hat{\lambda}^-_m(k)| \vee 1, \] (7.2.13)
where we use that for finite-dimensional matrices, the operator norm is equal to the maximal eigenvalue, and for \(x, y \in \mathbb{R}\), we write \((x \vee y) = \max\{x, y\}\). Thus, we arrive at
\[ |\hat{b}_n(k)| \leq m(|\hat{\lambda}^+_m(k)| \vee |\hat{\lambda}^-_m(k)| \vee 1)^{n-1}. \] (7.2.14)
By Lemma 7.2.2 and since \(m \geq 2\) so that \(\sqrt{m-1} \geq 1\),
\[ \frac{|\hat{\lambda}^\pm_m(k)|}{m - 1} \leq (m - 1)^{-1/2} \vee \left(1 - [1 - D(k)] \frac{m - 1}{m - 2}\right) \leq (m - 1)^{-1/2} \vee |\hat{D}(k)|. \] (7.2.15)
Substitution into (7.2.12) yields the claim. \(\square\)

### 7.2.2 Asymptotics for NBW on the torus

In this section, we study the convergence towards equilibrium of NBW on tori of width \(r \geq 3\). We focus on two different examples. The first is random walk on products of complete graphs, where
\[ V_0 = \{x: \exists i \in \{1, \ldots, d\} \text{ such that } x_i \neq 0\}. \] (7.2.16)
Our second example is NBW on the nearest-neighbor torus. The reason why we study these cases separately is that NBW on products of complete graphs is aperiodic, while nearest-neighbor NBW is periodic. Therefore, the stationary distribution for NBW on products of complete graphs equals the uniform distribution on the torus, while for nearest-neighbor NBW, the parity of the position after \(n\) steps always equals that of \(n\). In Section 7.2.3, we further study random walk on the \(m\)-dimensional hypercube.

For any small \(\xi > 0\), we write \(T_{\text{mix}}(\xi)\) for the \(\xi\)-uniform mixing time of NBW, that is,
\[ T_{\text{mix}}(\xi) = \min \left\{ n: \max_{x,y} \frac{p_n(x,y) + p_{n+1}(x,y)}{2} \leq (1 + \xi)V^{-1} \right\}, \] (7.2.17)
where \(V = r^d\) is the volume of the torus. We start to investigate NBW on products of complete graphs:
**NBW on products of complete graphs.** Our main result is as follows:

**Lemma 7.2.3.** For every $d \geq 1$, $r \geq 3$, $n > \frac{d(r-1)}{r} \log((r-1)/([1 + \xi]^{1/d} - 1))$, NBW on products of complete graphs satisfies that

$$\max_{x \in \mathbb{T}_{r,d}} |p_n(x) - r^{-d}| \leq (m-1)^{(n-1)/2} + \xi r^{-d}.$$  

(7.2.18)

In particular, for every $\varepsilon > 0$, there exists $V_0$ such that when $r^d \geq V_0$,

$$T_{\text{mix}}(\xi) \leq (1 + \varepsilon) \frac{d(r-1)}{r} \log((r-1)/([1 + \xi]^{1/d} - 1)).$$

When $\xi$ is quite small, we obtain that $T_{\text{mix}}(\xi) \leq (1 + 2\varepsilon) \frac{r}{r-1} \log(d(r-1)/\xi)$.

**Proof.** The inverse Fourier transform on $\mathbb{T}_{r,d}$ is given by

$$f(x) = \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*} \hat{f}(k)e^{ik \cdot x},$$

so that

$$b_n(x) = \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*} \hat{b}_n(k)e^{ik \cdot x} = r^{-d} \hat{b}_n(0) + \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*: k \neq 0} \hat{b}_n(k)e^{ik \cdot x}. \quad (7.2.19)$$

Therefore,

$$|p_n(x) - r^{-d}| \leq \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*: k \neq 0} \frac{|\hat{b}_n(k)|}{\hat{b}_n(0)}. \quad (7.2.20)$$

To bound $|\hat{b}_n(k)|$, we rely on Theorem [7.2.1] and start by computing

$$\hat{D}(k) = \frac{1}{d(r-1)} \sum_{i=1}^d \sum_{j=1}^{r-1} e^{ik_i j}. \quad (7.2.21)$$

Since $k_i \in \frac{2\pi}{r} \{-\left\lfloor \frac{r-1}{2} \right\rfloor, \ldots, \left\lceil \frac{r-1}{2} \right\rceil\}$, we have that $\sum_{j=0}^{r-1} e^{ik_i j} = 0$ for $k_i \neq 0$. Therefore,

$$\hat{D}(k) = \frac{1}{d(r-1)} \sum_{i=1}^d \sum_{j=1}^{r-1} e^{ik_i j} = \frac{1}{d(r-1)} \sum_{i=1}^d (r \mathbb{1}_{k_i = 0} - 1) = 1 - \frac{r}{d(r-1)} a(k), \quad (7.2.22)$$

where $a(k) = \sum_{i=1}^d \mathbb{1}_{k_i \neq 0}$ denotes the number of non-zero coordinates of $k$. By this observation, $\hat{D}(k(j)) = 1 - \frac{r_j}{d(r-1)}$ for any $k(j) \in \mathbb{T}_{r,d}^*$ for which $a(k(j)) = j$. Then, by Theorem [7.2.1] and the fact that there are $\binom{d}{j}(r-1)^j$ values of $k \in \mathbb{T}_{r,d}^*$ for which
\[a(k(j)) = j,\]

\[
|p_n(x) - r^{-d}| = r^{-d} \sum_{j=1}^{d} \left( \frac{1}{\sqrt{m-1}} \vee |\hat{D}(k(j))| \right)^{n-1} \left( \frac{d}{j} \right) (r-1)^j \tag{7.2.23}
\]

\[
\leq r^{-d} \sum_{j=1}^{d} \left( \frac{d}{j} \right) (r-1)^j \left[ |1 - \frac{r^j}{d(r-1)}|^{n-1} + (m-1)^{-\frac{(n-1)}{2}} \right]
\]

\[
\leq (m-1)^{-\frac{(n-1)}{2}} + r^{-d} \sum_{j=1}^{d} \left( \frac{d}{j} \right) (r-1)^j e^{-r j(n-1)/|d(r-1)|},
\]

where we use that \(|1 - \frac{r^j}{d(r-1)}| \leq e^{-r j(n-1)/|d(r-1)|}\) for any \(j = 1, \ldots, d\). Thus,

\[
|p_n(x) - r^{-d}| \leq (m-1)^{-\frac{(n-1)}{2}} + r^{-d} \left[ (1 + (r-1)e^{-r(n-1)/|d(r-1)|})^d - 1 \right]
\]

\[
\leq (m-1)^{-\frac{(n-1)}{2}} + \xi r^{-d},
\tag{7.2.24}
\]

when \(n > \frac{d(r-1)}{r} \log \left( (r-1)/[(1 + \xi)^{1/d} - 1] \right)\). The result on \(T_{\text{mix}}(\xi)\) follows immediately. \(\square\)

**NBW on the nearest-neighbor torus.** Our main result is as follows:

**Lemma 7.2.4.** For every \(d \geq 1\), \(r \geq 3\) and \(n > \log \left( 2/[(1 + \xi/2)^{1/d} - 1]/[1 - \cos(2\pi/r)] \right)\), NBW on the \(d\)-dimensional nearest-neighbor torus satisfies that

\[
\max_{x \in \mathbb{T}_{r,d}} |p_n(x) - [1 + (-1)^{\|x\|_1 + n}] r^{-d}| \leq (2d - 1)^{-n/2} + \xi r^{-d}.
\tag{7.2.25}
\]

**In particular, for every \(\varepsilon > 0\), there exists \(V_0\) such that whenever \(r^d \geq V_0\),

\[
T_{\text{mix}}(\xi) \leq (1 + \varepsilon) \log \left( 2/[(1 + \xi/2)^{1/d} - 1]/[1 - \cos(2\pi/r)] \right).
\]

When \(r^d\) is large and \(\xi\) small, the above implies that

\[
T_{\text{mix}}(\xi) \leq (1 + 2\varepsilon)(r^2/(2\pi^2))(\log(2d/\xi)).
\tag{7.2.26}
\]

**Proof.** We adapt the proof of Lemma 7.2.3 to this setting. We have

\[
b_n(x) = \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*} \hat{b}_n(k)e^{ik \cdot x} = r^{-d} \left[ \hat{b}_n(0) + (-1)^{\|x\|_1} \hat{b}_n(\vec{\pi}) \right] + \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*; \ k \neq \vec{0}, \vec{\pi}} \hat{b}_n(k)e^{ik \cdot x}.
\tag{7.2.27}
\]

Further, \(\hat{D}(\vec{\pi}) = -1\), so that, by (7.1.19) \(\hat{b}_n(\vec{\pi}) = \hat{b}_n(0)(-1)^n\). Therefore,

\[
|p_n(x) - [1 + (-1)^{\|x\|_1 + n}] r^{-d}| = \left| \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*; \ k \neq \vec{0}, \vec{\pi}} \hat{b}_n(k)e^{ik \cdot x} \right| \leq \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*; \ k \neq \vec{0}, \vec{\pi}} \left| \hat{b}_n(k) \right|
\]

\[
\leq (m-1)^{\frac{(n-1)}{2}} + \frac{1}{r^d} \sum_{k \in \mathbb{T}_{r,d}^*; \ k \neq \vec{0}, \vec{\pi}} |\hat{D}(k)|^{n-1}.
\]
In the nearest-neighbor case, \( \hat{D}(k - \vec{\pi}) = -\hat{D}(k) \), so we may restrict to \( k \) for which \( \hat{D}(k) \geq 0 \). Let
\[
\mathbb{T}_{r,d,+}^* = \{ k \in \mathbb{T}_{r,d}^* : \hat{D}(k) \geq 0 \}
\]
denote the set of \( k \)'s for which \( \hat{D}(k) \geq 0 \). Then,
\[
|p_n(x) - [1 + (-1)^{\|x\|_1 + n}]r^{-d}| \leq (m - 1)^{(n-1)/2} + \frac{2}{r^d} \sum_{k \in \mathbb{T}_{r,d,+}^* : k \neq 0} \hat{D}(k)^{n-1}. \tag{7.2.29}
\]
We use that
\[
\hat{D}(k) = 1 - [1 - \hat{D}(k)] = e^{-[1 - \hat{D}(k)]}, \tag{7.2.30}
\]
so that
\[
|p_n(x) - [1 + (-1)^{\|x\|_1 + n}]r^{-d}| \leq (m - 1)^{(n-1)/2} + \frac{2}{r^d} \sum_{k \in \mathbb{T}_{r,d}^* : k \neq 0} e^{-(n-1)[1 - \hat{D}(k)]} \tag{7.2.31}
\]
\[
= (m - 1)^{(n-1)/2} + \frac{2}{r^d} \left[ \sum_{k \in \mathbb{T}_{r,d}^*} e^{-(n-1)[1 - \hat{D}(k)]} - 1 \right]
\]
\[
= (m - 1)^{(n-1)/2} + \frac{2}{r^d} \left[ \left( \sum_{k \in \mathbb{T}_{r,1}^*} e^{-(n-1)[1 - \cos(k)/d]} \right)^d - 1 \right].
\]
We use that the dominant contributions to the sum over \( k \in \mathbb{T}_{r,1}^* \) comes from \( k = 0 \) and \( k = \pm 2\pi/r \), so that
\[
\sum_{k \in \mathbb{T}_{r,1}^*} e^{-(n-1)[1 - \cos(k)/d]} = 1 + 2e^{-(n-1)[1 - \cos(2\pi/r)/d]}(1 + o(1)), \tag{7.2.32}
\]
so that
\[
|p_n(x) - [1 + (-1)^{\|x\|_1 + n}]r^{-d}| \leq (m - 1)^{(n-1)/2}
\]
\[
+ \frac{2}{r^d} \left[ \left( 1 + 2e^{-(n-1)[1 - \cos(2\pi/r)/d]}(1 + o(1)) \right)^d - 1 \right]
\]
\[
\leq (2d - 1)^{-(n-1)/2} + \xi r^{-d} \tag{7.2.33}
\]
when \( n > (\log(2/[(1 + \xi/2)^{1/d} - 1])/[1 - \cos(2\pi/r)]. \)

\[\square\]

**7.2.3 NBW on the hypercube**

In this section we specialize the results of Sections \[7.1\] and Sections \[7.2.1\]-\[7.2.2\] to the hypercube \( \mathbb{T}_{2,m} = \{0, 1\}^m \). The results in this section are an important ingredient to the analysis of percolation on the hypercube in \[51\]. We start with some notation.
It will be convenient to let the Fourier dual space of \( \{0,1\}^m = \{0,1\}^m \) be \( \mathbb{Q}_m^* = \{0,1\}^m \), so that the Fourier transform of a summable function \( f: \{0,1\}^m \to \mathbb{C} \) is given by

\[
\hat{f}(k) = \sum_{x \in \{0,1\}^m} f(x) e^{i \pi k \cdot x} = \sum_{x \in \{0,1\}^m} f(x)(-1)^{k \cdot x}, \quad (k \in \{0,1\}^m).
\]

For \( k \in \{0,1\}^m \), let \( a(k) \) be its number of non-zero entries. Then, the SRW step distribution on \( \{0,1\}^m \) satisfies

\[
\hat{D}(k) = \frac{1}{m} \sum_{j=1}^{m} (-1)^j = 1 - 2a(k)/m. \quad (7.2.34)
\]

The main result of this section is as follows:

**Theorem 7.2.5 (NBW on hypercube).** *For NBW on the hypercube \( \{0,1\}^m \),

\[
\hat{p}_n(k) \leq \left( |\hat{D}(k)| \vee 1/\sqrt{m-1} \right)^{n-1}. \quad (7.2.35)
\]

Consequently, for every \( \varepsilon > 0 \), there exists \( m_0 \) such that for all \( m \geq m_0 \),

\[
T_{\text{mix}}(\xi) \leq \frac{m(1+\varepsilon)}{2} \log(2m/\xi). \quad (7.2.36)
\]

Random walks on hypercubes have attracted considerable attention, in fact, the bound on the uniform mixing time for NBW in discrete time closely matches the one for SRW in continuous time (see [5, Lemma 2.5(a)]).}

**Proof.** The bound in (7.2.35) follows directly from Theorem 7.2.1. We continue to investigate the convergence of the NBW transition probabilities \( x \mapsto p_n(x) \) to its quasi-stationary distribution, which is \( 2 \times 2^{-m} \) when \( n \) and \( x \) have the same parity and 0 otherwise. Here we say that \( n \) and \( x \) have the same parity when there exists an \( n \)-step path from 0 to \( x \).

**Lemma 7.2.6 (Convergence to equilibrium for NBW on hypercube).** *For NBW on \( \{0,1\}^m \) and every \( n > d(\log d + \log \xi)/2 \),

\[
\max_{x \in \{0,1\}^m} |p_n(x) - 2^{-m} [1 + (1/2)^{\|x\|_1 + n}]| \leq (m - 1)^{-(n-1)/2} + \xi 2^{-m}. \quad (7.2.37)
\]

Consequently, for every \( \varepsilon > 0 \), there exists \( m_0 \) such that for every \( m \geq m_0 \),

\[
T_{\text{mix}}(\xi) \leq -\frac{m(1+\varepsilon)}{2} \log([1 + \xi/2]^{1/m} - 1).
\]

**Proof.** The inverse Fourier transform on the \( m \)-dimensional hypercube is given by

\[
f(x) = 2^{-m} \sum_{k \in \{0,1\}^m} \hat{f}(k)(-1)^{k \cdot x},
\]
so that, using $\hat{b}_n(\mathbf{1}) = (-1)^n \hat{b}_n(0)$,

$$b_n(x) = 2^{-m} \sum_{k \in \{0,1\}^m} \hat{b}_n(k)(-1)^{k \cdot x}$$

$$= 2^{-m} \left[ 1 + (-1)^{\|x\|_1 + n} \right] \hat{b}_n(0) + \sum_{k \in \{0,1\}^m : k \neq \mathbf{1}, \mathbf{0}} \hat{b}_n(k)(-1)^{k \cdot x}. \quad (7.2.38)$$

Substituting the bound (7.2.35) in (7.2.38) leads to

$$|p_n(x) - 2^{-m} [1 + (-1)^{\|x\|_1 + n}]| \leq 2 \cdot 2^{-m} \sum_{j=1}^{m/2} \binom{m}{j} \left[ (1 - 2j/m)^{n-1} + (m-1)^{-n/2} \right]$$

$$\leq (m-1)^{-n/2} + 2 \cdot 2^{-m} \sum_{j=1}^{m/2} \binom{m}{j} e^{-2j(n-1)/m}$$

$$\leq (m-1)^{-n/2} + 2 \cdot 2^{-m} \left[ (1 + e^{-2(n-1)/m})^{m-1} - 1 \right]$$

$$\leq (m-1)^{-n/2} + \xi 2^{-m}, \quad (7.2.39)$$

when $n > -\frac{m}{2} \log (\{1 + \xi/2\}^{1/m} - 1)$. \hfill \square


[76] Y. Miranda and G. Slade. Expansion in high dimension for the growth constants of lattice trees and lattice animals.


In September 2008, Robert moved to the Netherlands and started as a Ph.D. student at Technische Universiteit Eindhoven and EURANDOM. Under the guidance of Prof. Remco van der Hofstad, he worked on mean-field behavior of statistical mechanical models. The results of this study are presented in this thesis. He presented his research in Vancouver, Saint-Flour, Hilversum, Berlin, Istanbul, Leipzig, Utrecht, Stockholm and Eindhoven. Starting October 2013, Robert will work as a postdoctoral researcher at the Stockholm University.
Acknowledgements

It took Remco and me four and a half years to realise the NoBLE project and even at this stage the project is not fully completed. The size of the project made working on it sometimes hard. In particular, the last year, in which I put all notes together to create this document and finished the accompanying implementations, was very exhausting. Nevertheless, it has been a great pleasure to create the NoBLE, to learn so many new things about math, me and life in general and to get to know so many outstanding people. Over the last years, numerous people supported me. I would like to express my deep gratitude for this to all of them, and want to address now only a few of them.

First of all, I would like to thank my fiancée Marina for her support in the recent years. She always encouraged me and allowed me to find and to keep a healthy balance between work and private life. Further, I want to thank my daughter Sophie. She is pure joy and is for me the perfect example of a deep belief of mind: “You should never stop exploring the world and learn new things.” Seeing how she learned to walk, to talk and to ask “why”, motivated me even in the darkest hours of the PhD.

Markus Heydenreich deserves a prominent place in the list of people that I need to thank. Five years ago I approached him with the question of what to take into consideration when choosing a PhD-position. Instead of answering my question he brought me to Eindhoven and ever since share with me this experience of probability, research, academic life and Remco as supervision. For this I am very grateful.

Considering all the people from whom I learned a lot in the last years I especially want to thank two researchers, who helped me and the NoBLE project to an large extent. Guido Janssen gave us essential advice in the early stages on the project, when we still wanted to use matrix perturbation argument to analyze the equations. His experience was very helpful for us and made us realise that this was not a suitable approach.

Takashi Hara deserves many thanks for sharing his experiences concerning the lace expansion with us. The computation of Section 3.6 are based upon joint discussions
during our stay in Istanbul (2012) and the analysis of Section 3.5 could not have been finished in time without his help.

Next, I would like to thank my office mates, Sander Dommers and Tim Hulshof, with whom I presumably spent more time in the last years, than with my fiancée. It was a great time sharing the office with you. I want to thank Sander for always knowing who to do stuff at the university and in Eindhoven and Tim for the many fruitful discussions and second opinions on the small things. Moreover, I would like to thank Tim and Stella Kapodistria for all their advices on the typesetting and formatting of the thesis. Stella was of great assistance at the later stages of the PhD, including also here help with applications for a followup position. Thank you for that.

Regarding this document I would also like to thank Sabrina Schröder and my father for helping me correcting uncountably many language mistakes that I have been in the early versions of the thesis.

Also, I would also like to thank my room mate and colleague Botond Szabó, for always staying interesting and coming up with the right questions at the right time. On many occasion he challenged me to explain my topic and techniques to him, a mathematician from a different field of research. Further, Botond kept our small social group of young mathematicians in Eindhoven together, and prevented all - or at least most - of us from going insane during their PhD.

Finally and foremost, I want to thank my supervisor Remco van der Hofstad for this guidance and never ending support during the last years. He created an excellent environment for all of his PhD-students. He always found time for our questions and gave us the freedom to travel and to find our own way to do research. It was a pleasure working with him and I hope that we will continue collaborating also after the completion of the NoBLE project.