Nonsmooth dynamical systems

on stability of hybrid trajectories and bifurcations of discontinuous systems

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Chapter 1

Introduction

1.1 Nonsmooth systems

Nonsmooth systems are used to model various dynamical behaviours in physics, engineering, biology, and economy. In these models, nonsmooth elements are added such that complex dynamical characteristics can be captured. In this manner, nonsmooth systems are obtained that can provide a dynamical model at a level of abstraction that is appropriate for design, analysis or control purposes. In this thesis, the qualitative dynamical behaviour and the stability of nonsmooth systems are analysed to support the application of such systems.

To illustrate one of the dynamical characteristics that are effectively modelled with nonsmooth elements, observe that a printed copy of this thesis does not show any motion shortly after it has been placed on a desk, even though the desk its surface is probably not perfectly horizontal. Hence, the thesis arrives at an equilibrium position in a finite time. If Newton’s second law is used to describe this dynamics, then a discontinuous force model for the friction between the desk and the thesis, e.g. Coulomb’s friction law, can accurately model the observed behaviour, without the infinitely long transient behaviour that is expected in smooth differential equations, and without the need to model the detailed profile of the surface of the desk.

Next to the possibility to attain equilibrium positions in finite time, this example illustrates another reason to employ nonsmooth dynamical models. If the desk with the thesis is slightly tilted, no motion is expected, and the thesis will stick to its original position. Consequently, the equilibrium position is completely robust with respect to certain perturbations, including small tilting of the desk. Equilibrium points of smooth differential equations will not show such robustness properties, whereas a discontinuous dynamical model that uses
Coulomb’s friction law does effectively represent this behaviour.

In addition to finite-time convergence and different robustness properties, various other properties of dynamical systems have motivated the use of non-smooth models. For example, non-smooth systems provide an effective method to model dynamical behaviour that combines slow and fast motions. If the timescales of the slow and fast motions are far apart, then smooth differential equations will have steep gradients. This makes their numerical simulation cumbersome, and complicates the application of conventional results on stability, control and robustness. A way to proceed is to model the fast phenomena to occur instantaneously. In mechanical systems with unilateral constraints, this approach leads to non-smooth dynamical models with idealised contact and impact laws. Such models are called hybrid systems, and contain both continuous-time evolution and discrete events, i.e. instantaneous jumps in the system states, that represent the fast phenomena. Hybrid systems form an important subset of the class of non-smooth systems. Hybrid models can also be employed to model systems that feature both continuous evolution in time and discrete events, as is e.g. apparent in switching control systems.

Non-smooth systems have been used to model dynamical behaviour in various disciplines. In mechanical engineering, the interaction between rigid bodies can be modelled using non-smooth force laws for contact, impact and friction, leading to non-smooth dynamical systems, cf. [66, 103, 104, 125, 159]. In control engineering, sliding mode controllers [144, 154], controller saturation [150], switching piecewise affine controllers [50, 109], and networked control systems [78] are usually modelled with non-smooth dynamical systems. In various control systems, including sliding mode controllers and control systems with hysteresis elements, non-smooth elements are intentionally introduced in the closed-loop system to exploit properties that are specific to non-smooth systems, such as finite-time transients or increased robustness, cf. [136, 144]. In electrical engineering, switches (relays) and diodes can induce non-smooth dynamical behaviour, cf. e.g. [3, 14, 65]. Non-smooth dynamical models are also used in other disciplines, such as physics [89], biology [113] and economics [124].

This thesis is focussed on non-smooth systems that display continuous-time evolution, and dynamical systems described by difference equations (i.e. iterations of maps) are only used as an analysis tool for non-smooth systems with continuous-time evolution. The class of continuous-time systems with non-smooth behaviour, as studied in this thesis, can be classified in the following three cases with increasing degree of non-smoothness, cf. [15, 102, 103, 143]:

- **Non-differentiable systems** – Systems described by a continuous differential equation that is piecewise smooth, such that it is non-differentiable only at some \((n-1)\)-dimensional surfaces in an \(n\)-dimensional state space. An example is given by a mechanical system with a one-sided spring, cf. Fig. 1.1. Despite the non-differentiability of the right-hand side, these differential equations still satisfy a local Lipschitz’s condition, see e.g. [91],
1.1 Nonsmooth systems

Non-differentiable systems

\[ M\ddot{x} + c\dot{x} + g = \begin{cases} 0, & x \geq 0 \\ -kx, & x < 0 \end{cases} \]

Discontinuous systems

\[ M\ddot{x} + kx + c\dot{x} \in \begin{cases} -\mu Mg, & \dot{x} > 0 \\ [-\mu Mg, \mu Mg], & \dot{x} = 0 \\ \mu Mg, & \dot{x} < 0 \end{cases} \]

Hybrid systems

\[ \begin{cases} M\ddot{x} + c\dot{x} = -Mg, & x > 0 \\ \text{impact+unilateral contact law}, & x = 0 \end{cases} \]

Fig. 1.1. Examples of the three types of nonsmooth systems.

such that the standard definition of solutions for differential equations can be applied. Consequently, for a given initial condition, a unique solution curve is found that is differentiable with respect to time. The lack of differentiability of the vector field renders existing results on stability and robustness for smooth systems inapplicable when they rely on local linearisation. For example, the indirect method of Lyapunov to study stability, cf. [91], and the Hartman-Grobman theorem to study robustness, cf. [74], can no longer be employed.

- **Discontinuous systems** – Systems described by an \( n \)-dimensional differential equation that is piecewise smooth and contains discontinuities at some \( (n - 1) \)-dimensional surfaces in the state space. For example, Coulomb’s
law, that describes the dry friction force in mechanical systems, has a discontinuous dependency on the slip velocity, such that, in the presence of Coulomb’s friction, application of Newton’s second law yields a discontinuous system. For discontinuous differential equations, the standard notion of solutions can not be used, and, in general, solutions can not always be continued uniquely from their initial conditions. Various definitions of solutions have been given for discontinuous systems, see, e.g., [7,45,58,154]. Throughout this thesis, the solution concept of Filippov [58] is employed, that replaces the discontinuous differential equation by a differential inclusion, yielding solution curves that are continuous, but not necessarily differentiable with respect to time. Discontinuous systems are illustrated in Fig. 1.1 by the example of a mechanical system that experiences dry friction described with a friction coefficient $\mu$. The depicted solution shows that the velocity is not always differentiable with respect to time and that convergence to an equilibrium position occurs in a finite time period, which contrasts the behaviour that can be observed in any smooth dynamic counterpart.

- **Hybrid systems** – Systems that show both continuous evolution in time and jumps of the state (i.e., discrete events). Hence, models of hybrid systems contain both a differential equation, a description of the jumps, and information when jumps or continuous-time evolution occurs. Mechanical systems with impacts, for example, can be described by the combination of a differential equation, that gives the continuous-time evolution, and an impact law, that relates post- and pre-impact velocities, cf. Fig. 1.1. In the literature, various frameworks have been presented to describe hybrid systems, including the hybrid system framework in [67,68], complementarity systems [77], automata [112], the framework presented in [139], and measure differential inclusions [119]. Hybrid systems are sometimes referred to as jumping, impulsive or impacting systems.

Note that non-differentiable systems are contained in the set of discontinuous systems, and discontinuous systems are a subclass of hybrid systems. In addition, the class of nonsmooth systems contains smooth differential equations. Nonsmooth systems can therefore display richer behaviour than what is possible in smooth systems.

In principle, a nonsmooth model can be approximated by a smooth system when nonsmooth functions in the differential equation are replaced by smooth functions that have steep gradients, and jumps of hybrid systems are modelled by smooth dynamics on a fast time scale. However, the smooth approximation (which is sometimes called regularisation, cf. [38, 71, 151]) of nonsmooth systems can introduce three disadvantages. Firstly, the nonsmooth system can have different limiting behaviour than the smooth approximation. For example, mechanical systems with dry friction contain equilibrium sets that consist of a
1.2 Motivation and objectives

In this thesis, two different problems are considered in nonsmooth dynamical systems. Consequently, the thesis consists of two parts. In Part I, that contains Chapters 2-5, the effect of small parameter variations on the qualitative behaviour of nonsmooth systems is studied and, in particular, bifurcations in several classes of nonsmooth systems are investigated. In Part II, that consists of Chapter 6, the tracking control problem for hybrid systems is considered. In the present section, the motivation for both research topics is given. This introductory chapter is written in a concise fashion to highlight the motivation and contributions of the thesis. Detailed literature surveys are presented in the individual chapters.

1.2.1 Bifurcation analysis

The qualitative behaviour of a dynamical system is mainly determined by its limit sets, the stability of the limit sets, and the trajectories converging to, or diverging from these sets. Limit sets consist of the points in the state space to which trajectories tend when time evolves to infinity, or from which trajectories have diverged in the past. Examples of limit sets are equilibria and periodic orbits. In general, small changes of a dynamical system will only change the dynamics quantitatively, and such changes will not induce qualitative changes of the limit sets. Loosely speaking, changes in the qualitative behaviour of a dynamical system, such as the loss of stability of an equilibrium point or the creation of a new limit set, are called bifurcations when they are caused by a variation in a system parameter. A formal definition of bifurcations will be given in Chapter 2. We note that bifurcations induce structural changes in the phase portrait of a dynamical system.

A dynamical system is called structurally stable when small perturbations of the system cannot induce qualitative changes in its behaviour. Hence, an arbitrary small variation of the system parameters may induce qualitatively different dynamics only when a system is not structurally stable. Consequently, structural stability excludes the occurrence of bifurcations, and vice versa.

The study of structural stability and bifurcations in dynamical systems is motivated in two ways. Firstly, structural stability and bifurcation results give important insight in the robustness to parameter variations in the analysis, design
and modelling of dynamical systems. When the amplitude of an external disturbance is considered as a system parameter, knowledge on the possible bifurcations can be used to assess the robustness for disturbances. Secondly, the study of bifurcations allows to explore the types of behaviour that can be displayed by a given dynamical system, thereby giving insight in the overall dynamics. Such insight can, for example, be instrumental in supporting the design of robust systems.

The theory of bifurcations and structural stability in smooth dynamical systems is well developed, cf. [72, 95, 127, 132, 142, 145], and hinges on dimension reduction by methods such as the center manifold theorem, cf. [43] and normal hyperbolicity, cf. [57, 142], and the analysis of low-dimensional systems using normal forms, which, in the simplest cases, can be obtained using linearisation, cf. [74].

In nonsmooth dynamical systems, both dimension reduction techniques and bifurcations in low-dimensional systems are not well understood and are still subject to ongoing research. In Part I of this thesis, bifurcations of low-dimensional systems are discussed. The development of dimension reduction techniques falls outside the scope of this thesis.

A large portion of the current literature on bifurcation theory for nonsmooth dynamical systems deals with non-differentiable or discontinuous systems of low dimensions (in particular, systems with a two or three dimensional state space), see e.g. [15,103,114]. Moreover, most results are focussed on bifurcations of the local dynamical behaviour near limit cycles and isolated equilibrium points. It is common to assume that the nonsmooth behaviour is localised to one surface in the state space, and away from this surface, trajectories are described by a locally smooth differential equation.

In the neighbourhood of limit cycles, bifurcation scenarios have been identified where, due to parameter variation, a limit cycle becomes tangential to a surface where the vector field is nonsmooth, or when a limit cycle arrives at a point where this surface is not-differentiable, e.g. makes a sharp corner. Bifurcations induced by such interactions are called grazing and boundary crossing bifurcations, respectively, cf. [8,15,47,58,123,155]. Normally, such bifurcations are investigated using a Poincaré return map, see e.g. [138], that reduces the problem to the analysis of the iterates of a non-differentiable or discontinuous difference equation.

Various bifurcations have been investigated that can occur in the neighbourhood of equilibrium points for nonsmooth systems, see e.g. [14,56,58,103]. Classification of these bifurcations has yielded lists of bifurcations in non-differentiable and discontinuous systems, as can be found in [14,71,96,143,151]. In these studies, systems are considered where the discontinuity of the vector field is restricted to an $n-1$-dimensional surface in the $n$-dimensional state space. The vector fields on both sides of this surface are chosen to be two arbitrary vector fields with no relation between them, (in contrast to, e.g., models of mechanical systems
1.2 Motivation and objectives

with dry friction, where the difference between the vector field on both sides of the discontinuity is constant). Certain behaviour that does occur in physical systems, (e.g. the existence of non-isolated equilibrium positions in mechanical systems with friction) will be considered highly non-generic in this approach. On the other hand, some bifurcations that occur generically according to the lists presented in [14, 71, 96, 143, 151] have not yet been observed in discontinuous system models of physical systems.

The bifurcation analysis presented in Chapter 2 can be considered as an extension of this line of research towards the case where, in the neighbourhood of an equilibrium point, the nonsmoothness is not completely located on a single smooth 1-dimensional surface (i.e. a smooth curve) in the 2-dimensional state space, but either a single non-differentiable curve, or multiple intersecting curves are required to describe the set where the vector field is non-differentiable. Hence, the approach presented in Chapter 2 can be used to study the case where three or more distinct domains are present in which the trajectories are described with different smooth differential equations. The results in Chapter 2 also illustrate that a complete classification of all possible bifurcations will be very hard to achieve, as this classification should contain very diverse bifurcation scenarios, even when restricted to the simple case of non-differentiable two-dimensional systems.

In [86, 163] and Chapters 3-5, an alternative approach is pursued, namely, structural stability and bifurcations of specific nonsmooth systems are investigated, that model mechanical systems with dry friction. This restricts the class of vector fields under study, such that dynamical behaviour appears that is rather non-generic in the full class of discontinuous systems, but which does correspond to behaviour that is highly relevant in mechanical systems. In particular, equilibrium points are, in general, not isolated for mechanical systems with dry friction, but appear in connected sets of non-isolated equilibrium points, which, in this thesis, are called equilibrium sets. In Chapter 3, structural stability and bifurcations are investigated of the local vector field near such equilibrium sets. In Chapter 4, the effect on the dynamics of this system is discussed of periodic perturbations, and a new type of limit set is discovered that can be induced by the perturbation of such discontinuous differential equations. The new limit set has some interesting analogies with chaotic saddles found in smooth systems, cf. [98, 145]. In Chapter 4, the properties of the limit set are analysed by construction of a Poincaré return map. Motivated by this argument, in Chapter 5, we analyse the dynamics of such maps in more detail.

1.2.2 Tracking control for hybrid systems

In Part II of this thesis, the tracking control problem for hybrid systems is studied. In tracking control problems, a control law should be designed such that a given physical system, that is called the plant and that is modelled as a
Chapter 1. Introduction

dynamical system with inputs, performs a task by following a pre-described reference trajectory. Since initial conditions of the plant will not be exactly known, the controller should induce asymptotic stability of the reference trajectory. In words, the stability of a trajectory implies that, if another trajectory is initially close, it remains close in the future. Asymptotic stability implies, in addition to stability, that both trajectories tend towards each other over time. In this part, a notion of stability of trajectories is formulated that can be applied to hybrid systems with state-triggered jumps, and show that this notion is instrumental in the design of a control law solving the tracking problem.

In smooth systems, the stability of a trajectory is conventionally analysed by evaluation of the difference between the given trajectory and trajectories with nearby initial conditions, cf. [130, 157]. Usually, the evolution of this difference over time is called the error system. The asymptotic stability of the given trajectory is equivalent to the asymptotic stability of the origin for the error system, which is commonly defined in the sense of Lyapunov, cf. [91].

In non-differentiable or discontinuous dynamical systems, this stability notion can directly be employed. However, for hybrid systems, this approach is not always satisfactory. Namely, the jump times of two trajectories of a hybrid system do not necessarily coincide, i.e. they may display a small timing mismatch. During this time interval, the Euclidean distance between the states of both trajectories is expected to be large. Hence, the conventional Euclidean error between both trajectories cannot be made arbitrarily small by selecting the initial conditions sufficiently close, contradicting asymptotic stability in terms of the Euclidean distance, as observed e.g. in [75,104,116]. A small timing mismatch of the jumps (converging to zero over time), however, does not always contradict desired behaviour. For this reason, asymptotic stability, evaluated in terms of the Euclidian distance between the states of two hybrid systems, can be infeasible, even though intuitively desirable behaviour can still be achieved.

In Part II of this thesis, we propose a novel notion for the stability of trajectories for hybrid systems, that does not require that jump times coincide, but still corresponds to an intuitive notion of stability for jumping solutions. Using this notion of stability, in Chapter 6, a tracking control problem is formulated that is feasible for a larger class of hybrid systems, including hybrid systems with state-triggered jumps. For two exemplary systems, controllers are designed that solve this problem.

1.3 Main contributions

Based on the discussion in the previous section, the main contributions of this thesis with respect to the bifurcation analysis of nonsmooth systems (Part I of this thesis) can be summarised as follows:

- For two-dimensional non-differentiable systems, in Chapter 2, a procedure
1.3 Main contributions

is presented to identify all limit sets that can be created when under parameter variation, an equilibrium arrives at a point where the vector field is non-differentiable. This procedure does not require the nonsmoothness of the vector field to be restricted to a single differentiable surface in the state space, and allows to study bifurcations in which an equilibrium point arrives at the intersection of multiple surfaces where the vector field is non-differentiable, or where such a surface displays a ‘kink’. The procedure is based on a local approximation that represents the dynamics in the neighbourhood of the equilibrium point with affine differential equations that are employed in cones of the state space. Conditions are provided that guarantee that the topological structure of the vector field of the original system is accurately represented by the approximation.

The applicability of this procedure is illustrated in a few examples. In these examples, using this procedure, new bifurcations are shown that can change the dynamics near isolated equilibrium points.

• Sufficient conditions for the structural stability of the dynamics near equilibrium sets in discontinuous systems describing mechanical systems with dry friction are presented in Chapter 3. Focussing on systems where dry friction acts only in one direction, it is shown that equilibrium sets are intervals of curves in the state space. It is shown that parameter changes can only induce qualitative changes of the vector field near the endpoints of these equilibrium sets. This result significantly simplifies the further study of structural stability and bifurcations of equilibrium sets for this class of mechanical systems with friction.

Since the obtained conditions for structural stability are only sufficient, the conservatism of these conditions is studied in examples. In these examples, no conservatism is observed, since violation of the conditions for structural stability directly induces bifurcations of the discontinuous system in the neighbourhood of the equilibrium set.

Furthermore, it is argued in Chapter 3 that the class of allowed perturbations should be tailored to the application of the discontinuous system under study. When arbitrary perturbations of the right-hand side are considered, as is common in smooth dynamical systems, then such perturbations will generically destroy the equilibrium sets, whereas, from a practical point of view, equilibrium sets in mechanical systems with friction are expected to be persistent at most system parameters.

• In Chapters 4, the effect of time-dependent perturbations is studied in discontinuous differential equations describing mechanical systems with friction. It is shown that small time-dependent perturbations can induce unexpected dynamical behaviour when the discontinuous system contains a homoclinic trajectory that emanates from an equilibrium set. A new
Chapter 1. Introduction

A type of limit set is discovered in Chapter 4 that governs this behaviour and is an analogue of a chaotic saddle in smooth systems. By construction of a Poincaré return map, the properties of the newly found limit set are analysed, and many similarities are found between this chaotic saddle and the chaotic saddles in smooth systems. In contrast to smooth chaotic saddles, the discontinuous behaviour of friction makes the future dynamics qualitatively different from the dynamical behaviour in backward time. In addition, the fractal geometry of the newly discovered limit set is not similar to the smooth chaotic saddle: the discontinuous effect of friction has simplified this geometry in one direction.

- To analyse the dynamics in the limit set discovered in Chapter 4 in more detail, in Chapter 5, a class of discrete-time systems is studied which we expect to describe the phenomena discussed in Chapter 4. A rigorous description is given of the geometry of the limit set of this class of discrete-time systems. In addition, a symbolic dynamical system is presented that is topologically conjugate to the discrete-time system under study. Using this topological conjugacy, it is proven that the limit set contains an infinite number of periodic orbits and, in addition, that the limit set is transitive.

The contributions of this thesis on tracking control for hybrid systems (Part II of this thesis) are summarised as follows:

- Tracking control problems are naturally formulated by requiring asymptotic stability of a given reference trajectory. The stability of jumping trajectories of hybrid systems, however, can not always be studied using the Euclidean distance measure. This problem is overcome in Chapter 6 by the formulation of a new distance function. Using this distance function, a definition of stability of trajectories is formulated that does correspond to an intuitively correct notion of stability. This stability concept allows to study stability problems of jumping trajectories in various disciplines, including tracking control, synchronisation, and observer design.

- The applicability of the new notion of stability is illustrated for tracking control problems in two examples of hybrid systems, including a mechanical system with a unilateral position constraint. For these systems, it is shown that the new distance function greatly facilitates the design of controllers that achieve asymptotic stability of a jumping reference trajectory, and that the closed-loop system indeed shows desirable tracking behaviour.

1.4 Outline of this thesis

As mentioned above, the contributions of this thesis are divided between results on bifurcation theory, and results on the stability of trajectories. Consequently, the thesis consists of two parts.
1.4 Outline of this thesis

In Part I, bifurcations and structural stability are investigated in nonsmooth systems. In Chapter 2, bifurcations of non-differentiable systems near an equilibrium point are considered and a procedure is presented to find all limit sets that are created in such bifurcations.

In Chapter 3, equilibrium sets are studied in discontinuous systems describing mechanical systems with dry friction. Focussing on trajectories in the neighbourhood of the equilibrium set, structural stability and bifurcations are investigated. In addition, it is shown that perturbations of the full right-hand side of the differential equation are not appropriate to model perturbations of mechanical systems with dry friction. At the end of this chapter, it is recommended to further study bifurcations of specific classes of discontinuous systems relevant in applications, in contrast to generic discontinuous systems, where little structure is imposed on the vector field.

In Chapters 4 and 5, the effect of time-dependent perturbations is investigated in discontinuous systems describing mechanical systems with dry friction. A phenomenological description of the resulting dynamics is presented in Chapter 4, whereas a mathematically more rigorous study is presented in Chapter 5. In particular, in Chapter 5, the topological properties of the trajectories in the newly discovered limit set are studied with a dynamical representation given by a map between infinite strings of symbols. In particular, under a technical assumption, it is proven that this symbolic dynamics is topologically conjugate to the dynamics occurring in the discontinuous mechanical system under study.

In Part II, that consists of Chapter 6, the tracking control problem for hybrid systems is studied and the stability of jumping trajectories is investigated. A new notion of stability is formulated, based on a non-Euclidean tracking error measure. For two exemplary systems, this tracking error measure is designed and the resulting tracking problem is formulated. Moreover, based on this tracking error measure, controllers are designed solving the tracking control problem.

Conclusions of this research are presented in Chapter 7. In addition, this chapter presents recommendations for future research on nonsmooth and hybrid systems.
Part I

Bifurcation analysis of nonsmooth systems
Chapter 2

Nonsmooth bifurcations of equilibria in planar continuous systems

Abstract – In this chapter, we present a procedure to find all limit sets near bifurcating equilibria in a class of hybrid systems described by continuous, piecewise smooth differential equations. For this purpose, the dynamics near the bifurcating equilibrium is locally approximated as a piecewise affine system defined on a conic partition of the plane. To guarantee that all limit sets are identified, conditions for the existence or absence of limit cycles are presented. Combining these results with the study of return maps, a procedure is presented for a local bifurcation analysis of bifurcating equilibria in continuous, piecewise smooth systems. With this procedure, all limit sets that are created or destroyed by the bifurcation are identified in a computationally feasible manner.

2.1 Introduction

In this chapter, local bifurcations are studied for a class of hybrid systems described by continuous, piecewise smooth differential equations. This type of system models can be used to describe mechanical, electrical, biological or economical systems, see e.g. [48, 103, 109, 124]. These systems can exhibit the so-called discontinuity-induced bifurcations, see [15, 103]. In this chapter, we study discontinuity-induced bifurcations of equilibria in planar systems. We present a procedure to find all limit sets which are created or destroyed by the bifurcation of an equilibrium point. Using this procedure, all these limit sets are identified in a computationally feasible manner.

This chapter is based on [23], and parts have appeared in [22].
The state space of piecewise smooth systems can be partitioned in a number of domains where the dynamics is smooth, and their boundaries, where the dynamics is nonsmooth. Discontinuity-induced bifurcations are topological changes in behaviour when system parameters are varied around the values where a limit set collides with such a boundary. Although the effect of such bifurcations is observed both in simulations and experiments, [15, 103], no complete theory is available to describe these bifurcations.

In planar autonomous systems, limit sets can be equilibria, periodic orbits (including limit cycles), homoclinic or heteroclinic orbits. Discontinuity-induced bifurcations of periodic orbits and homoclinic or heteroclinic orbits can be studied by taking a Poincaré section transversal to these orbits and analysing the resulting return map. In this manner, bifurcations of limit cycles in piecewise smooth dynamical systems are rather well understood, cf. [15, 123].

Several studies exist in which bifurcations of equilibria are investigated, where at the bifurcation point the equilibrium is positioned on a single, smooth boundary, see [14, 15, 101]. However, no theoretical result is available when this equilibrium is positioned on multiple boundaries, or when the boundary is a locally nonsmooth curve in state space. Existence of such bifurcations was recognized in numerical simulations of exemplary systems in [101, 102].

The main contribution of this chapter is a procedure for a class of planar hybrid systems, namely systems described by continuous, piecewise smooth differential equations, to find all limit sets that can be created or destroyed during a bifurcation of an equilibrium. Using this procedure, all limit sets that are created or destroyed during a bifurcation are identified in a computationally feasible manner.

To analyse the dynamics near the bifurcation point, we construct a local approximation of the dynamics in a neighbourhood of the bifurcating equilibrium, such that we obtain an approximate system, where the dynamics is affine with respect to the bifurcation parameter in each smooth domain, that is a cone. Furthermore, the dynamics is dependent on the bifurcation parameter in the affine term. These systems are called conewise affine systems, and also represent a class of hybrid systems. We derive criteria under which the limit set of the nonsmooth systems are accurately described by the approximated system.

To exclude closed orbits in certain regions of state space, Bendixson’s Theorem and index theory are used. To obtain all closed orbits in the remaining part of the state space, return maps are derived, whose Poincaré sections are chosen at locations that are selected by the investigation of specific trajectories. Fixed points of these return maps determine the existence, location and stability of limit cycles or closed orbits.

We derive general conditions for the existence of a halfline in the conewise affine system, that can not be traversed by closed orbits. Using these conditions, one can guarantee that all limit sets can be found in a computationally feasible manner with the given procedure. According to index theory, closed orbits,
including limit cycles, should encircle at least one equilibrium point. Derivation of all possible return maps for the trajectories that cross a line between the equilibria and the halfline mentioned above will obtain all existing closed orbits. The domain of these return maps is bounded, such that all fixed points can be detected efficiently with numerical methods.

Although the Poincaré-Bendixson theorem can be used to give sufficient conditions for the existence of limit cycles, cf. [74], we will use a different approach to guarantee that all limit cycles are identified.

This chapter is organized as follows. In Section 2.2, some preliminary results are given, including the local approximation of the piecewise smooth system by a conewise affine system. Subsequently, in Section 2.3, the stability of an equilibrium of the resulting conewise affine system at the bifurcation point is investigated. In Section 2.4, the main theoretical results of this chapter are presented, together with the procedure to find all limit sets near the bifurcation point. Subsequently, in Section 2.5, the effect of the used approximation is studied. The presented procedure is illustrated with examples in Section 2.6. Finally, conclusions are formulated in Section 2.7.

2.2 Preliminaries

Throughout this chapter, for the sake of brevity, we will adopt the term non-differentiable systems to annotate the class of continuous piecewise smooth systems. These systems can be described by the ordinary differential equation:

\[
\dot{x} = F(x, \nu),
\]

\[
F(x, \nu) = F_i(x, \nu), \quad x \in D_i \subset \mathbb{R}^2,
\]

with open, non-overlapping domains \(D_i, i = 1, \ldots, \bar{m}\), such that \(\bigcup_{i \in \{1, \ldots, \bar{m}\}} D_i = \mathbb{R}^2\), all functions \(F_i\) are smooth in \(x\) for all \(x \in \mathbb{R}^2\), and smooth in \(\nu\) for all \(\nu \in \mathbb{R}\), which is a single system parameter. Throughout this chapter, let \(\bar{D}\) denote the closure of an open set \(D\). We assume that the domains \(D_i, \ i = 1, \ldots, \bar{m}\), are independent on the system parameter \(\nu\). These domains are separated by the boundaries \(C_{ij}\) between \(D_i\) and \(D_j\). Note that the boundaries \(C_{ij}\) can be nonsmooth curves in \(\mathbb{R}^2\). Let the domains \(D_i, \ i = 1, \ldots, \bar{m}\), and boundaries \(C_{ij}\) be such that every finite line segment in \(\mathbb{R}^2\) traverses each boundary \(C_{ij}\) a finite number of times. Similar to the approach given in [61], one can prove that \(F(x, \nu)\) is Lipschitz continuous in \(x\).

In this chapter, we adopt the following assumptions:

**Assumption 2.1.** At \(\nu = 0\), a single isolated equilibrium coincides with one or more boundaries \(C_{ij}\).

Without loss of generality, we will assume that this equilibrium point is positioned at the origin \(x = 0\) for \(\nu = 0\).
Assumption 2.2. The derivative $\frac{\partial E}{\partial \nu} |_{(x,\nu) = (0,0)} \neq 0$.

Under these assumptions, a local analysis of the dynamics around the equilibrium is constructed. We will make a local approximation of system (2.1) that accurately represents the existence and stability of equilibria and limit cycles of the original system, as we will show in Section 2.5. Hence, bifurcations that change the limit sets of the system (2.1) will be accurately represented in this local approximation.\footnote{Formally, bifurcations of parameterised dynamical systems can be defined as follows.}

Definition 2.1. A parameterised dynamical system $\dot{x} = F(x, \nu)$, with $x \in \mathbb{R}^n$ and function $F$ smoothly depending on the parameter $\nu \in \mathbb{R}$, is said to undergo a bifurcation at parameter $\nu = \nu^*$ if there does not exist a neighbourhood $N$ of $\nu^*$ such that for each pair $\nu_1, \nu_2$ with $\nu_1, \nu_2 \in N$, there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that maps the trajectories of the system for $\nu = \nu_1$ to the trajectories of the system for $\nu = \nu_2$, and, in addition, preserves the direction of time along the trajectories.

This definition corresponds to the definitions used in [95] and [103]. In the literature, alternative definitions for bifurcations of non-differentiable or discontinuous systems have also been proposed, see, e.g. [14, 71]. In [103], a comparison is given between various definitions for bifurcations. In general, the existence of such a homeomorphism can be hard to prove. However, the appearance or disappearance of limit cycles or equilibria at a system parameter $\nu = \nu^*$ provide a sufficient condition that guarantees that a bifurcation occurs.
\[ \Sigma_{m-1,m}, \Sigma_{m1}, \text{respectively. Define } \rho_{01} := \rho_{m1} \text{ and } \Sigma_{01} := \Sigma_{m1}, \text{ such that each } \Sigma_i \text{ is bounded by } \Sigma_{i-1,i} = \{x \in \mathbb{R}^2 \mid x = c \rho_{i-1,i}, c \in [0, \infty)\} \text{ and } \Sigma_{i,i+1} = \{x \in \mathbb{R}^2 \mid x = c \rho_{i,i+1}, c \in [0, \infty)\}. \] For \( \mu = 0 \), the system is called conewise linear.

In this chapter, the following definition of a cone is used, that is an adapted version of the definition given in [41].

**Definition 2.2.** Consider a region \( S \subset \mathbb{R}^n \). If \( x \in S \) implies \( cx \in S, \forall c \in (0, \infty) \) and \( S \setminus \{0\} \) is connected, then \( S \) is a cone.

Note that when the bifurcating equilibrium is positioned on a single boundary \( \Sigma_{ij} \) that is nonsmooth at the origin, then the conewise affine system contains one convex cone, and one nonconvex cone. To assess the validity of the approximation, the relation between limit sets of the non-differentiable system (2.1) and the conewise affine approximation (2.3) will be discussed in Section 2.5.

Similar to [4], we define visible eigenvectors.

**Definition 2.3.** Let \( \dot{x} = A_i x + \mu b \) be the dynamics on an open cone \( S_i \subset \mathbb{R}^2, i = 1, \ldots, m \). An eigenvector of \( A_i \) is visible if it lies in \( \bar{S}_i \).

Based on the index theory presented in [46], we can formulate the following theorem.

**Theorem 2.1.** Inside a closed orbit \( C \) of the planar dynamical system \( \dot{x} = f(x) \), where \( f : E \to \mathbb{R}^2 \) is a Lipschitz continuous function on \( E \), at least one equilibrium point exists. If all equilibria inside \( C \) are hyperbolic nodes, saddles, or foci, then there must be an odd number \( 2n + 1 \) of equilibria, where \( n \) is an integer, such that \( n \) equilibria are saddles and \( n + 1 \) equilibria are nodes or foci.

**Proof.** The proof of this theorem is given in Appendix A.1.2.

Isolated closed orbits are limit cycles. According to the definition in [82], all closed orbits are limit sets. The following extension of Bendixson’s Theorem is used.

**Theorem 2.2 ([27]).** Suppose \( E \) is a simply connected domain in \( \mathbb{R}^2 \) and \( f(x) \) is a Lipschitz continuous vector field on \( E \), such that the quantity \( \nabla f(x) := \frac{\partial f_1}{\partial x_1}(x) + \frac{\partial f_2}{\partial x_2}(x) \) is not zero almost everywhere over any subregion of \( E \) and is of the same sign almost everywhere in \( E \). Then \( E \) does not contain closed trajectories of \( \dot{x} = f(x) \), where \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) and \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \).
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2.3 Stability of an equilibrium at the bifurcation point

For $\mu = 0$, the dynamics of the system (2.3) is described by the continuous, conewise linear system:

$$\dot{y} = f(y),$$

$$f(y) = A_i y, \ y \in S_i, \ i = 1, \ldots, m.$$  \(2.5\)

$$2.6$$

To analyse the dynamics of the conewise affine system (2.3), the stability of the equilibrium $y = 0$ of the conewise linear system (2.5) is important. The stability result presented here provides necessary and sufficient conditions for the stability of the origin of (2.5) and is an extension of a result presented in [4], since in that work all cones are required to be convex. For the sake of brevity, in this chapter, we restrict ourselves to the case of systems described by differential equations with continuous right-hand side. We note that the stability result presented here can readily be extended to obtain necessary and sufficient conditions for exponential stability or to allow for discontinuous functions $f(\cdot)$ in (2.5). Here, we refrain from treating such extensions since the focus of the current chapter is on bifurcation analysis.

To assess the stability of the equilibrium point $y = 0$ of (2.5), we distinguish systems with, or without, visible eigenvectors, as defined in Definition 2.3. In Section 2.3.1, the case of systems with visible eigenvectors is discussed. Subsequently, in Section 2.3.2, the case of systems without visible eigenvectors is studied. Finally, in Section 2.3.3, necessary and sufficient conditions for asymptotic stability of (2.5) are derived.

2.3.1 Systems with visible eigenvectors

Here, conewise linear systems of the form (2.5) with visible eigenvectors are studied. When a closed cone $\bar{S}$ does contain a visible eigenvector, the following result holds for trajectories inside this cone.

**Lemma 2.3.** Let $\bar{S}$ be a closed cone, in which the dynamics is described by $\dot{y} = Ay$, and let there exist a visible eigenvector $v$ in $\bar{S}$, corresponding to the eigenvalue $\lambda < 0$. Suppose no visible eigenvectors exist in this cone, associated with $\lambda \geq 0$. Then, all trajectories inside $\bar{S}$ converge to $y = 0$ or leave $\bar{S}$ in finite time.

**Proof.** This lemma is proven in Appendix A.1.2. \(\Box\)

In Fig. 2.1, two possible phase portraits have been depicted schematically in a cone with a visible eigenvector. A similar result is obtained for trajectories inside cones, that do not contain visible eigenvectors.
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Fig. 2.1. Schematic representation of two possible phase portraits of (2.5), (2.6) in a cone \( \bar{S}_i \) with a visible eigenvector corresponding to an eigenvalue \( \lambda < 0 \). This eigenvector is depicted with a dashed line. In (b), the visible eigenvector coincides with the vertical boundary of \( \bar{S}_i \).

Fig. 2.2. Schematic representation of two possible phase portraits of (2.5), (2.6) in a cone \( \bar{S}_i \) without a visible eigenvector. (a) Two real eigenvectors of the matrix \( A_i \) are positioned outside the cone \( \bar{S}_i \). These eigenvectors have been depicted with dashed lines. (b) The eigenvalues of \( A_i \) are complex.

Lemma 2.4. Let \( \bar{S} \) be a closed cone in \( \mathbb{R}^2 \). Suppose no eigenvectors of \( A \in \mathbb{R}^{2 \times 2} \) are visible in \( \bar{S} \). Then for any initial condition \( y_0 \in \bar{S} \), with \( y_0 \neq 0 \), there exists a time \( t \geq 0 \) such that \( e^{At}y_0 \not\in \bar{S} \).

Proof. This lemma is proven in Appendix A.1.2. \hfill \Box

In Fig. 2.2, two possible phase portraits have been depicted schematically in a cone without a visible eigenvector. Using the foregoing lemmas, the following result is proven, providing necessary and sufficient conditions for asymptotic stability of the origin of conewise linear systems (2.5) with visible eigenvectors.

Lemma 2.5. Consider a continuous, conewise linear system described by (2.5). When this system contains one or more cones with visible eigenvectors, then \( y = 0 \) is an asymptotically stable equilibrium of (2.5) if and only if all visible eigenvectors correspond to real eigenvalues \( \lambda < 0 \).

Proof. This lemma is proven in Appendix A.1.2. \hfill \Box
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2.3.2 Systems without visible eigenvectors

In conewise linear systems (2.5) without visible eigenvectors, trajectories exhibit a spiralling motion around the origin, visiting each region $S_i$, $i = 1, \ldots, m$, once per rotation, as shown in Fig. 2.3. Stability results are obtained for the spiralling motion by the computation of a return map.

In the absence of visible eigenvectors, a trajectory in the region $S_i$, $i = 1, \ldots, m$, will traverse this region in finite time, cf. Lemma 2.4. The position $y_0$ where a trajectory enters this region at time $t_0 = 0$ is located on the boundary $\Sigma_{i-1,i}$, such that $y_0$ can be expressed as $y_0 = p_i^i \rho_{i-1,i}$. Furthermore, this trajectory will cross $\Sigma_{i,i+1}$ in a finite time $t_i$. The position of this crossing can be given as: $y(t_i) = p_i^{i+1} \rho_{i,i+1}$. Since the dynamics inside the cone are linear, the time $t_i$ can be solved for, such that $y(t_i)$ is parallel to $\rho_{i,i+1}$. In this manner, in [4], expressions for the traversal time and crossing positions are derived. The crossing positions are linear in $p_i$. Using such analysis, we can derive expressions for a scalar $M_i$, such that $p_i^{i+1} = M_i p_i^i$. Note, that similar expressions have been derived in [4] for systems (2.5) with cones, that are convex.

First, the position vectors $y$ and tangency vectors $\rho$ are represented in a new coordinate frame:

$$\tilde{y}^i = P_i^{-1} y, \quad \text{for } \tilde{y}^i \in \tilde{S}_i := \{ \tilde{y}^i \in \mathbb{R}^2| \tilde{y}^i = P_i^{-1} y, y \in \bar{S}_i \}, \quad (2.7)$$

where $P_i$ is given by the real Jordan decomposition of $A_i$, yielding $A_i = P_i J_i P_i^{-1}$. This decomposition distinguishes three cases.

**Case 1:** If $A_i$ has complex eigenvalues, then $J_i = \begin{bmatrix} a_i & -\omega_i \\ \omega_i & a_i \end{bmatrix}$, where $a_i$ and $\omega_i$ are real constants and $\omega_i > 0$. Define $\Psi(a_1, a_2)$ to be the angle in anticlockwise direction from vector $a_1$ to vector $a_2$. Herewith,

$$M_i = \frac{\| \tilde{\rho}_{i-1,i} \|}{\| \tilde{\rho}_{i,i+1} \|} e^{\frac{a_1}{\omega_i} \Psi(\tilde{\rho}_{i-1,i}, \tilde{\rho}_{i,i+1})}. \quad (2.8)$$

**Fig. 2.3.** Illustration of a trajectory of (2.5) that traverses each cone $S_i$, $i = 1, \ldots, 4$, once per rotation.
2.3 Stability of an equilibrium at the bifurcation point

Case 2: If $A_i$ has two distinct real eigenvalues $\lambda_{ai}$ and $\lambda_{bi}$ and two distinct eigenvectors, then $J_i = \begin{bmatrix} \lambda_{ai} & 0 \\ 0 & \lambda_{bi} \end{bmatrix}$ and

$$
M_i = \begin{vmatrix}
\frac{e_2^T \tilde{\rho}_{i,i+1}^i}{e_2^T \tilde{\rho}_{i-1,i}^i} & \frac{\lambda_{ai}}{\lambda_{bi} - \lambda_{ai}} \\
\frac{e_1^T \tilde{\rho}_{i,i+1}^i}{e_1^T \tilde{\rho}_{i-1,i}^i} & \frac{\lambda_{bi}}{\lambda_{ai} - \lambda_{bi}} 
\end{vmatrix},
$$

(2.9)

where $e_1 := (1 \ 0)^T$ and $e_2 := (0 \ 1)^T$.

Case 3: If $A_i$ has two equal real eigenvalues $\lambda_{ai}$ with geometric multiplicity 1, then $J_i = \begin{bmatrix} \lambda_{ai} & 1 \\ 0 & \lambda_{ai} \end{bmatrix}$ and

$$
M_i = \begin{vmatrix}
\frac{e_2^T \tilde{\rho}_{i,i+1}^i}{e_2^T \tilde{\rho}_{i-1,i}^i} & e_1^T \tilde{\rho}_{i,i+1}^i \\
\frac{e_2^T \tilde{\rho}_{i,i+1}^i}{e_2^T \tilde{\rho}_{i-1,i}^i} & e_1^T \tilde{\rho}_{i-1,i}^i 
\end{vmatrix} \lambda_{ai} - \frac{e_1^T \tilde{\rho}_{i,i+1}^i}{e_1^T \tilde{\rho}_{i-1,i}^i}.
$$

(2.10)

By computation of the scalars $M_i$ with (2.8), (2.9) or (2.10) for each cone $S_i$, $i = 1, \ldots, m$, one can compute the return map between the positions $y_k$ and $y_{k+1}$ of two subsequent crossings of the trajectory $y(t)$ with the boundary $\Sigma_{m1}$:

$$
y_{k+1} = \Lambda y_k,
$$

(2.11)

where

$$
\Lambda = \prod_{i=1}^m M_i.
$$

(2.12)

2.3.3 Stability result

Using the results given in Sections 2.3.1 and 2.3.2, we can derive necessary and sufficient conditions for the global asymptotic stability of the origin of the conewise linear system (2.5).

Theorem 2.6. The origin of the continuous, conewise linear system (2.5) is globally asymptotically stable if and only if

(i) in each cone $S_i$, $i = 1, \ldots, m$, all visible eigenvectors are associated with eigenvalues $\lambda < 0$,

(ii) in case no visible eigenvectors exist, it holds that $\Lambda < 1$, with $\Lambda$ defined in (2.8), (2.9), (2.10) and (2.12).

Proof. The proof of this theorem is given in Appendix A.1.2. □
2.4 Bifurcation analysis of a conewise affine system

The limit sets that can occur in planar continuous systems are equilibria, closed orbits and homoclinic or heteroclinic orbits. To analyse the occurring bifurcations in (2.3), we are interested in characterisation of these limit sets, including their local stability, for different values of the system parameter $\mu$. The relationship between these limit sets and the limit sets of (2.1) will be discussed in Section 2.5. The following assumption is adopted to study the conewise affine system (2.3).

**Assumption 2.3.** All matrices $A_i$, $i = 1, \ldots, m$, of (2.3) are invertible.

Note that this assumption implies that for given bifurcation parameter $\mu$, all equilibrium points $x_{eq}(\mu)$ of (2.3), that satisfy $f(x_{eq}(\mu), \mu) = 0$, are isolated. Solutions of conewise affine systems as given in (2.3) scale linearly with the bifurcation parameter $\mu$, as formalised in the following lemma.

**Lemma 2.7.** Consider two continuous conewise affine systems $\dot{x} = f(x) + \mu_i b$, $\mu_i \in (0, \infty)$, $i = 1, 2$, where $f(\cdot)$ is piecewise linear with cone-shaped regions. If $\phi_1(t)$ is a solution of $\dot{x} = f(x) + \mu_1 b$, then $\phi_2(t) = \frac{\mu_2}{\mu_1} \phi_1(t)$ is a solution of $\dot{x} = f(x) + \mu_2 b$.

**Proof.** The proof is omitted for the sake of brevity and follows from the observation that $f(x)$ is homogeneous of degree 1.

From this lemma, we conclude that a complete bifurcation diagram can be obtained by finding all existing limit sets at an arbitrary negative, and an arbitrary positive parameter $\mu$, and at the bifurcation point with $\mu = 0$. Subsequently, with Lemma 2.7, the limit sets for all parameters $\mu$ can be found. The conewise affine system (2.3) is conewise linear if $\mu = 0$. The dynamical behaviour of (2.3) at $\mu = 0$ has been analysed in the previous section.

In continuous, conewise affine systems with $\mu \neq 0$, the trajectories are tangent to a specific boundary $\Sigma_{ij}$ at zero, one, or all points on this boundary. When, at this boundary, there exists an isolated point where the trajectories are tangent to the boundary, then such a point will be called a **tangency point** and denoted with $T_{ij}$. We determine all tangency points of the conewise affine system and compute trajectories in forward and backward time through these tangency points and through the origin. When the vector $f(x, \mu)$ of (2.3) is parallel to a boundary at all points of this boundary, then a trajectory exists that is parallel to the boundary.

In addition, when a node or saddle point exists, the stable and unstable manifolds of this point are computed. Computation of this finite number of trajectories yields insight in the possible behaviour of all trajectories. With these manifolds and trajectories, for each domain $S_i$, $i = 1, \ldots, m$, we can
identify which subsets of $S_i$ contain trajectories that leave or enter this domain and through which boundary. Therewith, one can identify what sequence of boundaries and cones can possibly be visited by closed orbits.

For each of these sequences, a return map is derived. Hence, finding fixed points in these maps is equivalent to finding closed orbits of (2.3). However, the domain of these maps may be unbounded, such that no feasible computational approach would exist to find all fixed points in the map. Below, we present two theorems, that can be used to find a halfline in state space, that cannot be traversed by any closed orbit. Existence of such a halfline reduces the domain of the return map in which fixed points may exist to a bounded domain.

**Theorem 2.8.** Consider the continuous, conewise affine system (2.3) with constant $\mu \neq 0$. Suppose the system does not contain visible eigenvectors. Construct a system

$$\dot{y} = f(y),$$

$$f(y) = A_i y, \; y \in S_i, \; i = 1, \ldots, m,$$

by setting $\mu = 0$ in (2.3). Let $\Lambda$ for this system be defined in (2.8), (2.9), (2.10) and (2.12). When $\Lambda \neq 1$, there exists an $x_F \in \Sigma_{m1} \setminus \{0\}$, such that all points in the halfline $R := \{x \in \Sigma_{m1} \mid \|x\| \geq \|x_F\|\}$ are not part of a closed orbit of (2.3).

**Proof.** The proof of this theorem is given in Appendix A.1.2.

A similar result is formulated for systems with visible eigenvectors.

**Theorem 2.9.** Consider system (2.3), satisfying Assumption 2.3. If visible eigenvectors exist and all boundaries $\Sigma_{ij}$ do not contain a visible eigenvector of $A_i$ or $A_j$, then there exists a halfline $H \subset \mathbb{R}^2$, such that closed orbits can not contain a point $x_0 \in H$.

**Proof.** The proof of this theorem is given in Appendix A.1.2.

The case where the boundary $\Sigma_{ij}$ contains visible eigenvectors of the matrices $A_i$ or $A_j$ is excluded in this theorem, such that in this case, no halfline $H$ can be constructed with the properties given in the theorem.

When analysing systems described by (2.3), Theorems 2.1, 2.2, 2.8 and 2.9 can be exploited to exclude closed orbits in specific regions of state space. However, in certain cases the existence of closed orbits, including limit cycles, can not be excluded in some parts of the domain $\mathbb{R}^2$.

To find all closed orbits, return maps are constructed for all possible sequences of cones and boundaries. A logical choice of the position of the Poincaré section on which the return maps are defined is the position of a boundary crossed by the trajectories. This is possible for all closed orbits that traverse multiple cones. Closed orbits in a single cone encircle a center, since the dynamics in that cone is affine. In the following section, partial maps are constructed.
A partial map describes the position of a trajectory before and after the visit of a specific cone \( S_i, \ i = 1, \ldots, m \). Subsequently, we discuss how to combine these partial maps to obtain the return map.

### 2.4.1 Trajectories visiting a cone \( S_i \)

In Sections 2.3.1 and 2.3.2, a trajectory of a conewise linear system is followed inside a specific cone \( S_i \) during the traversal of this cone. During this traversal, the trajectory is described by the linear differential equation \( \dot{y} = A_i y_i \), such that an analytical expression for the trajectory \( y(t) \) with initial position \( y_0 \in \Sigma_{i-1,i} \) has been derived. With this expression, the traversal time \( t_i \) and final position \( y(t_i) \) are obtained. Here, a similar approach is used for the conewise affine system (2.3), where we focus our attention on trajectories that traverse cones \( S_i, \ i = 1, \ldots, m \).

For a given cone \( S_i, \ i = 1, \ldots, m \), and given boundaries where the trajectory enters or leaves this domain, the partial map is constructed that gives the exit position as a function of the position, where \( S_i \) is entered. Since (2.3) is autonomous, we can assume without loss of generality that the domain \( S_i \) is entered at the time \( t = 0 \). We study a trajectory traversing \( S_i \) from the boundary \( \Sigma_- \) towards the boundary \( \Sigma_+ \) in a finite time \( t_i \). Therefore, the trajectory \( x(t) \) satisfies \( x(t) \in S_i, \ t \in (0, t_i), \ x(0) \in \Sigma_- \) and \( x(t_i) \in \Sigma_+ \). We define the maps \( g_i : D_i \subset \Sigma_- \to I_i \subset \Sigma_+ \), describing the position \( x(t_i) \) as a function of \( x(0) \). Expressions for \( g_i \) are derived in Appendix A.1.1.

### 2.4.2 Construction of the return map

The stable or unstable manifolds of nodes and saddle points and the trajectories through tangency points and the origin are computed. Therewith, for each domain \( S_i \), we can identify what subsets of \( S_i \) contain trajectories that leave or enter this domain and through which boundary. Combining these domains, one can identify what sequences of boundaries and cones can contain closed orbits. A return map is computed for each sequence to find all closed orbits and their stability properties.

For example, suppose we want to study whether there exist one or more closed orbits that traverse the regions and boundaries \( S_1, \Sigma_{12}, S_2, \Sigma_{23}, S_3, \Sigma_{31} \) in this order. A Poincaré section is taken at the moments where trajectories cross \( \Sigma_{31} \), the corresponding return map is denoted as \( M : D_M \subset \Sigma_{31} \to I_M \subset \Sigma_{31} \). Therewith, \( M(x_k) \) describes the first crossing of a trajectory \( x(t), \ t > 0 \) with boundary \( \Sigma_{31} \), where \( x(t) \) corresponds to the initial condition \( x(0) = x_k \in D_M \).

Define \( g_1 : D_1 \subset \Sigma_{31} \to I_1 \subset \Sigma_{12} \), which can be computed with Appendix A.1.1, where \( \Sigma_- = \Sigma_{31} \) and \( \Sigma_+ = \Sigma_{12} \). In addition, define \( g_2 : D_2 \subset \Sigma_{12} \to I_2 \subset \Sigma_{23} \) and \( g_3 : D_3 \subset \Sigma_{23} \to I_3 \subset \Sigma_{31} \) in a similar fashion. From a combination of \( g_1 \), \( g_2 \) and \( g_3 \), one obtains the return map \( M(x_k) = g_3 \circ g_2 \circ g_1(x_k) \).
2.4 Bifurcation analysis of a conewise affine system

Since $M$ is a return map, every fixed point of this map is on a closed orbit. If this fixed point is isolated, then the periodic orbit is a limit cycle. Furthermore, each closed orbit of (2.3) that traverses the boundaries and regions in the sequence $S_1, \Sigma_1, S_2, \Sigma_2, S_3, \Sigma_3, S_1$, yields a fixed point in $M$.

The return map $M$ can be computed for the possible sequences of cones and boundaries. By determining the fixed points of such maps, the existence or absence of limit cycles and closed orbits can be investigated. Each return map is continuous, since (2.3) is Lipschitz continuous, and trajectories of this class of systems are continuous with respect to initial conditions, see [91], Theorem 3.4. Furthermore, the Euclidean norm of the map, $\|M(x)\|$, is monotonously increasing in $\|x\|$. Monotonicity follows from the fact that the time-reversed system of (2.3) is Lipschitz as well, such that the inverse of $M$ should exist and should be unique. The norm $\|M(x)\|$ has to be increasing in $\|x\|$. Otherwise, there would have to exist points $x_a, x_b \in D_M$, where $\|x_a\| < \|x_b\|$ and $\|M(x_a)\| > \|M(x_b)\|$. Note that $M$ is a return map, such that there exist a trajectory from $x_a$ to $M(x_a)$ and a trajectory from $x_b$ towards $M(x_b)$. The positions $x_a, M(x_a), x_b$ and $M(x_b)$ should all be positioned on the same boundary, that is a halfline. If $\|x_a\| < \|x_b\|$ and $\|M(x_a)\| > \|M(x_b)\|$, in planar systems, the trajectories from $x_a$ and $x_b$ have to cross each other before they return to the Poincaré section. Such a crossing is not possible in systems that are Lipschitz. The fact that the return map is continuous and monotonously increasing can be used in the computational approach to find all fixed points.

2.4.3 Procedure to obtain all closed orbits

In this section, a stepwise procedure is developed such that all limit sets of (2.3) are found for negative, positive and zero bifurcation parameter $\mu$. With this procedure, the bifurcations of the continuous, conewise affine system (2.3) can be described entirely.

Lemma 2.7 implies that only an arbitrary positive and negative $\mu$, and $\mu = 0$, should be studied to obtain the full bifurcation diagram. Theorems 2.1 and 2.2 are used to exclude the existence of closed orbits. For systems without visible eigenvectors, Theorem 2.8 supplies a bound to exclude closed orbits far away from the origin. If visible eigenvectors exist, Theorem 2.9 can be applied to bound the domain, in which closed orbits can occur. When Theorem 2.8 or 2.9 can be applied, a bounded domain for the return map remains, such that it is computationally feasible to find all fixed points of the return map with a numerical method. When certain sequences of boundaries and cones may contain closed orbits, return maps will be constructed.

The following procedure yields a bifurcation diagram of (2.3) that contains all limit sets:

1. Identify all equilibria for positive and negative $\mu$, i.e. the points $x \in \mathbb{R}^2$ where $f(x, \mu) = 0$, with $f(x, \mu)$ given in (2.3).
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2. Study the stability of the equilibrium point \( x = 0 \) at \( \mu = 0 \) using Theorem 2.6. Identify all visible eigenvectors of \( A_i \), \( i = 1, \ldots, m \).

3. For both an arbitrary fixed \( \mu < 0 \) and \( \mu > 0 \):
   
   a. Compute points where the vector field is tangent to the boundaries. Subsequently, compute trajectories through these tangency points and through the origin for a finite time. In addition, compute the eigenvalues of the matrices \( A_i \), \( i = 1, \ldots, m \), when an equilibrium exists inside the corresponding cone \( S_i \). When an equilibrium with real eigenvalues exist, compute the stable and unstable manifolds by simulating a trajectory emanating from or converging to this equilibrium in the direction of the eigenvectors. To check whether homoclinic or heteroclinic orbits exist, investigate whether stable and unstable manifolds coincide.
   
   b. Identify, if possible, certain domains that cannot contain closed orbits. First, identify the value of \( \text{tr}(A_i) \), \( i = 1, \ldots, m \) for each region \( S_i \), i.e. the trace of the matrices \( A_i \). According to Theorem 2.2, a closed orbit should visit regions \( S_i \) where the traces \( \text{tr}(A_i) \) have opposite sign or are zero, since \( \nabla f(x) = \text{tr}(A_i) \) for \( x \in S_i \).

   Second, identify the character of the existing equilibria. With Theorem 2.1 one can guarantee that no closed orbits exist in specific domains. For example, according to this theorem, no closed orbits are possible that encircle one hyperbolic saddle and one focus. Subsequently, determine which equilibria should be encircled by possibly existing closed orbits.

   Finally, when an unbounded domain remains that may contain closed orbits, identify halflines \( R \) or \( H \) as defined in Theorem 2.8 or Theorem 2.9. Such halflines will reduce the domain, in which closed orbits can occur, to a bounded domain. Investigate the sequences of cones and boundaries that may be traversed by closed orbits. For these sequences of cones and boundaries, a return map will be constructed.

   c. Compute the maps \( g_i : \Sigma_{-,i} \rightarrow \Sigma_{+,i} \) of the individual cones \( S_i \) that may be traversed by a closed orbit from \( \Sigma_{-,i} \) towards \( \Sigma_{+,i} \). The derivation of these maps is given in Appendix A.1.1. Combination of the maps yields the return maps for the possible sequences of cones and boundaries. Note that when a halfline \( R \) or \( H \), as defined in Theorem 2.8 or 2.9, respectively, is found that can not be crossed by a closed orbit, the domains of these maps where fixed points may exist will be bounded. Determine the fixed points of all possible return maps in a numerical manner. Compute the local derivative of the return map at this fixed point, since this determines the stability of the limit cycle or closed orbit.

4. Identify what limit sets appear, disappear or change their local stability for changing \( \mu \). Application of Lemma 2.7 with respect to the limit sets for a
given $\mu < 0$ or $\mu > 0$ yields all limit sets for $\mu \neq 0$. Combination with the piecewise linear stability result for the case $\mu = 0$ gives a bifurcation diagram, containing all changes in limit sets and their stability.

The procedure given above yields all changes in the limit sets of the conewise affine system (2.3). Completeness of the obtained limit cycles follows from the fact that for each conewise affine system (2.3), a finite number of return maps can be determined, that may contain fixed points. Computation of each of these return maps yields all limit cycles.

\section{2.5 Approximation effects}

In this section, the effect of the conewise affine approximation of a non-differentiable system is studied, as introduced in Section 2.2. Results are presented for the existence and stability of equilibria (Theorem 2.10) and limit cycles (Theorem 2.11) in the non-differentiable system (2.1) when such limit sets exist in the conewise affine system (2.3) and vice versa. With these theorems, we show the applicability of the procedure for the bifurcation analysis as presented in Section 2.4 for non-differentiable systems of the form (2.1).

We will use the following assumptions in addition to Assumptions 2.1 and 2.2.

\begin{assumption}
For all functions $F_i(x, \nu)$, $i = 1, \ldots, \bar{m}$, the Jacobian at the origin, i.e. $\frac{\partial F_i}{\partial x}(x, \nu)=(0,0)$, is invertible.
\end{assumption}

Note that this assumption is stricter than Assumption 2.3, in which only the vector fields $F_i(x, \nu)$, $i = 1, \ldots, m$, are considered. In a neighbourhood around the origin, Assumption 2.4 excludes the existence of non-isolated equilibria in domains $D_i$ that are cusp-shaped at the origin.

\begin{assumption}
The equilibria of the nonlinear system (2.1) do not move locally tangential to the boundaries when $\nu$ is varied around 0.
\end{assumption}

We illustrate Assumption 2.5 in Fig. 2.4. This assumption implies that $n_{i-1,i}A_i^{-1}b \neq 0, \forall i \in \{1, \ldots, m\}$.

\begin{remark}
Paths of equilibria that approach the origin through a cusp-shaped region are excluded by Assumption 2.5.
\end{remark}

Without loss of generality, we may assume that $\left\| \frac{\partial F}{\partial \nu}(x, \nu=(0,0)) \right\| = 1$, yielding $\mu = \nu$. In the following assumption, the occurrence of center-like behaviour is excluded.

\begin{assumption}
In a neighbourhood around the origin, the non-differentiable system (2.1) and conewise affine system (2.3) at the bifurcation point $\nu = 0$ or $\mu = 0$ do not contain trajectories that are encircling the equilibrium without converging to the origin for $t \to \infty$ or $t \to -\infty$.
\end{assumption}
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Fig. 2.4. Paths of equilibria of (2.1), depicted with dashed lines, where boundaries $C_{ij}$ are depicted with solid lines. Assumption 2.5 excludes equilibria emanating tangentially to a boundary, for example the equilibrium path in $D_3$. In addition, the same assumption excludes equilibria that emanate in a cusp-shaped domain, as depicted in domain $D_1$.

For the conewise affine system, this assumption corresponds to $\Lambda \neq 1$, with $\Lambda$ given in (2.12). The following result for the existence of equilibria and the local stability of these equilibria is obtained.

**Theorem 2.10.** Let Assumptions 2.1, 2.2, 2.4, and 2.5 be satisfied. There exists a neighbourhood $N \subset \mathbb{R}$ of 0, such that for every equilibrium of the nonlinear system (2.1) that exists for some $\nu \in N$ and converges to the origin for $\nu \to 0$, there exists an equilibrium in the conewise affine system (2.3). Moreover, for every equilibrium of (2.3) there will exist an equilibrium in the full nonlinear system for $\nu \in N$.

When in addition to these assumptions for a given $\nu \in \bar{N}$, with a neighbourhood $\bar{N} \subseteq N$ of 0 chosen small enough, an equilibrium of (2.1) exists in $D_i$ or at the origin and the following three conditions hold:

(i) when the equilibrium of (2.1) exists in $D_i$, then the eigenvalues of the corresponding matrix $A_i$ have nonzero real part, and

(ii) when this equilibrium has an unstable and stable manifold, no homoclinic or heteroclinic orbit connected to this equilibrium point exist, and

(iii) Assumption 2.6 holds,

then for every $\nu \in \bar{N}$ the stability properties of the equilibrium of system (2.1) in $D_i$ or at the origin and the equilibrium of (2.3) in $S_i$ or at the origin, with $\mu = \nu$, are equal.

**Proof.** The proof of this theorem is given in Appendix A.1.2.

**Remark 2.2.** The combination of conditions (i) and (iii) of Theorem 2.10 for the non-differentiable system (2.1) can be seen as a counterpart for the assumption
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of hyperbolic dynamics near equilibria of smooth systems, as would be required to apply the Hartman-Grobman Theorem, cf. [74].

In this work, we will consider limit cycles to be stable when, on both sides of the limit cycle, trajectories are converging to the limit cycle. We will refer to limit cycles as unstable when, on both sides of the limit cycle, trajectories are diverging from the limit cycle. Limit cycles that are attracting from one side and repelling from the other side are semi-stable, cf. [2]. Note that semi-stable limit cycles are unstable in the sense of Lyapunov.

We introduce the following assumptions on the closed orbits of (2.1) and (2.3).

**Assumption 2.7.** For every closed orbit of (2.1) and for every closed orbit of (2.3), a Poincaré map taken transversal to this closed orbit only has isolated fixed points.

Note that this assumption implies that all closed orbits are limit cycles. However, they are allowed to be nonhyperbolic. To state a result on the existence of limit cycles when the bifurcation parameter is perturbed.

**Assumption 2.8.** For all limit cycles, denoted with \( \gamma \), of the non-differentiable system (2.1) there exists a parameter range \( \nu \in (0, \nu^*) \) with \( \nu^* > 0 \) (or \( \nu \in (\nu^*, 0) \) with \( \nu^* < 0 \)) and constants \( c_1, c_2 > 0 \) such that \( \|x\| > c_1|\nu| \wedge \|x\| < c_2|\nu|, \forall x \in \gamma \) holds for \( \nu \in (0, \nu^*) \) (or \( \nu \in (\nu^*, 0) \)).

This assumption implies that the curve in the bifurcation diagram, depicting the ”amplitude” of a limit cycle, has a nonzero finite derivative with respect to the parameter \( \mu \) at the bifurcation point, cf. Fig. 2.5. For example, limit cycles \( \gamma \) that show a square-root dependency with respect to the bifurcation parameter \( (\|x\| \sim \sqrt{\nu}) \) are excluded; such behaviour occurs for example in the Hopf bifurcations of smooth systems.

The following theorem describes the relationship between limit cycles of (2.1) and (2.3).

**Theorem 2.11.** Let Assumptions 2.1, 2.2, 2.7, and 2.8 be satisfied. There exists a neighbourhood \( N \) of 0, such that the number of limit cycles in the non-differentiable system (2.1) for \( \nu \in N \) that are not semi-stable, is equal to the number of limit cycles in the approximation (2.3), which are not semi-stable. In addition, their stability properties are equal.

**Proof.** The proof of this theorem is given in Appendix A.1.2.

No results are obtained for homoclinic and heteroclinic orbits, or the closed orbits around a center. When such limit sets are not present, then the cone-wise affine approximation (2.3) serves as a good local approximation of (2.1). Theorem 2.10 shows that equilibria of (2.1) are accurately represented in the
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Fig. 2.5. Exemplary bifurcation diagram with curves for an equilibrium and a limit cycle. Assumption 2.8 implies that there exist finite \( c_1, c_2 > 0 \) such that all curves representing limit cycles of (2.1) are contained in the shaded sectors.

approximation (2.3). Furthermore, by Theorem 2.11, limit cycles are accurately represented as well. Therefore, the procedure developed in Section 2.4 for conewise affine systems is an appropriate tool to study the bifurcations of non-differentiable systems (2.1). This will be illustrated in Section 2.6.2 with an example.

2.6 Illustrative examples

In Section 2.6.1, we will present a complete bifurcation analysis for a conewise affine system, using the results in Sections 2.2-2.4. Subsequently, in Section 2.6.2, we present an example of a non-differentiable system that undergoes two discontinuity-induced bifurcations of an equilibrium point using the results in Section 2.5.

2.6.1 Example for a conewise affine system

Consider the continuous system:

\[
\dot{x} = \begin{cases}
A_1 x + \mu b, & x \in S_1 := \{x \in \mathbb{R}^2 | n_{41}^T x < 0 \land n_{12}^T x > 0\}, \\
A_2 x + \mu b, & x \in S_2 := \{x \in \mathbb{R}^2 | n_{12}^T x < 0 \land n_{23}^T x > 0\}, \\
A_3 x + \mu b, & x \in S_3 := \{x \in \mathbb{R}^2 | n_{23}^T x < 0 \land n_{34}^T x > 0\}, \\
A_4 x + \mu b, & x \in S_4 := \{x \in \mathbb{R}^2 | n_{34}^T x < 0 \land n_{41}^T x > 0\},
\end{cases}
\]  

(2.15)

where the normal vectors are chosen as \( n_{12} = [0 \ 1]^T \), \( n_{23} = \frac{1}{\sqrt{2}} [-1 \ -1]^T \), \( n_{34} = [0 \ -1]^T \), \( n_{41} = \frac{1}{\sqrt{2}} [1 \ 1]^T \). The vector \( b = [\cos(0.375\pi) \ \sin(0.375\pi)]^T \) and \( \mu \in \mathbb{R} \) is the bifurcation parameter. The phase portrait of this system
with $\mu = -0.5$ is shown in Fig. 2.6. The matrices $A_i$ are $A_1 = \begin{bmatrix} -0.5 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.91 \\ -1 & 0.58 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & 0.41 \\ 0.5 & 2.08 \end{bmatrix}$, $A_4 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$. System (2.15) will be analysed with the procedure given in Section 2.4.3:

1. For $\mu < 0$, two equilibria exist, with positions $x = -\mu A_2^{-1}b$ in $S_2$ and $x = -\mu A_4^{-1}b$ in $S_4$. For $\mu > 0$, no equilibria exist.

2. At $\mu = 0$, the conewise linear dynamics is unstable, since the visible eigenvector in $S_4$ corresponds to an unstable eigenvalue. In addition, one visible eigenvector in $S_3$ exists, that corresponds to a stable eigenvalue.

3. Now, the system will be analysed for an arbitrarily chosen negative, and an arbitrarily chosen positive parameter $\mu$. Subsequently, with application of Lemma 2.7 the complete bifurcation diagram is obtained. For $\mu = -0.5$, we obtain:

   a. On $\Sigma_{12}$, $\Sigma_{34}$ and $\Sigma_{41}$, there exist points where the vector field is tangent to the boundary, i.e. points $\mathcal{T}_{12}$, $\mathcal{T}_{34}$ and $\mathcal{T}_{41}$, respectively. Trajectories through these points and the origin are shown in Fig. 2.6. An unstable
focus exist in $S_2$, since the related eigenvalues of $A_2$ are $0.42 \pm 0.79i$, where $i^2 = -1$. A saddle point exist in $S_4$ with eigenvalues $-1.10$ and $1.60$. The stable and unstable manifolds of this point are shown and do not form a homoclinic orbit.

b. The trace $\text{tr}(A_1) < 0$, whereas all other traces satisfy $\text{tr}(A_i) > 0$, $i = 2, 3, 4$. Therefore, application of Theorem 2.2 yields that each possible closed orbit visits $S_1$. To satisfy Theorem 2.1, closed orbit(s) should encircle the focus without encircling the saddle point.

By studying the depicted trajectories, one can conclude, that no closed orbit can traverse $\Sigma_{12} \setminus [O, a]$, since these trajectories cannot encircle the focus without encircling the saddle point, which is required according to Theorem 2.1. Furthermore, closed orbits can not traverse the interior of the line $[a, b]$, since trajectories through this open line will arrive at the line $[c, d]$ in finite time, and enter the positively invariant region that is depicted gray in Fig. 2.6. Now, one can conclude, that possible closed orbits visit only the regions $S_1$ and $S_2$, such that they should be contained in the domain, that is depicted gray. This implies, that all closed orbits traverse the line $[T_{12}, e]$.

c. Existing closed orbits should traverse the line $[T_{12}, e]$. We construct a map $g_2 : [T_{12}, e] \rightarrow [d, T_{12}]$, that yields the position $g_2(x)$ where a trajectory leaves the cone $S_2$ when this cone was entered at $x$. Similarly, the map $g_1 : [b, T_{12}] \rightarrow [T_{12}, e]$ is computed. The maps are computed as presented in Appendix A.1.1. The combined map $M := g_1 \circ g_2(x)$ is the return map and is shown in Fig. 2.7(a). It contains one fixed point. Apparently, a unique stable limit cycle exists that contains $x = (-0.55 \ 0)^T$.

For $\mu = 0.5$, no equilibrium point of (2.15) exists, such that according to Theorem 2.1, no closed orbits can exist.

4. With the analysis above and application of Lemma 2.7, the bifurcation diagram is constructed, as given in Fig. 2.7(b). Both the limit cycle, focus and saddle exist only for $\mu < 0$. For $\mu = 0$, unstable behaviour is observed. We note that this bifurcation can not occur in smooth dynamical systems and is explicitly induced by the nonsmoothness of the system.

### 2.6.2 Example for a piecewise smooth system

In the following example, a piecewise smooth system is studied that undergoes two bifurcations of an equilibrium. Using the procedure presented in this chapter, a local analysis of these bifurcations is performed. For the conewise affine approximations, this analysis guarantees to find all equilibria and limit cycles that are created or destroyed locally during the bifurcations. According
2.6 Illustrative examples

Fig. 2.7. (a) Combined map $M$ of (2.15) with $\mu = -0.5$. (b) Bifurcation diagram of (2.15) with the bifurcation parameter $\mu$.

...to Theorems 2.10 and 2.11, the local bifurcations of the piecewise smooth system are accurately described.

We consider the following continuous piecewise smooth system:

$$
\dot{x} = F_i(x, \nu), \quad x \in D_i, \ i = 1, 2, 3, \quad (2.16)
$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \in \mathbb{R}^2$, $\nu \in \mathbb{R}$ is the bifurcation parameter and

$$
F_1(x, \nu) = A_1 x + \nu b + 0.1 \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix}, \quad (2.17)
$$

$$
F_2(x, \nu) = A_2 x + \nu b + 0.1 \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix}, \quad (2.18)
$$

$$
F_3(x, \nu) = A_3 x + \nu b, \quad (2.19)
$$

and $A_1 = \begin{bmatrix} 0.5 & -1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.5 & -1 \\ 1 & -0.75 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 0 \end{bmatrix}$. The vector $b = \begin{bmatrix} \cos(0.375\pi) \\ \sin(0.375\pi) \end{bmatrix}^T$. The domains $D_i$, $i = 1, 2, 3$, are given by

$$
D_1 = \{ x \in \mathbb{R}^2 | n_{31}^T x < 0 \land n_{12}^T x > 0 \}, \quad (2.20)
$$

$$
D_2 = \{ x \in \mathbb{R}^2 | n_{12}^T x < 0 \lor h_{23}(x) < 0 \}, \quad (2.21)
$$

$$
D_3 = \{ x \in \mathbb{R}^2 | n_{31}^T x > 0 \land h_{23}(x) > 0 \}, \quad (2.22)
$$

where the function $h_{23}(x)$ is defined to describe the boundary between $D_2$ and $D_3$:

$$
h_{23}(x) := x_2 - \frac{4}{30} x_1^3 - 2 x_1, \quad (2.23)
$$

$n_{31} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$ and $n_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$. The boundary between $D_i$ and $D_j$ is called $C_{ij}$. The partitioning of the state space of this piecewise smooth system is depicted in Fig. 2.8(a).

First, we study at what parameters an equilibrium point is coinciding with one or more boundaries. An equilibrium point is positioned at the origin at...
Fig. 2.8. (a) State space of (2.16) with boundaries $C_{ij}$ and domains $D_i$, $i = 1, 2, 3$. (b) Limit cycles of (2.24), where $\mu = 1$. The inner limit cycle is unstable, the outer limit cycle is stable. The stable focus is depicted with an asterisk. According to Theorems 2.10 and 2.11, this phase portrait is locally a good approximation for the system (2.16) for $\nu$ in a neighbourhood of 0.

$\nu = 0$. The origin is coinciding with all boundaries. Furthermore, at $\nu = \nu_1$, where $\nu_1 \approx 1.88$, the equilibrium point is positioned on $x = x^* \in C_{12}$, where $x^* \approx (-1.44, 0)^T$.

For $\nu < \nu_1$, $\nu \neq 0$, an isolated equilibrium exists in $D_2$. For $\nu > \nu_1$, an equilibrium point exists in $D_1$. No equilibrium point can exist in $D_3$.

The equilibria in these domains will not undergo smooth bifurcations, since the Jacobian $J_1(x, \nu) = \frac{\partial F_1(x, \nu)}{\partial x}$ has eigenvalues $\frac{1}{4} \pm i\frac{1}{2}\sqrt{3.75 + 1.2x_1^2}$, such that if an equilibrium exists in $D_1$, then it always is an unstable focus. Furthermore, the Jacobian $J_2(x, \nu) = \frac{\partial F_2(x, \nu)}{\partial x}$ of $F_2$ has eigenvalues $-\frac{1}{8} \pm i\frac{1}{2}\sqrt{2.4375 + 1.2x_1^2}$, such that if an equilibrium exists in $D_2$, then it always is a stable focus.

The nonsmooth bifurcations around $\nu = 0$ and $\nu = \nu_1$ will be studied locally with a conewise affine approximation of the vector field.

2.6.2.1 Local analysis around $\nu = 0$

For $\nu = 0$ an equilibrium of (2.16) exists at the origin and the partial derivative $\frac{\partial F}{\partial \nu} = b$, such that Assumptions 2.1 and 2.2 are satisfied. Therefore, we approximate the boundaries $C_{12}, C_23, C_{31}$ with the halflines $\Sigma_{12}, \Sigma_{23}$ and $\Sigma_{31}$, respectively. Here, the vectors $\rho_{12} = (-1, 0)^T$, $\rho_{23} = \frac{1}{\sqrt{5}} (1, 2)^T$ and $\rho_{31} = (0, 1)^T$ define the halflines $\Sigma_{ij} := \{ x \in \mathbb{R}^2 | x = c \rho_{ij}, c \in [0, \infty) \}$. Definition of the nor-
2.6 Illustrative examples

Introducing Taylor expansions of the vector fields $F_i$, $i = 1, 2, 3$, near the origin, equation (2.16) is locally approximated with:

$$\dot{x} = \begin{cases} A_1 x + \mu b, & x \in S_1 = \{ x \in \mathbb{R}^2 | n_{13}^T x < 0 \land n_{12}^T x > 0 \}, \\ A_2 x + \mu b, & x \in S_2 = \{ x \in \mathbb{R}^2 | n_{12}^T x < 0 \land n_{23}^T x > 0 \}, \\ A_3 x + \mu b, & x \in S_3 = \{ x \in \mathbb{R}^2 | n_{23}^T x < 0 \land n_{13}^T x > 0 \}, \end{cases}$$  \hspace{1cm} (2.24)

where $A_i = A_i$, $i = 1, 2, 3$; $\mu = \nu$ is the bifurcation parameter and the affine vector is given by $b = [\cos(0.375\pi) \sin(0.375\pi)]^T$.

The conewise affine system (2.24) is analysed with the procedure presented in Section 2.4.3. This analysis is completely analogous to the analysis of the example given in Section 2.6.1. For the sake of brevity we omit the detailed analysis here and focus on the discussion of the results, see [18] for further details.

At $\mu = 0$, a stable spiralling motion is observed, such that the origin is a stable equilibrium point. For all $\mu \neq 0$, only one equilibrium exists, that is positioned in $S_3$ and is a stable focus. For negative parameters $\mu$, using a halfline $R$ as defined in Theorem 2.8, we obtain that no periodic orbits exist, such that only a single, stable focus exists. For positive parameters $\mu$, the system (2.24) contains two limit cycles and a stable focus, that are depicted in Fig. 2.8(b) for $\mu = 1$. With the return maps that are derived in this chapter, we obtain that the inner limit cycle is unstable, the outer limit cycle is stable.

The bifurcation diagram of (2.24) is depicted in Fig. 2.9(a) for the parameter range $\mu \in [-0.8, 0.8]$, that corresponds to the same range of the system parameter $\nu$ of (2.16).

2.6.2.2 Local analysis around $\nu = \nu_1$

For $\nu = \nu_1$, an equilibrium point $x^*$ exists that satisfies $x^* \in \Sigma_{12}$. We consider the neighbourhood around the bifurcation point, and therefore introduce $\mu = \nu - \nu_1$ and $y := x - x^*$, where $y = (y_1, y_2)^T$. The partial derivative $\frac{\partial E}{\partial \nu} = b$, such that Assumptions 2.1 and 2.2 are satisfied.

We approximate the boundary $\mathcal{C}_{12}$ with the halflines $\Sigma_{12p}$, and $\Sigma_{12n}$, where $\Sigma_{12p}$ and $\Sigma_{12n}$ describes the boundary $\mathcal{C}_{12}$ for $x_1 \geq x_1^*$ and $x_1 \leq x_1^*$, respectively. Let the unit vectors $\rho_{12p} = (1, 0)^T$ and $\rho_{12n} = (-1, 0)^T$ define the halflines $\Sigma_{12p} := \{ y \in \mathbb{R}^2 | y = c\rho_{12p}, c \in [0, \infty) \}$ and $\Sigma_{12n} := \{ y \in \mathbb{R}^2 | y = c\rho_{12n}, c \in [0, \infty) \}$. The normal vectors are $n_{12p} = (0, -1)^T$ and $n_{12n} = (0, 1)^T$.

Introducing a Taylor expansion of the vector fields $F_i$, $i = 1, 2$, near the point $(x^*, \nu_1)$, system (2.16) is locally approximated with:

$$\dot{y} = \begin{cases} A_1 y + \mu b, & y \in S_1 = \{ y \in \mathbb{R}^2 | n_{12p}^T y < 0 \}, \\ A_2 y + \mu b, & y \in S_2 = \{ y \in \mathbb{R}^2 | n_{12p}^T y > 0 \}, \end{cases}$$  \hspace{1cm} (2.25)
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Fig. 2.9. Application of conewise affine approximations for the bifurcation analysis of (2.16). In panel (a), the limit sets are shown of the conewise affine approximations at $\nu = 0$ and $\nu = \nu_1$. In panel (b), the equilibrium path of (2.16) is shown, which is computed analytically. Furthermore, the limit cycles are computed by repeated application of the shooting method. Since the stable limit cycle grows in amplitude with increasing $\nu$, the contents of panel (b) is repeated in panel (c) with a larger y-scale.

where $A_1 = \left( A_1 + \begin{pmatrix} 0 & 0.3(x_1^*)^2 & 0 \end{pmatrix} \right)$ and $A_2 = \left( A_2 + \begin{pmatrix} 0 & 0.3(x_1^*)^2 & 0 \end{pmatrix} \right)$, where the affine vector is given by $b = \begin{bmatrix} \cos(0.375\pi) & \sin(0.375\pi) \end{bmatrix}^T$ and $\mu \in \mathbb{R}$ is the bifurcation parameter.

Once more, for details of the analysis of bifurcations of (2.25) using the procedure of Section 2.4, we refer to [18]. Here we focus on discussing the results. For $\mu < 0$, one equilibrium point exists in $S_2$, that is a stable focus. This focus is encircled by an unstable limit cycle, as depicted in Fig. 2.10 for $\mu = -1$. At $\mu = 0$, an expanding spiralling motion is observed around the origin, such that the equilibrium at the origin is unstable. For $\mu > 0$, one equilibrium point exist in $S_2$, and no limit cycles exist.

The bifurcation diagram is depicted in Fig. 2.9(a) for the parameter range $\mu \in [-0.8, 0.8]$, that corresponds to the range of the system parameters $\nu \in$
Fig. 2.10. Unstable limit cycle of (2.25) with $\mu = -1$. The stable focus is depicted with an asterisk.

[1.08, 2.68] of system (2.16). This type of bifurcation was identified in [101] as a discontinuous Hopf bifurcation.

2.6.2.3 Bifurcations of the non-differentiable system

The nonsmooth bifurcations around $\nu = 0$ and $\nu = \nu_1$ are approximated locally in the previous sections, yielding the bifurcation diagram as depicted in Fig. 2.9.a. According to Theorems 2.10 and 2.11, similar limit sets exist in the smooth system (2.16). Using the approximations, the bifurcation diagram of (2.16) is given in Fig. 2.9.b and 2.9.c. The limit cycles created or destroyed by the bifurcations, as found by the conewise affine approximations, are followed for varying $\nu \neq 0$ and $\nu \neq \nu_1$ using a sequential implementation of the shooting algorithm, cf. [129]. The path of the equilibrium point is computed analytically.

The local bifurcations of the equilibrium at $\nu = 0$ and $\nu = \nu_1$ are accurately described. However, since the conewise affine approximation uses a local approximation of (2.16) near $(\nu, x) = (\nu_1, x^*)$, the stable limit cycle induced at $\nu = 0$ is not identified by the local approximation around $\nu = \nu_1$.

Note that the approach of this chapter does not yield information on bifurcations of limit cycles. For example, with the presented analysis we can not exclude a bifurcation of the stable limit cycle for $\nu > 0$.

2.7 Conclusions

A procedure is presented that yields a complete analysis of bifurcating equilibria in a class of hybrid systems, described by a continuous, piecewise smooth differential equations. This procedure is a computationally feasible method to
identify all limit sets that are created or destroyed during the bifurcation of an equilibrium point.

To analyse the bifurcation with the given procedure, under certain assumptions, continuous, piecewise smooth hybrid systems can be approximated near the bifurcation point by a conewise affine system. The bifurcation of the approximated, conewise affine system accurately describes the local bifurcation of the equilibrium point in the continuous, piecewise smooth system.

For the conewise affine system, we study what limit sets appear or disappear, change in character or in their stability properties under change of a system parameter. Existing equilibria, homoclinic and heteroclinic orbits of the conewise affine system can be found in a relatively straightforward manner. The other limit sets that are possible in autonomous planar systems are closed orbits, including limit cycles. To find these, we study the Poincaré return maps.

To be able to find all limit sets in a computationally feasible manner, new theoretical results are presented. With these results, one can exclude the existence of closed orbits far away from the equilibria. Using these theoretical results, the presented procedure is able to identify, in a computationally efficient manner, all limit sets, which appear or disappear in the discontinuity-induced bifurcation of the equilibrium of the approximated, conewise affine system. The procedure will identify all closed orbits, when multiple closed orbits are created at the bifurcation point. In two examples, the theoretical results are illustrated.

The procedure presented in this chapter is useful to assess the parameter dependency of the dynamics of planar piecewise smooth systems. Knowledge on what limit sets can appear or disappear under parameter changes of a system can be useful in the design of a system that is non-differentiable, and can be used to evaluate the robustness of the system, which is generally an important desired characteristic for non-differentiable control systems.

The results of this work can be extended to a more general class of hybrid systems, described by piecewise smooth systems with a discontinuous right-hand side, which are relevant, for example, in the context of mechanical systems with Coulomb friction. Analogue to the conewise affine approximation of non-differentiable systems, in the following chapter, we employ the linearisation of one smooth vector field near a discontinuity to study trajectories near equilibrium sets in discontinuous systems describing mechanical systems with dry friction.
Bifurcations of equilibrium sets in mechanical systems with dry friction

Abstract – The presence of dry friction in mechanical systems induces the existence of an equilibrium set, consisting of infinitely many equilibrium points. The local dynamics of the trajectories near an equilibrium set is investigated for systems with one frictional interface. In this case, the equilibrium set will be an interval of a curve in phase space. It is shown in this chapter that local bifurcations of equilibrium sets occur near the endpoints of this curve. Based on this result, sufficient conditions for structural stability of equilibrium sets in planar systems are given and two new bifurcations are identified. The results are illustrated by application to a controlled mechanical system with friction.

3.1 Introduction

Many mechanical systems experience stick due to dry friction, such that trajectories converge to an equilibrium set, that consists of a continuum of equilibrium points, rather than to an isolated equilibrium point. Dry friction appears at virtually all physical interfaces that are in contact. The presence of equilibrium sets in engineering systems compromises positioning accuracy in motion control systems, such as robot positioning control, see e.g. [115, 120, 133]. Dry friction may also cause other effects that deteriorate performance of motion control systems, such as the occurrence of periodic orbits, cf. [79, 133]. In this chapter, sufficient conditions for structural stability of equilibrium sets are given, and bifurcations of equilibrium sets are studied.

This chapter is based on [26], and parts have appeared in [24, 25].
The dry friction force is modelled with a set-valued friction law that depends on the slip velocity, such that the friction law is set-valued only at zero slip velocity. Using such a set-valued friction law, the systems are described by a differential inclusion. Such friction laws can accurately describe the existence of an equilibrium set, see e.g. [66, 104, 164]. This class of friction models contains the models of Coulomb and Stribeck. Dynamic friction models as discussed in [125] are not considered, since these models increase the dimension of the phase space, such that the study of bifurcations becomes more involved. In this chapter, the dynamics of mechanical systems are studied where dry friction is present in one interface.

The equilibrium sets of systems with dry friction may be stable or unstable in the sense of Lyapunov. In addition, equilibrium sets may attract all nearby trajectories in finite time, cf. [104]. A natural question is to ask how changes in system parameters may influence these properties. To answer this question, structural stability and bifurcations of equilibrium sets are studied.

For this purpose, the local phase portrait near an equilibrium set is studied and possible bifurcations are identified. Under a non-degeneracy condition, the local dynamics is shown to be always structurally stable near the equilibrium set except for two specific points, namely, the endpoints of the equilibrium set. Hence, analysis of the trajectories near these points yields a categorisation of the bifurcations that are possible. In this manner, particularly, bifurcations of equilibrium sets of planar systems are studied in this chapter.

Although quite some results exist on the asymptotic stability and attractivity of equilibrium sets of mechanical systems with dry friction, see [1, 39, 85, 104, 105, 159], few results exist that describe bifurcations of equilibrium sets, see e.g. [86, 163]. In [104, 105, 159], Leine and van de Wouw derive sufficient conditions for attractivity and asymptotic stability of equilibrium sets using Lyapunov theory and invariance results. In the papers of Adly, Attouch and Cabot [1, 39], conditions are presented under which trajectories converge to the equilibrium set in finite time. Using Lyapunov functions, the attractive properties of individual points in the equilibrium sets are analysed in [85]. Existing results on bifurcations of systems with dry friction are either obtained using the specific properties of a given model, or they do not consider trajectories in the neighbourhood of an equilibrium set. In [163], a van der Pol system is studied that experiences Coulomb friction. By constructing a Poincaré return map, Yabuno et al. show that the limit cycle, created by a Poincaré-Andronov-Hopf bifurcation for the system without friction, can not be created near the equilibrium set in the presence of friction. In [86], the appearance or disappearance of an equilibrium set is studied by solving an algebraic inclusion.

Bifurcations of the larger family of differential inclusions, that contains models of mechanical systems with friction, are studied e.g. in [15, 16, 47, 58, 71, 94, 96, 101, 103, 151]. Bifurcations of limit cycles of discontinuous systems are studied using a return map, see [15, 47, 94, 103]. However, in these systems the fric-
tion interface is moving, such that the discontinuity surface does not contain equilibria. Bifurcations of equilibria in two- or three-dimensions are studied in e.g. [16,58,96,151]. Here, the dynamics is understood by following the trajectories that become tangent to a discontinuity boundary. Guardia et al. present in [71] a generic classification of bifurcations with codimension one and two in planar differential inclusions. However, the special structure of differential inclusions describing mechanical systems with dry friction, which we will analyse in this chapter, is considered to be non-generic by Guardia et al. In the class of systems studied in this chapter, equilibrium sets occur generically, and persist when physically relevant perturbations are applied. Due to the difference in allowed perturbations, the scenarios observed in this chapter are not considered in [71]. In [15,101], the authors study bifurcations of equilibria in continuous systems, that are not differentiable. Due to the assumptions posed in these papers, all equilibria are isolated points.

In this chapter, a more general class of mechanical systems with friction is studied. Both the existence of an equilibrium set and the local phase portrait are investigated. Sufficient conditions for structural stability of this phase portrait are given. Such conditions are derived for equilibrium points in smooth systems, among others, by Hirsch, Smale, Hartman, and Pugh, see [74,82,132]. Parallel to their approach in smooth systems, we analyse the structural stability of differential inclusions restricted to a neighbourhood of the equilibrium set. At system parameters where the conditions for structural stability are not satisfied, two bifurcations are identified that do not occur in smooth systems.

The outline of this chapter is as follows. First, we introduce a model of a mechanical system with friction in Section 3.2 and present the main result. This theorem states that local bifurcations of equilibrium sets occur in the neighbourhood of two specific points, which are the endpoints of the equilibrium set. In Section 3.3, classes of systems are identified that are structurally stable. In Section 3.4, two bifurcations of the equilibrium set of planar systems are presented. In addition, it is shown that no limit cycles can be created by a local scenario similar to the Poincaré-Andronov-Hopf bifurcation. In Section 3.5, the results of this chapter are illustrated with an example of a controlled mechanical system with dry friction. Conclusions are given in Section 3.6. In Section 3.7, we present an additional discussion on the relation to and differences with existing results in the literature on bifurcations of discontinuous systems.

3.2 Modelling and main result

Consider a mechanical system that experiences friction on one interface between two surfaces that move relative to each other in a given direction. Let $x$ denote the displacement in this direction and $\dot{x}$ denote the slip velocity, see Fig. 3.1 for an example. For an $n-$dimensional dynamical system this implies that $n−2$ other states $y$ are required besides $x$ and $\dot{x}$. These states contain the other
positions and velocities of the mechanical system, and, for example, controller and observer states in the case of a feedback-controlled motion system. The system given in Fig. 3.1 can be modelled with the additional states \( y = \begin{pmatrix} y_1 \\ \dot{y}_1 \end{pmatrix} \). In general, using the states \( x, \dot{x} \) and \( y \), the dynamics are described by the following differential inclusion, cf. [58]:

\[
\ddot{x} - f(x, \dot{x}, y) \in -F_s \text{Sign}(\dot{x}), \\
\dot{y} = g(x, \dot{x}, y),
\]

where \( f \) and \( g \) are sufficiently smooth, \( F_s \neq 0 \), and \( \text{Sign}(\cdot) \) denotes the set-valued sign function 
\[
\text{Sign}(p) = \begin{cases} 
\frac{p}{|p|}, & p \neq 0, \\
[-1, 1], & p = 0.
\end{cases}
\]

Note that (3.1) also encompasses systems with other nonlinearities than dry friction, e.g. robotic systems. Introducing the state variables \( q = (x \ \dot{x} \ y^T)^T \), the dynamics of (3.1) can be reformulated as:

\[
\dot{q} \in F(q),
\]

\[
F(q) = \begin{cases} 
F_1(q), & q \in S_1 := \{q \in \mathbb{R}^n : h(q) < 0\}, \\
F_2(q), & q \in S_2 := \{q \in \mathbb{R}^n : h(q) > 0\}, \\
\text{co}\{F_1(q), F_2(q)\}, & q \in \Sigma := \{q \in \mathbb{R}^n : h(q) = 0\},
\end{cases}
\]

where \( q \in \mathbb{R}^n \), \( \text{co}(a, b) \) denotes the smallest convex hull containing \( a \) and \( b \), and \( F_1, F_2 \) and \( h \) are given by the smooth functions:

\[
F_1(q) = \begin{pmatrix} \dot{x} \\ f(x, \dot{x}, y) + F_s \\ g(x, \dot{x}, y) \end{pmatrix}; \\
F_2(q) = \begin{pmatrix} \dot{x} \\ f(x, \dot{x}, y) - F_s \\ g(x, \dot{x}, y) \end{pmatrix}; \\
h(q) = \dot{x}.
\]

In most existing bifurcation results for differential inclusions, see e.g. [16, 58, 96, 151], parameter changes are considered that induce perturbations of the function \( F \) in (3.2). Hence, in these studies the first component of \( F \) is perturbed, which implies that the case where the discontinuity surface coincides with the set where
the first element of $F$ is zero is considered non-generic by these authors. This implies that the existence of an equilibrium set in (3.2) is non-generic. However, parameter changes for the specific system (3.1) will only yield perturbations of $f$ and $g$ in (3.2). Namely, for such mechanical systems, the perturbation of the kinematic relationship between position and velocity is not realistic. Using this insight, we show that for the class of systems under study, i.e., mechanical systems with set-valued friction, equilibrium sets will occur, generically.

To study trajectories at the discontinuity surface $\Sigma$, the solution concept of Filippov is used, see [58]. Three domains are distinguished on the discontinuity surface. If trajectories on both sides arrive at the boundary, then we have a stable sliding region $\Sigma^s$. If one side of the boundary has trajectories towards the boundary, and trajectories on the other side leave the boundary, this domain is called the crossing region $\Sigma^c$ (or transversal intersection). Otherwise, we have the unstable sliding motion on the domain $\Sigma^u$. The mentioned domains are identified as follows:

\[
\begin{align*}
\Sigma & : = \{ q \in \mathbb{R}^n : h(q) = 0 \} \\
\Sigma^s & : = \{ q \in \Sigma : L_{F_1}h > 0 \land L_{F_2}h < 0 \}, \\
\Sigma^u & : = \{ q \in \Sigma : L_{F_1}h < 0 \land L_{F_2}h > 0 \}, \\
\Sigma^c & : = \{ q \in \Sigma : (L_{F_1}h)(L_{F_2}h) > 0 \},
\end{align*}
\] (3.5)

where $L_{F_i}h, i = 1, 2$, denotes the directional derivative of $h$ with respect to $F_i$, i.e., $L_{F_i}h = \nabla h F_i(q)$.

The vector field $\dot{q} = F^s(q)$, that is active during sliding motion for $q \in \Sigma^u \cup \Sigma^s$, is defined by employing Filippov’s solution concept as follows. For each $q$, the vector $F^s(q)$ is the vector on the segment between $F_1(q)$ and $F_2(q)$ that is tangent to $\Sigma$ at $q$:

\[
\dot{q} = F^s(q) := \frac{L_{F_1}h(q) F_2(q) - L_{F_2}h(q) F_1(q)}{L_{F_1}h(q) - L_{F_2}h(q)} = \begin{pmatrix} 0 \\ 0 \\ g(x, 0, y) \end{pmatrix}. \] (3.6)

Since $L_{F_1}h = L_{F_2}h + 2F_s$, it follows from (3.5) that $F_s > 0$ implies that no unstable sliding occurs, and $F_s < 0$ implies that no stable sliding occurs. The resulting phase space is shown schematically in Fig. 3.2 for the case $n = 3$.

In Lemma A.7, which is presented in Appendix A.2.1, we show that the equilibrium set is a segment of a curve on the discontinuity surface $\Sigma$ when the following assumption is satisfied.

**Assumption 3.1.** The functions $f$ and $g$ are such that $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$ and \( \begin{pmatrix} \frac{\partial f(x, 0, y)}{\partial x} & \frac{\partial f(x, 0, y)}{\partial y} \\ \frac{\partial g(x, 0, y)}{\partial x} & \frac{\partial g(x, 0, y)}{\partial y} \end{pmatrix} \) is invertible for all $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-2}$. Furthermore, the map $\begin{pmatrix} f \\ g \end{pmatrix}$ is proper\(^1\).

\(^1\)A continuous map is proper if the inverse image of any compact set is compact.
Chapter 3. Bifurcations of equilibrium sets in mechanical systems

For systems satisfying this assumption, the equilibrium set $\mathcal{E}$ of (3.1) is a one-
dimensional curve as shown in Fig. 3.2. The equilibrium set of a differential in-
clusion is given by $0 \in F(q)$, which is equivalent with $(q \in \Sigma^s \cup \Sigma^u \wedge L_{F_2}h F_1(q) - L_{F_1}h F_2(q) = 0)$, since $0 \in \left(\cap \{f(q) - F_s, f(q) + F_s\}\right)$ is equivalent with $q \in \Sigma^s \cap \Sigma^u$ and $g(q) = 0$ is equivalent with $L_{F_2}h F_1(q) - L_{F_1}h F_2(q) = 0$ for $q \in \Sigma^s \cap \Sigma^u$, see (3.6).

The equilibrium set is divided into interior points and the two endpoints as
follows:

$$\mathcal{E} : = \{q \in \Sigma^u \cup \Sigma^s : L_{F_2}h F_1 - L_{F_1}h F_2 = 0\},$$

$$I : = \{q \in \mathcal{E} : F_1 \neq 0 \wedge F_2 \neq 0\},$$

$$E_i : = \{q \in \mathcal{E} : F_i = 0\}, \ i = 1, 2. \quad (3.7)$$

Note that interior points are called pseudo-equilibria in [15]. The endpoints $E_1$ and $E_2$ satisfy $L_{F_1}h = 0$ or $L_{F_2}h = 0$, respectively, hence they are positioned on the boundary of the stable or unstable sliding mode as given by (3.5).

In this chapter, the trajectories near the equilibrium set will be studied. The
influence of perturbations of (3.1) on the phase portrait of a system is studied.
For this purpose, we define the topological equivalence of phase portraits of
(3.1) in Definition 3.1. We note that this definition is equal to the definition for
smooth systems, see e.g. [6,32,72].

\textbf{Definition 3.1} ([58]). \textit{We say that two dynamical systems in open domains $G_1$ and $G_2$, respectively, are topologically equivalent if there exists a homeo-
morphism from $G_1$ to $G_2$ which carries, as does its inverse, trajectories into
trajectories.}

This equivalence relation allows for a homeomorphism that does not pre-
serve the parameterisation of the trajectory with time, as required for \textit{topological conjugacy} defined in [72]. Throughout this chapter, we assume that $f$ and
$g$ smoothly depend on system parameters. When a parameter variation of a
dynamical system $A$ yields a system $\tilde{A}$ which is not topologically equivalent to $A$, then the dynamical system experiences a bifurcation.

With the definitions given above, we can formulate our main result in the following theorem.

**Theorem 3.1.** Assume (3.1) satisfies Assumption 3.1. If $\frac{\partial g}{\partial y} \bigg|_p$ has no eigenvalue $\lambda$ with $\text{real}(\lambda) = 0$ for any $p \in \mathcal{E}$, then the dynamical system (3.1), in a neighbourhood of the equilibrium set, can only experience bifurcations near the endpoints $E_1$ or $E_2$.

**Proof.** The proof is given in Appendix A.2.2. □

In the next section, we will first introduce the concept of structural stability, which is used in the proof of Theorem 3.1. We will employ a structural stability definition that is tailored to systems of the form (3.1) and differs from the commonly used notion of structural stability for generic systems of the form (3.2). In particular, only perturbations on the right-hand side of (3.1) should be considered. Subsequently, we will discuss the similarities between the line of reasoning used in the proof of Theorem 3.1 and the standard arguments that prove structural stability of hyperbolic equilibrium points in smooth systems.

### 3.3 Structural stability of the system near the equilibrium set

To prove Theorem 3.1, the influence of perturbations on systems (3.1) are studied. If perturbations of $f$ and $g$ of (3.1) can not yield a dynamical system which is not topologically equivalent to the original system, then the occurrence of bifurcations is excluded. Hence, structural stability of (3.1) is investigated, which is defined as follows.

**Definition 3.2.** A system $A$ given by (3.1) is structurally stable for perturbations in $f$ and $g$ if there exists an $\epsilon > 0$ such that the system $\tilde{A}$ given by (3.1) with $\tilde{f}$ and $\tilde{g}$ such that

$$|f - \tilde{f}| < \epsilon, \left\| \frac{\partial (f - \tilde{f})}{\partial q} \right\| < \epsilon, \|g - \tilde{g}\| < \epsilon, \left\| \frac{\partial (g - \tilde{g})}{\partial q} \right\| < \epsilon,$$

is topologically equivalent to system $A$.

Note, that this definition corresponds to $C^1$-structural stability as defined by [146], and is tailored to dynamical systems described using second-order time derivatives of the state $x$.

Note that perturbations of (3.1) in $f$ and $g$ do not cause perturbations of the first component of $F(\cdot)$ in (3.2), as observed e.g. in [146] or [58, page 226].
One consequence of this fact is that equilibrium sets occur generically in systems (3.1), although they are non-generic in systems (3.2). In experiments on mechanical systems with dry friction, such equilibrium sets are found to occur generically, see e.g. [125]. For this reason, perturbations of the class (3.8) are used throughout this chapter. System $A$ can be structurally stable for perturbations in $f$ and $g$, whereas the corresponding system (3.2) is not structurally stable for general perturbations of $F$. Small changes of system parameters cause small perturbations of $f$ and $g$ and their derivatives. However, the first equation of (3.2) will not change under parameter changes. Namely, this equation represents the kinematic relationship between position and velocity of a mechanical system, such that perturbation of this equation does not make sense for the class of physical systems under study. Hence, structural stability for perturbations in $f$ and $g$ excludes the occurrence of bifurcations near the system parameters studied.

In smooth systems, structural stability of dynamical systems in the neighbourhood of equilibrium points is studied four decades ago by, among others, Hirsch, Smale, Hartman, Grobman, and Pugh, see [74, 82, 132]. It is now well known that, restricted to the neighbourhood of an equilibrium point, smooth dynamical systems are structurally stable when the equilibrium point is hyperbolic. For hyperbolic equilibrium points of smooth systems, the inverse function theorem implies that the equilibrium point is translated over a small distance when the vector field is perturbed, and the perturbed vector field near this equilibrium point is 'close' to the original vector field near the unperturbed equilibrium. Hence, the Hartman-Grobman theorem shows that there exists a topological equivalence for the phase portrait in the neighbourhoods of both equilibria. In this chapter, the structural stability of system (3.1) in the neighbourhood of equilibrium sets is studied analogously.

Small perturbations of system (3.1) will cause the equilibrium set $E$ to deform, but the equilibrium set of the perturbed system remains a smooth curve in state space. Hence, there exists a smooth coordinate transformation (analogous to the translation for smooth systems) that transforms the original equilibrium set $E$ to the equilibrium set of the perturbed system, as shown in Appendix A.2.1. Furthermore, the vector field near both equilibrium sets are 'close', as shown in Lemma A.10. Using this coordinate transformation, Theorem 3.1 is be proven in Appendix A.2.2.

### 3.3.1 Structural stability of planar systems

In this section, sufficient conditions are presented for the structural stability of planar systems restricted to a neighbourhood of equilibrium sets, which form an extension of the conditions presented in Theorem 3.1. In the planar case, (3.1) and (3.2) reduce to, respectively:

$$
\ddot{x} - f(x, \dot{x}) \in -F_x \text{Sign}(\dot{x}),
$$

(3.9)
3.3 Structural stability of the system near the equilibrium set

\[ \dot{q} = F_2(q), \quad h(q) = q_2, \] near an equilibrium set \( E \) with endpoints \( E_1 \) and \( E_2 \).

\[ \dot{q} \in F(q) = \begin{cases} F_1(q) = \left( \frac{q_2}{f(q_1, q_2) + F_s} \right), & h(q) < 0, \\ F_2(q) = \left( \frac{q_2}{f(q_1, q_2) - F_s} \right), & h(q) > 0, \end{cases} \quad (3.10) \]

where \( q = (x, \dot{x})^T \) and \( h(q) = q_2 \). In this case, the Filippov solution \( \dot{q} = F_s(q) = 0 \) for \( q \in \Sigma^s \cup \Sigma^u \), such that the set of interior points of the equilibrium point satisfies \( I = \Sigma^s \cup \Sigma^u \).

In this section, it is assumed that \( F_s > 0 \), which corresponds to the practically relevant case where dry friction dissipates energy. The assumption \( F_s > 0 \) does reduce the number of topological distinct systems of (3.9). However, the case \( F_s < 0 \) yields topologically equivalent systems when time is reversed. The case \( F_s > 0 \) implies \( \Sigma^u = \emptyset \) such that trajectories remain unique in forward time. The local phase portrait in the neighbourhood of an equilibrium set is depicted in Fig. 3.3 for an exemplary system.

According to Theorem 3.1, structural stability of (3.9), restricted to a neighbourhood of the equilibrium set, is determined by the trajectories of (3.9) near the endpoints. Analogously to the Hartman-Grobman theorem, which derives sufficient conditions for structural stability of trajectories near an equilibrium point in smooth systems based on the linearised dynamics near this point, sufficient conditions for structural stability of (3.9) will be formulated based on the linearisation of \( F_1 \) and \( F_2 \) near the endpoints of the equilibrium set.

For ease of notation, we define: \( A_k := \frac{\partial F_k}{\partial q} \bigg|_{q = E_k} \), \( k = 1, 2 \), which determines the linearised dynamics in \( S_k \) near the endpoints of the equilibrium set. In the other domain, i.e. \( S_{3-k} \), it follows from (3.10) that the vector field is pointing towards the discontinuity surface. To study the structural stability of planar systems (3.9), we adopt the following assumption.

**Assumption 3.2.**

(i) The dry friction force satisfies \( F_s > 0 \).

(ii) The eigenvalues of \( A_k \), \( k = 1, 2 \), are distinct and nonzero.
Table 3.1. Possible systems (3.9), categorised by the eigenvalues of the Jacobian matrix near the endpoints, which are locally structurally stable in a neighbourhood of the equilibrium set. Note that the indices of $A_1$ and $A_2$ can be interchanged. Eigenvalues are denoted with c, -, or + when the eigenvalues complex, real and positive, or negative, respectively.

Observe that (i) implies that the equilibrium point $E$ persists, whereas (ii) concerns the linearised vector field near the endpoints of $E$. Furthermore, (ii) implies that $A_k$ is invertible. The following theorem presents sufficient conditions for structural stability of (3.9), restricted to a neighbourhood of the equilibrium set.

**Theorem 3.2.** Consider a system $A$ given by (3.9) satisfying Assumptions 3.1 and 3.2. Restricted to a neighbourhood of the equilibrium set, system $A$ is structurally stable for perturbations in $f$.

**Proof.** The proof is given in Appendix A.2.3. □

This theorem implies that one can identify 10 different types of systems (3.9) with a stable sliding mode which, restricted to a neighbourhood of the equilibrium set, are locally structurally stable, as shown in Table 3.1. Note, that an unstable sliding mode yields analogous types of equilibrium sets.

If $f$ satisfies the symmetry property: $f(x,0) = -f(-x,0)$, then only the Source-Source, Sink-Sink, Saddle-Saddle and Focus-Focus types are possible.

### 3.4 Bifurcations

Due to the special structure of (3.1), general dynamical systems of the form (3.2) have a richer dynamics than systems (3.1). For example, all codimension-one
bifurcations of (3.2) as observed in [96] can not occur in (3.1). In general, the sliding motion of the system (3.2) yields a nonzero sliding vector field, whereas the sliding motion of (3.1) contains a set of equilibria. Therefore, in this section bifurcations of (3.1) are studied, restricting ourselves to planar systems as given in (3.9). Such bifurcations occur in systems (3.2) as well, but will have a higher codimension.

Using Assumption 3.2, in Section 3.3 several types of topologically distinct planar systems (3.9) are identified, which are structurally stable in a neighbourhood of the equilibrium set. Hence, it seems a reasonable step to consider parameter-dependent systems and study the parameters where the given conditions on the differential inclusion (3.9) no longer hold. In this manner, two bifurcations of planar systems (3.1) are presented. At the bifurcations points, Assumption 3.2.(ii) is violated.

3.4.1 Real or complex eigenvalues

Consider system (3.9) where eigenvalues of $A_1$ change from real to complex eigenvalues under a parameter variation. From (3.9) it follows that

$$A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix},$$

hence the eigenvalues of $A_1$ are distinct when $a_{21} \neq -\frac{1}{4}a_{22}^2$ and the eigenvalues are both nonzero given $a_{21} \neq 0$. Now, let the first part of Assumption 3.2(ii) be violated, such that the eigenvalues are equal. In that case, we obtain $a_{21} = -\frac{1}{4}a_{22}^2$.

Arbitrarily small variations of the system parameter $a_{21}$ near $a_{21} = -\frac{1}{4}a_{22}^2$ can create topologically distinct systems, since a system where $A_1$ has complex eigenvalues is topologically distinct from a system where $A_1$ has real eigenvalues. This follows from the observation that there exists a stable or unstable manifold containing $E_1$ if and only if $A_1$ has real eigenvalues. Namely, there can not exist a homeomorphism that satisfies the conditions in Definition 3.1 and maps a trajectory on this manifold to a trajectory of a system where $A_1$ has complex eigenvalues, since in that case only one trajectory converges towards $E_1$, which originates from $S_2$.

This bifurcation is illustrated with the following exemplary system:

$$\ddot{x} - a_{21}x - a_{22}\dot{x} \in -F_s\text{Sign}(\dot{x}), \quad (3.11)$$

with $F_s = 1$, $a_{22} = -0.1$ and varying $a_{21}$. In this example, the matrices $A_1$ and $A_2$ are equal, such that both endpoints undergo a bifurcation at the same time. This system shows a bifurcation when $a_{21} = -0.0025$, as shown in Fig. 3.4. We refer to this bifurcation as a focus-node bifurcation. According to [39], all trajectories of system (3.11) will arrive in the equilibrium set $E$ in finite time if and only if $a_{21} < -\frac{1}{4}a_{22}^2 = -0.0025$. For $a_{21} \geq 0.0025$, the matrices $A_1$ and $A_2$ have a real eigenvector corresponding to an eigenvalue $\lambda$. The span of this eigenvector contains trajectories that converge exponentially according to
3.4.2 Zero eigenvalue

Consider system (3.9) where an eigenvalue of $A_1$ becomes zero under parameter variation, where $A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$. This matrix has an eigenvalue equal to zero when $a_{21} = 0$ becomes zero.

At the point $E_1$ of the equilibrium set, the vector field satisfies $F_1(E_1) = 0$. By definition, the point $E_1$ is an endpoint of $E$, such that trajectories cross $\Sigma$ outside $E$. This implies that the second component of $F_1$, denoted $F_1^\dot{x}$, evaluated on the curve $\Sigma$ changes sign at $E_1$. Since $F_1$ is smooth and $F_1^\dot{x}$ changes sign at $E_1$, we obtain $\frac{\partial^k F_1^\dot{x}}{\partial x^k}\Big|_{E_1} \neq 0$ for an odd integer $k \geq 3$, and $\frac{\partial^i F_1^\dot{x}}{\partial x^i}\Big|_{E_1} = 0$, for $i = 1, \ldots, k - 1$. The equilibrium set $E$ on $\Sigma$ is given by $F_2^\dot{x} \leq 0$ and $F_1^\dot{x} \geq 0$. A change of a system parameter can create two distinct domains where $F_1^\dot{x} \geq 0$ near $E_1$, such that two equilibrium sets are created.

This bifurcation is illustrated with the following exemplary system:

$$\ddot{x} - a_{21} x - a_{22} \dot{x} + F_s + x^3 \in -F_s \text{Sign}(\dot{x}), \quad (3.12)$$

with $F_s = 1$, $a_{22} = -1$ and varying $a_{21}$. The system is designed such that the origin is always the endpoint of an equilibrium set. The resulting phase portrait
3.4 Bifurcations

![Fig. 3.5](image)

Fig. 3.5. System (3.12) with $F_s = 1$, $a_{22} = -1$ and varying $a_{21}$, showing a bifurcation where an eigenvalue becomes zero. A neighbourhood of the origin is depicted, that does not contain the complete equilibrium set. The equilibrium set $E$ is given by a bold line, the eigenvectors of stable or unstable eigenvalues of $\left. \frac{\partial F_2}{\partial q} \right|_{q=0}$ are represented with dashed lines.

is given in Fig. 3.5, and shows the mentioned bifurcation. For $a_{21} = -0.1$, one compact equilibrium set exists. For $a_{21} = 0$, an eigenvalue of the system becomes zero, and the corresponding eigenvector is parallel to the equilibrium set. Note that this implies that both Assumptions 3.1 and 3.2.(ii) are violated. For $a_{21} > 0$, the equilibrium set splits in two separated, compact, equilibrium sets, cf. Fig. 3.5(c).

Another bifurcation occurs when $F_1$ and $F_2$ are linear systems. In that case, the equilibrium set grows unbounded when $a_{21} \to 0$, and becomes the complete line satisfying $x \in \mathbb{R}$, $\dot{x} = 0$. This bifurcation is illustrated in Fig. 3.6 using system (3.11) with $a_{22} = -0.1$, $F_s = 1$ and varying $a_{21}$.

3.4.3 Closed orbits

In smooth systems, the Poincaré-Andronov-Hopf bifurcation can create a small closed orbit near an equilibrium point. A similar scenario can not occur in planar systems (3.9) when $F_s \neq 0$, as shown in the following lemma.

Lemma 3.3. Consider system (3.9) with $F_s \neq 0$ satisfying Assumption 3.1 which has closed orbit $\gamma$ with period $T_\gamma$. Given a period time $T_\gamma$, there exists an $\epsilon > 0$ such that $E + B_\epsilon$ does not contain parts of the closed orbit $\gamma$.

Proof. The proof is given in Appendix A.2.4. □

This lemma contradicts the appearance of a limit cycle with finite period near the equilibrium set, as appears close to a smooth Poincaré-Andronov-Hopf bifurcation point in smooth systems. Note, that the appearance of limit cycles is not excluded, when changes of system parameters causes the dry friction force $F_s$.
Chapter 3. Bifurcations of equilibrium sets in mechanical systems

Fig. 3.6. System (3.11) with $F_s = 1$, $a_{22} = -0.1$ and varying $a_{21}$, showing a bifurcation when an eigenvalue becomes zero and the system is linear. The equilibrium set $E$ is given by a bold line, the stable or unstable manifolds are represented with dashed lines. At $a_{21} = 0$, the equilibrium set coincides with the line $x \in \mathbb{R}, \dot{x} = 0$.

to change sign, which is physically unrealistic. When the discontinuous nature of the system is introduced by other effects than dry friction, the appearance of limit cycles can occur in physical systems, see e.g. [165].

Remark 3.1. A heteroclinic orbit may exist that connects the endpoints $E_1$ to $E_2$ through two trajectories, one positioned in the smooth domain $S_1$ and the other positioned in the opposite smooth domain $S_2$. Small perturbations of this system can be expected to cause the appearance of limit cycles with an arbitrary large period time. $\ltimes$

The result given in Lemma 3.3 is derived using the specific structure of the vector field near the equilibrium set. Non-local events such as the appearance of homoclinic or heteroclinic orbits are not considered in this chapter and will be studied in Chapters 4 and 5.

3.5 Illustrative example

The applicability of Theorem 3.1 for higher-dimensional systems is illustrated with an observer-based control system, where a single mass is controlled using a velocity observer. The system is given by:

$$\dot{q} = Aq + B(u + f(q)), \quad (3.13)$$

with $q = (x \ x')^T \in \mathbb{R}^2$, measurement $z = x$, control input $u$ and friction force $f(q) \in -F_s \text{Sign}(q_2)$, as shown schematically in Fig. 3.7. We assume $M = 1$. The matrix $A$ is given by $A = \begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}$, with $c > 0$ and $B = (0 \ 1)^T$. For this system,
a linear state feedback controller of PD-type is designed, yielding \( u = -k_p x - k_d \dot{v} \), with proportional gain \( k_p \), differential gain \( k_d \), and \( v \) an estimate of the velocity \( \dot{x} \). This estimate is obtained with the following reduced-order observer, that is designed for the linear system without friction, see [64]:

\[
\dot{v} = -cv + u, \quad (3.14)
\]

After the notational substitution of \( v = y \), the resulting closed-loop system is given by

\[
\begin{pmatrix}
\dot{x} \\
\dot{\dot{x}} \\
\dot{y}
\end{pmatrix} = A_c
\begin{pmatrix}
x \\
\dot{x} \\
y
\end{pmatrix} + \begin{pmatrix}
0 \\
-F_s \text{Sign}(\dot{x}) \\
0
\end{pmatrix},
\quad (3.15)
\]

\[
A_c = \begin{pmatrix}
0 & 1 & 0 \\
-k_p & -c & -k_d \\
-k_p & 0 & -c - k_d
\end{pmatrix},
\quad (3.16)
\]

which is equivalent with (3.2), where \( f(x, \dot{x}, y) = -k_p x - c \dot{x} - k_d y \) and \( g = -k_p x - (c + k_d) y \). Assumption 3.1 implies \( ck_p \neq 0 \). If this is satisfied, system (3.16) has the equilibrium set \( \{ q = (x, \dot{x}, y) \in \mathbb{R}^3 : (x, \dot{x}, y) = (-\left(\frac{1}{k_p} + \frac{k_d}{ck_p}\right)\alpha, 0, \frac{\alpha}{c}) \}, \alpha \in [-F_s, F_s] \).

Since \( \frac{\partial q}{\partial y} = -c - k_d \), Theorem 3.1 shows that when \( -c - k_d \neq 0 \), no bifurcations occur away from the endpoints \( E_1 \) and \( E_2 \), given by \( (x, \dot{x}, y) = \pm \left(\frac{1}{k_p} + \frac{k_d}{ck_p}\right) F_s 0 \frac{F_s}{c} \).

The structural stability of trajectories near the endpoints of an equilibrium set is studied in this chapter only for planar systems, while the current example is 3-dimensional. However, we will still present a bifurcation of trajectories near the endpoints. Similar to the approach used in Section 3.4, the linearisation of the vector field near the endpoints is used. Here, matrices \( A_1 \) and \( A_2 \) coincide with \( A_c \), which has eigenvalues \( \lambda_1 = -c \) and \( \lambda_{2,3} = -\frac{c+k_d}{2} \pm \frac{1}{2} \sqrt{(c+k_d)^2 - 4k_p} \).

The eigenvalues \( \lambda_{2,3} \) change from real to complex when \( k_d = -c + 2\sqrt{k_p} \). At this point a bifurcation occurs similar to the focus-node bifurcation observed in
Section 3.4.1. When two eigenvalues are complex, for both endpoints $E_i$, $i = 1, 2$, there exists only one trajectory that converges to the endpoints $E_i$ from domain $S_i$ for $t \to \infty$ or $t \to -\infty$. When eigenvalues $\lambda_{2,3}$ are real, more trajectories exist with this property. Hence, a bifurcation occurs when $k_d$ crosses the value $-c + 2\sqrt{k_p}$. This bifurcation is illustrated in Fig. 3.8 where the parameters $c = 0.5$, $k_p = 1$, $F_s = 2$ are used. At these parameters, the mentioned bifurcation occurs at $k_d = 1.5$. For the used system parameters, the eigenvalues of $A_c$ have negative real part. In Fig. 3.8, only trajectories near the endpoint $E_2$ are shown. Since the system is symmetric, the same bifurcation occurs near the endpoint $E_1$.

These results suggest that using the linearisation of the dynamics near the endpoints, sufficient conditions for structural stability of trajectories can be constructed for higher-dimensional systems, analogously to the results in Sections 3.3.1 and 3.4 for planar systems.

3.6 Conclusion

In this chapter, bifurcations and structural stability of a class of nonlinear mechanical systems with dry friction are studied in the neighbourhood of equilibrium sets. It has been shown in Theorem 3.1 that local bifurcations of equilibrium sets of a class of nonlinear mechanical systems with a single frictional interface can be understood by studying the trajectories of two specific points in phase space, which are the endpoints of the equilibrium set. Hence, local techniques can be applied in a neighbourhood of these points. For differential inclusions given by (3.1), the linearisation of vector fields is only applicable to the part of the state space where the vector field is described by a smooth function. A careful study of this linearisation has given insight in the topological nature of solutions of the differential inclusion near the equilibrium set. Hence, in the neighbourhood of equilibrium sets, the result of Theorem 3.1 significantly simplifies the further study of structural stability and bifurcations for this class of mechanical systems with friction.

Using this approach, sufficient conditions are derived for structural stability of planar systems given by (3.1), restricted to a neighbourhood of the equilibrium set. Furthermore, two types of bifurcations of the equilibrium set of this class of systems are identified, which do not occur in smooth systems.
Fig. 3.8. System (3.16) with $c = 0.5$, $k_p = 1$ and $F_s = 2$, showing a bifurcation near the endpoints at $k_d = 1.5$. The equilibrium set $\mathcal{E}$ is given by a dotted line, and the real eigenvectors of $A_2$ are represented with thick lines.
3.7 Comparison with existing results on bifurcations and structural stability of discontinuous systems

Sections 3.1-3.6 have been published as the paper [26]. Here, we aim to add a further discussion on the relation to and differences with existing results in the literature on bifurcations of systems of the form (3.2).

The results in this chapter are tailored to the class of systems (3.1), that form a specific subclass of the differential inclusions given by (3.2)-(3.3). In the literature, various results are available on bifurcations of the latter class of systems, where smooth functions $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are considered, and the only assumption imposed is that $h$ satisfies $\partial h / \partial x \neq 0$ along the discontinuity boundary.

Since the behaviour that can be displayed by these systems is very diverse, few results exist for systems of arbitrary dimension. Exceptions are bifurcations of limit cycles, see e.g. [13], and the results presented in [14, 51], where $n-$dimensional systems are considered and the appearance or disappearance of equilibria is studied. However, in $n$-dimensional systems, structural stability results are not yet available.

Focussing on two- and three-dimensional systems, various local bifurcation scenarios have been identified where isolated equilibrium points interact with the discontinuity boundary, cf. [15, 17, 51, 53, 58, 71, 88, 96, 103]. Most notably, in [71, 96], classifications are presented of the bifurcations that can occur near equilibria in planar discontinuous systems.

In contrast to Definition 3.2 used in this chapter, in these studies, structural stability is considered with respect to arbitrary perturbations of $F_i$, $i = 1, 2$, that are small in $C^1$ metric, that is, where the perturbed vector fields $\tilde{F}_i$ satisfy:

$$
|F_i(q) - \tilde{F}_i(q)| < \epsilon,
\left\| \frac{\partial(F_i(q) - \tilde{F}_i(q))}{\partial q} \right\| < \epsilon, \quad \forall q \in \mathbb{R}^n, \quad i = 1, 2.
$$

(3.17)

At the system parameters where the vector field is not structurally stable for perturbations satisfying (3.17), bifurcations are observed.

Analogously to smooth systems, bifurcations can be classified based on their codimension, which gives the smallest dimension of a parameter space such that the bifurcation occurs persistently, even when perturbations of $F_i$, $i = 1, 2$, satisfying (3.17) are allowed. In [15, 88, 96], codimension one bifurcations are considered, and in [15, 17, 51, 53, 71], codimension two phenomena are studied.

Most of the recent contributions to the bifurcation analysis of discontinuous systems aim to develop a categorisation of bifurcations in two- and three-dimensional systems of the form (3.2)-(3.3) for bifurcations of low codimension. In the literature, nonsmooth models can be found where some of these bifurcations occur, see, for example, models of superconducting resonators [89], pop-
ulation models in ecology [53], and switching electronic circuits [14]. However, other bifurcations that should occur generically in (3.2)-(3.3), have, to the best of the author’s knowledge, not yet been observed in models of physical systems.

In order to apply the results from bifurcation analysis in nonsmooth models that describe physical systems, we are convinced that bifurcation results for more restricted classes of dynamical systems than (3.2)-(3.3) are highly desirable. Namely, in various nonsmooth models that occur in mechanical and control engineering, some structure is directly imposed on the vector fields $F_1$ and $F_2$ in (3.2)-(3.3). In these cases, it is not desirable to define structural stability based on perturbations given in (3.17), as this may not correspond to behaviour that is persistent in the physical system.

Mathematically, one may think here about the classes of Hamiltonian systems [28] and systems with time-reversal symmetry [99], that have attracted specific attention in the study of smooth dynamical systems. For Hamiltonian systems, next to arbitrary small perturbations of the right-hand side of the vector fields (see e.g. [34]), also perturbations of the Hamiltonian function [30,33], and perturbations that preserve spatial, or spatio-temporal, symmetries of the system have been considered [29,31]. For systems with a time-reversal symmetry, as discussed in [99], reversibility-preserving or non-reversible perturbations also lead to different definitions of structural stability.

In the existing literature of discontinuous systems, in addition to the class of systems (3.1) studied in this chapter, special attention has been paid to discontinuous dynamical systems described by a piecewise-affine Hamiltonian function [149], and systems with a time-reversing symmetry [87]. Next to these system classes, one could identify subclasses of discontinuous systems that are relevant for the modelling of physical systems by imposing the following conditions on $F_1$ and $F_2$ for the discontinuous system (3.2).

Various nonsmooth models exist where the difference $F_1(x) - F_2(x)$ is constant. For example, this occurs in control system where a switching controller is employed, cf. [144], and in mechanical systems with friction, as studied in this chapter. Another example is the buck-converter introduced in [15]. If the discontinuity boundary is linear, then a constant difference $F_1(x) - F_2(x)$ directly implies that either a stable or unstable sliding motion is not feasible, i.e. either $\Sigma^s$ or $\Sigma^u$ are empty sets.

Alternatively, for some classes of physical systems, equilibrium points are only expected to occur on the discontinuity boundary, and no equilibrium points are possible in the domains where (3.2) is smooth. For example, sliding mode controllers are typically employed in order to achieve fast convergence towards an equilibrium position at the origin, such that the controller design guarantees that all equilibrium points have to lie on the discontinuity boundary. Furthermore, the mechanical systems studied in Sections 3.1-3.6 of this chapter can only have equilibrium points at the discontinuity boundary, since Coulomb’s friction law is discontinuous at the boundary in state space where the slip velocity is zero.
We expect two advantages from results on bifurcations and structural stability results that are specific to a smaller class of systems than the full class of systems given by (3.2). Firstly, since one focusses on a more structured class of systems, the number of possible bifurcations will be reduced, and the remaining bifurcations are more likely to be found in nonsmooth models in, for example, engineering. Secondly, restrictions on the vector field (3.2) will simplify the dynamics, which may be of help to explore structural stability and bifurcations in higher-dimensional systems. In fact, the structure imposed by (3.1) is used in this chapter to derive Theorem 3.1, which forms a particular step in this direction.

Discontinuous systems modelled by (3.2) have been studied in [38,147] using a smooth approximation of the discontinuity, followed by the use of singular perturbation theory to obtain the dynamics on a slow manifold. However, the interpretation of the results of this analysis for the discontinuous system is not yet fully understood. For example, if this approach is followed for system (3.1), then the equilibrium set is represented by an isolated equilibrium point on the slow manifold. An interesting question is whether the result of this approach can be interpreted in a more direct way for more restricted classes of vector fields.

In conclusion, the study of structural stability and bifurcations of discontinuous vector fields described by (3.2) should be supplemented by the dynamical analysis of particular discontinuous vector fields that have a more restricted structure than the full class described by (3.2), but that are relevant in nonsmooth models of physical systems.
Chapter 4

Dynamical collapse of trajectories

Abstract – Friction induces unexpected dynamical behaviour. In the paradigmatic pendulum and double-well systems with friction, modelled with differential inclusions, distinct trajectories can collapse onto a single point. Transversal homoclinic orbits display collapse and generate chaotic saddles with forward dynamics that is qualitatively different from the backward dynamics. The space of initial conditions converging to the chaotic saddle is fractal, but the set of points diverging from it is not: friction destroys the complexity of the forward dynamics by generating a unique horseshoe-like topology.

4.1 Introduction

Friction remains one of the poorly understood ‘f’ problems in physics, fracture and fatigue being the others. Throughout history, friction has been dealt with in different ways. The tire industry aims at maximising the friction between the car and the road. Galileo, in his famous inclined plane experiment to measure the acceleration of gravity experimentally, made every effort to minimise friction. Though friction is ever present in many natural and man-made systems, it induces complex dynamical behaviour which is not yet fully understood. Friction forces are notoriously hard to model, since they are influenced by surface imperfections, wear and debris, crack formation, creep, local stress distribution, and material properties \[12, 42, 54, 80, 125, 153\]. However, qualitative dynamical behaviour of physical oscillators with friction, such as stick-slip behaviour, can be represented by empirical models where the friction force only depends on the slip velocity, and a dry friction (or Amontons-Coulomb) element is used to include a stick-phase. Using such models, we show that friction induces novel dynamical behaviour in physical oscillators, and in particular, creates a horseshoe-like topology.

This chapter is based on [19].
object with unexpected dynamics.

Dry friction is described by a force $F$, which, for non-zero velocities, is constant in magnitude, and its direction is opposite to the velocity. For zero velocities, $F$ can take a range of values. This singular behaviour has very important consequences for the dynamics of physical oscillators. Because $F$ has constant value in magnitude for any non-zero velocity, trajectories of systems with friction can come to rest in finite time, without the exponential approach one is used to in smooth systems. This finite-time convergence causes the violation of the uniqueness of trajectories in backward time: different initial conditions can end up in the same point \textit{in finite time}. Equilibria are typically not isolated: an equilibrium point is turned into a one-dimensional manifold of degenerate equilibria by the addition of friction, see e.g. [1, 39, 85, 104, 105, 159, 164] and Chapter 3 of this thesis.

To visualise this in a classical physical oscillator, consider a pendulum with constant dry friction torque of magnitude $T_f$ acting in the joint, and moment of inertia $I$. Near the bottom position and for nonzero velocities, the friction torque is dominant and acts in opposite direction of the motion, i.e. $I\ddot{\theta} \approx -T_f\text{sign}(\dot{\theta})$, where $\theta$ denotes the angle of the pendulum. Hence, if the initial velocity is small, then the acceleration is approximately constant and the pendulum comes to rest in finite time. An interval of equilibria exists near the top and bottom position of the pendulum, and all points in these sets attract an infinite set of trajectories in finite time.

In more complex systems displaying chaos, such as the paradigmatic case of the pendulum with forcing, we expect these singular properties of friction to have deep consequences for the global dynamics and the chaotic saddle in transient systems [98, 145]. In particular, the collapse of trajectories suggests that the future and past dynamics are qualitatively different: only in forward time, trajectories can be continued uniquely from their initial conditions. We will show that this induces a complete asymmetry in the geometry of the stable and the unstable sets of the chaotic saddle. The stable set, corresponding to the set of initial conditions whose orbits converge to the chaotic saddle, is fractal, whereas the unstable set is a regular, non-fractal set. This feature has no counterpart in smooth systems, where the stable and unstable sets are either both fractal (with the same fractal dimensions) or both non-fractal [98]. We will explain how this comes about by adapting the Smale horseshoe [145] to the case of frictional oscillators. The broad impact of our findings is shown in simulations of a paradigmatic system in physics, the perturbed pendulum.

In this chapter, we show that in physical systems with friction, homoclinic orbits \textit{collapse} in finite time onto the equilibrium set, creating a homoclinic tangle with a surprising geometry. To show this phenomenon, we consider planar autonomous oscillators with a homoclinic orbit from an equilibrium set and add a small periodic perturbation. As expected from standard dynamical system theory, the separatrix breaks up, creating infinitely many crossings between the
set of points converging to the equilibrium set and the set of points emanating from its neighbourhood. We show that the created homoclinic tangle has entirely novel properties compared to homoclinic sets in conventional systems, due to the collapse of trajectories. A return map shows that this new homoclinic tangle generates a chaotic saddle whose forward dynamics is qualitatively different from the dynamics in backward time.

The singular behaviour displayed by frictional oscillators cannot be modelled with ordinary differential equations. Instead, differential inclusions are used, where time-derivatives of the state are elements of a set-valued right-hand side \[66,131\]. Solutions of differential inclusions (as defined by Filippov \[58,89\]) depend continuously on initial conditions. However, differential inclusions are essentially distinct from differential equations in two ways. First, the collapse of trajectories at the discontinuity surface implies that not all solutions are defined uniquely in backward time. Second, differential inclusions exhibit equilibrium sets, i.e., intervals of non-isolated equilibria, e.g. a pendulum with friction has an interval of equilibrium points near its top or bottom position. Previous results show that physical systems with friction can be modelled using differential inclusions (see e.g. \[37,148\] and also Chapter 3). Although stability and bifurcations of isolated equilibria or limit cycles in these systems have been studied extensively, \[14, 49\], and homoclinic orbits from isolated equilibria are studied e.g. in \[107\] for a planar autonomous system, the global phenomenon presented in this chapter is still unexplored to this date.

### 4.2 Model of a pendulum with dry friction

Our results are generic for planar systems with friction that have unstable equilibrium sets, such that homoclinic or heteroclinic orbits can occur. We illustrate them with a periodically perturbed pendulum experiencing friction, see Fig. 4.1(a), modelled by the differential inclusion:

\[
\ddot{\theta} \in g \sin(\theta) - \delta \dot{\theta} - T_f \text{Sign}(\dot{\theta}) + \gamma \cos(t),
\]  

(4.1)

with \((T_f, g) = (2, 10)\), parameters \(\delta, \gamma\) and \(\text{Sign}(\dot{\theta}) = \dot{\theta}/|\dot{\theta}|\) for \(\dot{\theta} \neq 0\), and \(\text{Sign}(0) = [-1, 1]\). Fig. 4.1(b) shows the equilibrium sets \(\mathcal{E}\) and \(\bar{\mathcal{E}}\) of (4.1) and their basins of attraction without perturbations (i.e. for \(\gamma = 0\)). Points near the endpoints of an equilibrium set display complex dynamics, while trajectories near other, interior, points of this set behave in a simple fashion, as discussed in Chapter 3: locally, friction dominates the vector field, such that these trajectories collapse onto the equilibrium set in a finite time. Hence, all equilibrium sets in frictional systems, which are line intervals, are attractive and have a two-dimensional basin of attraction, cf. Fig. 4.1.
4.3 Local behaviour

Trajectories $\theta(t)$ of the differential inclusion (4.1) follow smooth vector fields when $\dot{\theta} < 0$ or $\dot{\theta} > 0$; let such trajectories be given by $\dot{\theta} = f_-(t, \theta, \dot{\theta})$ or $\dot{\theta} = f_+(t, \theta, \dot{\theta})$, respectively, see Fig. 4.2(a). Trajectories stick to the surface where $\dot{\theta} = 0$, denoted $\Sigma$, when the vector fields $f_-$ and $f_+$ point towards $\Sigma$, otherwise, trajectories instantly cross $\Sigma$. We denote with $\Sigma^s(t)$ and $\Sigma^c(t)$ the points where trajectories can stick to $\Sigma$, or cross through $\Sigma$, respectively.

The origin of system (4.1) corresponds to the top position of the pendulum, see Fig. 4.2(b)-(e). Trajectories can leave the neighbourhood of $\Sigma^s$ only along...


4.4 Homoclinic or heteroclinic orbits

We argue that a homoclinic or a heteroclinic orbit is generically created from an unstable equilibrium set in physical systems with friction. Such an equilibrium set has an unstable set $M_u$, such that, typically, sufficiently large perturbations induce a transversal intersection of $M_u$ with the stable set $M_s$. This intersection creates a homoclinic or heteroclinic tangle, whose unexpected geometry we will

---

1 Similar basin boundaries can be observed in Fig. 7.7 of [104] for a different system.

Chapter 4. Dynamical collapse of trajectories

Fig. 4.3. Pictorial illustration of the stable set $M^s$ (dashed) and unstable set $M^u$ (solid) of the endpoint $E_-$.

explain by studying the break-up of a separatrix by a small periodic perturbation. For the pendulum, typically, a heteroclinic orbit connects the right and left endpoint of the ‘top’ equilibrium set. For ease of exposition, we first study a simpler homoclinic case, which occurs in a perturbed double-well oscillator experiencing friction, modelled with:

$$\ddot{x} \in x - x^3 - \delta \dot{x} - \gamma \cos(t) - F \text{Sign}(\dot{x}),$$  \hspace{1cm} (4.2)

with $F = 0.1$ and parameters $\delta$, $\gamma$, and introduce $f_+$, $f_-$, $\Sigma$, $\Sigma^s$, $\Sigma^c$ analogously to the pendulum case. For this system, with $\gamma = 0$, a homoclinic orbit occurs when $\delta = \delta^* \approx -0.208$. In this autonomous case, the stable and unstable set coincide to form a homoclinic orbit that is a separatrix. A small perturbation is expected to break up this separatrix, creating a homoclinic tangle of the stable and unstable set, as shown in Fig. 4.3.

The finite-time convergence of trajectories on the stable set dramatically affects the shape of the homoclinic tangle. Analogously to the analysis of homoclinic tangles in periodic differential equations, (see e.g. [11]), consider a stroboscopic map of (4.2), whose period, denoted $T$, coincides with the period of the perturbation. Any crossing between the stable and unstable set, e.g. point $p$ in Fig. 4.3, corresponds to a homoclinic trajectory that collapses onto the endpoint in finite time. The stroboscopic representation of this trajectory yields a collection of points, that are both contained in $M^s$ and $M^u$. Due to the finite time convergence, the curve along $M^s$ between the intersection $p$ and the endpoint $E_-$ is crossed only a finite number of times by the unstable set. Trajectories from initial conditions on the unstable set and outside the stable set $M^s$ (e.g. point $p_1$ in Fig. 4.3) remain outside $M^s$ and collapse onto the equilibrium set. Initial conditions on the unstable set and inside the stable set (e.g. point $p_2$ in Fig. 4.3) have trajectories that collapse onto $\Sigma^s(t)$ when they arrive at $\Sigma$ close enough to the equilibrium set. Further away, such trajectories cross $\Sigma$ instantly.
The forward dynamics of the differential inclusion collapses parts of the ‘tongues’ of the homoclinic tangle (e.g. domains $L_1$ and $L_2$ in Fig. 4.3) onto sticking trajectories, or onto the equilibrium set. Since trajectories on the unstable set $M^u$ emanate from the endpoint without a finite-time property, infinitely many crossings occur between the stable and unstable sets. Hence, the homoclinic tangle looks qualitatively as shown in Fig. 4.3.

4.5 Return map of trajectories around a homoclinic orbit

We show the appearance of a new type of chaotic saddle by studying trajectories from a closed domain $Q$ near the endpoint of the equilibrium set, where $Q$ is bounded on one side by $M^u$, cf. Fig. 4.3. Trajectories with initial conditions in $Q$ tend to the unstable set $M^u$, such that the homoclinic tangle implies that some trajectories from $Q$ return to the same set after a time $kT$, where an integer $k$ is chosen. Let $H$ denote the map from initial conditions $q_0$ in $Q$ to the state of the trajectory from $q_0$ after time $kT$. To describe the nature of the chaotic saddle, we restrict our attention to trajectories that return to $Q$ during each iterate of $H$. This is the reformulation of Smale’s horseshoe map, see [145], taking the discontinuity and dynamical collapse of our case into account.

The image $HQ$ is described as follows. During the time interval $kT$, some trajectories from $Q$ experience dynamical collapse: they stick to $\Sigma^s$ and collide with other trajectories at this surface. At first, we focus on these trajectories. Dynamical collapse occurs when the trajectory arrives in the stick set $\Sigma^s(t)$. Trajectories from points $x \in \Sigma^s(t)$ outside the equilibrium set will slide at time $\tau_-(x)$, where $\tau_-(x)$ is the continuous function that gives the time when $f_-(t, x, 0)$ changes sign. In an extended phase space consisting of $(x, \dot{x}, t)$, these trajectories are released from stick at the curve $(x, 0, \tau_-(x))$ (denoted $\tilde{c}$ in Fig 4.2c for the pendulum). Solutions depend continuously on initial conditions, such that the stroboscopic section $H$ maps trajectories from this curve onto a curve $c$. Hence, the curve $c \subset HQ$ contains all trajectories that experienced collapse in the last perturbation period. Close to the equilibrium set, the curve $c$ coincides with the unstable set $M^u$ of the endpoint, representing trajectories that stick and slide repetitively, like trajectory $\diamond$ in Fig 4.2c. Further away, $c$ and $M^u$ separate, since not all trajectories in $M^u$ will have collapsed during the last period. Hence, trajectories from $Q$ that arrive at $\Sigma$ near $E_-$ experience dynamical collapse and are represented at the Poincaré section by the curve $c$, shown schematically in Fig. 4.4(a).

Since trajectories from $Q$ tend to the unstable set, which folds back onto itself in the homoclinic tangle, the trajectories that return to $Q$ after time $kT$ emanate from two connected domains in $Q$, one arriving slightly after the other at the surface $\Sigma$. Sufficiently far away from $E_-$, some trajectories from one of
these sets cross $\Sigma$ through $\Sigma^c(t)$ and remain unique, whereas the other domain collapses completely on $\Sigma^s(t)$. Taking a Poincaré section, the first domain is mapped on a 2-dimensional set $C$, whereas the latter is mapped to the curve $c$, cf. Fig 4.4(a). Hence, $HQ \cap Q$ consists of a line $c$ and a stripe $C$. The intersection of the forward iterates of this map, i.e. $M^u := \bigcap_{i=0}^{\infty} H^i Q$, shows a set of lines with an accumulation point, and not a fractal structure, see Fig. 4.4(b).

The set of initial conditions that return to $Q$ after time $kT$ is given by the preimage $H^{-1}Q$, where $H^{-1}q := \{q_0|q = H(q_0)\}$. Since the structure of $H^{-1}Q$ is governed by the stable set, analogous to the Smale horseshoe, $H^{-1}Q \cap Q$ contains two vertical stripes, see Fig. 4.4(a). One iteration further, both stripes are divided in two, such that $M^s := \bigcap_{i=-\infty}^{0} H^i Q$ is a Cantor set, see Fig. 4.4(b). The backward dynamics of $H$ behaves in a complex manner on a fractal geometry, whereas friction collapses the forward dynamics of $H$ onto a non-fractal structure.

4.6 Chaotic saddle

The intersection $X := M^s \cap M^u$ is the Cartesian product of a countable set of points and a Cantor set, and contains an infinite number of points. Since, in addition, any trajectory in $X$ has a stable and an unstable direction along $M^s$ and $M^u$, we refer to the set $X$ as a chaotic saddle. Note, that in the usual Smale horseshoe, the above intersection is the Cartesian product of two Cantor sets.

Trajectories close to $M^s$ approach the chaotic saddle $X$ and spend a long transient time near this set. During this time interval, the chaotic saddle determines their behaviour. Subsequently, they leave the chaotic saddle along the set $M^u$ which has a simple, non-fractal structure: dynamical collapse destroys the fractal nature of the forward dynamics. Despite this simplified forward dynamics of the physical system, the backward dynamics still behaves in a complex manner. For example, the time spent in the neighbourhood of a chaotic saddle is highly sensitive to initial conditions, cf. Fig. 4.5(a).
4.6 Chaotic saddle

Fig. 4.5. Residence time before trajectories from a grid of initial conditions arrive at an equilibrium set. Larger residence times are depicted in lighter shades of gray, equilibrium sets in white. (a) for (4.2), with \((\gamma, \delta) = (0.002, \delta^*)\), (b) for the pendulum (4.1) with \((\gamma, \delta) = (0.5, -0.4)\).

Heteroclinic orbits, that exist e.g. in the pendulum (4.1), will also generate a chaotic saddle that experiences collapse. Before a trajectory near this saddle returns to a domain close to its initial conditions, it will pass the neighbourhood of two equilibrium sets, and can experience collapse in both of these regions. Although the return map \(H\) is tailored to the homoclinic case with only one such domain, near the heteroclinic tangle, collapse also occurs and will induce the asymmetry between the forward and backward dynamics. The backward dynamics near the chaotic saddle remains complex, as illustrated in Fig. 4.5(b), where the time is shown before trajectories of the pendulum converge to one of the equilibrium sets.
4.7 Conclusion

In conclusion, we showed that friction in physical systems can induce a homoclinic or heteroclinic tangle with a surprising character. This tangle creates a novel type of chaotic saddle, which induces transient chaos for nearby trajectories. In physical oscillators, dynamical collapse of trajectories generates a qualitative difference between forward and backward dynamics of the saddle: the stable set is fractal, whereas the unstable set is not.

In [152], chaotic saddles in fluidic flows are shown to generate effective mixing, since the unstable manifold is fractal. Our results show, however, that dry friction may cause the final state of trajectories to have a simple geometry, such that chaotic saddles do not yield effective mixing. Apart from physical oscillators with friction studied in this chapter, collapse of trajectories is expected in electrical systems, control systems and biological models exhibiting discontinuities, cf. [65, 113, 154]. Hence, these systems will also show the asymmetry between forward and backward dynamics.
Chapter 5

Horseshoes in discrete-time systems described by non-invertible maps

Abstract – In this chapter, we will further investigate the dynamics of the limit set that has been described in the previous chapter. A class of discrete-time systems is presented with properties similar to the return map $H$ constructed in Chapter 4 and it is shown that these discrete-time systems are topologically conjugate to a symbolic dynamics. Using this topological conjugacy, it is proven that the limit set contains an infinite number of periodic orbits. Furthermore, we will show that the nonsmooth dynamics in the new limit set is closely related to the Smale horseshoe dynamics in smooth systems.

5.1 Introduction

Discontinuous dynamical systems have attracted considerable attention in the last decades and have been used with success to model dynamic behaviour occurring in mechanical engineering [66,104], control engineering [49,154] or electrical engineering [14]. In addition, discontinuous systems find applications in biology [113], physics [89] and economics [124]. Compared to the smooth dynamical system theory, analysis results on the limiting behaviour of discontinuous dynamical systems are still relatively limited, see [15, 58, 71, 103, 104, 114] for an overview of existing results. In this chapter, we explore a new type of limit set that can occur in discontinuous systems. We have discussed the geometrical properties of this limit set briefly in Chapter 4, and will now investigate the trajectories in this limit set in more detail. For this purpose, a symbolic dynamics is presented that is topologically conjugate to the discontinuous dy-
Chapter 5. Horseshoes in non-invertible maps

In smooth two-dimensional systems, the occurrence of transversal homoclinic orbits from isolated equilibrium points induces the presence of horseshoe-type limit sets, that contain an infinite number of periodic orbits, cf. [145]. Such limit sets exist near the stable and unstable manifold of the hyperbolic equilibrium point, and the geometry of the horseshoe is, to a large extent, determined by the geometry of the stable and unstable manifolds of the equilibrium point. This geometry has been analysed using the lambda- or inclination lemma [127] or with Melnikov functions [72]. As shown e.g. in [145], the trajectories in the horseshoe can be described by a topologically conjugate symbolic dynamics, consisting of a shift map formulated on a state space containing bi-infinite strings of two symbols. This symbolic dynamics has been employed by Smale to prove various characteristics of trajectories in the horseshoe-type limit set, e.g., that the limit set is transitive (which implies that it contains a dense orbit) and that it contains an infinite number of periodic orbits.

These results have been extended to the case of heteroclinic orbits [142] and to higher-dimensional systems [128]. In addition, a symbolic dynamics has been formulated to describe the trajectories in the neighbourhood of a homoclinic orbit when there exists an isolated point that attracts more than one trajectory in finite time, such that the image of a suitably defined return map from a rectangular domain is a subset of a cusp-shaped domain, see [52,83,84,97].

For discontinuous dynamical systems, in [10, 55, 70], results have recently been obtained for the case where the discontinuity of the vector field is located away from the isolated hyperbolic equilibrium point that has a homoclinic orbit. In this case, exponentially contracting and expanding directions can still be observed. In fact, in [10, 55, 70], the resulting horseshoe can be analysed by a suitably designed Poincaré return map, that remains invertible, and has the same properties as the Smale horseshoe map, which has been defined for smooth differential equations. In [9], the case is considered where a time-dependent perturbation is applied to an autonomous system with a homoclinic orbit that does experience sliding motion, and emanates from a hyperbolic equilibrium point. In Chapter 4, we have focussed on discontinuous dynamical systems where a homoclinic orbit exists that emanates from a non-isolated equilibrium point located at the discontinuity of the vector field. For these systems, a return map $H$ has been defined that shows some similarities with the Smale horseshoe map, although, in contrast to the Smale horseshoe map, the map $H$ is not invertible.

In the present chapter, we will study a class of continuous, non-invertible maps which we expect to contain the map $H$ defined in Chapter 4. However,
5.1 Introduction

A mathematically rigorous description of the homoclinic tangle, as sketched in Fig. 4.3, has not yet been obtained. As mentioned before, for smooth differential equations, such description has been obtained with tools such as the inclination-or lambda-lemma [127], and Melnikov theory [72]. However, since both results are not applicable for the class of discontinuous systems studied in the previous chapter, no formal description of the homoclinic tangle is available. Hence, in this chapter, no formal proof is presented that the map $H$ derived in Chapter 4 is contained in the class of systems studied here.

We note that the maps discussed in Chapter 4 and in the present chapter are essentially different from the Nordmark-type maps discussed in [123, 155, 156], which are used to study the dynamics of hybrid systems in the neighbourhood of periodic orbits that become tangential to a surface where an impact may occur. Nordmark-type maps contain a surface where the return map is non-differentiable, and on one side of this surface, the return map has a square-root dependency on the initial conditions. This non-differentiability of a Nordmark-type map generates a complex bifurcation scenario, which may, among others, create a chaotic attractor, as observed in [156]. In contrast, in the class of maps considered in the present chapter, the complexity of the return map is introduced by the horseshoe-like structure of the return map, in combination with the non-invertibility of this map in part of the state space. These maps exhibit a chaotic limit set that has a saddle-type nature, i.e. it is not an attractor, and nearby trajectories are attracted from one direction and repelled towards another direction.

The motivation to study the class of non-invertible maps, which is defined explicitly in the next section, mainly originates from the expectation that the map $H$ presented in Chapter 4 is contained in this class. In fact, in Section 5.2.1, numerical simulations are presented that suggest that this is the case for the exemplary system given in (4.2). As an additional motivation, we argue that if the collapse of trajectories discussed in Chapter 4 should be represented, then a natural adaptation of the standard Smale horseshoe map [145] leads to the class of maps analysed in the present chapter.

As we will show, the non-invertibility of the map strongly affects both the geometry of the limit set of this map, and the character of the trajectories contained in this limit set. Consequently, the standard symbolic dynamics, as introduced by Smale in [145], is no longer applicable. In particular, the map is invertible in parts of the limit set, but non-invertible at other points. In order to study the dynamics in the limit set, we will formulate a symbolic dynamics defined on a quotient space of the standard symbolic state space, and prove that this symbolic dynamics is topologically conjugate to the discrete-time system considered in this chapter. Using this topological conjugacy, we can prove that the newly found limit set contains an infinite number of periodic orbits.

The contributions of this chapter can be summarised as follows. Firstly, we describe the geometry of the limit sets of a class of discrete-time systems char-
acterised by non-invertible maps, and show that topologically, the limit set is described as the Cartesian product of a countable set of points and a Cantor set. The analysis of this class of maps is motivated by the analysis of the dynamics near homoclinic orbits emanating from non-isolated equilibrium points, which occur in discontinuous vector fields describing mechanical systems with dry friction, as discussed in Chapter 4. Secondly, we present the symbolic dynamics and prove that it is topologically conjugate to the considered non-invertible map. Finally, using this symbolic dynamics, we prove that the considered discrete-time system contains an infinite number of periodic orbits, and that the limit set is transitive.

The remainder of this chapter is outlined as follows. First, in Section 5.2.1, simulations are presented of an exemplary mechanical system with dry friction, and the map $H$ as defined in Chapter 4 is constructed numerically. These numerical results suggest that the return map $H$ is contained in the class of systems considered in the present chapter, that is formally defined in Section 5.2.2. In Section 5.2.3, the geometry of the limit sets of these discrete-time systems is described. Subsequently, in Section 5.3, a symbolic dynamics is introduced to analyse the trajectories of the discrete-time system. The main theorem of this chapter, which states that the considered map is topologically conjugate to the symbolic dynamics, is presented in Section 5.4. Using this result, we conclude that every discrete-time system that falls in the class of systems described in Section 5.2 has an infinite number of periodic orbits. Conclusions of this chapter are discussed in Section 5.5.

**Notation**

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of non-negative integers. Let $\log_2$ denote the logarithm with base 2, i.e. $\log_2(x) = \log(x)/\log(2)$. Given sets $A, B \subseteq \mathbb{R}^n$, let $A^2$ denote the Cartesian product $A \times A$, and $A + B = \{q \in \mathbb{R}^n, \, q = a + b, \, a \in A, b \in B\}$. Given a map $E : A \to B$ and a set $C \subseteq A$, let $E(C)$ denote $\{y \in B, \exists x \in C, \, y = E(x)\}$ and $E^{-1}(D) = \{x \in A, \, E(x) \in D\}$ for $D \subseteq B$. For points $y \in \mathbb{R}^n$, we denote the distance between $y$ and $A$ with $d(y, A) := \inf_{z \in A} |z - y|$. Given a scalar $a > 0$, $B_a$ denotes the closed ball with radius $a$, i.e. $B_a := \{q \in \mathbb{R}^n, \, |q| \leq a\}$.

### 5.2 Discrete-time dynamical system

In this section, firstly, in Section 5.2.1, simulation results are presented of a mechanical system with dry friction and the return map $H$ as introduced in Chapter 4 is constructed numerically. Motivated by these numerical results, in Section 5.2.2, a class of discrete-time systems is presented that is studied in the remainder of this chapter. In Section 5.2.3, the geometry of the limit sets of these systems is described.
5.2 Discrete-time dynamical system

5.2.1 Motivating numerical example

In this section, the map $H$ as defined in Chapter 4 will be constructed for the dynamical system

$$
\dot{x} \in x - x^3 - \delta \dot{x} - \gamma \cos(t) - F_s \text{Sign}(\dot{x}),
$$

(5.1)

where we select the system parameters $\gamma = 0.002$, $\delta = -0.2077$, $F_s = 0.1$ and $\omega = 1$. Since system (5.1) coincides with system (4.2), we recall from Chapter 4 that this system represents a periodically forced Duffing-type oscillator where dry friction is added. The system parameters have been selected such that a homoclinic tangle occurs. Simulations for this system have been performed using an event-driven method in which a 4th-order Runge-Kutta integration scheme with adaptive time steps is used, and event detections are used to handle the discontinuity in the differential equation.

In order to define the domain of the map $H$, we will first identify the equilibrium sets and numerically construct the homoclinic tangle. The equilibrium sets are found by solving the expression

$$
0 \in x - x^3 - \gamma \cos(t) - F_s \text{Sign}(0),
$$

(5.2)

numerically. We conclude that the equilibrium sets are given by $x \in [-1.0458, -0.9468]$, $x \in [-0.0990, 0.0990]$ or $x \in [0.9468, 1.0458]$ and $\dot{x} = 0$, and denote these sets with $E$ in Fig. 5.1. We will focus our attention to the endpoint $E_- = (E_-^x, 0)^T = (-0.0990, 0)^T$ of the middle equilibrium set, since this endpoint, as we will show, has stable and unstable sets with transversal intersections.

The unstable set $M^u$ is constructed by numerical evaluation of trajectories with initial conditions close to the point $E_-$. The intersection of these trajectories with the stroboscopic Poincaré section provide a numerical approximation of the unstable set $M^u$, that is depicted in Fig. 5.1. The Poincaré section has been chosen at $t_{\text{section}} + 2\pi k, k \in \mathbb{Z}$, with $t_{\text{section}} = 0.7(2\pi)$.

The unstable set $M^u$ has been obtained by following trajectories in reverse time from initial conditions near the equilibrium set. However, due to the finite-time convergence property (in forward direction of time) of trajectories on the stable set, the stable set cannot be found completely by following reverse-time trajectories from initial conditions near $E_-$ with a fixed initial time $t_0$. Hence, we have computed reverse-time trajectories with varying initial times $t_0$ from the fixed initial condition $E_- + (0 \delta \dot{x})^T$. Here, the scalar $\delta \dot{x} > 0$, which has been selected sufficiently small, is used to introduce a small offset $(0 \delta \dot{x})^T$ from the discontinuity surfaces that enables a unique reverse-time continuation of the trajectory from the point $E_- + (0 \delta \dot{x})^T$. In this manner, the stable set $M^s$ in Fig. 5.1 is obtained.

Using the numerical approximation of the unstable set $M^u$, we select a domain $\bar{Q}$ as follows. Firstly, by interpolation, we construct a function $y_{M^s}(x)$
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\[ \gamma = 0.002, \; \delta = -0.2077, \; F_s = 0.1 \text{ and } \omega = 1, \text{ depicted at time } t = 0.7(2\pi). \]

\( \mathcal{E} \) denotes the equilibrium sets, and \( M^s \) and \( M^u \) the stable and unstable set of the point \( E_- \), respectively.

whose graph coincides locally with \( M^u \). Subsequently, we select the domain of the map \( H \), denoted with \( \bar{Q} \), as \( \bar{Q} = \{(x, \dot{x}) \in \mathbb{R}^2, x \in [-0.115 - 0.100], \dot{x} \in [y_{M^u}(x), y_{M^u}(x) + 0.004]\} \). The boundaries of this set have been selected such that two domains in \( \bar{Q} \) exist whose trajectories return to \( \bar{Q} \) after \( 4\pi \) time units.

We note that in \( \dot{x} \)-direction, the set \( \bar{Q} \) is too narrow to be depicted in (the inset of) Fig. 5.1.

Now, given a point \( y_0 \in \bar{Q} \), the map \( H \) is constructed by numerical evaluation of the trajectory \( \varphi \) of (5.1), from the initial condition \( y_0 \) and initial time \( t_{\text{section}} \), over a time interval of \( 4\pi \) time units, such that \( H(y_0) = \varphi(t_{\text{section}} + 4\pi) \).

In order to represent \( \bar{Q} \) on a rectangular domain, we introduce new coordinates \( \bar{u}, \bar{v} \), where \( \bar{u} \) is a scaled version of \( E_-^x - x \), and \( \bar{v} \) is a nonlinearly scaled version of \( \dot{x} - y_{M^u}(x) \). This scaling is introduced, firstly, such that the domain \( \bar{Q} \) is mapped onto the square \( Q = [0, 1]^2 \), and secondly, such that the features in \( \bar{v} \)-direction of the return map can be represented in a clear picture. We care to highlight that the coordinate transformation \( (\bar{u} \; \bar{v})^T = \pi_{uv}(x \; \dot{x})^T \) is invertible.

The numerically constructed map \( \bar{R} : [0, 1]^2 \to [0, 1]^2 \), given by \( \pi_{uv} \circ H \circ \pi_{uv}^{-1} \), has been depicted in Fig. 5.2. In this figure, in gray, the points \( (\bar{u}, \bar{v}) \in [0, 1]^2 \) are depicted that return to the domain \( [0, 1]^2 \) in the next iterate of \( \bar{R} \). Hence, these points represent the pre-image \( \bar{R}^{-1}([0, 1]^2) \), which consists of two domains, denoted with \( P_0 \) and \( P_1 \). In black, the image of these points under the map \( \bar{R} \) is depicted. Clearly, as noted in Chapter 4, the image \( \bar{R}([0, 1]^2) \) is given by a curve \( c \) and a two-dimensional set \( C \). The first connected domain of \( \bar{R}^{-1}([0, 1]^2) \)
Fig. 5.2. Numerically constructed return map $\bar{R}$ for system (5.1) with $\gamma = 0.002$, $\delta = -0.2077$, $F_s = 0.1$ and $\omega = 1$. The gray domain denotes the pre-image $\bar{R}^{-1}([0,1]^2) \cap [0,1]^2$, the black domain the image $\bar{R}([0,1]^2) \cap [0,1]^2$ is crossed by the curve, but has an empty intersection with the strip.

Motivated by these numerical results, in the following section, a class of return maps is presented that has the same behaviour of the image and pre-image of the map $\bar{R}$, as shown in Fig. 5.2. In fact, one may expect that there exists a coordinate transformation such that the map $\bar{R}$ falls in the class of maps introduced in the next section.

### 5.2.2 Definition of the discrete-time system

In the remainder of this chapter, we consider the class of discrete-time dynamical systems given by a map $R : [0,1]^2 \rightarrow [0,1]^2$ which can be described as:

\begin{align}
R \left( \begin{pmatrix} u \\ v \end{pmatrix} \right) &= \begin{cases} 
\begin{pmatrix} R_u(u) \\ R_{vc}(u,v) \end{pmatrix}, & u \in p_0 \cup p_1, \\
\emptyset, & \text{otherwise},
\end{cases} \\
R_{vc}(u,v) &= \begin{cases} f_c(R_u(u)), & u \in p_0, \\
\min(R_v(v), f_c(R_u(u))), & u \in p_1,
\end{cases}
\end{align}

(5.3a) (5.3b)
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Fig. 5.3. Pictorial illustration of the functions \( R_u, R_v \) and \( f_c \).

for \((u, v)^T \in [0, 1]^2\). The sets \( p_0 \) and \( p_1 \) are closed intervals and are subsets of \((0, \frac{1}{2})\) and \((\frac{1}{2}, 1)\), respectively. In addition, the functions \( R_u, f_c, \) and \( R_v \) satisfy the properties formalised in the following assumption. These properties are pictorially illustrated in Fig. 5.3.

Assumption 5.1.

\(\begin{align*}
a. \text{The continuous function } R_u(u) \text{ maps both } p_0 \text{ and } p_1 \text{ onto } [0, 1]. \text{ Furthermore, there exist scalars } \lambda, \bar{\lambda} > 1 \text{ such that} \\
\lambda|u_0 - u_1| \leq |R_u(u_0) - R_u(u_1)| \leq \bar{\lambda}|u_0 - u_1| \\
\text{holds for any two points } u_0, u_1 \in p_i \text{ for } i = 0 \text{ or } i = 1.
\end{align*}\) (5.4)

\(\begin{align*}
b. \text{The function } f_c \text{ is continuous and non-decreasing, } f_c(u^+) = 0 \text{ for } u^+ \in p_0, \text{ and } f_c(u^+) = \frac{1}{2} \text{ for } u^+ \in p_1.
\end{align*}\)

\(\begin{align*}
c. \text{The function } R_v \text{ is continuous, strictly monotonically decreasing, satisfies} \\
0 < R_v(v) < \frac{1}{2} \text{ for all } v \in [0, 1], \text{ and there exists a } \lambda_v \in (0, 1) \text{ such that} \\
|R_v(v_1) - R_v(v_0)| < \lambda_v|v_1 - v_0|, \forall v_0, v_1 \in [0, 1].
\end{align*}\) (5.5)

Note that, if this assumption is satisfied, then the pre-image of the map \( R \), as defined in (5.3), is given by two sets \( P_0 = p_0 \times [0, 1] \) and \( P_1 = p_1 \times [0, 1] \), similar to the numerical results shown in Fig. 5.2. In addition, similar to the curve \( c \) and set \( C \) in this figure, the image of the map \( R \) consists of a curve (given by the graph of the function \( f_c \)), and a two dimensional set, that is given by the image of the map \( R \) from the points \((u, v) \in [0, 1]^2\) for which \( u \in p_1 \) and \( R_v(v) < f_c(R_u(u)) \).

In the following remark, a further motivation is presented for the study of the class of discrete-time systems given by (5.3) that satisfy Assumption 5.1.
5.2 Discrete-time dynamical system

Fig. 5.4. Pictorial illustration of the map $R$ and the Smale horseshoe map $R_{\text{Smale}}$.

Remark 5.1. The map $R$ given in (5.3) can be thought of as the composition of the Smale horseshoe map and a map representing the collapse of trajectories. This has been illustrated in Fig. 5.4(a)-(c) as follows.

Let the square $Q = [0,1]^2$ be depicted as the square $abcd$ in Fig. 5.4(a). Under the Smale horseshoe map $R_{\text{Smale}}$, cf. [145], this square is mapped onto the domain $a'b'c'd'$. Consequently, two domains in $abcd$ exist that are mapped into the square $abcd$; these domains are denoted with $P_0 = p_0 \times [0,1]$ and $P_1 = p_1 \times [0,1]$, respectively. The Smale horseshoe map is constructed in [145] such that in the domains $P_0$ and $P_1$, it is affine and expanding in $u$-direction and contracting in $v$-direction.

Now, let the map $R_{\text{collapse}} : [0,1]^2 \to [0,1]^2$ be given by

$$R_{\text{collapse}} \left( \begin{pmatrix} u \\ v \end{pmatrix} \right) := \begin{pmatrix} u \\ \min(v, f_c(u)) \end{pmatrix}.$$  \hfill (5.6)

The minimum in this expression induces ‘collapse’ of the trajectories as described in Chapter 4: namely, if $\min(v, f_c(u)) = f_c(u)$, then the map $R_{\text{collapse}}$ becomes independent on the $v$-coordinates, and, consequently, this map is non-invertible at these points.

In panel (b) of Fig. 5.4, the graph of the function $f_c$ is shown, together with the domains $P_0$ and $P_1$ and their images $R_{\text{Smale}}(P_0)$ and $R_{\text{Smale}}(P_1)$. If we construct the map $R$ as $R := R_{\text{collapse}} \circ R_{\text{Smale}}$, then we observe that the image of this set is given by the union of the graph of $f_c$ and the part of $R_{\text{Smale}}(P_1)$ corresponding to $v$-coordinates that are below this curve. The pre-image $R^{-1}([0,1]^2)$ is given by $P_0 \cup P_1$. The image and pre-image of the map $R$ are depicted in Fig. 5.4(c).

Since the map $R_{\text{Smale}}$ is affine, expanding in $u$-direction and contracting in $v$-direction in the domains $P_0$ and $P_1$, the properties mentioned in Assumption 5.1a. and c. hold for the map $R$. Consequently, if the map $R$ is constructed
as \( R := R_{\text{collapse}} \circ R_{\text{Smale}} \), then Assumption 5.1 is automatically satisfied.

We argue that this construction is a natural way to adapt the Smale horseshoe map to capture the occurrence of the ‘collapse’ of trajectories. Together with the simulation results presented in Section 5.2.1, this observation motivates the study of the iterates of the map \( R \) as presented in this chapter.

\begin{remark}
When smooth differential equations are considered, a rigorous description of the homoclinic tangle can be obtained using, for example, Melnikov functions [72] and the inclination, or lambda-lemma [127]. Together with a local description of the vector field near a hyperbolic equilibrium point, these results allow to prove that a local domain \( Q \) exists for which trajectories returning to the same set are described by the Smale horseshoe map after an appropriate coordinate transformation, cf. [128]. In Appendix A.4, we will analyse the local dynamics near equilibrium sets that occur in the class of discontinuous dynamical systems considered in Chapter 4. We envision that a version of the lambda-lemma could be formulated that is applicable for this class of differential inclusions. Indeed, further research is needed to formally attain such a result; once available, it would enable a rigorous description of the homoclinic tangle as sketched in Fig. 4.3. In that case, one may expect that the identification of vertical and horizontal stripes, as formalised in [128], can be adapted to apply in our case. This would result in a rigorous proof that the map \( H \) as presented in Chapter 4 and the map \( \bar{R} \) described in Section 5.2.1 are contained, after a continuous coordinate transformation, in the class of systems considered in the present chapter.
\end{remark}

### 5.2.3 Geometry of the limit set

In this section, we will describe the geometry of the invariant set of the map \( R \) defined in (5.3). For this purpose, firstly, we will study the forward iterates of \( R \), leading to a geometrical description of the set \( M^u := \bigcap_{i=0}^{\infty} R^iQ \), where we introduced the set \( Q := [0,1]^2 \) for notational convenience. Subsequently, we will employ an existing result that describes the set \( M^s := \bigcap_{i=-\infty}^{0} R^iQ \), and describe the intersection between both sets, leading to a description of the invariant set \( X \) of the map \( R \).

The contraction in \( v \)-direction, as given in Assumption 5.1c., implies that the forward iterates of \([0,1] \) under the map \( R \), form two-dimensional sets that are increasingly smaller in \( v \)-direction. From (5.3) we conclude that the set \( p_0 \times [0,1] \) is mapped onto a curve by the map \( R \). These observations are instrumental in the proof of the following lemma, that states that the set \( M^u := \bigcap_{i=0}^{\infty} R^iQ \) consists of a set of curves \( h^i, i \in \mathbb{N} \), that accumulates to a curve \( h^* \), and, consequently, the set \( M^u \) does not have a fractal structure. In addition, this lemma introduces a labeling of the curves \( h^i, i \in \mathbb{N} \), in \( M^u \) which facilitates the analysis of the map \( R \). This labelling is illustrated in Fig. 5.5. In the formulation of this lemma, we use the sets \( P_0 := p_0 \times [0,1] \) and \( P_1 := p_1 \times [0,1] \).
5.2 Discrete-time dynamical system

\textbf{Fig. 5.5.} Pictorial illustration of the set $\mathcal{M}^u$ where the curve $h^*$ and the labelling of the curves $h^i, \ i \in \mathbb{N}$, is shown.

\textbf{Lemma 5.1.} Consider the map $R$ given in (5.3) and let Assumption 5.1 be satisfied. The set $\mathcal{M}^u$ can be described by

$$\mathcal{M}^u = h^* \cup \bigcup_{i=0}^{\infty} h^i \quad (5.7)$$

where each $h^i, i \in \mathbb{N} \cup \{\star\}$, denotes a curve in $Q$ that can be described as the graph of a continuous function $\tilde{h}^i(u)$, that is defined for $u \in [0,1]$ and constant for $u \in p_0$ and $u \in p_1$. The curves $h^i, i \in \mathbb{N} \cup \{\star\}$, are such that $h^i \cap h^j \cap P_1 = \emptyset$ and $h^i \cap P_0 = h^j \cap P_0$ for all $i \neq j, i, j \in \mathbb{N} \cup \{\star\}$. In addition, the curves $h^i$ converge to the curve $h^*$ for $i \to \infty$. Finally, the map $R$ is such that $h^{i+1} = R(h^i \cap P_1)$ for $i \in \mathbb{N}$.

\textit{Proof.} The proof of this lemma is given in Appendix A.3. \hfill \Box

The backward iterates of the map $R$ behave similar to the backward dynamics of the Smale horseshoe map. Hence, we will only briefly describe the geometry of the set $\mathcal{M}^s = \bigcap_{i=-\infty}^{0} R^i Q$.

Observe that given Assumption 5.1, equation (5.3) implies that $R((u,v)^T) \in Q$ if $R_u(u) \in [0,1]$, and, in addition, that the dynamics in $u$-direction is not depending on the $v$-coordinate. By Assumption 5.1c., the map $R$ is contracting in $v$-direction, such that we conclude that $\bigcap_{i=-l}^{0} R^i Q$ is equal to $\left(\bigcap_{i=-l}^{0} R^i_u([0,1])\right) \times [0,1]$ for all $l \geq 1$. Hence, we may restrict our attention to the interval map $R_u : [0,1] \to [0,1]$.

If Assumption 5.1 is satisfied, then, using the approach of [90], one can show that $\bigcap_{i=-\infty}^{0} R^i_u([0,1])$ is a Cantor set. Consequently, the set $\mathcal{M}^s = \bigcap_{i=-\infty}^{0} R^i Q = \left(\bigcap_{i=-\infty}^{0} R^i_u([0,1])\right) \times [0,1]$ is described geometrically by the Cartesian product of a Cantor set and the interval $[0,1]$. 
Chapter 5. Horseshoes in non-invertible maps

Fig. 5.6. Markov Partitioning.

If we compare the forward iterates of \( R \) of the domain \( Q \), yielding \( \mathcal{M}^u \), with the backward iterates, yielding \( \mathcal{M}^s \), we observe that the backward dynamics of \( R \) behaves in a complex manner on a fractal geometry, whereas the non-invertibility of the map \( R \) collapses the forward dynamics of \( R \) onto a non-fractal structure.

From the definitions of \( \mathcal{M}^s \) and \( \mathcal{M}^u \), we conclude that

\[
\mathcal{X} := \mathcal{M}^s \cap \mathcal{M}^u
\]

is the largest invariant set for the map \( R \) given in (5.3) and, topologically, is the Cartesian product of a countable set of points and a Cantor set. For comparison, we note that in the usual Smale horseshoe for smooth differential equations, the invariant set is the Cartesian product of two Cantor sets, cf. [72,145].

In direct analogy with the stable and unstable sets of horseshoes in smooth homoclinic tangles, cf. [72,126], the sets \( \mathcal{M}^s \) and \( \mathcal{M}^u \) contain the set of points converging to, or diverging from the invariant set \( \mathcal{X} \), respectively. Hence, these sets represent the stable and unstable sets of the limit set \( \mathcal{X} \).

5.3 Symbolic dynamics

In this section, in analogy with the symbolic analysis of the Smale horseshoe map, cf. [110,145], a symbolic dynamics is formulated to investigate the properties of the dynamical behaviour of the map \( R \) restricted to \( \mathcal{X} \) (denoted \( R_{|\mathcal{X}} \)), with \( \mathcal{X} \) defined in (5.8). For this purpose, in this section, we will introduce a dynamical system \( \sigma_c \) on a state space consisting of infinite strings of the symbols \( \{0,1\} \). In Section 5.4, we will prove that the dynamics \( R_{|\mathcal{X}} \) on \( \mathcal{X} \) is topologically conjugate to this symbolic dynamics, and using this conjugacy, prove that \( R_{|\mathcal{X}} \) has an infinite number of periodic orbits.

To describe the dynamics of \( R_{|\mathcal{X}} \), we partition the set \( \mathcal{X} \subset R^{-1}Q \) using a partitioning with the two disjoint domains \( P_0 := p_0 \times [0,1] \) and \( P_1 := p_0 \times [0,1] \), as shown in Fig. 5.6. In the next section, we will represent a trajectory \( \{y^i\}_{i \in \mathbb{Z}} \) of the map \( R \) by keeping track of whether \( P_0 \) or \( P_1 \) is visited by this trajectory \( \{y^i\}_{i \in \mathbb{Z}} \). In this manner, an infinite sequence \( a = \{a_i\}_{i=-\infty}^{\infty} = \)
5.3 Symbolic dynamics

(...a_{-1}.a_0.a_1...) \in \{0,1\}^\mathbb{Z} is obtained, where a_i \in \{0,1\} for all i. In Theorem 5.2, which we will present in the next section, we show that each trajectory of R can be represented in such strings of symbols. This relation between trajectories of the map R and strings of symbols \( a \in \{0,1\}^\mathbb{Z} \) motivates the remainder of the present section, where we will formally introduce a dynamical system on the state space \( \{0,1\}^\mathbb{Z} \). Using the nomenclature of [158], we call every \( a \in \{0,1\}^\mathbb{Z} \) a point, and if we represent a point \( a \in \{0,1\}^\mathbb{Z} \) as \((...a_{-1}.a_0.a_1...)\), then the dot is followed by the symbol \( a_0 \) with index 0.

In [110, 145], the dynamics of the Smale horseshoe map, which is an invertible map, is shown to be topologically conjugate to an invertible dynamical system defined on \( \{0,1\}^\mathbb{Z} \), namely, the full 2-shift, which can be described by the edge shift on a graph with transition matrix \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) as formalised in [110].

In [52, 83], a cusp horseshoe map is considered, that is motivated by the singular perturbation of a smooth dynamical system. It is shown that this map is non-invertible at one particular point in state space, coinciding with the cusp position. In [52], a topological conjugacy is constructed between the cusp horseshoe map and a shift map defined on a quotient space of symbols. The main difference between the map R and the cusp horseshoe maps considered in [52,83] is that the map R considered in the present chapter is non-invertible at more than one point.

In fact, in our case, the return map R is not invertible for all points mapped into the set \( P_0 \). Hence, for points \( y_0 \in P_0 \cap X \), the pre-image \( R^{-1}(y_0) \) is set-valued, although \( R^{-1}(y_1) \) is a singleton for \( y_1 \in P_1 \cap X \).

Since R is not invertible at all points, we will define a symbolic dynamics on a quotient space of \( \{0,1\}^\mathbb{Z} \), which is defined with the following equivalence. Essentially, this equivalence implies that two points \( a, b \in \{0,1\}^\mathbb{Z} \) are equivalent when their current and future symbols \( a_i, a_i \in [0, \infty) \), coincide, and their past symbols \( a_i, a_i \in (-\infty, -1) \), coincide for \( i \geq -k \), where \( k \geq 0 \) denotes the minimum index such that \( a_{-k} = 0 \). If a shift map \( a_i \rightarrow a_{i+1}, i \in \mathbb{Z} \), defines the discrete-time system on this state space, then the trajectories with distinct ‘histories’ collapse onto each other when \( a_0 = 0 \), whereas no collapse occurs when \( a_0 = 1 \).

**Definition 5.1.** Points \( a, b \in \{0,1\}^\mathbb{Z} \) are equivalent, denoted \( a \sim b \), when \( a = b \) or when there exists an integer \( k \geq 0 \) such that

\[
\begin{align*}
a_i &= b_i, & \forall i \in [1, \infty), \\
a_i &= b_i = 1, & \forall i \in [-k + 1, 0], \\
a_{-k} &= b_{-k} = 0,
\end{align*}
\]

holds.

This is indeed an equivalence relation, since \( a \sim a, \forall a \in \{0,1\}^\mathbb{Z} \), the relation is symmetric as (5.9)-(5.11) are symmetric for each index \( i \in \mathbb{Z} \), and \( a \sim b \) and
b ∼ c imply a ∼ c. Hence, we can formulate the quotient space, cf. [93]:

\[ S_c := \{0, 1\}^\mathbb{Z} / \sim, \]  

(5.12)

which implies that two points \(a_1, a_2 \in \{0, 1\}^\mathbb{Z}\) are identified with each other when \(a_1 \sim a_2\). On the space \(S_c\), we define the metric \(\rho_c\) as follows, where we adopt the convention \(2^{-\infty} = 0\).

**Definition 5.2.** Given two points \(a, b \in S_c\), let \(k\) be a nonnegative integer or \(\infty\), such that \(a_i = 1\) for all \(i \in [-k+1, 0]\) and \(a_{-k} = 0\) if \(k\) is finite. If \(\ell\) denotes the maximal integer or \(\infty\) such that \(a_i = b_i\) for all \(i \in [\max(-k, -\ell), \ell]\), then we define:

\[ \rho_c(a, b) = 2^{-\ell}. \]  

(5.13)

**Remark 5.3.** Observe that \(\rho_c(a, b) = \inf_{c \in \{0, 1\}^\mathbb{Z}, c \sim b} \rho(a, c)\), with \(\rho(a, c) = 2^{-m}\), and \(m\) the maximal integer such that \(a_i = c_i\) for \(i \in [-m, m]\). The metric \(\rho(a, c)\), for \(a, c \in \{0, 1\}^\mathbb{Z}\) is the metric presented in [110].

We observe that \(\rho_c\) is a metric on \(S_c\), as it satisfies the symmetry requirement \(\rho_c(a, b) = \rho_c(b, a)\), triangle inequality \(\rho_c(a, c) \leq \rho_c(a, b) + \rho_c(b, c)\) and, finally, \(\rho_c(a, b) = 0\) if and only if \(a \sim b\).

We define the forward dynamics on \(S_c\) as a shift \(\sigma_c : S_c \to S_c\) on the symbols as follows:

\[ \sigma_c((\ldots a_{-1}a_0a_1a_2\ldots)) = (\ldots a_{-1}a_0a_1a_2\ldots), \]  

(5.14)

such that \((\sigma_c(a))_i = a_{i+1}, \forall i \in \mathbb{Z}\).

We note that the shift map \(\sigma_c : S_c \to S_c\) is a continuous map which is not invertible. The pre-image of \(\sigma_c\) is given by:

\[ \sigma_c^{-1}((\ldots a_{-2}a_{-1}a_0a_1\ldots)) \in \begin{cases} (\ldots a_{-2}a_{-1}a_0a_1a_2\ldots), & a_0 = 1, \\ \{b \in S_c, \sigma_c(b) \sim a\}, & a_0 = 0. \end{cases} \]  

(5.15)

The pre-image \(\sigma_c^{-1}\) is set-valued for certain points in \(S_c\), which enables a comparison between the map \(\sigma_c\) on \(S_c\) and the map \(R_{\mathcal{X}}\) defined on \(\mathcal{X}\), which also is non-invertible at the points \(x\) where \(R(x) \in P_0\). Therefore, both the dynamical systems \((\mathcal{X}, R_{\mathcal{X}})\) and \((S_c, \sigma_c)\) are continuous and, for part of their domain of definition, the maps \(R_{\mathcal{X}}\) and \(\sigma_c\) are not invertible. In the next section, the collapse of trajectories in the map \(R_{\mathcal{X}}\) will be related to iterations of the shift map \(\sigma_c\) such that the symbol \((\sigma_c(a))_0\) becomes 0.

### 5.4 Topological conjugacy between symbolic dynamics and discrete-time system

In this section, the symbolic dynamics presented in the previous section is shown to be topologically conjugate to the return map \(R_{\mathcal{X}}\), where we adopt the definition of topological conjugacy from [142, Definition 8.1], which is given as follows.
**Definition 5.3.** Let $M$ and $N$ be two topological spaces and $\Phi : M \to M$ and $\Psi : N \to N$ two continuous maps. A homeomorphism $\tau : M \to N$ is a topological conjugacy if $\tau \circ \Phi = \Psi \circ \tau$, that is, if the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\Phi} & M \\
\downarrow \tau & & \downarrow \tau \\
N & \xrightarrow{\Psi} & N
\end{array}
\]

Two dynamical systems $(M, \Phi)$ and $(N, \Psi)$ are topologically conjugate if there is a topological conjugacy between them.

We can now present the main theorem of this chapter. This theorem states that $(S_c, \sigma_c)$ is topologically conjugate to $(X, R|_X)$ and, in addition, presents a mapping $\tau$ that is a topological conjugacy between both systems.

**Theorem 5.2.** Consider the map $R|_X : X \to X$, which is defined as the restriction of the map $R$ given in (5.3) to the invariant set $X$ as in (5.8), and let the map $\sigma_c : S_c \to S_c$ be given in (5.14). In addition, let Assumption 5.1 be satisfied. Consider $h^i$, $i \in \mathbb{N}$ as defined in Lemma 5.1 and let the function $\bar{\kappa}(y)$, for $y \in X$, denote the minimum integer such that $y \in h^{\bar{\kappa}(y)}$, or $\bar{\kappa}(y) = \infty$ if $y \not\in h^i$ for all $i \in \mathbb{N}$. Let $\tau : X \to S_c$ be given by:

\[
(\tau(y))_i = \begin{cases} 
0, & i \in (-\infty, -\bar{\kappa}(y) - 1], \\
1, & i \in [-\bar{\kappa}(y), -1], \\
0, & i \in [0, \infty) \land R^i(y) \in P_0, \\
1, & i \in [0, \infty) \land R^i(y) \in P_1.
\end{cases}
\]

(5.16)

The function $\tau$ is a topological conjugacy, such that $(S_c, \sigma_c)$ is topologically conjugate to $(X, R|_X)$.

**Proof.** The proof of this theorem is given in Appendix A.3. \qed

Analogously to the standard argument for the full two-shift on $\{0, 1\}^\mathbb{Z}$, as analysed e.g. in [145], we observe that the dynamical system $(S_c, \sigma_c)$ contains an infinite number of periodic orbits. Initial conditions of such periodic orbits are given by $(0)^i$, $(1)^i$, $(01)^i$, $(001)^i$, $(011)^i$, ..., where $(z)^i$ denotes an infinite repetition of the block $z$. The period of these orbits is equal to the length of the block $z$. Since the topological conjugacy $\tau$ given in Theorem 5.2 connects each of these periodic orbits of $(S_c, \sigma_c)$ to a periodic orbit of $(X, R|_X)$, we directly obtain the following corollary.

**Corollary 5.3.** The dynamical system $(X, R|_X)$ contains an infinite number of periodic orbits.
We will now show that the dynamical system \((\mathcal{X}, R|_{\mathcal{X}})\) is topologically transitive. To define this property, we call an orbit \(\{y_i\}_{i \in \mathbb{Z}}\) dense in a set \(Q\) if it is arbitrarily close to any point \(p \in Q\).

**Definition 5.4** (\cite{90}). A dynamical system \((\mathcal{X}, R|_{\mathcal{X}})\) is topologically transitive if there exists a point \(q \in \mathcal{X}\) such that its orbit \(\{R_i|_{\mathcal{X}}(q)\}_{i \in \mathbb{N}}\) is dense in \(\mathcal{X}\).

**Lemma 5.4.** The dynamical system \((\mathcal{X}, R|_{\mathcal{X}})\) is topologically transitive.

**Proof.** The proof of this lemma is given in Appendix A.3. \(\square\)

Corollary 5.3 and Lemma 5.4 imply that the dynamical system \((\mathcal{X}, R|_{\mathcal{X}})\) has an infinite number of periodic orbits, and that the system is topologically transitive. Both of these properties are known to hold as well for the Smale horseshoe map, cf. \cite{145}. For this reason, we argue that the map \(R\) is closely related to the Smale horseshoe map. The geometry of the limit set \(\mathcal{X}\), however, is essentially different from the smooth horseshoe, as discussed in Section 5.2.3. In addition, in contrast to the smooth horseshoe dynamics, the trajectories in \(\mathcal{X}\) cannot be uniquely continued in the backward direction of time.

### 5.5 Conclusion

In this chapter, a class of discrete-time systems is studied that consists of non-invertible maps that are closely related to the Smale horseshoe map. We expect that this class of maps describes the trajectories of mechanical systems with dry friction and periodic perturbation given in Chapter 4. The discrete-time systems considered here are given by maps that are continuous, but not invertible. We have described the geometry of the limit sets of these maps, have shown that the limit set is transitive, and have proven that an infinite number of periodic orbits exist.

The analysis of the trajectories in the limit set was performed by formulating a new type of symbolic dynamics, where a symbolic state space is designed with a quotient structure. Due to this structure, a shift map on this state space can represent the non-invertibility of the horseshoe-like map. We have proven that this symbolic dynamics is topologically conjugate to the horseshoe-like map. In this manner, we have identified a dense orbit in the limit set of this map, and have proven that the map has an infinite number of periodic orbits. Due to these similarities with the limit set of the Smale horseshoe map, we refer to this limit set as a horseshoe. However, in comparison with the limit set of the Smale horseshoe map, the geometry of this limit set is qualitatively different, and trajectories in the horseshoe can not be continued uniquely in the reversed direction of time.
Part II

Tracking control for hybrid systems
Chapter 6

Tracking control for hybrid systems with state-triggered jumps

Abstract — This chapter addresses the tracking problem in which the controller should stabilise time-varying reference trajectories of hybrid systems. Despite the fact that discrete events (or jumps) in hybrid systems can often not be controlled directly, as, e.g., is the case in impacting mechanical systems, the controller should still stabilise the desired trajectory. A major complication in the analysis of this hybrid tracking problem is that, in general, the jump times of the plant do not coincide with those of the reference trajectory. Consequently, the conventional Euclidean tracking error does not converge to zero, even if trajectories converge to the reference trajectory in between jumps, and the jump times converge to those of the reference trajectory. Hence, standard control approaches can not be applied. We propose a novel definition of the tracking error that overcomes this problem and formulate Lyapunov-based conditions for the global asymptotic stability of the hybrid reference trajectory. Using these conditions, we design hysteresis-based controllers that solve the hybrid tracking problem for two exemplary systems, including the well-known bouncing ball problem.

6.1 Introduction

Hybrid systems, such as for example robotic systems with impacts, digitally controlled physical systems, electrical circuits with switches, and models of chemical reactors with valves, can be characterised by the interaction between continuous-time dynamics and discrete events, cf. [67, 111, 139]. Due to this interaction, hybrid systems can show more complex behaviour than can occur in ordinary

*This chapter is based on [21], and parts have appeared in [20].*
differential control for hybrid systems with state-triggered jumps

Most existing results in the literature on hybrid control systems deal with the stability of time-independent sets (especially with equilibrium points), such that the stability can be analysed using Lyapunov functions, see, e.g., [40, 67, 76, 104, 109, 112, 117, 139, 141]. Essentially, such a set is asymptotically stable when a Lyapunov function decreases both during flow and jumps (i.e., discrete events), see, e.g., [40, 67, 117]. Extensions of these results allow for Lyapunov functions that increase during jumps, as long as this increase is compensated for by a larger decrease during flow, or vice versa. Such results are reviewed in [81, 141]. Using these Lyapunov-based stability results, several control strategies have been developed to stabilise time-independent sets for hybrid systems, see, e.g., [67, 76].

Few results exist where controllers are designed to make a system track a given, time-varying, reference trajectory, that exhibits both continuous-time behaviour and jumps. In this chapter, we consider reference trajectories that are solutions of the plant for a given, time-dependent reference input. When the jump times of the plant trajectories can be guaranteed to coincide with jumps of the reference trajectory, then stable behaviour of the Euclidean tracking error is possible and several tracking problems have been solved in this setting, see, e.g., [35, 106, 134, 135, 137, 160, 161]. In [75], observer problems are considered for a class of hybrid systems where a similar condition is exploited, namely, that the jumps of the plant and the observer coincide. When jump times of the plant trajectory $x$ and reference trajectory $r$ can be ensured to coincide, standard Lyapunov tools are applicable to study the evolution of $x - r$ along trajectories. However, requiring the jump times of plant and reference trajectories (or plant and observer) to coincide is a strong condition that limits the applicability of these results. For example, this can not be ensured for general hybrid systems with state-triggered jumps, such as models of mechanical systems with unilateral constraints, cf. [66, 104].

In hybrid systems with state-triggered jumps, the jump times of the plant and the reference trajectory are in general not coinciding. To illustrate this behaviour, we consider the trajectories of a scalar hybrid system with state $x \in [0, 1]$, where the continuous-time evolution is given by

$$\dot{x} = 1 + u(t), \quad x \in [0, 1],$$

(6.1a)

where $u$ is a bounded control input and jumps occur according to:

$$x^+ = 0, \quad x = 1.$$  

(6.1b)

Now, consider the signal $r = t \mod 1$ as a reference trajectory, where mod denotes the modulus operator, and observe that $r$ is the solution of (6.1) from the initial condition $r_0 = 0$ with $u(t) \equiv 0$. Suppose that a control signal $u$ is constructed such that a plant trajectory $x$ tracks the reference trajectory $r$ (in fact,
such a controller will be designed in Section 6.5.1), then we expect behaviour as given in Fig. 6.1, where the state $x$ and reference trajectory $r$ converge to each other away from the jump times, and the jump times show a vanishing mismatch. During the time interval caused by this jump-time mismatch, the Euclidean error $|x - r|$ is large, as shown in Fig. 6.1(b). Since this behaviour also occurs for arbitrarily small initial errors $|x - r|$, the Euclidean error displays unstable behaviour in the sense of Lyapunov. This “peaking behaviour” was observed in [35, 75, 104, 116, 134, 135], and is expected to occur in all hybrid systems with state-triggered jumps when considering tracking or observer design problems. However, although the Euclidean error may display undesirable properties, from a control engineering point of view, the trajectories shown in Fig. 6.1 are considered to exhibit desirable behaviour. Therefore, it seems that the evaluation of the tracking error using non-Euclidean distance functions might be advantageous for a class of hybrid systems, such as the example in (6.1). For this reason, we formulate a different notion of tracking in this chapter, that considers the behaviour shown in Fig. 6.1 as a proper solution, since the jump times of the plant converge to the jump times of the reference and the distance between the plant and reference trajectories converges to zero during time intervals without jumps. This tracking notion is less restrictive than notions requiring stability of the Euclidean error (cf. [35, 106, 134, 135, 137]), such that the class of hybrid systems that can be considered is widened significantly.

Several approaches have already been presented in the literature to formalise tracking notions where controllers that solve the resulting tracking problem are allowed to induce behaviour as shown in Fig. 6.1. However, in these approaches it is not clear how to formulate conditions under which such tracking problems are solved. In [63, 116], the tracking of a billiard system is considered using the concept of “weak stability”, which implies that the position of the ball is always required to be close to the reference trajectory, but the error in velocity is not studied for a small time interval near the jump instances. In addition, the convergence of jump times is required. In [62], this approach is extended to a larger class of hybrid systems. However, since no requirements are imposed close to the
jump times, such a tracking problem definition needs knowledge of complete trajectories. Alternatively, the notion of weak stability is employed in [36, 121] for unilaterally constrained mechanical systems with reference trajectories where all impacts, if they occur, show accumulation points (i.e., Zeno behaviour), followed by a time interval where the constraint is active. In the very recent conference papers [59, 60], tracking control problems for billiard systems are formulated by requiring asymptotic stability of a set of trajectories, consisting of the reference trajectory and its mirror images, when reflected in the boundaries of the billiard. This independent research effort resulted in a related control problem formulation and controller design approach as those given for the bouncing ball example in Section 6.5.2 of this thesis. In this chapter, we aim to present a general framework for addressing tracking problems for a relatively generic class of hybrid systems (not focusing on a class of mechanical systems with unilateral constraints as in [59, 60]). Alternatively, in [116], it is suggested to employ the stability concept of Zhukovsky (see [108]). Using this stability concept, the plant trajectory is compared with the reference trajectory after a rescaling of time for the plant trajectory, i.e., the error \( x(\rho(t)) - r(t) \) should behave asymptotically stable, where a function \( \rho(t) \) is used with \( \lim_{t \to \infty} (\rho(t) - t) = 0 \). As a second alternative, a Hausdorff-type metric between the graphs of the reference and plant trajectory is suggested in [118]. Both the rescaling function \( \rho \) for Zhukovsky stability and the Hausdorff-type metric require complete knowledge of the trajectories, and, consequently, it is not clear how these concepts can be used to solve the design problem of tracking controllers.

In order to study tracking problems with non-matching jump times, we propose an alternative approach using a non-Euclidean distance between the plant and reference states, where convergence of this distance measure corresponds to the desired notion of tracking. Since this distance measure incorporates information on the “closeness” of the reference state and plant state at each time instant, the tracking problem can be formulated based on the time evolution of the distance measure evaluated along trajectories of the closed-loop system. This fact is instrumental in our approach, as it allows us to derive sufficient conditions under which the tracking problem is solved, that are formulated using the instantaneous state, its time-derivatives, and the jumps that can occur. Since such information is encoded directly in the hybrid system description, this property is an advantage of our approach when compared to the analysis of [62, 63, 116], where convergence of jump times is proven using complete trajectories. In addition to this new formulation of the tracking problem for hybrid systems, we present sufficient conditions that guarantee that this problem is solved. In this manner, we will provide a general framework for the formulation and analysis of tracking problems for hybrid systems. Although we do not address the synthesis problems of tracking controllers in its full generality, we are convinced that the results of this chapter provide an indispensable stepping stone towards such a synthesis procedure. In fact, the applicability of the presented framework for the
design of tracking controllers will be demonstrated for two exemplary systems, including a mechanical system with a unilateral constraint.

The main contributions of this chapter can be summarised as follows. First, the proposed reformulation of the tracking notion using a non-Euclidean tracking error measure allows to state and analyse tracking problems for a large class of hybrid systems, and these tracking problems are not rendered infeasible by the “peaking” of the Euclidean tracking error. Second, existing Lyapunov-type stability conditions, both with and without an additional average dwell-time condition, are extended to allow non-Euclidean distance functions, yielding sufficient conditions for the global asymptotic stability of time-invariant sets. This result allows to formulate conditions that ensure that the new tracking problem is solved. Third, in two examples we show that the new tracking error measure can be used to design controllers that solve the tracking problem.

This chapter is organised as follows. In Section 6.2, the hybrid system model and the corresponding solution concept are introduced. Subsequently, in Section 6.3, requirements are formulated for the design of appropriate tracking error measures, and the tracking problem is formulated. Section 6.4 contains Lyapunov-type conditions that are sufficient for the tracking problem to be solved. The results of this chapter are illustrated with examples on controller synthesis for the hybrid tracking problem in Section 6.5, and conclusions are formulated in Section 6.6.

**Notation:** \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space; \( \mathbb{R} \) the set of real numbers; \( \mathbb{R}_{\geq 0} \) the set of nonnegative real numbers; \( \mathbb{N} \) the set of natural numbers including 0. Let \( \text{Int}(S) \) denote the interior of a set \( S \subset \mathbb{R}^n \), \( \partial S \) the boundary of the set, \( \text{cl}(S) \) its closure, \( \text{co}(S) \) the smallest closed convex hull containing \( S \), and \( \mu(S) \) denotes its Lebesgue measure. Let \( f : \mathbb{R}^m \supseteq \mathbb{R}^n \) denote a set-valued mapping from \( \mathbb{R}^m \) to subsets of \( \mathbb{R}^n \). Given vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \), \( |x| \) denotes the Euclidean vector norm, \( \text{col}(x, y) \) denotes \( (x^T y^T)^T \) and \( \nabla_x \) denotes \( \frac{\partial}{\partial x} \). A function \( \alpha : [0, \infty) \rightarrow [0, \infty) \) is said to belong to class-\( K_{\infty} \) if it is continuous, zero at zero, strictly increasing and unbounded. Given a Lipschitz function \( V : \mathbb{R}^m \rightarrow \mathbb{R}, \partial_C V \) denotes the generalized differential of Clarke, i.e., \( \partial_C V = \text{co}(\lim \nabla_x V(x_i) | x_i \rightarrow x, x_i \not\in \Omega_V) \), where \( \Omega_V \) denotes the set of measure zero where \( \nabla_x V \) is not defined.

6.2 Modelling of hybrid control systems

In this chapter, we employ the framework of hybrid inclusions described in [67], allowing the continuous-time dynamics (flow) to be time-dependent, such that the hybrid system is given by

\[
\begin{align*}
\dot{x} &\in F(t, x, u), & x \in C \subseteq \mathbb{R}^n, \\
\bar{x} &\in G(x), & x \in D \subseteq \mathbb{R}^n,
\end{align*}
\]
Chapter 6. Tracking control for hybrid systems with state-triggered jumps

with $F : \mathbb{R}_{\geq 0} \times C \times \mathcal{U} \Rightarrow \mathbb{R}^n$, $G : D \Rightarrow \mathbb{R}^n$, where $\dot{x} \in F(t, x, u)$ describes the continuous-time (flow) dynamics that is feasible when states $x$ are in the flow set $C \subseteq \mathbb{R}^n$, and jumps can occur according to $x^+ \in G(x)$ when states are in the jump set $D \subseteq \mathbb{R}^n$. The control inputs $u \in \mathcal{U} \subseteq \mathbb{R}^m$ are assumed to be contained in the compact set $\mathcal{U}$. Moreover, only trajectories from initial conditions in $C \cup D$ are considered. Note that the reset map $G$ in (6.2b) is not dependent on time or the actuator inputs $u$, and models purely state-triggered jumps. We adopt the convention that $G(x) = \emptyset$ when $x \notin D$.

The following technical assumption is imposed on the data of the hybrid system (6.2).

**Assumption 6.1.** For any bounded set $C_1 \subset \text{cl}(C)$, the set $F(t, x, u)$ is non-empty, measurable and essentially bounded for all $(t, x, u) \in \mathbb{R}_{\geq 0} \times C_1 \times \mathcal{U}$ and $G(x) \subset C \cup D$ is non-empty and bounded for all $x \in D$.

In order to define solutions of the hybrid system (6.2), we assume that the input $u$ satisfies $u \in \bar{U}(t, x) \subseteq \mathcal{U}$, which allows, firstly, to evaluate solutions of the hybrid systems when the input is a time signal $U(t, x) = U(t)$, and secondly, to consider discontinuous, state-dependent control laws, e.g., sliding mode controllers. We consider solutions $\varphi$ of the hybrid system (6.2) in the sense of [67], such that $\varphi$ is defined on a hybrid time domain $\text{dom} \varphi \subset [0, \infty) \times \mathbb{N}$ as follows. A hybrid time instant is given as $(t, j) \in \text{dom} \varphi$, where $t$ denotes the continuous time lapsed, and $j$ denotes the number of experienced jumps. The arc $\varphi$ is a solution of (6.2) associated to $u$ when jumps satisfy (6.2b) and $\varphi$ is a Filippov solution of (6.2a) during flow, cf. [58]. This implies $\varphi(t, j + 1) \in G(\varphi(t, j))$ for all $(t, j) \in \text{dom} \varphi$ such that $(t, j + 1) \in \text{dom} \varphi$ and\footnote{We employ, with a slight abuse of notation, the convention that $\bar{F}(t, \varphi(t, j), \mathcal{U})$ denotes \{f $\in \mathbb{R}^n$ $| f = \bar{F}(t, \varphi(t, j), u), \ u \in \mathcal{U}$\}.} $\frac{\text{d}}{\text{d}t} \varphi(t, j) \in \bar{F}(t, \varphi(t, j), U(t, \varphi(t, j)))$ for almost all $t \in I_j := \{t \mid (t, j) \in \text{dom} \varphi\}$ and all $j$ such that $I_j$ has non-empty interior, where $\bar{F}(t, x, U(t, x)) = \bigcap_{\delta \geq 0} \bigcap_{N = 0} \text{co}\{F(t, \tilde{x}, U(t, \tilde{x})), | \tilde{x} - x| \leq \delta, \tilde{x} \notin N\}$ represents the convexification of the vector field as defined by Filippov, where sets $N$ of Lebesgue measure zero are excluded. The solution $\varphi$ is said to be complete if $\text{dom} \varphi$ is unbounded, which, for example, holds for all trajectories of (6.2) if $C \cup D$ is invariant under the dynamics of (6.2). The hybrid time domain $\text{dom} \varphi$ is called unbounded in $t$-direction when for each $T \geq 0$ there exist a $j$ such that $(T, j) \in \text{dom} \varphi$. In this chapter, we only consider maximal solutions, i.e., solutions $\varphi$ for which the domain $\text{dom} \varphi$ can not be extended.

Analogous to the common approach in tracking control for ODEs, we consider reference trajectories $r$ that are unique solutions to (6.2), i.e., solutions to $\dot{r} \in F(t, r, u_{\text{ref}}(t))$, $r \in C$, $r^+ \in G(r)$, $r \in D$, for a given input signal $u = u_{\text{ref}}(t) \in \mathcal{U}$ and initial condition $r_0$. We design a control law for $u$ to obtain asymptotic tracking, in an appropriate sense, of the reference trajectory $r$ by the resulting closed-loop plant. We consider feedback controllers that are static, where $u =$
6.2 Modelling of hybrid control systems

\( u_c(t, r, x) \in U \), or dynamic, where the (possibly hybrid) controller has an internal state \( \eta \in \mathbb{R}^p \) and is described by

\[
\begin{align*}
\dot{\eta} & \in F_c(t, r, x, \eta), \ (r, x, \eta) \in C_c, \\
\eta^+ & \in G_c(r, x, \eta), \ (r, x, \eta) \in D_c, \\
u & = u_c(t, r, x, \eta),
\end{align*}
\]

and assume that this controller satisfies Assumption 6.1. In order to study the stability of the closed-loop system, we create an extended hybrid system with state \( q = \text{col}(r, x, \eta) \). The dynamics of this extended hybrid system is given by

\[
\begin{align*}
\dot{q} & \in F_e(t, q), \quad q \in C_e := (C^2 \times \mathbb{R}^p) \cap C_c, \\
q^+ & \in G_e(q) = \begin{cases} 
\text{col}(G(r), x, \eta), & q \in (D \times (C \cup D) \times \mathbb{R}^p) \cap (C_c \cup D_c) \\
\text{col}(r, G(x), \eta), & q \in ((C \cup D) \times D \times \mathbb{R}^p) \cap (C_c \cup D_c) \\
\text{col}(r, x, G_c(r, x, \eta)), & q \in D_c,
\end{cases}
\end{align*}
\]

where \( F_e(t, q) := \text{col}(F(t, r, u_{\text{ref}}(t)), F(t, x, u_c(t, r, x, \eta)), F_c(t, r, x, \eta)) \), and we denote the domain of \( G_e \) in (6.4b) with \( D_c \). We refer to \( F_e, G_e, C_e \) and \( D_e \) as the data of system (6.4).

The main advantage of considering this extended hybrid system (6.4) is that a joint hybrid time domain is created, where hybrid times \( (t, j) \in \text{dom} q \) denote the continuous time \( t \) lapsed, and \( j \) gives the total number of jumps that occurred in \( x, r \) and \( \eta \). Hence, one can compare the reference state \( r \) with the plant state \( x \) for each time instant \( (t, j) \in \text{dom} q \). Let \( I_{i,i} \) denote the unit matrix of dimension \( i \times i \), and let \( O_{i,j} \) be a zero matrix of dimension \( i \times j \). Defining

\[
\begin{align*}
\bar{r}(t, j) & := (I_{n,n} \ O_{n,n} \ O_{n,p}) \ q(t, j) \\
\bar{x}(t, j) & := (O_{n,n} \ I_{n,n} \ O_{n,p}) \ q(t, j) \\
\bar{\eta}(t, j) & := (O_{p,n} \ O_{p,n} \ I_{p,p}) \ q(t, j),
\end{align*}
\]

allows to introduce a tracking error \( d(r, x) \) and formulate a tracking problem by requiring, firstly, that \( d(\bar{r}(t, j), \bar{x}(t, j)) \) remains small provided that the initial error \( d(\bar{r}(0,0), \bar{x}(0,0)) \) is small, and secondly, that \( d(\bar{r}(t, j), \bar{x}(t, j)) \) converges to zero for \( t+j \to \infty \). \( (t, j) \in \text{dom} q \), provided that this limit exists. In fact, we will formulate the tracking problem by requiring asymptotically stable behaviour of \( d(\bar{r}(t, j), \bar{x}(t, j)) \), and, in two exemplary systems, we will design controllers that guarantee that the domain \( \text{dom} q \) is unbounded in \( t \)-direction, such that the limit \( t \to \infty \) exists and, for example, no accumulation of jump times (known as Zeno behaviour, cf. [67]) will occur.

Remark 6.1. If two distinct trajectories \( x \) and \( r \) from (6.2) are considered, then, in general, \( \text{dom} r \neq \text{dom} x \). In this case, if one would define a time-dependent tracking error at time \( (t, j) \in \text{dom} x \), then it would not be not clear what time \( (t_r, j_r) \in \text{dom} r \) is appropriate to use in a comparison of \( x \) and \( r \). Such problems
are avoided by studying the extended dynamics in (6.4), where the functions $\tilde{x} : \text{dom } q \to C \cup D$, $\tilde{r} : \text{dom } q \to C \cup D$ and $\tilde{\eta} : \text{dom } q \to \mathbb{R}^p$ are reparameterisations of the functions $x : \text{dom } x \to C \cup D$, $r : \text{dom } r \to C \cup D$ and $\eta : \text{dom } \eta \to \mathbb{R}^p$, respectively.

6.3 Formulation of the tracking control problem

In Section 6.3.1, we introduce distance functions $d(r, x)$ suitable to compare the plant trajectory with the reference trajectory, such that asymptotically stable behaviour of $d(\tilde{r}(t, j), \tilde{x}(t, j))$ corresponds to appropriate tracking. Subsequently, in Section 6.3.2 we introduce asymptotic stability with respect to the tracking error $d(r, x)$, and formalise the hybrid tracking problem.

6.3.1 Definition of the tracking error measure

In hybrid systems where jumps of the plant are state-triggered, as in (6.2), asymptotically stable behaviour of the Euclidean error $|x - r|$ is generally impossible to achieve due to the peaking phenomenon, see Fig. 6.1, which, even when $x$ and $r$ converge to each other away from the jump instances, occurs when jump times of the reference and plant trajectories show a small, possibly asymptotically vanishing, mismatch. To illustrate this in the exemplary system (6.1), observe that if $|x - r|$ converges to zero during continuous-time evolution and jumps are not exactly coinciding, then, by the structure of the jump map (6.1b), $|x - r| \approx 1$ directly after a jump of either $x$ or $r$. Since this peaking phenomenon renders all tracking problems infeasible that require stable behaviour of the Euclidean error $|x - r|$, in this chapter, we present a novel approach to compare trajectories of the plant with a reference trajectory. For hybrid systems with state-triggered jumps given in (6.2), we will show that the exact properties of the jumps can be used to compare a reference trajectory with a plant trajectory, when one of them just experienced a jump, and the other did not. A distance function between two trajectories that enables such a comparison incorporates the structure of the jumps, as described in (6.2b), and hence is tailored to the specific hybrid system. In this chapter, we employ distance functions, denoted as $d(r, x)$, to formulate and solve the tracking problem.

We consider distance functions $d(r, x)$ that are not sensitive to jumps of the plant and the reference trajectory, i.e. $d(r, x) = d(g_r, x)$ for $r \in D$, $g_r \in G(r)$ and $d(r, x) = d(r, g_x)$ for $x \in D$, $g_x \in G(x)$. In this manner, stability with respect to the distance function $d(r, x)$ is not influenced by the jumps of the plant or the reference trajectory. As we will show below, a distance function $d(r, x)$ is an appropriate measure to compare a reference trajectory $r$ with a plant trajectory $x$ when it satisfies the following conditions. We adopt the notation $G^0(x) = \{x\}$, $\forall x \in C \cup D$ and $G^{k+1}(x) = G(G^k(x))$, $k = 0, 1, 2, \ldots$. Recall that $G(x) = \emptyset$ when $x \notin D$. 
Definition 6.1. Consider a hybrid system $H$ given by (6.2) that satisfies Assumption 6.1. A non-negative function $d : \text{cl}((C \cup D) \times (C \cup D)) \to \mathbb{R}_{\geq 0}$ is called a distance function compatible with $H$ when it is continuous and satisfies

$$d(r, x) = 0 \iff (\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(x) \neq \emptyset), \forall (r, x) \in \text{cl}(C \cup D)^2,$$

(6.6a)

$$\{x \in C \cup D | d(r, x) < \beta\} \text{ is bounded, } \forall r \in C \cup D, \beta \geq 0,$$

(6.6b)

$$d(r, x) = d(r, g_x), \forall x \in D, g_x \in G(x), r \in C \cup D,$$

(6.6c)

$$d(r, x) = d(g_r, x), \forall r \in D, g_r \in G(r), x \in C \cup D.$$  

(6.6d)

In this chapter, we will study stability of the set where $d(r, x)$ is zero for system (6.4). Using (6.6a), in this set, $(\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(x) \neq \emptyset)$ holds true, such that the distance $d(r, x)$ is zero if and only if either $r = x$ (such that the right-hand side of the implication in (6.6a) holds with $k_1 = k_2 = 0$), or $x$ and $r$ can be mapped onto each other instantaneously by $k_1$ jumps of $x$ and $k_2$ jumps of $r$, and, hence, $G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset$. For example, if jumps of (6.2b) cannot directly follow each other, i.e., when $D \cap G(D) = \emptyset$, and $G$ is invertible, then (6.6a) becomes $d(r, x) = 0 \iff (x = r \lor x = G(r) \lor r = G(x)), \forall (r, x) \in \text{cl}(C \cup D)^2$.

The condition (6.6b) implies that, for every given $r$, $d(r, x)$ is radially unbounded. This property will be instrumental to prove that convergence of $d$ to zero implies convergence of $|x - r|$ to zero, away from the jump instances, i.e., when $r$ is not close to the sets $D$ or $G(D)$.

Finally, (6.6c)-(6.6d) guarantee that $d(r, x)$ remains constant over jumps, such that the evaluation of the function $d$ along a trajectory of a closed-loop system (6.4), i.e., the function $d(\bar{r}(t, j), \bar{x}(t, j))$, is a continuous function with respect to $t$, and is not affected when $j$ changes. Consequently, $d(\bar{r}(t, j), \bar{x}(t, j))$ does not show the ‘peaking behaviour’, as occurs in the Euclidean distance $|\bar{r}(t, j) - \bar{x}(t, j)|$. This property is illustrated in Fig. 6.2, where the function $d(r, x) = \min (|x - r|, |x - r + 1|, |x - r - 1|)$ is shown when evaluated along the trajectories depicted in Fig. 6.1. In Section 6.5.1, we will show that this tracking error definition for $d(r, x)$ is indeed compatible with the hybrid system (6.1) in the sense of Definition 6.1.

In this chapter, we will formulate a tracking problem that requires asymptotic convergence of $d(\bar{r}(t, j), \bar{x}(t, j))$ to zero along trajectories. The following theorem states that such a convergence property of $d(\bar{r}(t, j), \bar{x}(t, j))$ implies, at least for the time instances where $\bar{r}(t, j)$ is bounded away from the jump set $D$ and its image $G(D)$, that $|\bar{r}(t, j) - \bar{x}(t, j)|$ converges to zero.

Theorem 6.1. Let the distance $d(\bar{r}(t, j), \bar{x}(t, j))$ converge asymptotically to zero along solutions $q$ of the closed-loop system (6.4), i.e., $\lim_{t + j \to \infty} d(\bar{r}(t, j), \bar{x}(t, j)) = 0$, 

$$\lim_{t + j \to \infty} d(\bar{r}(t, j), \bar{x}(t, j)) = 0,$$
Fig. 6.2. Tracking error $d(r, x) = \min(|x - r|, |x - r + 1|, |x - r - 1|)$ evaluated along trajectories of (6.1).

let these solutions be complete, let $\bar{r}(t, j)$ be bounded for all $(t, j) \in \text{dom } q$ and let $d$ be compatible with (6.2). For each trajectory $q$ and all $\epsilon > 0$, there exists a $T > 0$ such that $|\bar{x}(t, j) - \bar{r}(t, j)| < \epsilon$ holds when $t + j > T$ and

$\inf_{y \in D \cup G(D)} |\bar{r}(t, j) - y| > \epsilon.$

(6.7)

Proof. See Appendix A.5.1. \hfill \Box

We note that in Section 6.5, we present examples of hybrid systems and reference trajectories where the time intervals where (6.7) is violated can be made arbitrarily small by selecting $\epsilon > 0$ sufficiently small. Consequently, in these cases, asymptotic convergence of $d(\bar{r}(t, j), \bar{x}(t, j))$ to zero implies that the time intervals where ‘peaking’ may occur (i.e. the time intervals where the Euclidean distance $|x - r|$ can be large), become shorter over time. We are convinced that more general (sufficient) conditions can be formulated that guarantee that the time intervals where the Euclidean error may display ‘peaking behaviour’ vanish asymptotically over time.

Inspired by these observations, in the following section, we will formulate the tracking problem by requiring asymptotically stable behaviour of $d(\bar{r}(t, j), \bar{x}(t, j))$.

6.3.2 Tracking problem formulation

In this section, we discuss the stability of reference trajectories, and restrict our attention to bounded reference trajectories $r$ that satisfy the following assumption.

Assumption 6.2. The reference trajectory $r$ is bounded, $\text{dom } r$ is unbounded in $t$-direction, and $r$ is the unique solution of (6.2) for an input $u = u_{\text{ref}}(t)$ and given initial condition $r(0, 0) = r_0$. 
Below, we formulate the tracking problem which requires all trajectories of (6.4) to be such that \( d(r, x) \) behaves asymptotically stable. Hereeto, we combine the definitions of (asymptotic) stability of trajectories, cf. [108, 157], with existing stability notions for hybrid systems, cf. [67], and employ distance functions \( d(r, x) \) compatible with system (6.2) to express the distance between \( r \) and \( x \), as introduced in Definition 6.1. To create a common hybrid time domain, as mentioned before, we consider solutions of the embedded system (6.4), such that \( \bar{x} \) and \( \bar{r} \) are both defined on the hybrid time domain \( \text{dom} \ q \) of trajectories of (6.4), where \( \bar{x} \) and \( \bar{r} \) are defined in (6.5). Let us now formalise the stability properties of the reference trajectory.

**Definition 6.2.** Given a distance function \( d: (C \cup D) \times (C \cup D) \to \mathbb{R}_{\geq 0} \) compatible with system (6.2), a reference trajectory \( r \) satisfying Assumption 6.2 is

- stable with respect to \( d \) if for all \( T_0 \geq 0 \) and \( \epsilon > 0 \) there exists a \( \delta(T_0, \epsilon) > 0 \) such that
  \[
  d(\bar{r}(0, 0), \bar{x}(0, 0)) < \delta(T_0, \epsilon) \Rightarrow d(\bar{r}(t, j), \bar{x}(t, j)) < \epsilon, \forall t + j \geq T_0; \quad (6.8)
  \]

- asymptotically stable with respect to \( d \) if it is stable and one can choose \( \delta > 0 \) such that
  \[
  d(\bar{r}(0, 0), \bar{x}(0, 0)) < \delta \Rightarrow \lim_{t+j \to \infty, (t,j) \in \text{dom} \ q} d(\bar{r}(t, j), \bar{x}(t, j)) = 0 \quad (6.9)
  \]
  holds if \( \text{dom} \ q \) is unbounded;

- globally asymptotically stable with respect to \( d \) if it stable with respect to \( d \) and
  \[
  \lim_{t+j \to \infty, (t,j) \in \text{dom} \ q} d(\bar{r}(t, j), \bar{x}(t, j)) = 0, \quad (6.10)
  \]
  holds for all trajectories \( q = \text{col}(r, x, \eta) \) of (6.4) such that \( \text{dom} \ q \) is unbounded.

As a special case of this definition, the (global) asymptotic stability of an equilibrium point \( r_{\text{eq}} \) with respect to \( d \) can be evaluated by using \( r(t, j) = r_{\text{eq}} \). If the Euclidean distance \( d(r, x) = |x - r| \) would be used, then the given definition reduces to the classical definition of asymptotic stability of trajectories in the sense of Lyapunov, see e.g., [108].

Using Definition 6.2, we formalise the tracking problem as follows.

**Problem 6.1 ((Global) tracking problem).** Given a hybrid system (6.2) satisfying Assumption 6.1, a compatible distance function \( d \) and a reference trajectory \( r \) satisfying Assumption 6.2, design a controller (6.3) such that the trajectory \( r \) is (globally) asymptotically stable with respect to \( d \).
This tracking problem is not affected by the peaking phenomenon of the Euclidean error, as depicted in Fig. 6.1(b), since the trajectory \( x \) of the plant is compared with the reference trajectory using a distance function compatible with (6.2), as given in Definition 6.1. As stated in Theorem 6.1, convergence of \( d \) to zero implies that, away from the jump instances, \(|x - r|\) converges to zero. By formulation of the tracking problem using the distance \( d(r, x) \), the tracking problem in Problem 6.1 embeds a more intuitive and less restrictive notion of the closeness of jumping trajectories. Note that the tracking problem defined here considers a controller (6.3) that does not induce jumps of the plant state \( x \) directly, cf. (6.2b).

Remark 6.2. The tracking problem given in Problem 6.1 only considers asymptotically stable behaviour of the tracking error \( d(r, x) \). However, transient performance requirements of the closed-loop system in terms of decay rates can also be formulated using the distance function \( d(r, x) \) given in Definition 6.1.

Remark 6.3. Problem 6.1 implies that \( d(r, x) = 0 \) has to be an invariant set for the closed-loop system (6.2), (6.3), and for this reason, it requires \( u_c(t, r, x, \eta) = u_{ref}(t) \) a.e. when \( d(r, x) = 0 \) and \( r, x \not\in D \cup G(D) \). Although, for example, PI-controllers naturally do not have this property, a relevant class of static and dynamic hybrid controllers can be considered, including hysteresis-based controllers, of which an example will be given in Section 6.5.1.

6.4 Sufficient conditions for stability with respect to \( d \)

In this section, we will use Lyapunov-like functions to study the behaviour of \( d \) and to analyse whether a given controller (6.3) solves the tracking problem formulated in Problem 6.1. First, we will show that the solutions of the closed-loop system can be considered as Filippov solutions during continuous-time evolution (Section 6.4.1). Subsequently, in Section 6.4.2 we present two theorems with Lyapunov-type stability conditions for the stability of a set, and apply these results to obtain sufficient conditions under which the tracking problem formulated in Problem 6.1 is solved.

6.4.1 Closed-loop solutions

By construction of the extended hybrid system (6.4), if we adopt Assumption 6.1 for the plant (6.2) and controller (6.3), then the following property directly follows for the extended hybrid system (6.4).

Property 6.1. For any bounded set \( C_1 \subset cl(C_e) \), the set \( F_e(t, q) \) is non-empty, measurable and essentially bounded for all \((t, q) \in \mathbb{R}_{\geq 0} \times C_1 \), and \( G_e(q) \) is non-empty and bounded for all \( q \in D_e \).
Note, in particular, that this property implies that the solution concept of Filippov can be applied over those segments of the hybrid trajectories of (6.4) where flow occurs. As noted before, Filippov’s solutions are defined using the convexification of the vector field of (6.4a):

$$\bar{F}_e(t,q) := \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \{\text{co}(F_e(t,\tilde{q})) \cup |q - \tilde{q}| \leq \delta, \tilde{q} \not\in N\},$$  \hspace{1cm} (6.11)

such that $\bar{F}_e(t,q)$ is non-empty, bounded, closed, and convex for all $t$ and all $q$ from a bounded set, and upper semi-continuous in $q$, as shown in [58, p.85]. Hence, trajectories satisfy $\frac{d}{dt} \varphi(t,j) \in \bar{F}_e(t,\varphi(t,j))$ for almost all $t$ and fixed $j$.

### 6.4.2 Lyapunov-type stability conditions

In order to formulate Lyapunov-type conditions for the tracking problem given in Problem 6.1, we first present conditions for the asymptotic stability of the set $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e | \rho(q) = 0\}$ for a continuous function $\rho : C_e \cup D_e \to \mathbb{R}_{\geq 0}$. The considered stability properties of this set for the dynamics (6.4), using $\rho(q) = d(r,x)$, directly imply asymptotic stability of the reference trajectory $r$ for the closed-loop trajectory of the hybrid system (6.2), (6.3). Analogously to Definition 6.2, the set $\{q \in C_e \cup D_e | \rho(q) = 0\}$ is said to be stable with respect to a continuous function $\rho$, when, for each $T_0, \epsilon > 0$, there exists a $\delta(T_0, \epsilon) > 0$ such that $\rho(q(t,j)) < \epsilon$, $\forall(t,j) \in \text{dom } q$, $t + j > T_0$, holds for all trajectories $q(t,j)$ of system (6.2) with $\rho(q(0,0)) < \delta(T_0, \epsilon)$. The set is asymptotically stable with respect to the function $\rho$ when, in addition, there exist a $\delta > 0$ such that, for all complete solutions $q$ with initial conditions $\rho(q(0,0)) < \delta$, $\rho(q(t,j))$ converges to zero for $t + j \to \infty$. We first formulate a basic Lyapunov-function-based result guaranteeing asymptotic stability with respect to $\rho$ (Theorem 6.2). In contrast to existing results on stability of sets, see e.g. [67], firstly, non-Euclidean errors measures are used, and secondly, stability of unbounded sets is considered. Furthermore, we present Theorem 6.3 which allows the increase of the Lyapunov function over jumps, as long as this increase is compensated for by a larger decrease over continuous-time evolution, a characteristic which we will employ in Section 6.5 to prove stability of a reference trajectory when a hysteresis-based controller is used.

Recall that Property 6.1 holds naturally for the system (6.4) when both the plant (6.2) and controller (6.3) satisfy Assumption 6.1, and that the data of (6.4) is designed to model the closed-loop dynamics, as discussed in Section 6.2. For this reason, we will now proceed as follows. In Theorems 6.2 and 6.3, we present sufficient conditions for the stability of the set $\{q \in C_e \cup D_e | \rho(q) = 0\}$ with respect to $\rho$ for hybrid systems (6.4) with Property 6.1. Subsequently, these conditions are used in Theorem 6.4 to present conditions that guarantee that the tracking problem presented in Problem 6.1 is solved. Throughout this chapter, Lyapunov functions are considered that are Lipschitz functions, which,
in addition, are regular as defined in Definition 2.3.4 in [44]. Recall that, without any further reference, we will only consider solutions which are maximal.

**Theorem 6.2.** Consider the hybrid system (6.4), and let Property 6.1 hold. In addition, suppose there exist a continuous function $\rho : \text{cl}(C_e \cup D_e) \to \mathbb{R}_{\geq 0}$, a regular and Lipschitz function $V : \text{cl}(C_e \cup D_e) \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in K_\infty$ and scalar $c < 0$, such that:

$$\alpha_1(\rho(q)) \leq V(q) \leq \alpha_2(\rho(q)), \quad \forall q \in \text{cl}(C_e \cup D_e) \quad (6.12a)$$

$$\max_{\zeta \in \partial_{C_e} V, f \in \bar{F}_e(t,q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e) \quad (6.12b)$$

$$V(g) \leq V(q), \quad \forall g \in G_e(q), \forall q \in D_e, \quad (6.12c)$$

with $\bar{F}_e$ given in (6.11), then $\{q \in C_e \cup D_e | \rho(q) = 0\}$ is a stable set of (6.4) with respect to $\rho$. If, in addition, for all solutions $\varphi$ of (6.4) with initial conditions in $C_e \cup D_e$, $\text{dom} \varphi$ is unbounded in $t$-direction, then the set $\{q \in C_e \cup D_e | \rho(q) = 0\}$ is globally asymptotically stable with respect to $\rho$.

**Proof.** See Appendix A.5.2. \qed

Theorem 6.2 is valuable for the design of controllers solving the tracking problem formulated in Problem 6.1. For example, if static state-feedback controllers are employed, where $u = u_c(t,r,x)$, and a Lyapunov function $V$ is selected, then (6.12b) directly gives sufficient conditions for the closed-loop differential inclusion (6.4a) after substitution of the control law. In Section 6.5.2, a bouncing-ball system is presented as an example, where a static control law $u = u_c(t,r,x)$ is designed in this manner.

For various hybrid control systems, along discrete events (i.e., jumps), the Lyapunov function might increase (i.e. might not satisfy condition (6.12c) of Theorem 6.2), while this increase is compensated for by a larger decrease of the Lyapunov function over a sufficiently long continuous-time period without jumps. An exemplary system with this behaviour will be given in Section 6.5.1, where the increase of the Lyapunov function is induced by the switches of a hysteresis-based controller. To study the behaviour of such systems, in Theorem 6.3 below, we present sufficient conditions for stability of trajectories that have an average inter-jump time of at least $\tau$ time units, as defined in the following definition. This definition is formulated by adapting the average dwell time conditions of [81] to the hybrid system framework, allowing trajectories that have a finite number of subsequent jumps at the same continuous-time instant $t$.

**Definition 6.3.** We say that a trajectory $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ for $\tau, \kappa > 0$, if for all $(t,j) \in \text{dom} \varphi$ and all $(T,J) \in \text{dom} \varphi$ where $T + J \geq t + j$, the relation $J - j \leq \kappa + \frac{T - t}{\tau}$ holds.
Note that $\varphi \in S_{\text{avg}}(\tau, 1)$ implies that the trajectory satisfies a minimal inter-jump time restriction of $\tau$ time units. However, the state of the extended hybrid system (6.4) is designed such that it embeds the plant state $x$ and reference state $r$. Hence, if the jump times of the plant converge to the jump times of the reference trajectory, as shown, e.g., in Fig. 6.1, then the extended hybrid system (6.4) directly violates such a dwell time restriction $\varphi \in S_{\text{avg}}(\tau, 1)$. For this reason, we will focus on trajectories satisfying the less restrictive condition $\varphi \in S_{\text{avg}}(\tau, \kappa)$, with $\kappa > 1$. For the example presented in Section 6.5.1, we will show that explicit expressions for $\tau$ and $\kappa$ can be derived, such that trajectories $\varphi$ of the extended hybrid system (6.4) indeed satisfy $\varphi \in S_{\text{avg}}(\tau, \kappa)$.

We present sufficient conditions for global asymptotic stability of the set $\{ q \in C_e \cup D_e \mid \rho(q) = 0 \}$ with respect to the distance $\rho$ in the following theorem, where we require all trajectories $\varphi$ of a system (6.4) to satisfy $\varphi \in S_{\text{avg}}(\tau, \kappa)$.

**Theorem 6.3.** Let trajectories $\varphi$ of (6.4) from all initial conditions in $C_e \cup D_e$ satisfy $\varphi \in S_{\text{avg}}(\tau, \kappa)$, for some $\tau, \kappa > 0$, and let (6.4) satisfy Property 6.1. In addition, suppose there exist a continuous function $\rho : \text{cl}(C_e \cup D_e) \to \mathbb{R}_{\geq 0}$, a regular and Lipschitz function $V : \text{cl}(C_e \cup D_e) \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in K_\infty$ and scalars $c < 0$, $\mu \geq 1$ such that

$$\alpha_1(\rho(q)) \leq V(q) \leq \alpha_2(\rho(q)), \quad \forall q \in \text{cl}(C_e \cup D_e),$$

$$\max_{\zeta \in \partial_{C_e} V, \ f \in \bar{F}_e(t, q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e),$$

$$V(g) \leq \mu V(q), \quad \forall g \in G_e(q), \quad \forall q \in D_e,$$

with $\bar{F}_e$ given in (6.11), then $\{ q \in C_e \cup D_e \mid \rho(q) = 0 \}$ is a stable set of (6.4) with respect to $\rho$ if

$$\mu e^{c\tau} < 1$$

holds. If, in addition, $\text{dom} \varphi$ is unbounded in $t$-direction, then $\{ q \in C_e \cup D_e \mid \rho(q) = 0 \}$ is globally asymptotically stable with respect to $\rho$.

**Proof.** See Appendix A.5.3. \qed

Note that in the proof of this theorem we evaluate an upper bound of the Lyapunov function for all $t$. For this reason, in contrast to the approach in [62], our analysis does not require periodicity of reference trajectories.

Using the previously presented Theorems 6.2 and 6.3, we now formulate sufficient conditions under which the tracking control problem, i.e. Problem 6.1, is solved. We note that, for a relevant class of hybrid systems (6.2), controllers can be designed such that the closed-loop system trajectories $\varphi$, which are modelled by the extended hybrid system (6.4), satisfy $\varphi \in S_{\text{avg}}(\tau, \kappa)$, with $\tau, \kappa > 0$, where the values for $\tau$ and $\kappa$ follow directly from the plant dynamics (6.2) and the controller (6.3). In Section 6.5.1, we present an example where the derivation of $\tau$ and $\kappa$ is shown.
Theorem 6.4. Consider the global tracking problem for the hybrid system (6.2) with compatible tracking error \( d \) (according to Definition 6.1) and reference trajectory \( r \) satisfying Assumption 6.2, let the controller (6.3) be given, let (6.2) and (6.3) satisfy Assumption 6.1, and let the hybrid time domain \( \text{dom} \varphi \) of any trajectory \( \varphi \) of system (6.4) be unbounded in \( t \)-direction. Suppose there exist a regular and Lipschitz function \( V : \text{cl}(C_e \cup D_e) \to \mathbb{R}_{\geq 0} \), functions \( \alpha_1, \alpha_2 \in K_\infty \) and scalars \( c < 0, \mu \geq 1 \) such that:

\[
\alpha_1(d(r, x)) \leq V(q) \leq \alpha_2(d(r, x)), \quad \forall q \in \text{cl}(C_e \cup D_e) \tag{6.14a}
\]

\[
\max_{\zeta \in \partial C V, f \in \bar{F}_e(t, q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e) \tag{6.14b}
\]

\[
V(g) \leq \mu V(q), \quad \forall g \in G_e(q), \quad \forall q \in D_e, \quad \tag{6.14c}
\]

with \( \bar{F}_e \) given in (6.11), such that \( q = \text{col}(r, x, \eta) \) is the state of the extended hybrid system (6.4) with data \( F_e, G_e, C_e, D_e \). If one of the following conditions hold:

(i) the expression (6.14c) holds with \( \mu = 1 \), or

(ii) there exist \( \tau, \kappa > 0 \) such that all solutions of (6.4) satisfy \( \varphi \in S_{\text{avg}}(\tau, \kappa) \), and

\[
\mu e^{c\tau} < 1, \quad \tag{6.14d}
\]

then the global tracking problem (given in Problem 6.1) is solved.

Proof. See Appendix A.5.4. \qed

6.5 Tracking controller design for exemplary systems

In this section, two examples are given where a controller is designed to solve the tracking problem formulated in Problem 6.1. For these two examples, we show that the design of a distance function compatible with the exemplary hybrid system not only allows to formulate the tracking problem, but in addition, can be used to design a controller solving the tracking problem.

After designing the tracking error measure \( d \) and the controller (6.3), for the examples presented in the following sections, we employ Theorem 6.4 to show that the closed-loop system renders the reference trajectory asymptotically stable.

6.5.1 Global tracking for a scalar hybrid system

In this section, we consider the tracking problem for the scalar hybrid system (6.1) with a reference trajectory \( r \), that is the solution to (6.1) for \( u(t) = u_{\text{ref}}(t) = \frac{1}{2} \cos(t) \) and initial condition \( r(0, 0) = 0.95 \in C \cup D \). Since \( 1 + u_{\text{ref}}(t) > 0 \) for all
6.5 Tracking controller design for exemplary systems

$t$, the continuous-time dynamics, described by (6.1a), yields trajectories that can only leave $C$ by arriving at $D$ and thus experience a jump. Integrating equation (6.1a) over time $t$, we obtain $v(t) = r(0,0) + t + \frac{1}{2} \sin(t)$, such that we can write the reference trajectory $r$ as

$$r(t, j) = v(t) - j,$$

with $(t, j) \in \text{dom } r = \text{cl} \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} | j = [v(t)]\}$, where $[a]$ denotes the largest integer such that $[a] < a$. Any jump of $r$ will induce $r = 0$, such that subsequently, flow is possible. A next jump can only occur when $r$ has increased to 1, which takes at least $\frac{1}{\max_{s}(1+u_{\text{sat}}(t))} = \frac{2}{3}$ time units. For this reason, the time domain $\text{dom } r$ is unbounded in $t$-direction and the reference trajectory $r$ satisfies Assumption 6.2.

To evaluate the tracking error between a plant trajectory $x$ and the reference trajectory $r$, we employ the distance function

$$d(r, x) = \min \{|x - r|, |x - r + 1|, |x - r - 1|\}. \quad (6.16)$$

To show that this function satisfies the conditions of Definition 6.1, first, we show that (6.6a) holds. We observe that for the reset map (6.1b), $G^{2}(x) = \emptyset$ for all $x$, hence, $G^{k_{1}}(x) \cap G^{k_{2}}(r) \neq \emptyset$ can only hold for $k_{1}, k_{2} \in \{0, 1\}$. As a consequence, $(\exists k_{1}, k_{2} \in \mathbb{N}, \text{such that } G^{k_{1}}(x) \cap G^{k_{2}}(r) \neq \emptyset) \text{ is equivalent to } (\{x\} \cup \{x\} = G(r) \cup G(x) = \{r\} \cup G(x) = G(x)).$ Considering the jump map (6.1b), we observe that $x^{+} \in G(x) = \{0\}$ for $x = 1$, such that, equivalently, we can write $G(s) = \{s - 1\}$. Substituting this function in the foregoing logical expression yields $(x - r = 0 \lor x = r - 1 = 0 \lor x = r - 1 = 0)$, which is equivalent to $d(r, x) = \min \{|x - r|, |x - r + 1|, |x - r - 1|\} = 0$, such that indeed (6.6a) holds.

The requirement formulated in (6.6b) holds since $C \cup D$ is bounded. For $x \in D = \{1\}$ and arbitrary $r \in C \cup D = [0, 1]$, we observe that $d(r, x) = d(r, 1) = \min\{|1 - r|, |r|\}$, and since $G(x) = 0$ for $x \in D$, we also obtain $d(r, G(x)) = d(r, 0) = \min\{|1 - r|, |1 - r|\}$, such that $d(r, x) = d(r, G(x))$ holds, as required in (6.6c). An analogous argument shows that (6.6d) holds. As continuity of $d$ is obvious, this show that the conditions in Definition 6.1 are satisfied and the distance function $d(r, x)$ in (6.16) is compatible with the hybrid system (6.1).

Now, we will design a controller that solves the global tracking problem as formulated in Problem 6.1 using the distance function $d(r, x)$ given in (6.16). Since during flow, solutions are considered in the sense of Filippov, static controllers $u = u_{c}(t, r, x)$ cannot solve the global tracking problem, as can be observed intuitively as follows. Let $r$ represent an analog 12-hours clock, such that $r = 0$ corresponds to 0h00 or 12h00, let $x$ denote a different pointer that should track the hour hand of the clock, such that the jumps in $x$ and $r$ of plant (6.1) correspond to the 12-hour time jump of the clock. If $x$ is just behind $r$, asymptotically stable behaviour requires “speeding up” the pointer to converge to the reference time $r$. However, if $x$ is just ahead of $r$, $x$ should run slower than the
reference hour hand, and thus slow down and “wait” for the reference trajectory \( r \). However, at some point, say, if the difference between \( r \) and \( x \) is 6 hours, \( x \) should arbitrarily decide to “wait for \( r \)”, or “accelerate to catch up with \( r \)”. If the evolution of \( x \) is described by a Filippov solution, then such a decision is not possible at all points: the Filippov solution will always have a trajectory, where \( \dot{x} = \dot{r} \), that corresponds to an anti-phase synchronised state, meaning that the difference between \( r \) and \( x \) remains 6 hours.

Essentially, this behaviour is induced by the fact that Brockett’s necessary conditions for stabilisability hold both for smooth differential equations and Filippov systems, cf. e.g., [45]. Since the observed phenomenon can be translated directly to system (6.1), global tracking is not feasible using a static controller \( u = u(t,r,x) \). As observed e.g. in [136], one approach to avoid this problem, which we will employ in this chapter, is to introduce a hysteresis-based controller with discrete variable \( \eta \in \{-1,0,1\} \) that will ensure the global tracking problem to be solved.

The rationale behind the controller design is as follows. First, we observe that the distance (6.16) is given by the minimum of three functions \( \rho_1, \rho_2 \) and \( \rho_3 \), where \( \rho_1(r,x) = |x - r|, \rho_2(r,x) = |x - r + 1|, \) and \( \rho_3(r,x) = |x - r - 1| \). For each of these functions, a controller \( u_i, i = 1,2,3 \) is designed that enforces converging behaviour of \( \rho_i \) to zero during flow, i.e., controls \( x - r \), or \( x - r + 1 \) or \( x - r - 1 \) to zero. Subsequently, the function

\[
\tilde{V}(r,x,\eta) = \frac{1}{2}(x - r + \eta)^2, \quad (6.17)
\]

is used to determine in which part of the state space, which of the three control inputs \( u_i, i = 1,2,3 \), is applied. We design the updates of the hysteretic state \( \eta \in \{-1,0,1\} \) such that \( \tilde{V}(r,x,\eta) \) becomes zero if either \( x - r - 1 = 0, x - r = 0 \) or \( x - r + 1 = 0 \). For this purpose, updates of \( \eta \) are triggered by a violation of \( \tilde{V}(r,x,\eta) \leq \mu \frac{c_2}{2}d(r,x)^2 \), with \( \mu > 1 \), which may occur when the plant or reference trajectory experiences a jump. At these time instances, \( \eta \) is reset to ensure \( \tilde{V}(r,x,\eta) \leq \frac{c_2}{2}d(r,x)^2 \) again. The parameter \( \mu > 1 \) determines the hysteretic domain. Using this reasoning, the following hysteresis-based controller is designed:

\[
\begin{align*}
\dot{\eta} &= 0, \quad (r,x,\eta) \in C_c := \{(r,x,\eta) \mid \tilde{V}(r,x,\eta) \leq \frac{c_2}{2}d(r,x)^2\}, \quad (6.18a) \\
\eta^+ &= \arg \min_{i \in \{-1,0,1\}} \tilde{V}(r,x,i), \quad (r,x,\eta) \in D_c, \quad (6.18b)
\end{align*}
\]

with \( D_c := \{(r,x,\eta) \mid \tilde{V}(r,x,\eta) = \frac{c_2}{2}d(r,x)^2\} \cup \{(r,x,\eta) \mid x = 0, \tilde{V}(0,x,\eta) \geq \frac{c_2}{2}d(0,x)^2\} \cup \{(r,x,\eta) \mid r = 0, \tilde{V}(0,x,\eta) \geq \frac{c_2}{2}d(0,x)^2\} \), where \( (r,x,\eta) \in [0,1]^2 \times \{-1,0,1\} \) is used implicitly. The resets of \( \eta \) are designed to ensure that a violation of \( \tilde{V}(r,x,\eta) \leq \frac{c_2}{2}d(r,x)^2 \) is directly corrected by a reset in \( \eta \), and, as we will show in the proof of Theorem 6.5, such jumps only occur directly after jumps of \( x \) or \( r \). We set \( \mu = 1.125 \) and design the controller output as

\[
u_c(t,r,x,\eta) = u_{\text{ref}}(t) - \alpha(x - r + \eta), \quad (6.18c)
\]
where we select $\alpha = \frac{1}{2}$, which is a parameter that determines the convergence rate of the closed-loop tracking error $d$.

The following theorem proves that this controller solves the global tracking problem given in Problem 6.1 for the reference trajectory $r$, by employing case (ii) of Theorem 6.4.

**Theorem 6.5.** Given the hybrid system (6.1) with reference trajectory (6.15) and distance function (6.16), the hybrid controller (6.18) solves the global tracking problem stated in Problem 6.1.

**Proof.** See appendix A.5.5. \qed

In Fig. 6.3, simulation results are shown that illustrate the controller for a reference trajectory $r$ with initial condition $r(0,0) = 0.95$. The plant shows intuitively correct behaviour and converges to the reference away from the jump instances, as predicted by Theorem 6.1. Note that the introduced hysteresis causes the Lyapunov function $V$ to increase at the first jump of the reference trajectory. The Lyapunov function decreases monotonically to zero during flow, and, after the first jump, is not affected by jumps of the hybrid system. After the first hysteretic reset, the control feedback action $-\alpha(x - r + \eta)$, i.e., $u_c - u_{\text{ref}}(t)$, is always negative, such that the plant trajectory “waits” for the reference trajectory, as explained using the analogy between system (6.1) and a clock hand.

The tracking error evaluated in $d(r, x)$, depicted in Fig. 6.3(d), does not display the “peaking” of the Euclidean tracking error of these trajectories, shown in Fig. 6.1.

### 6.5.2 Tracking control for the bouncing ball

We consider the bouncing trajectories of a ball on a table, see Fig. 6.4(a), as an elementary though representative model in the class of hybrid models for mechanical systems with impacts. Assuming that non-impulsive forces $u$ can be applied on the ball with unit mass, the flow of the system is described by:

$$\dot{x} = \text{col}(x_2, -g + u + \lambda(x_1, x_2)), \quad x_1 \geq 0,$$

(6.19a)

where $x = \text{col}(x_1, x_2)$ contains the vertical position $x_1$ and velocity $x_2$ of the ball, respectively, $g$ is the gravitational acceleration, $u$ is a force that can be applied to the system, and the contact force $\lambda$ between the ball and the table, with $\lambda(x_1, x_2) = 0$, for $x_1 > 0$, and $\lambda(x_1, x_2) \in [0, \infty)$, for $x_1 = 0$, avoids penetration of the table by the ball, cf. [66].

Motion according to (6.19a) is only possible when the distance $x_1$ between the table and the ball is non-negative. If the ball arrives at the surface $x_1 = 0$, then a Newton-type impact law with restitution coefficient equal to one is assumed, modelled as

$$x^+ = \text{col}(x_1, -x_2), \quad x_1 = 0 \text{ and } x_2 \leq 0.$$

(6.19b)
Fig. 6.3. Simulation results for system (6.1) with controller (6.18), where $u_{\text{ref}}(t) = 0.5 \cos(t)$ and reference trajectory $r$ is the solution from initial condition $r(0, 0) = 0.95$. The plant trajectory emanates from initial condition $x(0, 0) = 0.455$. (a) Reference and plant trajectory. (b) Feedback action. (c) Lyapunov function evaluated along the trajectory. (d) Error measure $d(r, x)$. (e) Hysteretic variable $\eta$. (f) Control input $u_c(t, r, x, \eta)$. Details near $t = 0$ are shown in the insets of panels (b)-(d).

Fig. 6.4. (a) Bouncing ball system (6.21). (b) Sets $\{x \in C \cup D | d(r^i, x) < \delta\}$ for two points $r^1$ and $r^2$, where $d$ is given in (6.22).
We consider the following reference trajectory:

\[ r(t) = \text{col}(\tau - \frac{g}{2}\tau^2, 1 - g\tau), \quad \tau = t \ mod \ \frac{2}{g}, \]  

(6.20)

where \( \text{mod} \) denotes the modulus operator. This trajectory is a solution to (6.19) for initial conditions \( r(0, 0) = \text{col}(0, 1) \) and \( u_{\text{ref}} \equiv 0 \). In this example, we focus on the local tracking problem given in Problem 6.1 and design a static control law \( u \) for initial conditions \( r \) where \( \text{mod} \) denotes the modulus operator. This trajectory is a solution to equation 6.1, first, note that

\[ G(x) := \text{col}(x_1, -x_2), \quad x \in D := \{0\} \times (-\infty, 0]. \]  

(6.21b)

In order to define a tracking error measure \( d(r, x) \), we use the property that the velocity \( x_2 \) changes sign at impacts, and the position \( x_1 \) is zero, see (6.21b). Hence, if we want to compare a reference state \( r \) with plant state \( x \) when one of them just experienced a jump, then the distance \( |x + r| \) is appropriate. Away from jump instances, typically, the conventional distance \( |x - r| \) can be used. To distinguish when the distance \( |x - r| \) or \( |x + r| \) should be considered, we use the minimum of both, such that the novel distance measure is given by

\[ d(r, x) = \min(|x - r|, |x + r|). \]  

(6.22)

To recognise that this distance function satisfies the conditions posed in Definition 6.1, first, note that \( G(D) \cap D = \{0\} \), such that \( G^k(x) = \emptyset, \forall x \in C \cup D \setminus \{0\} \) when \( k \geq 2 \), implying that \( \exists k_1, k_2 \in \mathbb{N}, \text{such that } G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset \) is equivalent to \( x = r \lor \{x\} = G(r) \lor G(x) = \{r\} \). Since \( G(x) = \{-x\} \) for \( x \in D \), the condition \( x = r \lor \{x\} = G(r) \lor G(x) = \{r\} \) can be rewritten to \( x - r = 0 \lor x + r = 0 \), such that \( d(r, x) = 0 \) directly follows and relation (6.6a) is satisfied. As required in (6.6b), for given \( r, \beta \), the set \( \{x \in C \cup D| d(r, x) < \beta\} \) is compact, as it is the union of the bounded sets \( \{x \in C \cup D| |x - r| < \beta\} \) and \( \{x \in C \cup D| |x + r| < \beta\} \). Since \( G(x) = -x \) for \( x \in D \), relation (6.6c) holds, as \( d(r, G(x)) = d(r, -x) = \min(|r - x|, |r + x|) \), which equals \( d(r, x) \). An analogue argument shows that (6.6d) holds. Since, in addition, \( d \) is continuous, the tracking error measure \( d(r, x) \) is compatible with the hybrid system (6.19).

In Fig. 6.4(b), the neighbourhoods \( \{x \in C \cup D| d(r^i, x) < \delta\} \) of two different points \( r^i, \ i = 1, 2 \), are shown. Essentially, the tracking error measure \( d \) allows to compare a reference trajectory with a plant trajectory, “as if” both of them already jumped. For example, in Fig. 6.4(b), the gray domain with positive \( x_2 \) is considered close to \( r^2 \), since \( r^2 \) will experience a jump soon, and after this jump, \( r^2 \) will arrive in this domain.

We design a tracking control law \( u = u_d(t, r, x) \) for system (6.21) using a reasoning that exploits the design of the tracking error measure \( d \) in (6.22).
Analogously to the design approach in the previous section, observe that $d(r, x)$ in (6.22) is given by the minimum between the two functions, $\rho_1 = |x - r|$ and $\rho_2 = |x + r|$. When the trajectory $x$ is sufficiently close to $r$ and neither of them experiences a jump in the near future or past, then the tracking error $d(r, x)$ given in (6.22) is given by $\rho_1 = |x - r|$. Along solutions of the differential equation (6.19a), this error could accurately be controlled towards zero using a controller with PD-type feedback, given by

$$u_1 = -[k_p, k_d](x - r),$$

(6.23)

where $k_p, k_d > 0$. Implementation of this controller yields the error dynamics $\ddot{x}_1 = -k_p(x_1 - r_1) - k_d(\dot{x}_1 - \dot{r}_1)$, such that col($x_1 - r_1, \dot{x}_1 - \dot{r}_1$) = col$(0, 0)$ is an asymptotically stable equilibrium point of the flow dynamics with $u$ given in (6.23).

However, if either the reference trajectory or the plant just experienced a jump, $d(r, x)$ as given in (6.22) is given by $\rho_2 = |x + r|$. The continuous-time behaviour of $x + r$ is stable when the closed-loop dynamics satisfy $\ddot{x}_1 + \dot{r}_1 = -k_p(x_1 + r_1) - k_d(\dot{x}_1 + \dot{r}_1)$, which is obtained by selecting the controller as

$$u_2 = 2g - [k_p, k_d](x + r),$$

(6.24)

with $k_p, k_d > 0$. Based on these insights, we propose a controller that switches between (6.23) and (6.24). To choose the partitioning of the state space where either (6.23) or (6.24) are applied, the following candidate Lyapunov function $V(r, x)$ is considered:

$$V(r, x) = \min(V_d(r, x), V_m(r, x)),
$$

(6.25)

with

$$V_d(r, x) = \frac{1}{2}(x - r)^TP(x - r), \quad V_m(r, x) = \frac{1}{2}(x + r)^TP(x + r),$$

(6.26)

where a symmetric, positive definite matrix $P$ and scalar $c > 0$ are chosen such that

$$A^TP + PA \prec -cP,$$

(6.27)

with $A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}$, where for symmetric and real matrices $M, N$, we adopt the notation $M < 0$ when $M$ is negative definite, and $M \prec N$ when $M - N < 0$. For strictly positive $k_p, k_d$, the matrix $A$ is Hurwitz, which implies that such $P$ and $c$ exist, since [91, Theorem 4.6] implies that for any symmetric, positive definite matrix $Q$ there exist a symmetric, positive definite $P$ such that $A^TP + PA \prec -Q$, and $c > 0$ can be chosen sufficiently small, such that $-Q \prec -cP$. Based on the Lyapunov function candidate $V$, the following control law $u = u_d(t, r, x)$ is designed, such that $\rho_1$ or $\rho_2$ decreases along continuous-time solutions described by (6.19a):

$$u_d(t, r, x) = \begin{cases} -[k_p, k_d](x - r), & V_d \leq V_m \\ 2g - [k_p, k_d](x + r), & V_d > V_m. \end{cases}$$

(6.28)
In the next theorem we show that the control law \((6.28)\) indeed solves the tracking problem formulated in Problem 6.1.

**Theorem 6.6.** Consider the bouncing ball system \((6.21)\), tracking error \(d\) given in \((6.22)\) and reference trajectory \(r\) given in \((6.20)\) for \(u_{ref}(t) \equiv 0\). Application of the control law \(u_d(t, r, x)\) as defined in \((6.28)\), with \(V_d, V_m\) given in \((6.26)\) and \(k_p, k_d > 0\), to the hybrid system \((6.21)\) makes the reference trajectory \(r\) asymptotically stable with respect to \(d\). In addition, the set \(\{x \in C \cup D | V(r(0, 0), x) \leq K\}\) is contained in the basin of attraction of \(r\), where \(K\) is chosen to satisfy

\[
K < \min_{(t, j) \in \text{dom } r} V(r(t, j), 0). \tag{6.29}
\]

**Proof.** See Appendix A.5.6.

The control law \(u_d(t, r, x)\) given in \((6.26),(6.28)\) with \([k_p k_d] = [1 \ 0.5]\) and \(P = \begin{bmatrix} 2.25 & 0.5 \\ 0.5 & 2 \end{bmatrix}\) is applied to system \((6.21)\) with \(g = 9.81\). The trajectory from the initial condition \(x(0, 0) = \text{col}(4, 3)\) is shown with the dashed line in Fig. 6.5(a)-(b). Clearly, the hybrid trajectory \(x\) converges to \(r\) during flow, and the jump instances of \(r\) and \(x\) converge to each other. The Euclidean distance \(|x - r|\) and the distance \(d(r, x)\) between both trajectories are shown in Fig. 6.5(c) and (d), respectively. Although the Euclidean distance displays the unstable “peaking” behaviour, the tracking error expressed using the distance \(d(r, x)\) remains continuous over trajectories, and converges to zero. Hence, the local tracking problem as formulated in Problem 6.1 is solved, and the trajectories shown in Fig. 6.5(a) show desirable tracking behaviour. Indeed, as predicted by Theorem 6.1, \(|x - r|\) decreases to zero for \(r\) away from \(D \cup G(D)\), where \(r_1 = 0\).

Although the controller designs for the examples in Sections 6.5.1 and 6.5.2 have been tailored to the specific examples, we care to highlight that we employed a common rationale for the controller design. First, the tracking error measure \(d(r, x)\) is designed as the minimum of functions \(\rho_i, \ i = 1, 2, 3\) corresponding to \(|x - r|, |x - \tilde{G}(r)|, |\tilde{G}(r) - x|\), where the functions \(\tilde{G}\) are designed such that \(\tilde{G}(x) = G(x)\) for \(x \in D\). (For the bouncing ball example given in \((6.21)\), the functions \(\rho_2\) and \(\rho_3\) coincide due to the structure of \(G\) and \(D\).) Subsequently, for each function \(\rho_i, \ i = 1, 2, 3\), a control law \(u_i\) is designed that would stabilise the set where \(\rho_i = 0\) along flowing solutions. Finally, a Lyapunov candidate function is employed to determine which control law \(u_i\) is applied in what part of the state space.

### 6.6 Conclusion

In this chapter, the problem of (global) tracking of time-dependent reference trajectories is studied for hybrid systems with state-triggered jumps. We formulated
the tracking problem in such a way that it corresponds to the intuitive notion of tracking for hybrid systems: the plant trajectories tend asymptotically to the reference trajectory, such that away from the time instances where the reference trajectory jumps, the Euclidean tracking error becomes small. To formalise this notion of tracking, the tracking error is evaluated using a novel, non-Euclidean distance measure. It is shown in this chapter that such distance functions have three advantages. First, it facilitates the formulation of a tracking problem that is feasible for a large class of hybrid system, including mechanical systems with impacts, and does not require the jumps of the plant to coincide with the jumps of the reference trajectory. Second, the formulated tracking problem can be analysed by evaluating Lyapunov functions along closed-loop trajectories, and is feasible for a large class of reference trajectories, which are not required to
be periodic. Third, as shown in the examples, the new tracking error measure can be used to design controllers solving the tracking problem. Using exemplary systems, including the well-known bouncing ball system, we illustrate that the tracking problem is feasible for hybrid systems with state-triggered jumps and that the presented results support the design of tracking controllers for such hybrid systems.

Further research should be directed to the development of a synthesis procedure for generic hybrid systems that leads to, firstly, a tracking error measure that is tailored to the hybrid system under study and, secondly, a control law that solves the tracking problem formulated in this chapter. The requirements on the distance measure and the stability analysis presented in this chapter form important stepping stones towards such a generic synthesis procedure.
7.1 Conclusions

In this thesis, nonsmooth dynamical systems have been investigated and results are obtained in two domains. Firstly, in Part I of this thesis, the qualitative dynamical behaviour and the effect of parameter variations are discussed for non-differentiable and discontinuous systems. Secondly, the tracking control problem for hybrid systems is studied in Part II.

In Part I, we discuss the qualitative dynamical behaviour and identify possible bifurcations for two classes of nonsmooth systems, namely, for non-differentiable and discontinuous systems. In both system classes, the dynamics is described by a differential equation that, away from some surfaces in the state space, is a smooth vector field. If non-differentiable systems are considered, then the differential equation is continuous at the mentioned surfaces. Non-differentiable systems can describe, for example, control systems with actuator saturation or mechanical systems with one-sided flexibilities. In discontinuous systems, the differential equation can be discontinuous at the mentioned surfaces. Discontinuous systems can be used, for example, to model mechanical systems with friction, where the friction force is modelled with Coulomb’s friction law.

In this thesis, we present a procedure to analyse local bifurcations of the dynamics near equilibria in planar non-differentiable systems. For a class of discontinuous systems describing mechanical systems with dry friction, sets of non-isolated equilibrium points are identified. Sufficient conditions are presented for the structural stability of the dynamics near such equilibrium sets, and non-smooth bifurcations are identified. Furthermore, the effect of a homoclinic orbit
from an endpoint of an equilibrium set has been investigated. When transversal homoclinic orbits from such points exist, then a horseshoe-like limit set is found that exhibits features that are characteristic to nonsmooth systems.

In Part II, the tracking control problem for hybrid systems has been investigated. Hybrid systems show both continuous evolution in time and jumps of the states (i.e., discrete events). For example, hybrid systems can be used to model mechanical systems with impacts. Due to the jumps of hybrid systems, the standard tracking control problem for smooth dynamical systems can be too restrictive. We have formulated a new tracking control problem, based on a non-Euclidean tracking error measure, and showed that this problem can be solved for a larger class of hybrid systems.

The contributions of this thesis are discussed in more detail below.

Local bifurcations of equilibria are studied in Chapter 2 for the class of planar non-differentiable systems. For these systems, bifurcations may occur when, due to a parameter variation, an equilibrium point arrives at a surface, or at the intersection of multiple surfaces, where the vector field is non-differentiable. A procedure is presented that can identify all limit sets that are created or destroyed at such bifurcations. This procedure uses a local approximation of the smooth vector fields that govern the dynamics away from the mentioned surfaces, resulting in a cone-wise affine system that experiences the same bifurcation scenario as the local bifurcation of the non-differentiable system. With the presented procedure, in an example, a new type of non-smooth bifurcation has been identified.

In Chapters 3 and 4, a class of discontinuous dynamical systems has been considered that represents mechanical systems with dry friction. In Chapter 3, it is shown that these systems generically have equilibrium sets, which are sets of non-isolated equilibrium positions that form intervals of curves in the state space. We showed that, in general, local bifurcations near these equilibrium sets can only occur around the endpoints of these sets. Using this result, sufficient conditions are presented for the local structural stability near equilibrium sets in two-dimensional systems. In examples where these conditions are violated, local bifurcations are observed that change the dynamics near the equilibrium set.

In Chapter 4, for the same class of systems, homoclinic orbits are studied that emanate from an endpoint of an equilibrium set. A periodic perturbation of such systems induces a complex geometry of the sets of points converging to, or diverging from the endpoint of the equilibrium set. This geometry, which is called the homoclinic tangle, is strongly affected by the discontinuity of the vector field. Using a return map, the dynamics near the homoclinic tangle is studied, and a limit set is found whose geometry and dynamical behaviour are essentially different from those of the limit set that appears near homoclinic tangles in smooth systems.

To attain a more detailed description of the newly found limit set, in Chap-
Chapter 5, we study the dynamics of a class of return maps with similar properties as the return map derived in Chapter 4. In particular, the considered maps are continuous and non-invertible. It is shown that these maps have a limit set that is geometrically described as the Cartesian product of a Cantor set and a countable set of points. Furthermore, topologically conjugate symbolic dynamics are introduced, that are defined on a quotient space of infinite strings of symbols. Using this symbolic dynamics, it is proven that the limit sets of the class of maps considered in Chapter 5 are topologically transitive and contain an infinite number of periodic orbits. Both of these properties are known to hold as well for the Smale horseshoe map, which describes the dynamics near homoclinic orbits of smooth systems. For this reason, we argue that the non-invertible maps considered in Chapter 5 are closely related to the Smale horseshoe map.

In Chapter 6, the tracking control problem for hybrid systems has been considered. In this problem, a controller should be designed that forces a hybrid system to follow a pre-described (jumping) reference trajectory asymptotically. In smooth dynamical systems, conventionally, the tracking controller is required to induce asymptotic stability, in the Euclidean error measure, of the reference trajectory. However, if jumps of the system are state-triggered and the control input does not affect these jumps, then the jump instants of the plant may become close to the jump instants of the reference trajectory, but these will generically not coincide. During the small time intervals of such a jump-timing mismatch, a peaking phenomenon is observed in the Euclidean tracking error, rendering asymptotic stability of the Euclidean tracking error infeasible. This problem is addressed in Chapter 6 by reformulating the tracking problem such that it is not affected by the peaking phenomenon and still corresponds to an intuitively correct notion of tracking. The new tracking problem requires asymptotic stability of the reference trajectory with respect to a non-Euclidean distance measure. This problem is less restrictive, and, consequently, for a larger class of hybrid systems, controllers can be designed that solve this tracking problem. We formulated sufficient conditions for the stability of a trajectory with respect to the new error measure, and, in two examples, we designed controllers solving the tracking problem.

7.2 Recommendations

In this section, recommendations are presented for further research on the dynamics and control of nonsmooth systems. First, three recommendations are given that can be directly linked to the contents of the previous chapters. Subsequently, recommendations are formulated which have a broader scope.

• The effect of the discontinuous behaviour of dry friction on homoclinic tangles has been discussed only briefly in Chapter 4. A mathematically rigourous description of this homoclinic tangle could perhaps be attained.
when the lambda-lemma could be adapted such that it applies in the discontinuous case discussed in Chapter 4. For smooth discrete-time systems, roughly speaking, the lambda-lemma states that a small disc that has a transversal intersection with the stable manifold of a hyperbolic equilibrium point becomes arbitrarily close to the unstable manifold of this point under forward iterations of the map that defines the discrete-time system, cf. [127]. This result implies that a single transversal intersection between the stable and unstable manifolds generates an infinite number of transversal intersections between these manifolds. Furthermore, if a transversal intersection is present, then the stable and unstable manifolds accumulate on themselves, such that a homoclinic tangle is created. Hence, the lambda-lemma is instrumental to describe the geometry of the homoclinic tangle in smooth systems. The numerical results presented in Section 5.2.1 suggest that a result similar to the lambda-lemma does hold in an exemplary system. Hence, we suggest to further investigate the dynamics near the points that are attracted to, or have diverged from, the endpoints of the equilibrium set. Such investigations may lead to a nonsmooth variant of the lambda-lemma and could enable a rigorous description of the homoclinic tangle discussed in Chapter 4.

- In Chapter 6, the tracking control problem for hybrid systems has been studied, and this problem is formulated using a non-Euclidean tracking error measure. From a control perspective, it is desirable to develop a synthesis procedure for classes of hybrid systems that, firstly, designs an error measure that is tailored to the hybrid system under study, and secondly, designs a control law that solves the tracking problem presented in Chapter 6.

- The results in Chapter 6 are formulated in the framework of [68]. However, the new formulation of the tracking control problem and the definition of stability with respect to a non-Euclidean error measure can also be applied in different hybrid system frameworks. If these ideas are transferred to the framework of measure differential inclusions, then tracking control problems could be investigated for models of mechanical systems that use the more realistic contact and impact laws that are available in this framework.

- Most results on bifurcations and structural stability of nonsmooth systems that exist in the literature, including the results in Chapters 2, 4 and 5, have been derived only for systems with a low-dimensional (i.e. two, or three-dimensional) state space. The development of dimension-reduction techniques for non-differentiable and discontinuous systems would significantly increase the applicability of these results.

For smooth dynamical systems, dimension reduction can be achieved using, e.g., center manifolds [43] or normal hyperbolic invariant manifolds [57].
similar dimension reduction techniques could be attained for discontinuous systems, then, next to the increased applicability of existing bifurcation results, the dynamics of higher-dimensional nonsmooth models could be better understood. As a consequence, higher-dimensional nonsmooth systems could also be more effectively used to model and analyse dynamical phenomena in engineering, physics, biology or economy.

- The mathematical literature on bifurcations and structural stability in nonsmooth systems has a strong emphasis on behaviour that is generic under arbitrary perturbations of the right-hand side of the differential equations. For various applications, this definition of generic behaviour excludes physically relevant phenomena, such as non-isolated equilibrium points in mechanical systems with friction, as discussed in Chapter 3. If perturbations are considered that are physically relevant, but that preserve some of the qualitative properties of the vector field (e.g. perturbations preserving the kinematic relation between position and velocity in models of mechanical systems), then results on structural stability and bifurcations can be achieved that are important for a range of physical applications of the nonsmooth models. In this manner, bifurcation and structural stability results for restricted classes of perturbations can form an important addition to the existing theory of nonsmooth dynamical systems.

- The approximation of nonsmooth dynamical systems by smooth differential equations (which is also called regularisation) may lead to important insights in the dynamical behaviour of the nonsmooth system. However, the limit sets of the approximating system, and its bifurcation diagram, can be qualitatively different from those of the nonsmooth dynamical system. In addition, approximating smooth models will not show the collapse (i.e. the loss of uniqueness) of trajectories that can occur in discontinuous systems.

While acknowledging these fundamental differences between smooth and nonsmooth dynamical systems, we note that some of the main differences between nonsmooth systems and their smooth approximations only manifest themselves when trajectories are considered over infinitely long time horizons. For example, we expect that a set of non-isolated equilibrium points, that occurs in the class of systems discussed in Chapter 3, is represented in a smooth approximation by a set in the state space where the vector field is very small. If trajectories in this set are considered on a finite time horizon, then little differences can be observed between these trajectories, and those of the discontinuous vector field.

Motivated by this example, we expect that if trajectories are only considered on finite-time horizons, then the dynamics of nonsmooth systems can be more directly linked with their smooth approximations. Using such
a link, various results that are obtained for smooth dynamical systems may 
be of significant relevance in the domain of nonsmooth systems.

• The notion of stability of hybrid trajectories, as presented in Chapter 6 
to study the tracking control problem, can also be applied to different 
problems, e.g. in the fields of synchronisation, or observer design. In fact, 
the non-Euclidean distance measure introduced in Chapter 6 provides a 
versatile tool to study the differences between time-varying and jumping 
trajectories of hybrid systems. As such, this type of distance measures can 
also be used, for example, to study the properties of numerical simulation 
routines for hybrid systems.
Appendix A

Proofs and technical results
A.1 Appendices of Chapter 2

In the following section, the computation of the maps \( g_i, \ i = 1, \ldots, m, \) is given which are defined in Section 2.4.2. Subsequently, in Section A.1.2, the proofs are given of the theorems and lemmas presented in Chapter 2.

A.1.1 Computation of maps \( g_i, \ i = 1, \ldots, m. \)

We study a trajectory traversing \( S_i \) from the boundary \( \Sigma_- \) towards the boundary \( \Sigma_+ \) in a finite time \( t_i. \) Therefore, the trajectory \( x(t) \) satisfies \( x(t) \in S_i, \ t \in (0, t_i), \ x(0) \in \Sigma_- \) and \( x(t_i) \in \Sigma_+. \) To analyse this trajectory in the cone \( S_i, \) a new coordinate frame will be used, whose origin is shifted to the point where \( A_i x + \mu b = 0. \) In addition, other basis vectors are chosen to describe positions in \( \mathbb{R}^2. \) The relations between coordinates in the original frame, denoted as \( x, \) with coordinates in the new frame, denoted as \( \tilde{x}, \) are:

\[
\tilde{x} = P_i^{-1} x + \mu P_i^{-1} A_i^{-1} b, \quad x = P_i \tilde{x} - \mu A_i^{-1} b, \tag{A.1}
\]

where \( P_i \) is found by the real Jordan decomposition, such that \( A_i = P_i J_i P_i^{-1}. \) The dynamics expressed in the new coordinate frame is given by:

\[
\dot{\tilde{x}} = J_i \tilde{x}, \quad \text{for} \ t \in [0, t_i]. \tag{A.2}
\]

Consider an initial condition with coordinates \( x_0 = p^i \rho_- \in \Sigma_. \) In the new coordinate frame, one finds \( \tilde{x}_0 = P_i^{-1} (p^i \rho_- + \mu A_i^{-1} b). \) The direction of the vector tangent to the boundary \( \Sigma_- \) is given by \( \tilde{\rho}_- := P_i^{-1} \rho_- \) such that \( \tilde{x}_0 = p^i \tilde{\rho}_- + \mu P_i^{-1} A_i^{-1} b. \)

There exists a crossing of the trajectory with the boundary \( \Sigma_+ \) at time \( t_i. \) Suppose this crossing occurs at \( x(t_i) = p^{i+1} \tilde{\rho}_+. \) In the new coordinate frame, this position is given by \( \tilde{x}(t_i) = p^{i+1} \tilde{\rho}_+ + \mu P_i^{-1} A_i^{-1} b, \) where a vector \( \tilde{\rho}_+ := P_i^{-1} \rho_+ \) is introduced, that is parallel with \( \Sigma_+. \) Defining a vector orthogonal to \( \tilde{\rho}_+ \) yields \( \tilde{n}_+ := (e_1 e_2^T - e_2 e_1^T) \tilde{\rho}_+, \) such that:

\[
\tilde{n}_+^T \tilde{x}(t_i) = \mu \tilde{n}_+^T P_i^{-1} A_i^{-1} b. \tag{A.3}
\]

Substitution of the solution \( \tilde{x}(t) = e^{J_i t} \tilde{x}_0 \) of (A.2) in (A.3) yields an expression which is evaluated at the time \( t_i: \)

\[
\tilde{n}_+^T e^{J_i t_i} \tilde{x}_0 = \tilde{n}_+^T \tilde{x}_T, \tag{A.4}
\]

where we defined the translation vector \( \tilde{x}_T := \mu P_i^{-1} A_i^{-1} b. \)

When the traversal time \( t_i \) satisfying (A.4) is found, this time can be used to obtain the traversal position. Integrating (A.2) over a time interval \([0, t_i]\) yields \( \tilde{x}(t_i) = e^{J_i t_i} \tilde{x}_0. \) In the original coordinate frame, this yields:

\[
x(t_i) = P_i e^{J_i t_i} \tilde{x}_0 - \mu A_i^{-1} b. \tag{A.5}
\]
Case 1: If $A_i$ has complex eigenvalues, then $J_i = \begin{bmatrix} a_i & -\omega_i \\ \omega_i & a_i \end{bmatrix}$, where $a_i$ and $\omega_i$ are real and $\omega_i > 0$. Hence, $e^{J_i t} = e^{\alpha_i t} \begin{bmatrix} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix}$. Herewith, (A.4) yields:

$$e^{\alpha_i t} \cos(\omega_i t) \tilde{n}_+^T \tilde{x}_0 + e^{\alpha_i t} \sin(\omega_i t) \rho_+^T \tilde{x}_0 = \tilde{n}_+^T \tilde{x}_T.$$  

This equation can be solved with a numerical solver to obtain the time $t_i$. This time yields the position:

$$x(t_i) = -e^{\alpha_i t_i} \sin(\omega_i t_i) P_1 (e_1^T e_2 - e_2^T e_1) \tilde{x}_0 + e^{\alpha_i t_i} \cos(\omega_i t_i) P_1 \tilde{x}_0 - \mu A_i^{-1} b.$$  

(A.7)

Case 2: If $A_i$ has two real eigenvalues $\lambda_{ai}$ and $\lambda_{bi}$ whose eigenvectors are distinct, then $J_i = \begin{bmatrix} \lambda_{ai} & 0 \\ 0 & \lambda_{bi} \end{bmatrix}$. Using $e^{J_i t} = e^{\lambda_{ai} t} e_1^T + e^{\lambda_{bi} t} e_2 e_2$, we obtain: $\tilde{x}^i(t) = e^{\lambda_{ai} t} e_1 e_1^T e_1 + e^{\lambda_{bi} t} e_2 e_2^T e_2$, such that (A.4) becomes:

$$e^{\lambda_{ai} t} \tilde{n}_+^T e_1 e_1^T e_1 \tilde{x}_0 + e^{\lambda_{bi} t} \tilde{n}_+^T e_2 e_2^T e_2 \tilde{x}_0 = \tilde{n}_+^T \tilde{x}_T,$$  

that can be solved with a numerical solver to obtain the smallest time $t_i > 0$. Evaluating (A.5) on this time yields:

$$x(t_i) = e^{\lambda_{ai} t_i} P_1 e_1 e_1^T \tilde{x}_0 + e^{\lambda_{bi} t_i} P_1 e_2 e_2^T \tilde{x}_0 - \mu A_i^{-1} b.$$  

(A.9)

Case 3: If $A_i$ has two equal real eigenvalues $\lambda_{ai}$ with geometric multiplicity 1, then $J_i = \begin{bmatrix} \lambda_{ai} & 1 \\ 0 & \lambda_{ai} \end{bmatrix}$ and $e^{J_i t} = e^{\lambda_{ai} t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Substituting this expression in (A.4) yields:

$$e^{\lambda_{ai} t} \tilde{n}_+^T \tilde{x}_0 + t_i e^{\lambda_{ai} t} \tilde{n}_+^T e_1 e_1^T \tilde{x}_0 = \tilde{n}_+^T \tilde{x}_T.$$  

(A.10)

When the smallest $t_i > 0$ satisfying (A.10) is found with a numerical solver, this can be substituted in (A.5), yielding:

$$x(t_i) = e^{\lambda_{ai} t_i} P_1 \tilde{x}_0 + t_i e^{\lambda_{ai} t_i} P_1 e_1 e_1^T \tilde{x}_0 - \mu A_i^{-1} b.$$  

(A.11)

### A.1.2 Proofs

**Proof of Theorem 2.1.** First we prove the existence of at least one equilibrium point. For this purpose, the index of a point and the index of a Jordan curve are introduced. Next, we will prove the second part of the theorem.

Let $\Delta \theta$ be the total change in the angle $\theta$ that the vector $f(x)$ makes with some fixed direction as $x$ traverses a Jordan curve $J$ once in the positive direction.
Recall from [74] that a Jordan curve is defined as a topological image of a circle, i.e. \( J \) is an \( x \)-set of points \( x = x(t), a \leq t \leq b \), where \( x(t) \) is continuous, \( x(a) = x(b) \) and \( x(s) \neq x(t) \), for \( a \leq s < t < b \). Define the index of \( J \) with respect to \( f \) as \( I_f(J) := \frac{\Delta \phi}{2\pi} \).

Define the index of an isolated equilibrium point \( P \) with respect to \( f \) as the index of any Jordan curve enclosing \( P \), and no other equilibrium points. This index will be denoted by \( I_f(P) \). For certain equilibria, the index is known; \( I_f(P) = 1 \) if \( P \) is a hyperbolic node or focus and \( I_f(P) = -1 \) if \( P \) is a hyperbolic saddle.

According to Theorem 4.4, p. 400, [46], since \( C \) is a periodic orbit, \( I_f(C) = 1 \). The periodic orbit \( C \) is a Jordan curve, such that Theorem 4.1, p. 398, [46] yields that \( C \) encircles at least one equilibrium point. This proves the first statement of the theorem.

Suppose that all equilibrium points \( P_1, \ldots, P_m \) that are encircled by the closed orbit are hyperbolic. Application of Corollary 2, p. 400, [46] yields that the sum of indices of these equilibria equals one, i.e. \( I_f(C) = \sum_{i=1}^{n} I_f(P_i) = 1 \). The value of the indices of a node, focus and saddle imply, that there must be an odd number \( 2n + 1 \) of equilibria, of which \( n \) are saddles and \( n + 1 \) are nodes or foci.

\[ \square \]

**Proof of Lemma 2.3.** Three cases are distinguished according to the eigenvalues of the matrix \( A \), that describes the dynamics \( \dot{y} = Ay \) in region \( \tilde{S} \). The existence of a visible eigenvector, corresponding to a real eigenvalue, implies that all eigenvalues of \( A \in \mathbb{R}^{2 \times 2} \) are real.

In the first case, the eigenvalues of \( A \) are equal and this eigenvalue has geometric multiplicity two. The trajectories inside \( \tilde{S} \) converge to the origin since \( \text{det}(\lambda I-A) = 0 \). In the second case, the matrix \( A \) has an eigenvalue \( \lambda < 0 \) with algebraic multiplicity two and geometric multiplicity one. Trajectories starting in \( \tilde{S} \) can be written as \( y(t) = Pe^{\lambda t}P^{-1}y_0 \) as long as \( y(\tau) \in \tilde{S} \forall \tau \in [0,t], \) where the Jordan canonical form \( A = PJP^{-1} \) is used, with \( J = \lambda(e_1e_1^T + e_2e_2^T) + e_1e_1^T, \) \( e_1 := (1 0)^T \) and \( e_2 := (0 1)^T \). Consequently, the orientation of the vector \( e^{\lambda t}(y_0 + tPe_1e_1^TP^{-1}y_0) \) converges, for \( t \rightarrow \infty \), to the orientation of \( Pe_1 \), which is the eigenvector corresponding to eigenvalue \( \lambda \). With \( \lambda < 0 \), either convergence of the trajectory to the origin is obtained, or the trajectory leaves \( \tilde{S} \) in finite time.

In the third case, the matrix \( A \) has two distinct eigenvalues \( \lambda_a < 0 \) and \( \lambda_b \in \mathbb{R} \setminus \{ \lambda_a \} \). Trajectories in the region \( \tilde{S} \) are either positioned on the equilibrium, such that \( y(t) = 0, \forall t > 0 \), or the trajectories can be written as \( y(t) = Pe^{\lambda t}P^{-1}y_0 = Pe_1e_1^TP^{-1}y_0e^{\lambda_a t} + Pe_2e_2^TP^{-1}y_0e^{\lambda_b t}, \forall y_0 \in \tilde{S} \) as long as \( y(\tau) \in \tilde{S}, \forall \tau \in [0,t] \), where the Jordan canonical form \( A = PJP^{-1} \) is used, with \( J = \lambda_a e_1e_1^T + \lambda_b e_2e_2^T \) and \( y_0 \in \tilde{S} \). The eigenvectors of \( A \) are given by the vectors \( v_a = Pe_1 \) and \( v_b = Pe_2 \). The trajectory \( y(t) \) converges to either \( I_{va} = \{ y \in \tilde{S} \} \).
\( \mathbb{R}^2 | y = cv_a, c \in (0, \infty) \) when \( \lambda_a > \lambda_b \) or \( I_{vb} = \{ y \in \mathbb{R}^2 | y = cv_b, c \in (0, \infty) \} \) when \( \lambda_a < \lambda_b \). If trajectories converge to the set \( I_{vk}, k = a, b \), corresponding to a visible eigenvector, i.e. \( I_{vk} \in \bar{S}, k = a, b \), they will approach the origin, since the corresponding eigenvalue \( \lambda_k < 0, k = a, b \). Otherwise, they will leave the region \( \bar{S} \) in finite time.

**Proof of Lemma 2.4.** When \( \bar{S} \) is a halfline, the vector field \( Ay \) is pointing out of \( \bar{S} \), since otherwise, all \( y \in \bar{S} \setminus \{0\} \) are eigenvectors, that are visible. Every nonempty closed cone \( \bar{S} \subset \mathbb{R}^2 \) that is not a halfline can be written as the union of two nonempty closed convex cones \( \bar{S}_1 \) and \( \bar{S}_2 \) that do not contain a subspace of \( \mathbb{R}^2 \), i.e. \( \bar{S} = \bar{S}_1 \cup \bar{S}_2 \), where these cones intersect only at the halfline \( \Sigma_{12} = \bar{S}_1 \cap \bar{S}_2 \). Due to the assumption in Lemma 2.4, both regions \( \bar{S}_j, j = 1, 2 \), and the boundary \( \Sigma_{12} \) do not contain an eigenvector of \( A \).

According to Theorem 3 of [4], for any \( y_0 \in \bar{S}_j \), \( j = 1, 2 \), with \( y_0 \neq 0 \), there exists a time \( t_1 \geq 0 \) such that \( e^{At_1}y_0 \notin \bar{S}_j \). Trajectories can traverse the boundary \( \Sigma_{12} \) only in one direction. Therefore, trajectories from a point \( y_0 \in \bar{S}_j \) leave \( \bar{S} \) in a finite time, possibly after traversing the other cone \( \bar{S}_k, k \in \{1, 2\}, k \neq j \).

**Proof of Lemma 2.5.** Necessity is trivial. To prove sufficiency of the stability requirements, we will prove that for every trajectory \( y(t) \), with initial condition \( y_0 \in \mathbb{R}^2 \setminus 0 \), there exist a finite time \( t_f \) and closed cone \( \bar{S}_k \), such that \( y(t) \in \bar{S}_k, \forall t > t_f \). Subsequently, we will prove that the trajectories converge to the origin for \( t \to \infty \).

Consider the trajectory from any initial condition \( y_0 \in \mathbb{R}^2 \setminus 0 \). When \( y_0 \) is inside a cone \( \bar{S} \) containing no visible eigenvectors, Lemma 2.4 guarantees that the trajectory will leave this cone in a finite time. When \( y_0 \) is inside a cone \( \bar{S} \) containing visible eigenvectors, then, according to Lemma 2.3, trajectories may converge to the origin asymptotically and remain in this cone. In that case, choose \( t_f = 0 \). Otherwise, they will leave the cone \( \bar{S} \) in a finite time.

Trajectories can traverse a boundary \( \Sigma_{i,i+1} = \{ y \in \mathbb{R}^2 | y = c\rho_{i,i+1}, c \in (0, \infty) \} \) only in one direction, either from \( \bar{S}_i \) to \( \bar{S}_{i+1} \) or vise versa, since the vector field \( f(y) \) on \( y = c\rho_{i,i+1} \) is given by \( \dot{y} = cA_i\rho_{i,i+1} \) and \( c > 0 \). This means that possibly after leaving or traversing of some regions, all trajectories will arrive at a cone \( \bar{S}_k \) containing a visible eigenvector and remain there for all \( t > t_f \). Here, \( t_f \) is the sum of the finite time it took to escape the first region, and the finite times for the traversal of regions without visible eigenvectors. Since these times are all finite according to Lemmas 2.3 and 2.4, the sum of these, which is defined as \( t_f \), is also finite.

By Lipschitz continuity, the trajectory \( y(t) \) remains bounded for \( t \in [0, t_f] \). Subsequently, it remains in a specific cone \( \bar{S}_k \) for all \( t > t_f \), where \( \bar{S}_k \) contains a visible eigenvector. Hence, Lemma 2.3 guarantees asymptotic stability of the origin.
Proof of Theorem 2.6. When visible eigenvectors are present, then necessity and sufficiency of (i) is given in Lemma 2.5. In absence of visible eigenvectors, necessity of (ii) is proven by contradiction. Let $T$ be the period time of the spiralling motion of (2.5), as given in [4]. When $\Lambda \geq 1$, then a trajectory from $y_0 \in \Sigma_{m1}$ visits the sequence of positions $y(kT) = \Lambda^k y_0$, $k = 1, 2, \ldots$, contradicting asymptotic stability.

Sufficiency of (ii) can be obtained by proving that all trajectories will cross the boundary $\Sigma_{m1}$ in finite time. After this finite time, the contraction property of the return map $y_{k+1} = \Lambda y_k$, with $\Lambda < 1$, implies asymptotic stability.

For trajectories starting with initial conditions positioned in one of the regions $S_i$, $i = 1, \ldots, m$, according to Lemma 2.4, there exists a finite time $t_1 \in [0, \infty)$, such that the trajectory is not in the region $S_i$. Since no state jumps can exist in (2.5), the trajectory will therefore cross a boundary $\Sigma_{i,i+1}$ in a finite time $t_s \in [0, t_1]$. Each boundary can only be traversed in one direction. Therefore, only a finite number of regions $S_i$ can be traversed, before $\Sigma_{m1}$ is reached in a finite time. In this finite time, the trajectory does not grow unbounded since the right-hand side of (2.5) is globally Lipschitz.

After the trajectory has reached $\Sigma_{m1}$ for the first time, the trajectory converges to the origin in a spiralling motion, as described by the return map $y_{k+1} = \Lambda y_k$, due to the fact that $\Lambda < 1$. Herewith, sufficiency of (ii) is proven.

Proof of Theorem 2.8. To prove the theorem, first, a scaling law for conewise linear systems is presented in Lemma A.1. Second, a relationship between trajectories of the conewise affine system with $\mu \neq 0$ and conewise linear systems with $\mu = 0$ is given in Lemma A.2. Using that result, Theorem 2.8 is proven.

Lemma A.1. Consider two trajectories of the continuous, conewise linear system:

$$\dot{y} = f(y),$$  
$$f(y) := A_i y, \ y \in S_i, \ i = 1, \ldots, m,$$

where $S_i$, $i = 1, \ldots, m$, are cones. Let $y_0$ and $\tilde{y}_0$ be two initial conditions, that satisfy $\tilde{y}_0 = cy_0$, $c \in [0, \infty)$. If $y(t)$ is a trajectory for the system with $y(0) = y_0$, then $\tilde{y}(t) = cy(t)$ is a trajectory emanating from $\tilde{y}(0) = \tilde{y}_0$.

Proof. The proof is omitted for the sake of brevity and follows from the observation that $f(y)$ is homogeneous of degree 1.

Lemma A.2. Consider the continuous, conewise affine system given in (2.3), with constant $\mu \neq 0$. Suppose this system does not contain visible eigenvectors. Define a Poincaré section for this system at the moments where the trajectory $x(t)$ traverses $\Sigma_{m1}$ with a specified direction (i.e. either from $S_m$ to $S_1$ or vise versa). Define the return map $M : D_M \subset \Sigma_{m1} \rightarrow I_M \subset \Sigma_{m1}$, such that $M(x_k)$
denotes the position of the first crossing of \( x(t) \), \( t > 0 \), for the initial condition \( x(0) = x_k \in \Sigma_m \).

Construct a conewise linear system (A.12) by setting \( \mu = 0 \) in (2.3). Let \( \Lambda \) for the conewise linear system (A.12) be defined by (2.8), (2.9), (2.10) and (2.12). The following two statements hold for the trajectories \( x(t) \) of the conewise affine system (2.3):

(i) When \( \Lambda < 1 \), there exists an \( x_F \in \Sigma_m \) such that \( \| M(x_k) \| < \| x_k \| \), \( \forall x_k \in R := \{ x \in \Sigma_m \mid \| x \| \geq \| x_F \| \} \).

(ii) When \( \Lambda > 1 \), there exists an \( x_F \in \Sigma_m \) such that \( \| M(x_k) \| > \| x_k \| \), \( \forall x_k \in R := \{ x \in \Sigma_m \mid \| x \| \geq \| x_F \| \} \).

Proof. An analytical expression for \( x_F \) is obtained as follows. The conewise affine system (2.3) is considered as a perturbed conewise linear system, where \( \mu b \) is considered as the perturbation. Using the fact that the system is globally Lipschitz, the difference between the trajectory \( x(t) \) of the conewise affine system and the trajectory \( \tilde{y}(t) \) of the conewise linear system (A.12) can be bounded for a given time period. A trajectory of \( \tilde{y}(t) \) from \( \tilde{y}_0 \in \Sigma_m \) is studied, that encircles the origin and crosses the boundary \( \Sigma_m \) after one spiralling motion. We will use an initial condition \( \tilde{y}_0 \), with \( \| \tilde{y}_0 \| \) large enough, such that the trajectories \( \tilde{y}(t) \) and \( x(t) \) emanating from this initial position will encircle the origin and traverse the boundary \( \Sigma_m \), independent of the direction of the bounded perturbation. To find such an initial condition \( \tilde{y}_0 \) we first study an arbitrary initial position \( y_0 \in \Sigma_m \), for which the trajectory \( y(t) \) of (A.12) is followed during one spiralling period.

Let \( T \) be the period time of the spiralling motion of (A.12), as given in [4]. Consider a trajectory \( y(t), t \in [0, T] \), of (A.12) with an arbitrary initial condition \( y_0 \in \Sigma_m \setminus \{0\} \) at time \( t = 0 \). From the stability analysis of the system (A.12), we obtain \( y(T) = \Lambda y_0 \), where \( \Lambda \) is defined in (2.8), (2.9), (2.10) and (2.12).

Define the open set \( C(y_0) \) as:

\[
C(y_0) := \begin{cases} 
(S_m \cup \Sigma_m \cup S_1) \cap \{ y \in \mathbb{R}^2 \mid 0 < \rho_m y < \| y_0 \| \}, \\
(S_m \cup \Sigma_m \cup S_1) \cap \{ y \in \mathbb{R}^2 \mid \rho_m y > \| y_0 \| \},
\end{cases}
\tag{A.14}
\]

Note that \( \rho_m \) is a unit vector. The set \( C(y_0) \) is shown in Fig. A.1. Without loss of generality, assume that for all \( y \in \Sigma_m \setminus \{0\} \) the vector field \( f(y) \) of (A.12) is pointing in direction of \( S_1 \). Here, no generality is lost, since \( f(\cdot) \) is homogeneous, \( \Sigma_m \setminus \{0\} \) is a halfline from the origin and the regions \( \{ S_i \} \) can be renumbered such that \( f(y) \) is pointing in direction of \( S_1 \) for all \( y \in \Sigma_m \setminus \{0\} \). Since \( y(T) \in \Sigma_m \) and \( f(y(T)) \) is pointing in direction of \( S_1 \), one can conclude that there exist small times \( \tau_-, \tau_+ \in (0, T) \) such that the trajectory \( y(t) \) satisfies
Appendix A. Proofs and technical results

Fig. A.1. Graphical representation of the conditions (A.18)-(A.20) for $\Lambda < 1$ and $\Lambda > 1$. The set $C(y_0)$ is shown as the dashed region. Note that the domain around $y(t)$, $t \in [T - \tau_-, T + \tau_+]$ is contained in $C(y_0)$.

the following three conditions:

\begin{align}
    y(t) &\in C(y_0), \forall t \in [T - \tau_-, T + \tau_+], \quad (A.15) \\
    y(T - \tau_-) &\in C(y_0) \cap S_m, \quad (A.16) \\
    y(T + \tau_+) &\in C(y_0) \cap S_1. \quad (A.17)
\end{align}

Here, the following facts are used; the trajectory $y(t)$ is continuous in time, the point $y(T) \in C(y_0)$ and the trajectory traverses $\Sigma_{m1}$ from $S_m$ towards $S_1$ at the time instant $t = T$. Since $C(y_0)$, $S_m$ and $S_1$ are open sets, there exists an $\epsilon \in (0, \infty)$, such that for all vectors $\delta$, with $\|\delta\| \leq \epsilon$ the conditions

\begin{align}
    y(t) + \delta &\in C(y_0), \forall t \in [T - \tau_-, T + \tau_+], \quad (A.18) \\
    y(T - \tau_-) + \delta &\in C(y_0) \cap S_m, \quad (A.19) \\
    y(T + \tau_+) + \delta &\in C(y_0) \cap S_1. \quad (A.20)
\end{align}

are satisfied. These conditions are illustrated in Fig. A.1. A new initial condition $\tilde{y}_0 = ky_0$, with $k$ a positive constant, is chosen for the system (A.12). Application of Lemma A.1 yields that $\tilde{y}(t) = ky(t)$. We introduce $\zeta = k\delta$, such that combination of the conditions (A.18)-(A.20) with definition (A.14) yields:

\begin{align}
    \tilde{y}(t) + \zeta &\in C(\tilde{y}_0), \forall t \in [T - \tau_-, T + \tau_+], \quad (A.21) \\
    \tilde{y}(T - \tau_-) + \zeta &\in C(\tilde{y}_0) \cap S_m, \quad (A.22) \\
    \tilde{y}(T + \tau_+) + \zeta &\in C(\tilde{y}_0) \cap S_1. \quad (A.23)
\end{align}

for the trajectory $\tilde{y}(t)$ with initial condition $\tilde{y}_0 = ky_0$ and for all $\zeta$ with $\|\zeta\| \leq k\epsilon$.

Now, consider the affine term of (2.3), i.e. $\mu b$, as a constant disturbance of the system (A.12). We study the trajectory $x(t)$ of (2.3) and the trajectory $\tilde{y}(t)$...
of (A.12), both with the same initial condition \( x_0^k = \tilde{y}_0 = ky_0 \) at \( t = 0 \). Since system (2.3) is globally Lipschitz with Lipschitz constant \( L \), Theorem 3.4 in [91] can be applied, yielding

\[
\|\tilde{y}(t) - x(t)\| \leq \frac{|\mu|}{L} \left( e^{L(T+\tau_+)} - 1 \right), \quad \forall t \in [0, T + \tau_+], \tag{A.24}
\]

where \( \mu \) is given in (2.3), \( \|b\| = 1 \) is used, \( T \) the period of the spiralling motion of (A.12) and \( \tau_+ \) is introduced above. If we choose \( \tilde{y}_0 \in R := \left\{ \tilde{y}_0 \in \mathbb{R}^2 | \tilde{y}_0 = ky_0, \quad k \geq \frac{|\mu|}{cL} \left( e^{L(T+\tau_+)} - 1 \right) \right\} \), with \( y_0 \) and \( \epsilon \) introduced above, then (A.24) yields \( \|x(t) - \tilde{y}(t)\| \leq k\epsilon, \forall t \in [0, T + \tau_+] \) and (A.21)-(A.23) yield:

\[
x(t) \in C(x_0^k), \forall t \in [T - \tau_-, T + \tau_+], \tag{A.25}
\]

\[
x(T - \tau_-) \in C(x_0^k) \cap \mathcal{S}_m, \tag{A.26}
\]

\[
x(T + \tau_+) \in C(x_0^k) \cap \mathcal{S}_1, \tag{A.27}
\]

where we used \( x_0^k = \tilde{y}_0 \). From these conditions we conclude that the continuous trajectory \( x(t) \) of (2.3) crosses \( \Sigma_{m1} \cap C(x_0^k) \) at a time \( t_c \in (T - \tau_-, T + \tau_+) \). The definition of the map \( M \) in Lemma A.2 yields \( M(x_0^k) = x(t_c) \in \Sigma_{m1} \cap C(x_0^k) \), where \( x(t) \) denotes the trajectory of (2.3) with initial condition \( x_0^k \). For \( \Lambda < 1 \), the intersection \( \Sigma_{m1} \cap C(x_0^k) \) equals \( \{ x \in \Sigma_{m1} | ||x|| \in (0, ||x_0^k||) \} \). For \( \Lambda > 1 \), the intersection \( \Sigma_{m1} \cap C(x_0^k) \) equals \( \{ x \in \Sigma_{m1} | ||x|| > ||x_0^k|| \} \).

Consequently, the set \( R \), as presented under (A.24), satisfies the conditions of the lemma. The set \( R \) can be written as \( R = \{ x \in \Sigma_{m1} | ||x|| \geq ||x_F|| \} \), with

\[
x_F := \frac{|\mu|}{cL} \left( e^{L(T+\tau_+)} - 1 \right) y_0, \quad y_0 \in \Sigma_{m1} \setminus \{0\}. \tag{A.28}
\]

Hence, for \( \Lambda < 1 \), we obtain \( ||M(x_0)|| < ||x_0||, \forall x_0 \in R = \{ x \in \Sigma_{m1} | ||x|| \geq ||x_F|| \} \). For \( \Lambda > 1 \), we obtain \( ||M(x_0)|| > ||x_0||, \forall x_0 \in R = \{ x \in \Sigma_{m1} | ||x|| \geq ||x_F|| \} \). \( \square \)

Using Lemma A.2, we will now prove Theorem 2.8. Consider a trajectory from the initial condition \( x_0 \in R = \{ x \in \Sigma_{m1} | ||x|| \geq ||x_F|| \} \), with \( x_F \) derived in Lemma A.2, and assume \( \Lambda \neq 1 \). Define the return map \( M \) on \( \Sigma_{m1} \) according to Lemma A.2, and choose \( x_F \) as given in that lemma. The return map of the trajectory through the point \( x_0 \) satisfies \( ||M(x_0)|| \neq ||x_0|| \).

It remains to be proven that the trajectory from \( M(x_0) \) can not return to \( x_0 \) after the first crossing of \( \Sigma_{m1} \) at position \( M(x_0) \). Without loss of generality, suppose the trajectory traverses \( \Sigma_{m1} \) from \( \mathcal{S}_m \) towards \( \mathcal{S}_1 \) at \( t = 0 \). The vector field (2.3) on the boundary \( \Sigma_{m1} \) can be described by \( \dot{x} = A_1 x + \mu b, \ x \in \Sigma_{m1} \), which is autonomous and affine. Hence, all trajectories traversing \( [x_0, M(x_0)] \) cross this line from \( \mathcal{S}_m \) to \( \mathcal{S}_1 \). Since the trajectory from \( M(x_0) \) can not cross his own trajectory or \( [x_0, M(x_0)] \), it not return to \( x_0 \). No closed orbit can exist that contains a point \( x_0 \in R \). \( \square \)
Proof of Theorem 2.9. Consider a visible eigenvector \( v \in \tilde{S}_i \), \( i = 1, \ldots, m \), corresponding to the real eigenvalue \( \lambda_u \) of a system matrix \( A_i \). Define a unit vector \( n_{i,i+1} \) normal to the boundary \( \Sigma_{i,i+1} \) and a unit vector \( n_{i-1,i} \) normal to the boundary \( \Sigma_{i-1,i} \), where both vectors point towards \( S_i \). This implies that \( S_i = \{ x \in \mathbb{R}^2 | x^T n_{i,i+1} > 0 \land x^T n_{i-1,i} > 0 \} \).

Now, we will prove that there exists a scalar \( c_H \in [0, \infty) \), such that the halfline \( H = \{ x \in \mathbb{R}^2 | x = -\mu A_i^{-1} b + c v, c \in [c_H, \infty) \} \) is included completely in \( \tilde{S}_i \), that is the closure of \( S_i \). Taking the inner product of the vectors \( x = -\mu A_i^{-1} b + c v, c \in (0, \infty) \) with \( n_{i-1,i} \) and \( n_{i,i+1} \), we find that \( x^T n_{i-1,i} > 0 \) if \( c > \frac{\mu n_{i-1,i}^T A_i^{-1} b}{n_{i-1,i}^T v} \) and \( x^T n_{i,i+1} > 0 \) if \( c > \frac{\mu n_{i,i+1}^T A_i^{-1} b}{n_{i,i+1}^T v} \). Both denominators are nonzero and positive, since the visible eigenvector \( v \) is not contained in either of the boundaries \( \Sigma_{i-1,i} \) or \( \Sigma_{i,i+1} \), as stated in the theorem, and \( v \in \tilde{S}_i \) according to Definition 2.3. Consequently, the set \( H \) is contained in the closure of \( S_i \), i.e. \( H \subset \tilde{S}_i \), when \( c_H = \max(\frac{\mu n_{i-1,i}^T A_i^{-1} b}{n_{i-1,i}^T v}, \frac{\mu n_{i,i+1}^T A_i^{-1} b}{n_{i,i+1}^T v}) \).

Consider a trajectory of (2.3) with initial position \( x_0 \in H \). Any position in \( H \) can be written as \( x = -\mu A_i^{-1} b + c v \). However, since \( v \) is an eigenvector, we obtain \( \dot{x} = c \lambda_u v \). This vector field is tangent to the halfline \( H \). This implies, that trajectories can only enter or leave the halfline \( H \) at \( x = -\mu A_i^{-1} b + c_H v \), the end point of the halfline. When the vector field at this point is into (out of) \( H \), then \( H \) is a positively (negatively) invariant set.

Uniqueness of solutions of (2.3) excludes closed orbits inside the halfline \( H \). In addition, a closed orbit cannot traverse \( H \), since in that case, the closed orbit should enter and leave \( H \). This is not possible, since \( H \) is either a positively or negatively invariant set. No closed orbits can exist that either are contained in \( H \), or contain a point in \( H \).

Proof of Theorem 2.10. The proof of this theorem is given in two parts. In Part 1, the theorem is proven for all \( \nu \in N \setminus \{0\} \). Subsequently, in Part 2, the case \( \nu = 0 \) is discussed.

Part 1
In this part we will first discuss the existence of equilibria for \( \nu \neq 0 \), subsequently their local stability properties are discussed. Let \( J \subseteq \{1, \ldots, m\} \) contain the indices, such that the domains \( \tilde{D}_i, i \in J \) contain the origin. According to Assumption 2.1, \( F_i(0,0) = 0 \), \( \forall i \in J \). With Assumption 2.4, we can apply the Implicit Function Theorem, cf. [122], which yields that for each \( i \in J \) locally there exists a smooth path \( x_{i,1}(\nu) \), that satisfies \( F_i(x_{i,1}(\nu), \nu) = 0 \). Note that this path may be positioned outside \( \tilde{D}_i \). When \( x_{i,1}(\nu) \in \tilde{D}_i \) for given \( \nu \in N \), then an equilibrium of (2.1) exists in this domain. Assumption 2.5 excludes the possibility that there exists an equilibrium in the domain \( D_i \) for \( \nu \in N \) when \( D_i \) is a cusp-shaped region at the origin. (Note that a cusp-shaped region is not represented by a cone in (2.3)). Therefore, we may restrict ourselves to the
paths of possible equilibria \( x_{i,1}(\nu), \ i \in \{1, \ldots, m\} \subseteq J \).

Differentiating \( F_i(x_{i,1}(\nu), \nu) = 0 \) with respect to \( \nu \), we obtain with Assumptions 2.2 and 2.4:

\[
\frac{\partial F_i}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} + \frac{\partial F_i}{\partial x} \bigg|_{(x,\nu)=(0,0)} \frac{\partial x_{i,1}(\nu)}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} = 0,
\]

\[
\frac{\partial x_{i,1}(\nu)}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} = -A_i^{-1}b. \quad (A.29)
\]

To prove the first statement of the theorem for \( \nu, \mu \neq 0 \), without loss of generality we may assume there exists a positive \( \nu^* \), such that \( x_{i,1}(\nu) \in D_i \) for all \( \nu \in (0, \nu^*) \) for a given \( i \in \{1, \ldots, m\} \), where an open set \( D_i \) is chosen since we adopted Assumption 2.5. This implies that \( \frac{\partial x_{i,1}(\nu)}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} = -A_i^{-1}b \in D_i \), and since \( D_i \) is locally approximated by \( S_i \) and \( \mu = \nu > 0 \), we obtain \( -\mu A_i^{-1}b \in S_i \), for \( \mu > 0 \). In combination with the fact that equilibria of (2.3) are positioned on \( x_{i,2}(\mu) = -\mu A_i^{-1}b \), we obtain the following result: when (2.1) contains an equilibrium in \( D_i \) for an \( i \in \{1, \ldots, m\} \) and \( \nu \) in a range \((0, \nu^*)\), then (2.3) contains an equilibrium point in \( S_i \) for all \( \mu > 0 \).

To prove the converse statement, we note that with Assumption 2.5, all paths of equilibria of (2.3) will be positioned in a cone \( S_i \), \( i = 1, \ldots, m \). Therefore, we assume without loss of generality that (2.3) contains an equilibrium path \( x_{i,2}(\mu) \in S_i \) for \( \mu > 0 \) and given \( i \in \{1, \ldots, m\} \), such that \( f_i(x_{i,2}(\mu), \mu) = 0 \). Solving this equation yields \( x_{i,2}(\mu) = -\mu A_i^{-1}b \in S_i, \ \forall \mu > 0 \). Using (A.29), we obtain: \( \frac{\partial x_{i,1}(\nu)}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} \in S_i \), where \( x_{i,1}(\nu) \) denotes a path such that \( F_i(x_{i,1}(\nu), \nu) = 0 \) holds. The domain \( D_i \) is locally approximated by \( S_i \), such that we have: \( \frac{\partial x_{i,1}(\nu)}{\partial \nu} \bigg|_{(x,\nu)=(0,0)} \in D_i \). Since \( x_{i,1}(\nu) \) is a smooth path and \( D_i \) is an open set, there exists a \( \nu^* > 0 \) such that \( x_{i,1}(\mu) \in D_i, \forall \nu \in (0, \nu^*) \).

For \( \nu = \mu \neq 0 \), the final statement of Theorem 2.10, which states that the stability properties of the equilibria of (2.1) and (2.3) are identical, is obtained from the Andronov-Pontryagin condition, cf. Theorem 2.5 of [95]. Without loss of generality, we assume there exists an equilibrium of (2.1), that follows an equilibrium path \( x_{i,1}(\nu) \in D_i \) for \( \nu \in (0, \nu^*) \subseteq \tilde{N}, \nu^* > 0 \) with \( i \in \{1, \ldots, m\} \). We define \( \hat{F}_i(x, \nu) := F_i(x-x_{i,1}(\nu), \nu) \) for \( \nu \in (0, \nu^*) \), which is a smooth function, such that \( \hat{x} = \hat{F}_i(x, \nu) \) describes the dynamics of (2.1) in a neighbourhood \( \tilde{N}(\nu) \) near the equilibrium. For each \( \nu \in (0, \nu^*) \), \( \tilde{N}(\nu) \ni 0 \) is chosen such, that for all \( \hat{x} \in \tilde{N}(\nu), \hat{x} + x_{i,1}(\nu) \) lies inside \( D_i \).

According to the Andronov-Pontryagin condition the system \( \hat{x} = \hat{F}_i(x, \nu) \) is structurally stable in \( \tilde{N}(\nu) \). Therefore, there exists an \( \epsilon > 0 \) such that for all vector fields \( G_i \) that satisfy

\[
\sup_{x \in \tilde{N}(\nu)} \left( \| \hat{F}_i(x) - G_i(x) \| + \left\| \frac{\partial \hat{F}_i}{\partial x} - \frac{\partial G_i}{\partial x} \right\| \right) \leq \epsilon, \quad (A.30)
\]
the systems \( \dot{x} = G_i(x) \) and \( \dot{x} = \bar{F}_i(x, \nu) \) are topologically equivalent. Choosing \( G_i(x) = A_i x \), and observing that \( \bar{F}_i(x, \nu) \) and \( x_{i,1}(\nu) \) are smooth functions satisfying \( x_{i,1}(0) = 0 \) and \( \frac{\partial \bar{F}_i}{\partial x}(x, \nu) = \frac{\partial G_i}{\partial x}(x, \nu) = A_i \), we can choose \( \nu \) and \( \tilde{\nu}(\nu) \) small enough such that (A.30) is satisfied. Since \( \dot{x} = G_i(x) \) describes the dynamics in the neighbourhood of an equilibrium of (2.3), the systems (2.1) and (2.3) near their equilibria are locally topologically equivalent for \( \nu \in (0, \nu^*) \) when \( \nu^* \) is chosen small enough. In addition, the stability properties of the equilibria of (2.3) and (2.1) are equal. Hence, the theorem is proven for the case \( \nu \neq 0 \).

**Part 2**

In this part, we will prove the theorem for the case \( \nu = 0 \). Existence of an isolated equilibrium at the origin of (2.1) for \( \nu = 0 \) is given by Assumption 2.1. By construction, an equilibrium at the origin exist in (2.3) for \( \mu = 0 \), which is isolated since all matrices \( A_i \) are invertible, cf. Assumption 2.4. It remains to be proven that the local stability properties of the equilibrium at the origin of (2.1) with \( \nu = 0 \) and (2.3) with \( \mu = 0 \) are equal.

By Assumption 2.4, either a stable or unstable manifold of this equilibrium exist, or all trajectories near an equilibrium of (2.1) encircle this point. Therefore, we will study both cases, yielding Lemmas A.3 and A.4, respectively.

We study manifolds of the non-differentiable systems (2.1) and (2.3) similar to the stable and unstable manifolds of nodes and saddle points in smooth dynamical systems. However, we allow these manifolds to be defined only in a domain \( \bar{S}_i \), \( i = 1, \ldots, m \) or \( \bar{D}_i \), \( i = 1, \ldots, \bar{m} \).

**Lemma A.3.** Under Assumptions 2.1 and 2.4, the equilibrium at the origin of (2.1) with \( \nu = 0 \) has a stable (unstable) manifold if and only if the equilibrium at the origin of (2.3) with \( \mu = 0 \) has a stable (unstable) manifold, that is tangent to the manifold of (2.1) at the origin.

**Proof.** First, we will prove the necessity part of the lemma. By Assumption 2.1, an equilibrium exists at the origin for the non-differentiable system (2.1). Assume this equilibrium of (2.1) has a stable (unstable) manifold \( C \) that emanates from the origin towards a domain \( \bar{D}_i, i = 1, \ldots, \bar{m} \). We distinguish two cases. In Case I, we will prove that this manifold of (2.1) is represented in (2.3) when the manifold emanates from the origin towards a domain \( \bar{D}_i, i = 1, \ldots, m \), that does not form a cusp. Subsequently, we will prove this statement when the manifold emanates into a cusp-shaped domain in Case II, which, after a possible renumbering of the domains \( \bar{D}_i \), can be denoted with \( \bar{D}_n, n \in \{ m + 1, \ldots, \bar{m} \} \).

To prove necessity in Case I, we assume that the manifold \( C \in \bar{D}_i, i = 1, \ldots, m \), is not positioned in a cusp of boundaries near the origin. Then, the domain \( \bar{D}_i \) in (2.1) is represented by the cone \( \bar{S}_i \) in (2.3). When \( C \) is emanating tangentially to a boundary \( C_{ij} \) of (2.1), then we choose the index \( i \) such that on the intersection of the manifold \( C \) with a neighbourhood of the origin, the
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dynamics of system (2.1) is described by \( \dot{x} = F_i(x, 0) \). This implies that there exists a \( \lambda \) such that \( \frac{\partial F_i}{\partial x}(x, \nu) = (0, 0) \) \( r = \lambda r \), where \( r \) is the vector tangent to \( C \) at the origin, with the direction chosen to satisfy \( r \in D_i \). Note that \( \lambda < 0 (\lambda > 0) \) for the stable (unstable) manifold. Since \( A_i \) is defined as \( \frac{\partial F_i}{\partial x}(x, \nu) = (0, 0) \), \( A_i \) therefore has eigenvector \( r \). Since \( \bar{D}_i \) is locally approximated with \( \bar{S}_i \), the eigenvector \( r \) is visible in \( \bar{S}_i \). This implies that the set \( c := \{ x \in \mathbb{R}^2 : x = sr, s \in [0, \infty) \} \) is a manifold of the conewise affine system (2.3), on which the dynamics is given by \( \dot{x} = A_i x = \lambda x \), such that a stable (unstable) manifold of (2.3) corresponds to a stable (unstable) manifold of (2.1).

To prove necessity in Case II, we assume that the manifold \( C \) is emanating from the origin towards a cusp of boundaries into a domain \( D_n \), \( n \in \{ m + 1, \ldots, \bar{m} \} \) of (2.1), and \( r \) is the vector pointing into this cusp from the origin. Then, there exists a \( \lambda \) such that \( \frac{\partial F_n}{\partial x}(x, \nu) = (0, 0) \) \( r = \lambda r \). Without loss of generality, we assume that we do not have two or more adjoining cusp-shaped regions. Note that \( \lambda < 0 (\lambda > 0) \) for the stable (unstable) manifold. Since the vector field of (2.1) is continuous at each boundary \( C_{ij} \), we observe that \( \frac{\partial F_i}{\partial x} \rho_{ci,j}(x) = \frac{\partial F_j}{\partial x} \rho_{cj,i}(x), \forall x \in C_{ij} \), where \( \rho_{ci,j}(x) \) denotes a vector tangent to \( C_{ij} \) at the point \( x \). Therefore, \( \frac{\partial F_i}{\partial x}(x, \nu) = (0, 0) \) \( r = \lambda r \) implies that \( \frac{\partial F_i}{\partial x}(x, \nu) = (0, 0) \) \( r = \lambda r \), with the indices \( i, j \in \{1, \ldots, m \} \) chosen such that the domains \( D_i \) and \( D_j \) have boundaries tangent to the cusp at the origin, although the domains \( D_i \) and \( D_j \) do not form a cusp at the origin. Since \( A_i \) and \( A_j \) are, respectively, defined as \( \frac{\partial F_i}{\partial x}(x, \nu) = (0, 0) \) and \( \frac{\partial F_j}{\partial x}(x, \nu) = (0, 0) \), both \( A_i \) and \( A_j \) have eigenvector \( r \). The domains \( \bar{D}_i \) and \( \bar{D}_j \) are locally approximated with, respectively, \( \bar{S}_i \) and \( \bar{S}_j \), such that the eigenvector \( r \) is visible and lies on the boundary \( \Sigma_{ij} \) between \( S_i \) and \( S_j \). This implies that the set \( c := \{ x \in \mathbb{R}^2 : x = sr, s \in [0, \infty) \} \) is a manifold of the conewise affine system (2.3), on which the dynamics is given by \( \dot{x} = A_i x = A_j x = \lambda x \), such that a stable (unstable) manifold of (2.3) corresponds to the stable (unstable) manifold of (2.1). In both cases, the necessity part of the lemma is obtained.

Now, we will prove the sufficiency part of the lemma by assuming that there exists a stable (unstable) manifold \( c \in \bar{S}_i \), \( i = 1, \ldots, m \), of the system (2.3). Since the dynamics in this cone is given by \( \dot{x} = A_i x \), we can denote this manifold with \( c = \{ x \in \mathbb{R}^2 : x = sr, s \in [0, \infty) \} \), where \( r \in \bar{S}_i \) is an eigenvector of \( A_i \). Let \( \lambda \) be the eigenvalue corresponding to this eigenvector. Since \( c \) is a stable (unstable) manifold, we obtain \( \lambda < 0 (\lambda > 0) \). Again, two cases are distinguished. First, we will study the case where \( c \) is positioned in an open cone \( S_i \), subsequently we will prove the sufficiency part of the theorem in case \( c \) is positioned on a boundary.
To prove sufficiency in the first case, we assume that the stable (unstable) manifold \( c \) is positioned in an open cone \( S_i \). Then the Hartman-Grobman Theorem, [74], guarantees that the system \( \dot{x} = F_i(x,0) \) locally has the same stable and unstable manifolds as the system \( \dot{x} = A_i x \). Therefore, the vector field \( F_i(x,0) \) satisfies \( \frac{\partial F_i}{\partial x}(x,0) = r = \lambda r \). Since the trajectories of (2.3) in the cone \( S_i \) coincide with the trajectories of \( \dot{x} = A_i x \) and the trajectories of (2.1) in the domain \( D_i \) coincide with the trajectories of \( \dot{x} = F_i(x,0) \), the non-differentiable system (2.1) has a stable (unstable) manifold corresponding to \( c \) if \( \lambda < 0 \) (\( \lambda > 0 \)).

To prove sufficiency in the second case, we assume the stable (unstable) manifold of (2.3) lies on a boundary \( \Sigma_{ij} \). Recall that \( \rho_{ij} \) denotes the vector tangent to \( \Sigma_{ij} \), pointing towards this ray. Let \( K \subseteq \{1, \ldots, \tilde{m}\} \) denote the set of indices, such that \( D_n \) has a boundary tangent to \( \Sigma_{ij} \) at the origin if and only if \( n \in K \). (The domain \( D_n \) may or may not be a cusp-shaped region). We assumed the existence of a stable (unstable) manifold \( c \) of (2.3) on a boundary \( \Sigma_{ij} \). This implies that there exists a \( \lambda < 0 \) (\( \lambda > 0 \)) such that \( A_i \rho_{ij} = \lambda \rho_{ij} = \frac{\partial F_i}{\partial x}(x,0) \rho_{ij} \). For the sake of contradiction, suppose (2.1) has no stable (unstable) manifold tangent to \( \Sigma_{ij} \). Then \( \forall n \in K, \forall \lambda_n < 0 \) (\( \lambda_n > 0 \)) : \( \frac{\partial F_i}{\partial x}(x,\nu)=(0,0) \rho_{ij} \neq \lambda_n \rho_{ij} \). However, since \( F(x,\nu) \) is continuous, we obtain \( \frac{\partial F_i}{\partial x}(x,\nu)=(0,0) \rho_{ij} = \frac{\partial F_i}{\partial x}(x,\nu)=(0,0) \rho_{ij} \), \( \forall n \in K \). Hence, \( A_i = \frac{\partial F_i}{\partial x}(x,\nu)=(0,0) \) has no eigenvalue \( \lambda < 0 \), (\( \lambda > 0 \)) with the eigenvector \( \rho_{ij} \), contradicting the existence of a stable (unstable) manifold with eigenvalue \( \lambda \) in (2.3). A contradiction is obtained, such that (2.1) has a stable (unstable) manifold, which is tangent to \( \Sigma_{ij} \) at the equilibrium. In both cases, the sufficiency part of the lemma is proven.

**Lemma A.4.** Under Assumptions 2.1, 2.4, and 2.6, all trajectories of (2.1) with \( \nu = 0 \) in a neighbourhood around the equilibrium at the origin are spiralling around the origin and converge towards the origin for \( t \to \infty \) \( (t \to -\infty) \) if and only if all trajectories of (2.3) with \( \mu = 0 \) are spiralling around the origin and converge to the origin for \( t \to \infty \) \( (t \to -\infty) \).

**Proof.** First we prove the sufficiency part of the statement, subsequently the necessity part of the statement is proven. Assume a spiralling motion exists in the system (2.3). Since system (2.3) is Lipschitz continuous, the time-reversed system can be studied, and since, in addition, the chosen coordinate frame may be mirrored along a coordinate axis, we may assume, without loss of generality, that the spiralling motion is counter clockwise and trajectories of (2.3) are converging to the origin for \( t \to \infty \). We consider a trajectory of (2.3) starting at \( t = 0 \) from a point \( x_0 \) which is positioned on the positive vertical axis. There exists a time \( T \) such that the trajectory \( x(t) \) of (2.3) encircles the origin \( O \) once and returns to a point \( x(T) \) on the interval of the line segment \([O, x_0]\). We will compare the trajectories \( x(t) \) of (2.3) with \( \tilde{x}(t) \) of (2.1), where both are starting from \( x(0) = \tilde{x}(0) = x_0 \). With the same reasoning as used in the proof of Theorem 2.8,
we can find finite $\tau, E > 0$, such that $\tilde{x}(t)$ traverses the interior of $[O, x_0]$ at a time $t_0 \in (T - \tau, T + \tau)$ if $\|x(t) - \tilde{x}(t)\| \leq E, \forall t \in [0, T + \tau]$. Let $L$ be a Lipschitz constant of both $F(x, 0)$, given in (2.1) and $f(x, 0)$, given in (2.3). Theorem 3.4 of [91] guarantees that the requirement $\|x(t) - \tilde{x}(t)\| \leq E, \forall t \in [0, T + \tau]$ is met when $\|F(x, 0) - f(x, 0)\| < D := \frac{LE}{2(\triangle S) - 1}, \forall x \in R(x_0), \text{where the open, bounded set } R(x_0) \text{ is chosen such that the previously mentioned trajectories satisfy } x(t), \tilde{x}(t) \in R(x_0), \forall t \in [0, T + \tau].$ Next, we will prove that, by choosing $x_0$ small, $\|F(x, 0) - f(x, 0)\| < D$ holds for all $x \in R(x_0).

From Lemma A.1, we find that trajectories of the conewise affine system with $\mu = 0$ scale linearly with initial conditions, such that $E$ and $R(x_0)$ can be chosen such that they scale linearly with $\|x_0\|$. We will prove that $\|F(x, 0) - f(x, 0)\| = O(L\|x\|^2, \|x\|^2)$, such that by decreasing $\|x_0\|, \|F(x, 0) - f(x, 0)\| < D, \forall x \in R(x_0)$ can be satisfied. By the Taylor expansion of $F_i(x, 0)$, we obtain:

$$\|F_i(x, 0) - f_i(x, 0)\| = O(\|x\|^2). \tag{A.31}$$

Define $D_i \triangle S_i := (D_i \cup S_i) \setminus (D_i \cap S_i)$, which is given in Fig. A.2. The width $\|w\|$ of this set, as graphically defined in Fig. A.2, is locally quadratic with $\|x\|$. Since on one of the boundaries of $D_i \triangle S_i$, the dynamics is described by $\dot{x} = F_i(x, 0)$, we apply the Lipschitz property of (2.1) to obtain:

$$\|F(x, 0) - F_i(x, 0)\| = O(L\|x\|^2), \forall x \in D_i \triangle S_i. \tag{A.32}$$

Combination of (A.31) with (A.32) yields:

$$\|F(x, 0) - f_i(x, 0)\| = O(L\|x\|^2, \|x\|^2), \forall x \in D_i \triangle S_i \cup (D_i \cap S_i), \tag{A.33}$$

such that we obtain

$$\|F(x, 0) - f(x, 0)\| = O(L\|x\|^2, \|x\|^2), \forall x \in \mathbb{R}^2. \tag{A.34}$$

By choosing $x_0$ small enough, $R(x_0)$ becomes small enough, such that $\|F(x, 0) - f(x, 0)\| < D, x \in R(x_0)$. Given this set $R(x_0)$, we conclude that the
trajectory $\dot{x}(t)$ from $x_0 \in R(x_0)$, starting from the positive vertical axis, crosses the interior of the line $[O, x_0]$ in the time interval $[T - \tau, T + \tau]$. Consequently, the non-differentiable system (2.1) exhibits a spiralling motion, and trajectories of (2.1) in the neighbourhood $R(x_0)$ do converge to the origin for $t \to \infty$, when $\|x_0\|$ is chosen small enough.

The necessity part of the statement is equivalent with the statement that the absence of a stable (unstable) spiralling motion of (2.3) with $\mu = 0$ excludes a stable (unstable) spiralling motion of (2.1) with $\nu = 0$. To prove the latter statement, assume the conewise linear system (2.3) with $\mu = 0$ does not exhibit spiralling motion, converging to the origin for $t \to \infty$ ($t \to -\infty$). Considering Assumptions 2.4 and 2.6, system (2.3) should either exhibit spiralling motion converging to the origin for $t \to -\infty$ ($t \to \infty$), or should contain visible eigenvectors. In the first case, the spiralling motion, converging to the origin for $t \to -\infty$ ($t \to \infty$), implies that in a neighbourhood of the origin, a spiralling motion of (2.1) with $\nu = 0$ exist, and trajectories converge to the origin for $t \to -\infty$ ($t \to \infty$). This follows directly from the sufficiency part of the proof. In case visible eigenvectors exist in (2.3) with $\mu = 0$, Lemma A.3 guarantees that a manifold exist in (2.1) for $\nu = 0$, emanating from the origin. This manifold excludes a spiralling motion of (2.1) for $\nu = 0$. Hence, we have proven that the absence of a stable (unstable) spiralling motion of (2.3) with $\mu = 0$ excludes a stable (unstable) spiralling motion of (2.1) with $\nu = 0$.

Given Assumption 2.4 an equilibrium is unstable when either an unstable manifold exist, or a diverging spiralling motion occurs. An equilibrium point is locally asymptotically stable when a converging spiralling motion exist or when a stable manifold exist, and no unstable manifolds. Trajectories encircling an equilibrium, which are not converging to this point for $t \to \infty$ or $t \to -\infty$, are, in a neighbourhood around the origin, excluded by Assumption 2.6. Therefore, application of Lemma A.3 guarantees that at $\nu = \mu = 0$, the stability properties of the equilibrium at the origin of (2.1) and (2.3) are equal when visible eigenvectors exist in (2.1). When no visible eigenvectors exist in (2.1), then the stability properties of the equilibria at the origin of (2.1) and (2.3) are equal according to Lemma A.4.

\textbf{Proof of Theorem 2.11.} To prove Theorem 2.11, first we will introduce a new coordinate system, and derive the technical Lemma A.5 for the obtained system. Subsequently, in Lemma A.6 we will state an intermediate result about the number of limit cycles, which will be used to prove the theorem.

In a new coordinate system $z = \frac{x}{\mu}$, defined for $\mu \neq 0$, the conewise affine system (2.3) is represented as:

\begin{align*}
\dot{z} &= \tilde{f}(z), \quad \text{(A.35)} \\
\tilde{f}(z) &= A_i z + b, \quad z \in S_i. \quad \text{(A.36)}
\end{align*}
Clearly, this transformation does not change the existence and stability of limit sets of the conewise affine system. Expressing (2.1) in the coordinates $z = \frac{x}{\nu}$, defined for $\nu \neq 0$, we obtain:

$$\tilde{z} = \tilde{F}(z, \nu),$$

(A.37)

$$\tilde{F}(z, \nu) = \tilde{F}_i(z, \nu) := \frac{1}{\nu} F_i(\nu z, \nu), \ z \in \tilde{D}_i,$$

(A.38)

where $\tilde{D}_i := \{z \in \mathbb{R}^2 | z = \frac{x}{\nu}, x \in D_i\}$.

For the difference between $\tilde{f}(z)$ in (A.36) and $\tilde{F}(z, \nu)$ in (A.38), we obtain the following lemma.

**Lemma A.5.** Consider $\tilde{F}(z, \nu)$ as given in (A.38) and $\tilde{f}(z)$ as given in (A.36). For every domain $R(\nu) := \{z \in \mathbb{R}^2 ||z|| \leq c|\nu|, c > 0\}$, and for all $D > 0$ there exists a $\tilde{\nu} > 0$, such that

$$\|\tilde{F}(z, \nu) - \tilde{f}(z)\| \leq D, \ \forall \nu \in (-\tilde{\nu}, \tilde{\nu}), \ \forall z \in R(\nu).$$

(A.39)

**Proof.** We observe that

$$\|\tilde{F}_i(z, \nu) - \tilde{f}_i(z)\| = \frac{1}{\nu} \mathcal{O}(\nu^2 \|z\|^2, \nu^2 \|z\|, \nu^2),$$

(A.40)

$$= \mathcal{O}(\nu^3 c^2, \nu^3 c, \nu), \ \forall z \in R(\nu),$$

(A.41)

since $f_i$ is a first-order approximation of $F_i$, such that $\|F_i(x, \nu) - f_i(x, \nu)\| = \mathcal{O}(\|x\|^2, \nu \|x\|, \nu^2)$.

The set $D_i \Delta S_i := (D_i \cup S_i) \setminus (D_i \cap S_i)$ is given in Fig. A.2. The width $\|w\|$ of this set, as graphically defined in Fig. A.2, is locally quadratic with $\|x\|$. At least at one of the boundaries of $D_i \Delta S_i$, we know that $F_i = F_j$, due to continuity of the vector field $F$, where $j \in \{1, \ldots, m\}$ is chosen such that $D_j$ adjoins $D_i$. Hence, we obtain $\|F_j(x, \nu) - F_i(x, \nu)\| = \mathcal{O}(\|w\|) = \mathcal{O}(\|x\|^2)$, where (2.1) and (2.3) both have a Lipschitz constant $L$. This implies:

$$\|\tilde{F}_j(z, \nu) - \tilde{F}_i(z, \nu)\| = \mathcal{O}(\nu^3 L c^2), \ \forall z \in (D_i \Delta S_i) \cap S_i \cap R(\nu),$$

(A.42)

where $\tilde{D}_i \Delta S_i := (\tilde{D}_i \cup S_i) \setminus (\tilde{D}_i \cap S_i)$.

Note that the state space can be partitioned as: $\mathbb{R}^2 = \bigcup_{i=1,\ldots,m} (\tilde{D}_i \cap S_i) \cup \bigcup_{i=1,\ldots,m} (\tilde{D}_i \Delta S_i)$. This implies that combination of (A.41) and (A.42) yields

$$\|\tilde{F}(z, \nu) - \tilde{f}(z)\| = \mathcal{O}(\nu^3 c^2, \nu^2 c, \nu, \nu^3 L c^2), \ \forall z \in R(\nu),$$

such that for all $D > 0$ and bounded domain $R(\nu)$ defined in the lemma, we can find a $\tilde{\nu} > 0$ such that

$$\|\tilde{F}(z, \nu) - \tilde{f}(z)\| \leq D, \ \forall \nu \in (-\tilde{\nu}, \tilde{\nu}), \forall z \in R(\nu).$$

□
Appendix A. Proofs and technical results

Fig. A.3. (a) Stable limit cycle $\gamma$ of (A.43) with two nearby trajectories and Poincaré section $P$. (b) Invariant set $I$ of (A.44), depicted shaded, that is bounded by trajectories of (A.44) from points $a$ and $e$.

Lemma A.6. If the dynamical system

$$\dot{z} = f(z), \quad (A.43)$$

exhibits a stable or unstable limit cycle, denoted $\gamma \in \Theta$, with $f : \mathbb{R}^2 \to \mathbb{R}^2$ a Lipschitz continuous function and $\Theta \subset \mathbb{R}^2$ an open set containing no other limit sets, then there exists a $D > 0$ such that the dynamical system

$$\dot{\tilde{z}} = f(\tilde{z}) + g(\tilde{z}), \quad (A.44)$$

has at least one closed orbit in $\Theta$ when $\|g(\tilde{z})\| \leq D, \forall \tilde{z} \in \Theta$. When the Poincaré return map taken transversal to the closed orbits of (A.44) in $\Theta$ does not have non-isolated fixed points, then at least one of these orbits is a limit cycle with the same stability properties as $\gamma$.

Proof. Consider a line $P$ that is locally transversal to the set $\gamma$ of (A.43), as depicted in Fig. A.3(a). Let the point $c$ be the point of intersection of $\gamma$ with $P$. First, we assume $\gamma$ is asymptotically stable, such that there should exist trajectories $\delta_i \in \Theta$ and $\delta_o \in \Theta$ of (A.43) as depicted in Fig. A.3(a), that are converging to the limit cycle and encircle the same equilibria. Notice that the intersection $b$ of $\delta_i$ with $P$ lies in the interior of $[a, c]$ and the intersection $d$ of $\delta_o$ with $P$ lies in the interior of $[c, e]$.

Now, we will determine a maximum difference $E > 0$ and time interval $[0, T + \tau_i]$, such that $\|z(t) - \tilde{z}(t)\| < E, \forall t \in [0, T + \tau_i]$ would imply similar behaviour for the trajectories of (A.43) and (A.44). Any such difference can be obtained by choosing $D$ small enough, since both systems are Lipschitz continuous.
The following argument is similar as in the proof of Theorem 2.8. Let $T$ be chosen such that $\pi(T) = b$ when $\pi(0) = a$. We choose a small $\tau_i > 0$ such that the trajectory $\pi(t)$ from $\pi(0) = a$ satisfies $\|\pi(t) - b\| < \min(\|a - b\|, \|c - b\|), \forall t \in [T - \tau_i, T + \tau_i]$. When the trajectory $\pi(t), t \in [0, T + \tau_i]$ of (A.43) from $\pi(0) = a$ and $\tilde{\pi}(t), t \in [0, T + \tau_i]$ of (A.44) from $\tilde{\pi}(0) = a$ satisfy $\|\pi(t) - \tilde{\pi}(t)\| \leq E_i$, for $t \in [0, T + \tau_i]$, with $E_i > 0$ small enough, then the trajectory $\tilde{\pi}(t)$ from $\tilde{\pi}(0) = a$ traverses the interior of $[a, c]$ in a time interval $t \in (T - \tau_i, T + \tau_i)$.

In a similar fashion, we can derive $\tau_o$ and $E_o$, such that when the trajectory $\pi(t), t \in [0, T + \tau_o]$ of (A.43) from $\pi(0) = b$ and $\tilde{\pi}(t), t \in [0, T + \tau_o]$ of (A.44) from $\tilde{\pi}(0) = e$ satisfy $\|\pi(t) - \tilde{\pi}(t)\| \leq E_o$, $t \in [0, T + \tau_o]$, with $E_o > 0$ small enough, then the trajectory $\tilde{\pi}(t)$ from $\tilde{\pi}(0) = e$ traverses the interior of $[c, e]$ in a time interval $t \in (T - \tau_o, T + \tau_o)$.

Now, let $E = \min(E_i, E_o)$. When $L$ is a Lipschitz constant of $f(x)$, Theorem 3.4 of [91] implies $\|\pi(t) - \tilde{\pi}(t)\| \leq E$, $t \in [0, \max(T + \tau_i, T + \tau_o)]$ is satisfied for the trajectories from $\pi(0) = \tilde{\pi}(0) = e$ and $\pi(0) = \tilde{\pi}(0) = e$, when $E = \frac{D}{\tau} \left(e^{L \max(T + \tau_i, T + \tau_o)} - 1\right)$, such that choosing a positive $D \leq \hat{D} := \frac{\min(E_i, E_o)}{e^{L \max(T + \tau_i, T + \tau_o)} - 1}$ renders $E > 0$ small enough. In that case, the orbits of (A.43) and (A.44) from the point $a$ both show, during one rotation, a diverging spiralling motion with respect to the origin. Similarly, the orbits of (A.43) and (A.44) from the point $e$ show a converging spiralling motion during one rotation.

Since (A.43) is Lipschitz continuous, when $a$ and $e$ are chosen close enough to $c$ and a positive $D \leq \hat{D}$ is chosen small enough, the function $n_p^T \tilde{\pi}$ is of constant sign on the line segment $[a, e]$, where a normal vector $n_p$ of $P$ is introduced. This implies that the domain $I$ is a positively invariant set, with $I$ bounded by two trajectories of (A.44) and two line segments of $[a, e]$, as depicted in Fig. A.3(b).

In addition, the points $a$ and $e$ can be chosen such, that the domain $I$ does not contain equilibria of (A.44) and all trajectories of (A.44) remain in $\Theta$ for $t \in [0, T + \tau]$, with small $\tau > 0$. Therefore, the Poincaré-Bendixson theorem, see [74], guarantees the existence of a closed orbit in $I$ for system (A.44).

Since $I$ is a positively invariant set, a stable limit set of (A.44) exists in $I$. This follows from studying the return map of the perturbed system (A.44), which should be monotonous, to allow uniqueness of solutions in reverse time. Denoting the return map of the perturbed system with $\tilde{M}$, we observe that $\|\tilde{M}(a)\| > \|a\|$ and $\|\tilde{M}(e)\| < \|e\|$. Since $\|a\| < \|e\|$, this implies that the monotonous function $x_{k+1} = \tilde{M}(x_k)$ has to cross the line $x_{k+1} = x_k$ from $\|\tilde{M}(x_k)\| > \|x_k\|$ towards $\|\tilde{M}(x_k)\| < \|x_k\|$ in the interval $x_k \in [a, e]$. When no non-isolated fixed points in this return map can occur, as stated in the lemma, then the perturbed system (A.44) exhibits a limit cycle which is asymptotically stable. The case of an unstable limit cycle follows analogously by studying the time-reversed systems. 

Using this lemma, Theorem 2.11 can be proven. By Assumption 2.7, all closed orbits of (2.1) are limit cycles. At given system parameter $\nu$, for each
stable or unstable limit cycle \( \gamma_i, i = 1, \ldots, l \), of system (2.1), we can define an open set \( R_i(\nu) \ni \gamma_i \), containing no other limit sets. We apply the coordinate transformation \( z = \frac{x}{|\nu|} \), relating (2.3) to (A.35) and (2.1) to (A.37), since we assumed \( \mu = \nu \). Using Assumption 2.8, all limit cycles \( \gamma_i, i = 1, \ldots, l \), of (2.1) are represented in the system (A.37), such that (A.37) contains the same number \( l \) of limit cycles, denoted \( \bar{\gamma}_i, i = 1, \ldots, l \), and the stability properties of \( \gamma_i \) and \( \bar{\gamma}_i \) correspond for \( i = 1, \ldots l \).

Combination of Assumption 2.7 and Lemma A.6 yields that, since \( R_i(\nu) \) is an open set and the system (2.1) is Lipschitz continuous, there exists a \( D > 0 \) such that when
\[
\|\tilde{F}(z, \nu) - \tilde{f}(z)\| \leq D, \forall z \in R_i(\nu),
\]
(A.45)
holds, then \( k_i \) limit cycles of (A.35) exists in \( R_i(\nu) \), of which at least one has the same stability properties as \( \gamma_i \). According to Lemma A.5, a neighbourhood \( N \) of \( \nu = 0 \) exists, such that condition (A.45) is satisfied. Applying the reversed transformation \( x = |\mu|z \), we obtain \( k_i \) limit cycles of the conewise affine system (2.3), of which at least one has the same stability properties as \( \gamma_i \).

The proof of the theorem is concluded by proving \( k_i = 1, \forall i = 1, \ldots, l \), which is proven by contradiction. Suppose that there exists a domain \( R_i(\nu) \) with \( k_i \geq 2 \) stable or unstable limit cycles \( \gamma_{i,j}, j = 1, \ldots, k_i \), existing in the system (2.3) in the domain \( R_i(\nu) \), \( i = 1, \ldots, l \). For all these sets \( \gamma_{i,j}, j = 1, \ldots, k_i \), we can find non-overlapping domains \( R_{i,j}(\nu) \subset R_i(\nu), j = 1, \ldots, k_i \), containing only one limit cycle. Using the same reasoning as above, we obtain that therefore, every set \( R_{i,j}(\nu) \) contains a limit cycle of the non-differentiable system (2.1), implying \( R_i(\nu) \) contains at least \( k_i \) limit cycles. By construction however, \( R_i(\nu) \) contains only one such set, yielding a contradiction. Therefore, \( k_i = 1 \) for all \( i \in \{1, \ldots, l\} \). Hence, the numbers of stable or unstable limit cycles of (2.1) and (2.3) are equal. In addition, their stability properties are equal. \( \square \)
A.2 Appendices of Chapter 3

A.2.1 Existence of an equilibrium set

In this section, the existence of an equilibrium set is shown for system $A$ given by (3.1) and a perturbed system $\tilde{A}$. Subsequently, the existence is proven of a smooth coordinate transformation that maps the equilibrium set of $A$ onto the equilibrium set of a perturbed system $\tilde{A}$. The section is concluded with a technical result on the dynamics of $A$ expressed in the new coordinates. This result will show that we may assume that the equilibrium sets of $A$ and $\tilde{A}$ coincide, without influencing the conditions posed in Theorem 3.1.

Let $\tilde{A}$ be a perturbed system given by (3.1) with $\tilde{f}$ and $\tilde{g}$ perturbed versions of $f$ and $g$, respectively. Let the sets $E$, $I$, $S_1$, $S_2$, $\Sigma$, $\Sigma^c$, $\Sigma^s$, functions $F(\cdot)$, $I(\cdot)$, $F_2(\cdot)$, $F_s(\cdot)$, respectively, be defined analogous to the sets, functions, and points $E_1$ and $E_2$ of system $A$ be defined analogous for system $A$ in Section 3.2.

The following result shows that the equilibrium sets $E$ and $\tilde{E}$ are curves in the state space.

**Lemma A.7.** Consider system $A$ and $\tilde{A}$ given by (3.1) with $f, g, \tilde{f}, \tilde{g}$ satisfying (3.8) for $\epsilon > 0$ sufficiently small. Furthermore, let Assumption 3.1 be satisfied. The equilibrium sets $E$ and $\tilde{E}$ of systems $A$ and $\tilde{A}$, respectively, are curves in state space that can be parameterised by smooth functions $c$ and $\tilde{c}$, such that $E = \{c(\alpha), \alpha \in [-F_s, F_s]\}$, $f(c(\alpha)) = \alpha$ and $g(c(\alpha)) = 0$, resp. $\tilde{E} = \{\tilde{c}(\alpha), \alpha \in [-F_s, F_s]\}$, $\tilde{f}(\tilde{c}(\alpha)) = \alpha$ and $\tilde{g}(\tilde{c}(\alpha)) = 0$.

**Proof.** Define $Z(x, y) = \left(\begin{array}{c} f(x, 0, y) \\ g(x, 0, y) \end{array}\right)$ and observe that $0 \in F(q)$ for all $q \in E$ implies $Z(x, y) = \left(\begin{array}{c} \alpha \\ 0 \end{array}\right)$, with $\alpha \in [-F_s, F_s]$. Using Assumption 3.1, the global inverse function theorem, cf. [162], can be applied, which states that $Z(x, y)$ is a homeomorphism. Application of the corollary following Lemma 2 of [162] shows that the inverse of the function $Z$ is smooth. Hence, there exist smooth functions $X(\beta)$ and $Y(\beta)$ defined by $Z(X(\beta), Y(\beta)) = \beta$.

Since (3.8) is satisfied by $A$ and $\tilde{A}$ with $\epsilon > 0$ sufficiently small, we find that for sufficiently small $\epsilon$, the matrix

$$
\begin{pmatrix}
\frac{\partial f(x, 0, y)}{\partial x} & \frac{\partial f(x, 0, y)}{\partial y} \\
\frac{\partial g(x, 0, y)}{\partial x} & \frac{\partial g(x, 0, y)}{\partial y}
\end{pmatrix}
$$

is invertible. Define

$$
\tilde{Z}(x, y) := \left(\begin{array}{c} \tilde{f}(x, 0, y) \\ \tilde{g}(x, 0, y) \end{array}\right).
$$

The functions $\tilde{f}$ and $\tilde{g}$ are smooth, such that application of the global inverse function theorem yields that there exists an $\epsilon > 0$ such that (3.8) implies that there exist smooth functions $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{Z}(\tilde{X}(\beta), \tilde{Y}(\beta)) = \beta$.

Now, we define $c_x(\alpha) = X(\left(\begin{array}{c} \alpha \\ 0 \end{array}\right))$, $c_y(\alpha) = Y(\left(\begin{array}{c} \alpha \\ 0 \end{array}\right))$ and $c(\alpha) = \left(\begin{array}{c} c_x(\alpha) \\ 0 \end{array}\right)$,
such that \( Z(c_x(\alpha), c_y(\alpha)) = (\alpha, 0) \). The curve \( c(\cdot) \) is a parameterisation of \( \mathcal{E} \), such that \( \mathcal{E} = \{ q \in \mathbb{R}^n : q = c(\alpha), \alpha \in [-F_s, F_s] \} \). Analogously, a parameterisation \( \tilde{c}(\cdot) \) of \( \tilde{\mathcal{E}} \) can be constructed.

Using this lemma, a map is constructed that maps the equilibrium set \( \mathcal{E} \) of \( A \) onto the equilibrium set \( \tilde{\mathcal{E}} \) of \( \tilde{A} \).

**Lemma A.8.** Consider system \( A \) and \( \tilde{A} \) given by (3.1) with \( f, g, \tilde{f}, \tilde{g} \) satisfying (3.8) for \( \epsilon > 0 \) sufficiently small. Furthermore, let Assumption 3.1 be satisfied and let \( \frac{\partial g}{\partial y} \bigg|_p \) be invertible for all \( p \in \mathcal{E} \). There exists a smooth map \( H_e = \begin{pmatrix} H^x_e \\ H^y_e \end{pmatrix} \) with smooth inverse in a neighbourhood of \( \mathcal{E} \), such that \( H^x_e = H^x_e(x) \) and \( H^y_e = \frac{\partial H^x_e}{\partial x} x \), that maps \( \mathcal{E} \) onto \( \tilde{\mathcal{E}} \). Furthermore, for any \( \delta > 0 \) there exists an \( \epsilon > 0 \) such that for any system \( \tilde{A} \) for which (3.8) holds for that \( \epsilon \), the resulting map \( H_e \) satisfies \( \|H_e(q) - q\| < \delta \) and \( \|\frac{\partial H^e}{\partial y} - I\| < \delta \), where \( I \) denotes the \( n \)-dimensional identity matrix.

**Proof.** In this proof, the map \( H_e \) is constructed as follows. Using Lemma A.7, it will be shown that the equilibrium sets \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) can be parameterised as functions of \( x \) and \( \tilde{x} \), respectively. Using this parameterisation, a map \( H_e \) is constructed that maps the curve \( \mathcal{E} \) onto \( \tilde{\mathcal{E}} \).

Consider the parameterisations \( c(\cdot) \) and \( \tilde{c}(\cdot) \) of \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \), respectively, as given in Lemma A.7. In order to construct a map \( H_e \) that is defined in a neighbourhood of \( \mathcal{E} \), we extend \( c(\alpha) \) and \( \tilde{c}(\alpha) \) such that they are defined in a neighbourhood of a closed set \( \alpha \in [\alpha_1, \alpha_2] \), with \( \alpha_1 < -|F_s| \) and \( \alpha_2 > |F_s| \), which is possible according to the extension lemma, see [100].

Let \( c(\alpha) = \begin{pmatrix} c_x(\alpha) \\ 0 \\ c_y(\alpha) \end{pmatrix} \) and \( \tilde{c}(\alpha) = \begin{pmatrix} \tilde{c}_x(\alpha) \\ 0 \\ \tilde{c}_y(\alpha) \end{pmatrix} \). Differentiating the defining expressions \( f(c_x(\alpha), 0, c_y(\alpha)) = \alpha \) and \( g(c_x(\alpha), 0, c_y(\alpha)) = 0 \) with respect to \( \alpha \) yields \( \frac{\partial f}{\partial x} \frac{dc_x}{d\alpha} + \frac{\partial f}{\partial y} \frac{dc_y}{d\alpha} = 1 \), and \( \frac{\partial g}{\partial x} \frac{dc_x}{d\alpha} + \frac{\partial g}{\partial y} \frac{dc_y}{d\alpha} = 0 \), which should be satisfied at the equilibrium set. Since both equations should be satisfied, \( \frac{dc_x}{d\alpha} \neq 0 \) has to hold along the equilibrium curve, since \( \frac{dc_x}{d\alpha} = 0 \) and the first equation would require \( \frac{dc_y}{d\alpha} \neq 0 \), such that the second equation could not be satisfied, as \( \frac{\partial g}{\partial y} \) is invertible. Smoothness of \( g \) and \( c \) imply that one can pick \( \alpha_1 < -|F_s| \) and \( \alpha_2 > |F_s| \) sufficiently close to \( -F_s \) and \( F_s \), such that \( \frac{dc_x}{d\alpha} \bigg|_{\alpha} \neq 0 \) and \( \frac{\partial g}{\partial y} \bigg|_{c(\alpha)} \) is invertible for all \( \alpha \in (\alpha_1, \alpha_2) \). The function \( c_x(\alpha) \) is smooth, such that the relation \( \frac{dc_x}{d\alpha} \bigg|_{\alpha} \neq 0 \) implies that the inverse function theorem can be applied, which yields that there exists a smooth inverse \( c_x^{-1}(x) \) defined on the interval \( x \in \text{co}(c_x(\alpha_1), c_x(\alpha_2)) \). Furthermore, \( \frac{dc_x}{d\alpha} \neq 0 \) implies that \( c_x \) is either monotonously increasing or decreasing, such that the \( x \)-variables of \( \mathcal{E} \) are positioned in
co\{c_x(-F_s), c_x(F_s)\}. Now, let \( y = \Psi(x) := c_y(c_x^{-1}(x)) \) denote the \( y \)-coordinate such that the equilibrium set \( \mathcal{E} \) can be parameterised as follows:

\[
\mathcal{E} = \{ q \in \mathbb{R}^n : q = (x \ 0 \ \Psi^T(x))^T, \ x \in \text{co}\{c_x(-F_s), c_x(F_s)\} \}. \tag{A.46}
\]

We observe that \( c_x^{-1}(x) \) is defined on the interval \( x \in \text{co}\{c_x(\alpha_1), c_x(\alpha_2)\} \) and \( c_y(\alpha) \) is defined on \( (\alpha_1, \alpha_2) \). Hence, the map \( \Psi(x) \) is defined in a neighbourhood of \( \text{co}\{c_x(-F_s), c_x(F_s)\} \).

To construct a function \( \tilde{\Psi}(\cdot) \) for the equilibrium set \( \tilde{\mathcal{E}} \) analogous to \( \Psi(\cdot) \), first we will prove that \( \frac{\partial \tilde{g}}{\partial y}\bigg|_{\tilde{p}} \) is invertible for \( \tilde{p} \in \tilde{\mathcal{E}} \) if \( \epsilon \) is sufficiently small. Since the functions \( f, g, \tilde{f}, \) and \( \tilde{g} \) are smooth and satisfy Assumption 3.1 and (3.8), the inverse function theorem implies that for each \( \epsilon_1 > 0 \) there exists an \( \epsilon > 0 \) such that for each \( p \) satisfying \( (p_2 \ f(p) \ g(p))^T = (0 \ \alpha) \) there exists a \( \tilde{p} \), such that \( (\tilde{p}_2 \ \tilde{f}(\tilde{p}) \ \tilde{g}(\tilde{p}))^T = (0 \ \alpha) \) and \( ||p - \tilde{p}|| < \epsilon_1 \). Hence, there exists an \( \epsilon > 0 \), such that, firstly, \( ||p - \tilde{p}|| < \epsilon_1 \) and secondly, invertibility of \( \frac{\partial \tilde{g}}{\partial y}\bigg|_{\tilde{p}} \) implies invertibility of \( \frac{\partial \tilde{g}}{\partial y}\bigg|_{p} \). If \( \epsilon \) is sufficiently small, this implies that \( \frac{\partial \tilde{g}}{\partial y}\bigg|_{\tilde{p}} \) is invertible.

Therefore a function \( \tilde{\Psi} \) can be constructed for the equilibrium set \( \tilde{\mathcal{E}} \) analogously to the function \( \Psi \) for equilibrium set \( \mathcal{E} \) when \( \epsilon > 0 \) is sufficiently small, such that the equilibrium set \( \tilde{\mathcal{E}} \) can be parameterised as follows:

\[
\tilde{\mathcal{E}} = \{ q \in \mathbb{R}^n : q = (\tilde{x} \ 0 \ \tilde{\Psi}^T(\tilde{x}))^T, \ \tilde{x} \in \text{co}\{\tilde{c}_x(-F_s), \tilde{c}_x(F_s)\} \}. \tag{A.47}
\]

Let \( E_1^x, E_2^x \) denote the \( x \)-component of \( E_1 \) and \( E_2 \), respectively, such that \( E_1^x = c_x(-F_s) \), and \( E_2^x = c_x(F_s) \). Hence, \( E_1^x \) and \( E_2^x \) are the \( x \)-coordinates of the endpoints of the equilibrium set \( \mathcal{E} \). The values \( E_1^x \) and \( E_2^x \) are defined analogously. The smooth invertible map \( H_e \) is constructed as follows. Let \( H_e^x(x) = \tilde{E}_1^x + \frac{(x-E_1^x)(E_2^x-E_1^x)}{E_2^x-E_1^x} \) and \( H_e^x(\tilde{x}) = \frac{(E_2^x-E_1^x)}{E_2^x-E_1^x} \tilde{x} \). Furthermore, let \( H_e^y(x, y) = y + \tilde{\Psi}(H_e^x(x)) - \Psi(x) \).

Since \( \Psi(x) \) and \( \tilde{\Psi}(\tilde{x}) \) are defined in neighbourhoods of \( x \in \text{co}\{E_1^x, E_2^x\} \) and \( \tilde{x} \in \text{co}\{\tilde{E}_1^x, \tilde{E}_2^x\} \), respectively, we observe that \( H_e \) is defined in a neighbourhood of \( \mathcal{E} \). The maps \( \Psi \) and \( \tilde{\Psi} \) are smooth since \( c_y \) and \( c_x^{-1} \) are smooth functions. It follows that the map \( H_e \) and its inverse are smooth. Clearly, \( \tilde{x} = H_e^x(x) \) maps the interval \( x \in \text{co}\{E_1^x, E_2^x\} \) onto the interval \( \tilde{x} \in \text{co}\{\tilde{E}_1^x, \tilde{E}_2^x\} \). Combination of this fact with equations (A.46) and (A.47) yields that \( \tilde{q} = H_e(q) \in \tilde{\mathcal{E}} \) if and only if \( q \in \mathcal{E} \).

To prove the final statement of the lemma, observe that for each \( \delta_1 > 0 \), we may choose an \( \epsilon > 0 \) such that \( ||\tilde{E}_i^x - E_i^x|| < \delta_1, \ i = 1, 2, ||c(\alpha) - \tilde{c}(\alpha)|| < \delta_1 \) and \( ||\frac{\partial c}{\partial q} - \frac{\partial \tilde{c}}{\partial q}|| < \delta_1 \). The statement \( ||\tilde{E}_i^x - E_i^x|| < \delta_1, \ i = 1, 2 \) implies that given an arbitrarily small \( \delta_2 > 0 \) there exists a \( \delta_1 > 0 \) such that \( H_e^x(q) - x < \delta_2 \), \( H_e^x(q) - \tilde{x} < \delta_2 \) for \( q \) in a neighbourhood of \( \mathcal{E} \). The statement \( ||c(\alpha) - \tilde{c}(\alpha)|| < \delta_1 \) implies that for each \( \delta_3 > 0 \) there exists a \( \delta_1 > 0 \) such that \( ||\tilde{\Psi}(x) - \Psi(x)|| < \delta_3 \).
Hence, for each $\delta > 0$ there exists an $\epsilon > 0$ small enough such that $\|H_e(q) - q\| < \delta$.

The statement $\|\partial \bar{c} / \partial q - \partial \bar{c} / \partial q\| < \delta_1$ implies that $|\partial \bar{y} / \partial \bar{x} - \partial \bar{y} / \partial \bar{x}|$ can be chosen small. Choosing $\epsilon > 0$ small enough one obtains that $|(E_2^2 - E_1^2) - 1|$ becomes arbitrarily small. Hence, $\partial H_e / \partial q = \left(\begin{array}{cc}
(E_2^2 - E_1^2) & 0 \\
0 & (E_2^2 - E_1^2) \nabla \bar{y} / \partial \bar{x} - \partial \bar{y} / \partial \bar{x}
\end{array} \right)$ satisfies the last statement of the lemma.

In the next result new coordinates are introduced for system $A$. The equilibrium set $\mathcal{E}$ of $A$, expressed in these coordinates, will be shown to coincide with the equilibrium set $\tilde{\mathcal{E}}$ of $\tilde{A}$.

**Lemma A.9.** Consider systems $A$ and $\tilde{A}$ given by (3.1) with $f, g, \tilde{f}, \tilde{g}$ satisfying (3.8) for $\epsilon > 0$ sufficiently small. Furthermore, let Assumption 3.1 be satisfied, let $\partial \bar{q} / \partial y$ be invertible for all $p \in \mathcal{E}$ and let the map $H_e$ satisfy the conditions in Lemma A.8. The dynamics of system $A$ near the equilibrium set can be described in new coordinates $\bar{q} = (\bar{x}, \bar{\dot{x}}, \bar{y})^T = H_e(q)$, such that

$$
\begin{align*}
\bar{x} - \tilde{f}(\bar{x}, \bar{\dot{x}}, \bar{y}) & \in -F_s \text{Sign}(\bar{\dot{x}}), \\
\dot{\bar{y}} & = \tilde{g}(\bar{x}, \bar{\dot{x}}, \bar{y}),
\end{align*}
$$

(A.48)

where the functions $\tilde{f}$ and $\tilde{g}$ are smooth, given that $\epsilon$ is sufficiently small. The equilibrium sets of (A.48) coincides with the equilibrium set of $A$.

**Proof.** Let $\bar{q} = H_e(q)$, with $H_e(q)$ given in Lemma A.8. Observe that $q \in S_i$, $i = 1, 2$, when $\epsilon$ is small enough, since $(E_2^2 - E_1^2)$ becomes close to one according to the last statement of Lemma A.8. Smoothness of $\tilde{f}$ and $\tilde{g}$ follows from $\bar{q} = \tilde{F}_i(\bar{q}) = \partial H / \partial q \bar{q} = \partial H / \partial q F_1(H_0^{-1}(q))$ and the fact that $H_e$ and $H_0^{-1}$ are smooth, see Lemma A.8, where $\tilde{F}_i(\bar{q}) = (\bar{q}_2 \tilde{f}(\bar{q}) + (-1)^{i+1} F_s \bar{g}^T(\bar{q}))^T$.

In order to study the structural stability of system (3.1), the functions $f, g$ and their Jacobian matrices will be important. Hence, the following technical result on $f, g, \tilde{f}$ and $\tilde{g}$ will be used.

**Lemma A.10.** Consider systems $A$ and $\tilde{A}$ given by (3.1) and let the conditions of Lemma A.9 be satisfied. Consider the coordinates $\bar{q} = H_e(q)$ and functions $\tilde{f}, \tilde{g}$ given in Lemma A.9. For each $\delta > 0$ there exists an $\epsilon > 0$ such that (3.8) implies that, firstly, $\|\partial \bar{g} / \partial y - \partial \bar{y} / \partial y\| < \delta$ near the equilibrium set, and secondly,

$$
\left\| \frac{\partial \tilde{F}_i}{\partial q} \bigg|_{E_i} - \frac{\partial \tilde{F}_i}{\partial q} \bigg|_{E_i} \right\| < \delta, \quad i = 1, 2,
$$

where $\tilde{F}_i(\bar{q}) = (\bar{q}_2 \tilde{f}(\bar{q}) + (-1)^{i+1} F_s \bar{g}^T(\bar{q}))^T$. 


Proof. Given \( \delta_1 > 0 \), Lemma A.8 states that there exists an \( \epsilon > 0 \) such that (3.8) implies \( \| \frac{\partial H}{\partial q} - I \| < \delta_1 \). It follows that \( \frac{\partial H^{-1}}{\partial q} \) becomes close to identity as well, such that for each \( \delta_2 > 0 \), there exists an \( \epsilon > 0 \) such that (3.8) implies, firstly, that \( \| \frac{\partial H}{\partial q} - I \| < \delta_2 \) and secondly, \( \| \frac{\partial H^{-1}}{\partial q} - I \| < \delta_2 \). Hence, the properties of \( \bar{F} \) follow from \( \frac{\partial F_i}{\partial q}\big|_{E_i} = \frac{\partial^2 H}{\partial q^2} F_i(H_e^{-1}(\bar{E}_i)) + \frac{\partial H_e}{\partial q} \frac{\partial F_i}{\partial q} \frac{\partial H_e^{-1}}{\partial q}(\bar{E}_i) \), \( i = 1, 2 \), where \( F_i(H_e^{-1}(\bar{E}_i)) = F_i(E_i) = 0 \).

In order to prove \( \| \frac{\partial g}{\partial y} - \frac{\partial q}{\partial y} \| < \delta \), we observe that \( \bar{g}(q) = (O \ O \ I_{n-2}) \hat{\bar{q}} = (O \ O \ I_{n-2}) \frac{\partial H_e}{\partial q} F_i(H_e^{-1}(\bar{q})) \), with \( I_{n-2} \) the \( (n-2) \)-dimensional identity matrix and \( O \) a zero matrix of dimension \( (n-2) \times 1 \). Hence, \( \frac{\partial \bar{g}}{\partial q} \) is given by \( \frac{\partial \bar{g}}{\partial q} = (O \ O \ I_{n-2}) \frac{\partial H_e}{\partial q} \frac{\partial F_i}{\partial q} \frac{\partial H_e^{-1}}{\partial q}(O \ O \ I_{n-2})^T \). Since \( (O \ O \ I_{n-2}) \frac{\partial F_i}{\partial q} (O \ O \ I_{n-2})^T = \frac{\partial q}{\partial y} \), and using the fact that we can make \( \| \frac{\partial H}{\partial q} - I \| < \delta_2 \) sufficiently small, we can select \( \epsilon \) sufficiently small to obtain \( \| \frac{\partial \bar{g}}{\partial y} - \frac{\partial q}{\partial y} \| < \delta \). \qed

We now obtained a coordinate transformation \( H_e \), which can be applied to guarantee that the equilibrium sets of \( A \) and \( \tilde{A} \) coincide. Hence, assuming that the equilibrium sets of \( A \) and \( \tilde{A} \) coincide will not introduce a loss of generality. The properties of the dynamical equation (3.1) that will be used in the proof of Theorem 3.1 are not changed by this coordinate transformation. Note, that a similar argument is used to study perturbations of a hyperbolic equilibrium point for a smooth dynamical system, see e.g. [127].

### A.2.2 Proof of Theorem 3.1

To prove Theorem 3.1, first the structural stability of the sliding dynamics given by (3.6) is investigated. Subsequently, it is shown that a topological map, i.e. a homeomorphism satisfying the conditions given in Definition 3.1, can be extended orthogonal to this boundary if it exists for the sliding trajectories on \( \Sigma \). An eigenvalue \( \lambda \) will be considered critical if \( \text{real}(\lambda) = 0 \). The following lemma presents conditions for structural stability, as defined in Definition 3.2, for the sliding dynamics.

**Lemma A.11.** Let system (3.1) satisfy Assumption 3.1 and let \( F_s > 0 \). If \( \frac{\partial q}{\partial y} \big|_p \) does not have critical eigenvalues for all \( p \in \mathcal{E} \), then for any closed set \( J \subset \Gamma \) the sliding trajectories of system (3.1) in a neighborhood \( N(J) \subset \Sigma \) of \( J \) are structurally stable for perturbations of \( f \) and \( g \) as defined in Definition 3.2.

**Proof.** We consider a system \( A \) given by (3.1) and a perturbed system \( \tilde{A} \) described by (3.1) with \( \tilde{f}, \tilde{g} \) satisfying (3.8) for sufficiently small \( \epsilon > 0 \). By Lemma A.10, we may assume without loss of generality that the equilibrium set of \( A \) and \( \tilde{A} \) coincide.
The sliding solutions of $A$ are described by (3.6). Restricting this dynamics to the boundary $\Sigma$, one obtains:

\[
\begin{align*}
\dot{x} &= 0, \\
\dot{y} &= g(x,0,y),
\end{align*}
\]  

(A.49)

for system $A$ and

\[
\begin{align*}
\dot{x} &= 0, \\
\dot{y} &= \tilde{g}(x,0,y),
\end{align*}
\]  

(A.50)

for system $\tilde{A}$. Note that $\epsilon > 0$ can be chosen such that $\frac{\partial \tilde{g}}{\partial y}$ is arbitrarily close to $\frac{\partial g}{\partial y}$, see Lemma A.10. Hence, the Jacobian matrix $\left| \frac{\partial \tilde{g}}{\partial y} \right|_p$ for $p \in E$ has the same number of positive and negative eigenvalues as $\left| \frac{\partial g}{\partial y} \right|_p$. This implies that trajectories of $\dot{y} = \tilde{g}(x,0,y)$ are locally topologically equivalent near $p$ to the trajectories dynamics of $\dot{y} = g(x,0,y)$, cf. Theorem 5.1 of [127, page 68]. From the reduction theorem, cf. [5, page 15], we conclude that (A.49) and (A.50) are topologically equivalent.

The importance of the topological nature of sliding trajectories will be shown using the following lemma. To prove this lemma, the following notation is used. Let $q(t) = \varphi(t,q_0)$ denote a solution of system $A$ given by (3.1) in the sense of Filippov with initial condition $q(0) = \varphi(0,q_0) = q_0$. Furthermore, let $q(t) = \varphi^i(t,q_0)$, $i = 1,2$, denote a trajectory of $\dot{q} = F_i(q)$ satisfying $q(0) = \varphi^i(0,q_0) = q_0$. Analogously, the functions $\tilde{\varphi}(\cdot,\cdot)$, $\tilde{\varphi}^i(\cdot,\cdot)$, $i = 1,2$, are defined for the perturbed system $\tilde{A}$. We note that this notation can also be used for trajectories in reverse time when $t < 0$.

**Lemma A.12.** Let system (3.1) satisfy Assumption 3.1 and let $F_s > 0$. For every interior point $p \in I$ there exists a neighbourhood $N(p)$ and a finite $\tau > 0$ such that any trajectory with an initial condition in $N(p)$ arrives in $\Sigma^s$ at time $t \in [0,\tau]$.

**Proof.** This proof follows the line of reasoning as described by Filippov, see [58, page 262]. Consider an interior point $p \in I$, and observe that $I \subset \Sigma^s$. Trajectories near $p$ on $\Sigma^s$ trivially satisfy the lemma for $t = 0$. Hence, we restrict our attention to trajectories in $S_1$ near $p$. The trajectories in $S_2$ can be handled analogously. The interior point $p \in I \subset \Sigma^s$, such that (3.5) implies $L_{F_1}h(p) > 0$. Since $F_1$ is smooth, there exist an $\delta > 0$ and neighbourhood $N_\delta(p)$ of $p$ such that $L_{F_1}h(q) > \delta, \forall q \in N_\delta$. Since $F_1$ is smooth, for all $q \in N_\delta(p) \cap \Sigma$ there exists a unique trajectory $\varphi^1(t,q)$ of $\dot{q} = F_1(q)$ that satisfies $\varphi^1(0,q) = q$. By $L_{F_1}h(q) > \delta$ there exists a $\tau > 0$ such that $\varphi^1(t,q) \in S_1, \forall t \in (-\tau,0)$. Hence, the trajectory $\varphi^1(t,q)$ coincides with a trajectory of (3.1) on this time interval.
From Theorem 3, [58, page 128], we conclude that these trajectories form a one-sided neighbourhood $N_1(p)$ of $p$. By studying the trajectories in $S_2$, analogously we find a one-sided neighbourhood $N_2(p)$. Since there exists a neighbourhood $N(p) \subset \Sigma^* \cup N_1(p) \cup N_2(p)$, the lemma is proven.

From this lemma, it follows that the qualitative nature of trajectories near interior points $p \in I$ can be described as follows. According to Lemma A.12, trajectories arrive at the discontinuity surface in finite time. Subsequently, these trajectories are described by the sliding vector field. Using this property, a topological map defined for sliding trajectories on $\Sigma$ can be extended towards a one-sided neighbourhood $N_1(p)$. Consider two systems $\Sigma$ and $\tilde{\Sigma}$, respectively. Hence, there exists a unique topological map $H_\Sigma : U \mapsto \tilde{U}$, where $U \subset \Sigma$ and $\tilde{U} \subset \tilde{\Sigma}$. If $(-1)^i L_{F_i} h(q) < 0$, $\forall q \in U$, and $(-1)^i L_{\tilde{F}_i} h(\tilde{q}) < 0$, $\forall \tilde{q} \in \tilde{U}$, for $i = 1$ or $i = 2$, then one can extend the topological map $H_\Sigma$ towards $S_i$ such that $H_\Sigma$ is defined in a closed $n-$dimensional subset of $U \cup S_i$ that contains $U$.

**Lemma A.13.** Consider two systems $A$ and $\tilde{A}$ and let there exist a topological map $H_\Sigma : U \mapsto \tilde{U}$, where $U \subset \Sigma$ and $\tilde{U} \subset \tilde{\Sigma}$. If $(-1)^i L_{F_i} h(q) < 0$, $\forall q \in U$, and $(-1)^i L_{\tilde{F}_i} h(\tilde{q}) < 0$, $\forall \tilde{q} \in \tilde{U}$, for $i = 1$ or $i = 2$, then one can extend the topological map $H_\Sigma$ towards $S_i$ such that $H_\Sigma$ is defined in a closed $n-$dimensional subset of $U \cup S_i$ that contains $U$.

**Proof.** To prove the lemma, we exploit the assumption that $L_{F_1} h(q) > 0$, $\forall q \in U$ and $L_{\tilde{F}_1} h(\tilde{q}) > 0$, $\forall \tilde{q} \in \tilde{U}$. The case $L_{F_1} h(q) < 0$, $\forall q \in U$ and $L_{\tilde{F}_1} h(\tilde{q}) < 0$, $\forall \tilde{q} \in \tilde{U}$, can be handled analogously. We observe that $L_{F_1} h(q) > 0$, $\forall q \in U$, implies that for any $q \in U$, there exists a unique trajectory of system $A$ that satisfies $\varphi(0, q) = q$ and $\varphi(t, q) \in S_1$ for $t \in [T_1, 0)$, with $T_1 < 0$. Choosing $\tilde{q} = H_\Sigma(q)$, $L_{\tilde{F}_1} h(H_\Sigma(q)) > 0$ analogously implies that there exists a unique trajectory of $\tilde{A}$ such that $\tilde{\varphi}(0, H_\Sigma(q)) = H_\Sigma(q) \in \tilde{U}$ and $\tilde{\varphi}(t, H_\Sigma(q)) \in \tilde{S}_1$ for $t \in [T_2, 0)$, where $T_2 < 0$. Introducing $T = \max(T_1, T_2)$ yields $\varphi(t, q) \in S_1$ and $\tilde{\varphi}(t, H_\Sigma(q)) \in \tilde{S}_1$, $\forall t \in [T, 0)$, $\forall q \in U$.

From Theorem 3, [58, page 128], we conclude that the union of these trajectories form compact, connected $n-$dimensional sets $V$ and $\tilde{V}$ containing $U$ and $\tilde{U}$, respectively. Hence, there exists a unique map $\psi : V \mapsto [T, 0] \times U$ such that $(\tau, \rho) = \psi(q)$ with inverse $q = \varphi(\tau, \rho)$. Here, $\rho$ denotes the first point where the trajectory with initial condition $q$ crosses $\Sigma$, the time lapse is denoted $-\tau$. Since $(-1)^i L_{F_i} h(q) < 0$, $\forall q \in U$, there exists a unique trajectory $\varphi^i(t, \rho)$ of $\dot{q} = F_i(q)$ with initial condition $\varphi(0, \rho) = \rho \in U$ at time $t = 0$ that crosses $\Sigma$ non-tangentially (i.e. transversally) at time $t = 0$ and satisfies $\varphi(t, q) \in S_1$, $\forall t \in (T, 0)$. For this time interval, the trajectory $\varphi^i$ of $\dot{q} = F_i(q)$ coincides with the trajectory $\varphi$ of $A$. Hence, we observe that $\psi$ is continuous and unique, cf. [82, page 242].

Now, we define $H_\Sigma(q)$ for $q \notin \Sigma$ as $H_\Sigma(q) = \tilde{\varphi}(\tau, H_\Sigma(\rho))$, where $(\tau, \rho) = \psi(q)$. We observe that $H_\Sigma : V \mapsto \tilde{V}$ satisfies the conditions of the lemma, and maps trajectories of $A$ onto trajectories of $\tilde{A}$. 

The following lemma gives sufficient conditions for the structural stability of trajectories near interior points of the equilibrium set. Since trajectories near
the endpoints $E_1$ and $E_2$ of the equilibrium set are not considered, the following lemma is restricted to any closed subset of $I$.

**Lemma A.14.** Let system (3.1) satisfy Assumption 3.1 and let $F_s > 0$. If $\frac{\partial g}{\partial y} \bigg|_p$ does not have critical eigenvalues for all $p \in E$, then for any closed set $J \subset I$ the trajectories of system (3.1) in a neighbourhood $N(J)$ of $J$ are structurally stable for perturbations of $f$ and $g$.

**Proof.** Consider system $A$ described by (3.1) and let the perturbed version be denoted by $\tilde{A}$. Under the conditions given in the lemma, the result of Lemma A.11 implies that there exists a topological map $H_\Sigma$ in a set $\bar{N}(J) \subset \Sigma$ containing $J$, that maps trajectories of $A$ onto trajectories of $\tilde{A}$. Observing that $\bar{N}(J) \subset \Sigma^s$, one can apply Lemma A.13, which proves that the map $H_\Sigma$ can be extended to subsets of $S_1$ and $S_2$. In this manner, a topological map is obtained from a neighbourhood of $J$ to a neighbourhood of $H_\Sigma(J)$, which proves the lemma, see Definition 3.2.

Under the conditions given in the foregoing lemma, we conclude that no changes can occur in the topological nature of trajectories around interior points of the equilibrium set. Hence, we are able to prove Theorem 3.1.

**Proof of Theorem 3.1.** Theorem 3.1 is proven by contradiction. Suppose there exists a system $A(\mu)$ smoothly depending on parameter $\mu$, that undergoes a local bifurcation at parameter $\mu = 0$ which does not occur at the endpoints. Furthermore, we use the assumption in the theorem that $\frac{\partial g}{\partial y} \bigg|_p$ does not have critical eigenvalues for system $A(0)$. A sufficiently small change of $\mu$ near 0 can be chosen such that systems $A(0)$ and $A(\mu)$ satisfy (3.8) for arbitrarily small $\epsilon$. Reversing the direction of time if necessary\(^1\), we may assume $F_s > 0$ such that Lemma A.14 can be applied. This lemma contradicts the occurrence of a local bifurcation of the equilibrium set at an interior point.

\(^1\)We note that time-reversal of a system yields a topologically different system, but both systems have identical structural stability properties.

**A.2.3 Proof of Theorem 3.2**

In this section, Theorem 3.2 is proven. First, we study the trajectories in the neighbourhood of an individual endpoint $E_k$ with $k = 1$ or $k = 2$, where the eigenvalues of $A_k$ are complex. Subsequently, endpoints are studied where $A_k$ has real eigenvalues. Recall that a topological map is a homeomorphism satisfying the conditions given in Definition 3.1.

**Lemma A.15.** Consider a planar system $A$ given by (3.9) and satisfying Assumptions 3.1 and 3.2, where $A_k$, with $k = 1$ or $k = 2$, has complex eigenvalues $\lambda = \alpha \pm i\omega$, $\omega \neq 0$. Furthermore, let $\tilde{A}$ be a perturbed system satisfying (3.8)
with sufficiently small $\epsilon > 0$. Then, there exist a topological map $H_c$ and neighbourhoods $N(E_k)$ of $E_k$ and $\tilde{N}(\tilde{E}_k)$ of $\tilde{E}_k$ such that $H_c : N(E_k) \mapsto \tilde{N}(\tilde{E}_k)$.

**Proof.** By Lemma A.10, we may assume that the equilibrium sets of $A$ and $A$ coincide. From (3.9) it follows, together with $F_\epsilon > 0$ of Assumption 3.2, that the equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide with stable sliding motion, i.e. $\mathcal{E} = \Sigma^s$ and $\tilde{\mathcal{E}} = \tilde{\Sigma}^s$.

In this proof we consider the case $k = 1$, the proof for $k = 2$ can be derived analogously. Let the real matrix $P$ be given by the real Jordan decomposition of $A_1$ given by $A_1 = PJP^{-1}$, where $J = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$. Hence, the linear system $\dot{w} = A_1 w$ can be represented by new coordinates $r, \theta$ such that $\dot{w}_1 = r \cos(\theta)$ and $\dot{w}_2 = r \sin(\theta)$, with $(\dot{w}_1 \ \dot{w}_2)^T = P^{-1} \dot{w}$. In these coordinates, the dynamics $\dot{w} = A_1 w$ yields $\dot{r} = \alpha r$ and $\dot{\theta} = \omega$. Hence, all trajectories of this system encircle the origin, and cross every line through the origin every $T = \frac{\pi}{\omega}$ time units. If we let $w = q - E_1$, then $\dot{w} = A_1 w$ serves as a linear approximation of $\dot{q} = F_1(q)$ near $E_1$. Using the same coordinates $r$ and $\theta$, we obtain $\dot{\theta} = \omega + \tilde{\omega}(r, \theta)$, where $\tilde{\omega}$ is determined by the error introduced by the linearisation. Since $F_1$ is a smooth function, this linearisation will introduce an error of order 2, that is, $|F_1(q) - A_1(q - E_1)| = O(|q - E_1|^2)$. Hence, the function $\tilde{\omega}$ is smooth and $\tilde{\omega}(r, \theta) = O(|q - E_1|^2) = O(r^2)$. For each $\delta > 0$, there exists a small enough neighbourhood $N_1(E_1)$ of $E_1$ such that $\theta \in [\omega - \delta, \omega + \delta]$ and $\dot{r} \in [\alpha r - \delta, \alpha r + \delta]$.

This implies that there exists a neighbourhood $N_2(E_1) \subset N_1(E_1)$ of $E_1$, such that trajectories of $\dot{q} = F_1(q)$ in $N_2(E_1)$ cross every line through $E_1$ in a transversal manner, see Fig. A.4. Hence, for each $q \in \Sigma^s \cap N_2(E_1)$ there exists a $T(q)$ near $-\frac{\pi}{\omega}$ such that $\varphi^1(T(q), q) \in \Sigma^c$, where $\varphi^1(t, q)$ denotes the trajectory of $\dot{q} = F_1(q)$ with initial condition $\varphi^1(0, q) = q$ at time $t = 0$. The function $T(q)$ is continuous, see [82, page 242]. Since $L_{F_1} h(q) > 0$ for $q \in \Sigma^s$, the trajectory $\varphi^1(t, q)$ crosses $\Sigma^s$ from $S_1$, such that $\varphi^1(t, q) \in S_1, \forall t \in (T(q), 0)$. By Filippov’s solution convention, the trajectory $\varphi^1(t, q)$ coincides with a trajectory of (3.9) for $t \in [T(q), 0]$. Furthermore, any point $q \in S_1 \cap N_2(E_1)$ can be described with the coordinates $(\tau, \rho), \rho \in \Sigma^s, \tau \in (T(\rho), 0)$, such that $q = \varphi^1(\tau, \rho)$.

Now, consider a perturbed system $\tilde{A}$ satisfying (3.8) with $\epsilon$ sufficiently small. 

---

**Fig. A.4.** Sketch of trajectories near $E_1$ for a 2-dimensional system when $A_1$ has complex eigenvalues. The left panel shows trajectories of $A$, the right panel shows trajectories of a perturbed system $\tilde{A}$. 

---
This implies the eigenvalues of $\tilde{A}_1$ are arbitrarily close to the eigenvalues of $A$, and hence are complex. Analogous to the reasoning given in the foregoing paragraph for system $A$, one can show that there exists a continuous function $\tilde{T}(\tilde{\rho})$ for $\tilde{\rho} \in \Sigma^*$, such that $\tilde{\varphi}^1(\tilde{T}(\tilde{\rho}), \tilde{\rho}) \in \Sigma^c$ and any point $\tilde{q} \in \tilde{S}_1 \cap N_2(\tilde{E}_1)$ can be described with the coordinates $(\tilde{\tau}, \tilde{\rho})$, $\tilde{\tau} \in (\tilde{T}(\tilde{\rho}), 0)$, such that $\tilde{q} = \tilde{\varphi}^1(\tilde{\tau}, \tilde{\rho})$.

Let $H_{c1}$ map points $(\tau, \rho) \to (\tilde{\tau}, \tilde{\rho}) = (\frac{\tilde{T}(\rho)}{T(\rho)}, \tau, \rho)$. This map and its inverse are locally continuous since $T(\rho)$ and $\tilde{T}(\rho)$ are nonzero, and the functions $T(\cdot), \tilde{T}(\cdot)$ are continuous.

Now, let $H_c$ map $\varphi^1(\tau, \rho) \to \tilde{\varphi}^1(\tilde{\tau}, \tilde{\rho})$, where $(\tilde{\tau}, \tilde{\rho}) = H_{c1}(\tau, \rho)$. This map is continuous away from the point $E_1$ since $H_{c1}$ is continuous and both $\varphi^1$ and $\tilde{\varphi}^1$ are trajectories of a smooth system, hence $\varphi^1$ and $\tilde{\varphi}^1$ are continuous.

The domain of definition of $H_c$ is extended from $S_1$ towards $\Sigma$ as follows. For points $q \in \Sigma$ we choose a sequence $\{q_i\}$ with $\lim_{i \to \infty} q_i = q \in \Sigma$ and $q_i \in S_1$ and define $H_c(q) = \lim_{i \to \infty} H_c(q_i)$, where this limit exist, at least away from $E_1$, due to continuity of $H_c$ on $S_1 \cap N_2(E_1)$. In this manner, the domain of $H_c$ becomes $(S_1 \cup \Sigma) \cap N_2(E_1)$.

Continuity of $H_c$ is trivial away from the point $E_1$. Now, continuity of $H_c$ at $E_1$ is proven. Consider two arbitrary sequences $\{q_i\} \in S_1$ and $\{\tilde{q}_i\} \in S_1$ with $\lim_{i \to \infty} q_i = \lim_{j \to \infty} \tilde{q}_j = E_1$. The two sequences correspond to different coordinates $\{\tau_i, \rho_i\}$ and $\{\tilde{\tau}_j, \tilde{\rho}_j\}$. We observe that both $\lim_{i \to \infty} \rho_i = E_1$ and $\lim_{j \to \infty} \tilde{\rho}_j = E_1$, whereas the limits of the $\tau$ and $\tilde{\tau}$-sequences may differ. However, $\tilde{E}_1 = E_1$ is an equilibrium of $\tilde{q} = \tilde{F}_1(\tilde{q})$, such that $\tilde{\varphi}^1(t, \tilde{E}_1) = \tilde{E}_1$, is independent on $t$. Hence, we obtain $\tilde{E}_1 = H_c(E_1) = \lim_{i \to \infty} H_c(q_i) = \lim_{j \to \infty} H_c(\tilde{q}_j)$. Continuity of $H_c$ at $E_1$ is proven.

At $E_1$ one finds $F_1(E_1) = 0$, such that $L_{F_2}h(E_1) = -2F_a$, see (3.4). Hence, according to Lemma A.13 the domain of definition of $H_c$ can be extended towards a neighbourhood of $E_1$ in $S_2$, such that $H_c : N(E_1) \to \tilde{N}(\tilde{E}_1)$, where $N(p)$ denotes a neighbourhood of $p$ and $\tilde{N}(\tilde{p})$ denotes a neighbourhood of $\tilde{p}$. The map $H_c$ is a topological map. \hfill \Box

Lemma A.15 proves that the trajectories near an endpoint $E_k$, $k = 1$ or $k = 2$ are structurally stable when the matrix $A_k$ has complex eigenvalues, such that all trajectories in $S_k$ near $E_k$ will encircle this point and either enter $\Sigma^*$, or cross the boundary $\Sigma^c$ in finite time.

In the following lemma, structural stability is studied of trajectories in the neighbourhood of an endpoint $E_k$ when the matrix $A_k$ has real eigenvalues. For this purpose, separatrices of this system, consisting of trajectories that converge asymptotically to the endpoint $E_k$, are introduced as follows.

In case $A_k$, $k = 1$ or $k = 2$, has real nonzero eigenvalues, Assumption 3.2 implies that the eigenvectors $v_j$ of $A_k$ satisfy $\nabla h v_j \neq 0$, since $A_k = \begin{pmatrix} 0 & 1 \\ \frac{\partial f}{\partial x} |_{E_k} & \frac{\partial f}{\partial x} |_{E_k} \end{pmatrix}$. To study the case where $A_k$ has real eigenvalues, separatrices of the system are
studied. Assume $A_k$ has nonzero, real eigenvalues and let $H_{HG}$ denote the local topological map that maps trajectories of (3.9) in $S_k$ to trajectories of $\dot{q} = A_k(q - E_k)$, which exists according to the Hartman-Grobman theorem. Given an eigenvalue $\lambda_j$, let $v_j$ denote the unique corresponding unit eigenvector of $A_k$ that points towards $S_i$. The set $M_j := \{ q \in \mathbb{R}^2 : q = cv_j, \ c \in (0, \infty) \}$, $j = 1, 2$, is invariant for $\dot{q} = A_k q$, such that the set $M_j := \{ E_k \} + H_{HG}^{-1}(M_j)$ is invariant for $\dot{q} = F_k(q)$. Since the set $M_j$ consist of a single trajectory of the smooth differential equation $\dot{q} = F_k(q)$, $M_j$ is a smooth curve. The invariant manifold $M_j$ is a separatrix for system (3.9) that is tangent to $\bar{E}$ at $E_k$. If $A_k$ has two distinct real eigenvalues, the separatrices $M_1$ and $M_2$ correspond to the eigenvalues $\lambda_1$ and $\lambda_2$ of $A_k$. Note that $M_1$ and $M_2$ coincide with the stable and unstable manifold of $E_k$ positioned in $S_k$ if the eigenvalues of $A_k$ satisfy $\lambda_1 < 0 < \lambda_2$. For perturbed systems $A$, the separatrices $\tilde{M}_1$ and $\tilde{M}_2$ are defined analogously.

**Lemma A.16.** Consider a planar system $A$ given by (3.9) satisfying Assumptions 3.1 and 3.2, where $A_k$, with $k = 1$ or $k = 2$, has real, nonzero eigenvalues. Furthermore, let $\tilde{A}$ be a perturbed system satisfying (3.8) with sufficiently small $\epsilon > 0$. There exist a topological map $H_r$ and neighbourhoods $N(E_k)$ of $E_k$ and $\tilde{N}(E_k)$ of $\tilde{E}_k$ such that $H_r : N(E_k) \rightarrow \tilde{N}(E_k)$.

**Proof.** In this proof we consider the case $k = 1$; the proof for $k = 2$ can be derived analogously. Since the eigenvalues and eigenvectors of a real, nonsingular matrix are continuous functions of parameters, the eigenvalues and eigenvectors of $\tilde{A}_1$ are close to those of $A_1$. Hence, $\tilde{A}_1$ has real, nonzero eigenvalues, and separatrices $\tilde{M}_1$ and $\tilde{M}_2$ of $\tilde{A}$ are locally close to $M_1$ and $M_2$. From (3.9) it follows, together with $F_s > 0$ of Assumption 3.2, that the equilibrium sets $E$ and $\tilde{E}$ near the points $E_1$ or $\tilde{E}_1$ coincide with stable sliding motion, hence $E$ and $\tilde{E}$ coincide with $\Sigma^s$ and $\tilde{\Sigma}^s$, respectively.

For the system $A$ the separatrices $M_1$ and $M_2$ partition the domain $N(E_1) \cap S_1$ into three sectors $c_1, c_2, c_3$, see Fig. A.5. The index of $c_1$ is chosen such that $E$ is a subset of the boundary of $c_1$ and the boundary of $c_3$ contains $\Sigma^c$, as shown in Fig. A.5. Similar, we partition $\tilde{N}(E_1)$ into three domains $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$, bounded by the manifolds $\tilde{M}_1, \tilde{M}_2, \tilde{\Sigma}$ and $\tilde{E}_1$.

Trajectories in $c_2 \cup M_1 \cup M_2$ of $A$, are described by $\dot{q} = F_1(q)$ and trajectories of $\tilde{A}$ in $c_2 \cup M_1 \cup M_2$ are described by $\dot{\tilde{q}} = \tilde{F}_1(\tilde{q})$. Near $E_1$, these trajectories are locally equivalent according to Lemma 9 and 10 of [2, page 306-307]. Hence, there exists a topological map $H^2_{r}$ mapping $c_2 \cup M_1 \cup M_2$ unto $\tilde{c}_2 \cup M_1 \cup M_2$.

For a sufficiently small neighbourhood $N(E_1)$, by Lemma 3 of [58, page 194], there exist topological maps $H^1_{r}$ and $H^2_{r}$ from trajectories of $A$ in $c_1 \cap N(E_1)$ and $c_3 \cap N(E_1)$, respectively, onto trajectories of $\tilde{A}$ in sectors $\tilde{c}_1 \cap \tilde{N}(E_1)$ and $\tilde{c}_3 \cap \tilde{N}(E_1)$, respectively. According to [58, page 196], the topological maps $H^1_{r}$ and $H^2_{r}$ can be chosen to coincide at separatrices $M_1, M_2$ with $H^2_{r}$. In this manner, we obtain a topological map $H_r : N(E_1) \cap (\Sigma \cup S_1) \rightarrow \tilde{N}(E_1) \cap (\tilde{\Sigma} \cup \tilde{S}_1)$.

At $E_1$ one finds $F_1(E_1) = 0$, such that $L_{F_2} h(E_1) = -2F_s$, see (3.4). Hence,
Lemma A.13 is applied. The domain of definition of $H_r$ is extended towards a subset of $S_2$ such that $H_r$ maps $N(E_1)$ onto $\tilde{N}(\tilde{E}_1)$.

Lemmas A.15 and A.16 guarantee structural stable properties for system \((3.9)\) in neighbourhoods of the endpoints of an equilibrium set under certain conditions on the linearised dynamics around these endpoints. These results are combined with Theorem 3.1, to prove Theorem 3.2.

**Proof of Theorem 3.2.** Let $\tilde{A}$ be an arbitrary system satisfying \((3.8)\) for $\epsilon > 0$ sufficiently small. By Lemma A.10, we may assume that the equilibrium sets of $A$ and $\tilde{A}$ coincide. According to Assumption 3.2, the eigenvalues of $A_1$ and $A_2$ are either complex or real, nonzero and distinct. In the first case, Lemma A.15 guarantees that there exists a topological map $H_k$, $k = 1, 2$, from a neighbourhood $N(E_k)$ of $E_k$ to a neighbourhood of $\tilde{E}_k$; in the case of real eigenvalues this is guaranteed by Lemma A.16. Now, in the neighbourhood $N(E_1)$ we select two arbitrary points $p^a_1, p^b_1$ such that $p^a_1, p^b_1 \in I \cap N(E_1)$ and $0 < |p^a_1 - E_1| < |p^b_1 - E_1|$, see Fig. A.6. Analogously, we select two points $p^a_2, p^b_2$. Choosing $J = \text{co}\{p^a_1, p^b_1\}$ yields $J \subset I$, such that Lemma A.14 implies there exists a topological map $H_J$ defined in a neighbourhood $N(J)$ of $J$. According to \cite[page 196]{2}, the topological maps $H_k$, $k = 1, 2$, can be chosen to coincide with $H_J$ at the equilibrium points, such that $H_k(p) = H_J(p)$, $\forall p \in \text{co}\{p^a_k, p^b_k\}$ for $k = 1, 2$.

To obtain a topological map that is continuous and coincides with $H_k$ near $E_k$, $k = 1, 2$, and coincides with $H_J$ for points further away from these endpoints, transition regions $P_1$ and $P_2$ are introduced, given by $P_k(J) := \{q \in N(J) \cap N(E_k) : \lim_{t \to \infty} \varphi(q, t) \in \text{co}\{p^a_k, p^b_k\}\}$, $k = 1, 2$, see Fig. A.6. In these regions, new topological maps are introduced that connect $H_J$ and $H_k$, $k = 1, 2$, in a continuous fashion.

In this manner, a topological map is constructed in a neighbourhood of $\mathcal{E}$, which is constructed as follows. One can select a subset $N'(J) \subset N(J)$ containing $J$, and $N'(E_k) \subset N(E_k)$, $k = 1, 2$, such that $N'(E_1) \cup N'(E_2) \cup P_1 \cup P_2 \cup N'(J)$
contains a neighbourhood of \( E \), the interiors of the sets \( N'(E_k), k = 1, 2 \), and \( N'(J) \) have an empty intersection and the domains \( N'(E_k), k = 1, 2 \), and \( N'(J) \) intersect with \( P_k \) only at a one-dimensional set, cf. Fig. A.7. For \( q \in P_k, k = 1, 2 \), we will construct a topological map \( H_{P_k} \) that coincides with \( H_k \) for \( q \in N'(E_k) \) and coincides with \( H_J \) for \( q \in N'(J) \).

We will now proceed to construct the map \( H_{P_1} \) that connects \( H_1 \) and \( H_J \) in a continuous fashion. Analogously, a map \( H_{P_2} \) can be constructed. Since \( H_J \) and \( H_1 \) coincide for \( q \in P_1 \cap J = \text{co}\{p_1^a, p_1^b\} \), we need to construct a map \( H_{P_1} \) that, firstly, coincides with \( H_J \) at the intersection of the closures of \( N'(J) \) and \( P_1 \) and, secondly, coincides with \( H_1 \) at the boundary between \( N'(E_1) \) and \( P_1 \).

Let the function \( H_{P_1} \) coincide with \( H_J \) for \( q \in P_1 \cap J \). Since \( P_1 = (P_1 \cap S_1) \cup (P_1 \cap J) \cup (P_1 \cap S_2) \), firstly we will construct the map \( H_{P_1} \) for points in \( P_1 \cap S_1 \) such that it coincides with \( H_J \) at the boundary between \( P_1 \) and \( N'(J) \), which is denoted with \( B \), and coincides with \( H_1 \) at the boundary between \( P_1 \) and \( N'(E_1) \), which is denoted with \( C \), cf. Fig. A.7.

Both maps \( H_1 \) and \( H_J \) satisfy \( H_1(q) \in \tilde{S}_1 \) and \( H_J(q) \in \tilde{S}_1 \) when \( q \in S_1 \). Hence, we will introduce new coordinates in \( \tilde{P}_1 := \{ H_1(q), q \in P_1 \} \subset \tilde{S}_1 \). Similar to the proof of Lemma A.13, for points \( \tilde{q} \in \tilde{P}_1 \) we introduce the new coordinates \( (\tau, \rho) \) given by \( \psi : \tilde{P}_1 \to [T, 0] \times [p_1^a, p_1^b] \) with \( T < 0 \), such that

\[
(\tau, \rho) = \psi(\tilde{q}), \quad \tilde{\varphi}(t, \tau, \rho) = \tilde{q}
\]

and \( \rho \in [p_1^a, p_1^b] \subset J \), where \( \tilde{q}(t) = \tilde{\varphi}(t, \tilde{q}_0) \) is a trajectory of \( \tilde{q} = \tilde{F}_1(\tilde{q}) \) with initial condition \( \tilde{q}(0) = \tilde{q}_0 \). For \( q \in P_1 \cap S_1 \), both the maps \( H_1(q) \) and \( H_J(q) \) can be expressed in the coordinates \( (\tau, \rho) \), such that

\[
(H_1^\tau, H_1^\rho) = \psi(H_1(q)) \quad \text{and} \quad (H_J^\tau, H_J^\rho) = \psi(H_J(q)).
\]

The functions \( H_1^\tau(q) \) and \( H_J^\rho(q) \) denote the position of the first crossing with the boundary \( \Sigma \) of trajectories of system \( A \) with initial conditions \( q \). At \( \Sigma \cap P_1 \), the maps \( H_1 \) and \( H_J \) coincide since \( \Sigma \cap P_1 \) is part of the equilibrium set. Since both \( H_1 \) and \( H_J \) map trajectories onto trajectories and the maps coincide for \( q \in \Sigma \cap P_1 = \text{co}\{p_1^a, p_1^b\} \), we find that \( H_1^\rho(q) = H_J^\rho(q), \forall q \in P_1 \).

Recall that the map \( H_{P_1} \) should coincide with \( H_J \) at curve \( B \) and with \( H_1 \) at curve \( C \). For the variable \( \tau \), we introduce a linear interpolation \( H_{P_1}^\tau \) given by

\[
H_{P_1}^\tau(q) = H_1^\tau(q) + \frac{H_1^\rho(q) - p_1^\rho}{p_1^\rho - p_1^\alpha}(H_J^\rho(q) - H_1^\rho(q)),
\]

which clearly is a function that blends \( H_1^\tau \) and \( H_J^\tau \) in set \( P_1 \cap S_1 \). Transforming \( (H_1^\tau(q), H_{P_1}^\tau(q)) \) to the coordinates \( \tilde{q} \), a topological map \( H_{P_1} \) is defined in \( P_1 \cap S_1 \). Expressed in \( (\tau, \rho) \) coordinates, it immediately follows that, firstly, \( H_{P_1} \) coincides with \( H_1 \) at curve \( B \), secondly, it coincides with \( H_1 \) at curve \( C \) and, finally, \( H_{P_1}(p) = H_1(p) = H_J(p), \forall p \in P_1 \cap J \). Analogously, \( H_{P_2} \) is defined in \( P_1 \cap S_2 \), such that \( H_{P_2}(q) \) is defined for all \( q \in P_1 \). The map \( H_{P_2} \) can be constructed analogously.

Now, we define

\[
H(q) = \begin{cases} 
H_1(q), & q \in N'(E_1), \\
H_J(q), & q \in N'(J), \\
H_2(q), & q \in N'(E_2), \\
H_{P_1}(q), & q \in P_1,
\end{cases}
\] (A.51)
Appendix A. Proofs and technical results

Fig. A.6. Schematic construction of domains $P_1$ and $P_2$, which are shown hatched.

Fig. A.7. Schematic construction of a neighbourhood of $E$.

which is a topological map, proving the theorem. □

A.2.4 Proof of Lemma 3.3

Proof of Lemma 3.3. If system (3.9) contains a closed orbit $\gamma$ with period time $T_\gamma$, then reversing the direction of time does neither alter the graph of this periodic trajectory nor its period time. Consequently, without loss of generality, we assume $F_s > 0$. The existence of a minimum distance between interior points of the equilibrium set and a limit cycle is trivial, since trajectories close to the interior points are attracted to the equilibrium set in finite time, cf. Lemma A.12. Hence, we need to prove that any limit cycle remains away from the endpoints $E_1$ and $E_2$. In this proof we will show that the limit cycle does not come close to the equilibrium point $E_1$, the second case (of $E_2$) can be excluded analogously. For the sake of contradiction, we assume that a limit cycle $\gamma$ comes arbitrarily close to $E_1$ and has finite period $T_\gamma$. Since the vector field $F_2$ is pointing towards $S_1$ in a neighbourhood of $E_1$, the limit cycle should leave the neighbourhood of $E_1$ in direction of $S_1$. Since the dynamics in $S_1$ is described by the smooth differential equation $\dot{q} = F_1(q)$, we observe that $\|\dot{q}\|$ becomes arbitrarily small near $E_1$. Hence, a trajectory on $\gamma$ will spend an arbitrarily long period of time in the neighbourhood of $E_1$. This implies that the period time of the closed orbit
γ becomes arbitrarily long, yielding a contradiction. The lemma is proven.  \qed
A.3 Proofs of theorems and lemmas of Chapter 5

Proof of Lemma 5.1. First, we will investigate the geometry of $S^1 := R(Q)$ and of $S^2 := R^2(Q)$. Subsequently, using induction, we will describe the geometry of $S^l := R^l(Q)$ for arbitrary $l \in \mathbb{N}$. Using the definition $M^u := \bigcap_{i=0}^{\infty} R^i(Q)$, we observe that $M^u = R^\infty(Q) =: S^\infty$. Namely, since $R(y) = \emptyset$ if $y \notin R^{-1}(Q)$, it can directly be observed that $R^{l+1}(Q) \subset R^l(Q)$ for $l \in \mathbb{N}$.

First, we will investigate the set $S^1 = R(Q)$. Introducing $P_0 := p_0 \times [0,1]$ and $P_1 := p_1 \times [0,1]$, with $p_0, p_1$ as given in (5.3), we observe that $S^1 = R(P_0) \cup R(P_1)$, since $R(y) = \emptyset$, $\forall y \notin R^{-1}(Q) = P_0 \cup P_1$. Using (5.3) and the property that $R_u$ maps $\mathbb{P}$ into the graph of the function $f_c$. We denote this curve with $c$.

We will now proceed to describe the geometry of $R(P_1)$. From (5.3), we observe that if $(u,v) \in P_1$, then the map $R$ can be written as:

$$R(((u,v)^T) = \begin{cases} 
    \left( R_u(u) \ R_u(v) \right)^T, & R_v(v) < f_c(R_u(u)) \land u \in p_1, \\
    \left( R_u(u) \ f_c(R_u(u)) \right)^T, & R_v(v) \geq f_c(R_u(u)) \land u \in p_1.
\end{cases} \quad (A.52)$$

From Assumption 5.1, we observe that on the set $p_1$, the map $R_u$ is continuous and satisfies (5.4), and, consequently, there exists a unique inverse $(R_u|_{p_1})^{-1} : [0,1] \to p_1$ in this domain. In addition, from Assumption 5.1c., we observe that $R_v$ is invertible. Therefore, if $(u,v)^T \in P_1$ and the map $R((u,v)^T)$ is given by the first case in (A.52), then the map $R$ is locally invertible, i.e. $(u,v)^T = \left( (R_u|_{p_1})^{-1}(u^+) \ R_v^{-1}(v^+) \right)^T$ gives the unique point in $P_1$ such that $R((u,v)^T) = (u^+ v^+)^T$. Due to this invertibility property, the points given in the first case of (A.52) are mapped from $P_1$ on a two-dimensional domain, which we denote with $C$. Hence, for all $(u^+ v^+)^T \in C$, the pre-image of the map $R$, i.e. $R^{-1}(u^+ v^+)^T$, is single-valued.

If points $(u,v)^T \in P_1$ fall in the second case of (A.52), then these points are mapped into the graph of the function $f_c$, which has been denoted before with $c$. For this reason, the image of $P_1$ consists of a two-dimensional set $C$ (for those points $(u,v)^T \in P_1$ for which $R_v(v) < f_c(R_u(u)))$ and a subset of the curve $c$. Hence, $S^1 = R(P_0) \cup R(P_1) = C \cup c$. For future reference, we note that Assumption 5.1 (in particular, that $f_c(u) = 0$ for $u \in p_0$ and $R_v(v) \geq 0$ for all $v \in [0,1]$) implies that the set $C$ has an empty intersection with $P_0$. Since $f_c(R_u(u)) = \frac{1}{2}$ for $R_u(u) \in p_1$ and since $R_v(v) < \frac{1}{2}$ for all $v$, we observe that if
A.3 Proofs of theorems and lemmas of Chapter 5

$u \in p_1$ and $R_u(u) \in p_1$, then $R$ is given by the first case of (A.52) and

$$R(P_1) \cap P_1 = C \cap P_1.$$  \hspace{1cm} (A.53)

Now, we will study $S^2 := R^2(Q)$. Since the map $R$ maps into $Q$, we may write $S^2 = R(R(Q) \cap R^{-1}(Q))$. Hence, we obtain $S^2 = R(R(Q) \cap P_0) \cup R(R(Q) \cap P_1)$. Using the relation $R(Q) = C \cup c$ and the fact that $C \cap P_0 = \emptyset$, we obtain

$$S^2 = R(c \cap P_0) \cup R(c \cap P_1) \cup R(C \cap P_1).$$  \hspace{1cm} (A.54)

We will first describe the geometry of $R(c \cap P_0)$ and then describe $R(c \cap P_1)$. As stated above, $c$ coincides with the graph of the function $f_c$, such that $c \cap P_0$ is given by $p_0 \times \{0\}$, which follows from Assumption 5.1b. Since $R_u$ maps $p_0$ onto $[0,1]$, as stated in Assumption 5.1a, we conclude that $R(c \cap P_0) = \{(R_u(u) \ f_c(R_u(u)))^T, u \in p_0\} = \{(u^+ \ f_c(u^+))^T, u^+ \in [0,1]\} = c$. In what follows, we will refer to this curve as $h^0 := c$. Introducing the function $\tilde{h}^0(u) := f_c(u)$, we observe that $\tilde{h}^0$ is defined for $u \in [0,1]$, is continuous and is constant for $u \in p_0$ and $u \in p_1$, as stated in the lemma.

Using Assumption 5.1b., we observe that $c \cap P_1$ is described as:

$$c \cap P_1 = h^0 \cap P_1 = \{(u \ v)^T, u \in p_1, v = \frac{1}{2}\},$$  \hspace{1cm} (A.55)

which is a line interval. The fact that $R_u$ maps $p_1$ onto the interval $[0,1]$, (as stated in Assumption 5.1a.) implies that the $u$-coordinates of the curve $h^1 := R(h^0 \cap P_1)$ cover the entire range from 0 to 1. Since $R$ is continuous in $P_1$, we observe that $h^1$ is a connected curve, and, consequently, can be described as the graph of a continuous function, which we denote with $\tilde{h}^1(u)$. Using (A.55) and the assumption that $R_v(v) < \frac{1}{2}$ for all $v$, as given in Assumption 5.1c., we observe that the function $\tilde{h}^1$ satisfies $\tilde{h}^1(u) = R_v(\frac{1}{2})$ for $u \in p_1$. Hence, the function $\tilde{h}^1$ is defined for $u \in [0,1]$, is continuous and is constant for $u \in p_0$ and $u \in p_1$, as stated in the lemma.

Using the results above, we may rewrite (A.54) as:

$$S^2 = h^0 \cup h^1 \cup R(C \cap P_1) = \bigcup_{k=0}^{1} h^k \cup R \left( \bigcap_{k=0}^{1} R^k(P_1) \right),$$  \hspace{1cm} (A.56)

where, in the last equality, we used (A.53).

We will now employ an induction argument and describe the geometry of $S^{n+1}$, for $n > 2$, using the assumed geometry of the set $S^n$. Hence, to apply an induction argument, assume that given $n \geq 2$, the set $S^n$ can be described by:

$$S^n = \bigcup_{k=0}^{n-1} h^k \cup R \left( \bigcap_{k=0}^{n-1} R^k(P_1) \right),$$  \hspace{1cm} (A.57)
with each curve $h^k, k = 0, \ldots, n - 1$, given by graphs of a function $\tilde{h}^k$ that is
defined for $u \in [0, 1]$, is continuous and is constant for $u \in p_0$ and $u \in p_1$, as
stated in the lemma. To employ induction, observe that this assumption implies,
using the same arguments as mentioned above, that the set $S^{n+1}$ can be written as:
\[
S^{n+1} = \bigcup_{k=0}^{n} h^k \cup R \left( \bigcap_{k=0}^{n} R^k(P_1) \right),
\]
(A.58)
with the curve
\[
h^n := R(h^{n-1} \cap P_1).
\]
(A.59)
From the properties of the map $R$ as stated in Assumption 5.1, it directly follows
that the curve $h^n$ can be described by the graph of a function $\tilde{h}^n$ that is defined
for $u \in [0, 1]$, is continuous and is constant for $u \in p_0$ and $u \in p_1$, as stated in
the lemma.

We can now apply induction, proving that
\[
S^{l+1} = \bigcup_{k=0}^{l} h^k \cup R \left( \bigcap_{k=0}^{l} R^k(P_1) \right), \forall l \in \{2, 3, \ldots\}.
\]
(A.60)
Using the fact that $\mathcal{M}^u = S^\infty$, we obtain:
\[
\mathcal{M}^u = \bigcup_{k=0}^{\infty} h^k \cup R \left( \bigcap_{k=0}^{\infty} R^k(P_1) \right).
\]
(A.61)
Clearly, the set $\bigcup_{k=0}^{\infty} h^k$ is a union of a countable set of curves. We will
now show that $h^* := R \left( \bigcap_{k=0}^{\infty} R^k(P_1) \right)$ also is a curve. Assumption 5.1c.
states that the map $R$ is contracting in $v$-direction for $(u, v) \in P_1$. Hence, under the
forward iterates of $R$, the set $C$ will be contracted in $v$-direction: the width
in $v$-direction of the set $\bigcap_{k=0}^{\infty} R^k(P_1)$ is not larger than $\lambda_u$, with $\lambda_u$ given in
Assumption 5.1. Here, we used (A.53) and observed that $R(P_1) \cap P_1$ is a subset
of the image of the first case in (A.52), such that the size in $v$-direction of
$R(P_1) \cap P_1$ is at most $\lambda_u$. We can now conclude that $\bigcap_{k=0}^{\infty} R^k(P_1)$ is a curve
and, using continuity of $R$, find that $h^* = R \left( \bigcap_{k=0}^{\infty} R^k(P_1) \right)$ is a curve. Hence,
we observe that $\mathcal{M}^u = h^* \cup \bigcup_{k=0}^{\infty} h^k$, proving the first statement of the lemma.

For all $i > 1$, $h^i \in R \left( \bigcap_{k=0}^{i} R^k(P_1) \right)$. As mentioned before, the latter set
contracts to the curve $h^*$ for $i \to \infty$, such that we observe that the curves $h^i$
converge to the line $h^*$ for $i \to \infty$.

In order to prove that $h^i \cap h^j \cap P_1 = \emptyset$ for $i, j \in \mathbb{N} \cup \{\star\}$, $i \neq j$, first, observe that
\[
h^0 \cap h^k \cap P_1 = \emptyset, \forall k \in \mathbb{N} \cup \{\star\}.
\]
(A.62)
since $h^0 \cap P_1$ is given in (A.55), $h^k \cap P_1 \subset R(P_1) \cap P_1 \subset C$, for $k \in \mathbb{N} \cup \{\star\}$, $k \neq 0$.
(the first and second inclusion follow from (A.59) and (A.53), respectively), and
all \(\nu\)-coordinates of points \((u \nu)^T \in C\) satisfy \(\nu < \frac{1}{2}\). Namely, since \(R_\nu(v) < \frac{1}{2}\) for all \(v \in [0, 1]\), as stated in Assumption 5.1c., the construction of the set \(C\) implies that \(\nu < \frac{1}{2}\) for \((u \nu)^T \in C\).

Recall that for all \((u^+ \nu^+)^T \in C\), the pre-image of the map \(R\) is single-valued, i.e. \(R^{-1}(u^+ \nu^+)^T\) is a singleton. Now, in order to obtain a contradiction, assume that there exists \(i, j \in \mathbb{N} \cup \{\ast\}\), \(i, j \neq 0\) and \(i \neq j\), and a point \(y \in P_1\) such that \(y \in h^i \cap h^j \cap P_1\). Without loss of generality, assume that \(i \neq \ast\), and \(j > i\) if both \(i, j \in \mathbb{N}\). Observe that \(h^k \cap P_1 \subset C\), \(\forall k \geq 1\), and, consequently, the first \(i\) iterates of the inverse map of \(R\), i.e. \(R^{-k}(y)\) for \(k \in \{0, \ldots, i\}\), are single-valued.

Using \(i \in \mathbb{N}\) and the definitions of the curves \(h^k\), \(k \in \mathbb{N} \cup \{\ast\}\), and the assumption \(y \in h^i \cap P_1\) which was posed in the previous paragraph, one finds

\[
R^{-i}(y) \subset h^0 \cap P_1. \tag{A.63}
\]

If \(j \neq \ast\), such that \(j \in \mathbb{N}\), \(j > i\), then the expression \(y \in h^j \cap P_1\) implies

\[
R^{-i}(y) \subset h^{j-i} \cap P_1. \tag{A.64}
\]

A contradiction is now obtained, since the singleton \(R^{-i}(y)\) cannot be positioned both in \(h^0 \cap P_1\) and in \(h^{j-i} \cap P_1\), with \(j - i > 0\), since these sets have an empty intersection according to (A.62). If \(j = \ast\), then \(y \in h^* \cap P_1\) implies \(R^{-i}(y) \subset h^* \cap P_1\), since, from continuity of \(R\) and the definition of \(h^*\), it follows that \(h^* = R(h^* \cap P_1)\). Again, the singleton \(R^{-i}(y)\) cannot simultaneously satisfy \(R^{-i}(y) \subset h^* \cap P_1\) and \(R^{-i}(y) \subset h^0 \cap P_1\) due to (A.62).

Hence, for both \(j \in \mathbb{N}\), \(j > i\) and for \(j = \ast\), a contradiction is obtained, such that \(h^i \cap h^j \cap P_1 = \emptyset\) when \(i, j \in \mathbb{N} \cup \{\ast\}\), \(i, j \neq 0\), and \(i \neq j\). From (A.62), we observe that we can select \(i = 0\) in this expression as well, such that the statement \(h^i \cap h^j \cap P_1 = \emptyset\) for \(i, j \in \mathbb{N} \cup \{\ast\}\), \(i \neq j\), as presented in the lemma, is obtained.

The last statement of the lemma follows directly from the construction of the curves \(h^i, i \in \mathbb{N}\), as given in (A.59), such that the lemma is proven. \(\square\)

Proof of Theorem 5.2. First, we prove that \(\tau : \mathcal{X} \rightarrow S_c\) is surjective and continuous. Subsequently, we will prove invertibility of \(\tau\), and continuity of its inverse. Finally, we show \(\sigma_c(\tau(y)) = \tau(R_{\mid \mathcal{X}}(y))\), such that the map \(\tau\) is a topological conjugacy as in Definition 5.3, with \(\Phi = R_{\mid \mathcal{X}}\) and \(\Psi = \sigma_c\).

Surjectiveness of \(\tau\)
To prove that \(\tau\) is surjective, we prove that for each \(a \in S_c\) there exists an \(y \in \mathcal{X}\) such that \(a = \tau(y)\).

First, we will show that given any \(a \in S_c\), there exists a \(u \in [0, 1]\) such that for appropriate \(v\), the points \(\tau((u \nu)^T) \in S_c\) and \(a \in S_c\) have identical elements \(\tau((u \nu)^T) = a\) for \(i \in \mathbb{N}\). For \(i \geq 0\), from (5.16), we conclude that \(\tau((u \nu)^T)\) is determined by the \(u\)-coordinate of \(R^i((u \nu)^T)\). From (5.3), we observe that
these points are described by the iterates of the map \( R_u \). By construction of the sets \( P_0, P_1, \) a point \( R^i((u, v)^T) \), for \( i \geq 0 \), satisfies \( R^i((u, v)^T) \in P_0 \), or \( R^i((u, v)^T) \in P_1 \) if and only if \( R_u^i(u) \in p_0 \) or \( R_u^i(u) \in p_1 \), respectively.

By Assumption 5.1a., the map \( R_u \) maps both \( p_0 \) and \( p_1 \) onto \([0, 1]\). Consequently, \( R_u^{-1}(p_i) \cap p_j \neq \emptyset \), for \( i, j \in \{0, 1\} \).

We observe that in two iterates, the set \( R_u^{-1}(p_i) \cap p_j \) is mapped onto the complete line interval \([0, 1]\). Using this property iteratively, given a point \( a \in S_c \), we observe that the set \( s_j(a) \) defined by

\[
s_j(a) := \bigcap_{m=0}^{j} R_u^{-1}(p_{a_m}), \quad j \in \mathbb{N}
\]

is nonempty for every finite \( j \). In addition, the iterates of the map \( R_u \) will satisfy \( R_u^m(u) \in p_{a_m} \) for all \( m \in \{0, \ldots, j\} \) if \( u \in s_j(a) \).

Observe that \( \{s_j(a)\}_{j \in \mathbb{N}} \) is a sequence of nested nonempty closed intervals on \( \mathbb{R} \), and for this reason, \( s_\infty(a) := \lim_{j \to \infty} s_j(a) \) is nonempty and closed, as stated in the nested sphere theorem in [93, p60]. Hence, given any point \( (u, v)^T \in \mathcal{X} \) with \( u \in s_\infty(a) \), we observe that \( R^i((u, v)^T) \in P_{a_i}, \forall i \geq 0 \). Consequently, \( \tau((u, v)^T) = a_i, \forall i \geq 0 \), cf. (5.16).

Now, given \( a \in S_c \), we will select \( v \) such that \( \tau((u, v)) = a \) if \( u \in s_\infty(a) \). For this purpose, let \( k \in \mathbb{N} \cup \{\star\} \) either be the maximum integer such that \( a_i = 1, \quad i \in [-k, -1] \), or be given by \( k = \star \) if \( a_i = 1, \forall i \in (-\infty, -1] \). Let \( v = \tilde{h}^k(u) \), with \( \tilde{h}^k \) as defined in Lemma 5.1. Then \( (u, v)^T \in \mathcal{X} \) is such that \( a = \tau((u, v)^T) \). Since \( a \in S_c \) was taken arbitrarily, \( \tau : \mathcal{X} \to S_c \) is surjective.

\[\text{Continuity of } \tau\]

To prove continuity of \( \tau \) in (5.16), we show that for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( y_0, y_1 \in \mathcal{X} \) with \( \|y_0 - y_1\| < \delta \), \( \rho_c(\tau(y_0), \tau(y_1)) < \epsilon \), with \( \rho_c \) as in Definition 5.2. For this purpose, we first select an integer \( \ell \geq 1 \) satisfying

\[
\ell > -\log_2(\epsilon),
\]

and observe that if the distance \( \rho_c(\tau(y_0), \tau(y_1)) \) is evaluated and \( \ell \), as given in Definition 5.2, can be chosen to satisfy \( \ell \geq \tilde{\ell} \), then \( \rho_c(\tau(y_0), \tau(y_1)) < \epsilon \) is obtained. Hence, we will prove continuity of \( \tau \) by proving that given an arbitrarily large integer \( \ell \geq 1 \), we can select \( \delta > 0 \) such that \( \tilde{\ell} \leq \ell \) in the evaluation of \( \rho_c(\tau(y_0), \tau(y_1)) \).

Note that \( \|y_0 - y_1\| < \delta \) implies \( |u_0 - u_1| < \delta \) if we use the notation \( y_i = (u_i, v_i)^T, \quad i = 0, 1 \). Let \( G = \inf_{u_0 \in p_0, u_1 \in p_1} |\tilde{u}_0 - \tilde{u}_1| \) be the minimum distance between \( p_0 \) and \( p_1 \). If we select \( \delta < \tilde{\lambda}^{-\ell}G \), with \( \tilde{\lambda} \) given in Assumption 5.1a., then, using the same assumption, we conclude that \( |R_u^i(u_0) - R_u^i(u_1)| \leq \tilde{\lambda}^i\delta < G \),
for all \( i \in [0, \bar{\ell}] \), which, by the design of \( G \), implies that for each \( i \in [0, \bar{\ell}] \), \( R^i_0(u_0) \) and \( R^i_0(u_1) \) lie in the same set \( p_0 \) or \( p_1 \). Since the \( u \)-coordinates of the map \( R \), as given in (5.3), are described by the iterates of the map \( R_u \), we conclude that \( R^i(y_0) \) and \( R^i(y_1) \) lie in the same set \( P_0 \) or \( P_1 \) for each \( i \in [0, \bar{\ell}] \). Hence, (5.16) implies \((\tau(y_0))_i = (\tau(y_1))_i \) for \( i \in [0, \bar{\ell}] \).

First consider the case \((\tau(y_0))_0 = (\tau(y_1))_0 = 0 \) and afterwards, the case \((\tau(y_0))_0 = (\tau(y_1))_0 = 1 \).

If \((\tau(y_0))_0 = (\tau(y_1))_0 = 0 \), then, in the evaluation of \( \rho_c(\tau(y_0), \tau(y_1)) \) as given in Definition 5.2, we can set \( k < 1 \), such that the conclusion of the previous paragraph directly implies that \( \ell \), as given in Definition 5.2, can be chosen larger than, or equal to \( \bar{\ell} \). Hence, \( \rho_c(\tau(y_0), \tau(y_1)) = 2^{-\ell} \leq 2^{-\bar{\ell}} \) from (A.67), we conclude \( 2^{-\ell} < \epsilon \), such that we obtain \( \rho_c(\tau(y_0), \tau(y_1)) < \epsilon \).

If \((\tau(y_0))_0 = (\tau(y_1))_0 = 1 \), then the distance \( \rho_c(\tau(y_0), \tau(y_1)) \) also depends on the \( v \)-coordinates of the points \( y_0 \) and \( y_1 \), since \((\tau(y_0))_0 = (\tau(y_1))_0 = 1 \) imply \( y_0, y_1 \in P_1 \). The integers \( \bar{\kappa}(y_0) \) and \( \bar{\kappa}(y_1) \) as given the theorem may be different, since all curves \( h^i, i \in \mathbb{N} \cup \{\ast\} \), have a nonempty intersection with \( P_1 \), and, consequently, both points \( y_1 \) and \( y_2 \) can be positioned in any curve \( h^i, i \in \mathbb{N} \cup \{\ast\} \).

Given any finite integer \( \bar{\ell} > 0 \), there exists a \( \delta_{\bar{\ell}} > 0 \) such that for each \( i \in [0, \bar{\ell}] \), the curve \( h^i \cap P_1 \) is more than the distance \( \delta_{\bar{\ell}} \) away from all the other curves \( h^j \), with \( i, j \in \mathbb{N} \cup \{\ast\}, j \neq i \), i.e.

\[
P_1 \cap h^i + B_{\delta_{\bar{\ell}}} \cap h^j = \emptyset, \quad \forall j \in (\mathbb{N} \cup \{\ast\}) \setminus \{i\}, \quad \text{for each} \quad i \in [0, \bar{\ell}]. \tag{A.68}
\]

Here, we used the facts that for each finite \( i \geq 0 \), both the curve \( h^i \) and the set \( \{y \in Q, \exists j \in (\mathbb{N} \cup \{\ast\}) \setminus \{i\}, y \in h^j\} \) are closed, and \( h^i \cap h^j \cap P_1 = \emptyset \) for \( i \neq j \), \( i, j \in \mathbb{N} \cup \{\ast\} \), as stated in Lemma 5.1. From (A.68), we conclude that, if \( y_0, y_1 \in P_1 \cap X \) and \( |y_0 - y_1| < \delta_{\bar{\ell}} \), then either \( y_0 \) and \( y_1 \) are positioned in the same curve \( h^i, i \in [0, \bar{\ell}] \) (such that \( \bar{\kappa}(y_0) = \bar{\kappa}(y_1) \)), or \( y_0 \in h^{k_0} \) and \( y_1 \in h^{k_1} \), with \( k_0, k_1 \in [\bar{\ell} + 1, \infty) \). In both cases, from (5.16), we conclude that \((\tau(y_0))_i = (\tau(y_1))_i \) for \( i \in [\max(-\bar{\ell}, -\bar{\kappa}(y_0)), -1] \).

Consequently, if \( \delta > 0 \) is chosen smaller than \( \min(\delta_{\bar{\ell}}, \bar{\lambda} - \bar{\ell})G \), then we have obtained \((\tau(y_0))_i = (\tau(y_1))_i \) for \( i \in [\max(-\bar{\ell}, -\bar{\kappa}(y_0)), -1] \). We find \( \rho_c(\tau(y_0), \tau(y_1)) < \epsilon \), proving continuity of \( \tau \).

**Invertibility of \( \tau \)**

To prove invertibility of \( \tau \), assume that there exists an \( a \in S_0 \) and two distinct points \( y_0, y_1 \in X \) such that \( \tau(y_0) = \tau(y_1) = a \). Let \( y_i = (u_i, v_i)^T \), \( i = 0, 1 \). First, we will show that \( u_0 = u_1 \) if \( \tau(y_0) = \tau(y_1) = a \). Using the same construction as above, let the intervals \( s_i(a), i \in \mathbb{N} \), denote the \( u \)-coordinates of the initial conditions in \( Q \) whose trajectories visit the sequence \( a_0 \ldots a_i \) of domains \( P_0 \) and \( P_1 \) in the next \( i \) iterates of the map \( R \), such that \( s_0(a) = p_{a_0} \) and \( s_{i+1}(a) = \{u \in s_i(a), R^i_{u+1}(u) \in p_{a_{i+1}}\} \). From Assumption 5.1a. and the definition of \( s_i(a) \), for
\(i \in \mathbb{N}\) and \(a \in S_c\), given in (A.66), we observe that
\[
\sup_{u_0, u_1 \in s_i(a)} |u_0 - u_1| \leq \lambda^{-i}, \forall i \geq 0, \text{ with } \lambda > 1. \tag{A.69}
\]

Since \(\tau(y_0) = \tau(y_1)\), the symbols \(\tau(y_0)_i\) and \(\tau(y_1)_i\) coincide for all \(i \geq 0\), which implies that \(u_0, u_1 \in s_i(a), \forall i\). Hence, with (A.69), we obtain \(u_0 = u_1\).

To study the \(v\)-coordinates of the points \(y_0\) and \(y_1\), observe that the definitions of \(\tau\) and \(\tilde{\kappa}\) in the theorem imply that \(\tilde{\kappa}(y_0)\) and \(\tilde{\kappa}(y_1)\) have to be equal when \(\tau(y_0) = \tau(y_1)\), and hence, the \(v\)-coordinate of both \(y_0\) and \(y_1\) are given, for finite \(\tilde{\kappa}(y_0)\), by \(\tilde{h}^{\tilde{\kappa}(y_0)}(u_0)\), and, for \(\tilde{\kappa}(y_0) = \infty\), by \(\tilde{h}^\ast(u_0)\), with the functions \(\tilde{h}^i, i \in \mathbb{N} \cap \{\ast\}\), given in Lemma 5.1. Hence, we obtained \(v_0 = v_1\). Combination of this result with the conclusion of the previous paragraph, yields the conclusion that \(y_1 = y_0\), which contradicts distinctness of \(y_0\) and \(y_1\). A contradiction is obtained, such that for any two distinct points \(y_0, y_1 \in \mathcal{X}\), \(\tau(y_0) \neq \tau(y_1)\) holds. Hence, \(\tau\) is invertible.

**Continuity of \(\tau^{-1}\)**

Now, we will prove that \(\tau^{-1}\) is continuous by showing that for all \(\varepsilon > 0\) there exists a \(\delta > 0\) such that for \(a, b \in S_c, \rho_c(a, b) < \delta\) implies
\[
|\tau^{-1}(a) - \tau^{-1}(b)| < \varepsilon. \tag{A.70}
\]

In fact, we will prove the stronger requirement that both
\[
|\pi_u(\tau^{-1}(a)) - \pi_u(\tau^{-1}(b))| < \frac{\sqrt{2}}{2} \varepsilon, \tag{A.71}
\]
and
\[
|\pi_v(\tau^{-1}(a)) - \pi_v(\tau^{-1}(b))| < \frac{\sqrt{2}}{2} \varepsilon, \tag{A.72}
\]
hold when \(\pi_u(y)\) and \(\pi_v(y)\) denote the \(u\)-, or \(v\)-coordinates of points \(y \in \mathcal{X}\), respectively.

Given two points \(a, b \in S_c\), let \(k\) be as given in Definition 5.2 for the evaluation of \(\rho_c(a, b)\). We will prove continuity of \(\tau^{-1}\) by showing that, if we select a bounded \(L\) satisfying
\[
L > \frac{-\log \left(\frac{\sqrt{2}}{2} \varepsilon\right)}{\log(\lambda)}, \tag{A.73}
\]
and, in addition,
\[
a_i = b_i, \forall i \in [\max(-L, -k), L] \tag{A.74}
\]
holds, then (A.71) and (A.72) are satisfied. We note that for any bounded \(L\), (A.74) will be satisfied when \(\delta > 0\) is chosen sufficiently small and \(\rho_c(a, b) < \delta\) is satisfied.

Given \(a_i = b_i, \forall i \in [0, L]\), we observe that for each \(i \in [0, L]\), \(R^i(\tau^{-1}(a))\) and \(R^i(\tau^{-1}(b))\) are in the same set \(P_0\) or \(P_1\). Hence, \(\pi_u(\tau^{-1}(a))\) and \(\pi_u(\tau^{-1}(b))\) are contained in the same set \(s_L(a)\), with the interval \(s_L(a)\) as constructed above.
Combination of (A.69), (A.73) and \( \pi_u(\tau^{-1}(a)), \pi_u(\tau^{-1}(b)) \in s_L(a) \) proves that (A.71) is satisfied.

Now, we will prove for each \( \varepsilon > 0 \), there exists a finite \( L_\varepsilon > 0 \) such that if \( L > L_\varepsilon \), then (A.74) implies (A.72).

If \( \max(-L, -k) = -k \), then \( k \) is finite, \( \bar{\kappa}(\tau^{-1}(a)) = \bar{\kappa}(\tau^{-1}(b)) = k \), with the functions \( \bar{\kappa} \) as given in the theorem, and \( \tau^{-1}(a), \tau^{-1}(b) \in h^k \). From \( L \geq 1 \), we conclude that either \( \tau^{-1}(a) \in P_0 \) and \( \tau^{-1}(b) \in P_0 \), or \( \tau^{-1}(a), \tau^{-1}(b) \in P_1 \) hold. Hence, combination of the observations that \( \pi_v(\tau^{-1}(a)) = h^k(\pi_u(\tau^{-1}(a))) \), \( \pi_v(\tau^{-1}(b)) = h^k(\pi_u(\tau^{-1}(b))) \) and that every function \( h^k(u), k \in \mathbb{N} \cup \{\infty\} \), is constant for \( u \in P_0 \) or \( u \in P_1 \) leads to the conclusion that \( \pi_v(\tau^{-1}(a)) = \pi_v(\tau^{-1}(b)) \), such that (A.72) is satisfied when \( k \leq L \) is finite.

It remains to be proven that one can select \( L_\varepsilon > 0 \) such that (A.72) is satisfied when \( L > k \) and \( L > L_\varepsilon \). Again, we use the fact that \( L > 0 \), such that both \( \tau^{-1}(a) \) and \( \tau^{-1}(b) \) are positioned in the same set \( P_1 \) or \( P_0 \).

Since the curves \( h^i \) converge to \( h^* \) when \( i \) is increased to infinity, and, in addition, the functions \( h^i(u), i \in \mathbb{N} \cup \{\star\} \), are constant on the intervals \( u \in P_0 \) and \( u \in P_1 \), we observe that we can select a finite integer \( \bar{L}_\varepsilon \) such that \( |h^i(u_0) - h^i(u_1)| < \frac{\sqrt{2} \varepsilon}{2} \) for all \( i, j \in \{k \in \mathbb{N}, k \geq \bar{L}_\varepsilon \} \cup \{\star\} \), when \( u_0 \) and \( u_1 \) are positioned in the same interval \( P_0 \) or \( P_1 \). In addition, we note that \( L > \bar{L}_\varepsilon \), (A.74) and (5.16) together imply \( \tau^{-1}(a) \in h^{k_a} \) and \( \tau^{-1}(b) \in h^{k_b} \) with \( k_a, k_b \in \{k \in \mathbb{N}, k \geq \bar{L}_\varepsilon \} \cup \{\star\} \). Hence, we obtain

\[
|\pi_v(\tau^{-1}(a)) - \pi_v(\tau^{-1}(b))| = |h^{k_a}(\pi_u(\tau^{-1}(a))) - h^{k_b}(\pi_u(\tau^{-1}(b)))| < \frac{\sqrt{3}}{2} \varepsilon. \quad (A.75)
\]

We can now conclude that if \( L > \max(\bar{L}_\varepsilon, \frac{-\log(\sqrt{2} \varepsilon)}{\log(\lambda)}) \), then both (A.71) and (A.72) are guaranteed to hold, such that (A.70) is satisfied. Hence, if \( \delta = 2^{-L} \) holds, then \( |\tau^{-1}(a) - \tau^{-1}(b)| < \varepsilon \) is guaranteed when \( \rho_\varepsilon(a, b) < \delta \), such that continuity of \( \tau^{-1} \) is proven.

**Conjugacy**

We have now proven that the function \( \tau \) is a homeomorphism. It remains to be proven that \( \sigma_\varepsilon(\tau(y)) \sim \tau(R(y)) \). Let \( \bar{\kappa}(y) \) be as given in the theorem and observe that (5.16) implies

\[
(\sigma_\varepsilon(\tau(y)))_i = \tau(y)_{i+1} = \begin{cases} 
0, & i \in (-\infty, -\bar{\kappa}(y) - 2], \\
1, & i \in [-\bar{\kappa}(y) - 1, -2], \\
0, & i \in [-1, \infty) \cap R^{i+1}(y) \in P_0, \\
1, & i \in [-1, \infty) \cap R^{i+1}(y) \in P_1.
\end{cases} \quad (A.76)
\]

To show that this expression coincides with \( \tau(R(y))_i, i \in \mathbb{Z} \), we distinguish three cases, namely, \( R(y) \in P_0 \), \( R(y) \in P_1 \cap h^0 \) and \( R(y) \in P_1 \cap (h^* \cup \bigcup_{i=1}^{\infty} h^i) \).

In the first case, i.e. the case where \( R(y) \in P_0 \), from \( h^i \cap P_0 = h^j \cap P_0, \forall i, j \), as stated in Lemma 5.1, we conclude that \( \bar{\kappa}(R(y)) = 0 \). Hence, the definition of
\( \tau \) given in (5.16) gives, for \( \bar{\kappa}(R(y)) = 0 \),

\[
\tau(R(y))_i = \begin{cases}
0, & i \in (-\infty, -1], \\
0, & i \in [0, \infty) \cap R^{i+1}(y) \in P_0, \\
1, & i \in [0, \infty) \cap R^{i+1}(y) \in P_1,
\end{cases}
\tag{A.77}
\]

which coincides with (A.76), since we observe \( \tau(R(y))_0 = (\sigma_c(\tau(y)))_0 = 0 \) as \( R(y) \in P_0 \), and \( \tau(R(y))_i = (\sigma_c(\tau(y)))_i, \forall i \in [1, \infty) \), which, using Definition 5.1, directly imply \( \tau(R(y)) \sim \sigma_c(\tau(y)) \).

In the second case, i.e. where \( R(y) \in P_1 \cap h^0 \), we observe that \( \bar{\kappa}(R(y)) = 0 \). Hence, again, \( \tau(R(y)) \) is given by (A.77). Note that (A.76) and (A.77) clearly coincide for all indices \( i \neq -1 \). To evaluate (A.76) for \( i = -1 \), observe that \( R(y) \in h^0 \cap P_1 \), can only hold when \( y \in P_0 \), such that \( (\tau(y))_0 = 0 \), and therefore, \( (\sigma_c(\tau(y)))_{-1} = 0 \). Hence, (A.77) and (A.76) coincide for all \( i \), such that \( \tau(R(y)) \sim \sigma_c(\tau(y)) \).

The last case to be considered is \( R(y) \in P_1 \cap (h^* \cup \bigcup_{i=1}^{\infty} h^i) \). Since \( \bar{\kappa}(\tilde{y}) \), for \( \tilde{y} \in \mathcal{X} \), denotes the minimum integer such that \( \tilde{y} \in h^{\bar{\kappa}(\tilde{y})} \) if \( \bar{\kappa}(\tilde{y}) \) is finite, and \( \tilde{y} \in h^* \) if \( \bar{\kappa}(\tilde{y}) = \infty \), we conclude that \( \bar{\kappa}(R(y)) \in \{k \in \mathbb{N}, \ n \geq 1\} \cup \{\infty\} \).

It can be observed, for the case where \( \bar{\kappa}(y) \) is finite, that the last statement of Lemma 5.1 implies \( \bar{\kappa}(R(y)) = \bar{\kappa}(y) + 1 \). Hence, we find

\[
\tau(R(y))_i = \begin{cases}
0, & i \in (-\infty, -\bar{\kappa}(R(y)) - 1] = (-\infty, -\bar{\kappa}(y) - 2], \\
1, & i \in [-\bar{\kappa}(R(y)), -1] = [-\bar{\kappa}(y) - 1, -1], \\
0, & i \in [0, \infty) \cap R^{i+1}(y) \in P_0, \\
1, & i \in [0, \infty) \cap R^{i+1}(y) \in P_1,
\end{cases}
\tag{A.78}
\]

which clearly coincides with (A.76) for all \( i \neq -1 \). To see that (A.76) and (A.78) also coincide for \( i = -1 \), we note that \( R(y) \in h^{\bar{\kappa}(R(y))} \) implies \( y \in h^{\bar{\kappa}(R(y)) - 1} \cap P_1 \), cf. the last statement of Lemma 5.1. Hence, (A.76) implies \( \sigma_c(\tau(y))_{-1} = (\tau(y))_0 = 1 \), and, consequently, \( \sigma_c(\tau(y))_i = \tau(R(y))_i, \forall i \in \mathbb{Z} \).

From (5.8) and Lemma 5.1, we conclude that each \( R(y) \in \mathcal{X} \) falls in one of the cases described above. Therefore, we have proven that the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{R_1|_{\mathcal{X}}} & \mathcal{X} \\
\downarrow & & \downarrow \\
S_c & \xrightarrow{\sigma_c} & S_c
\end{array}
\]

commutes. Hence, the homeomorphism \( \tau \) is a topological conjugacy between the dynamical systems \( (\mathcal{X}, R_1|_{\mathcal{X}}) \) and \( (S_c, \sigma_c) \), which concludes the proof of the theorem.

\[\square\]

**Proof of Lemma 5.4.** Topological transitivity is invariant under topological conjugacies, cf. [110, p. 198], such that we will prove the equivalent statement that
$(S_c, \sigma_c)$ is topologically transitive. For this reason, using the approach given in [92], a point $a \in S_c$ is constructed whose orbit under $\sigma_c$ is dense in $S_c$.

Let $\{\epsilon_i\}_{i \in \mathbb{N}}$ be a sequence of strictly positive numbers monotonically converging to zero. First, for each $\epsilon_i > 0$, we will construct a finite block of symbols $a^i = a^0_i a^1_i \ldots a^n_i$, such that any trajectory of $\sigma_c$ from a point $b \in S_c$, where $b$ starts with the block $a^i$, comes in an $\epsilon_i$-neighbourhood of each point $r \in S_c$. By concatenating these blocks $a^i$, as we will show below, we will find a point $a = (\ldots, a^0 a^1 a^2 \ldots) \in S_c$ whose orbit is dense in $S_c$.

Namely, the orbit from the point $a$ under the shift map is dense if for each $\delta > 0$ and any point $p \in S_c$, we can find a time $T$ such that $\rho_c(\sigma_c^T(a), p) < \delta$. If the point $a$ is constructed as above, then, given $\delta$, we can select a finite $i > 0$ such that $\epsilon_i < \delta$. Since the point $a$ contains the block $a^i$ in its future elements, the orbit from $a$ will arrive in every $\epsilon_i$-neighbourhood in $S_c$, and, consequently, in an $\epsilon_i$-neighbourhood of the point $p$.

We will now proceed to construct the blocks $a^i$ with the properties given above. Given $\epsilon_i > 0$, observe that for each $r \in S_c$, $\{q \in S_c, \rho_c(r, q) < \epsilon_i\}$, consists of points $q$ with $r_i = q_i$ for $i \in [\max(-\ell^i, -k), \ell^i]$, where $k$ is the nonnegative integer as defined in Definition 5.2, and $\ell^i$ is the minimum number such that $2^{-\ell^i} < \epsilon_i$. Let $r_{[-\ell^i, \ell^i]}$ denote the block $r_{-\ell^i} r_{-\ell^i+1} \ldots r_{\ell^i}$.

Since each element $r_i, i \in [-\ell^i, \ell^i]$ of a point $r$ satisfies $r_i \in \{0, 1\}$, which contains two elements, we can select a finite number $n^i \leq 2^{2\ell^i+1}$ of different points $r^j$ such that $S_c = \bigcup_{j=1}^{n^i} \{q \in S_c, \rho_c(r^j, q) < \epsilon_i\}$. Hence, we can construct the block $a^i$, which has finite length, by concatenation of the blocks $r_{[-\ell^i, \ell^i]}$, such that $a^i = r^0_{[-\ell^i, \ell^i]} r^1_{[-\ell^i, \ell^i]} \ldots r^{n^i}_{[-\ell^i, \ell^i]}$. Any trajectory of $\sigma_c$ of a point $b \in S_c$ that contains the block $a^i$ in the sequence $b_i, i \in \mathbb{N}$, becomes $\epsilon_i$-close to every point in $S_c$, as this trajectory visits every $\epsilon_i$-neighbourhood in $S_c$. Namely, for every $j \in 1, \ldots, n^i$, after a finite number $\kappa$ of iterations, $(\sigma_c^\kappa(b))_{[-\ell^i, \ell^i]} = r^j_{[-\ell^i, \ell^i]}$ holds.

By concatenating these blocks $a^i$ for $i \in \mathbb{N}$, we obtain a point $a \in S_c$ given by $a = (\ldots, a^0 a^1 a^2 \ldots)$, whose orbit is dense in $S_c$. Hence, from Definition 5.4, we conclude that the dynamical system $(S_c, \sigma_c)$ is transitive. Since $(\mathcal{X}, R_{\mathcal{X}})$ is topologically conjugate to this system, $(\mathcal{X}, R_{\mathcal{X}})$ is topologically transitive as well, proving the lemma.
A.4 Local phase portrait near an equilibrium set of a periodically forced mechanical system with dry friction

In this section, we describe the local phase portrait near a periodically forced mechanical system with dry friction, which has been discussed briefly in Chapter 4. In this section, we aim to describe this local phase portrait in more detail.

As mentioned in Remark 5.2, we expect that the return maps as described in Chapter 4 form a subset of the class of discrete-time systems described in Section 5.2. This statement would lead to the conclusion that horseshoe-type limit sets of the nature presented in Chapter 5 can exist in periodically forced mechanical systems with dry friction. We provide the results in this appendix since we expect that this will form a first step in a proof of such a statement, which will further motivate the work in Chapter 5.

Throughout this section, we consider models of mechanical systems with one degree of freedom that experience dry friction, modelled with an Amontons-Coulomb friction element. In appropriate coordinates, these systems can be described by the following differential inclusion:

\[ \ddot{x} \in f(x, \dot{x}) - F_s \text{Sign}(\dot{x}) + \varepsilon g(t), \]

where \( x \in \mathbb{R} \), and solutions are considered in the sense of Filippov, cf. [58]. In the differential inclusion (A.79), the set-valued sign function

\[ \text{Sign}(\dot{x}) = \begin{cases} \frac{\dot{x}}{|\dot{x}|}, & \dot{x} \neq 0 \\ [-1, 1], & \dot{x} = 0 \end{cases} \]

is used, such that the dry friction force is contained in the set \( F_s \text{Sign}(\dot{x}) \), where \( F_s > 0 \). Other forces acting on the system, that can include gravitational forces, damping forces and forces due to flexibilities, are described by the force \( f(x, \dot{x}) \).

We assume that \( f(x, \dot{x}) \) is a smooth function. Time-dependent forces, e.g. perturbations or actuation forces, are modelled with the term \( \varepsilon g(t) \), with \( |g(t)| \leq 1 \) for all \( t \). Throughout this appendix, we assume that \( \varepsilon g(t) \) is a periodic function of time and that \( \varepsilon \) is small.

In the next section, we will apply some results presented in Chapter 3 to the system (A.79) when the perturbation is absent, i.e. when \( \varepsilon = 0 \).

A.4.1 Unperturbed system dynamics

We consider (A.79) as a periodically perturbed version of an autonomous differential inclusion with \( \varepsilon = 0 \), given by:

\[ \ddot{x} \in f(x, \dot{x}) - F_s \text{Sign}(\dot{x}). \]
We describe the phase portrait of this differential inclusion using the results of Chapter 3. In that chapter, we show that the phase space of system (A.81) contains equilibrium sets, i.e., line intervals consisting of non-isolated equilibria, and these line-shaped equilibrium sets are given by the points \((x, \dot{x})^T \in \mathbb{R}^2\), where \(f(x, 0) \in F_s \text{Sign}(0)\) and \(\dot{x} = 0\). Trajectories can converge in finite time to points in the equilibrium set, whereas asymptotic convergence (i.e. trajectories that approach the equilibrium set with an infinitely long transient time) or asymptotic divergence (i.e. trajectories that, after reversing the direction of time, show asymptotic convergence) can only occur at the endpoints of the equilibrium set.

To analyse the phase portrait of the autonomous system (A.81), we introduce the following first-order Filippov system:

\[
\dot{y} \in ^{a}F(y) := \left(\frac{y_2}{f(y_1, y_2) - F_s \text{Sign}(y_2)}\right), \tag{A.82}
\]

which is equivalent to (A.81) when selecting \(y = (y_1, y_2)^T := (x, \dot{x})^T \in \mathbb{R}^2\). The discontinuity of (A.82) is restricted to a single surface:

\[
^{a}\Sigma := \{y \in \mathbb{R}^2, y_2 = 0\}. \tag{A.83}
\]

We observe that away from the discontinuity, the function \(^{a}F\) in (A.82) satisfies

\[
^{a}F(y) = ^{a}F_{+}(y) := \left(\frac{y_2}{f(y_1, y_2) - F_s}\right), \text{ for } y_2 > 0
\]

\[
^{a}F(y) = ^{a}F_{-}(y) := \left(\frac{y_2}{f(y_1, y_2) + F_s}\right), \text{ for } y_2 < 0, \tag{A.84}
\]

where \(^{a}F_{+}\) and \(^{a}F_{-}\) are smooth functions. Using the approach of Filippov to study solutions to differential inclusions, cf. [58], we distinguish two subsets of \(^{a}\Sigma\) in (A.83), denoted by \(^{a}\Sigma^{c}\) and \(^{a}\Sigma^{s}\), where \(^{a}F_{+}\) and \(^{a}F_{-}\) allow the trajectories of (A.82) to cross \(^{a}\Sigma\) instantaneously, or where this is not possible, respectively. According to [58], these sets can be given as:

\[
^{a}\Sigma^{c} := \{y \in ^{a}\Sigma, ^{a}F_{+2}(y)^{a}F_{-2}(y) > 0\}, \tag{A.85}
\]

\[
^{a}\Sigma^{s} := \{y \in ^{a}\Sigma, ^{a}F_{+2}(y)^{a}F_{-2}(y) \leq 0\}, \tag{A.86}
\]

where \(^{a}F_{+2}(y)\) and \(^{a}F_{-2}(y)\) denote the second component of \(^{a}F_{+}(y)\) and \(^{a}F_{-}(y)\), respectively. If a trajectory \(\phi\) of (A.82) arrives in \(^{a}\Sigma^{c}\), that is, if for a given \(t\), \(\phi(t) \in ^{a}\Sigma^{c}\), then this trajectory can be continued by selecting either \(\dot{y} = ^{a}F_{+}(y)\) (when \(^{a}F_{+2}(\phi(t)) > 0\)) or \(\dot{y} = ^{a}F_{-}(y)\) (when \(^{a}F_{-2}(\phi(t)) < 0\)). If, for a given time \(t\), \(\phi(t) \in ^{a}\Sigma^{s}\) holds, then neither \(\dot{y} = ^{a}F_{+}(y)\) nor \(\dot{y} = ^{a}F_{-}(y)\) can be used to continue the solution from this point. Consequently, Filippov’s solutions concept requires such solutions to stay at the surface \(^{a}\Sigma\), such that the differential equation \(\frac{d\phi}{dt} = 0\) holds almost everywhere for \(\phi(t) \in ^{a}\Sigma^{s}\). Hence, the
set $a\Sigma^s$ coincides with a set of equilibrium points, and we introduce $E_0 := a\Sigma^s$ to refer to this equilibrium set.

In order to describe the effect of a small time-periodic perturbation on the stick set $a\Sigma^s$ and $a\Sigma^c$, we impose the following assumption on the differential inclusion (A.79).

**Assumption A.1.** Consider system (A.79) and let, for the autonomous case $\varepsilon = 0$, $E_0 = (E_0^x \ 0)^T$ denote the $y$-coordinates of an endpoint of an equilibrium set where $aF^-$ vanishes, i.e. where $aF^-(E_0) = 0$. In addition, the smooth function $f$ satisfies $\frac{\partial f(x,0)}{\partial x} \bigg|_{E_0^x} > 0$. Assume that the function $g(t)$ is smooth and periodic with minimal period $T > 0$.

As mentioned before, the analysis of system (A.79) is motivated by the analysis of a transversal homoclinic orbit emanating from the endpoint of an equilibrium set. We expect such homoclinic orbits to be present for small $\varepsilon > 0$ when the autonomous system (A.82) has a single homoclinic orbit from the endpoint $E_0$, and, consequently, contains trajectories that leave the neighbourhood of the point $E_0$ and emanate away from the equilibrium set. This motivates the assumption that $\frac{\partial f(x,0)}{\partial x} \bigg|_{E_0^x}$ is positive in Assumption A.1, such that trajectories of (A.82) may cross $\Sigma^c$ at $x$-values slightly below $E_0^x$ from the domain where $\dot{x}$ is positive to the domain where $\dot{x}$ is negative. Subsequently, since $\dot{x}$ is negative, such trajectories leave the neighbourhood of the point $E_0$.

In the following section, we study the effect of a small time-periodic perturbation of the system (A.82) on the trajectories near the point $E_0$. In fact, such a perturbation creates a complex geometrical structure of trajectories converging to, or diverging from the endpoint of the equilibrium set, similar to the homoclinic tangle in smooth differential equations.

**A.4.2 Effect of a time-dependent perturbation**

Now, we will consider the phase portrait of the system (A.79) for small $\varepsilon \neq 0$, which can be considered as a small perturbation of the dynamics of (A.81). For this purpose, firstly, we introduce an autonomous system representing the dynamics of (A.79) by embedding the time as an additional state. Subsequently, near the point $E_0$ given in Assumption A.1, we describe the geometrical properties of the two sets where Filippov solutions can cross or stick to the discontinuity surface, respectively, and note that equilibrium points are given by trajectories that stick to the discontinuity for all time. Hence, the geometrical description of the stick-set and crossing-set will yield a description of the equilibrium set of (A.79) as an additional result. This analysis will be made in Section A.4.2.1. Subsequently, in Section A.4.2.2, we will describe the trajectories that converge to, or diverge from the boundary of this equilibrium set, and, in this manner, obtain a non-smooth analogue of the stable and unstable manifolds of isolated equi-
librium points in smooth systems. This information is used in Section A.4.2.3 to
describe the local phase portrait near the endpoint $E_0$ given in Assumption A.1.

We reformulate the dynamics of (A.79) as an autonomous first-order differ-
ential inclusion by embedding time as an additional state, yielding state vector
$q = (t \ x \ \dot{x})^T \in \mathbb{R}^3$. In this manner, the dynamics of (A.79) is represented by

$$\dot{q} \in F(q) := \begin{pmatrix} 1 \\ q_3 \\ f(q_2, q_3) - F_s \text{Sign}(q_3) + \epsilon g(q_1) \end{pmatrix}.$$ (A.87)

Let $S(q_0)$ denote the set of solutions $\varphi$ with initial condition $q_0$, such that all
$\varphi \in S(q_0)$ satisfy (A.87) and $\varphi(0) = q_0$. Now, let

$$\Sigma := \{q \in \mathbb{R}^3, \ q_3 = 0\},$$ (A.88)

denote the surface where $F(q)$ in (A.87) is discontinuous. Trajectories can cross
$\Sigma$ instantaneously on

$$\Sigma^c := \{q \in \Sigma, \ (f(q_2, q_3) - F_s + \epsilon g(q_1))(f(q_2, q_3) + F_s + \epsilon g(q_1)) > 0\},$$ (A.89)
and sliding motion in the sense of Filippov occurs at the surface

$$\Sigma^s := \{q \in \Sigma, \ (f(q_2, q_3) - F_s + \epsilon g(q_1))(f(q_2, q_3) + F_s + \epsilon g(q_1)) \leq 0\}. $$ (A.90)

The sliding motion of (A.87) is parallel to $\Sigma$, cf. [58], and, since in addition
$\dot{q}_2 = q_3 = 0$ at this surface, the sliding motion is described by

$$\dot{q} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad q \in \Sigma^s,$$ (A.91)

for almost all $t$.

A.4.2.1 Description of the dynamics at the discontinuity surface

Note, that for $\epsilon = 0$, the boundary between $\Sigma^c$ and $\Sigma^s$ is given by one or more
straight lines. One of these lines is given by $\mathbb{R} \times \{E_0\}$, which represents the
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Fig. A.8. Pictorial of the sets $\Sigma^s$ and $\Sigma^c$ for system (A.87). (a) for $\varepsilon = 0$. (b) for $\varepsilon > 0$.

equilibrium point $E_0$ of (A.82) as a line in the phase space of system (A.87), as depicted in Fig. A.8(a). For small $\varepsilon > 0$, we expect that the boundaries between $\Sigma^c$ and $\Sigma^s$, given by lines for $\varepsilon = 0$, persist and deform to curves in the phase space of (A.87) due to the small term $\varepsilon g(q_1)$, as depicted in panel (b) of Fig. A.8. The following lemma describes the curve from this set that converges to $\mathbb{R} \times \{E_0\}$ for $\varepsilon \to 0$.

**Lemma A.17.** Suppose that Assumption A.1 is satisfied by system (A.79) with endpoint $E_0 = (E_0^x, 0)^T$. Let $T$ denote the minimal period of $g$.

For any $\delta > 0$ we can find an upper bound $\bar{\varepsilon}$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ there exists a function $c_p : \mathbb{R} \to [E_0^x - \delta, E_0^x + \delta]$ with the following property. Locally near $\mathbb{R} \times \{E_0\}$, the boundary between $\Sigma^c$ and $\Sigma^s$, as defined in (A.89) and (A.90), respectively, is given by

\[
\bar{c} := \{q \in \mathbb{R}^3, q = (q_1 \ c_p(q_1), 0), \ q_1 \in \mathbb{R}\},
\]

and the function $c_p$ is continuously differentiable and periodic with minimal period $T$.

**Proof.** According to Assumption A.1, the endpoint $E_0 = (E_0^x, 0)^T$ of (A.82) satisfies $^sF_-(E_0) = 0$, such that

\[
f(E_0^x, 0) + F_s = 0.
\]

Consequently, for sufficiently small $\varepsilon > 0$, the definition of $\Sigma^c$ and $\Sigma^s$ in (A.89)-(A.90) implies that the boundary between $\Sigma^s$ and $\Sigma^c$ near $\mathbb{R} \times E_0$ is given by the points $q \in \Sigma$ such that $F_-^3(q) = 0$, where $F_-^3$ denotes the third component of $F_-$ defined in (A.91). With (A.91), this implies that the expression

\[
f(q_2, 0) + F_s = -\varepsilon g(q_1)
\]
defines the boundary between $\Sigma^s$ and $\Sigma^c$ near $\mathbb{R} \times E_0$.

By Assumption A.1, $\frac{\partial f(x,0)}{\partial x}\bigg|_{x=E_0^x} \neq 0$, such that application of the inverse function theorem to (A.93) implies that there exists a diffeomorphism $\alpha(\gamma)$ such that $\alpha(0) = E_0^x$ and

$$f(\alpha(\gamma), 0) + F_s = -\gamma \quad \text{(A.95)}$$

holds for all $\gamma$ in a neighbourhood $U$ of zero. We now select $\varepsilon > 0$ small enough such that $\varepsilon g(q_1) \in U \forall q_1, \varepsilon \in (0, \bar{\varepsilon}]$.

Now, given a parameter $\varepsilon \in (0, \bar{\varepsilon}]$, we define the continuously differentiable function

$$c_p(q_1) := \alpha(\varepsilon g(q_1)), \quad \text{(A.96)}$$

such that for all $q \in \bar{c}$, with $\bar{c}$ defined in (A.92), the expression (A.94) is satisfied. As observed, (A.94) locally characterises the boundary between $\Sigma^s$ and $\Sigma^c$, such that the curve $\bar{c}$ defined in (A.92) gives the boundary between $\Sigma^c$ and $\Sigma^s$ near $\mathbb{R} \times \{E_0\}$. Since $g$ is periodic with minimal period $T$, the function $c_p(q_1)$ is periodic with the same period, proving the lemma.

The curve $\bar{c}$ given in (A.92) is depicted schematically in Fig. A.8(b). On the set $\Sigma^s$, solutions of (A.87) satisfy the Filippov solution $\dot{q} = (1 \ 0 \ 0)^T$. Recall that the first element of the state vector $q$ was added to represent the time $t$, such that trajectories of (A.87) that remain in $\Sigma^s$ for all $t$ correspond to equilibrium positions of (A.79). These trajectories are located in the set $\bar{E}_\varepsilon := \{q \in \mathbb{R}^3, \ q + (t \ 0 \ 0)^T \in \Sigma^s, \ \forall t \in \mathbb{R}\}$, as depicted in Fig. A.8(b). Observe that near $\mathbb{R} \times E_0$, the set $\bar{E}_\varepsilon \subset \Sigma$ is bounded by the line $E_\varepsilon := \mathbb{R} \times E_\varepsilon$, with $E_\varepsilon = (\max_{q \in E_\varepsilon} q_2 \ 0)^T$ and $\bar{c}$ given by (A.92). Trajectories on the set $\Sigma^s \setminus \bar{E}_\varepsilon$ will only stick temporarily to the surface $\Sigma$, and, subsequently, will leave this set. The set $\bar{E}_\varepsilon$ is the equilibrium set of the system (A.87), consisting of trajectories that stick permanently.

In the next section, we will describe the local dynamics near the equilibrium positions $E_\varepsilon$, by the introduction of the stable and unstable sets of trajectories converging to $E_\varepsilon$ in forward or backward direction of time, respectively. In smooth systems with hyperbolic equilibrium points with a saddle-type nature, the existence of a local coordinate system where the flow of (A.79) is contracting in one direction and expanding in the other can be obtained using an hyperbolic splitting of the phase space, cf. [145]. This result is not applicable in our case due to the discontinuity of (A.87). However, as we will show in the next section, the unstable set consists of a two-dimensional set where the dynamics is expanding away from the equilibrium position $E_\varepsilon$, and the stable set constitutes of a two-dimensional set in the phase space that contains trajectories that are contracted towards this equilibrium position. We note, however, that we can not conclude from the contracting and expanding characteristics of these two sets that, in the
rest of the neighbourhood of $E_\varepsilon$, the vector field is contracting in one direction, and expanding in the other.

### A.4.2.2 Local stable and unstable sets of the endpoint of the equilibrium set

In order to study the local phase portrait near $E_\varepsilon$ (a boundary of the equilibrium set $\mathcal{E}_\varepsilon$ of (A.87)), we focus on trajectories converging to, or diverging from, $E_\varepsilon$. In this manner, we identify two sets in the phase space that play the same role as stable and unstable manifolds of isolated equilibrium points in smooth systems. In this section, we will describe the local dynamical behaviour on these sets. Afterwards, we will use this description in Section A.4.2.3 to analyse the local dynamical behaviour near $E_\varepsilon$.

We consider the sets:

$$M^s := \{ q_0 \in \mathbb{R}^3, \forall \varphi \in S(q_0), \lim_{t \to \infty} d(\varphi(t), E_\varepsilon) = 0 \}, \quad (A.97a)$$

$$M^u := \{ q_0 \in \mathbb{R}^3, \exists \varphi \in S(q_0), \lim_{t \to -\infty} d(\varphi(t), E_\varepsilon) = 0 \}, \quad (A.97b)$$

where we recall that $S(q_0)$ denotes the set of solutions $\varphi$ of (A.87) from initial condition $\varphi(0) = q_0$. We will refer to $M^s$ and $M^u$ as the stable and unstable set of $E_\varepsilon$, respectively. The asymmetry in the definitions of $M^s$ and $M^u$ stems from the fact that trajectories of (A.87) are uniquely defined in forward time, but can be non-unique in backward time. Note that a stroboscopic projection of these sets coincide with the stable and unstable sets defined in [73].

**Remark A.1.** The definition in (A.97) directly implies that $M^s$ is invariant (both in forward and backward direction of time) under the dynamics (A.87). However, the set $M^u$ is only forward invariant, but not backward invariant under these dynamics.

In the neighbourhood of $E_\varepsilon$, the set $M^s$ (or $M^u$) may contain initial conditions of trajectories that leave the neighbourhood of $E_\varepsilon$ before converging to $E_\varepsilon$ under the flow of (A.87) in forward (or backward) direction of time. Consequently, analogously to the definition of local stable and unstable manifolds in smooth systems, cf. [142], we define the local stable and unstable sets of $E_\varepsilon$ as:

$$M^s_\beta := \{ q_0 \in \mathbb{R}^3, \forall \varphi \in S(q_0), \lim_{t \to \infty} d(\varphi(t), E_\varepsilon) = 0 \text{ and } d(\varphi(t), E_\varepsilon) \leq \beta, \forall t \geq 0 \}, \quad (A.98a)$$

$$M^u_\beta := \{ q_0 \in \mathbb{R}^3, \exists \varphi \in S(q_0), \lim_{t \to -\infty} d(\varphi(t), E_\varepsilon) = 0 \text{ and } d(\varphi(t), E_\varepsilon) \leq \beta, \forall t \leq 0 \}, \quad (A.98b)$$

where $\beta > 0$ is small.

We will now derive two results (Lemma A.18 and Lemma A.19) for the trajectories in the local stable and unstable sets.
The trajectories in $\mathbb{M}^s_\delta$ satisfy a finite-time convergence property, as formalised in the following lemma.

**Lemma A.18.** Consider trajectories $\varphi \in S(q_0)$ of system (A.87) and let Assumption A.1 be satisfied. If $\varepsilon > 0$ is sufficiently small, then there exists a finite $t_f > 0$ such that for all $q_0 \in \mathbb{M}^s_\delta$ and all $\varphi \in S(q_0)$,
\[
\varphi(t) \in \mathcal{E}_\varepsilon, \quad \forall t > t_f,
\] (A.99)
holds, provided that $\beta > 0$ is chosen sufficiently small.

**Proof.** First, we prove that trajectories of (A.87) near $D_+$, as defined in (A.91), in finite time.

Assumption A.1 states that at the endpoint $E_0 = (E^x_0, 0)^T$ of the equilibrium set of (A.82), $\Phi F_-(q) = 0$ holds and, consequently,
\[
f(E^x_0, 0) - F_s = -2F_s,
\] (A.100)
which implies that the third component of $F_+(q)$ in (A.91), denoted $F_{+3}(q)$, satisfies $F_{+3}((q_1 E^x_0, 0)^T) = f(E^x_0, 0) - F_s + \varepsilon g(q_1) \leq -2F_s + \varepsilon < 0$, $\forall q_1 \in \mathbb{R}$, provided $\varepsilon < \frac{1}{2}F_s$, where we use that fact that $|g(t)| \leq 1$, $\forall t$.

Since $F_{+3}$ is a continuous function, we can select $\delta > 0$ sufficiently small, such that
\[
F_{+3}(q) = f(q_2, 0) - F_s + \varepsilon g(q_1) \leq \frac{1}{2}(-2F_s + \varepsilon) < 0, \quad \forall q \in \mathbb{R} \times [E^x_0 - \delta, E^x_0 + \delta] \times \{0\}
\] (A.101)
holds. Application of Lemma A.17 implies that there exists an $\bar{\varepsilon} > 0$ such that $q_2 \in [E^x_0 - \delta, E^x_0 + \delta]$ is satisfied for all $(q_1 q_2 q_3)^T \in \bar{c}$, with $\bar{c}$ defined in (A.92), provided that the parameter $\varepsilon$ in (A.87) satisfies $\varepsilon \leq \bar{\varepsilon}$. Consequently, $\mathcal{E}_\varepsilon \subset \mathbb{R} \times [E^x_0 - \delta, E^x_0 + \delta] \times \{0\}$, such that (A.101) implies $F_{+3}(q) < \frac{1}{2}(-2F_s + \varepsilon) < 0$, $\forall q \in \mathcal{E}_\varepsilon$.

Since $F_{+3}(q)$ is continuous, restricting our attention to the compact domain $\mathcal{E}_\varepsilon \cap ([0, T] \times \mathbb{R}^2)$, we observe from the conclusion of the previous sentence that we can select $\beta$ sufficiently small such that
\[
F_{+3}(q) \leq \frac{1}{4}(-2F_s + \varepsilon) < 0, \quad \forall q \in \mathcal{B}_\beta + \mathcal{E}_\varepsilon \cap ([0, T] \times \mathbb{R}^2).
\] (A.102)
Since $F_{+3}(q)$ is periodic in the argument $q_1$, we observe that (A.102) implies
\[
F_{+3}(q) \leq \frac{1}{4}(-2F_s + \varepsilon) < 0, \quad \forall q \in \mathcal{B}_\beta + \mathcal{E}_\varepsilon.
\] (A.103)
Hence, (A.91) implies that all trajectories $\varphi$ of system (A.87), with initial conditions in $D_+ \cap \mathbb{M}^s_\beta \subset D_+ \cap (B_\beta + \mathcal{E}_\varepsilon)$, satisfy $\frac{d\varphi_3}{dt} \leq \frac{1}{4}(-2F_s + \varepsilon) < 0$, with $\varphi_3(t)$ the third element of $\varphi(t)$. Consequently, the trajectories arrive at the discontinuity surface $\Sigma$ in a finite time $t_b \geq 0$. 

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In addition, from (A.103), we can conclude that no trajectories from \( \Sigma \cap (B_\beta + E_\varepsilon) \) can flow towards \( D_+ \). Hence, for trajectories \( \varphi \) with initial conditions in the local stable set \( M^s_\beta \) (i.e., \( \varphi \in S(q_0), \ q_0 \in M^s_\beta \)), we observe that \( \varphi(t) \in (\Sigma \cup D_-) \cap (B_\beta + E_\varepsilon) \), \( \forall t \geq t_0 \). Hence all trajectories \( \varphi \in S(q_0) \), with \( q_0 \in M^s_\beta \), have to fall in one of the following three cases, where we distinguish trajectories based on their behaviour for large \( t \):

1) there exists a \( t_f \geq t_b \) such that \( \varphi(t) \in D_- \), \( \forall t > t_f \),

2) for all \( t_f \geq t_b \) there exists times \( t_1, t_2 > t_f \) such that \( \varphi(t_1) \in \Sigma \), \( \varphi(t_2) \in D_- \), and \( \varphi(t) \in D_- \cup \Sigma \), \( \forall t \geq \min(t_1, t_2) \),

3) there exists a \( t_f \geq t_b \) such that \( \varphi(t) \in \Sigma \), \( \forall t > t_f \),

where we select \( t_b = 0 \) if \( q_0 \in \Sigma \cup D_- \).

In the remainder of this proof, we will show that for trajectories \( \varphi \in S(q_0) \) with \( q_0 \in M^s_\beta \), cases 1) and 2) cannot occur, and show that case 3) implies the finite-time converge property claimed in the lemma.

Suppose that there exists an initial condition \( q_0 \in M^s_\beta \) and a trajectory \( \varphi \in S(q_0) \) that satisfies 1), and let \( t_b \) be constructed as above, such that \( \varphi(t) \in \Sigma \cup D_- \) for \( t \geq t_b \). From 1), we observe that the trajectory \( \varphi(t) \) is given by \( \dot{\varphi} = F_-(q) \) for all \( t > t_f \). Since \( g(t) \) is continuous and periodic, following [32, Remark 1, p. 16], we can consider the dynamical system \( \dot{\varphi} = F_-(q) \) on the phase space \( C_T := S_T \times \mathbb{R}^2 \), where \( S_T \) denotes the circle \( \mathbb{R}/\{T \mathbb{Z}\} \), where \( T \) denotes the quotient operator, such that two points \( t_1, t_2 \in S_T \) are identified with each other when \( t_1 - t_2 \mod T = 0 \). If \( \pi_{C_T} \) denotes the projection from \( \mathbb{R}^3 \) to \( C_T \), given by \( \pi_{C_T}((q_1, q_2, q_3)^T) = ((q_1 \mod T, q_2, q_3)^T) \), then the curve \( \pi_{C_T}(E_\varepsilon) \) is a compact set. Since \( q_0 \in M^s_\beta \) and \( \varphi \in S(q_0) \), we observe that the \( \omega \)-limit set of the trajectory \( \pi_{C_T}(\varphi) \), i.e., \( \omega(\pi_{C_T}(\varphi)) = \{ q \in C_T \times \mathbb{R}^2 : \exists \times \{s_i\} \in \mathbb{N}, \lim_{i \to \infty} s_i = \infty, \lim_{i \to \infty} \pi_{C_T}(\varphi(s_i)) = q \}, \) is compact and positioned in \( \pi_{C_T}(E_\varepsilon) \), since for \( q_0 \in M^s_\beta \), the trajectories \( \varphi \in S(q_0) \) of (A.87) converge to \( E_\varepsilon \), see the definition of \( M^s_\beta \) in (A.98a). If \( F^-|_{C_T}(q) \), denotes the restriction of \( F^-(q) \) to \( C_T \), then \( \pi_{C_T}(\varphi) \) is described by the differential equation \( \dot{\varphi} = F^-|_{C_T}(q) \), and \( F^-|_{C_T} \) is a smooth function. Consequently, by additionally using the compactness of \( \pi_{C_T}(E_\varepsilon) \), we observe that \( \omega(\pi_{C_T}(\varphi)) = \pi_{C_T}(E_\varepsilon) \) has to be invariant under \( \dot{\varphi} = F^-|_{C_T} \), cf. [72].

The fact that \( \varepsilon \neq 0 \) implies that \( F^-(q) = (1 \ 0 \ f_2(q_2, q_0) + F_s + \varepsilon g(q_1))^T \neq (1 \ 0 \ 0)^T \), \( \forall q \in E_\varepsilon \). Consequently, \( F^-|_{C_T}(q) \neq (1 \ 0 \ 0)^T \) for all points \( q \) in the set \( \omega(\pi_{C_T}(\varphi)) = \pi_{C_T}(E_\varepsilon) \). Hence, this set is not invariant, obtaining a contradiction. Consequently, trajectories from initial conditions in \( M^s_\beta \) cannot satisfy condition 1).

Now, in order to arrive at a contradiction, suppose there exists a trajectory \( \varphi \in S(q_0) \), with \( q_0 \in M^s_\beta \), that satisfies 2), and let \( t_b \) be constructed as above, such that \( \varphi(t) \in \Sigma \cup D_- \) for \( t \geq t_b \). Let \( t_2 > t_1 > t_b \) be such that \( \varphi(t_2) \in \Sigma \), \( \varphi(t_1) \in D_- \) and \( \varphi(t) \in D_- \cup \Sigma \), \( \forall t \geq t_1 \). By again applying property 2), where
we employ $t_f = t_2$, we can select a time $t_3 > t_f = t_2$ such that $\varphi(t_3) \in D_-$. Since $\varphi(t_1) \in D_-$ and $q_0 \in M^s_\beta$, we observe that $\varphi(t_2) \in \Sigma^s$. Namely, by definition of $M^s_\beta$, the trajectory $\varphi \in S(q_0)$ will not leave the $\beta$-neighbourhood of the set $E_\varepsilon$, and in this $\beta$-neighbourhood, trajectories from $D_-$ cannot arrive in $\Sigma^c$.

Near the time instant $t_2$, the solution $\varphi$ is described by $\dot{q} = F_0(q)$, and since $\varphi(t_3) \in D_-$, with $t_3 > t_2$, the trajectory $\varphi$ has to leave $\Sigma^s$, which implies that there exists a time $t_4 \in [t_2, t_3]$ such that $\varphi(t_4) \in \bar{c}$ and $\varphi(t_4 + \delta_t) \in D_-$ for $\delta_t > 0$ sufficiently small. Let the second component of $\varphi(t)$ be denoted as $\varphi_2(t)$ and let $E^x_\varepsilon$ be the second coordinate of the points $q \in E_\varepsilon$, which, according to the definition of $E_\varepsilon$, is uniquely defined. From $\frac{\partial f(x, 0)}{\partial x} |_{E^x_\varepsilon} > 0$, as stated in Assumption A.1, we observe that for small $\varepsilon$, near $E_\varepsilon$, the $x$-coordinates of $\Sigma^c$ are smaller than $E^x_\varepsilon$. Since $\varphi$ leaves $\Sigma$ at time $t_4$ at the boundary of $\Sigma^c$, we observe that there exists a small $\delta_x > 0$ such that $\varphi_2(t_4) \leq E^x_\varepsilon - \delta_x$ holds. Using (A.91), we observe that $\varphi(t) \in D_- \cup \Sigma$, $\forall t > t_4$, implies $\frac{d\varphi_2(t)}{dt} = \varphi_3(t) \leq 0$ for almost all $t$, where $\varphi_3(t)$ denotes the third component of $\varphi(t)$. Combination of this result with the fact that Filippov solutions are absolutely continuous and $\varphi_2(t_4) \leq E^x_\varepsilon - \delta_x$, we obtain $\varphi_2(t) \leq E^x_\varepsilon - \delta_x$, $\forall t > t_4$, contradicting convergence of $\varphi$ to $E_\varepsilon$ since for all $q \in E_\varepsilon$, $q_2 = E^x_\varepsilon$. This contradiction proves that no solutions can exist that satisfy case 2).

We will conclude the proof of the lemma by showing that 3) implies finite time convergence. Given a trajectory $\varphi \in S(q_0)$ with $q_0 \in M^s_\beta$, construct $t_b$ as above such that $\varphi(t) \in \Sigma \cup D_-$ for $t \geq t_b$, and let case 3) be satisfied. We observe that $\varphi$ is described by $\dot{q} = F_0(q)$ for $t > t_f$. In order to obtain a contradiction, we assume that $\varphi(t_f) \notin E_\varepsilon$, such that $d(\varphi(t_f), E_\varepsilon) > \delta_0 > 0$, for some $\delta_0 > 0$. Using $\dot{q} = F_0(q)$, we observe that $d(\varphi(t), E_\varepsilon) > \delta_0 > 0$, $\forall t > t_f$, contradicting the (asymptotic) convergence of $\varphi$ to $E_\varepsilon$, which was required for $\varphi \in S(q_0)$, with $q_0 \in M^s_\beta$ and the definition of $M^s_\beta$ in (A.84a). A contradiction is obtained, such that we have proven that $\varphi(t_f) \in E_\varepsilon$. Since $t_f$ is finite according to 3), the lemma is proven.

From this lemma, it follows that all trajectories in $M^s_\beta$ arrive at $E_\varepsilon$ in a finite time. Since finite-time convergence of Filippov solutions can only occur by trajectories arriving transversally at the discontinuity surface $\Sigma$, the set $M^s_\beta$ consists of two parts, one emanating from $D_-$ and one from $D_+$.

The local analysis presented in this appendix is motivated by the analysis of homoclinic structures as given, e.g., in Fig. 5.1. For these type of homoclinic structures, all trajectories arrive at the stable set $M^s_\beta$ at one side of the discontinuity. Hence, we will now focus our attention onto $M^s_\beta \cap D_+$ and show that this is a two-dimensional set. First, from Assumption A.1, we observe that $aF_-(E_0) = 0$. Since $aF_+ - aF_- = (0 - 2F_8)^T$, cf. (A.84), we find that $aF_+(E_0) = (0 - 2F_8)^T$. Hence, using the definitions of $aF_+$ in (A.84) and $F_+$ in (A.91), we conclude that the third component of $F_+$, denoted with $F_{+3}$, satisfies
Proof.
Let \( \phi \) it holds that, for all \( t \) Lemma A.19. following lemma.
\[ M \subseteq E \] and \( \phi \in M \) for \( t \)
differential equation \( \dot{q} = F(q) \) will be close
to \(-2F_s\). Hence, again using continuity of \( F_{+3} \), we observe that there exists a \( \delta_1 < 0 \) such that
\[ F_{+3}(q) < \delta_1, \forall q \in \mathbb{R} \times \{E_0\} \] holds, provided that both \( \varepsilon > 0 \) and \( \beta > 0 \) are sufficiently small. From (A.104), we conclude that all trajectories in \( M_{\beta}^{s} \cap D_{+} \) leave \( D_{+} \) in a finite time, and that there exists a \( t_1 < 0 \) such that \( M_{\beta}^{s} \cap D_{+} \subseteq \mathbb{R}^3, \exists \phi \in S_{+}(q_0), \exists t \in [t_1, 0), q = \phi(t), q_0 \in E_{\varepsilon} \}, \) where \( S_{+}(q_0) \) denotes the set of solution of the smooth differential equation \( \dot{q} = F_{+}(q) \) from initial condition \( q_0 \). Since \( F_{+} \) is smooth and \( E_{\varepsilon} \) is a line, \( \{q \in \mathbb{R}^3, \exists \phi \in S_{+}(q_0), \exists t \in [t_1, 0), q = \phi(t), q_0 \in E_{\varepsilon} \} \) is a two-dimensional set, and, consequently, \( M_{\beta}^{s} \cap D_{+} \) is two-dimensional.

We will now analyse the trajectories in the unstable set \( M_{\beta}^{u} \). In the limit for \( t \to -\infty \), trajectories in the unstable set \( M_{\beta}^{u} \) stick to \( \Sigma \) or slide (experience motion away from \( \Sigma \)) in an alternating fashion. To be precise, the trajectories from initial conditions in \( M_{\beta}^{u} \) are described, in the limit for \( t \to -\infty \), in the following lemma.

**Lemma A.19.** Let both \( \varepsilon > 0 \) and \( \beta > 0 \) be sufficiently small. For any solution \( \phi \in S(q_0) \), with initial condition \( q_0 \in M_{\beta}^{u} \setminus E_{\varepsilon} \), that satisfies
\[ \phi(t) \in E_{\varepsilon} + B_{\beta}, \forall t < 0 \] and \( \lim_{t \to -\infty} d(\phi(t), E_{\varepsilon}) = 0 \),
\[ (A.105) \] it holds that, for all \( t_f \leq 0 \), there exist finite times \( t_1, t_2 < t_f \) such that \( \phi(t_1) \in \Sigma, \phi(t_2) \in D_{-} \) and \( \phi(t) \in D_{-} \cup \Sigma, \forall t < \max(t_1, t_2) \).

**Proof.** Let \( q_0 \in M_{\beta}^{u} \setminus E_{\varepsilon} \) and consider solutions \( \phi \in S(q_0) \) satisfying (A.105).

First, we will prove that \( M_{\beta}^{u} \cap D_{-} = \emptyset \), such that the statement \( \phi(t) \in D_{-} \cup \Sigma, \forall t < \max(t_1, t_2) \) in the lemma directly follows. Using the same arguments as in the proof of Lemma A.18, for sufficiently small \( \varepsilon > 0 \), we observe that \( F_{+3}(q) \leq \frac{1}{2}(-2F_s + \varepsilon), \forall q \in E_{\varepsilon} \). Continuity of \( F_{+3} \) implies that one can choose \( \beta > 0 \) sufficiently small, such that
\[ F_{+3}(q) < \frac{1}{2}(-2F_s + \varepsilon) < 0, \forall q \in E_{\varepsilon} + B_{\beta}. \] (A.106)

Hence, any trajectory \( \phi \) with \( \phi(0) = q_0 \in D_{+} \cap (E_{\varepsilon} + B_{\beta}) \), in backward time, remains in \( D_{+} \) until it leaves \( E_{\varepsilon} + B_{\beta} \), which occurs at a time \( t \in [-\frac{\beta}{F_s - \frac{\beta}{2}\varepsilon}, 0] \).

Since trajectories that satisfy (A.105) remain in \( (E_{\varepsilon} + B_{\beta}) \) for all \( t < 0 \), we obtain \( M_{\beta}^{u} \cap D_{+} = \emptyset \).

Consider an initial condition \( q_0 \in M_{\beta}^{u} \) and trajectory \( \phi \in S(q_0) \) satisfying (A.105) and, in order to obtain contradiction, suppose there exists a \( t_f < 0 \) such that the trajectory \( \phi \) satisfies \( \phi(t) \in D_{-}, \forall t < t_f \). Then, the behaviour of this trajectory, for \( t < t_f \), is given by \( \dot{q} = F_{-}(q) \).
Since $g(t)$ is continuous and periodic, we can consider the dynamical system $\dot{q} = F_-(q)$ on the phase space $C_T := S_T \times \mathbb{R}^2$, where $S_T$ denotes the circle $\mathbb{R}/\{T\mathbb{Z}\}$, such that two points $t_1, t_2 \in S_T$ are identified with each other when $t_1 - t_2 \mod T = 0$. If $\pi_{C_T}$ denotes the projection from $\mathbb{R}^3$ to $C_T$ given by $\pi_{C_T}((q_1 q_2 q_3)^T) = ((q_1 \mod T q_2 q_3)^T)$, then the curve $\pi_{C_T}(E_\varepsilon)$ is a compact set. If $\varphi$ is a trajectory of $\dot{q} = F_-(q)$, then $\pi_{C_T}(\varphi)$ is a trajectory of the system $\dot{q} = F^-|_{C_T}(q)$, where $F^-|_{C_T}$ denotes the restriction of $F^-$ to $C_T$, which is a smooth function on $C_T$ since $f$ is smooth and $g$ is smooth according to Assumption A.1. From (A.105), we observe that the $α$-limit set of the trajectory $\pi_{C_T}(\varphi)$, i.e. $α(\pi_{C_T}(\varphi)) = \{q \in C_T \times \mathbb{R}^2, \exists$ times $\{s_i\}_{i \in \mathbb{N}}, \lim_{i \to \infty} s_i = -\infty, \lim_{i \to \infty} \pi_{C_T}(\varphi(s_i)) = q\}$, is compact and positioned in $\pi_{C_T}(E_\varepsilon)$. Since $F^-|_{C_T}$ is a smooth function, $α(\pi_{C_T}(\varphi))$ has to be invariant under the dynamics of $\dot{q} = F^-|_{C_T}(q)$. However, since $α(\varphi) \subseteq \pi_{C_T}(E_\varepsilon)$, invariance to the system $\dot{q} = F^-|_{C_T}(q)$ would require $F^-|_{C_T}(q) = (1 0 0)^T$. However, the fact that $\varepsilon \neq 0$ implies that $F^-(q) = (1 0 f_2(q_2, 0) + F_\varepsilon + \varepsilon g(q_1))^T \neq (1 0 0)^T$, $\forall q \in E_\varepsilon$. Consequently, $F^-|_{C_T}(q) \neq (1 0 0)^T$ for all points $q$ in the set $α(\pi_{C_T}(\varphi)) = \pi_{C_T}(E_\varepsilon)$. Hence, this set is not invariant, and a contradiction is obtained. Consequently, we have shown that for any initial condition $q_0 \in M^u_\beta$, and any $\varphi \in S(q_0)$ satisfying (A.105) and all times $t_f < 0$, there exists a finite time $t_1 < t_f$ such that $\varphi(t_1) \in \Sigma$.

Now, we will conclude the proof of this lemma by guaranteeing that, for any initial condition $q_0 \in M^u_\beta \setminus E_\varepsilon$, any trajectory $\varphi \in S(q_0)$ satisfying (A.105), and all times $t_f < 0$, there exists a finite $t_2 < t_f$ such that $\varphi(t_2) \in D_-$. Suppose that such a time $t_2$ does not exist. This implies $\varphi(t) \in \Sigma \cap D_+, \forall t < t_f$. However, as shown above, for sufficiently small $\beta > 0$, $\varphi(t) \in D_+$ would imply that $\varphi$ would leave $E_\varepsilon + B_\beta$ in backward time, contradicting (A.105). Hence, we may restrict our attention to trajectories $\varphi$ such that $\varphi(t) \in \Sigma$, $\forall t < t_f$. Since the set $\Sigma^s \supset E_\varepsilon$ is an attracting sliding mode, in backward time, the trajectory $\varphi$ cannot arrive at $E_\varepsilon$ in a finite time, such that $\varphi(t_f) \notin E_\varepsilon$. The trajectories $\varphi$ satisfying $\varphi(t) \in \Sigma$, $\forall t < t_f$, are described by $\dot{q} = F^0(q)$, such that $q_0$ remains constant, excluding convergence of $\varphi(t) \to E_\varepsilon$ for $t \to -\infty$ when $\varphi(t_f) \notin E_\varepsilon$. A contradiction with $\varphi \in S(q_0)$, $q_0 \in M^u_\beta$ is obtained with the definition of $M^u_\beta$ as given in (A.98b), since $\lim_{t \to -\infty} \varphi(t) \notin E_\varepsilon$. Consequently, we have proven that there exists a finite $t_2 < t_f$ such that $\varphi(t_2) \in D_−$. Hence, for all trajectories $\varphi \in S(q_0)$, with $q_0 \in M^u_\beta$, that satisfy (A.105), we have proven that for all times $t_f < 0$, there exists a finite $t_2 < t_f$ such that $\varphi(t_2) \in D_-$. Combination of this result with the conclusion of the previous paragraph yields the statement of the lemma.

This lemma directly shows that, in the limit for $t \to -\infty$, any trajectory $\varphi$ as mentioned in the lemma visits $\Sigma$ and $D_-$ in an alternating fashion. Namely, given any finite time $t_1$ (or $t_2$) as given in the lemma, one can again apply the statement of the lemma using a new time $\hat{t}_f < t_1$ (or $\hat{t}_f < t_2$), which guarantees
that prior to $t_1$ (or $t_2$), the trajectory visits both $\Sigma$ and $D_-$.

To understand this alternation of stick and sliding motion, observe that all trajectories in $\Sigma_s \setminus \bar{E}_\varepsilon$ satisfy $\dot{q} = F_0(q) = (1 \ 0 \ 0)^T$ only for a time interval with a length that is smaller than $T$, and, subsequently, will leave the discontinuity surface $\Sigma$ at the boundary $\bar{c}$ between $\Sigma_s$ and $\Sigma_c$. From a physical point of view, these trajectories experience stick, caused by the dry friction, for some period of time. Afterwards, the force $f(q_1, q_2) + \varepsilon g(t)$ will overcome the dry friction force temporarily, such that motion is possible. Subsequently, the perturbation $\varepsilon g(t)$ will again become small, such that the trajectory collapses onto $\Sigma_s$.

We will now show that the set $\mathcal{M}^{u}_\beta \cap D_+$ is a two-dimensional set. For all $q_0 \in \mathcal{M}^{u}_\beta \setminus E_\epsilon$, there exists a trajectory $\varphi \in S(q_0)$ and times $t_1, t_2 < 0$ such that $\varphi(t_1) \in \Sigma$ and $\varphi(t_2) \in D_-$, which directly implies that $\varphi$ has crossed the curve $\bar{c}$ in the time interval $[t_1, t_2]$. Since $\bar{c}$ is a curve and the Filippov solutions of (A.87) depend continuously on initial conditions, $\mathcal{M}^{u}_\beta \setminus E_\epsilon$ is a 2-dimensional set.

Using Lemmas A.18 and A.19 and the observations that both $\mathcal{M}^{s}_\beta$ and $\mathcal{M}^{u}_\beta$ are two-dimensional sets, we can now describe the local phase portrait near $E_\epsilon$ in the next section.

### A.4.2.3 Discussion of the local phase portrait

In this section, we will describe the behaviour of trajectories in the neighbourhood of the line $E_\epsilon$, see Fig. A.9. As noted in the previous sections, both the sets $\mathcal{E}_\epsilon$ and $\mathcal{M}^{u}_\beta$ are two-dimensional, and consist of Filippov solutions of (A.87). Consequently, the set $\mathcal{E}_\epsilon \cup \mathcal{M}^{u}_\beta$ is two-dimensional and forms a separatrix for the trajectories of (A.87). This separatrix divides the neighbourhood of $E_\epsilon$ in two distinct domains. We denote these domains with $N_+$ and $N_-$, where $N_+$ is the domain that contains $\mathcal{M}^{s}_\beta \cap D_+$ in its closure.

In the remainder of this section, we will restrict our attention to the discussion of the trajectories in $N_+$. This restriction is motivated by the fact that we aim to use the description of the local phase portrait of (A.87) to analyse the properties of the return map $H$, as defined in Chapter 4. All trajectories that are represented in the map $H$ will not visit domain $N_-$, since the domain $Q$ as constructed in Chapter 4 is bounded on one side by the unstable set.

Restricting our attention to $N_+$, we expect trajectories outside $\mathcal{M}^{s}_\beta$ and $\mathcal{M}^{u}_\beta$ to behave qualitatively as follows. In a finite time, such trajectories arrive either at the set $\mathcal{E}_\epsilon$ or at the set $\mathcal{M}^{u}_\beta$, which can be understood as follows. For almost all times $q_1$, $E_\epsilon$ lies in the interior of $\Sigma^s$, and, since $\Sigma^s$ is a stable sliding surface due to the dissipative effect of dry friction (i.e. since $F_s > 0$), $\Sigma^s$ attracts nearby trajectories. Those trajectories that come close to $E_\epsilon \cap \bar{c}$, with $\bar{c}$ as given in Lemma A.17, also have to collapse onto $\Sigma^s$. Namely, after a short period of time, due to the geometry of the curve $\bar{c}$ as depicted in Fig. A.8, such trajectories are close to $\Sigma^s$ and further away from $\Sigma^c$, and will therefore collapse to $\Sigma^s$. All trajectories near $E_\epsilon$ arrive in finite time in the set $\Sigma^s$, and afterwards, will
either remain inside $\bar{E}_\varepsilon \subset \Sigma^s$, or leave $\Sigma^s$ through the curve $\bar{c}$. Since locally, $\bar{c}$ is contained in $M^u_\beta$, the latter option implies that such trajectories have collapsed onto $M^u_\beta$. The trajectories subsequently stay inside $\bar{E}_\varepsilon$ or $M^u$, since these sets are invariant in forward time under the dynamics of (A.87), as noted in Remark A.1. Observe that the set $M^s_\beta$ forms a separatrix between trajectories colliding with $\bar{E}_\varepsilon$ or $M^u_\beta$, such that the set $M^s$ coincides with the boundary of the basin of attraction of $\bar{E}_\varepsilon$ for the dynamics given in (A.87).

The phase portrait of (A.87), with small $\varepsilon$, is illustrated in Fig. A.9. In this figure, the equilibrium set $\bar{E}_\varepsilon$, its boundary $E_\varepsilon$ and the set $\Sigma^s \setminus \bar{E}_\varepsilon$ are shown. In addition, the local stable set $M^s_\beta$ and unstable set $M^u_\beta$ are depicted. These sets are represented for $q_1 \in [-T, T]$ for the sake of clarity. Note, however, that $g(t)$ is periodic with minimal period $T$, and, consequently, the sets $\bar{E}_\varepsilon$, $E_\varepsilon$, $\Sigma^s$, $M^s$ and $M^u$ are periodic in $q_1$ with the same period.

Since Filippov solutions can be non-differentiable with respect to the initial conditions, the sets $M^s$ and $M^u$ are not necessarily smooth. The local stable set $M^s_\beta \cap D_+$, however, is a locally smooth surface. The local unstable set $M^u_\beta$ cannot be a smooth surface, since, locally near $E_\varepsilon$, it consists of trajectories that experience stick and sliding motion repetitively. When any of these trajectories arrives at $\Sigma^s$, then this trajectory is non-differentiable since (A.87) is discontinuous, and this non-differentiability of the trajectories renders $M^u_\beta$ non-differentiable. In Fig. A.9, we depicted the positions where $M^u_\beta$ is non-differentiable with a dashed line.

**A.4.2.4 Discussion**

In conclusion, in Appendix A.4, we have rigourously described the geometry of the local stable and unstable sets $M^s_\beta$ and $M^u_\beta$ and have discussed the trajectories near these sets. In the autonomous case $\varepsilon = 0$, the non-local sets $M^s$ and $M^u$ coincide and contain the homoclinic trajectory. As argued in Chapter 4, we expect that in the case $\varepsilon > 0$, some trajectories emanate from $M^u_\beta$ and arrive in $M^s_\beta$ before converging to $E_\varepsilon$. However, due to the time-dependent perturbation of the system (A.81), this will probably not be the case for most trajectories from initial conditions in $M^u_\beta$. Similar to the case of periodic perturbations of homoclinic orbits in smooth systems, this will result in a complex geometry of the stable and unstable set, which is called a homoclinic tangle. However, due to the finite-time convergence of trajectories in $M^s_\beta$, as described in Lemma A.18, and the non-trivial divergent behaviour of trajectories in $M^u_\beta$, given in Lemma A.19, the geometrical properties of the stable and unstable set will be essentially different from the homoclinic tangle occurring in smooth systems, which has been described, e.g., in [72].

A rigourous description of the geometry of the homoclinic tangle has not yet been achieved. If a result similar to the lambda-lemma for smooth systems could be applied to the case presented in this appendix, then this would be an
Fig. A.9. Phase portrait of (A.87) near the line $E_\varepsilon$ which is the boundary of $\tilde{E}_\varepsilon$. Trajectories are depicted that lie in $M^s_\beta$ and $M^u_\beta$. The depicted grid of points denotes the surface $\Sigma^c$, and the dashed lines represent the curves in $M^u_\beta$ where $M^u_\beta$ is locally not differentiable.
important step to achieve this description. Hence, the derivation of such results would be an interesting direction for future research.
A.5 Appendixes of Chapter 6

A.5.1 Proof of Theorem 6.1

Proof of Theorem 6.1. Consider an arbitrary trajectory \( q(t, j) \) of system (6.4) satisfying the hypotheses of the theorem and define \( \bar{x}, \bar{r} \) as in (6.5). Select \( R > 0 \) such that \( |\bar{r}(t, j)| \leq R, \forall (t, j) \).

The requirement (6.6a) implies that, given \( |r| \leq R \), the non-negative function \( d(r, x) \) is zero if and only if \( (r, x) \in \mathcal{A} := \{ \text{col}(r, x) \in \text{cl}(C \cup D)^2 | \exists k_1, k_2 \in \mathbb{N}, G^{k_1}(r) \cap G^{k_2}(r) \neq \emptyset, |r| \leq R \} \). The set \( \mathcal{A} \) is closed, since \( \mathcal{A} = \{ \text{col}(r, x) \in \text{cl}(C \cup D)^2 | d(r, x) = 0, |r| \leq R \} \) due to (6.6a), and \( d \) is continuous. Boundedness of \( \mathcal{A} \) follows from (6.6b), such that \( \mathcal{A} \) is compact. Let \( \rho_{\mathcal{A}}(p) := \inf_{y \in \mathcal{A}} |p - y| \) give the distance to the set \( \mathcal{A} \). We will construct a \( K_\infty \)-function \( \alpha \) such that

\[
\alpha(\rho_{\mathcal{A}}(\text{col}(r, x))) \leq d(r, x). \tag{A.107}
\]

For this purpose, analogous to the proof of Lemma 4.3 in [91], let \( \phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) be given as:

\[
\phi(s) := \inf_{\text{col}(r, x) \in \text{cl}(C \cup D)^2, |r| \leq R, \rho_{\mathcal{A}}(\text{col}(r, x)) \geq s} d(r, x), \tag{A.108}
\]

which is non-decreasing. Since (6.6b) is satisfied, we observe that \( \phi(0) = 0 \), \( \phi(s) > 0 \) for \( s > 0 \) and \( \phi(s) \to \infty \) for \( s \to \infty \). Hence, we can construct a strictly increasing and continuous function \( \alpha \) such that \( \alpha(s) \leq \phi(s) \) and \( \alpha \in K_\infty \). Since, in addition, (A.108) implies \( \phi(\rho_{\mathcal{A}}(\text{col}(r, x))) \leq d(r, x) \), we observe that (A.107) is satisfied.

By assumption, \( d(r, x) \) converges to zero along the closed-loop trajectory \( q(t, j) = \text{col}(\bar{r}(t, j), \bar{x}(t, j), \bar{\eta}(t, j)) \). Hence, for all \( \epsilon > 0 \) there exists a time \( T > 0 \) such that

\[
d(\bar{r}(t, j), \bar{x}(t, j)) \leq \alpha(\epsilon), \forall (t, j) \in \text{dom} q, t + j > T. \tag{A.109}
\]

The \( K_\infty \)-function \( \alpha \) is invertible, such that (A.109) and (A.107) imply

\[
\rho_{\mathcal{A}}(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) \leq \epsilon, \forall (t, j) \in \text{dom} q, t + j > T. \tag{A.110}
\]

We will conclude the proof of this theorem by proving that

\[
\rho_{\mathcal{A}}(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) \leq \epsilon \tag{A.111}
\]

implies \( |\bar{r}(t, j) - \bar{x}(t, j)| \leq \epsilon \) when (6.7) holds. We can rewrite \( \rho_{\mathcal{A}} \) as:

\[
\rho_{\mathcal{A}}(\text{col}(r, x)) = \inf_{\text{col}(p_r, p_x) \in \mathcal{A}} \left| \frac{r - p_r}{x - p_x} \right| \tag{A.112}
\]

\[
= \inf_{p_r, p_x \in \text{cl}(C \cup D), k_1, k_2 \in \mathbb{N}, G^{k_1}(p_r) \cap G^{k_2}(p_x) \neq \emptyset, |p_r| \leq R} \left| \frac{r - p_r}{x - p_x} \right| \tag{A.113}
\]

\[
= \inf_{k_1, k_2 \in \mathbb{N}} \rho_{k_1k_2}(r, x) \tag{A.114}
\]
with \( \rho_{k_1k_2}(r, x) := \inf_{p_r, p_x \in \text{cl}(D \cup C), |p_x| \leq R} \left| \frac{r - p_r}{x - p_x} \right| \). Since we adopted the convention that \( G(x) = \emptyset \) when \( x \notin D \), for \( k_1 \geq 1 \), \( G^{k_1}(p_r) \neq \emptyset \) can only hold when \( p_r \in D \). Since, in addition,

\[
\left| \frac{r - p_r}{x - p_x} \right| \geq |r - p_r|, \quad \text{with (6.7) we observe that} \quad \rho_{k_1k_2}(\bar{r}(t, j), \bar{x}(t, j)) > \epsilon \quad \text{when} \quad k_1 \geq 1. \quad \text{Hence, (6.7) and (A.111) imply that the infimum of (A.114) is attained with} \quad k_1 = 0.
\]

For \( k_1 = 0 \) and \( k_2 \geq 1 \), \( G^{k_1}(p_r) \cap G^{k_2}(p_x) \neq \emptyset \) implies \( G^{k_1}(p_r) = \{p_r\} \subset G^{k_2}(p_x) \subset C \), such that (6.7) implies \( |\bar{r}(t, j) - p_r| \geq \epsilon \). Hence, \( \rho_{k_1k_2}(\bar{r}(t, j), \bar{x}(t, j)) > \epsilon \), when \( k_1 = 0, k_2 \geq 1 \). Consequently, the combination of (6.7) and (A.111) implies that the infimum in (A.114) is attained with \( k_1 = k_2 = 0 \), yielding

\[
\rho_A(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) = \inf_{p_r \in \text{cl}(D \cup C), |p_r| \leq R} \left| \frac{\bar{r}(t, j) - p_r}{\bar{x}(t, j) - p_x} \right| = |\bar{r}(t, j) - \bar{x}(t, j)|. \tag{A.115}
\]

With (A.110), we conclude that \( |\bar{r}(t, j) - \bar{x}(t, j)| \leq \epsilon \) holds for all \( (t, j) \in \text{dom} q, t + j > T \) where (6.7) is satisfied, proving the theorem.

### A.5.2 Proof of Theorem 6.2

**Proof of Theorem 6.2.** To prove stability of the set \( \{q \in C_e \cup D_e | \rho(q) = 0\} \), consider the Lyapunov function \( V \), evaluated along a trajectory \( \varphi \) of (6.4), i.e., \( V(\varphi(t, j)) \). Inequality (6.12c) implies that \( V \) is non-increasing over jumps and, according to Theorem 2.2 of [140], (6.12b) guarantees

\[
\frac{d}{dt} V(\varphi(t, j)) \leq c V(\varphi(t, j)), \quad \text{for almost all} \quad (t, j) \in \text{dom} \varphi. \tag{A.116}
\]

Integration of this expression with respect to time \( t \) and using (6.12c) yields

\[
V(\varphi(t, j)) \leq e^{ct} V(\varphi(0, 0)), \quad \forall (t, j) \in \text{dom} \varphi, \tag{A.117}
\]

such that the bounds in (6.12a) imply

\[
\rho(\varphi(t, j)) \leq \alpha_1^{-1}(e^{ct} \alpha_2(\rho(\varphi(0, 0)))), \quad \forall (t, j) \in \text{dom} \varphi, \tag{A.118}
\]

which, together with \( c < 0 \), directly implies \( \rho(\varphi(t, j)) \leq \alpha_1^{-1}(\alpha_2(\rho(\varphi(0, 0)))) \), \( \forall (t, j) \in \text{dom} \varphi \), proving stability with respect to \( \rho \). If, additionally, \( \text{dom} \varphi \) is unbounded in \( t \)-direction, then asymptotic convergence with respect to \( \rho \) is obtained. Since this argument holds for all trajectories \( \varphi \), global asymptotic stability with respect to \( \rho \) is obtained, thereby proving the theorem. \( \square \)
A.5.3 Proof of Theorem 6.3

Proof of Theorem 6.3. We consider the Lyapunov function $V$, evaluated along a trajectory $\varphi$ of (6.4), i.e., $V(\varphi(t,j))$, to prove stability of the set $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$. Let $s_0 = 0$, and let $s_j$, $j = 1, 2, \ldots$ denote the continuous time $t$ when $\varphi$ experiences the $j$-th jump, that is, where $(s_j, j)$ and $(s_j, j-1) \in \text{dom} \varphi$. According to Theorem 2.2 of [140], (6.13b) guarantees

\[
\frac{d}{dt}V(\varphi(t,j)) \leq cV(\varphi(t,j)), \text{ for almost all } (t,j) \in \text{dom} \varphi. \tag{A.119}
\]

Integration of this expression with respect to time $t$ over the time intervals $[s_j, s_{j+1}] \times j$, with $s_j = s_j + 1$, yields $V(\varphi(s_{j+1}, j)) \leq e^{c(s_{j+1} - s_j)}V(\varphi(s_j, j))$, $\forall (t,j) \in \text{dom} \varphi$, which holds trivially when $s_j = s_{j+1}$. Using this expression and (6.13c), we obtain

\[
V(\varphi(t,j)) \leq \mu^j e^{ct}V(\varphi(0,0)), \tag{A.120}
\]

Substituting $j < \kappa + t/\tau$, which follows from $\varphi \in S_{\text{avg}}(\tau, \kappa)$, and using $\mu \geq 1$, we obtain $V(\varphi(t,j)) \leq \mu^\kappa \mu^{t/\tau} e^{ct}V(\varphi(0,0))$, $\forall (t,j) \in \text{dom} \varphi$. Using $\mu^{t/\tau} e^{ct} = ((\mu e^{c\tau})^{1/\tau})^t$, we find $V(\varphi(t,j)) \leq \mu^\kappa ((\mu e^{c\tau})^{1/\tau})^t V(\varphi(0,0))$, $\forall (t,j) \in \text{dom} \varphi$. Combining (6.13d) and $\tau > 0$, we observe that $(\mu e^{c\tau})^{1/\tau} < 1$, such that the previous expression implies that $V$ converges to zero along trajectories. Using (6.13a) yields

\[
\rho(\varphi(t,j)) \leq \alpha_1^{-1}(\mu^\kappa ((\mu e^{c\tau})^{1/\tau})^t \alpha_2(\rho(\varphi(0,0))))), \forall (t,j) \in \text{dom} \varphi \tag{A.121}
\]

proving convergence of $\rho$ to zero. Since $\varphi$ is chosen arbitrarily, globally asymptotically stability of $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$ with respect to $\rho$ is obtained, proving the theorem. \hfill \Box

A.5.4 Proof of Theorem 6.4

Proof of Theorem 6.4. Let $V, \alpha_1, \alpha_2, c, \mu, r, u_{\text{ref}}(t)$ be as given in the formulation of the theorem and Assumption 6.2. If, additionally, requirement (i) or (ii) is satisfied, then application of Theorem 6.2 or 6.3, respectively, implies that the set $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e \mid d(r, x) = 0\}$ is globally asymptotically stable with respect to $d(r, x)$ for the dynamics (6.4). If we select $q(0,0) = \text{col}(r(0,0), x(0,0), \eta(0,0))$, then we observe that $\bar{r}(t,j)$, as defined in (6.5), represents the reference trajectory $r$ on the hybrid time domain $\text{dom} q$, since this reference trajectory satisfies Assumption 6.2 and is the unique solution of the hybrid system from initial condition $r(0,0)$. Additionally, we observe that $\bar{x}(t,j)$, as defined in (6.5), represents the closed-loop plant trajectory from the initial condition $x(0,0)$. Since the set $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e \mid d(r, x) = 0\}$ is globally asymptotically stable with respect to $d(r, x)$, the trajectories $\bar{r}$ and $\bar{x}$ satisfy all the conditions imposed in Definition 6.1 for any initial condition of the plant.
trajectory, i.e., for any $x(0,0) \in C \cup D$. Hence, the reference trajectory $r$ is a globally asymptotically stable trajectory of the closed-loop plant dynamics and the tracking problem of Problem 6.1 is solved.

**A.5.5 Proof of Theorem 6.5**

*Proof of Theorem 6.5.* In this proof, we will show that Theorem 6.4(ii) is applicable, by firstly, showing that (6.14a)-(6.14c) holds for the following Lyapunov function:

$$V(r, x, \eta) = \begin{cases} 
\frac{1}{2} (x - r + \eta)^2, & \frac{1}{2} (x - r + \eta)^2 \leq \frac{\mu}{2} d(r, x)^2 \\
\frac{1}{2} d(r, x)^2, & \frac{1}{2} (x - r + \eta)^2 \geq \frac{\mu}{2} d(r, x)^2.
\end{cases}$$

(A.122)

Subsequently, we prove that the closed-loop trajectories are unbounded in $t$-direction, and finally, we derive expressions for $\tau, \kappa$ that satisfy condition (ii) of Theorem 6.4.

First, we consider the extended hybrid system (6.4), where, in this case,

$$F_e(t, q) = \text{col}(1 + \text{u}_{\text{ref}}(t), 1 + \text{u}_{\text{ref}}(t) - \alpha(x - r + \eta), 0),$$

(A.123)

$$G_e(q) = \begin{cases} 
\text{col}(0, x, \eta), & \text{col}(r, x, \eta) \in C_e \cup D_e, \ r = 1 \\
\text{col}(r, 0, \eta), & \text{col}(r, x, \eta) \in C_e \cup D_e, \ x = 1 \\
\text{col}(r, x, \arg \min_{i \in \{-1, 0, 1\}} \tilde{V}(r, x, i)), & \text{col}(r, x, \eta) \in C_e \cup D_e,
\end{cases}$$

(A.124)

$$C_e = \{\text{col}(r, x, \eta) \in [0, 1]^2 \times \{-1, 0, 1\} | \tilde{V}(r, x, \eta) \leq \frac{\mu}{2} d(r, x)^2\},$$

(A.125)

$$D_e = \{\text{col}(r, x, \eta) \in [0, 1]^2 \times \{-1, 0, 1\} | \tilde{V}(r, x, \eta) = \frac{\mu}{2} d(r, x)^2 \vee ((\tilde{V}(r, x, \eta) \leq \frac{\mu}{2} d(r, x)^2) \land (x = 1 \lor r = 1)) \lor (\tilde{V}(r, x, \eta) \geq \frac{\mu}{2} d(r, x)^2 \land rx = 0)\}. $$

(A.126)

Using (A.122), we observe that the relation $V(r, x, \eta) \leq \frac{\mu}{2} d(r, x)^2$ is satisfied and thus also (6.14a) holds with the $\mathcal{K}_\infty$ functions $\alpha_1(d) = \frac{1}{2} d^2$ and $\alpha_2(d) = \frac{\mu}{2} d^2$.

Observe that, for $q \in \text{cl}(C_e)$, we obtain $\partial_C V(q) = \{\nabla V(q)\}$, and since $\nabla V(q) = (-x + r - \eta x - r + \eta x - r + \eta)$ we find that (6.14b) reduces to $\nabla V(q) F_e(t, q) \leq -cV(q)$, which is satisfied for $c = -2\alpha < 0$, since:

$$\nabla V(q) F_e(t, q) = -\alpha(x - r + \eta)^2 = -2\alpha V(q).$$

(A.127)

By verification of the three different jumps described in (A.124), it follows directly that (6.14c) is satisfied.

Now, we will derive lower and upper bounds for the right-hand side of (6.1a) to select $\tau, \kappa > 0$ such that $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$, and to show that the time domain of trajectories of (6.4) is unbounded in $t$-direction. Observe that $|x - r + \eta| \leq
\( \forall \frac{\alpha}{2}, \forall \text{col}(r, x, \eta) \in C_e, \) and \( \dot{x} = 1 + u_{\text{ref}}(t) - \alpha(x - r + \eta), \) with \( u_{\text{ref}}(t) = \frac{1}{2} \cos(t), \) such that \[
1 - \frac{1}{2} - |\alpha(x - r + \eta)| \leq \dot{x} \leq 1 + \frac{1}{2} + |\alpha(x - r + \eta)|
\]
and using the selection of \( \alpha \) and \( \mu, \) we obtain \( 0 < \dot{x} < 2. \) Observe that \( \dot{x} > 0 \) directly implies that trajectories of (6.1a) can only leave \( C \) by arriving at \( D \) and experiencing a jump, which, after one jump, can only be followed by flow. Hence, the time domain of \( x \) is unbounded in \( t \)-direction.

Now, we design \( \tau, \kappa \) such that trajectories \( \varphi \) of the embedded system (6.4) satisfy \( \varphi \in S_{\text{avg}}(\tau, \kappa). \) Observe that jumps in \( x \) (or jumps in \( r \)) can only occur after it increased from 0 to 1 since the last jump. Using the upper bound \( \dot{x} < 2, \) this will take at least 1/2 time units. Hence, every 1/2 time units, both \( x \) and \( r \) can jump once according to (6.1b). The controller input (6.18c) is designed such that continuous-time behaviour always decreases \( V(q) \), such that jumps in \( \eta \) only can be triggered by jumps in \( x \) or \( \eta, \) or to be more explicit, jumps of \( \eta \) will not be triggered by reaching the set where \( V(r, x, \eta) = \frac{\mu}{2} d(r, x)^2 \) when \( x \) and \( r \) are described by (6.1a). Hence, system (6.4) can exhibit at most \( \kappa = 4 \) jumps during every continuous-time interval of length 1/2, i.e. we can select \( (\tau, \kappa) = (1/8, 4) \) and obtain \( \varphi \in S_{\text{avg}}(1/8, 4). \) Since \( r \) satisfies Assumption 6.2, the time domain of \( x \) is unbounded and \( \varphi \in S_{\text{avg}}(1/8, 4), \) we directly obtain that the time domain of \( \varphi \) is unbounded in \( t \)-direction.

Evaluating (6.14d) with \( c = -2\alpha = -1 \) yields \( \mu e^{c\tau} = 1.125 e^{-1/8} \approx 0.993 < 1, \) such that Theorem 6.4\( (ii) \) proves that the global tracking problem given in Problem 6.1 is solved.

### A.5.6 Proof of Theorem 6.6

**Proof of Theorem 6.6.** This theorem is proven by application of case \((i)\) of Theorem 6.4 with the Lyapunov function candidate \( V \) defined in (6.25)-(6.27). Since we are interested in a local tracking problem, we restrict our attention to the given reference trajectory \( r \) and plant trajectories \( x \) satisfying \( V(r, x) \leq K, \) with \( K \) selected as required in the theorem. Such trajectories are described by system (6.4) when we select:

\[
F_e(t, \text{col}(r, x)) = \begin{cases} \text{col}(r_2, -g, x_2, -g - [k_p k_d](x - r)), & \text{for } V_d < V_m, \\ \text{col}(r_2, -g, x_2, +g - [k_p k_d](x + r)), & \text{for } V_d > V_m, \end{cases} \tag{A.128}
\]

\[
G_e(\text{col}(r, x)) = \begin{cases} \text{col}(-r, x), & \text{for } r \in \{0\} \times (-\infty, 0] \\ \text{col}(r, -x), & \text{for } x \in \{0\} \times (-\infty, 0], \end{cases} \tag{A.129}
\]

\[
C_e := \{ \text{col}(r, x) \in ([0, \infty) \times \mathbb{R})^2 | V(r, x) \leq K \}, \tag{A.130}
\]

\[
D_e := \{ \text{col}(r, x) \in (\{0\} \times (-\infty, 0]) \times [0, \infty) \times \mathbb{R}) \cup ([0, \infty) \times \mathbb{R} \times \{0\} \times (-\infty, 0]) | V(r, x) \leq K \}. \tag{A.131}
\]
We will apply Theorem 6.4 to prove global asymptotic stability of this system, which directly implies that the local tracking problem defined in Theorem 6.6 is solved. First, we show that the set $C_e \cup D_e$ does not contain points where $V_d(r, x) = V_m(r, x)$. This follows directly from (6.29), since, if $V_d = V_m$, then $V(r, x) = \frac{1}{2}x^TPx + \frac{1}{2}r^TPr \geq \frac{1}{2}r^TPr = V(r, 0) > K$. To find a lower bound for $V(r, x)$, observe that $V(r, x) = \min(V_d(r, x), V_m(r, x))$, with $V_d, V_m$ given in (6.26), and both $V_d$ and $V_m$ can be bounded from below using the minimum eigenvalue of $P$, which we denote with $\lambda_{\text{min}}(P)$, such that we observe $\alpha_1(d(r, x)) = \frac{1}{2}\lambda_{\text{min}}(P)d(r, x)^2 \leq V(r, x), \ \forall \ col(r, x) \in C_e \cup D_e$.

We obtain an upper bound for $V$ by separately studying the sets $\{\col(r, x) \in C_e \cup D_e \mid |x - r| < |x + r|\}, \{\col(r, x) \in C_e \cup D_e \mid |x - r| > |x + r|\}$, and $\{\col(r, x) \in C_e \cup D_e \mid |x - r| = |x + r|\}$, using the upper bounds $V_d(r, x) \leq \frac{\lambda_{\text{max}}(P)}{2}|x - r|^2$ and $V_m(r, x) \leq \frac{\lambda_{\text{max}}(P)}{2}|x + r|^2$, where $\lambda_{\text{max}}(P)$ is the maximum eigenvalue of the matrix $P$.

First, we observe that the set where $|x - r| < |x + r|$ contains the points where $x - r = 0$, and for this reason, $V_d(r, x) - V_m(r, x) < 0$ holds at these points. Since the domain $\{\col(r, x) \in C_e \cup D_e \mid |x - r| < |x + r|\}$ is connected, the function $V_d(r, x) - V_m(r, x)$ is continuous, and, additionally, $V_d(r, x) - V_m(r, x) = 0$ cannot occur, we find that $V_d(r, x) - V_m(r, x) < 0$ holds for all points in this domain, and hence, we can employ the upper bound $V(r, x) = V_d(r, x) \leq \frac{\lambda_{\text{max}}(P)}{2}d(r, x)^2$.

Via analogous reasoning, we obtain $V(r, x) = V_m(r, x) \leq \frac{\lambda_{\text{max}}(P)}{2}d(r, x)^2$ for the domain $\{\col(r, x) \in C_e \cup D_e \mid |x - r| > |x + r|\}$. Finally, in the domain where $|x - r| = |x + r|$, we observe that $V(r, x) \leq \min(V_d(r, x), V_m(r, x)) \leq \min(\frac{\lambda_{\text{max}}(P)}{2}|x - r|^2, \frac{\lambda_{\text{max}}(P)}{2}|x + r|^2)$, and since $d(r, x) = |x - r| = |x + r|$, we directly obtain $V(r, x) \leq \frac{\lambda_{\text{max}}(P)}{2}d(r, x)^2$. Therefore, in $C_e \cup D_e$ we observe that (6.14a) is satisfied with $\alpha_1(d(r, x)) = \frac{1}{2}\lambda_{\text{min}}(P)d(r, x)^2$ and $\alpha_2(d(r, x)) = \frac{1}{2}\lambda_{\text{max}}(P)d(r, x)^2$.

It can directly be observed that for the function $V$ given in (6.25), the requirement (6.14c) holds with equality for $\mu = 1$. Hence, it only remains to be shown that (6.14b) holds, and in addition, that the time domain of the trajectories in $C_e \cup D_e$ is unbounded in $t$-direction.

According to (6.28), the discontinuity of $u$ is restricted to the surface where $V_d(r, x) = V_m(r, x)$, which, as observed already, is not contained in $C_e \cup D_e$. Hence, firstly, $V$ is continuously differentiable and, secondly, $u$ is continuous for all $x \in \{x \in C \cup D \mid V(r(t, j), x) \leq K\}$ at any time $(t, j) \in \text{dom } r$, and one can write $F_e(t, q) = F_e(t, q)$.

Since $\nabla V(\col(r, x))$ is given by

\[
\nabla V(\col(r, x)) = \begin{cases}
-(x - r)^TP \begin{pmatrix} x - r \end{pmatrix}, & \text{for } V_d < V_m \\
(x + r)^TP \begin{pmatrix} x + r \end{pmatrix}, & \text{for } V_d > V_m, 
\end{cases}
\]
we obtain
\[
\nabla V(\text{col}(r, x)) F_e(t, \text{col}(r, x)) = \\
\begin{cases}
\frac{1}{2} (x - r)^T (A^T P + PA)(x - r), & \text{for } V_d < V_m \\
\frac{1}{2} (x + r)^T (A^T P + PA)(x + r), & \text{for } V_d > V_m.
\end{cases}
\]

Hence, (6.27) yields
\[
\dot{V} \leq \begin{cases}
-\frac{1}{2} c(x - r)^T P(x - r), & \text{for } V_d < V_m \\
-\frac{1}{2} c(x + r)^T P(x + r), & \text{for } V_d > V_m,
\end{cases}
\]

for all \((t, j) \in \text{dom } r\) and all \(x \in \{x \in C \cup D \mid V(r(t, j), x) \leq K\}\), which, given (6.25), directly implies \(\dot{V} \leq cV(r, x)\), such that (6.14b) holds. This implies that \(V(r, x)\) decreases during flow. Since \(F_e\) is not explicitly dependent on time, we can apply Proposition 2.4 of [69] which yields that all trajectories have an unbounded hybrid time domain. For any \((t, j) \in \text{dom } r\), the set \(\{x \in C \cup D \mid V(r(t, j), x) \leq K\}\) does not contain the origin, which follows directly from \(V(r(t, j), 0) = \frac{1}{2} r(t, j)^T Pr(t, j)\) and the choice of \(K\) as given in (6.29). Hence, after each jump in \(x\), the trajectory \(x\) has to experience flow and travel at least a distance \(\epsilon > 0\), which will take at least a continuous-time duration \(\tau > 0\), such that the time domain of \(\text{col}(r, x)\) is unbounded in \(t\)-direction.

Hence, Theorem 6.4(i) is applicable, such that the reference trajectory \(r\) is locally asymptotic stable for the dynamics (6.4). Since in the above analysis, convergence is shown for all initial conditions in \(\{x \in C \cup D \mid V(r(0, 0), x) \leq K\}\), this domain is contained in the basin of attraction of the reference trajectory \(r\), thereby proving the theorem. \(\square\)
Bibliography


Abstract

Nonsmooth and hybrid dynamical systems can represent complex behaviour occurring in engineering, physics, biology, and economy. The benefits of nonsmooth and hybrid systems are two-fold. Firstly, certain phenomena are inherently nonsmooth, such as switching control actions. Secondly, nonsmooth systems can represent dynamical behaviour at desirable abstraction levels, for example, rigid body models with impulsive contact laws can describe the motion of unilaterally constrained mechanical systems in an efficient manner. The goal of this research is to make contributions to the theory of stability and robustness for nonsmooth systems and therewith support the analysis, design and control of such systems.

The contributions are structured in two parts.

In the first part of this thesis, the robustness of dynamical systems is studied for the class of non-differentiable and discontinuous differential equations. Small parameter variations can induce complex bifurcations in two-dimensional differential equations with non-differentiable right-hand sides. In this thesis, bifurcations are studied where multiple equilibria and periodic solutions are created from an isolated equilibrium point. We present a procedure to identify all limit sets that can be created or destroyed by a bifurcation. This procedure is an important tool to assess what behaviour can occur in planar non-differentiable dynamical models.

In addition to non-differentiable systems, a class of discontinuous systems is studied, which exhibits even more diverse behaviour. We focus on bifurcations of discontinuous systems describing mechanical systems with dry friction acting in one interface. We show that equilibria of these systems are non-isolated and form a line interval in the state space. Parameter variations can only induce bifurcations by changing the dynamics near the endpoints of this line interval. Using this insight, we present sufficient conditions for structural stability of the vector field near the equilibrium set and identify various bifurcations. These results make it possible to assess the robustness to parameter variation in the modelling and control of mechanical systems with friction. Furthermore, time-
varying force perturbations are shown to induce unexpected behaviour in this type of systems. A new type of chaotic limit set is discovered that is created by the perturbation of a discontinuous system that has a homoclinic solution. The forward dynamics of trajectories in this limit set is shown to be qualitatively different from the reversed-time dynamics. Since many trajectories spend a long transient time in the neighbourhood of this limit set, the properties of this limit set will strongly affect the behaviour of the dynamical system under study.

In the second part of this thesis, the stability of jumping trajectories is studied for hybrid systems, in which solutions display both continuous-time evolution and jumps (i.e. discrete events). This research is motivated by tracking control problems for hybrid systems, e.g. for mechanical systems with unilateral contacts and impacts.

A major complication in the analysis of the stability of trajectories in hybrid systems is that, in general, the jump times of two trajectories do not coincide. Consequently, the conventional Euclidean distance between two trajectories does not converge to zero, even if the trajectories converge to each other in between jumps, and the jump time mismatch tends to zero. We propose a novel stability formulation that overcomes this problem by comparing the jumping trajectories using a new distance function. This leads to sufficient conditions for asymptotic stability of hybrid system trajectories. In two examples, we show that these conditions support the design of tracking controllers that achieve desired behaviour in hybrid systems.

The contributions of this thesis lead to a better understanding of nonsmooth dynamics, such that the advantages of nonsmooth system models can be more effectively exploited.
Samenvatting


In het eerste deel van dit proefschrift worden de robuustheidseigenschappen onderzocht van dynamische systemen die vallen in de klasse van niet-differentieerbare en discontinue differentiaalvergelijkingen.

Kleine parameterveranderingen kunnen leiden tot complexe bifurcaties in tweedimensionale differentiaalvergelijkingen met een continu, maar niet-differentieerbaar rechterlid. In dit proefschrift worden bifurcaties bestudeerd waar meerdere evenwichtspunten en periodieke oplossingen ontstaan uit een geïsoleerd evenwichtspunt. Een procedure wordt gepresenteerd om alle limietoplossingen te vinden die kunnen ontstaan of verdwijnen in een bifurcatie. Deze procedure kan worden gebruikt om het gedrag te analyseren van niet-differentieerbare dynamische modellen in het vlak.

Naast niet-differentieerbare systemen is een klasse van discontinue systemen onderzocht, die nog complexer gedrag kunnen vertonen. Dit onderzoek is beperkt tot discontinue systemen die mechanische systemen met droge wrijving beschrijven waarbij de wrijving aangrijpt in één bewegingsrichting. In dit proefschrift is aangetoond dat evenwichtsposities in deze systemen niet geïsoleerd zijn,
maar zijn gepositioneerd in een lijninterval in de toestandsruimte. Parameter-
variabelen kunnen alleen lokale bifurcaties veroorzaken als zij het vectorveld nabij
der eindpunten van dit lijninterval kwalitatief veranderen. Aan de hand van deze
constatering zijn verschillende bifurcaties waargenomen en ook zijn voldoende
voorwaarden geformuleerd voor de structurele stabiliteit van het vectorveld in
de omgeving van de evenwichtsverzameling. Dit resultaat maakt het mogelijk
de parametergevoeligheid te bestuderen tijdens het modelleren of regelen van
mechanische systemen met droge wrijving. Bovendien wordt aangetoond dat
tijdsvariërende verstoringen onverwacht en onbekend gedrag kunnen induceren
in deze klasse van systemen. Een nieuw type limietoplossing is gevonden die cha-
otisch is en die kan worden verwacht indien een tijdsvariërende verstoring wordt
angebracht bij een discontinu systeem die een homocliene baan als oplossing
heeft. In dit proefschrift is aangetoond dat de dynamica in de limietoplossing
voorwaarts in de tijd kwalitatief verschilt van de dynamica achterwaarts in de
tijd. Aangezien veel oplossingsbanen een lange tijd in de buurt van deze chaoti-
sche limietoplossing zullen verblijven, zullen de eigenschappen hiervan het totale
gedrag van het dynamische systeem sterk beïnvloeden.

In het tweede deel van dit proefschrift is de stabiliteit van springende oploss-
singsbanen onderzocht van hybride systemen, waar oplossingsbanen zowel spron-
gen (discrete gebeurtenissen) als een continue evolutie in de tijd vertonen. De
motivatie van dit onderzoek komt van het regelprobleem waarbij hybride syste-
men (bijvoorbeeld mechanische systemen met botsingen) een vooraf opgegeven
referentiesignaal moeten volgen.

Een belangrijke complicatie in de stabiliteitsanalyse van hybride oplossings-
banen is dat in het algemeen bij twee oplossingsbanen niet exact
op hetzelfde tijdstip plaatsvinden. Hierdoor zal de Euclidische fout tussen beide
oplossingsbanen niet naar nul convergeren, zelfs als deze banen naar elkaar toe
neigen buiten de sprongtijden en het verschil in sprongtijden na nul conver-
geeft. In dit proefschrift wordt een nieuwe stabiliteitsformulering voorgesteld,
waarbij bovengenoemd probleem wordt voorkomen door de verschillende oploss-
singsbanen te vergelijken met behulp van een niet-Euclidische foutmaat. Deze
formulering leidt tot voldoende voorwaarden voor de stabiliteit van oplossings-
banen. Met twee voorbeelden wordt aangetoond dat deze voorwaarden kunnen
helpen bij het ontwerp van trajectie-volgende regelaars. Bovendien laten deze
voorbeelden zien dat gewenst gedrag wordt geïnduceerd door de ontworpen re-
gelaars.

Concluderend leiden de bijdragen van dit proefschrift tot een beter begrip
van niet-gladde dynamica, zodat niet-gladde dynamische modellen effectiever
kunnen worden toegepast.
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List of publications

Peer-reviewed journal articles


Peer-reviewed articles in conference proceedings


Curriculum vitae

Benjamin Biemond was born on July 16, 1985 in Rotterdam, the Netherlands. In 2003, he completed his secondary education at the Norbertuscollege in Roosendaal. Subsequently, he studied Mechanical Engineering at the Eindhoven University of Technology, the Netherlands. As part of this curriculum, he visited Celso Grebogi during an internship at the University of Aberdeen in Scotland, where he studied the onset of chaotic advection in open flows. In 2009, he obtained his Master’s degree (cum laude) in Mechanical Engineering on the thesis entitled “Bifurcations in planar nonsmooth systems,” that was written under the supervision of Nathan van de Wouw and Henk Nijmeijer.

In 2009, Benjamin started a PhD project at the Dynamics and Control group of Henk Nijmeijer at the Eindhoven University of Technology. His research was focussed on dynamics, bifurcations and control of nonsmooth systems. His research was part of the NWO project “Stability, bifurcations and stabilisation of invariant sets in differential inclusions”, that was a collaborative research effort with the Dynamical Systems group of Henk Broer at the Department of Mathematics of the University of Groningen, the Netherlands. In 2010, he visited Celso Grebogi and Alessandro de Moura at the Institute of Complex Systems and Mathematical Biology at the Department of Physics of the University of Aberdeen. Furthermore, he collaborated on the tracking control problem for hybrid systems with Maurice Heemels of the Hybrid and Networked Systems group at the Eindhoven University of Technology and with Ricardo Sanfelice of the Department of Aerospace and Mechanical Engineering at the University of Arizona. The main results of his PhD research are presented in this dissertation.